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Homework No. 2

4.7.23 Statement: "The negative of any irrational number is irrational."
$\equiv$ "If a number is irrational, its negative is also irrational."

a) Proof by Contraposition: Let r be some number, the negative n of which is rational. By the definition of *rational*, r can be expressed as the ratio of two integers, where $r = \frac{a}{b}$ for some a, b. By the definition of *negative*, the negative of r can be expressed as $n = (-1)r$, or consequently as $n = (-1)\frac{a}{b} = -\frac{a}{b}$. Because n is expressible as a ratio of two integers, by the definition of rational, n is rational. Thus, if the negative of a number is rational, the number itself must also be rational. Therefore, by contraposition, the negative of any irrational number is irrational. $\square$

b) Proof by Contradiction: Suppose not. Let r be some irrational number, but which has a rational negative. By the definition of *rational*, $-r$ must be expressible as the ratio of two integers, $\frac{a}{b}$ for some $a, b \in \mathbb{Z}$. By the definition of *negative*, the negative of such a number is $(-1)r$, or $(-1)\frac{a}{b} = -\frac{a}{b}$. By the definition of *rational*, that number is rational, which contradicts our premise. Therefore, if a number is irrational, its negative is also irrational. $\square$

4.7.25 Statement: "For every integer n, if n^2 is odd then n is odd."

a) Proof by Contraposition: Let n be an even integer. By the definition of *even*, n can be represented as twice another integer, $n = 2k$ for some $k \in \mathbb{Z}$. The square of n, then, can be rewritten $n^2 = n \cdot n = (2k) \cdot (2k) = 2(2k^2)$, where $(2k^2)$ is an integer. By the definition of *even*, if n^2 can be expressed as twice another integer, then n^2 is even. So, if an integer n is even, its square n^2 is also even. Hence, by contraposition, if n^2 is odd, then n must also be odd. $\square$

b) Proof by Contradiction: Suppose not. Let n represent some integer, the square of which is odd, but which itself is even. By the definition of *even*, n^2 must be expressible as twice another integer, or $2k$ for some $k \in \mathbb{Z}$. Then, n^2 can be rewritten as $(2k)^2 = (2k) \cdot (2k) = 4k^2 = 2(2k^2)$, where $2k^2$ is an integer. Thus, by the definition of *even*, n^2 is also even, which contradicts the premise. Therefore, by contradiction, for every integer n, if n^2 is odd, n is also odd. $\square$

4.7.26 Statement: "For all integers a, b, and c, if $a \nmid bc$, then $a \nmid b$."
Proof: Let a, b, and c be any three integers, particularly such that $a \mid b$. By the definition of *divisibility*, b must be expressible as a product of a and another integer, so that $b = ak$, for some $k \in \mathbb{Z}$. In this case, the product of b and c can be written $bc = akc$. Because (kc) is an integer, bc can be expressed as the product of a and another integer. Therefore, by the definition of *divisibility*, if $a \mid b$, then $a \mid bc$. Thus, by contraposition, if $a \nmid bc$, then $a \nmid b$. $\square$

4.7.29 Statement: "For all integers m and n, if $m+n$ is even, then m and n are both even or m and n are both odd."

Proof: Suppose this is not the case - that is, suppose there exist some integers m and n, such that $m+n$ is even, and either m is even and n is odd, or m is odd and n is even. By the definitions of *even* and *odd*, this would mean that either m or n can be written as twice some other integer, and the other as twice another integer plus one. That is, $m = 2j$ and $n = 2k+1$. This being so, $m+n = 2j+(2k+1)$, which by the transitivity of addition can be rewritten $(2j+2k)+1 = 2(j+k)+1$, where $(j+k)$ is an integer. By the definition of *odd*, then, $m+n$ is odd, which contradicts the stated premise. Hence, if the sum of integers m and n is even, then either both m and n are even, or they are both odd. □

4.8.6 Statement: "$6-7\sqrt{2}$ is irrational."

Proof: Assume this is not so, and that $6-7\sqrt{2}$ is rational. By the definition of *rational*, this means that $6-7\sqrt{2}$ can be expressed as the ratio of two integers, i.e. $6-7\sqrt{2} = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$. Then, by basic algebra, that statement can be rewritten as $\sqrt{2} = -\frac{1}{7} \cdot \left(\frac{m}{n} - 6\right) = -\frac{1}{7} \cdot \left(\frac{m-6n}{n}\right) = -\frac{m-6n}{7n}$. Because $(m-6n)$ and $(7n)$ are integers, this shows $\sqrt{2}$ to be expressible as the ratio of two integers, which by the definition of *rational* means that $\sqrt{2}$ is rational. This directly contradicts the known property that $\sqrt{2}$ is irrational; therefore, by contradiction, $6-7\sqrt{2}$ is irrational. □