UNIT-V

GAME THEORY...

* **Introduction:-**

A compitative situation is called a Game. The term game represents conflict between two or more participents. A situation is termed a Game when it posses the following properties.

1. The Number of Compitaters is finite.
2. There is a conflict between the participents.
3. Each of the participents has a finite set of possible course of action.
4. The rules governing these choice are specified and known to all players.

The Game begins when each player chooses a single course of action from the list of courses available to him.

5. The outcome of the Game is effected by the choice made by all the players.
6. The outcome for all specific set of choices by all the players is known in advance and numerically defines.

* **Basic Terminologies:-**

Players:-

Each participent of a game is called a player.

Strategy:-

The strategy of a players is the pre-determine rule by which a player decides his course of action from the ~~first~~ list of courses of action during the game.

A strategy may be of two types:-

1. Pure strategy:-

It is a decision in advance of all place always choose a particular course of action.

2. Mixed strategy:-

It is a decision in advance of all place to choose a course of action for each play in accordence with some particular probability distribution.

* **Optimal strategy:-**

The course of action which maximises the profit of a player (or) minimises his loss is called an optimal strategy.

Pay-Off:- Pay-off is the outcome of playing the game. A pay-off matrix is a table showing the amount received by the player named at the left hand side after all possible place of the Game. The payment is made by the player named at the top of the table.

If a player-A has 'm' courses of action and player-B has 'n' cources then a pay-off matrix may be constructed using the following steps.

Step-1: Row designation for each matrix are the course of action available to A.

Step-2: Column designation for each matrix are the course of action available to B.

Step-3: With a two persons zero's sum games the cell entire's in B's pay-off matrix will be the Negative of the corresponding entires in A's pay-off matrix and the matrices will be as shown below.

Player-B

$$\text{player A}\quad \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & \cdots & j & \cdots & n \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

⇒ player-A pay-off matrix

Player-B

$$\text{player-A}\quad \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & \cdots & j & \cdots & n \end{array} \\ \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1j} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2j} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{ij} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mj} & \cdots & -a_{mn} \end{bmatrix} \end{array}$$

⇒ Player-B playoff matrix

* Types of Games :-

1. Two person Games and 'n' person Games :-

In two person game the players may have many possible choice open to them for each play of the game. But the no. of player remain only to hence it is called a two person game.

In case of more than two persons the game is generally called 'n' person Game.

2. Zero Sum Game :-

A zero sum game is one in which the sum of the payment to all the competitors is zero for every possible outcome of the game is in a game if the sum of the points won, equals the sum of the points loss.

3. **Two person zero sum Game:-** A Game with two players is called two person zero sum Game. If the losses of one player are equalant to the Games of the other. So, that the sum of their Net Gains is zero, this Gain is also known as Rectangular Game.

In a two person Game, suppose that the player as m courses of action and player B as n courses of action, then a pay-off matrix may be constructed by using the following steps.

step-1: Row designation for each matrix are the course of action available to A.

step-2: Column designation for each matrix are the course of action available to B.

Step-3: Cell entry A_{ij} is the payment to the player-A in A's pay-off matrix when A chooses the course of action-i and B chooses the course of action-j.

Step-4: For a zero sum game the cell entry in the player-B's pay-off matrix will be negative corresponding to the cell entry A_{ij} in the player A's pay-off matrix, so that the sum of pay-off matrices for the player A and B ultimatly zero.

Player-B

$$\text{player-A}\quad \begin{array}{c c} & \begin{array}{ccccccc} 1 & 2 & \cdots & j & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

player-A payoff matrix

Player-B

$$\text{Player-A}\quad \begin{array}{c c} & \begin{array}{ccccccc} 1 & 2 & \cdots & j & \cdots & n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ i \\ \vdots \\ m \end{array} & \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1j} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2j} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{i1} & -a_{i2} & \cdots & -a_{ij} & \cdots & -a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mj} & \cdots & -a_{mn} \end{bmatrix} \end{array}$$

player-B payoff matrix.

Consider, a two person coin tossing game, each player tosses an unbiased coins simultaneously each player selects either a head (or) a tail. If the out comes matches i.e, [H;H (or) T;T] then A wins ₹4 from B otherwise B wins ₹3 from A. player A's payoff matrix is,

Player B

Player A	H	T
H	4	-3
T	-3	4

This game is a two person zero sum Game. Since, the winning of one player is taken as losses for the other.

* **The MAXIMIN – MINIMAX Principle:-** This principle is used for the selection of optimal strategies by two players.
Consider two players A & B. A is a player who wishes to maximise his gains, while player-B wishes to minimise to his losses. Since, A would like to maximise his minimum gain, The value called MAXIMIN value and the corresponding strategy is called the MAXIMIN strategy.
On the otherhand since player-B wishes to minimise his losses, a value called the MINIMAX value, which is the minimum of the maximum losses will found. The corresponding strategy is called the MINIMAX strategy.

When these two are equal [MAXIMIN value = Minimax value] The corresponding strategies are called optimal strategy. The game is said to have a saddle point. The value of the game given by the saddle point.

The selection of MAXIMIN and MINIMAX by A & B is based on the MAXIMIN – MINIMAX principle.

* **Saddle Point:-**

A saddle point is a position in the pay-off matrix where the maximum of row minima coinsides with the minimum of column maxima.

The payoff at the saddle point is called the value of the game.

MAXIMIN value is denoted by $\underline{\gamma}$, The MINIMAX value is denoted by $\overline{\gamma}$ and the value of the game is denoted by R.

* Note:-
1. A Game is said to be fail if MAXIMIN value = MINIMAX value = 0, i.e if $\overline{\gamma} = \underline{\gamma} = 0$ (or) $\underline{\gamma} = \overline{\gamma} = 0$
2. A Game is said to be strictly determinable if MAXIMIN value = MINIMAX value $\neq 0$

*5M

* **Rules for Determining a Saddle Point:-**
1. Select the minimum element of each row of the pay-off matrix.
2. Select the maximum element of each column of the pay-off matrix.
3. Take the maximum of all minimum row elements and take the minimum value among all maximum column elements. When MAXIMIN = MINIMAX then, saddle point exists.

1. Solve the game whose pay-off matrix is given by

Player-B

$$\text{Player-A}\quad \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{c} \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \end{array}$$

Sol.

$$\begin{array}{cc} & \text{Player-B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \quad \text{Row minima} \\ \text{Player-A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix} \begin{array}{c} 1 \\ -4 \\ -1 \end{array} \\ \text{Column maxima} & \begin{array}{ccc} 1 & 5 & 1 \end{array} \end{array}$$

$\text{Max(Minima)} = \text{Max}(1, -4, -1) = 1$

$\text{Min(Maxima)} = \text{Min}(1, 5, 1) = 1$

∴ Saddle point exist. The value of the game is the saddle point which is 1. The optimal strategy is the position of the saddle point and is given by (A_1, B_1)

2. Solve the game whose pay-off matrix is given below.

$$\begin{array}{cc} & \text{player-B} \\ & \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \\ \text{Player-A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix} \end{array}$$

Sol.

$$\begin{array}{cc} & \text{Player-B} \\ & \begin{array}{ccccc} B_1 & B_2 & B_3 & B_4 & B_5 \end{array} \quad \text{Row minima} \\ \text{Player-A} \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{bmatrix} -2 & 0 & 0 & 5 & 3 \\ 3 & 2 & 1 & 2 & 2 \\ -4 & -3 & 0 & -2 & 6 \\ 5 & 3 & -4 & 2 & -6 \end{bmatrix} \begin{array}{c} -2 \\ 1 \\ -4 \\ -6 \end{array} \\ \text{Column maxima} & \begin{array}{ccccc} 5 & 3 & 1 & 5 & 6 \end{array} \end{array}$$

$\text{Max(Minima)} = \text{Max}(-2, 1, -4, -6) = 1 = \underline{r}$

$\text{Min(Maxima)} = \text{Min}(5, 3, 1, 5, 6) = 1, \overline{r}$

Since, $\underline{r} = \overline{r} = 1 = r$ there exists a saddle point, value of the game is 1. The optimal strategy is given by (A_2, B_3)

3. Determine the optimal minmax strategies to each player in the following game.

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 5 & 2 & 0 & 7 \\ 5 & 6 & 4 & 8 \\ 4 & 0 & 2 & -3 \end{bmatrix} \end{array}$$

Sol.

$$\begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \quad \text{Row minima} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} -5 & 2 & 0 & 7 \\ 5 & 6 & 4 & 8 \\ 4 & 0 & 2 & -3 \end{bmatrix} \begin{array}{c} 0 \\ 4 \\ -3 \end{array} \\ \text{Column maxima} & \begin{array}{cccc} 5 & 6 & 4 & 8 \end{array} \end{array}$$

$\text{Max(Minima)} = \text{Max}(0, 4, -3) = 4$

$\text{Min(Maxima)} = \text{Min}(5, 6, 4, 8) = 4$

Since $\underline{r} = \overline{r} = 4 = r$ there exist a saddle point value of game is 4. The optimal strategy is given by (A_2, B_3)

4. Determine which of the following two persons zero sum game are strictly determinable and fail. Given the optimum strategy for each player in the case of strictly determinable games.

①

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{Row minima} \\ \hline A_1 & -5 & 2 & -5 \\ A_2 & -7 & -4 & 7 \\ \hline \text{Column maxima} & -5 & 2 & \end{array}$$

(Player-A rows, Player-B columns)

$\therefore$ Max(Minima) = Max$(-5,-7) = -5 = \underline{\gamma}$

Min(Maxima) = Min$(-5, 2) = -5 = \overline{\gamma}$

$\therefore \underline{\gamma} = \overline{\gamma} \neq 0$.

The given game is strictly determinable game. There exist a saddle point and the value of game is -5, The optimum strategy is $(A_1; B_1)$

②

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{Row minima} \\ \hline A_1 & 1 & 1 & 1 \\ A_2 & 4 & -3 & -3 \\ \hline \text{Column maxima} & 4 & 1 & \end{array}$$

(Player-A rows, Player-B columns)

$\therefore$ Max(Minima) = Max$(1,-3) = 1$

Min(Maxima) = Min$(4,1) = 1$

$\therefore \underline{\gamma} = \overline{\gamma} = 1 \neq 0$

$\therefore$ The given game is strictly deteminable. There exist a saddle point and the value of game is 1, The optimum strategy is (A_1, B_2)

5. For a game with the following pay-off matrix

$$\begin{array}{c|ccc} \text{Player-B} \backslash \text{Player-A} & A_1 & A_2 & A_3 \\ \hline B_1 & -1 & 2 & -2 \\ B_2 & 6 & 4 & -6 \end{array}$$

Sol.

$$\begin{array}{c|ccc|c} & A_1 & A_2 & A_3 & \text{Row minima} \\ \hline B_1 & -1 & 2 & -2 & -2 \\ B_2 & 6 & 4 & -6 & -6 \\ \hline \text{Column maxima} & 6 & 4 & -2 & \end{array}$$

(Player-B rows, Player-A columns)

$\therefore$ Max(Minima) = Max$(-2,-6) = -2 = \underline{\gamma}$

$\therefore$ Min(Maxima) = Min$(6, 4, -2) = -2 = \overline{\gamma}$

$\therefore \underline{\gamma} = \overline{\gamma} \neq 0$

The given game is strictly determinable. There exists a saddle point, value of the game is -2, The optimal strategy is given by (A_3, B_2)

* Games without Saddle Point (Mixed Strategies) :-

A game without saddle point canbe solved by various solution methods.

1. 2X2 games with saddle point :-

Consider 2X2 two person zero sum game without any saddle point, having the pay-off matrix for player-A as

$$\begin{array}{c|cc} \text{Player-A} \backslash \text{Player-B} & B_1 & B_2 \\ \hline A_1 & a_{11} & a_{12} \\ A_2 & a_{21} & a_{22} \end{array}$$

The optimum mixed strategies $S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$. where,

$$P_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \quad \text{and } P_1+P_2=1 \Rightarrow P_2 = 1-P_1$$

$$P_2 = \frac{a_{11}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} \quad \text{and } q_1+q_2=1, \; q_2 = 1-q_1$$

$$q_2 = \frac{a_{11}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$\Rightarrow$ value of the game v or $\gamma = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

1. Solve the following pay-off matrix, also determine the optimal strategies and value of the game.

Player – B

$$\text{Player-A}\quad\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

Sol. It is clear that the pay-off matrix doesn't posses any saddle point. The players will use mixed strategies. The optimum mixed strategy for the players are,

$$S_A = \begin{pmatrix} A_1 & A_2 \\ P_1 & P_2 \end{pmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$P_1 + P_2 = 1 \qquad q_1 + q_2 = 1$$

$$\Rightarrow P_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4-3}{(5+4)-(1+3)} = \frac{1}{5}$$

$$P_2 = 1 - P_1 = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\Rightarrow q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4-1}{(5+4)-(1+3)} = \frac{3}{5}$$

$$q_2 = 1 - q_1 = 1 - \frac{3}{5} = \frac{2}{5}$$

$\therefore$ the optimum mixed strategies are $S_A = \begin{pmatrix} A_1 & A_2 \\ 1/5 & 4/5 \end{pmatrix}$ & $S_B = \begin{bmatrix} B_1 & B_2 \\ 3/5 & 2/5 \end{bmatrix}$

The value of the game $(\gamma) = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \dfrac{(5)(4)-(1)(3)}{(5+4)-(1+3)}$

$$\gamma = \frac{17}{5}$$

2. Solve the following game and determine its value.

P-B

$$\text{P-A}\quad\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{c} \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array}$$

Sol. It is clear that the pay-off matrix does not posses any saddle point. The players will ~~be~~ use mixed strategies. The optimum mixed strategy for the players are,

$$S_A = \begin{pmatrix} A_1 & A_2 \\ P_1 & P_2 \end{pmatrix} \text{ and } S_B = \begin{pmatrix} B_1 & B_2 \\ q_1 & q_2 \end{pmatrix}$$

$$P_1 + P_2 = 1 \qquad q_1 + q_2 = 1$$

$$\Rightarrow P_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4+4}{(4+4)-(-4-4)} = \frac{8}{8-(-8)} = \frac{8}{8+8} = \frac{\cancel{8}^{1}}{\cancel{16}_{2}} = \frac{1}{2}$$

$$P_2 = 1 - P_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{4+4}{(4+4)-(-4-4)} = \frac{\cancel{8}^{1}}{\cancel{16}_{2}} = \frac{1}{2}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$\therefore$ The optimum mixed strategies are $S_A = \begin{bmatrix} A_1 & A_2 \\ 1/2 & 1/2 \end{bmatrix}$ & $S_B = \begin{bmatrix} B_1 & B_2 \\ 1/2 & 1/2 \end{bmatrix}$

The value of the game $(\gamma) = \dfrac{(a_{11}a_{22})-(a_{12}a_{21})}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \dfrac{(4)(4)-(-4)(-4)}{(4+4)-(-4-4)} = 0$

3. In a game of matching points with two players, suppose A wins 1 unit of value when there are two heads, wins nothing when there are one head and one tail and last $\frac{1}{2}$ unit of values when there are one head and one tail. Determine the pay-off matrix, the best strategies for each player and the value of the game to A.

Sol. The pay-off matrix for given problem is

$$\begin{array}{cc} & P\text{-}B \\ & \begin{array}{cc} H & T \end{array} \\ A \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 1 & -1/2 \\ -1/2 & 0 \end{bmatrix} \end{array}$$

It is clear that the pay-off matrix does not posses any saddle point. The players will use mixed strategies. The optimum mixed strategy for the players are.

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$P_1 + P_2 = 1 \qquad q_1 + q_2 = 1$$

$$\Rightarrow P_1 = \frac{a_{22} - a_{21}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{0-(-1/2)}{(1+0)-(-1/2-1/2)} = \frac{1/2}{1+1} = \frac{1/2}{2} = \frac{1}{4}$$

$$P_2 = 1 - P_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\Rightarrow q_1 = \frac{a_{22} - a_{12}}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \frac{0-(-1/2)}{(1+0)-(-1/2-1/2)} = \frac{1/2}{1+1} = \frac{1/2}{2} = \frac{1}{4}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\therefore S_A = \begin{bmatrix} A_1 & A_2 \\ 1/4 & 3/4 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ 1/4 & 3/4 \end{bmatrix}$$

then, the value of game $(\sigma) = \dfrac{(a_{11}a_{22}) - (a_{12}a_{21})}{(a_{11}+a_{22}) - (a_{12}+a_{21})} = \dfrac{(1)(0) - (-1/2)(-1/2)}{(1+0) - (-1/2 - 1/2)}$

$$\sigma = \frac{0 - 1/4}{2} = \frac{-1}{4}\left(\frac{1}{2}\right) = \frac{-1}{8}$$

4. Two players A and B match points if the coins match then A wins 2 units of value if coins do not match, the B wins 2 units of value. Determine the optimum strategies for the players and find the value of the game.

Sol. The pay-off matrix for given problem is

$$\begin{array}{cc} & P\text{-}B \\ A & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array}$$

then, it is clear that the pay-off matrix doesnot posses any saddle point. The players will use mixed strategies. The optimum mixed strategies for the players are,

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$\Rightarrow P_1 = \frac{2+2}{(2+2)-(-2-2)} = \frac{4}{4+4} = \frac{\cancel{4}^1}{\cancel{8}_2} = \frac{1}{2} \text{ and } P_2 = 1 - P_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow q_1 = \frac{2+2}{(2+2)-(-2-2)} = \frac{4}{4+4} = \frac{\cancel{4}^1}{\cancel{8}_2} = \frac{1}{2} \text{ and } q_2 = 1 - q_1 = 1 - \frac{1}{2} = \frac{1}{2}$$

then, the value of game $(\gamma) = \frac{(2)(2)-(-2)(-2)}{(2+2)-(-2-2)} = \frac{\cancel{4}-\cancel{4}}{8} = 0$

5. $\begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix}$

Sol. then, it is clear that the pay-off matrix doesnot posses any saddle point. The players will use mixed strategies. The optimum mixed strategy for the players are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$$P_1+P_2=1 \qquad q_1+q_2=1$$

$\Rightarrow P_1 = \frac{3+3}{(6+3)-(-3-3)} = \frac{6}{9+6} = \frac{\cancel{6}2}{\cancel{15}5} = \frac{2}{5}$ and $P_2 = 1-P_1 = 1-\frac{2}{5} = \frac{3}{5}$

$\Rightarrow q_1 = \frac{3+3}{(6+3)-(-3-3)} = \frac{6}{9+6} = \frac{2}{5}$ and $q_2 = 1-q_1 = 1-\frac{2}{5} = \frac{3}{5}$

$\therefore S_A = \begin{bmatrix} A_1 & A_2 \\ 2/5 & 3/5 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_1 & B_2 \\ 2/5 & 3/5 \end{bmatrix}$

then, the value of game $(\gamma) = \frac{(6)(3)-((-3)(-3))}{(6+3)-(-3-3)} = \frac{18-9}{9+6} = \frac{\cancel{9}3}{\cancel{15}5} = \frac{3}{5}$

6. $\begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}$

Sol. then, it is clear that the pay-off matrix doesnot posses any saddle point. The players will use mixed strategies. The optimum mixed strategy for the players are,

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} \text{ and } S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$$

$\Rightarrow P_1 = \frac{1-4}{(2+1)-(4+5)} = \frac{-3}{3-9} = \frac{+\cancel{3}}{+\cancel{6}2} = \frac{1}{2}$ and $P_2 = 1-P_1 = 1-\frac{1}{2} = \frac{1}{2}$

$\Rightarrow q_1 = \frac{1-5}{(2+1)-(4+5)} = \frac{-4}{3-9} = \frac{+\cancel{4}2}{+\cancel{6}3} = \frac{2}{3}$ and $q_2 = 1-q_1 = 1-\frac{2}{3} = \frac{1}{3}$

then, the value of game $(\gamma) = \frac{(2)(1)-(5)(4)}{(2+1)-(5+4)} = \frac{2-20}{3-9} = \frac{+\cancel{18}3}{+\cancel{6}1} = 3$

* Graphical Method for 2xn (or) mx2 Games :-

Consider a $2 \times n$ game

$$\begin{array}{cc} & \text{Player-B} \\ & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \text{Player-A} \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{bmatrix} \end{array}$$

Let the mixed strategy of player-A is given by, $S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$ such that $P_1+P_2=1$.

Now, for each of the pure strategies available to B, expected play-off for player-A would be as follows.

B's pure move	A's Expected pay-off $E(P)$
B_1	$E_1(P) = a_{11}P_1 + a_{21}P_2$
B_2	$E_2(P) = a_{12}P_1 + a_{22}P_2$
⋮	⋮
B_n	$E_n(P) = a_{1n}P_1 + a_{2n}P_2$

The player-B would like to choose that pure move B_j against S_A for which $E_j(P)$ expected pay-off for A is minimum for $j = 1, 2 --- n$

Let us denote this minimum expected pay-off for A as

$$\gamma = Min[E_j(P)] ; j = 1, 2 --- n$$

The objective of player-A is to select P_1 & P_2 in such a that 'γ' is as large as possible. This may be done by plotting the straight lines.

$$E_j(P) = a_{1j}P_1 + a_{2j}P_2$$
$$= a_{1j}P_1 + a_{2j}(1-P_1)$$
$$= P_1(a_{1j} - a_{2j}) + a_{2j} \quad ; j = 1, 2 -- n$$ as linear function of P_1

The highest point on the lower boundary of these lines will give the maximum value among the minimum expected pay-off's on the lower boundary (lower envelop) as well as the optimum value of probability P_1 and P_2.

Now the 2 strategies of player-B corresponding to the lines that passes through the MAXMIN point can be determined. It helps in reducing the size of the game.

Similarly, we can treat $m \times 2$ games in the same way and get the MINIMAX point which will be the lowest point on the upper boundary (upper envelop).

1. Solve the following by using Graphical method.

Player-B

$$\text{Player-A} \quad \begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ A_1 & 1 & 3 & 11 \\ A_2 & 8 & 5 & 2 \end{array}$$

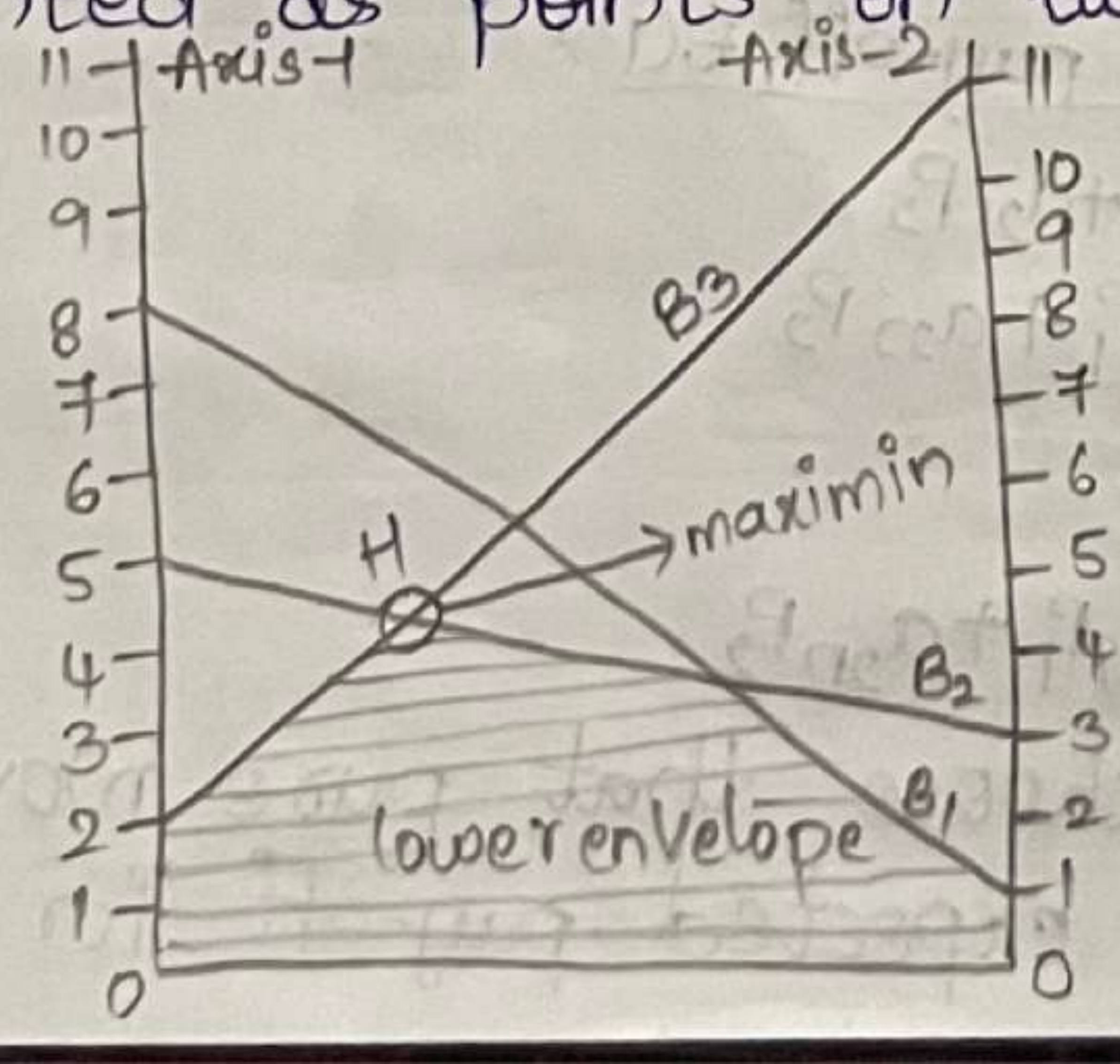
Sol. Since, the problem doesnot posses any saddle point.

Let, player-A play by the mixed strategy, $S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$ such that $P_1 + P_2 = 1 \Rightarrow P_2 = 1 - P_1$, against player-B

A's expected pay-off against B's pure move is given by

B's pure move	A's expected payoff
B_1	$E_1(P) = P_1 + 8P_2 = P_1 + 8(1-P_1) = 8 - 7P_1$
B_2	$E_2(P) = 3P_1 + 5P_2 = 5 - 2P_1$
B_3	$E_3(P) = 11P_1 + 2P_2 = 9P_1 + 2$

B's expected pay-off equation are then plotted as a functions of P_1 which shows the pay-offs of each column represented as points on two vertical axis-1 and axis-2



Now, since player-A wishes to maximize his minimum expected pay-off, we consider the highest point of intersection 'H' on the lower envelop of A's expected pay-off equation. The lines B_2 & B_3 passing through 'H' define the relevant moves that B_2 & B_3 alone need to play. The solution to the original 2X3 game & reduces to

$$\begin{array}{c c} & \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 3 & 11 \\ 5 & 2 \end{bmatrix} \end{array}$$

$$\Rightarrow P_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-5}{3+2-(11+5)} = \frac{-3}{-11} = \frac{3}{11}$$

$$P_2 = 1-P_1 = 1-\frac{3}{11} = \frac{11-3}{11} = \frac{8}{11}$$

$$\Rightarrow q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-11}{3+2-(11+5)} = \frac{+9}{+11} = \frac{9}{11}$$

$$q_2 = 1-q_1 = 1-\frac{9}{11} = \frac{2}{11}$$

The optimal mixed strategies are,

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 3/11 & 8/11 \end{bmatrix}$$

$$S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & q_1 & q_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 9/11 & 2/11 \end{bmatrix}$$

The value of game $(\gamma) = \dfrac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \dfrac{(3)(2)-(11)(5)}{3+2-(11+5)} = \dfrac{6-55}{5-16}$

$$\gamma = \frac{+49}{+11} = \frac{49}{11}$$

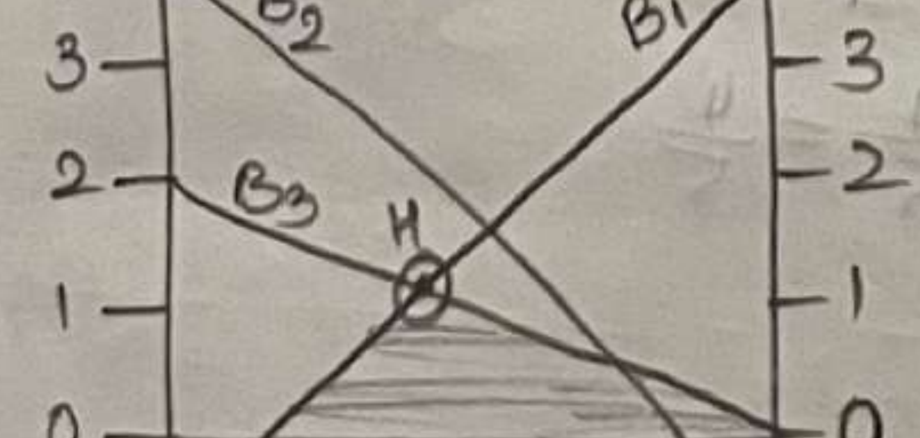
2. Solve by Graphical Method.

$$\begin{array}{c c} & \text{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & 2 \end{bmatrix} \end{array}$$

Sol. Since, the problem doesnot posses any saddle point.
Let, player-A play by the mixed strategy, $S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$
such that $P_1+P_2=1 \Rightarrow P_2 = 1-P_1$ against player-B.
A's expected pay-off againist B's pure move is given by,

B's pure move	A's expected pay off
B_1	$E_1(P) = 4P_1 - P_2 = 4P_1 - 1 + P_1 = 5P_1 - 1$
B_2	$E_2(P) = -P_1 + 4 - 4P_1 = 4 - 5P_1$
B_3	$E_3(P) = 0P_1 + 2P_2 = 2 - 2P_1$

B's expected pay-off equation are then plotted as a functions of P_1 which shows the pay-offs of each column represented as points on two vertical axis-1 and axis-2



Now, since player-A wishes to maximize his minimum expected pay-off, we consider the highest point of intersection 'H' on the lower envelop of A's expected pay-off equation. The lines B_1 and B_3 passing through 'H' define the relevant moves that B_1 & B_3 alone need to play.

The solution to the original 2X3 game & reduce to

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 4 & 0 \\ -1 & 2 \end{bmatrix} \end{array}$$

$\Rightarrow p_1 = \frac{2-(-1)}{2+4-(0-1)} = \frac{3}{7}$ and $p_2 = 1-p_1 = 1-\frac{3}{7} = \frac{4}{7}$

$\Rightarrow q_1 = \frac{2-0}{2+4-(0-1)} = \frac{2}{7}$ and $q_2 = 1-q_1 = 1-\frac{2}{7} = \frac{5}{7}$

$\therefore$ The optimal mixed strategies are, $S_A = \begin{pmatrix} A_1 & A_2 \\ p_1 & p_2 \end{pmatrix} = \begin{bmatrix} A_1 & A_2 \\ 3/7 & 4/7 \end{bmatrix}$

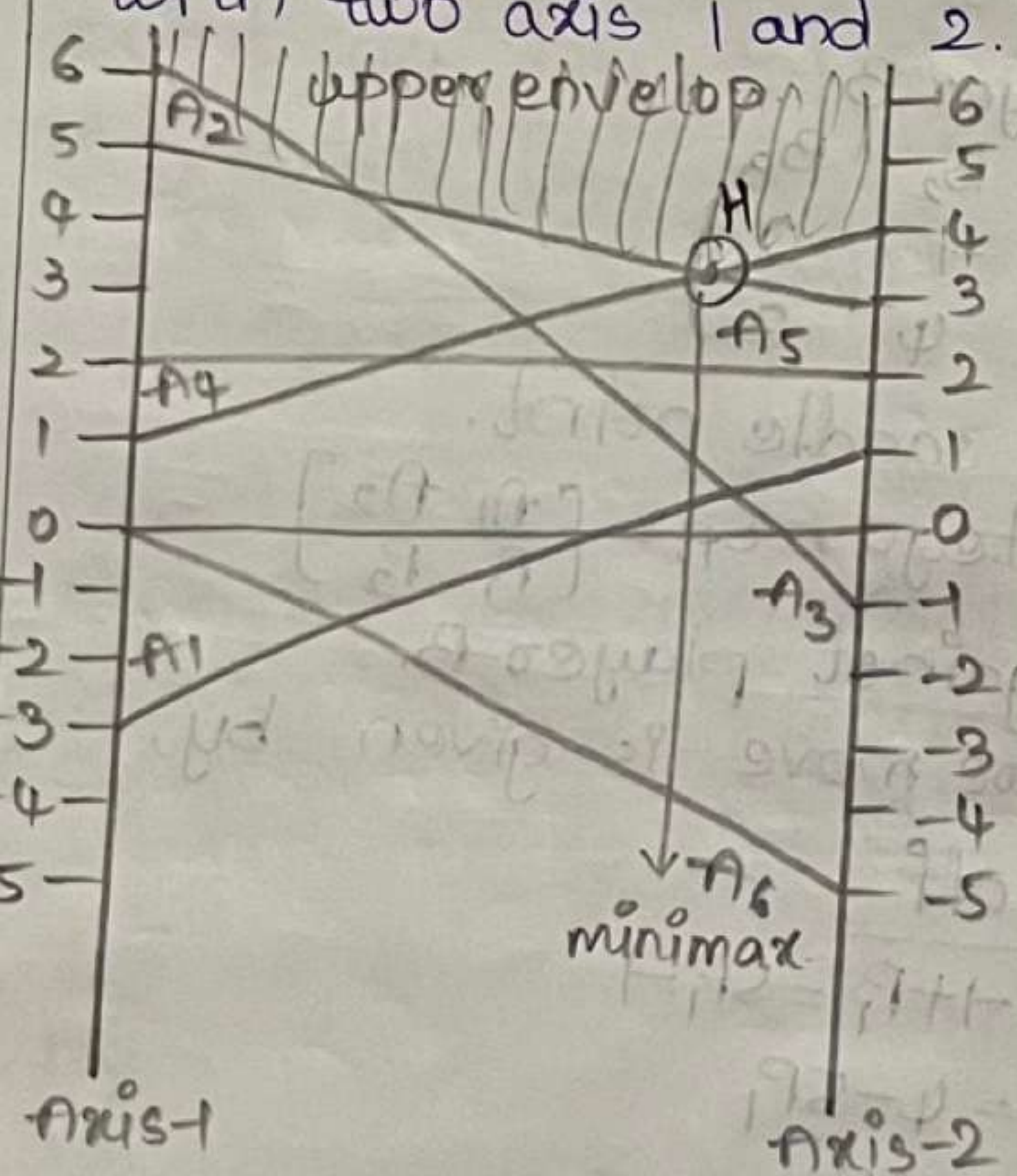
and $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & 0 & q_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 \\ 2/7 & 0 & 5/7 \end{bmatrix}$

$\therefore$ The value of game, $\gamma = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{(4)(2)-(0)(-1)}{2+4-(0-1)} = \frac{8}{7}$

$\gamma = \frac{8}{7}$

3. Solve by Graphical Method. Player-A $\begin{array}{c} \text{Player-B} \\ \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 6 \\ 4 & 1 \\ 2 & 2 \\ -5 & 0 \end{bmatrix} \end{array}$

Sol. Since, the problem does not posses any saddle point.
Let, player-B play by the mixed strategy.
$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ Such that $q_1 + q_2 = 1 \Rightarrow q_2 = 1 - q_1$ against Player-A
The expected pay-off equations are plotted in the graph with two axis 1 and 2.



Since, player-B minimise his maximum expected pay-off, we consider the lowest point of the upper boundary of B's expected pay-off equation, the point of intersection 'H' of A_2 & A_4 represents the MINIMAX expected value of the game player-B. Hence the ~~slou~~ solution of original 6×2 game is reduced to 2×2 game as

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_2 & 3 & 5 \\ A_4 & 4 & 1 \end{array}$$

$\Rightarrow p_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{1-4}{3+1-(5+4)} = \frac{-3}{-5} = \frac{3}{5}$

$p_2 = 1-p_1 = 1-\frac{3}{5} = \frac{2}{5}$

$\Rightarrow q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{1-5}{3+1-(5+4)} = \frac{-4}{-5} = \frac{4}{5}$

$q_2 = 1-q_1 = 1-\frac{4}{5} = \frac{1}{5}$

$\Rightarrow \gamma = \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{(3)(1)-(5)(4)}{3+1-(5+4)} = \frac{3-20}{-5} = \frac{-17}{-5} = \frac{17}{5}$

$\therefore$ The optimal mixed strategies are,

$S_A = \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & p_1 & 0 & p_2 & 0 & 0 \end{pmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ 0 & 3/5 & 0 & 2/5 & 0 & 0 \end{bmatrix}$

$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ 4/5 & 1/5 \end{bmatrix}$

$\therefore$ The value of game is, $\gamma = \frac{17}{5}$

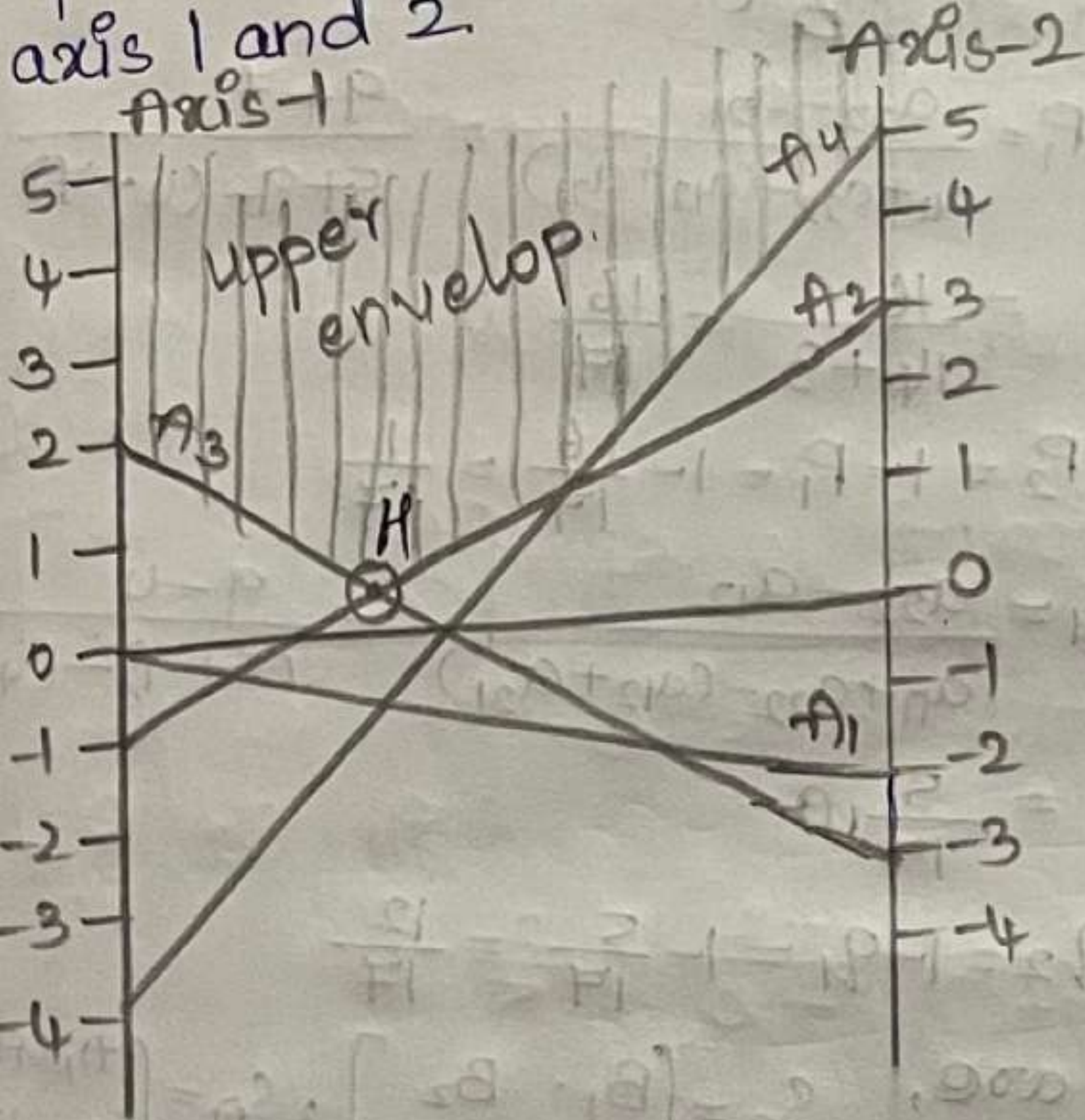
4. Then Solve By Graphical method Player-A $\begin{bmatrix} -2 & 0 \\ 3 & -1 \\ -3 & 2 \\ 5 & -4 \end{bmatrix}$ (Player-B)

Sol. Since, the problem does not posses any saddle point.

Let, player-B play by the mixed strategy.

$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ such that $q_1+q_2=1 \Rightarrow q_2 = 1-q_1$ against Player-A

The expected pay-off equations are plotted in the graph with two axis 1 and 2.



Here, the solution of original 4×2 game is reduced to 2×2 game as

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_2 & 3 & -1 \\ A_3 & -3 & 2 \end{array}$$

$\Rightarrow p_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{2+3}{3+2-(-3-1)} = \frac{5}{5+4} = \frac{5}{9}$

$p_2 = 1-p_1 = 1-\frac{5}{9} = \frac{4}{9}$

$\Rightarrow q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{2+1}{3+2-(-3-1)} = \frac{3}{5+4} = \frac{\cancel{3}}{\cancel{9}3} = \frac{1}{3}$

$q_2 = 1-q_1 = 1-\frac{1}{3} = \frac{2}{3}$

$\Rightarrow \gamma = \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{(3)(2)-(-3)(-1)}{3+2-(-3-1)} = \frac{6-3}{5+4} = \frac{\cancel{3}}{\cancel{9}3} = \frac{1}{3}$

$\therefore$ The optimal mixed strategies are,

$S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & p_1 & p_2 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 \\ 0 & 5/9 & 4/9 & 0 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ 1/3 & 2/3 \end{bmatrix}$

$\therefore$ The value of game is, $\gamma = \frac{1}{3}$

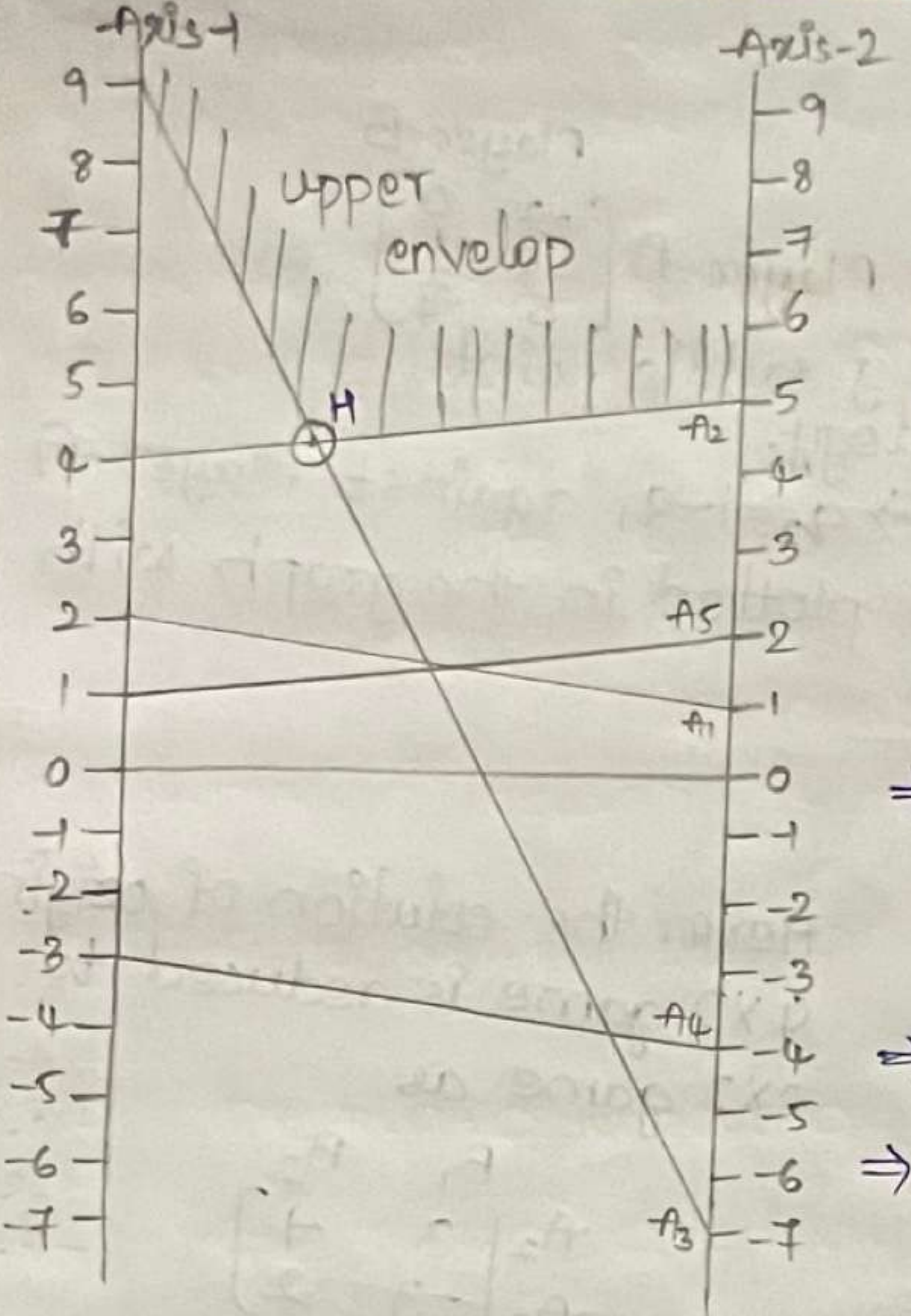
4. Solve by Graphical Method Player-A $\begin{bmatrix} 1 & 2 \\ 5 & 4 \\ -7 & 9 \\ -4 & -3 \\ 2 & 1 \end{bmatrix}$ (Player-B)

Sol. Since, the problem doesnot posses any saddle point.

Let, player-B play by the mixed strategy.

$S_B = \begin{bmatrix} B_1 & B_2 \\ q_1 & q_2 \end{bmatrix}$ such that $q_1+q_2=1 \Rightarrow q_2=1-q_1$ against Player-A

The expected pay-off equations are plotted in the graph with two axis 1 and 2



Here, the solution of original 5x2 game is reduced to 2x2 game as

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_2 & 5 & 4 \\ A_3 & -7 & 9 \end{array}$$

$$\Rightarrow P_1 = \frac{a_{22}-a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{9+7}{5+9-(4-7)}$$

$$= \frac{16}{14+3} = \frac{16}{17}$$

$$\Rightarrow P_2 = 1-P_1 = 1-\frac{16}{17} = \frac{1}{17}$$

$$\Rightarrow q_1 = \frac{a_{22}-a_{12}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{9-4}{(5+9)-(4-7)}$$

$$= \frac{5}{17}$$

$$\Rightarrow q_2 = 1-q_1 = 1-\frac{5}{17} = \frac{12}{17}$$

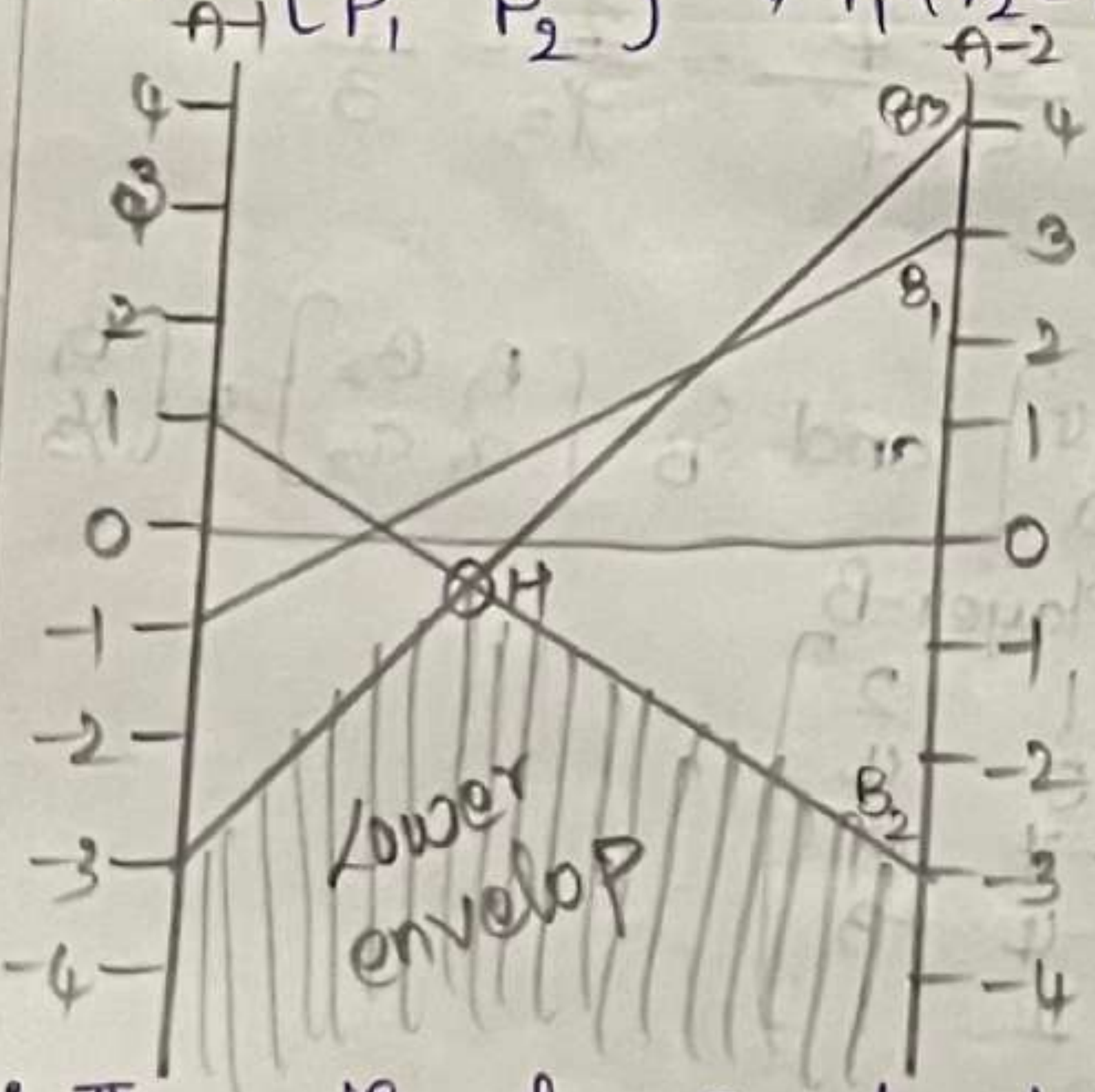
∴ The optimal mixed strategies are, $S_B = \begin{bmatrix} B_1 & B_2 \\ 5/17 & 12/17 \end{bmatrix}$, $S_A = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \\ 0 & \frac{16}{17} & \frac{1}{17} & 0 & 0 \end{bmatrix}$

∴ The value of game, $\gamma = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{(5)(9)-(4)(-7)}{(5+9)-(4-3)} = \frac{45+28}{17}$

$$\gamma = \frac{73}{17}$$

5. Solve $\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 3 & -3 & 4 \\ A_2 & -1 & 1 & -3 \end{array}$

Sol. $S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$; $P_1+P_2=1 \Rightarrow P_2 = 1-P_1$



∴ reduced game:- $\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -3 & 4 \\ A_2 & 1 & -3 \end{array}$

$$\Rightarrow P_1 = \frac{(-3)-1}{(-3-3)-(4+1)} = \frac{-4}{-6-5} = \frac{+4}{+11} = \frac{4}{11}$$

$$P_2 = 1-P_1 = 1-\frac{4}{11} = \frac{7}{11}$$

$$\Rightarrow q_1 = \frac{(-3)-(4)}{-6-5} = \frac{+7}{+11} = \frac{7}{11}$$

$$q_2 = 1-q_1 = 1-\frac{7}{11} = \frac{4}{11}$$

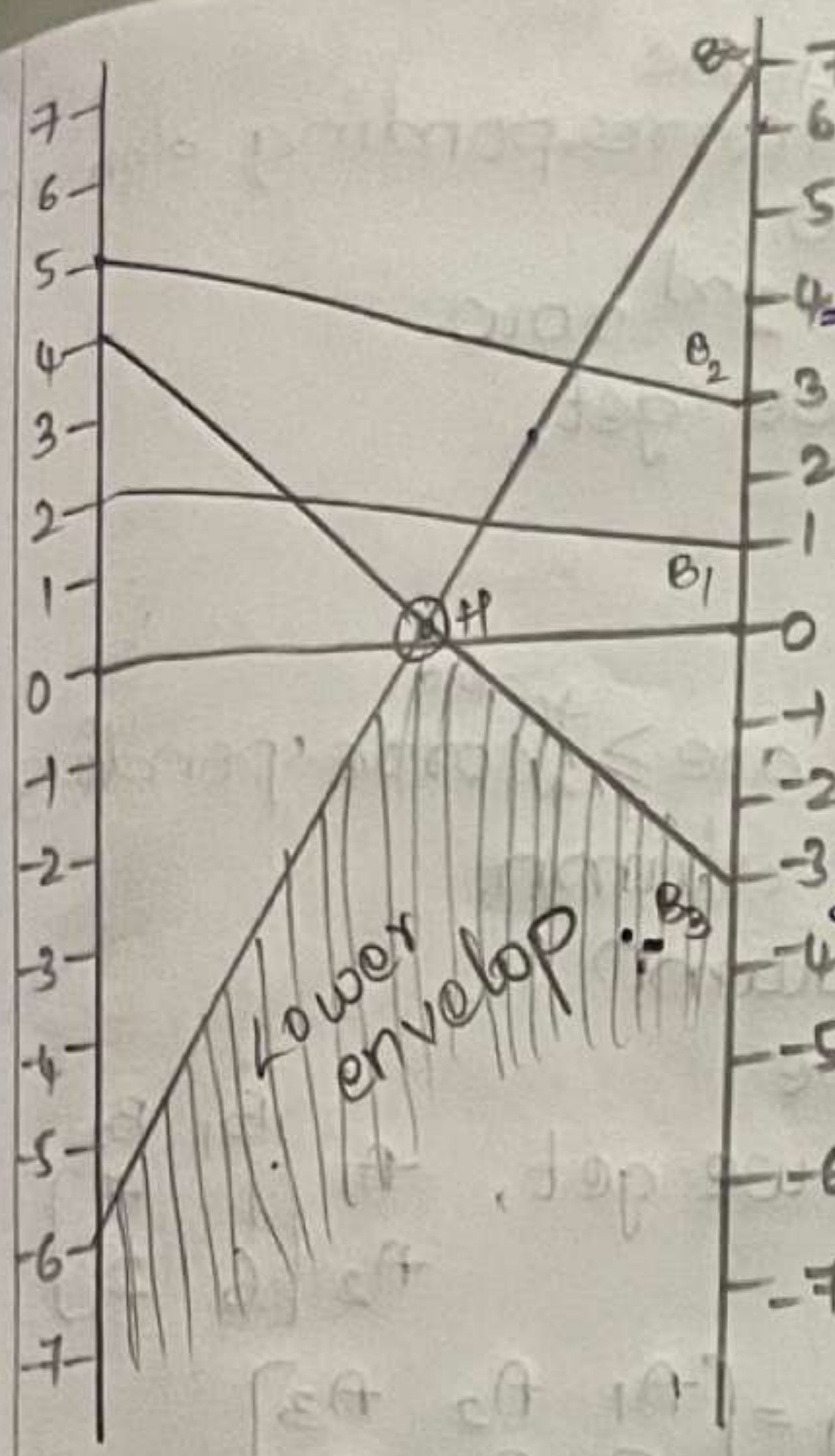
∴ The optimal mixed strategies are : $S_A = \begin{pmatrix} A_1 & A_2 \\ 4/11 & 7/11 \end{pmatrix}$ and $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 0 & 7/11 & 4/11 \end{bmatrix}$

The value of game, $\gamma = \frac{a_{11}a_{22}-a_{12}a_{21}}{a_{11}+a_{22}-(a_{12}+a_{21})} = \frac{(-3)(-3)-(4)(1)}{(-3-3)-(4+1)} = \frac{9-4}{-11}$

$$\gamma = \frac{-5}{11}$$

6. Solve $\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 1 & 3 & -3 & 7 \\ A_2 & 2 & 5 & 4 & -6 \end{array}$

Sol. $S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix}$ i.e, $P_1+P_2=1 \Rightarrow P_2 = 1-P_1$ against player-B



∴ reduced game is,

$$\begin{array}{c} \\ A_1 \\ A_2 \end{array}\begin{array}{cc} B_3 & B_4 \\ \left[\begin{matrix} -3 & 7 \\ 4 & -6 \end{matrix}\right] \end{array}$$

$\Rightarrow p_1 = \frac{-6-4}{(-3-6)-(7+4)} = \frac{-10}{-9-11} = \frac{+10}{+20} = \frac{1}{2}$; $p_2 = 1-p_1 = 1-\frac{1}{2}$

$p_2 = 1/2$

$\Rightarrow q_1 = \frac{-6-7}{(-3-6)-(7+4)} = \frac{-13}{-9-11} = \frac{+13}{+20} = \frac{13}{20}$

$q_2 = 1-q_1 = 1-\frac{13}{20} = \frac{7}{20}$

∴ The optimal mixed strategies are,

$S_A = \begin{bmatrix} A_1 & A_2 \\ 1/2 & 1/2 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & q_1 & q_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \\ 0 & 0 & \frac{13}{20} & \frac{7}{20} \end{bmatrix}$

The value of game, $v = \frac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})}$

$\therefore v = \frac{(-3)(-6)-(7)(4)}{(-3-6)-(7+4)} = \frac{18-28}{-20} = \frac{+10}{+20} = \frac{1}{2}$

* Dominance Property:-

Sometimes, it is observed that one of the pure strategies of either player is always inferior to atleast one of the remaining. The superior strategies are said to be dominate the inferior. In such cases of dominance we reduce the size of the pay-off matrix by deleting those strategies which are dominated by others. The general rules for dominance are

1. If all the elements of the row say kth row are less than or equal to the (≤) corresponding elements of any other row say rth row. then, kth row is dominated by rth row.
2. If all the elements of the column say kth column are greater than or equal to the (≥) corresponding elements of any other column say rth column. then, kth column is dominated by the rth column.
3. Dominated rows and columns may be deleted to reduce the size of the pay-off matrix as the optimal strategies will remain unaffected.
4. If some linear combinations of some rows dominate ith row, then the ith row will be deleted. Similar arguments follow for column.

1. Using the principle of Dominance solve the following game.

Player B

$$\text{Player A}\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \end{array}\begin{array}{ccc} B_1 & B_2 & B_3 \\ \left[\begin{matrix} 1 & 7 & 2 \\ 6 & 2 & 7 \\ 5 & 1 & 6 \end{matrix}\right] \end{array}\begin{array}{c} \text{Row Min} \\ 1 \\ 2 \\ 1 \end{array}$$

Sol. Column max $\;6 \quad 7 \quad 7$

Maximin = Max (1, 2, 1) = 2

Minimax = Min (6, 7, 7) = 6

∴ Maximin ≠ Minimax

There is no saddle point.

Since, all the elements of the 3rd row are $\leq$ to the corresponding elements of 2nd row.

Hence, the 3rd row is dominated by the 2nd row.

we have to delete the 3rd row then we get,

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 1 & 7 & 2 \\ A_2 & 6 & 2 & 7 \end{array}$$

Since, all the elements of the 3rd column are $\geq$ to the corresponding elements of the 1st column.

Hence, the 3rd column is dominated by the 1st column

we have to delete the 3rd column then we get,

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 1 & 7 \\ A_2 & 6 & 2 \end{array}$$

then, the mixed strategy for A is $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ P_1 & P_2 & 0 \end{bmatrix}$

the mixed strategy for B is $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ q_1 & q_2 & 0 \end{bmatrix}$

$$\Rightarrow P_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-6}{(1+2)-(7+6)} = \frac{-\cancel{4}^2}{-\cancel{10}_5} = \frac{2}{5}$$

$$P_2 = 1-P_1 = 1-\frac{2}{5} = \frac{3}{5}$$

$$\Rightarrow q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-7}{(1+2)-(7+6)} = \frac{+\cancel{5}^1}{+\cancel{10}_2} = \frac{1}{2}$$

$$q_2 = 1-q_1 = 1-\frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \text{The value of game, } \gamma = \frac{(a_{11})(a_{22})-(a_{12})(a_{21})}{(a_{11}+a_{22})-(a_{12}+a_{21})}$$

$$\gamma = \frac{(1)(2)-(7)(6)}{(1+2)-(7+6)} = \frac{2-42}{-10} = \frac{+4\cancel{0}}{+1\cancel{0}} = 4$$

$\therefore$ The optimal strategies are, $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 2/5 & 3/5 & 0 \end{bmatrix}$ and $S_B = \begin{bmatrix} B_1 & B_2 & B_3 \\ 1/2 & 1/2 & 0 \end{bmatrix}$

2. Using the principle of Dominance solve the following game.

Player B

$$\begin{array}{cc|ccc|c} & & B_1 & B_2 & B_3 & \text{row min} \\ \hline & A_1 & 3 & -2 & 4 & -2 \\ \text{player A} & A_2 & -1 & 4 & 2 & -1 \\ & A_3 & 2 & 2 & 6 & 2 \\ \hline & \text{Col max} & 3 & 4 & 6 & \end{array}$$

Maximin = Max(−2, −1, 2) = 2

Minimax = Min(3, 4, 6) = 3

$\therefore$ Maximin $\neq$ Minimax.

$\therefore$ There is no saddle point.

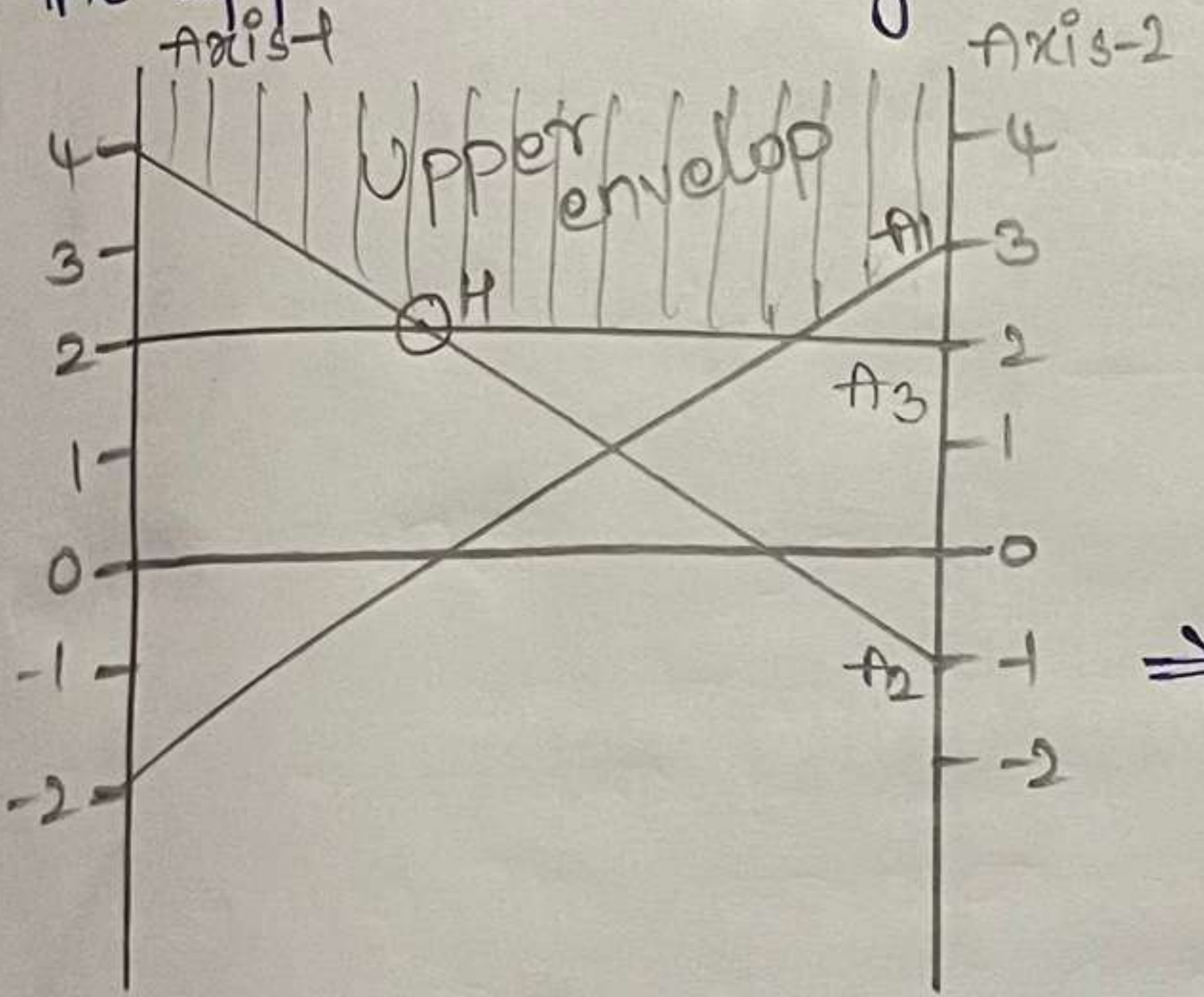
all the elements in the 3rd column are $\geq$ to the corresponding elements of the 1st column

Delete the 3rd column.

∴ The reduced pay-off matrix is,

$$\begin{array}{cc} & \text{Player } B \\ \text{Player-}A & \begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 3 & -2 \\ A_2 & -1 & 4 \\ A_3 & 2 & 2 \end{array} \end{array}$$

Since, no row dominate any another row and no column dominate the another column. the 3X2 game can now be solved by the graphical method. Since, player-B wishes to maximize his minimum loss. we find the lowest point of the upper boundary.



Hence, the solution of original 3X2 game is reduced to 2X2 game as,

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_2 & -1 & 4 \\ A_3 & 2 & 2 \end{array}$$

$$\Rightarrow P_1 = \frac{a_{22}-a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-2}{(-1+2)-(4+2)} = 0$$

$$P_2 = 1-P_1 \Rightarrow P_2 = 1$$

$$\Rightarrow q_1 = \frac{a_{22}-a_{12}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{2-4}{(-1+2)-(4+2)} = \frac{-2}{1-6} = \frac{+2}{+5} = \frac{2}{5}$$

$$q_2 = 1-q_1 = 1-\frac{2}{5} = \frac{3}{5}$$

∴ The optimal mixed strategies are, $S_A = \begin{bmatrix} A_1 & A_2 & A_3 \\ 0 & 0 & 1 \end{bmatrix}$

and $S_B = \begin{bmatrix} B_1 & B_2 \\ 2/5 & 3/5 \end{bmatrix}$

The value of game, $$v = \frac{a_{11}a_{22}-a_{12}a_{21}}{(a_{11}+a_{22})-(a_{12}+a_{21})} = \frac{(-1)(2)-(4)(2)}{(-1+2)-(4+2)} = \frac{-2-8}{1-6}$$

$$v = \frac{+\cancel{10}^{2}}{+\cancel{5}} = 2$$