I. Approximation and Non Elementary Integral

Use Trapezoidal rule and Simpson’s rule to approximate the values of the integrals below for n = 10, 20, 30, 40, 50 and 100. Please construct a table containing the result similar to Table 8.5 in the lecture slides of Chapter 8. Each group must choose the exercise according to the group number.

$$\text{VI. } \int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\, dx$$

Trapezoidal rule:

Tn= $\frac{\Delta x}{2}$[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) +.....+ 2f(x_{n-1}) + f(x_n)]

For better readability, we can group f(x_0) and f(x_n) and any function that is multiply by 2.

Tn= $\frac{\Delta x}{2}$[f(x_0) + f(x_n) +2[f(x_1) + f(x_2) + f(x_3) +.....+ f(x_{n-1})]]

Δx is the base length of each trapezoid and n is the subinterval between the lower and upper bound

$\Delta x = \frac{b-a}{n}$

x_0=a x_n=b

2[f(x_1) + f(x_2) + f(x_3) +.....+ f(x_{n-1})] =2 $\sum_{i=1}^{n-1}$ f[a + (i * Δx)]

We can rewrite it as:

Tn= $\frac{\Delta x}{2}$[f(a) + f(b) +2 $\sum_{i=1}^{n-1}$ f[a + (i * Δx]]

Simpson's rule:

Sn= $\frac{\Delta x}{3}$[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) +.....+ 2f(x_{n-2})+ 4f(x_{n-1}) + f(x_n)]

For better readability, we can group f(x_0) and f(x_n) and any function that is multiply by 2 and 4.

Sn= $\frac{\Delta x}{3}$[f(x_0) + f(x_n) +4[f(x_1) + f(x_3) + f(x_5) +.....+ f(x_{n-1})] +2[f(x_2) + f(x_4) + f(x_6) +.....+ f(x_{n-2})]]

4[f(x_1) + f(x_3) + f(x_5) +.....+ f(x_{n-1})]= 4*Sum*[f[a+i(Δx)] ,{ i, 1, n-1, 2}]

2[f(x_2) + f(x_4) + f(x_6) +.....+ f(x_{n-2})]= 2*Sum*[f[a+i(Δx)] ,{ i,2,n-1,2}]

We can rewrite it as:

Sn= $\frac{\Delta x}{2}$[f(x_0) + f(x_n) + 4Sum[f[a+i(Δx)] ,{ i, 1, n-1, 2}]+2Sum[f[a+i(Δx)] ,{ i,2,n-1,2 }]]

Trapezoidal rule:

In[∘]:= $f[x\_] = \frac{1}{\sqrt{1 - x^2}};$

In[∘]:=
```
a = 0;
b = 0.5;
```

In[∘]:=
```
DeltaX[n_] := (b - a) / n;
```

In[∘]:= $Tn[n\_] = \frac{DeltaX[n]}{2} \left((f[a] + f[b]) + \left(2 \sum_{i=1}^{n-1} f[a + i * DeltaX[n]] \right) \right);$

In[∘]:=
```
Tn[10]
```

Out[∘]=

0.523759026413492

Simpson's rule:

In[∘]:= $Sn[n\_] := \left(\frac{DeltaX[n]}{3}\right) * (f[a] + f[b] + 4 * Sum[f[a + i * (DeltaX[n])], \{i, 1, n - 1, 2\}] + 2 * Sum[f[a + i * (DeltaX[n])], \{i, 2, n - 1, 2\}]);$

In[∘]:=
```
Sn[10]
```

Out[∘]=

0.523599265267644

Calculate the Margin of Error (Trapezoid and Simpson)

$$E_T \leq \frac{K(b-a)^3}{12\,n^2}$$

K is the maximum value of the second derivative f”(x) within the bound [a,b]

In[ ]:=
```
f2[x_] = D[f[x], {x, 2}]
```
Out[ ]=

$$\frac{3\,x^2}{\left(1-x^2\right)^{5/2}}+\frac{1}{\left(1-x^2\right)^{3/2}}$$

In[ ]:=
```
K = MaxValue[f2[x], {0 ≤ x ≤ 0.5}, x]
```
Out[ ]=
```
3.0792014213084
```
In[ ]:=
```
f2[0]
```
Out[ ]=
```
1
```
In[ ]:=
```
f2[0.5]
```
Out[ ]=
```
3.079201435678
```
In[ ]:=

$$\text{ET[n\_]} = \frac{K\,(0.5-0)^3}{12 * (n)^2};$$

In[ ]:=
```
ET[10]
```
Out[ ]=
```
0.000320750148052958
```

$$E_s \leq \frac{K(b-a)^5}{180\, n^4}$$

K is the maximum value of the fourth derivative f””(x) within the bound [a,b]

In[ ]:=
```
f4[x_] = D[f[x], {x, 4}]
```
Out[ ]=

$$\frac{105\, x^4}{\left(1-x^2\right)^{9/2}}+\frac{90\, x^2}{\left(1-x^2\right)^{7/2}}+\frac{9}{\left(1-x^2\right)^{5/2}}$$

In[ ]:=
```
M = MaxValue[f4[x], {0 ≤ x ≤ 0.5}, x];
```

In[ ]:=
```
f4[0]
```
Out[ ]=
```
9
```

In[ ]:=
```
f4[0.5]
```
Out[ ]=
```
104.008581827346
```

In[ ]:=

$$\mathrm{ES}[n\_] = \frac{M\,(0.5-0)^5}{180*(n)^4};$$

In[ ]:=
```
ES[10]
```
Out[ ]=

$1.80570448626533 \times 10^{-6}$

In[ ]:=
```
Integral = N[Integrate[f[x], {x, 0, 0.5}]]
```
Out[ ]=
```
0.523598775598293
```

Table

In[∘]:=
```
TableForm[
  Table[{n, N[Tn[n], 20], N[ET[n], 20] , N[Sn[n], 20], N[ES[n], 20]},
   {n, {10, 20, 30, 40, 50, 100}}], TableHeadings → {{Integral},
    {"n", "Trapezoid", "Et", "Simpson", "Es"}}] // N // Magnify[#, 1.1] &
```

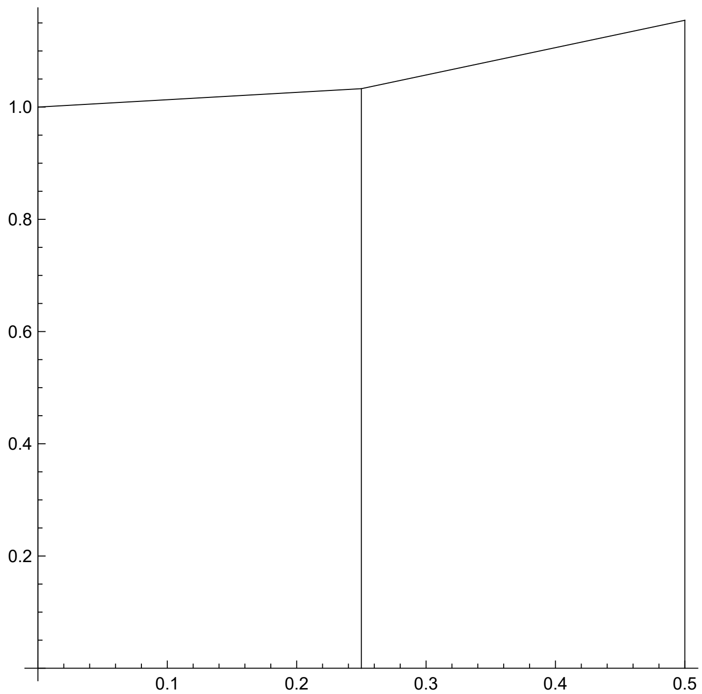
Out[∘]=

	n	Trapezoid	Et	Simpso
0.523598775598293	10.	0.523759026413492	0.000104166666666667 K	Sn[10.
	20.	0.523638861578478	0.0000260416666666667 K	Sn[20.
	30.	0.523616593511758	0.0000115740740740741 K	Sn[30.
	40.	0.523608798553339	$6.51041666666667 \times 10^{-6}$ K	Sn[40.
	50.	0.523605190401743	$4.16666666666667 \times 10^{-6}$ K	Sn[50.
	100.	0.523600379336574	$1.04166666666667 \times 10^{-6}$ K	Sn[100

Trapezoidal Rule Graphical

In[ ]:=
```
visualTrapezoidRule[func_, xmin_, xmax_, steps_, aspect_ : Automatic] := Module[
  {pts, verticals, caps}, pts = Table[{x, func[x]}, {x, Subdivide[xmin, xmax, steps]}];
  verticals = Line[{{#[[1]], 0}, #}] & /@ pts;
  caps = Line@pts;
  Graphics[{verticals, caps}, AspectRatio → aspect, Axes → True]]
Trapezoidal = visualTrapezoidRule[f[x] &, 0, 1/2, 2, 1]
```

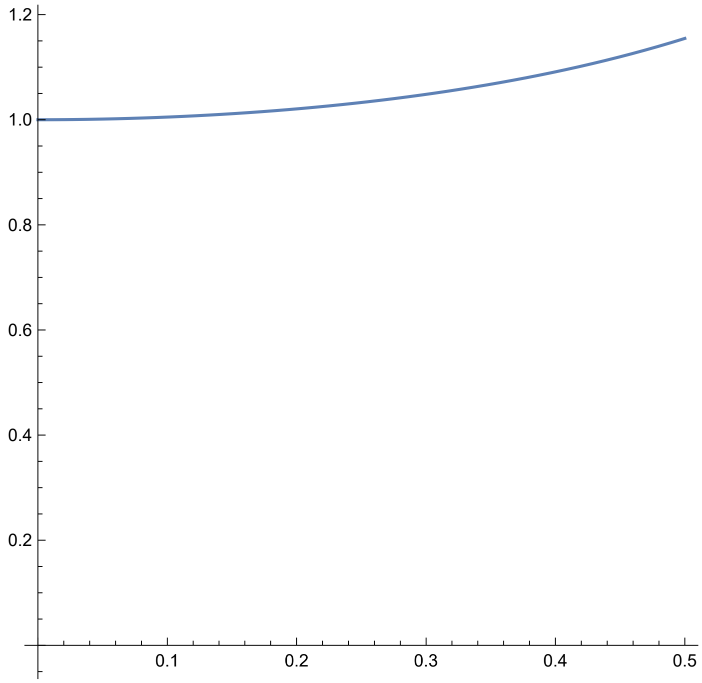
Out[ ]=



In[ ]:=

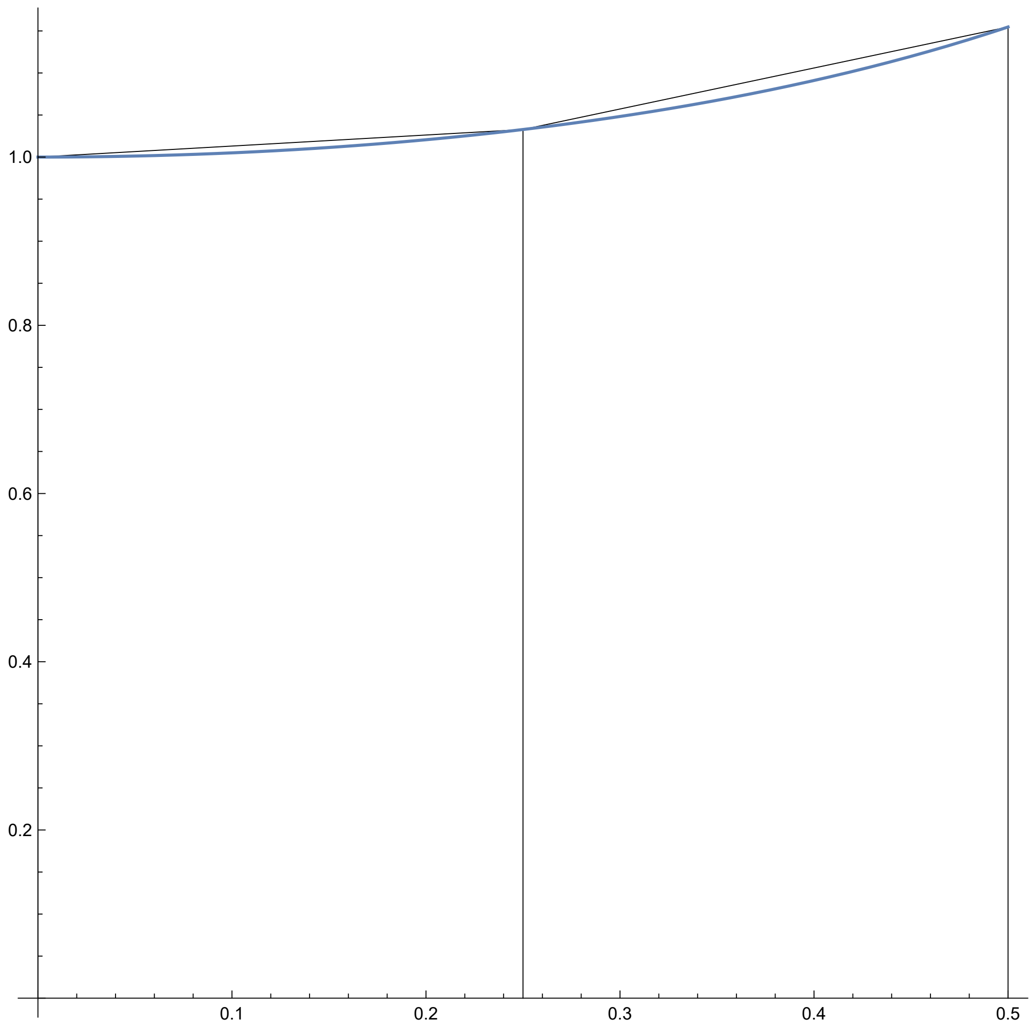
```
line = Plot[{1/Sqrt[1 - x^2]}, {x, 0, 0.5}, AspectRatio → 1, Axes → True, AxesOrigin → {0, 0}]
```

Out[ ]=



In[ ]:= `Show[Trapezoidal, line]`

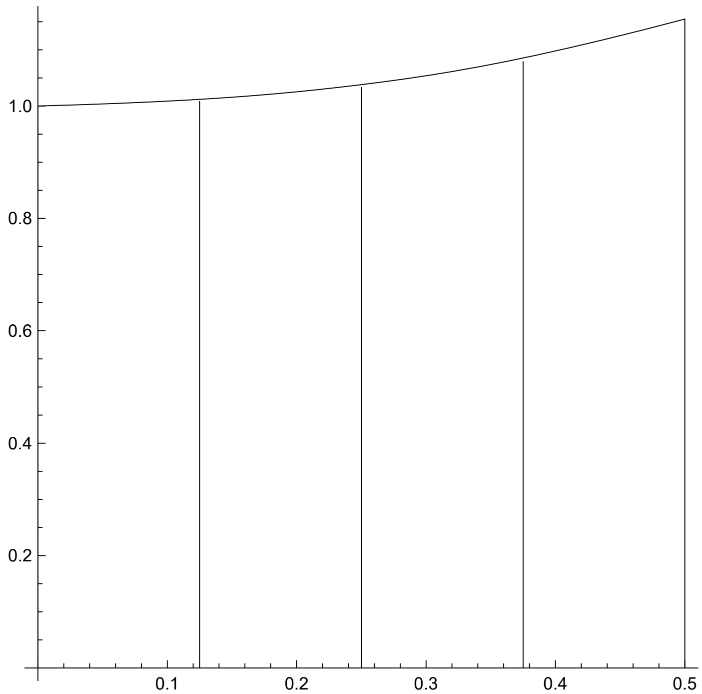
Out[ ]=



Simpson’s rule visualization

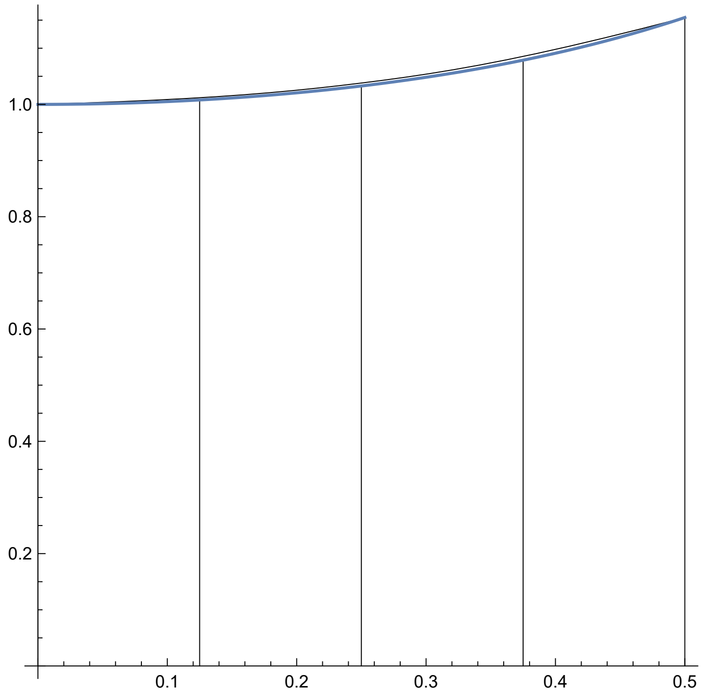
In[ ]:=
```
visualSimpsonRule[func_, xmin_, xmax_, steps_, aspect_ : Automatic] := Module[
  {pts, verticals, caps}, pts = Table[{x, func[x]}, {x, Subdivide[xmin, xmax, steps]}];
  verticals = Line[{{#[[1]], 0}, #}] & /@ pts;
  caps = BSplineCurve[pts];
  Graphics[{verticals, caps}, AspectRatio → aspect, Axes → True]]
Simpson = visualSimpsonRule[f[#] &, 0, 1 / 2, 4, 1]
```

Out[ ]=



In[ ]:=
```
Show[Simpson, line]
```

Out[ ]=



II. Problem 1: Drug assimilation

An average adult under the age of 60 assimilates a 12-h cold medicine into his or her system at a rate modeled by

$$\frac{dy}{dt}=6-\ln(2t^2-3t+3)$$

where y is measured in milligrams and t is the time in hours since the medication was taken. What amount of medicine is absorbed into a person's system over a 12-h period?

$dy=6-\ln(2t^2-3t+3)dt$

$\int 1\, dy=\int 6-\ln\left(2t^2-3t+3\right) dt$

$y=\int 6-\ln\left(2t^2-3t+3\right) dt$

Let $y(t)=6-\ln\left(2t^2-3t+3\right)$

```
In[∘]:= y[t_] = 6 - Log[3 + (2 * t^2) - (3 * t)];
```

Simpson's rule

$$Sn=\frac{\Delta y}{3}[\, y(t_0)+y(t_n)+4[y(t_1)+y(t_3)+y(t_5)+.....+y(t_{n-1})]+2[y(t_2)+y(t_4)+y(t_6)+.....+y(t_{n-2})]]$$

```
In[∘]:= DeltaY[a_, b_, n_] := (b - a) / n;
```

```
In[∘]:= Sn[f_, a_, b_, n_] :=
         DeltaY[a, b, n]/3 * (f[a] + f[b] + 4 * Sum[f[a + i * (DeltaY[a, b, n])], {i, 1, n - 1, 2}] +
           2 * Sum[f[a + i * (DeltaY[a, b, n])], {i, 2, n - 1, 2}]);
```

```
In[∘]:= N[Sn[y[#] &, 0, 12, 10]];
```

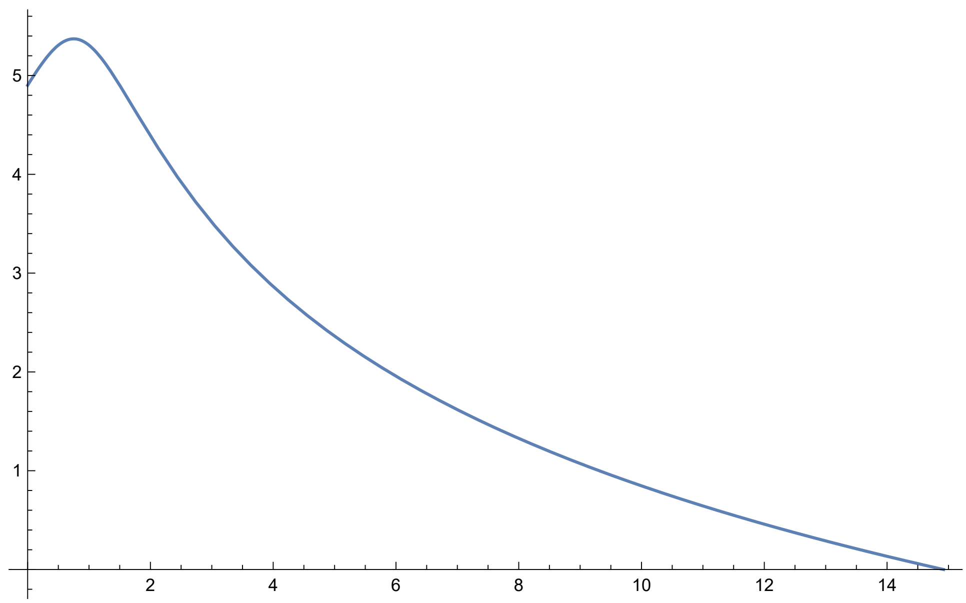
```
In[∘]:= Exact = NIntegrate[y[t], {t, 0, 12}];
```

Graph of overall medicine consumed (mg)

In[ ]:=
```
NSolve[6 - log(2 t^2 - 3 t + 3) == 0, t]
```
Out[ ]=
```
{{t → -13.4195764490816}, {t → 14.9195764490816}}
```

In[ ]:=
```
Plot[y[t], {t, 0, 14.91957645}, Axes → True]
```

Out[ ]=



In[ ]:=
```
Overall = N[Sn[y[#] &, 0, 14.91957645, 100]]
```
Out[ ]=
```
29.3292251306686
```

In[ ]:=
```
Percent[f_, a_, b_, n_] := (Sn[f, a, b, n] / Overall) * 100;
```

In[ ]:=
```
Percent[y[#] &, 0, 12, 100]
```
Out[ ]=
```
97.8034388061954
```

Table

In[ ]:=
```
TableForm[Table[{N[Sn[y[#] &, a, b, n], 20]},
   {n, {10}}, {a, {0}}, {b, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}}],
  TableHeadings → {None, {"Time  Medicine Absorbed(mg)"},
    {"1hr", "2hr", "3hr", "4hr", "5hr", "6hr", "7hr", "8hr",
     "9hr", "10hr", "11hr", "12hr"}}] // N // Magnify[#, 2] &
```

Out[ ]=

Time	Medicine Absorbed (mg)
1hr	5.23718304801414
2hr	10.1224302956036
3hr	14.0539364793833
4hr	17.2283090387805
5hr	19.8325637812797
6hr	21.9897876711085
7hr	23.7818171486917
8hr	25.2639946728299
9hr	26.4748001461995
10hr	27.4422634719643
11hr	28.1881806726499
12hr	28.7306917236096

So after 12hr we can estimated that 28.73mg of medicine is being absorbed (n=10). However, we can increase the accuracy by increasing the value of n.

In[ ]:=
```
TableForm[Table[{N[Sn[y[#] &, a, b, n], 20]},
   {n, {100}}, {a, {0}}, {b, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}}],
  TableHeadings → {None, {"Time  Medicine Absorbed(mg)"},
    {"1hr", "2hr", "3hr", "4hr", "5hr", "6hr", "7hr", "8hr",
     "9hr", "10hr", "11hr", "12hr"}}] // N // Magnify[#, 2] &
```

Out[ ]=

Time	Medicine Absorbed (mg)
1hr	5.23718038011257
2hr	10.1224093881091
3hr	14.0538536130321
4hr	17.2282996177513
5hr	19.8319781558569
6hr	21.9851016641027
7hr	23.767914154394
8hr	25.2370371396063
9hr	26.4343419755234
10hr	27.3919674361229
11hr	28.1353243056343
12hr	28.6849907530048

The table above uses n=100.

Percentage of medicine the body absorbed

In[∘]:=
```
TableForm[Table[{N[Percent[y[#] &, a, b, n], 20]}, {n, {100}},
   {a, {0}}, {b, {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14.919}}],
  TableHeadings → {None, {"Time  Percentage of medicine absorbed"},
    {"1hr", "2hr", "3hr", "4hr", "5hr", "6hr", "7hr", "8hr", "9hr",
     "10hr", "11hr", "12hr", "13hr", "14hr"}}] // N // Magnify[#, 2] &
```

Out[∘]=

Time	Percentage of medicine absorbed
1hr	17.8565248716244
2hr	34.5130474569696
3hr	47.9175755595957
4hr	58.7410664311627
5hr	67.6184865692862
6hr	74.9597085028804
7hr	81.0383296814095
8hr	86.0474050274747
9hr	90.129697793761
10hr	93.3947873293116
11hr	95.9293134417455
12hr	97.8034388061954
13hr	99.0751106263018
14hr	99.9999999175326