TRANSPORTATION PROBLEM

* **Introduction:-** The transportation problem is one of the sub classes of LPP in which the objective is to transport various quantities of a single homogenous commodity that are initially store at various origins to different destinations in such a way that the transportations cost is minimum. To achieve this objective we must know the amount and location of available supplies and the quantities demand. In addition we must know the cost that result from transporting of one unit of commodity from various origins to various destinations.

* **Mathematical Formulations:-**

Consider a transportation problem from 'm' origins (rows) and 'n' destinations (columns). Let c_{ij} be the cost of transporting one unit of the product from i^{th} origin to j^{th} destination. 'a_i' be the quantity of commodity available at origin 'i', 'b_j' be the quantity of commodity needed at destination 'j'. x_{ij} or x_{ij} is the quantity transportated from i^{th} origin to j^{th} destination.

The above transportation problem can be stated in the following tabular form.

$i \backslash j$	1	2	----	j	----	n	Supply or cost
1	$\boxed{x_{11}}$ c_{11}	$\boxed{x_{12}}$ c_{12}	----	$\boxed{x_{ij}}$ c_{ij}	----	$\boxed{x_{1n}}$ c_{1n}	a_1
2	$\boxed{x_{21}}$ c_{21}	$\boxed{x_{22}}$ c_{22}	----	$\boxed{x_{2j}}$ c_{2j}	----	$\boxed{x_{2n}}$ c_{2n}	a_2
⋮	⋮	⋮		⋮		⋮	⋮
i	$\boxed{x_{i1}}$ c_{i1}	$\boxed{x_{i2}}$ c_{i2}	----	$\boxed{x_{ij}}$ c_{ij}	----	$\boxed{x_{in}}$ c_{in}	a_i
⋮	⋮	⋮		⋮		⋮	⋮
m	$\boxed{x_{m1}}$ c_{m1}	$\boxed{x_{m2}}$ c_{m2}	----	$\boxed{x_{mj}}$ c_{mj}	----	$\boxed{x_{mn}}$ c_{mn}	a_m
Demand	b_1	b_2	----	b_j	----	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The linear programming model representing the transportation problem is given by,

$$\text{Minimum } z = \sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} c_{ij}$$

s. to contraints, $\sum_{j=1}^{n} x_{ij} = a_i$

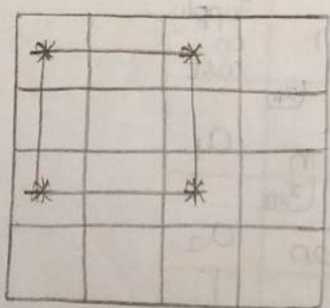
$$\sum_{i=1}^{m} x_{ij} = b_j \text{ and } x_{ij} > 0$$

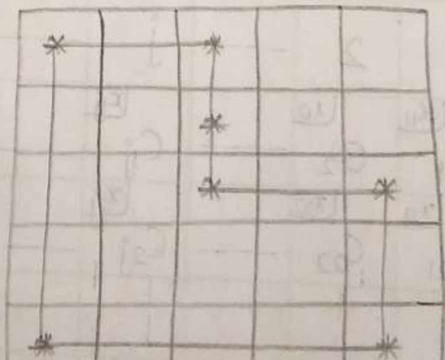
The given transportation problem is said to be balanced if $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ i.e, Row total = Column total.

* **Feasible Solution :-** Any set of non-negativity allocations which satisfies the row and column sum (Rim requirements) is called feasible Solution.
* **Basic Feasible Solution :-** A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to $m+n-1$ where 'm' is the no. of rows and 'n' is the no. of columns in a transportations problem.
* **Non-Degenerate Basic Feasible Solution :-**

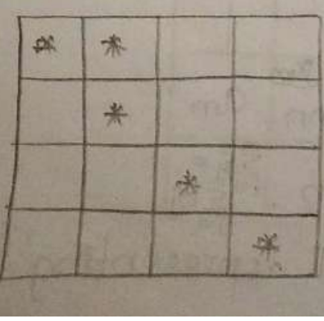
Any Feasible solution to a transportation problem contains 'm' origins and 'n' destinations is said to be non-degenerate if it contains $m+n-1$ occupied cells and each allocation is in independent position.

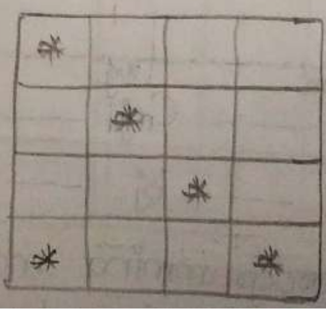
The allocations are said to be independent position if it is impossible to form a closed path. Closed path means by allowing horizontal and vertical lines and all the corner cells are occupied. The Allocations in the following table are not in independent positions.





The Allocations in the following table are in independent positions.





* **Optimal Solution**:- Optimal solution is a feasible solutions which minimize the total cost. The solution of a transportation problem can be obtained in two stages namely

1. Initial Solution.
2. Optimal Solution.

Initial solution can be obtained by using any one of the methods

1. North-west Corner rule (NWCR)
2. Least Cost Method (LCM) or Matrix Minima Method.
3. Vogel's Approximation Method (VAM).

1. North-West Corner Rule (NWCR):-

Step-1:- Starting with the cell at the upper left corner (North-west) of the transportation matrix, we allocate as much as possible so that either the capacity of the first row is exhausted (or) the destination of the requirement of the first column is satisfied.

Step-2:- If $b_1 > a_1$, move down vertically to the 2^{nd} row and make the 2^{nd} allocation of magnitude $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2,1).

If $b_1 < a_1$, move right horizontally to the 2^{nd} column and make the 2^{nd} allocation of magnitude $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1,2).

If $b_1 = a_1$ then, there is a tie for 2^{nd} allocation, to make the 2^{nd} allocation of magnitude $x_{12} = \min(a_1 - x_{11}, b_2) = 0$ in the cell (1,2) or $x_{21} = \min(a_2, b_1 - x_{11}) = 0$ in the cell (2,1)

Step-3:- Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

1. Obtain the Initial Basic Feasible solution of a transportation problem by using North-west Corner rule.

Destinations

Origins		D_1	D_2	D_3	Supply
	O_1	2	7	4	5
	O_2	3	3	1	8
	O_3	5	4	7	7
	O_4	1	6	2	14
	Demand	7	9	18	

Sol. Since $\Sigma a_i = \Sigma b_j = 34$ there exist a feasible solution.

The first allocation is made in the cell (1,1) the magnitude being $x_{11} = \min(a_1, b_1) = \text{Min}(5,7) = 5$.

The second allocation is made in the cell (2,1) the magnitude being $x_{21} = \min(a_2, b_1 - x_{11}) = \min(8,2) = 2$

The third allocation is made in the cell (2,2) the magnitude beign $x_{22} = \min(a_2 - x_{12}, b_2) = \min(6,9) = 6$

The fourth allocation is made in the cell (3,2) and is given by $x_{32} = \min(a_3, b_2 - x_{22}) = \min(7, 9-3) = 3$

The fifth allocation is made in the cell (3,3) and is given by $x_{33} = \min(a_3 - x_{32}, b_3) = \min(\underset{=4}{7-3}, 18) = 4$

The final allocation is made in the cell (4,3) and is given by $x_{43} = \min(a_4, b_3 - x_{33}) = \min(14, 18-4) = 14$

Destinations

	D_1	D_2	D_3	Supply
O_1	2 $\boxed{5}$	7	4	$\cancel{8}$ 0
O_2	3 $\boxed{2}$	3 $\boxed{6}$	1	$\cancel{8}\ \cancel{6}$ 0
O_3	5	4 $\boxed{3}$	7 $\boxed{4}$	$\cancel{7}\ \cancel{4}$ 0
O_4	1	6	2 $\boxed{14}$	$\cancel{14}$ 0
Demand	$\cancel{7}\ \cancel{2}$ 0	$\cancel{9}\ \cancel{3}$ 0	$\cancel{18}\ \cancel{14}$ 0	34

Rim Requirement Satisfied ↑

$\therefore$ The no. of non-negative allocations $= m+n-1 = 4+3-1 = 6$

Hence, the solution is non-degenerate feasible solution.

Transportation cost $= 2\times5 + 3\times2 + 3\times6 + 4\times3 + 7\times4 + 2\times14$
$= 10+6+18+12+28+28 = 102$,

2. Obtain the initial basic feasible solution to the given transportation problem by NWCR.

Destinations

Origin		D_1	D_2	D_3	D_4	Supply
	O_1	6	4	1	5	14
	O_2	8	9	2	7	16
	O_3	4	3	6	2	5
	Demand	6	10	15	4	

Sol. Since, $\Sigma a_i = \Sigma b_j = 35$, there exist a feasible solution.

The first allocation is made in the cell (1,1) the magnitude being $x_{11} = \min(a_1, b_1) = \min(6, 14) = 6$

The second allocation is made in the cell (1,2) the magnitude being ~~$x_{12} = \min(8,10) = 8$ $\min(a_1, b_2$~~

$x_{12} = \min(a_1 - x_{11}, b_2) = \min(8,10) = 8$

The third allocation is made in the cell (2,2) the magnitude being $x_{22} = (a_2, b_2 - x_{12}) = (16, 2) = 2$

The fourth allocation is made in the cell (2,3) and is given by $x_{23} = \min(a_2 - x_{22}, b_3) = (14, 15) = 14$

The fifth allocation is made in the cell (3,3) and is given by $x_{33} = \min(a_3, b_3 - x_{23}) = (5, 1) = 1$

The sixth allocation is made in the cell (3,4) and is given by $x_{34} = (a_3 - x_{33}, b_4) = (4, 4) = 4$

Destinations.

Origin	D_1	D_2	D_3	D_4	Supply
O_1	6 [6]	4 [8]	1	5	~~14~~ ~~8~~ 0
O_2	8	9 [2]	2 [14]	7	~~16~~ ~~14~~ 0
O_3	4	3	6 [1]	2 [4]	~~5~~ ~~4~~ 0
Demand	~~6~~ 0	~~10~~ ~~2~~ 0	~~15~~ ~~1~~ 0	~~4~~ 0	$\Sigma a_i = \Sigma b_j = 35$

∴ The no. of non-negative allocations $= m+n-1$

$6 = 3+4-1$

$6 = 6$ (Rim requirement satisfied)

Hence, the solution is non-degenerate feasible solution.

Transportation cost $= 6\times6 + 4\times8 + 9\times2 + 2\times14 + 6\times1 + 2\times4$

$= 36 + 32 + 18 + 28 + 6 + 8$

$= 128$

$=$ Rs. 128/-

3. Obtain the initial Basic Feasible solution to the given transportation problem by NWCR

Destinations

Origins	A	B	C	D	E	supply
P	2	11	10	3	7	4
Q	1	4	7	2	1	8
R	3	9	4	8	12	9
Demand	3	3	4	5	6	

Sol Since, $\Sigma a_i = \Sigma b_j = 21$, there exist a Feasible Solution.

The First allocation is made in the cell (1,1) the magnitude being $x_{11} = \min(a_1, b_1) = \min(4, 3) = 3$

Origin		Destinations				
	A	B	C	D	E	supply
P	2 [3]	11 [1]	10	3	7	~~4~~ ~~1~~ 0
Q	1	4 [2]	7 [4]	2 [2]	1	~~8~~ ~~6~~ ~~2~~ 0
R	3	9	4	8 [3]	12 [6]	~~9~~ ~~6~~ 0
Demand	~~3~~ 0	~~3~~ ~~2~~ 0	~~4~~ 0	~~5~~ ~~3~~ 0	~~6~~ 0	$\Sigma a_i = \Sigma b_j = 21$

$x_{12} = 1$, $x_{22} = 2$, $x_{23} = 4$, $x_{24} = 2$, $x_{34} = 3$ and $x_{35} = 6$

$\therefore$ The no. of non-Negativity allocations $= m+n-1$

$7 = 3+5-1$ $\quad\therefore$ Rim Requirement Satisfied.

$7 = 8-1$

$7 = 7$

Hence, the solution is non-generate feasible solution.

Transportation cost $= 2\times3 + 11\times1 + 4\times2 + 7\times4 + 2\times2 + 8\times3 + 12\times6$

$= 6+11+8+28+4+24+72$

$= 153,$

$\therefore$ Transportation cost is Rs. 153/-

4. Find the Initial Basic Feasible solution to the given transportation problem by NWCR.

Destination

Origin	A	B	C	Supply
O_1	1	2	6	7
O_2	0	4	2	12
O_3	3	1	5	11
Demand	10	10	10	

Sol. Since, $\Sigma a_i = \Sigma b_j = 30$, there exist a Feasible solution

	A	B	C	Supply
O_1	1 $\boxed{7}$	2	6	$\cancel{7}$ 0
O_2	0 $\boxed{3}$	4 $\boxed{9}$	2	$\cancel{12}$ $\cancel{9}$ 0
O_3	3	1 $\boxed{1}$	5 $\boxed{10}$	$\cancel{11}$ $\cancel{10}$ 0
Demand	$\cancel{10}$ $\cancel{3}$ 0	$\cancel{10}$ $\cancel{1}$ 0	$\cancel{10}$ 0	$\Sigma a_i = \Sigma b_j$ 30

$x_{11} = 7$, $x_{21} = 3$, $x_{22} = 9$, $x_{32} = 1$ and $x_{33} = 10$

$\therefore$ The no. of non-Negativity allocations $= m+n-1$

$5 = 3+3-1$

$5 = 5$ $\quad\therefore$ Rim Requirement Satisfied.

Hence, the solution is non-generate feasible Solution

Transportation cost $= 1\times7 + 0\times3 + 4\times9 + 1\times1 + 5\times10$

$= 7+0+36+1+50$

$= 44+50$

$= 94/-$

$\therefore$ Transportation cost is Rs. 94/-

5. Find the minimum transportation cost (NWCR and Modi)

Sol.

Ware house

Factory	D_1	D_2	D_3	D_4	Supply
F_1	19 (5)	30 (2)	50	10	~~7~~ ~~2~~ 0
F_2	70	30 (6)	40 (3)	60	~~9~~ ~~3~~ 0
F_3	40	8	70 (4)	20 (14)	~~18~~ ~~14~~ 0
Demand	~~5~~ 0	~~8~~ ~~6~~ 0	~~7~~ ~~4~~ 0	~~14~~ 0	$\Sigma a_i = \Sigma b_j = 34$

15

12
10
22+22
38+45×5

Since, $\Sigma a_i = \Sigma b_j = 34$, there exist a basic feasible solution to the transportation problem.

Being RIM Requirement satisfied the no. of non-negative allocations $= m+n-1 = 3+4-1 = 6$

$x_{11} = 5,\ x_{12} = 2,\ x_{22} = 6,\ x_{23} = 3,\ x_{33} = 4$ and $x_{34} = 14$

Transportation Cost $= 19\times5 + 30\times2 + 30\times6 + 40\times3 + 70\times4 + 20\times14$

$= 95 + 60 + 180 + 120 + 280 + 280$

$= 1015$ Rs/-

* Least Cost Method (LCM):- LCM

Step-1:- Determine the smallest cost in the cost matrix of the transportation table.
Let, it be 'c_{ij}' allocate $x_{ij} = \text{Min}(a_i, b_j)$ in the cell (i,j).

Step-2:- If 'x_{ij}' $= a_i$ cross of the i^{th} row of the transportation table and decrease 'b_j' by a_i. then go to step-3
If '$x_{ij} = b_j$' cross of the j^{th} column of the transportation table and decrease 'a_i' by 'b_j' then go to step-3.
If '$x_{ij} = a_i = b_j$' cross of either i^{th} row or j^{th} column but not both.

Step-3:- Repeat step-1 and step-2 for the resulting reduced transportation table untill all the rim requirements are satisfied. When ever the minimum cost is not unique make an orbitry choice among the minima.

1. Obtain the Initial Basic Feasible solution for the following transportation problem by Least Cost Method.

	A	B	C	Supply
O_1	1	2	6	7
O_2	0	4	2	12
O_3	3	1	5	11
Demand	10	10	10	

(Destinations across the top, Origin down the side)

Sol. Since, $\Sigma a_i = \Sigma b_j = 30$, the given transportation problem is balanced we get the IBFS.

By least cost Method minimum cost at $c_{21} = 0$ we allocate

$x_{21} = \min(a_2, b_1) = \min(12, 10) = 10$

And cross of the first column

	A	B	C	supply
O_1	1	2	6	7
O_2	0 [10]	4	2	~~12~~ 2
O_3	3	1	5	11
Demand	~~10~~ 0	10	10	

By reduced transportation table minimum cost is at $c_{32} = 1$

Here we allocate $x_{32} = \min(11, 10) = 10$, we cross of the 2nd column

	B	C	Supply
O_1	2	6	7
O_2	4	2	2
O_3	1 [10]	5	~~11~~ 1
Demand	~~10~~ 0	10	

From the reduced transportation table Minimum cost at $c_{23} = 2$

here we allocate $x_{23} = \min(2, 10) = 2$

we cross of the 2nd row

	C	Supply
O_1	6	7
O_2	2 [2]	~~2~~ 0
O_3	5	1
Demand	~~10~~ 8	

From the reduced transportation table minimum cost at $c_{33} = 5$

here we allocate $x_{33} = \min(1, 8) = 1$

and we cross of the 3rd row

	C	Supply
O_1	6	7
O_3	5	~~1~~ 0
Demand	~~8~~ 7	

From the reduced transportation table Minimum cost at $c_{13} = \min(7, 7) = 7$ and cross of the first row

	C	
O_1	6 [7]	~~7~~ 0
	~~7~~ 0	

∴ The resultant allocated transportation table is

	A	B	C
O_1	1	2	6 [7]
O_2	0 [10]	4	2 [12]
O_3	3	1 [10]	5 [1]

The rim requirement satisfied i.e, the no. of non-negative allocation $= m+n-1 = 3+3-1 = 5$

Hence, the solution is non-degenerate feasible solution

The allocations are $x_{21}=10$, $x_{32}=10$, $x_{23}=2$, $x_{33}=1$, $x_{13}=7$

The initial (~~temperature~~) transportation cost is $=$

$= 0\times10 + 1\times10 + 2\times2 + 5\times1 + 6\times7$

$= 0+10+4+5+42$

$= 61$ Rs/-

2. Obtain the IBFS for the following transportation problem by using Least Cost Method.

Sol.

Destinations

Origin	D_1	D_2	D_3	D_4	Supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demands	4	6	8	6	

Sol. Since, $\Sigma a_i = \Sigma b_j = 24$ the given transportation problem is balanced, we get IBFS

	D_1	D_2	D_3	D_4	Supply
O_1	1	2 (6)	3	4	~~6~~ 0
O_2	4	3	2 (2)	0 (6)	~~8~~ ~~2~~ 0
O_3	0 (4)	2 (0)	2 (6)	1	~~10~~ ~~6~~ 0
Demand	~~4~~ 0	~~6~~ 0	~~8~~ ~~6~~ 0	~~6~~ 0	24

The rim requirement is satisfied i.e, the no. of non-negative allocation $= m+n-1 = 3+4-1 = 6$

Hence, the solution is non-degenerate feasible solution

The allocations are, $x_{12}=6$, $x_{23}=2$, $x_{24}=6$, $x_{31}=4$, $x_{32}=0$, $x_{33}=6$

The Transportation cost $= 2\times6 + 0\times4 + 2\times0 + 2\times6 + 2\times2 + 0\times6$

$= 12+0+0+12+4+0$

$= 28$/- Rs,

3. Obtain the IBFS for the following transportation problem by Least Cost Method.

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Demand	20	40	30	10	100

Sol. Since, $\Sigma a_i = \Sigma b_j = 100$ the given transportation problem is balanced, we get IBFS

		Destinations			
	D_1	D_2	D_3	D_4	Supply
Origins O_1	1 [20]	2	1 [10]	4	~~30~~ ~~10~~ 0
O_2	3	3 [20]	2 [20]	1 [10]	~~50~~ ~~40~~ ~~20~~ 0
O_3	4	2 [20]	5	9	~~20~~ 0
Demand	~~20~~ 0	~~40~~ ~~20~~ 0	~~30~~ ~~20~~ 0	~~10~~ 0	100

The rim requirement satisfied i.e, the no. of non-negative allocation $= m+n-1 = 3+4-1 = 6$
Hence the solution is non-degenerate feasible solution.
The allocations are $x_{11}=20$, $x_{13}=10$, $x_{24}=10$, $x_{22}=20$, $x_{23}=20$ and $x_{32}=20$

Transportation cost $= 1\times20+1\times10+3\times20+2\times20+1\times10+2\times20$
$= 20+10+60+40+10+40$
$= 180$ Rs. /-

4. Obtain the IBFS for the following transportation problem by Least Cost Method.

	A	B	C	Supply
1	2	7	4	5
2	3	1	8	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	34

Sol. Since, $\Sigma a_i = \Sigma b_j = 34$ the given transportation problem is balanced, we get IBFS

	A	B	C	Supply
1	2	7	4 [5]	~~5~~ 0
2	3	1 [8]	8	~~8~~ 0
3	5	4 [1]	7 [6]	~~7~~ ~~6~~ 0
4	1 [7]	6	2 [7]	~~14~~ ~~7~~ 0
Demand	~~7~~ 0	~~9~~ ~~1~~ 0	~~18~~ ~~11~~ ~~6~~ 0	34

The rim requirement satisfied i.e, the no. of non-negative allocation $= m+n-1 = 4+3-1 = 6$ ∴ Hence, the solution is non-degenerate Feasible sol^n.
The allocations are $x_{13}=5$, $x_{22}=8$, $x_{32}=1$, $x_{33}=6$, $x_{41}=7$ and $x_{43}=7$

Transportation cost $= 4\times5+1\times8+4\times1+7\times6+1\times7+2\times7$
$= 20+8+4+42+7+14$
$= 95$ Rs.

5. Obtain the IBFS for the following transportation problem by least Cost method.

	A	B	C	D	supply
P	5	4	2	6	20
Q	8	3	5	7	30
R	5	9	4	6	50
Demand	10	40	20	30	100

Sol. Since, $\Sigma a_i = \Sigma b_j = 100$ then given transportation problem is balanced, we get IBFS

	~~A~~	B	~~C~~	~~D~~	supply
~~P~~	5	4 [0]	2 [20]	6	~~20~~ 0
~~Q~~	8	3 [30]	5	7	~~30~~ 0
R	5 [10]	9 [10]	4	6 [30]	~~50~~ ~~40~~ ~~10~~ 0
Demand	~~10~~ 0	~~40~~ ~~10~~ 0	~~20~~ 0	~~30~~ 0	100

The rim requirements satisfied i.e, the no. of non-negative allocation $= m+n-1 = 3+4-1 = 6$

Hence, the solution is non-degenerate feasible solution

The allocations are $x_{12}=0$, $x_{13}=20$, $x_{22}=30$, $x_{31}=10$, $x_{32}=10$ and $x_{34}=30$

Transportation cost $= 4\times0+2\times20+3\times30+5\times10+9\times10+6\times30$
$= 0+40+90+50+90+180 = 450$
$=$ Rs. 450/-

Vogel's Approximation Method:- (VAM)

The steps involved in this method for finding the initial solution are as follows:

Step-1:- Find the penality cost namely the difference b/w the smallest and the next smallest cost in each row and each column.

Step-2:- Among the penalities as found in step-1, choose the maximum penality. If this maximum penality is ~~(maximum)~~ more than 1 i.e, If there is a tie then choose any one orbitarily.

Step-3:- In the selected row or column as by step-2, Find out the cell having the least cost. Allocate to the cell as much as possible depending on capacity and requirements.

Step-4:- Delete the row or column ~~(as by)~~ which is fully exhausted, again compute the row and column penalities for the reduced transportation table. and then go to step-2 until all the rim requirements are satisfied.

1. Find the initial Basic Feasible solution for the given transportation problem using VAM

	A	B	C	supply
1	2	7	4	5
2	3	1	8	8
3	5	4	7	7
4	1	6	2	14
Demand	7	9	18	34

Sol. Since, $\Sigma a_i = \Sigma b_j = 34$ the given problem is balanced. there exist intial basic Feasible solution.

	A	B	C	supply						
1	2 [5]	7	4	~~5~~ 0	2	2	2 ←	—	—	—
2	3	1 [8]	8	~~8~~ 0	2	—	—	—	—	—
3	5 [2]	4 [1]	7 [14]	~~7~~ ~~6~~ ~~7~~ 0	1	1	2	2	2	5
4	1	6	2 [14]	~~14~~ 0	1	1	1	1	—	—
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~1~~ 0	~~18~~ ~~4~~ 0	34						
	1	3 ↑	2							
	1	2 ↑	2							
	1	—	2							
	4	—	5 ↑							
	5	—	7 ↑							
	5 ↑	—	—							

The no. of non-negativity allocation $= m+n-1 = 4+3-1 = 6$

$\therefore$ The solution for the given transportation problem is non-degenerate feasible solⁿ.

The allocations are: $x_{11}=5$, $x_{22}=8$, $x_{31}=2$, $x_{32}=1$, $x_{33}=4$, $x_{43}=14$

$\therefore$ Transportation cost $= 2\times5+1\times8+5\times2+4\times1+7\times4+2\times14$

$= 10+8+10+4+28+28 = 88$

$=$ Rs. 88/-

2. Find the initial Basic Feasible solution for the given transportation problem using VAM

	A	B	C	D	supply
P	5	4	2	6	20
Q	8	3	5	7	30
R	5	9	4	6	50
Demand	10	40	20	30	100

Sol. Since, $\Sigma a_i = \Sigma b_j = 100$, the given problem is balanced. there exist initial basic feasible solution.

	A	B	C	D	Supply						
P	5	4	2 [20]	6	~~20~~ 0	2	—	—	—	—	—
Q	8	3 [30]	5	7	~~30~~ 0	2	2	—	—	—	—
R	5 [10]	9 [10]	4 [0]	6 [30]	~~50~~ ~~40~~ ~~10~~ 0	1	1	1	1	1	4
Demand	~~10~~ 0	~~40~~ ~~10~~ 0	~~20~~ 0	~~30~~ 0	100						
	0	1	2	0							
	3	6↑	1	1							
	5	9↑	4	6							
	5	—	4	6↑							
	5↑	—	4	—							
	—	—	4	—							

The no. of non-negativity allocation $= m+n-1 = 4+3-1 = 6$

$\therefore$ The solution for the given transportation problem is non-denerate feasible solⁿ

The allocations are: $x_{13}=20$, $x_{22}=30$, $x_{31}=10$, $x_{32}=10$, $x_{33}=0$, $x_{34}=30$

$\therefore$ Transportation cost $= 2\times20+3\times30+5\times10+9\times10+4\times0+6\times30$
$= 40+90+50+90+0+180 = 450$
$=$ Rs. 450/-

3. Find the IBFS for the given transportation problem using VAM

	D_1	D_2	D_3	D_4	Supply
O_1	11	13	17	14	250
O_2	16	18	14	10	300
O_3	21	24	13	10	400
Demand	200	225	275	250	950

Sol. Since, $\Sigma a_i = \Sigma b_j = 950$, the given problem is balanced. there exist initial basic feasible solution.

	D_1	D_2	D_3	D_4	Supply						
O_1	11 [200]	13 [50]	17	14	~~250~~ ~~50~~ 0	02	1	—	—	—	—
O_2	16	18 [175]	14	10 [125]	~~300~~ ~~125~~ 0	4	4	4	4	—	—
O_3	21	24	13 [275]	10 [125]	~~400~~ ~~125~~ 0	3	3	3	3	3	10
Demand	~~200~~ 0	~~225~~ ~~175~~ 0	~~275~~ 0	~~250~~ ~~125~~ 0	950						
	05↑	5	1	0							
	—	5↑	1	0							
	—	6↑	1	0							
	—	—	1	0							
	—	—	13↑	~~10~~							
	—	—	—	10							

The no. of non-negativity allocation $= m+n-1 = 3+4-1 = 6$

$\therefore$ The solution for the given transportation problem is non-degegerate feasible solⁿ

The allocations are: $x_{11}=200$, $x_{12}=50$, $x_{22}=175$, $x_{24}=125$, $x_{33}=275$ and $x_{34}=125$

$\therefore$ Transportation cost $= 11\times200+13\times50+18\times175+10\times125+13\times275+10\times$
$= 2200+650+3150+1250+3575+1250$
$= 12075/-$

Transportation cost is Rs. 12075/-

4. Find IBFS for the following Transportation problem by using NWCM, LCM and VAM destination.

Origins	P	Q	R	S	Supply
A	21	16	25	13	11
B	17	18	14	23	13
C	32	17	18	41	19
Demand	6	10	12	15	43

Sol. North-West Corner Method:-

Since, $\Sigma a_i = \Sigma b_j = 43$ the given problem exist a feasible solution.

	P	Q	R	S	Supply
A	21 $\boxed{6}$	16 $\boxed{5}$	25	13	~~11~~ ~~5~~ 0
B	17	18 $\boxed{5}$	14 $\boxed{8}$	23	~~13~~ ~~8~~ 0
C	32	17	18 $\boxed{4}$	41 $\boxed{15}$	~~19~~ ~~15~~ 0
Demand	~~6~~ 0	~~10~~ ~~5~~ 0	~~12~~ ~~4~~ 0	~~15~~ 0	43

$\therefore$ The no. of non-negativity allocations $= m+n-1 = 3+4-1 = 6$
$\therefore$ Rim Requirement satisfied.
Hence, the solution is non-degenerate feasible solⁿ
The allocations are: $x_{11}=6$, $x_{12}=5$, $x_{22}=5$, $x_{23}=8$, $x_{33}=4$ and $x_{34}=15$

$\therefore$ Transportation cost $= 21\times6+16\times5+18\times5+14\times8+18\times4+41\times15$
$= 126+80+90+112+72+615$
$= 1095.$ Rs

Least Cost Method:-

Since, $\Sigma a_i = \Sigma b_j = 43$ then, the given transportation problem is balanced we get IBFS

	P	Q	R	S	Supply
A	21	16	25	13 $\boxed{11}$	~~11~~ 0
B	17 $\boxed{1}$	18	14 $\boxed{12}$	23	~~13~~ ~~1~~ 0
C	32 $\boxed{5}$	17 $\boxed{10}$	18	41 $\boxed{4}$	~~19~~ ~~9~~ ~~4~~ 0
Demand	~~6~~ ~~5~~ 0	~~10~~ 0	~~12~~ 0	~~15~~ ~~4~~ 0	43

$\therefore$ The rim requirement satisfied.
The no. of non-negativity allocations $= m+n-1 = 3+4-1 = 6$
Hence, the solution is non-degenerate feasible solⁿ
The allocation are: $x_{14}=11$, $x_{21}=1$, $x_{23}=12$, $x_{31}=5$, $x_{32}=10$, $x_{34}=4$

∴ The Transportation cost $= 13\times11 + 17\times1 + 14\times12 + 32\times5 + 17\times10 + 41\times4$
$= 143 + 17 + 168 + 160 + 170 + 164 = 822/-$
$=$ Rs. 822/-

<u>Vogels Approximation Method</u>:-

Since, $\Sigma a_i = \Sigma b_j = 43$ then, the given transportation problem is balanced we get IBFS

	P	Q	R	S	Supply						
A	21	16	25	13 [11]	~~11~~ 0	3	—	—	—	—	—
B	17 [6]	18	14 [3]	23 [4]	~~13~~ ~~9~~ ~~3~~ 0	3	3	3	4	—	—
C	32	17 [10]	18 [9]	41	~~19~~ 10 0	1	1	1	1	1	17
Demand	~~6~~ 0	~~10~~ 0	~~12~~ 9 0	~~15~~ 4 0	43						
	4	1	4	10↑							
	15	1	4	18↑							
	↑15	1	4	—							
	—	1	4↑	—							
	—	1	4↑	—							
	—	17↑	—	—							

∴ the no. of non-negative allocation is $6 = m+n-1 = 3+4-1$
$= 6$

∴ Rim requirement satisfied.

Hence, the solution is non-denerate basic feasible sol^n

∴ The allocations are:

$x_{14} = 11$, $x_{21} = 6$, $x_{23} = 3$, $x_{24} = 4$, $x_{32} = 10$ and $x_{33} = 9$.

∴ Transportation Cost $= 11\times13 + 3\times14 + 4\times23 + 6\times17 + 17\times10 + 18\times9$
$= 143 + 42 + 92 + 102 + 170 + 162$
$= 711/-$ Rs

5. Find IBFS for the following Transportation problem by using NWCM, LCM and VAM

	D_1	D_2	D_3	D_4	Supply
O_1	2	2	2	1	3
O_2	10	8	5	4	7
O_3	7	6	6	8	5
Demand	4	3	4	4	15

Sol. <u>North-west Corner Method</u>:-

Since, $\Sigma a_i = \Sigma b_j = 15$ then, the given problem exist a feasible solution.

	D_1	D_2	D_3	D_4	Supply
O_1	2 [3]	2	2	1	~~3~~ 0
O_2	10 [1]	8 [3]	5 [3]	4	~~7~~ ~~6~~ ~~3~~ 0
O_3	7	6	6 [1]	8 [4]	~~5~~ ~~4~~ 0
Demand	~~4~~ ~~1~~ 0	~~3~~ 0	~~4~~ ~~1~~ 0	~~4~~ 0	15

∴ The no. of non-negativity allocations is $6 = 3+4-1$
$6 = 6$

∴ Rim Requirement satisfied

The allocations are:-

$x_{11} = 3$, $x_{21} = 1$, $x_{22} = 3$, $x_{23} = 3$ and $x_{33} = 1$, $x_{34} = 4$

∴ Hence, the solution is non-degenerate feasible sol^n.

$\therefore$ Transportation Cost $= 2\times3 + 10\times1 + 8\times3 + 5\times3 + 6\times1 + 8\times4$
$= 6+10+24+15+6+32 = 93$
$=$ Rs. 93/-

Least Cost Method:-

Since, $\Sigma a_i = \Sigma b_j = 15$ then, the given transportation problem is balanced we get IBFS

	D_1	D_2	D_3	D_4	Supply
O_1	2	2	2	1 [3]	~~3~~ 0
O_2	10 [2]	8	5 [4]	4 [1]	~~7~~ ~~3~~ ~~2~~ 0
O_3	7 [2]	6 [3]	6	8	~~5~~ ~~2~~ 0
Demand	~~4~~ ~~2~~ 0	~~3~~ 0	~~4~~ 0	~~4~~ ~~1~~ 0	15

The no. of non-negative allocations is $6 = m+n-1 = 3+4-1 = 6$
$\therefore$ Rim Requirements satisfied.
Hence, the solution is non-degenerate feasible sol^n
The allocations are: $x_{14}=3$, $x_{21}=2$, $x_{23}=4$, $x_{24}=1$, $x_{31}=2$, $x_{32}=3$
$\therefore$ Transportation Cost $= 1\times3 + 10\times2 + 20 + 4\times1 + 7\times2 + 6\times3$
$= 3+20+20+4+14+18 = 79$
$=$ Rs. 79/-

Vogels-Approximation Method:-

Since, $\Sigma a_i = \Sigma b_j = 43$ then, the given transportation problem is balanced we get, IBFS

	D_1	D_2	D_3	D_4	Supply						
O_1	2 [3]	2	2	1	~~3~~ 0	1	—	—	—	—	—
O_2	10	8	5 [3]	4 [4]	~~7~~ ~~3~~ 0	1	1	3 ←	—	—	—
O_3	7 [1]	6 [3]	6 [1]	8	~~5~~ ~~4~~ ~~1~~ 0	0	0	0	0	0	6 ←
Demand	~~4~~ ~~1~~ 0	~~3~~ 0	~~4~~ ~~1~~ 0	~~4~~ 0	15						
	↑5	4	3	3							
	3	2	1	4↑							
	3	2	1	—							
	7↑	6	6	—							
	—	6↑	6	—							
	—	—	6	—							

$\therefore$ The no. of non-negative allocation is $6 = m+n-1 = 3+4-1 = 6$
$6 = 6$
$\therefore$ Rim requirement satisfied.
Hence, the solution is non-degenerate basic Feasible sol^n

$\therefore$ The allocation are: $x_{11}=3$, $x_{23}=3$, $x_{24}=4$, $x_{31}=1$, $x_{32}=3$ and $x_{33}=1$
The Transportation cost $= 2\times3 + 5\times3 + 4\times4 + 7\times1 + 6\times3 + 6\times1$
$= 6+15+16+7+18+6 = 68$
$=$ Rs. 68/-

MODI METHOD :-

Step-1:- Find the Initial Basic Feasible solution of the given problem by North-west Corner Rule (or) Least Cost Method (or) Vogel's approximation Method.

Step-2:- Check the no. of occupied cells if these are less than (<) m+n-1 there exist degenerousy and we introduce a small (penality) positive assignment of 'Σ' (approx=0) in suitable independent positions so, that the no. of occupied cells is equal to (=) m+n-1.

Step-3:- Find the set of values u_i, v_j $(i=1,2,---m)(j=1,2,----n)$. from the relation $c_{ij} = u_i+v_j$ for each occupied cell (i,j), By starting initially with $u_i=0$ or $v_j=0$ Preferably for which the corresponding row or column has maximum number of individual allocations.

Step-4:- Find u_i+v_j for each unoccupied cells (i,j) and enter at the upper right corner of the corresponding cell (i,j).

Step-5:- Find the Net-Evaluation, $d_{ij} = c_{ij}-(u_i+v_j)$ for each unoccupied cell (i,j).

Examine the net-Evalution d_{ij} for all un-occupied cells.

1. If all $d_{ij}>0$ then, the solution is optimal and unique.
2. If all $d_{ij}\geq 0$ then, the solution is optimal but, there will be an alternative solution to the given transportation problem.
3. If atleast one $d_{ij}<0$ then, the solution is not optimal then, go to step-6.

Step-6:- From a new basic feasible sol^n by giving maximum allocation to the cell for which 'd_{ij}' is most negative by making an occupied cell empty. For that draw a closed path consisting of horizontal and vertical lines beginning and ending at the cell for which 'd_{ij}' is most negative and having its other corners at some allocated cells. Along this closed loop indicate $+\theta$ and $-\theta$ alternatively at the corners. Choose minimum of the allocations from the cells having '$-\theta$'. Add this minimum allocations to the cells with '$+\theta$' and substract the minimum allocation from the allocation to the cells '$-\theta$'.

Step-7:- Repeat the steps-2 to step-5, to test the optimality of this new Basic Feasible solution.

Step-8:- Continue the above procedure till an optimum solution is obtained.

1. Find the optimal solution for the following transportation problem.

Destination

		1	2	3	4	Supply
Origin	A	21	16	25	13	11
	B	17	18	14	23	13
	C	32	27	18	41	19
	Demand	6	10	12	15	43

Sol. Since, $\Sigma a_i = \Sigma b_j = 43$, the given transportation problem is balanced, $\therefore$ there exist an IBFS.

By using Vogel's Approximation Method we can find out the Initial Basic Feasible Solution:

	1	2	3	4	Supply						
A	21	16	25	13 [11]	~~11~~ 0	3	–	–	–	–	–
B	17 [6]	18 [3]	14	23 [4]	~~13~~ ~~9~~ ~~3~~ 0	3	3	3	4	18	18 ←
C	32	27 [7]	18 [12]	41	~~19~~ ~~7~~ 0	9	9	9	← 9	← 27	–
Demand	~~6~~ 0	~~10~~ ~~3~~ 0	~~12~~ 0	~~15~~ ~~4~~ 0	43						
	4	2	4	10 ↑							
	1~~0~~	2	4	18 ↑							
	↑15	9	4	–							
	–	9	4	–							
	–	9	–	–							
	–	18 ↑	–	–							

The no. of non-negative allocations is $6 = m+n-1 = 3+4-1 = 6$

$6 = 6$

Hence, the solution is non-degenerate feasible solution

The allocations are, $x_{14} = 11$, $x_{21} = 6$, $x_{22} = 3$, $x_{24} = 4$, $x_{32} = 7$ and $x_{33} = 12$

$\therefore$ Transportation cost $= 13\times11 + 17\times6 + 18\times3 + 23\times4 + 27\times7 + 18\times12$

$= 796/-$

$= \text{Rs. } 796/-$

To find the Optimal Solution:-

Consider, the above transportation table, Here $m+n-1 = 6$ we apply modi method.

Now, we determine a set of values u_i and v_j for each occupied cell (i,j) by using the relation $c_{ij} = u_i + v_j$, the 2nd row contains maximum no. of allocations, we choose $u_2 = 0$

$\Rightarrow c_{21} = u_2 + v_1 \Rightarrow 17 = 0 + v_1 \Rightarrow v_1 = 17$

$\Rightarrow c_{22} = u_2 + v_2 \Rightarrow 18 = 0 + v_2 \Rightarrow u_2 = 18$

$\Rightarrow c_{24} = u_2 + v_4 \Rightarrow 23 = 0 + v_4 \Rightarrow v_4 = 23$

$\Rightarrow c_{14} = u_1 + v_4 \Rightarrow 13 = u_1 + 23 \Rightarrow u_1 = 13 - 23 = -10$

$\Rightarrow c_{32} = u_3 + v_2 \Rightarrow 27 = u_3 + 18 \Rightarrow u_3 = 27 - 18 = 9$

$\Rightarrow c_{33} = u_3 + v_3 \Rightarrow 18 = 9 + v_3 \Rightarrow v_3 = 9$

Thus, we have the following TP Table:

21	16	25	13 [11]	$u_1=-10$
17 [6]	18 [3]	14	23 [4]	$u_2=0$
32	27 [7]	18 [12]	41	$u_3=9$
$v_1=17$	$v_2=18$	$v_3=9$	$v_4=23$	

Now, we find u_i+v_j for all unoccupied cells (i,j) and enter at upper right corner of the corresponding unoccupied cell (i,j)

Then, we find the net-evaluation for all unoccupied cell (i,j) and enter at lower right corner of the corresponding unoccupied cell (i,j)

Thus, we get the following Table:

7 21 14	8 16 8	-1 25 26	[11] 13	$u_1=-10$
[6] 17	[3] 18	9 14 5	[4] 23	$u_2=0$
26 32 6	[7] 27	[12] 18	32 41 9	$u_3=9$
$v_1=17$	$v_2=18$	$v_3=9$	$v_4=23$	

$\because d_{ij}=c_{ij}-(u_i+v_j)$

$d_{11}=c_{11}-(u_1+v_1)=21-7=14$

$d_{12}=c_{12}-(u_1+v_2)=16-8=8$

$d_{13}=c_{13}-(u_1+v_3)=25+1=26$

$d_{23}=c_{23}-(u_2+v_3)=14-9=5$

$d_{31}=c_{31}-(u_3+v_1)=32-26=6$

$d_{34}=c_{34}-(u_3+v_4)=41-32=9$

$\therefore$ Then, since all $d_{ij}>0$, the solution is optimum and unique.

The optimum allocations are $x_{14}=11, x_{21}=6, x_{22}=3, x_{24}=6, x_{32}=7$ and $x_{33}=12$

and Minimum (optimum) Transportation Cost $=13\times11+17\times6+18\times3+23\times4+27\times7+18\times12$

$=$ Rs. 796/-

2. Find the optimal Transportation cost of the following matrix using Least Cost Method for finding the critical solution.

		Market					
		A	B	C	D	E	Available
Factory	P	4	1	2	6	9	100
	Q	6	4	3	5	7	120
	R	5	2	6	4	8	120
Demand		40	50	70	90	90	

Sol. Since, $\Sigma a_i = \Sigma b_j = 340$, the given matrix problem is balanced, there exist an IBFS

By using Least Cost method we can find ou the Initial Basic Feasible solution.

	A	B ✗	C ✗	D ✗	E	
✗ P	4	1 [50]	2 [50]	6	9	~~100~~ 50 0
Q	6 [10]	4	3 [20]	5	7 [90]	~~120~~ 100 90 0
✗ R	5 [30]	2	6	4 [90]	8	~~120~~ 30 0
	~~40~~ 10 0	~~50~~ 0	~~70~~ 20 0	~~90~~ 0	~~90~~ 0	340

The no. of non-negativity independent allocations is $7 = m+n-1$
$= 3+5-1$
$7 = 7$

$\therefore$ The solution is non-degenerate IBFS

Allocation are $x_{12}=50$, $x_{13}=50$, $x_{21}=10$, $x_{23}=26$, $x_{25}=90$, $x_{31}=30$ and $x_{34}=90$

Transportation cost $= 1\times50+2\times50+6\times10+3\times20+7\times90+5\times30+4\times90$
$=$ Rs. 1410/-

<u>To find the optimal Solution:</u>

						u_i
	4 (5, −1)	1 [50]	2 [50]	6 (4, 2)	9 (6, 3)	$u_1=-1$
	6 [10]	4 (2, 2)	3 [20]	5 (5, 0)	7 [90]	$u_2=0$
	5 [30]	2 (1, 1)	6 (2, 4)	4 [90]	8 (6, 2)	$u_3=-1$
v_j	$v_1=6$	$v_2=2$	$v_3=3$	$v_4=5$	$v_5=7$	

$\therefore d_{11}=-1<0$, i.e, $d_{ij}<0$ then the current solution is not optimum.

Now, let us form a new basic feasible solution by given maximum allocation of the cell (i,j) for which d_{ij} is most negative by making an occupied cell empty. Here, the cell (1,1) having the negative value $d_{11}=-1$, we draw a closed path consisting of vertical and horizontal lines beginning and ending at this cell and having its other corners at some occupied cells along this closed loop indicate '$+\theta$' and '$-\theta$' alternatively at the corners.

4 $+\theta$	1 [50]	2 [50] $-\theta$	6	9
6 [10] $-\theta$	4	3 [20] $+\theta$	5	7 [90]
5 [30]	2	6	4 [90]	8

From the two cells [(1,3), (2,1)] having '$-\theta$' we find that the minimum of the allocations (50,10) is 10. Add this 10 to the cells with '$+\theta$' and Substract this 10 to the cell with '$-\theta$'.

Hence, the new Basic Feasible solution is as follows

4 [10]	1 [50]	2 [40]	6	9
6	4	3 [30]	5	7 [90]
5 [30]	2	6	4 [90]	8

Here, the no. of non-negative allocation $= m+n-1$, we apply modi method.

v_j	$v_1=4$	$v_2=1$	$v_3=2$	$v_4=3$	$v_5=6$	u_i
	$\boxed{10}$ 4	$\boxed{50}$ 1	$\boxed{40}$ 2	3 / 6 3	6 / 9 3	$u_1=0$
	6 1	4 2	$\boxed{30}$ 3	4 / 5 1	$\boxed{90}$ 7	$u_2=1$
	$\boxed{30}$ 5	2 / 2 0	3 / 6 3	$\boxed{90}$ 4	7 / 8 1	$u_3=1$

Since, all $d_{ij} \geq 0$ then solution is optimum but there will be alternative solution to the given transportation problem.

$\therefore$ The optimum allocations are : $x_{11}=10$, $x_{12}=50$, $x_{13}=40$, $x_{23}=30$, $x_{25}=90$, $x_{31}=30$, $x_{34}=90$

$\therefore$ The minimum Transportation cost $= 4\times10 + 1\times50 + 2\times40 + 3\times30 + 7\times90 + 5\times30 + 4\times90$
$= 40+50+80+90+630+150+360$
$=$ Rs. 1400/-

3. Find the Initial Basic Feasible solution by VAM and also Determine the optimal solution by modi Method.

	D_1	D_2	D_3	D_4	Supply
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Demand	5	8	7	15	$\Sigma a_i = \Sigma b_j = 35$

Sol. Since, $\Sigma a_i = \Sigma b_j = 35$, the given transportation problem is balanced. There exis an IBFS

By using Vogel's Approximation Method we can find out the Initial Basic Feasible solution.

	D_1	D_2	D_3	D_4	Supply						
P_1	$\boxed{5}$ 19	30	50	$\boxed{2}$ 12	~~7~~ ~~2~~ 0	7	18	38	38	—	—
P_2	70	30	$\boxed{7}$ 40	$\boxed{3}$ 60	~~10~~ ~~7~~ 0	10	10	20	20	20	40
P_3	40	$\boxed{8}$ 10	60	$\boxed{10}$ 20	~~18~~ ~~10~~ 0	10	10	← 40	—	—	—
Demand	~~5~~ 0	~~8~~ 0	~~7~~ 0	~~15~~ ~~5~~ ~~3~~ 0	35						
	21↑	20	10	8							
	—	20↑	10	8							
	—	—	10	8							
	—	—	10	48↑							
	—	—	40	60↑							
	—	—	40	—							

$\therefore$ The no. of non-negative allocations is $6 = 3+4-1 = 6$

Hence, the solution is non-degenerate solution.

The allocations are $x_{11}=5$, $x_{14}=2$, $x_{23}=7$, $x_{24}=3$, $x_{32}=8$ and $x_{34}=10$

$\therefore$ Transportation cost $= 19\times5 + 12\times2 + 40\times7 + 60\times3 + 10\times8 + 20\times10$
$= 95+24+280+180+80+200 = 859$/-

$\therefore$ Transportation Cost is Rs. 859/-

To find the optimal solution:-

Consider, the above transportation table, Here $m+n-1=6$ we apply modi Method.

Now, we determine a set of values u_i and v_j for each occupied cell (i,j). By using the relation $c_{ij}=u_i+v_j$

Here, the 4th column consists maximum no. of allocations we choose $v_4=0$

Thus, we have the following TP Table:-

				u_i
19 [5]	30	50	12 [2]	$u_1=12$
70	30	40 [7]	60 [3]	$u_2=60$
40	10 [8]	60	20 [10]	$u_3=20$
$v_1=7$	$v_2=-10$ (-10)	$v_3=-20$ (-20)	$v_4=0$	v_j

Now, we find u_i+v_j for all unoccupied cells (i,j) and enter at the upper right corner of the corresponding unoccupied cell (i,j)

Then, we find the net-evaluation for all unoccupied cell (i,j) and enter at the lower right corner of the corresponding unoccupied cell (i,j) by $d_{ij}=c_{ij}-(u_i+v_j)$

19 [05]	30 (2) 28	50 (-8) 58	12 [2]
70 (67) 3	30 (50) -20	40 [7]	60 [3]
40 (27) 13	10 [8]	60 (0) 60	20 [10]

$\therefore d_{22}=-20<0$, i.e, $d_{ij}<0$ then, the current solution is not optimal.

Now, let us form a new Basic Feasible solution by given maximum allocation of the cell (i,j) for which d_{ij} is most negative by making an occupied cell empty. Here, the cell $(2,2)$ having the negative value $d_{22}=-20$, we draw a closed path consisting of vertical and horizontal lines beginning and ending at this cell and having its other corners at some occupied cells along this closed loop indicate '$+\theta$' and '$-\theta$' altornatively at the corners.

19 [5]	30	50	12 [2]
70	30 $+\theta$	40 [7]	60 [3] $-\theta$
40	10 [8] $-\theta$	60	20 [10] $+\theta$

From the two cell $[(2,4),(3,2)]$ having '$-\theta$' we find that the minimum of the allocations $(3,8)$ is 3. Add this '3' to the cells with '$+\theta$' and substract this '3' from the cells with '$-\theta$'

Hence, the new Basic Feasible solⁿ is as follows:-

19 $\boxed{5}$	30	50	12 $\boxed{2}$
70	30 $\boxed{3}$	40 $\boxed{7}$	60
40	10 $\boxed{5}$	60	20 $\boxed{13}$

Here, the no. of non-negative allocation is $6 = m+n-1 = 6$, we apply MODI Method.

				u_i
19 $\boxed{5}$	30 2 28	50 12 38	12 $\boxed{2}$	$u_1=0$
70 47 23	30 $\boxed{3}$	40 $\boxed{7}$	60 40 20	$u_2=28$
40 27 13	10 $\boxed{5}$	60 20 40	20 $\boxed{13}$	$u_3=8$
v_j: $v_1=19$	$v_2=2$	$v_3=12$	$v_4=12$	

$\therefore$ All $d_{ij} > 0$, the solution is optimum and unique.

The allocations are $x_{11}=5$, $x_{14}=2$, $x_{22}=3$, $x_{23}=7$, $x_{32}=5$, $x_{34}=13$

$\therefore$ Minimum Transportation Cost $= 19\times5 + 12\times2 + 30\times3 + 40\times7 + 10\times5 + 20\times13$

$= 95+24+90+280+50+260$

$= 799/-$ Rs

4. Find the IBFS by NWCR and also determine the optimal solution by MODI method.

	D_1	D_2	D_3	Supply
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Demand	7	9	18	34

Sol. Since $\Sigma a_i = \Sigma b_j = 34$ the given transportation problem is balanced There exist a IBFS.

By using North-west corner rule we can find out the initial Basic feasible solution.

	D_1	D_2	D_3	supply
O_1	2 $\boxed{5}$	7	4	~~5~~ 0
O_2	3 $\boxed{2}$	3 $\boxed{6}$	1	~~8~~ ~~6~~ 0
O_3	5	4 $\boxed{3}$	7 $\boxed{4}$	~~7~~ ~~4~~ 0
O_4	1	6	2 $\boxed{14}$	~~14~~ 0
Demand	~~7~~ ~~2~~ 0	~~9~~ ~~3~~ 0	~~18~~ ~~14~~ 0	$\Sigma a_i = \Sigma b_j = 34$

$\therefore$ The no. of non-negative allocations $= m+n-1 = 4+3-1 = 6$

Hence, the solution is non-degenerate IBFS

Allocations are $x_{11}=5$, $x_{21}=2$, $x_{22}=6$, $x_{32}=3$, $x_{33}=4$, $x_{43}=14$

$\therefore$ Transportation Cost $= 2\times5 + 3\times2 + 3\times6 + 4\times3 + 7\times4 + 2\times14$

Transportation Cost $= 10+6+18+12+28+28 = 102$

$= 102/-$ Rs.

To find the Optimal Solution :-

Consider the above transportation table here $m+n-1 = 6$ ∴ we apply modi method.

Now we determine a set of values u_i and v_j for each occupied cell (i,j). By using the relation $c_{ij} = u_i + v_j$

Here, ~~the~~ we 2nd row as which consists maximum no. of allocations we choose $u_2 = 0$

Thus, we have the following TP table :-

2 [5]	7	4	$u_1 = -1$
3 [2]	3 [6]	1	$u_2 = 0$
5	4 [3]	7 [4]	$u_3 = 1$
1	6	2 [14]	$u_4 = -4$
$v_1 = 3$	$v_2 = 3$	$v_3 = 6$	

Now we find $u_i + v_j$ for all unoccupied cells (i,j) and enter at the URC of the corresponding unoccupied cells (i,j).

Then we find the net evaluation for all unoccupied cells (i,j) and enter at the LRC of the corresponding unoccupied cell (i,j) by $d_{ij} = c_{ij} - (u_i + v_j)$

2 [5]	7 +2 (5)	4 +5 −1	$u_1 = -1$
3 [2]	3 [6]	1 6 −5	$u_2 = 0$
5 4 1	4 [3]	7 [4]	$u_3 = 1$
1 −1 2	6 −1 7	2 [14]	$u_4 = -4$
$v_1 = 3$	$v_2 = 3$	$v_3 = 6$	

∴ $d_{23} = -5 < 0$, i.e, $d_{ij} < 0$ then, the current solution is not optimal.

Now, let us form a new Basic Feasible solution by given maximum allocation of the cell (i,j) for which d_{ij} is most negative by making an occupied cell empty. Here, the cell $(2,3)$ having the negative value $d_{23} = -5$, we draw a closed path consisting of vertical and horizontal lines beginning and ending at this cell and having its other corners at some occupied cells along this closed loop indicate '$+\theta$' and '$-\theta$' alternatively at the corners.

2 [5]	7	4
3 [2]	$-\theta$ 3 [6]	1 $+\theta$
5	$+\theta$ 4 [3]	7 $-\theta$ [4]
1	6	2 [14]

From the two cell [(2,2), (3,4)] having '$-\theta$' we find that the minimum of the allocation (6,4) is 4. Add this '4' to the cells with '$+\theta$' and substract this '4' from the cells with '$-\theta$'

Hence, the new Basic Feasible sol^n is as follows:-

2 [5]	7	4
3 [2]	3 [2]	1 [4]
5	4 [7]	7
1	6	2 [14]

Here, the no. of non-negative allocation is $6 = m+n-1 = 4+3-1 = 6$

we apply modi method

			u_i
2 [5]	7 2 5	4 0 4	$u_1 = -1$
3 $-\theta$ [2]	3	1 $+\theta$ [4]	$u_2 = 0$
5 4 1	4 [7]	7 2 5	$u_3 = 1$
1 $+\theta$ 4 -3	6 4 2	2 $-\theta$ [14]	$u_4 = 1$
$v_1 = 3$	$v_2 = 3$	$v_3 = 1$	v_j

Here, $d_{ij} < 0$, $d_{41} = -3 < 0$

$\therefore$ the current sol^n is not optimal

$\min(2, 14)$ is 2

We apply MODI method.

			u_i
2 [5]	7 5 2	4 3	$u_1 = 2$
3 0 3	3 [2]	1 [6]	$u_2 = 0$
5 1 4	4 [7]	7 2 5	$u_3 = 1$
1 [6]	6 4 2	2 [12]	$u_4 = 1$
$v_1 = 0$	$v_2 = 3$	$v_3 = 1$	v_j

$\therefore$ All $d_{ij} > 0$, the solution is optimum and unique

The allocations are:-

$x_{11} = 5$, $x_{22} = 2$, $x_{23} = 6$, $x_{32} = 7$

$x_{41} = 6$ and $x_{43} = 12$

$\therefore$ Minimum Transportation Cost $= 2\times5 + 3\times2 + 1\times6 + 4\times7 + 1\times6 + 2\times12$

$= 10+6+6+28+6+24 = 80$

$=$ Rs. 80/-

5. Solve the following Transportation problem.

	D_1	D_2	D_3	D_4	supply
P_1	23	27	16	18	30
P_2	12	17	20	51	40
P_3	22	28	12	32	53
Demand	22	35	25	41	123

sol. Since, $\Sigma a_i = \Sigma b_j = 123$ the given transportation problem is balanced. There exist an IBFS

By using Vogel's Approximation method. we can find out the IBFS.

	D_1	D_2	D_3	D_4	Supply						
P_1	23	27	16	18 [30]	~~30~~ 0	2	—	—	—	—	—
P_2	12 [5]	17 [35]	20	51	~~40~~ ~~5~~ 0	5	5	5	8	—	—
P_3	22 [17]	28	12 [25]	32 [11]	~~53~~ ~~42~~ ~~25~~ 0	10	10	10	10	10	12
Demand	~~22~~ ~~17~~ 0	~~35~~ 0	~~25~~ 0	~~41~~ ~~11~~ 0	123						
	10	10	4	14↑							
	10	11	8	19↑							
	10	11↑	8	—							
	10↑	—	8	—							
	10↑	—	12	—							
	—	—	12	—							

The no. of non-negative allocations is 6

$6 = m+n-1 \Rightarrow 6 = 3+4-1$

$6=6$

Since, the solution is non-degenerate IBFS

Allocations are $x_{14}=30$, $x_{21}=5$, $x_{22}=35$, $x_{31}=17$, $x_{33}=25$ and $x_{34}=11$

$\therefore$ Transportation Cost $= 18\times30+12\times5+17\times35+22\times17+12\times25+32\times11$

$= 540+60+595+374+300+352$

$TC = 2221.$ Rs/-

<u>To find Optimal solution</u> :-

Consider the above transportation table, here $m+n-1=6$ we apply MODI method.

Now, we determine a set of values u_i and v_j for each occupied cell (i,j) by using the relation $c_{ij}=u_i+v_j$, the third row contains maximum no. of allocations we choose $u_3=0$

Thus, we have the following transportation table :-

				u_i
23	27	16	18 [30]	$u_1=-14$
12 [5]	17 [35]	20	51	$u_2=-10$
22 [17]	28	12 [25]	32 [11]	$u_3=0$
$v_j \Rightarrow$ $v_1=22$	$v_2=27$	$v_3=12$	$v_4=32$	

Now, we find u_i+v_j for all unoccupied cells(i,j) and Enter upper right corner of the corresponding unoccupied cell.

Then we find the net-evaluation $d_{ij}=c_{ij}-(u_i+v_j)$ for each unoccupied cell (i,j) and enter at the lower right corner of the corresponding unoccupied cell (i,j)

Thus, we get the following table :-

8 / 23 15	13 / 27 14	-2 / 16 18	[30] 18
[5] 12	[35] 17	2 / 20 18	22 / 51 29
[17] 22	27 / 28 1	[25] 12	[11] 32

Since, all $d_{ij}>0$ then the solution is optimal and unique

$\therefore$ The optimal allocations are, $x_{14}=30$, $x_{21}=5$, $x_{22}=35$, $x_{31}=17$, $x_{33}=25$ and $x_{34}=11$

Minimum Transportation Cost = $18\times30 + 12\times5 + 17\times35 + 27\times17 + 12\times25 + 32\times11$

MTC = 2221. Rs/-

6. Solve the following transportation problem.

	D_1	D_2	D_3	supply
P	7	3	2	2
Q	2	1	3	3
R	3	4	6	5
Demand	4	1	5	10

Sol. Since, $\Sigma a_i = \Sigma b_j = 10$ then the given transportation problem is balanced. There exist on IBFS.

By using Vogel's approximation method. we can find out the IBFS

	D_1	D_2	D_3	Supply					
P	7	3	2 [2]	~~2~~ 0	1	5 ←	–	–	–
Q	2	1 [1]	3 [2]	~~3~~ ~~2~~ 0	1	1	1	–	–
R	3 [4]	4	6 [1]	~~5~~ ~~4~~ 0	1	3	3	3	3
Demand	~~4~~ 0	~~1~~ 0	~~5~~ ~~3~~ ~~1~~ 0	10					
	1	2 ↑	1						
	1	–	1						
	1	–	3 ↑						
	3	–	6 ↑						
	3	–	–						

The no. of non-negative allocations is 5

$5 = m+n-1 = 3+3-1 = 5$

Since, the solution is non-degenerate IBFS

Allocations are $x_{13}=2$, $x_{22}=1$, $x_{23}=2$, $x_{31}=4$ and $x_{33}=1$

$\therefore$ Initial Transportation Cost $= 2\times2 + 1\times1 + 3\times2 + 4\times3 + 6\times1$

$= 4+1+6+12+6 = 29$

$=$ Rs. 29/-

To find Optimal solution :-

Consider the above transportation table, here $m+n-1=5$. we apply MODI method.

			u_i
7 -1 8	3 0 3	2 [2]	$u_1 = 2$
2 0 2	1 [1]	3 [2]	$u_2 = 3$
3 [4]	4 4 0	6 [1]	$u_3 = 6$
v_j: $v_1 = -3$	$v_2 = -2$	$v_3 = 0$	

Since, all $d_{ij} > 0$ the solution is optimal and Unique.

The Minimum transportation Cost $= 2\times2 + 1\times1 + 3\times2 + 4\times3 + 6\times1$

$= 4+1+6+12+6$

$=$ Rs. 29/-

7. Solve the following transportation Cost table.

	D_1	D_2	D_3	D_4	Supply
O_1	3	3	4	1	100
O_2	4	2	4	2	125
O_3	1	5	3	2	75
Demand	120	80	75	25	300

Sol. Since, $\Sigma a_i = \Sigma b_j = 300$ then given transportation problem is balanced.

There exist an IBFS,

By using Vogel's Approximation Method we find out IBFS.

	D_1	D_2	D_3	D_4	Supply						
O_1	$\boxed{45}$ 3	3	$\boxed{30}$ 4	$\boxed{25}$ 1	~~100~~ ~~75~~ ~~30~~ 0	2	2	0	1	4	—
O_2	4	$\boxed{80}$ 2	$\boxed{45}$ 4	2	~~125~~ ~~45~~ 0	0	0	2	0	4	4
O_3	$\boxed{75}$ 1	5	3	2	~~75~~ 0	1	—	—	—	—	—
Demand	~~120~~ ~~45~~ 0	~~80~~ 0	~~75~~ ~~45~~ 0	~~25~~ 0	300						
	2↑	1	1	1							
	1	1	0	1							
	1	1	0	—							
	1	—	0	—							
	—	—	0	—							
	—	—	4	—							

The no. of non-negative allocations is $6 = m+n-1 = 3+4-1 = 6$

Hence, the solution is non-degenerate Basic Feasible solution

Allocations are $x_{11} = 45$, $x_{13} = 30$, $x_{14} = 25$, $x_{22} = 80$, $x_{23} = 45$ and $x_{31} = 75$

Transportation cost $= 3\times45 + 4\times30 + 1\times25 + 2\times80 + 4\times45 + 1\times75$

$= 135 + 120 + 25 + 160 + 180 + 75 = 695$

TC = Rs. 695/-

<u>To find Optimal solution</u> :-

Consider above transportation table, here $m+n-1 = 6$ we apply MODI method.

					u_i
	3 $\boxed{45}$	3 (2, 1)	4 $\boxed{30}$	1 $\boxed{25}$	$u_1 = 0$
	4 (3, 1)	2 $\boxed{80}$	4 $\boxed{45}$	2 (1, 1)	$u_2 = 0$
	1 $\boxed{75}$	5 (0, 5)	3 (2, 1)	2 (−1, 3)	$u_3 = -2$
v_j	$v_1 = 3$	$v_2 = 2$	$v_3 = 4$	$v_4 = 1$	

Since all $d_{ij} > 0$ then the solution is optimal and unique

The Minimum transportation cost $= 3\times45 + 4\times30 + 1\times25 + 2\times80 + 4\times45 + 1\times75$

$=$ Rs. 695/-

* Degeneracy in Transportation Problem:-

In a transportation problem whenever the no. of non-negative independent allocations is less than $m+n-1$, the transportation problem is said to be degenerate one. Degeneracy may occur either at the initial stage or at an intermediate stage at some subsequent iteration.

To resolve degeneracy, we allocate an extremely small amount (close to zero) to one more empty cells of transportation table (Generally minimum cost cell is possible) so that the total no. of occupied cells becomes $m+n-1$ at independent positions. We denote this small amount by 'ε'. The cell containing 'ε' are then treated like other occupied cells and the problem is solved in the usual way. The 'ε' are kept till the optimum solution is attained then we let each $\varepsilon \to 0$

1. Solve the following transportation problem by using VAM and find the optimal solution.

		Warehouses						
		A	B	C	D	E	F	Available
Factory	1	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
Requirement		4	4	6	2	4	2	22

Sol. Since, $\sum a_i = \sum b_j = 22$, then the given transportation problem is balanced. There exist an IBFS.

By using VAM we find out IBFS

	A	B	C	D	E	F	Available								
1	9	12	9 [5]	6	9	10	~~5~~ 0	3	3	0	0	0	0	0	9 ←
2	7	3 [4]	7	7	5	5 [2]	~~6~~ ~~4~~ 0	2	2	2	4 ←	0	0	0	—
3	6 [1]	5	9 [1]	11	3	11	~~2~~ ~~1~~ 0	2	2	2	1	3	3	3 ←	9
4	6 [3]	8	11	2 [2]	2 [4]	10	9 ~~7~~ ~~3~~ 0	0	0	4 ←	2	5 ←	—	—	—
Requirement	~~4~~ ~~1~~ 0	~~4~~ 0	~~6~~ ~~1~~ 0	~~2~~ 0	~~4~~ 0	~~2~~ 0	22								
	0	2	2	4	1	5 ↑									
	0	2	2	4 ↑	1	—									
	0	2	2	—	1	—									
	0	2	2	—	—	—									
	0	—	2	—	—	—									
	1	—	2	—	—	—									
	—	—	2	—	—	—									
	—	—	2	—	—	—									

Since, the no. of non-negtivity allocations = 8 which is less than
$(m+n-1) = 4+6-1 = 9$

This basic solution is degenerate one

To resolve degeneracy we allocate a small positive quantity 'ε' to the cell (3,2). So that the no. of occupied cell become $m+n-1$.

Hence, the non-degenerate basic feasible solution in the following table.

	A	B	C	D	E	F
1	9	12	9 [5]	6	9	10
2	7	3 [4]	7	7	5 [ε]	5 [2]
3	6 [1]	5 [ε]	9 [1]	11	3	11
4	6 [3]	8	11	2 [2]	2 [4]	10

$\therefore$ Transportation Cost $= 9\times5 + 3\times4 + 5\times2 + 6\times1 + 5\times\varepsilon + 9\times1 + 6\times3 + 2\times2 + 2\times4$

$= 45+12+10+6+\underbrace{5\varepsilon}_{5\times0=0}+9\times18\times4\times8$ $(\because \varepsilon \to 0)$

$=$ Rs. 112/-

To find Optimal Solution:-

Consider the above transportation table, here $m+n-1=9$ we apply MODI method.

	A	B	C	D	E	F	u_i
1	9, 6, 3	12, 5, 7	9 [5]	6, 2, 4	9, 2, 7	10, 7, 3	$u_1=0$
2	7, 4, 3	3 [4]	7, 7, 0	7, 0, 7	5, 0, 5	5 [2]	$u_2=-2$
3	6 [1]	5 [ε]	9 [1]	11, 2, 9	3, 2, 1	11, 7, 4	$u_3=0$
4	6 [3]	8, 5, 3	11, 9, 2	2 [2]	2 [4]	10, 7, 3	$u_4=0$
v_j	$v_1=6$	$v_2=5$	$v_3=9$	$v_4=2$	$v_5=2$	$v_6=7$	

Since, all $d_{ij} \geq 0$ with $d_{23}=0$ the no. of non-negative allocations $= m+n-1$

Allocations are $x_{13}=5,\ x_{22}=4,\ x_{26}=2,\ x_{31}=1,\ x_{32}=\varepsilon,\ x_{33}=1,$
$x_{41}=3,\ x_{44}=2$ and $x_{45}=4$

Optimal Cost $= 9\times5 + 3\times4 + 5\times2 + 6\times1 + 5\times\varepsilon + 9\times1 + 6\times3 + 2\times2 + 2\times4$

$= 45+12+10+6+5\varepsilon+9+18+4+8$

$=$ Rs. 112/- $(\because \varepsilon \to 0)$

2. Solve the following transportation problem by using VAM and find the optimal solution.

	To				Supply
From	1	2	3	4	6
	4	3	2	0	8
	0	2	2	1	10
Demand	4	6	8	6	

(6) $= 3+4-1 = 6$

1 [4]	2 [2]	3	4	2 0
4	3 [4]	2 [4]	0	8 4 0
0	2	2 [4]	1 [6]	10 6
4 0	6 2 0	8 4 0	6	

Sol. Since all $\Sigma a_i = \Sigma b_j = 24$ then, the given transportation problem is Balanced. There exist an IBFS.

By using Vogel's approxiation method we find out the IBFS.

1	2 [6]	3	4	~~6~~ 0	1	1	1 ←	—	—
4	3	2 [2]	0 [6]	~~8~~ ~~2~~ 0	2 ←	1	1	1 ←	—
0 [4]	2	2 [6]	1	~~10~~ ~~6~~ 0	1	2 ←	0	0	2
~~4~~ 0	~~6~~ 0	~~8~~ ~~6~~ 0	~~6~~ 0	24					
1	0	0	1						
1	0	0	—						
—	0	0	—						
—	—	0	—						
—	—	2	—						

Since, the no. of non-negative allocations is 5 which is less than $(m+n-1) = 3+4-1 = 6$

This basic solution is degenerate one. To resolve degeneracy we allocate a small positive quantity 'ε' to the cell (1,1) so, that the no. of occupied cell become $m+n-1$. Hence the non-degenerate basic feasible solution in the following table.

1 [ε]	2 [6]	3	4
4	3	2 [2]	0 [6]
0 [4]	2	2 [6]	1

$\therefore$ Transportation Cost $= 1\times\varepsilon + 2\times6 + 2\times2 + 0\times6 + 0\times4 + 2\times6$

$= \varepsilon + 12 + 4 + 12 \quad (\because \varepsilon \to 0)$

$TC = 28$. Rs/-

To find Optimal Solution:-

Consider the above table, here $m+n-1 = 6$ we apply MODI method.

				u_i
1 [ε]	2 [6]	3 (3) 0	4 (1) 3	$u_1 = 1$
4 (0) 4	3 (1) 2	2 [2]	0 [6]	$u_2 = 0$
0 [4]	2 (1) 1	2 [6]	1 (0) 1	$u_3 = 0$
$v_1 = 0$	$v_2 = 1$	$v_3 = 2$	$v_4 = 0$	v_j

Since, all $d_{ij} \geq 0$ with $d_{13} = 0$ the no. of non-negative allocations $= m+n-1$

Allocations are $x_{11} = \varepsilon$, $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$ and $x_{33} = 6$

Optimal Cost $= 1\times\varepsilon + 2\times6 + 2\times2 + 0\times6 + 0\times4 + 2\times6 = 0 + 28 \quad (\because \varepsilon \to 0)$

$=$ Rs. 28/-

3. Solve the following transportation problem to minimize the total cost of transportation problem.

	1	2	3	4	Supply
A	14	56	48	27	70
B	82	35	21	81	47
C	99	31	71	63	93
Demand	70	35	45	60	210

Sol. Since, $\sum a_i = \sum b_j = 210$ the given transportation problem is balanced. There exist an IBFS.
By using VAM we find out the initial basic Feasible solution

	1	2	3	4	Supply					
A	14 [70]	56	48	27	~~70~~ 0	13	21	8	8	8
B	82	35 [2]	21 [45]	81	~~47~~ ~~37~~	14	14	14	14	35 ←
C	99	31 [33]	71	63 [60]	~~93~~ ~~33~~ 0	32	32	40 ←	—	—
Demand	~~70~~ 0	~~35~~ ~~2~~ 0	~~45~~ 0	~~60~~ 0	210					
	68 ↑	4	27	36						
	—	4	27	36 ↑						
	—	21	27	—						
	—	21	27 ↑	—						
	—	21	—	—						
	—	—	—	—						

Since the non-negativity allocation = 5 which is less than $m+n-1 = 3+4-1 = 6$. This basic solution is degenerate one. To resolve degeneracy we allocate a small positive quantity 'ε' to the cell (1,4) so that the no. of occupied cell become $(m+n-1)$. Hence the non-degenerate basic feasible solution in the following table.

14 [70]	56	48	27 [ε]
82	35 [2]	21 [45]	81
99	31 [33]	71	63 [60]

Transportation Cost $= 14\times70 + 27\times\varepsilon + 35\times2 + 21\times45 + 31\times33 + 63\times60$
$= 980 + 27\varepsilon + 70 + 945 + 102[3] + 3780$

$TC = 6798 \ (\varepsilon \to 0)$

<u>To find optimal Solution</u>:- Consider the above transportation table here $m+n-1 = 6$ we apply MODI method.

				u_i
14 [70]	56 51 (−5)	48 67 (−19)	27 [ε]	$u_1 = -40$
82 28 (54)	35 [2]	21 [45]	81 14 (67)	$u_2 = 0$
99 49 (50)	31 [33]	71 54 (17)	63 [60]	$u_3 = -4$
v_j: $v_1 = 54$	$v_2 = 35$	$v_3 = 21$	$v_4 = 67$	

Since, all $d_{ij} > 0$ then, the solution is optimal and unique.
Allocations are $x_{11} = 70, x_{14} = \varepsilon, x_{22} = 2, x_{23} = 45, x_{32} = 33, x_{34} = 60$

Optimal Cost $= 14\times70 + 27\times\varepsilon + 35\times2 + 21\times45 + 31\times33 + 63\times60$
$= 6798.$ Rs/- $(\varepsilon \to 0)$

4. Solve the following transportation problem by using VAM and find the optimal solution.

	D_1	D_2	D_3	D_4	D_5	supply
O_1	4	7	3	8	2	4
O_2	1	4	7	3	8	7
O_3	7	2	4	7	7	9
O_4	4	8	2	4	7	2
Demand	8	3	7	2	2	22

Sol. Since $\sum a_i = \sum b_j = 22$ then the given Transportation problem is Balance. There exist an IBFS

By using VAM we find out the initial basic feasible solution.

	D_1	D_2	D_3	D_4	D_5	Supply							
O_1	4 [1]	7	3	8	2 [2]	~~4~~ 2	1	1	1	1	1	1	4
O_2	1 [7]	4	7	3	8	~~7~~ 0	2	2	–	–	–	–	–
O_3	7	2 [3]	4 [6]	7	7	~~9~~ ~~6~~ 0	2	2	2	← 3	–	–	–
O_4	4	8	2 [1]	4 [2]	7	~~2~~ 0	2	2	2	2	2	← 2	4
Demand	~~8~~ ~~1~~ 0	~~3~~ 0	~~7~~ ~~1~~ 0	~~2~~ 0	~~2~~ 0	22							
	3	2	1	1	5↑								
	↑3	2	1	1	–								
	0	5↑	1	3	–								
	0	–	1	3	–								
	0	–	1	4↑	–								
	0	–	1	–	–								
	0	–	–	–	–								

	D_1	D_2	D_3	D_4	D_5	Supply							
O_1	4 [1]	7	3 [1]	8	2 [2]	~~4~~ ~~2~~ ~~1~~ 0	1	1	1	1	1	1	3
O_2	1 [7]	4	7	3	8	~~7~~ 0	2	2	–	–	–	–	–
O_3	7	2 [3]	4 [6]	7	7	~~9~~ ~~6~~ 0	2	2	2	← 3	–	–	–
O_4	4	8	2	4 [2]	7	~~2~~ 0	2	2	2	2	2	–	–
Demand	~~8~~ ~~1~~ 0	~~3~~ 0	~~7~~ ~~1~~ 0	~~2~~ 0	~~2~~ 0	22							
	3	2	1	1	↑5								
	↑3	2	1	1	–								
	0	5↑	1	3	–								
	0	–	1	3	–								
	0	–	1	4↑	–								
	4↑	–	3	–	–								
	–	–	3	–	–								

Since, the non-negative allocation = 7 which is less than $m+n-1 = 4+5-1 = 8$ This basic solution is degenerate one.

To resolve degeneracy we allocate a small positive quantity 'ε' to the cell (4,3). so, that the no. of occupied cell become $(m+n-1)$.

Hence the non-degenerate basic feasible solution in the following table

	D_1	D_2	D_3	D_4	D_5
O_1	4 [1]	7	3 [1]	8	2 [2]
O_2	1 [7]	4	7	3	8
O_3	7	2 [3]	4 [6]	7	7
O_4	4	8	2 [ε]	4 [2]	7

Transportation cost =

$= 4\times1 + 3\times1 + 2\times2 + 1\times7 + 2\times3 + 4\times6 + 2\times\varepsilon + 4\times2$

$= 4+3+4+7+6+24+2(0)+8$

$=$ Rs. 56/-

<u>To Find Optimal Solution</u>:- Consider the above transportation table here $m+n-1 = 8$. we apply MODI method.

						u_i
	$\boxed{1}$ 4	1 7 6	$\boxed{1}$ 3	5 8 3	$\boxed{2}$ 2	$u_1=0$
	$\boxed{7}$ 1	-2 4 6	0 7 7	2 3 1	-1 8 9	$u_2=-3$
	5 7 2	$\boxed{3}$ 2	$\boxed{6}$ 4	6 7 1	3 7 4	$u_3=1$
	+3 4 1	0 8 8	$\boxed{\varepsilon}$ 2	$\boxed{2}$ 4	1 7 6	$u_4=-1$
v_j	$v_1=4$	$v_2=1$	$v_3=3$	$v_4=5$	$v_5=2$	

Since, all $d_{ij}>0$ the solution is optimal and unique

Allocation are $x_{11}=1$, $x_{13}=1$, $x_{15}=2$, $x_{21}=7$, $x_{32}=3$, $x_{33}=6$, $x_{43}=\varepsilon$ and $x_{44}=2$

Optimal solution $= 4\times1+3\times1+2\times2+1\times7+2\times3+4\times6+2\times\varepsilon+4\times2$

$= 4+3+4+7+6+24+2(0)+8 \quad (\varepsilon\to0)$

$=$ Rs. 56/-

5. Solve the following transportation problem by using NWCR and LCM and find the optimal solution

warehouses

		A	B	C	D	E	F	Available
Factory	1	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
	Requirment	4	4	6	2	4	2	22

Sol. <u>In Northwest corner Rule</u>:-

Since, $\Sigma a_i = \Sigma b_j = 22$ the given transportation problem is balanced. There exist an IBFS.

By using NWCR we find out IBFS:-

	A	B	C	D	E	F	Available
1	9 $\boxed{4}$	12 $\boxed{1}$	9	6	9	10	~~5~~ ~~1~~ 0
2	7	3 $\boxed{3}$	7 $\boxed{3}$	7	5	5	~~6~~ ~~3~~ 0
3	6	5	9 $\boxed{2}$	11	3	11	~~2~~ 0
4	6	8	11 $\boxed{1}$	2 $\boxed{2}$	2 $\boxed{4}$	10 $\boxed{2}$	~~9~~ ~~8~~ ~~6~~ ~~7~~ 0
Requirement	~~4~~ 0	~~4~~ ~~3~~ 0	~~6~~ ~~3~~ ~~1~~ 0	~~2~~ 0	~~4~~ 0	~~2~~ 0	22

The no. of allocation non-negative

$9 = m+n-1$

$= 4+6-1$

$9 = 9$

Hence, the solution is non-degenerate feasible solution

Transportation Cost $= 9\times4+12\times1+3\times3+7\times3+9\times2+11\times1+2\times2+2\times4+10\times$

$= 36+12+9+21+18+11+4+8+20 = 139$

$=$ Rs. 139/-

<u>To find Optimal Solution</u>:- Consider the above transportation table here, $m+n-1=9$. We apply MODI Method.

						u_i
9 $\boxed{4}$	−θ $\boxed{1}$ 12	+θ 16 9 −7	7 6 −1	7 9 2	15 10 −5	$u_1=5$
0 7 7	+θ $\boxed{3}$ 3	$\boxed{3}$ 7 −θ	−2 7 9	−2 5 7	6 5 −1	$u_2=-4$
2 6 4	5 5 0	$\boxed{2}$ 9	0 11 11	0 3 3	8 11 3	$u_3=-2$
4 6 2	7 8 1	$\boxed{1}$ 11	$\boxed{2}$ 2	$\boxed{4}$ 2	$\boxed{2}$ 10	$u_4=0$
v_j: $v_1=4$	$v_2=7$	$v_3=11$	$v_4=2$	$v_5=2$	$v_6=10$	

Here, $d_{ij} \leq 0$
i.e, most negative d_{ij} is $d_{13}=-7$
$\text{Min}(3,1)=1$
Add '1' to cells having '+θ'
Sub '1' to cells having '−θ'

						u_i
−θ $\boxed{4}$ 9	5 12 7	+θ $\boxed{1}$ 9	0 6 6	0 9 9	8 10 2	$u_1=-2$
7 7 0	$\boxed{4}$ 3	$\boxed{2}$ 7	−2 7 9	−2 5 7	6 5 −1	$u_2=-4$
9 6 −3	5 5 0	$\boxed{2}$ 9	0 11 11	0 3 3	8 11 3	$u_3=-2$
+θ 11 6 −5	7 8 1	$\boxed{1}$ 11 −θ	$\boxed{2}$ 2	$\boxed{4}$ 2	$\boxed{2}$ 10	$u_4=0$
v_j: $v_1=11$	$v_2=7$	$v_3=11$	$v_4=2$	$v_5=2$	$v_6=10$	

Here, $d_{ij} \leq 0$
i.e, most negative d_{ij} is $d_{41}=-5$
$\text{Min}(4,1)=1$
Add '1' to cells having '+θ'
Sub '1' to cells having '−θ'

						u_i
−θ $\boxed{3}$ 9	5 12 7	+θ $\boxed{2}$ 9	5 6 1	5 9 4	13 10 −3	$u_1=3$
7 7 0	$\boxed{4}$ 3	$\boxed{2}$ 7 −θ	3 7 4	3 5 2	+θ 11 5 −6	$u_2=1$
✓ 9 6 −3	5 5 0	$\boxed{2}$ 9	5 11 6	5 3 −2	13 11 −2	$u_3=3$
+θ $\boxed{1}$ 6	2 8 6	6 11 5	$\boxed{2}$ 2	$\boxed{4}$ 2	$\boxed{2}$ 10 −θ	$u_4=0$
v_j: $v_1=6$	$v_2=2$	$v_3=6$	$v_4=2$	$v_5=2$	$v_6=10$	

Here, $d_{ij} \leq 0$
i.e, Most negative d_{ij} is $d_{26}=-6$
$\text{Min}(2,3)=2$
Add '2' to cells having '+θ'
Sub '2' to cells having '−θ'

						u_i
$\boxed{1}$ 9	3 12 9	$\boxed{4}$ 9	5 6 +1	5 9 4	5 10 5	$u_1=0$
+θ 9 7 −2	$\boxed{4}$ 3	9 7 −2	5 7 2	−θ $\boxed{\varepsilon}$ 5	$\boxed{2}$ 5	$u_2=0$
9 6	3 5 2	$\boxed{2}$ 9	5 11 6	5 3 −2	3 11 6	$u_3=0$
$\boxed{3}$ 6 −θ	0 8 8	6 11 5	$\boxed{2}$ 2	$\boxed{4}$ 2 +θ	10 8	$u_4=-3$
v_j: $v_1=9$	$v_2=3$	$v_3=9$	$v_4=5$	$v_5=5$	$v_6=5$	

Since, the non-negativity allocation = 8
which is less than
$m+n-1=4+6-1=9$
This basic solution is degenerate one.

$\boxed{1}$ 9 −θ	3 12 9	+θ $\boxed{4}$ 9	5 6 1	5 9 4	5 10 5	$u_1=0$
9 7 −2	$\boxed{4}$ 3	9 7 −2	5 7 2	$\boxed{\varepsilon}$ 5	$\boxed{2}$ 5	$u_2=0$
9 6 +θ −3	3 5 2	$\boxed{2}$ 9 −θ	5 11 6	5 3 −2	5 11 6	$u_3=0$
$\boxed{3}$ 6	0 8 8	6 11 5	$\boxed{2}$ 2	$\boxed{4}$ 2	2 10 8	$u_4=-3$
$v_1=9$	$v_2=3$	$v_3=9$	$v_4=5$	$v_5=5$	$v_6=5$	

Unbalanced Transportation Problem:-

The given transportation problem is said to be unbalanced if $\Sigma a_i \neq \Sigma b_j$ i.e, Total supply is not equal to total demand.

There are 2 possible cases:

case-(i):- If $\Sigma a_i < \Sigma b_j$ i.e, If the total supply is less than total demand.

A dammi source (row) is included in the cost matrix with zero cost, The excess demand is entered as a rim requirement for this dammi source (origin). Hence, the unbalanced transportation problem can be converted into a balanced transportation problem.

case-(ii):- If $\Sigma a_i > \Sigma b_j$ i.e, If the total demand less than total supply, a dammi ~~(source)~~ destination (column) is included in the cost matrix with zero cost, the excess supply is entered as a rim requirement for this dammi destination. Hence, the given transportation problem can be converted into a balanced transportation problem.

1. Solve the following transportation problem.

	D_1	D_2	D_3	D_4	Supply
O_1	6	1	9	3	70
O_2	11	5	2	8	55
O_3	10	12	4	7	70
Demand	85	35	50	45	$\Sigma a_i = 195$ $\Sigma b_j = 215$

Sol. Since $\Sigma a_i \neq \Sigma b_j$

i.e, the given transportation problem is unbalance

First we convert the given unbalanced transportation problem into balanced transportation problem.

Here, $\Sigma a_i < \Sigma b_j$ i.e, Total supply < Total Demand. In this case we add dammi origin (row) with zero cost and with supply 20 then the transportation problem is balanced.

	D_1	D_2	D_3	D_4	Supply							
O_1	6 [65]	1 [5]	9	3	~~70~~ 5 0	2	2	2	–	–	–	–
O_2	11	5 [30]	2 [25]	8	~~55~~ 25 0	3	3	3	3	6	–	–
O_3	10	12	4 [25]	7 [45]	~~70~~ 45 0	3	3	3	3	3	3	4
O_4	0 [20]	0	0	0	~~20~~ 0	0	–	–	–	–	–	–
Demand	~~85~~ ~~65~~ 0	~~35~~ 30 0	~~50~~ 25 0	~~45~~ 0	215							
	6↑	1	2	3								
	4↑	4	2	4								
	–	4↑	2	4								
	–	7↑	2	1								
	–	–	2	1								
	–	–	4	7↑								
	–	–	4↑	–								

The no. of non-negativity allocation is $m+n-1 = 4+4-1 = 7$

The no. of non-negativity allocation is $m+n-1 = 4+4-1 = 7$,
Hence the solution is non-degenerate IBFS
Allocations are $x_{11}=65$, $x_{12}=5$, $x_{22}=30$, $x_{23}=25$, $x_{33}=25$,
$x_{34}=45$ and $x_{41}=20$

Transportation cost $= 6\times65 + 1\times5 + 5\times30 + 2\times25 + 4\times25 + 7\times45 + 0\times20$

$TC = 1010$- Rs/-

To find Optimal solution:-

Consider the above transportation problem, here $m+n-1=7$ we apply MODI method.

					u_i
	6, $-\theta$, [65]	1, $+\theta$, [5]	9, −2, 11	3, 1, 2	$u_1=0$
	11, 10, 1	5, [30], $-\theta$	2, $+\theta$, [25]	8, 5, 3	$u_2=4$
	10, 12, $+\theta$, −2	12, 7, 5	4, [25], $-\theta$	7, [45]	$u_3=6$
	0, [20]	0, −5, 5	0, −8, +8	0, −5, 5	$u_4=-6$
v_j	$v_1=6$	$v_2=1$	$v_3=-2$	$v_4=1$	

since some $d_{ij}<0$ the solution is not optimal

since $d_{ij}=-2$ is the most-negative. Let us form a new basic feasible solution by giving maximum allocation the corresponding allocation cell (3,1) by making an occupied cell empty. we draw a closed loop

					u_i
	6, [40]	1, [30]	9, −2, 11	3, 3, 0	$u_1=6$
	11, 10, 1	5, [5]	2, [50]	8, 7, 1	$u_2=10$
	10, [25]	12, 5, 7	4, 2	7, [45]	$u_3=10$
	0, [20]	0, −5, 5	0, −8, 8	0, +3, 3	$u_4=10$
v_j	$v_1=0$	$v_2=-5$	$v_3=-8$	$v_4=-3$	

i.e, $d_{14} \geq 0$

Since, $d_{ij} \geq 0$ then, the solution is optimum but there will be alternative solution to given transportation problem.

$\therefore$ Optimal allocations are
$x_{11}=40$, $x_{12}=30$, $x_{22}=5$,
$x_{23}=50$, $x_{31}=25$, $x_{34}=45$
and $x_{41}=20$

$\therefore$ Optimal allocations $=$
$= 6\times40 + 1\times30 + 5\times5 + 2\times50 + 10\times25 + 7\times45 + 0\times20$
$= 960$. Rs/-

					u_i
	6, [40]	1, [30]	9	3	$u_1=$
	11	5, [5]	2, [50]	8	$u_2=$
	10, [45]	12	4	7, [05]	$u_3=$
	0	0	0	0, [20]	$u_4=$
v_j	$v_1=$	$v_2=$	$v_3=$	v_4	

2. Solve the transportation problem.

Destinations

Origins	D_1	D_2	D_3	Supply
O_1	5	6	9	100
O_2	3	5	10	75
O_3	6	7	6	50
O_4	6	4	10	75
Demand	70	80	120	$\Sigma a_i=300$ $\Sigma b_j=270$

Sol. Since $\Sigma a_i \neq \Sigma b_j$ i.e, the given transportation problem is unbalanced to convert this into a balanced one. we Introduce a dammi destination with zero cost and having demand '30'.

	D_1	D_2	D_3	D_4	Supply						
O_1	5	6	9 [100]	0	~~100~~ 0	5	1	1	3	3	9 ←
O_2	3 [70]	5 [5]	10	0	~~75~~ ~~5~~ 0	3	2	2 ←	5	5 ←	—
O_3	6	7	6 [20]	0 [30]	~~50~~ ~~20~~ 0	6 ←	0	—	—	—	—
O_4	6	4 [75]	10	0	~~75~~ 0	4	2	2	6 ←	—	—
Demand	~~70~~ 0	~~80~~ ~~5~~ 0	~~120~~ ~~100~~ 0	~~30~~ 0	300						
	2	1	3	0							
	2	1	3 ↑	—							
	2	1	1	—							
	—	1	1	—							
	—	1	1	—							
	—	—	9	—							

Here, the no. of non-negative allocation = 6 which is less than $m+n-1 = 4+4-1 = 7$

This basic feasible solution is degenerate.

5	6	9 [100]	0
3 [70]	5 [5]	10	0 [ε]
6	7	6 [20]	0 [30]
6	4 [75]	10	0

To resolve degeneracy we allocate a small positive quantity 'ε' in the cell (2,4). so that the no. of occupied cells become 7.
Hence, the solution non-degenerate Basic Feasible solution is given in the following table.

Transportation Cost $= 9\times100 + 3\times70 + 5\times5 + 0\times\varepsilon + 6\times20 + 0\times30 + 4\times75$
$= 1555/-$ Rs.

Now the no. of non-negativity allocations at the independent positions $m+n-1$. we apply modi method to find optimal sol?

<u>To find Optimal solution</u>:-

<table>
<tr><td>6
5   −1</td><td>8
6   −2</td><td>[100]
9</td><td>3
0   −3</td><td>$u_1 = 3$</td></tr>
<tr><td>[70]
3</td><td>[5]
5</td><td>6
10   4</td><td>[ε]
0</td><td>$u_2 = 0$</td></tr>
<tr><td>3
6   3</td><td>5
7   2</td><td>[20]
6</td><td>[30]
0</td><td>$u_3 = 0$</td></tr>
<tr><td>2
6   4</td><td>[75]
4</td><td>5
10   5</td><td>−1
0   1</td><td>$u_4 = -1$</td></tr>
<tr><td>$v_1 = 3$</td><td>$v_2 = 5$</td><td>$v_3 = 6$</td><td>$v_4 = 0$</td><td>u_i / v_j</td></tr>
</table>

Since, some $d_{ij} < 0$ the solution is not optimal
since $d_{14} = -3$ is the most negative.
Let us form a new basic feasible solution by giving maximum allocation to corresponding cell (1,4) by making an occupied cell empty.

We draw a closed loop beginning and ending with the cell (1,4) and having its other corners at some occupied cells.
Along with this loop give '+θ' and '−θ' alternatively at the corners.

5	6	9 −θ [100]	0 +θ
3 [70]	5 [5]	10	0 [ε]
6	7	6 +θ [20]	0 −θ [30]
6	4 [75]	10	0

From the cells (1,3) (3,4) having '−θ' we find the minimum of the allocation (100, 30) is 30 add this 30 to '+θ' and substract this 30 to '−θ' and we have.

5	6	9 [70]	0 [30]
3 [70]	5 [5]	10	0 [ε]
6	7	6 [50]	0
6	4 [75]	10	0

we see that the above table satisfies the rim requirements with m+n-1 non-negative allocations at independent

Now, we apply MODI method.

				u_i
5, 3, 12	6, 5, 1	9 [70]	0 [30]	$u_1=0$
3 [70]	5 [5]	10, 9, 1	0 [ε]	$u_2=0$
6, 0, 6	7, 2, 5	6 [50]	0, -3, 3	$u_3=-3$
6, 2, 4	4 [75]	10, 8, 2	0, -1, 1	$u_4=-1$
v_j: $v_1=3$	$v_2=5$	$v_3=9$	$v_4=0$	

since, all $d_{ij}>0$, this is optimum and unique

Transportation cost =

$= 9\times70+0\times30+70\times3+5\times5+0\times\varepsilon+6\times50+4\times75$

$= 1465/-.$ Rs

3. Solve the following transportation problem.

	D_1	D_2	D_3	D_4	supply
O_1	11	20	7	8	50
O_2	21	16	20	12	40
O_3	8	12	18	9	70
Demand	30	25	35	40	$\Sigma a_i=160$ $\Sigma b_j=130$

Sol. Since, $\Sigma a_i \neq \Sigma b_j$ the given transportation problem is unbalanced. To convert this into a balanced one we introduced a dammi source with zero cost and having demand '30'.

	D_1	D_2	D_3	D_4	D_5	Supply							
O_1	~~11~~	~~20~~	~~7~~ [35]	~~8~~ [15]	~~0~~	~~50~~ ~~15~~ 0	7	1	3	3	← 3	—	—
O_2	~~21~~	16	20	12 [10]	0 [30]	~~40~~ ~~10~~ 0 ←	12	2	← 4	—	—	—	—
O_3	8 [30]	12 [25]	18	9 [15]	0	~~70~~ 45 ~~20~~ 0	8	1	1	1	1	1	8
Demand	~~30~~ 0	~~25~~ 0	~~35~~ 0	~~40~~ 30 ~~15~~ 0	~~30~~ 10 0	160							
	3	4	11	1	0								
	3	4	11↑	1	—								
	3	4	—	1	—								
	3	8↑	—	1	—								
	3	—	—	1	—								
	8	—	—	9↑	—								
	8↑	—	—	—	—								

Here, the no. of non-negative allocations is $7 = m+n-1 = 3+5-1 = 7$

Hence, the solution is non-degenerate IBFS

Allocations are $x_{13}=35$, $x_{14}=15$, $x_{24}=10$, $x_{25}=30$, $x_{31}=30$, $x_{32}=25$ and $x_{34}=15$

$\therefore$ Initial Transportation Cost $= 7\times35+8\times15+12\times10+0\times30+8\times30+25\times12+9\times15 = 1160$.Rs

$TC = 1160/-$

To find Optimal Solution:-

consider the above transportation problem, here $m+n-1=7$ we apply MODI method.

					u_i
7 11 4	11 20 9	$\boxed{35}$ 7	$\boxed{15}$ 8	-1 0 1	$u_1=8$
11 21 10	15 16 1	11 20 9	$\boxed{10}$ 12	$\boxed{30}$ 0	$u_2=12$
$\boxed{30}$ 8	$\boxed{25}$ 12	8 18 10	$\boxed{15}$ 09	0 0 0	$u_3=9$
v_j: $v_1=-1$	$v_2=3$	$v_3=-1$	$v_4=0$	$v_5=-9$	

Since, $d_{ij} \geq 0$ then the solution is optimum but, there will be alternative solution to given transportation problem

$\therefore$ Optimal Sol^n $= 7\times35+8\times15+12\times10+0\times30+8\times30+12\times25+9\times15$
$= 245+120+120+0+240+300+135 = 1160$
$=$ Rs. 1160/-

Maximization Case in Transportation Problem:-

Here, the objective is to maximize the total profit for which the profit matrix is given. For this first we have to convert the maximization problem into minimization. By substract all the elements from the highest element in the given transportation problem. This modified minimization problem can be solved in the usual manner.

1. Solve the following transportation problem to maximize profit

Destinations

Origin	D_1	D_2	D_3	D_4	Supply
O_1	40	25	22	33	100
O_2	44	35	30	30	30
O_3	38	38	28	30	70
Demand	40	20	60	30	

Sol. Since, the given problem is of maximization type. First convert this into minimization problem. By substracting with the cost elements from the maximum cost element.

	D_1	D_2	D_3	D_4	supply
O_1	4	19	22	11	100
O_2	0	9	14	14	30
O_3	6	6	16	14	70
Demand	40	20	60	30	$200=\Sigma a_i$ $150=\Sigma b_j$

Since, $\Sigma a_i \neq \Sigma b_j$ the given transportation problem is unbalanced. To convert this into the balanced one we introduce a dammi source with zero cost and having demand '50'.

	D_1	D_2 ✗	D_3 ✗	D_4 ✗	D_5 ✗	Supply							
O_1	4 [10]	19	22 [60]	11 [30]	0	~~100~~ ~~40~~ ~~10~~ 0	4	7	7	7	7	7	4
✗ O_2	0 [30]	9	14	14	0	~~30~~ 0	0	← 9	–	–	–	–	–
✗ O_3	6	6 [20]	16	14	0 [50]	~~70~~ ~~20~~ 0	← 6	0	0	8	–	–	–
Demand	~~40~~ ~~10~~ 0	~~20~~ 0	~~60~~ 0	~~30~~ 0	~~50~~ 0	200							
	4	3	2	3	0								
	4	3	2	3	–								
	2	13 ↑	6	3	–								
	2	–	6	3	–								
	4	–	22 ↑	11	–								
	4	–	–	11 ↑	–								
	4	–	–	–	–								

The no. of non-negative allocation is 6 is less than $m+n-1 = 3+5-1 = 7$

$6 < 7$

This basic solution is degenerate one.

To resolve degeneracy we allocate a small positive quantity 'ε' in the cell (1,5). So that the no. of occupied cells become 7. Hence the solution is non-degenerate basic feasible solution is given in the following table.

Allocations are $x_{11} = 10$, $x_{13} = 60$, $x_{14} = 30$, $x_{15} = \varepsilon$, $x_{21} = 30$, $x_{32} = 20$ and $x_{35} = 50$

4 [10]	19	22 [60]	11 [30]	0 [ε]
0 [30]	9	14	14	0
6	6 [20]	16	14	0 [50]

Transportation cost $= 4\times10 + 22\times60 + 11\times30 + 0\times\varepsilon + 0\times30 + 6\times20 + 0\times50$

$= 40 + 1320 + 330 + 120$

TC $=$ Rs. 1810/-

Now the no. of non-negativity allocations at the independent positions $m+n-1$ we apply modi method to find optimal solution.

<u>To find Optimal solution</u>:-

					u_i
4 [10]	6 19 13	$-\theta$ [60] 22	[30] 11	$+\theta$ [ε] 0	$u_1 = 0$
0 [30]	2 9 7	18 14 −4	7 14 7	−4 0 4	$u_2 = -4$
4 6 2	6 [20]	$+\theta$ 22 16 −6	11 14 3	[50] 0 $-\theta$	$u_3 = 0$
v_j: $v_1 = 4$	$v_2 = 6$	$v_3 = 22$	$v_4 = 11$	$v_5 = 0$	

Since, some $d_{ij} < 0$ the sol^n is not optimal.

Since, $d_{33} = -6$ is the most negative

Let us form a new basic feasible solution by giving maximum allocation to the corresponding cell (3,3) by making an occupied cell empty. We draw a closed loop beginning and ending with the cell (3,3) and having its other corners at some occupied cells

Along with this loop give '+θ' and '−θ' alternatively at the corners.

From the two cells (3,1)(3,5) having '−θ' we find the minimum of the allocations (60, 50) is 50. add this 50 to '+θ' and substract this 50 to '−θ' and we have. and apply modi method.

					u_i
4 $\boxed{10}$	19 12 7	22 $\boxed{10}$	11 $\boxed{30}$	0 $\boxed{50}$	$u_1=0$
0 $\boxed{30}$	9 8 1	14 18 −4	14 7 7	0 −4 4	$u_2=-4$
6 −2 8	6 $\boxed{20}$	16 $\boxed{50}$	14 5 9	0 −6 6	$u_3=-6$
v_j: $v_1=4$	$v_2=12$	$v_3=22$	$v_4=11$	$v_5=0$	

Since, $d_{ij}<0$ then the sol^n is not optimal.
$d_{23}=-4$ is the most negative.

Let us form a new basic feasible solution by giving maximum allocation to the corresponding cell(2,3) by making an occupied cell empty. we draw a closed loop beginning and ending with the cell (2,3) and having its other corners at some occupied cells. Along with this loop give '+θ' and '−θ' alternatively at the corners.

+θ 4 $\boxed{10}$	19	−θ 22 $\boxed{10}$	11 $\boxed{30}$	0 $\boxed{50}$
0 −θ $\boxed{30}$	9	14 +θ	14	0
6	6 $\boxed{20}$	16 $\boxed{50}$	14	0

From the two cells (2,1)(1,3) having '−θ' we find the minimum of the allocations (30, 10) is 10.
Add this 10 to '+θ' and substract this 10 to '−θ' and we have and apply MODI method.

					u_i
4 $\boxed{20}$	19 8 11	22 18 4	11 $\boxed{30}$	0 $\boxed{50}$	$u_1=0$
0 $\boxed{20}$	9 4 5	14 $\boxed{10}$	14 7 7	0 −4 4	$u_2=-4$
6 2 4	6 $\boxed{20}$	16 $\boxed{50}$	14 9 5	0 −2 2	$u_3=-2$
v_j: $v_1=4$	$v_2=8$	$v_3=18$	$v_4=11$	$v_5=0$	

Since, $d_{ij}>0$ then the solution is optimum and unique
Allocations are $x_{11}=20$, $x_{14}=30$, $x_{15}=50$, $x_{21}=20$, $x_{23}=10$, $x_{32}=20$ and $x_{33}=50$

The optimum profit is $=40\times20+33\times30+44\times20+30\times10+38\times20+28\times50$
$=$ Rs. 5130/-

2. Solve the following transportation problem to maximize profit.

	D_1	D_2	D_3	D_4	Supply
O_1	15	51	42	33	23
O_2	80	42	26	81	44
O_3	90	40	66	60	33
Demand	23	31	16	30	100

Sol. Since, the given problem is maximization type. First convert this into minimization problem. By substracting with the cost elements from the maximum cost element.

	D_1	D_2	D_3	D_4	Supply
O_1	75	39	48	57	23
O_2	10	48	64	9	44
O_3	0	50	24	30	33
Demand	23	31	16	30	100

since, $\Sigma a_i = \Sigma b_j$, the given transportation problem is Balanced.

By. vogel's approximation method we get the transportation cost.

	D_1	D_2	D_3	D_4	Supply						
O_1	75	39 $\boxed{23}$	48	57	~~23~~ 0	9	18	18	18	39	39
O_2	~~10~~ $\boxed{6}$	48 $\boxed{8}$	~~64~~	9 $\boxed{30}$	~~44~~ ~~38~~ 8 0	1	1	1	39	← 48	—
O_3	~~0~~ $\boxed{17}$	~~50~~	~~24~~ $\boxed{16}$	~~30~~	~~33~~ ~~17~~ 0	24	← 30	—	—	—	—
Demand	~~23~~ ~~6~~ 0	~~31~~ ~~23~~ 0	~~16~~ 0	~~30~~ 0	100						
	10	9	↑24	21							
	10	9	—	21							
	65↑	9	—	48↑							
	~~65~~	9	—	48↑							
	—	9	—	—							
	—	39	—	—							

The no. of non-negativity allocation is $6 = m+n-1 = 3+4-1 = 6$

Hence the solⁿ is non-degenerate IBFS

Allocations are $x_{12}=23$, $x_{21}=6$, $x_{22}=8$, $x_{23}=30$, $x_{31}=17$ and $x_{33}=16$

Transportation Cost $= 39\times23 + 10\times6 + 48\times8 + 9\times30 + 0\times17 + 24\times16$

$= 897 + 60 + 384 + 270 + 0 + 384 = 1995$

TC $=$ Rs. 1995/-

<u>To find Optimal Solution</u> :- Consider the above transportation table $m+n-1 = 6$ and we apply MODI method.

					u_i
	75 (1) 74	39 $\boxed{23}$	48 (25) 23	57 (0) 57	$u_1=-9$
	10 $\boxed{6}$	48 $\boxed{8}$	64 (34) 30	9 $\boxed{30}$	$u_2=0$
	0 $\boxed{17}$	50 (38) 12	24 $\boxed{16}$	30 (−1) 31	$u_3=-10$
v_j	$v_1=10$	$v_2=48$	$v_3=34$	$v_4=9$	

since all $d_{ij} > 0$ then, the solution is optimum and unique

Allocations are $x_{12}=23$, $x_{21}=6$, $x_{22}=8$, $x_{23}=30$, $x_{31}=17$ and $x_{33}=16$

Optimum profit $= 51\times23 + 80\times6 + 42\times8 + 81\times30 + 90\times17 + 66\times16$

Original Cost with allocation $=$ Rs. 7005/-

Assignment Problem

* **Introduction:-** The Assignment problem is a particular case of the transportation problem in which the objective is to assign. A number of tasks or jobs or origins or sources to an equal number of facilities or machiner or persons or destinations, at a minimum cost or maximum profit.

* **Definitions:-** Suppose there are n jobs to be performed and 'n' persons are available for doing these jobs. Assume that each person can do each job at a time. Thought with the varing degree of efficiency. Let 'c_{ij}' be the cost if the i^{th} person is assign to j^{th} job. The problem is to find an assignment so that the total cost of performing all jobs is minimum. Problems of this kind are known as Assignment Problems.

The Assignment Problem can be shown in the form of nxn cost matrix as given in the following table.

Persons (m persons)

$$\text{Jobs}\quad\begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ j \\ \vdots \\ n \end{array}
\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & \cdots & i & \cdots & n \end{array} \\
\begin{bmatrix} c_{11} & c_{21} & \cdots & c_{i1} & \cdots & c_{n1} \\ c_{12} & c_{22} & \cdots & c_{i2} & \cdots & c_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{1j} & c_{2j} & \cdots & c_{ij} & \cdots & c_{nj} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{1n} & c_{2n} & \cdots & c_{in} & \cdots & c_{nn} \end{bmatrix} \end{array}$$

* **Mathematical Formulation of an Assignment Problem:-**

Consider an assignment problem of assigning 'n' jobs to 'n' machines [1 job to 1 machine] Let, c_{ij} be the unit cost of assigning i^{th} machine to the j^{th} job and Let,

$$x_{ij} = \begin{cases} 1, & \text{if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\ 0, & \text{if } j^{th} \text{ job is not assigned to } i^{th} \text{ machine} \end{cases}$$

The assignment model is done by the following LPP:-

$$\text{Min } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$

s. to constraints $\sum_{i=1}^{n} x_{ij} = 1,\ j = 1, 2 \ldots n$

$$\sum_{j=1}^{n} x_{ij} = 1,\ i = 1, 2 \ldots n$$

and $x_{ij} = 0$ (or) 1

Difference between Transportation problem and Assignment Problem.

5M **

Transportation problem	Assignment Problem.
1. No. of sources and destinations need not be equal. Hence, the cost matrix is not a square matrix.	1. Since assignment is done on a one-one basis. the no. of sources and destinations are equal. Hence, the cost matrix must be a square matrix.
2. x_{ij}, the quantity to be transported from i^{th} origin to j^{th} destination can take any possible positive value and it satisfies the rim requirements.	2. x_{ij} the j^{th} job is to be assigned to the i^{th} person and can take either the value 1 or 0.
3. The capacity and the requirement value is equal to a_i and b_j for the i^{th} source and j^{th} destination ($i=1,2---m$, $j=1,2---n$)	3. The capacity and the requirement value is exactly '1' i.e, for each source of each destination the capacity and the requirement value is exactly 1
4. The problem is unbalanced. If the total supply and total demand are not equal.	4. The problem is unbalanced, if the cost matrix is not a square matrix.

Hungarian Algorithm or Method:-

First check whether the no. of rows is equals to no. of columns. The assignment problem is said to be Balanced. then, proceed to step-1. If it is not balanced then it should be balanced before applying the algorithm.

Step-1: Prepare a cost matrix, if the cost matrix is not a square matrix then add a dammi row or column with zero cost element.

Step-2: Substract the smallest (or) minimum element from each rows from all the elements from the respective rows.

Step-3: In the reduced matrix, locate the smallest element of each column and then substract the same from each element of the column. Each row and column now have atleast '1' zero.

Step-4: In the modified matrix obtained in step-3 search for an optimal assignment as follows.

4.1: Examine the rows successively until a row with a

* 5M ** Difference between Transportation problem and Assignment Problem.

Transportation problem	Assignment Problem
1. No. of sources and destinations need not be equal. Hence, the cost matrix is not a square matrix.	1. Since assignment is done on a one-one basis. the no. of sources and destinations are equal. Hence, the cost matrix must be a square matrix.
2. x_{ij}, the quantity to be transported from i^{th} origin to j^{th} destination can take any possible positive value and it satisfies the rim requirements.	2. x_{ij} the j^{th} job is to be assigned to the i^{th} person and can take either the value 1 or 0.
3. The capacity and the requirement value is equal to a_i and b_j for the i^{th} source and j^{th} destination ($i=1,2 \ldots m$, $j=1,2 \ldots n$)	3. The capacity and the requirement value is exactly '1' i.e, for each source of each destination the capacity and the requirement value is exactly 1
4. The problem is unbalanced. If the total supply and total demand are not equal.	4. The problem is unbalanced, if the cost matrix is not a square matrix.

* **Hungarian Algorithm or Method:-**

First check whether the no. of rows is equals to no. of columns. The assignment problem is said to be Balanced. then, proceed to step-1. If it is not balanced then it should be balanced before applying the algorithm.

Step-1: Prepare a cost matrix, if the cost matrix is not a square matrix then add a dammi row or column with zero cost element.

Step-2: Substract the smallest (or) minimum element from each rows from all the elements from the respective rows.

Step-3: In the reduced matrix, locate the smallest element of each column and then substract the same from each element of the column. Each row and column now have atleast '1' zero.

Step-4: In the modified matrix obtained in step-3 search for an optimal assignment as follows.

4.1: Examine the rows successively until a row with a single zero is obtain. Enrectangle (☐) is zero and Cross of all other zeroes in its column. Continue in this manner until all the rows have been taken care of.

4.2: Repeat the procedure for each column of the reduced matrix.

4.3: If a row or column has 2 or more 0's and One cannot be choosen by inspection then assign orbitrally any one cannot be zeroes and cross of all other 0's of that row or columns.

4.4: Repeat "4.1" to "4.3" above successively until the change of assigning or cross ends.

Step-5: If the no. of assignments i.e, N=order of the cost matrix (n) an optimal solution is reached. If the no. of assignments is less than 'n' then go to step-6

Step-6: Draw the minimum no. of horizontal or vertical lines to cover all the 0's of the reduced matrix. This can be alone by using a simple procedure.

6.1: Mark (✓) rows that do not have any assigned '0'.

6.2: Mark (✓) columns that have 0's in the marked rows.

6.3: Mark (✓) rows that have assigned 0's in the marked columns.

6.4: Repeat 6.2 and 6.3 until the above chain of marking is completed.

6.5: Draw lines through all the unmarked rows and marked columns.

This gives us the desired minimum no. of lines.

Step-7: Develop the revised cost matrix as follows.

7.1: Find the smallest element of the reduced matrix not covered by any '1' of the lines.

7.2: Substract this element from all the uncovered elements and add the some to all the celements lying at the intersection of any two lines.

Step-8: Go to step-4 and repeat the procedure until an optimal solution is obtain.

1. Solve the following Assignment Problem

Persons

$$
\begin{array}{cc}
 & \begin{array}{cccc} A & B & C & D \end{array} \\
\text{Jobs} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} &
\begin{bmatrix} 18 & 26 & 17 & 11 \\ 13 & 28 & 14 & 26 \\ 38 & 19 & 18 & 15 \\ 19 & 26 & 24 & 10 \end{bmatrix}
\end{array}
$$

Sol. Since, the no. of rows = no. of colums, the given assignment problem is balanced.

Substracting the smallest element of each from every element of corresponding row we get the reduced matrix

$$\begin{bmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

Substracting the smallest element of each column of reduced matrix. From every element of corresponding column we get the reduced matrix

$$\begin{bmatrix} 7 & 11 & 5 & 0 \\ 0 & 11 & 0 & 13 \\ 23 & 0 & 2 & 0 \\ 9 & 12 & 13 & 0 \end{bmatrix}$$

In the above matrix we orbitrary enrectangle a zero in column 1 because row 2 had 2 0's it may be noted that column 3 and row 4 do no have any assignment.

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \cancel{0} & 13 \\ 23 & \boxed{0} & 2 & \cancel{0} \\ 9 & 12 & 13 & \cancel{0} \end{bmatrix}$$ ∴ no. of assignments, N=3 order of matrix, n=4

N<n we goto next step

(i) Since, row 4 does not have any assignment to we mark this row (✓)

(ii) Now there is a zero in the 4th column of the marked row so we mark 4th column.

(iii) Further there is an assignment in the 1st row of the marked column so we mark first row.

(iv) Draw straight lines through all unmarked rows and marked column we have

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \cancel{0} & 13 \\ 23 & \boxed{0} & 2 & \cancel{0} \\ 9 & 12 & 13 & \cancel{0} \end{bmatrix} \begin{matrix} ✓ \\ \\ \\ ✓ \end{matrix}$$

The smallest element not covered by line i.e, 5 subtracting this element from all the uncovered element and adding the same through all the elements lying at the intersection of the line we obtain the following reduced matrix.

$$\begin{bmatrix} 2 & 6 & 0 & 0 \\ 0 & 11 & 0 & 18 \\ 23 & 0 & 2 & 5 \\ 4 & 7 & 8 & 0 \end{bmatrix}$$

Now search for optimal assignment to the reduced matrix we get

$$\begin{bmatrix} 2 & 6 & \boxed{0} & \cancel{0} \\ \boxed{0} & 11 & \cancel{0} & 18 \\ 23 & \boxed{0} & 2 & 5 \\ 4 & 7 & 8 & \boxed{0} \end{bmatrix}$$

Since each row and each column has only one assignment an optimum solution is reached.

Optimal assignments and optimal cost of assignments is

Jobs	Persons	Cost
1	C	17
2	A	13
3	B	19
4	D	0
		Rs.59

$1 \to C,\ 2 \to A,\ 3 \to B,\ 4 \to D$ with minimum cost Rs.59/-

2. Solve the following Assignment Problem.

$$\begin{array}{c|ccccc} & \multicolumn{5}{c}{\text{Jobs}} \\ & 1 & 2 & 3 & 4 & 5 \\ \hline A & 8 & 4 & 2 & 6 & 1 \\ B & 0 & 9 & 5 & 5 & 4 \\ C & 3 & 8 & 9 & 2 & 6 \\ D & 4 & 3 & 1 & 0 & 3 \\ E & 9 & 5 & 8 & 9 & 5 \end{array}$$

Sol. Since the no. of rows = no. of columns, the given assignment problem is balanced.

Substracting the smallest element of each from every element of the corresponding row we get the reduced matrix.

$$\begin{bmatrix} 7 & 3 & 1 & 5 & 0 \\ 0 & 9 & 5 & 5 & 4 \\ 1 & 6 & 7 & 0 & 4 \\ 4 & 3 & 1 & 0 & 3 \\ 4 & 0 & 3 & 4 & 0 \end{bmatrix}$$

Substracting the smallest element of each column of reduced matrix from every element of corresponding column we get the reduced matrix.

$$\begin{bmatrix} 7 & 3 & 0 & 5 & 0 \\ 0 & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & 0 & 4 \\ 4 & 3 & 0 & 0 & 3 \\ 4 & 0 & 2 & 4 & 0 \end{bmatrix}$$

Starting the row we get enrectangle a zero and cross of all other '0's in the column then we get.

$$\begin{bmatrix} 7 & 3 & \cancel{0} & 5 & \boxed{0} \\ \boxed{0} & 9 & 4 & 5 & 4 \\ 1 & 6 & 6 & \boxed{0} & 4 \\ 4 & 3 & \boxed{0} & \cancel{0} & 3 \\ 4 & \boxed{0} & 2 & 4 & \cancel{0} \end{bmatrix}$$

Jobs	Persons	Cost
1	B	0
2	E	5
3	D	1
4	C	2
5	A	1
		Rs.9/-

$1 \to B,\ 2 \to E,\ 3 \to D,\ 4 \to C,\ 5 \to A$ with minimum cost Rs.9/-

3. Solve the following Assignment Problem.

$$\begin{array}{c|cccc} & \multicolumn{4}{c}{\text{Persons}} \\ \text{Jobs} & A & B & C & D \\ \hline 1 & 5 & 7 & 11 & 6 \\ 2 & 8 & 5 & 9 & 6 \\ 3 & 4 & 7 & 10 & 7 \\ 4 & 10 & 4 & 8 & 3 \end{array}$$

Sol. Since the no. of rows = no. of columns the assignment problem is balanced.

Substracting the smallest element of each from every element of corresponding row we get the reduced matrix.

$$\begin{bmatrix} 0 & 2 & 6 & 1 \\ 3 & 0 & 4 & 1 \\ 0 & 3 & 6 & 3 \\ 7 & 1 & 5 & 0 \end{bmatrix}$$

Substracting the smallest element of each column of reduced matrix from every element of corresponding column we get the reduced matrix.

$$\begin{bmatrix} 0 & 2 & 2 & 1 \\ 3 & 0 & 0 & 1 \\ 0 & 3 & 2 & 3 \\ 7 & 1 & 1 & 0 \end{bmatrix}$$

In the above matrix we orbitrary enrectangle a '0' in column 1 because column 2 had 2 zero''s it may be noted that row 3 and column 3 do not have any assignment.

$$\begin{bmatrix} \boxed{0} & 2 & 2 & 1 \\ 3 & \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & \boxed{0} \end{bmatrix}$$ ∴ no. of assignment N=3, order of matrix n=4
N<n we go to next step.

$$\begin{bmatrix} \boxed{0} & 2 & 2 & 1 \\ 3 & \boxed{0} & \cancel{0} & 1 \\ \cancel{0} & 3 & 2 & 3 \\ 7 & 1 & 1 & \boxed{0} \end{bmatrix}$$ ✓ The smallest element not covered by line. i.e, '1' substracting this element from all the uncovered element and adding at the same through all the elements lying at the same of the line we get the following reduced matrix.

$$\begin{bmatrix} \cancel{0} & 1 & 1 & \cancel{0} \\ 4 & \boxed{0} & \cancel{0} & 1 \\ \boxed{0} & 2 & 1 & 2 \\ 8 & 1 & 1 & \boxed{0} \end{bmatrix}$$ N<n 3<4 ∴ no. of assignment N=3, order of matrix n=4
N<n we go to next step.

$$\begin{bmatrix} \cancel{0} & 1 & 1 & \cancel{0} \\ 4 & \boxed{0} & \cancel{0} & 1 \\ \boxed{0} & 2 & 1 & 2 \\ 8 & 1 & 1 & \boxed{0} \end{bmatrix}$$ ✓ The smallest element not covered by line i.e, '1' sub this ele from all the uncovered ele and add the same through all the ele lying at the inter-section of the line we get.

$$\begin{bmatrix} \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 5 & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & 1 & \boxed{0} & 1 \\ 8 & \cancel{0} & \cancel{0} & \boxed{0} \end{bmatrix}$$

Since each row and each column has only one assignment, an optimal solution is reached.
Optimal ~~solution is rea~~ assignments and optimal cost of assignment is,

Jobs	Persons	Cost
1	A	5
2	B	5
3	C	10
4	D	3
		Rs. 23/-

1→A, 2→B, C→3, 4→D with minimum cost Rs. 23/-

4. Solve the following Assignment Problem.

Machines

$$\begin{array}{c|ccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline J_1 & 9 & 22 & 58 & 11 & 19 \\ J_2 & 43 & 78 & 72 & 50 & 63 \\ J_3 & 41 & 28 & 91 & 37 & 45 \\ J_4 & 74 & 42 & 27 & 49 & 39 \\ J_5 & 36 & 11 & 57 & 22 & 25 \end{array}$$

Jobs

Sol. Since, the no. of rows = no. of columns, the given assignment problem is balanced.

Substracting the smallest element of each from every element corresponding row we get the reduced matrix.

$$\begin{bmatrix} 0 & 13 & 49 & 2 & 10 \\ 0 & 35 & 29 & 7 & 20 \\ 13 & 0 & 63 & 9 & 17 \\ 47 & 15 & 0 & 22 & 12 \\ 25 & 0 & 46 & 11 & 14 \end{bmatrix}$$

Substracting the smallest element of each column of reduced matrix from every element of corresponded column we get reduced matrix

$$\begin{bmatrix} \cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\ \boxed{0} & 35 & 29 & 5 & 10 \\ 13 & \boxed{0} & 63 & 7 & 7 \\ 47 & 15 & \boxed{0} & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix}$$

$$\begin{bmatrix} \cancel{0} & 13 & 49 & \boxed{0} & \cancel{0} \\ \boxed{0} & 35 & 29 & 5 & 10 \\ 13 & \boxed{0} & 63 & 7 & 7 \\ 47 & 15 & \boxed{0} & 20 & 2 \\ 25 & \cancel{0} & 46 & 9 & 4 \end{bmatrix} \quad \begin{bmatrix} \cancel{0} & 17 & 49 & \boxed{0} & \cancel{0} \\ \boxed{0} & 39 & 29 & 5 & 10 \\ 9 & \boxed{0} & 59 & 3 & 3 \\ 47 & 19 & \boxed{0} & 20 & 2 \\ 21 & \cancel{0} & 42 & 5 & \boxed{0} \end{bmatrix}$$

J	M	Cost
J_1	M_4	11
J_2	M_1	43
J_3	M_2	28
J_4	M_3	27
J_5	M_5	25
		Rs. 134/-

$J_1 \to M_4$, $J_2 \to M_1$, $J_3 \to M_2$, $J_4 \to M_3$, $J_5 \to M_5$ and with minimum cost is Rs. 134/-

Scanned with OKEN Scanner

5. The assignment cost of any one operator to any one machine it is given in the following table. Find optimal solution by hungarian method.

$$\begin{array}{c c} & \text{operators} \\ & \begin{array}{cccc} \text{I} & \text{II} & \text{III} & \text{IV} \end{array} \\ \text{machines} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 10 & 5 & 13 & 15 \\ 3 & 9 & 18 & 3 \\ 10 & 7 & 3 & 2 \\ 5 & 11 & 9 & 7 \end{bmatrix} \end{array}$$

Sol. Since, the no. of rows = no. of columns. the given assignment problem is balanced.

rows:- $\begin{bmatrix} 5 & 0 & 8 & 10 \\ 0 & 6 & 15 & 0 \\ 8 & 5 & 1 & 0 \\ 0 & 6 & 4 & 2 \end{bmatrix}$ column:- $\begin{bmatrix} 5 & \boxed{0} & 7 & 10 \\ \cancel{0} & 6 & 14 & \boxed{0} \\ 8 & 5 & \boxed{0} & \cancel{0} \\ \boxed{0} & 6 & 3 & 2 \end{bmatrix}$

Machines	Operators	Cost
A	II	5
B	IV	3
C	III	3
D	I	5
		Rs.16/-

$A \to \text{I}I$, $B \to \text{IV}$, $C \to \text{III}$, $D \to \text{I}$ and with minimum cost Rs.16/-

Unbalanced Assignment Problem:- If the no. of rows is not equal to the no. of columns in the cost matrix of the given assignment problem then the given assignment problem is said to be unbalanced.

First convert the unbalanced assignment problem into a balanced one by adding a dummy row or dummy column with zero unit element with the cost matrix depending upon $m<n$ (or) $m>n$, and then solve by the usual method.

1. A company has 4 machines to do 3 jobs. Each job can be assigned to one and only one machine. The cost of each job on each machine is given in the following table

$$\begin{array}{c c} & \text{Machines} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Jobs} \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \end{bmatrix} \end{array}$$

What are the job assignment which will minimize the cost?

Sol. Since, the no. of rows is less than the no. of columns, the problem is unbalanced. To convert unbalanced problem into balanced one by adding a dummy job 'D' with zero cost.

$$\begin{array}{c c} & \text{Machines} \\ & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \text{Jobs} \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 18 & 24 & 28 & 32 \\ 8 & 13 & 17 & 19 \\ 10 & 15 & 19 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

rows:- $\begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ column:- $\begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 11 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} \boxed{0} & 6 & 10 & 14 \\ \cancel{0} & 5 & 9 & 11 \\ \cancel{0} & 5 & 9 & 12 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} \end{bmatrix}$ ✓ ✓ ✓

No. of assignments N=2,

Order of matrix n=5

N<n we go to next step

The smallest element not covered by line. i.e, '5' sub this element from all the uncovered element and add the same through all the element lying at the intersection of the line we get.

$\begin{bmatrix} \boxed{0} & 1 & 5 & 9 \\ \cancel{0} & \boxed{0} & 4 & 6 \\ \cancel{0} & \cancel{0} & 4 & 7 \\ 5 & \cancel{0} & \boxed{0} & \cancel{0} \end{bmatrix}$ ✓ ✓ ✓

No. of assignments, N=3

Order of matrix, n=6

N<n we go to next step

$$\begin{bmatrix} \boxed{0} & 1 & 1 & 5 \\ \cancel{0} & \boxed{0} & \cancel{0} & 2 \\ \cancel{0} & \cancel{0} & \boxed{0} & 3 \\ 9 & 4 & \cancel{0} & \boxed{0} \end{bmatrix}$$

Since, no. of assignments N=4, order of matrix, n=4 ∴ N=n. Each row and each column has only one assignment an optimal solution is reached optimal assignments and optimal cost of assignments is

Jobs	Machines	Cost
A	1	18
B	2	13
C	3	19
D	4	0
		Rs.50/-

$A \to 1, B \to 2; C \to 3, D \to 4$ with minimum cost of Rs.50/-

2. Assign 4 trucks 1,2,3,4 to vacate spaces A,B,C,D,E,F so that the distance travelled is minimize. The matrix below shows the distance.

Trucks

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline A & 4 & 7 & 3 & 7 \\ B & 8 & 2 & 5 & 5 \\ C & 4 & 9 & 6 & 9 \\ D & 7 & 5 & 4 & 8 \\ E & 6 & 3 & 5 & 4 \\ F & 6 & 8 & 7 & 3 \end{array}$$

Vacant spaces

Sol. Since, the no. of rows ≠ no. of columns, the given problem is unbalanced. To convert an unbalanced to a balance one we add 2 dummy columns.

$$\begin{bmatrix} 4 & 7 & 3 & 7 & 0 & 0 \\ 8 & 2 & 5 & 5 & 0 & 0 \\ 4 & 9 & 6 & 9 & 0 & 0 \\ 7 & 5 & 4 & 8 & 0 & 0 \\ 6 & 3 & 5 & 4 & 0 & 0 \\ 6 & 8 & 7 & 3 & 0 & 0 \end{bmatrix}$$

rows:- $\begin{bmatrix} 4 & 7 & 3 & 7 & 0 & 0 \\ 8 & 2 & 5 & 5 & 0 & 0 \\ 4 & 9 & 6 & 9 & 0 & 0 \\ 7 & 5 & 4 & 8 & 0 & 0 \\ 6 & 3 & 5 & 4 & 0 & 0 \\ 6 & 8 & 7 & 3 & 0 & 0 \end{bmatrix}$

Columns:- $\begin{bmatrix} \cancel{0} & 5 & \boxed{0} & 4 & \cancel{0} & \cancel{0} \\ 4 & \boxed{0} & 2 & 2 & \cancel{0} & \cancel{0} \\ \boxed{0} & 7 & 3 & 6 & \cancel{0} & \cancel{0} \\ 3 & 3 & 1 & 5 & \boxed{0} & \cancel{0} \\ 2 & 1 & 2 & 1 & \cancel{0} & \boxed{0} \\ 2 & 6 & 4 & \boxed{0} & \cancel{0} & \cancel{0} \end{bmatrix}$

Since each row and each column has only one assignment an optimal solution is reached

Optimal assignments and optimal cost of assignments is,

Vspaces	Trucks	Cost
A	3	3
B	2	2
C	1	4
D	5	0
E	6	0
F	4	3
		Rs.12/-

$A \to 3, B \to 2, C \to 1, D \to 5, E \to 6, F \to 4$ with minimum cost of Rs.12/-

3. Solve the following unbalanced Assignment problem.

Machines

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline 1 & 4 & 3 & 6 & 2 & 7 \\ 2 & 10 & 12 & 11 & 14 & 16 \\ 3 & 4 & 3 & 2 & 1 & 5 \\ 4 & 8 & 7 & 6 & 9 & 6 \end{array}$$

Jobs

Sol. Since the no. of rows ≠ no. of columns, the given problem is unbalanced. To convert it into a balanced one we add one dummy row

$$\begin{bmatrix} 4 & 3 & 6 & 2 & 7 \\ 10 & 12 & 11 & 14 & 16 \\ 4 & 3 & 2 & 1 & 5 \\ 8 & 7 & 6 & 9 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

rows:- $\begin{bmatrix} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

columns:- $\begin{bmatrix} 2 & 1 & 4 & \boxed{0} & 5 \\ \boxed{0} & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & \cancel{0} & 4 \\ 2 & 1 & \boxed{0} & 3 & \cancel{0} \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$ ✓ ✓

Sub smallest element '1' & add inter section elements with 1

$$\begin{bmatrix} 1 & \boxed{0} & 3 & \cancel{0} & 4 \\ \boxed{0} & 2 & 1 & 5 & 6 \\ 2 & 1 & \cancel{0} & \boxed{0} & 3 \\ 2 & 1 & \boxed{0} & 4 & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & 1 & \boxed{0} \end{bmatrix}$$

Since, each row & ~~ele~~ each element of columns has only one assignment an optimal solution is reached

Jobs	M	Cost
1	B	3
2	A	10
3	D	1
4	C	6
5	E	0
		Rs.20/-

$1 \to B, 2 \to A, 3 \to D, 4 \to C, 5 \to E$ with minimum cost of Rs.20/-

Maximization in Assignment Problem:- In this case the objective is to maximize the profit. The conversion of the maximization problem into an equivalent minimization problem can be done by any one of the following methods

1. Since $\max z = -\min(-z)$; multiply all the cost elements c_{ij} of the cost matrix by '-1'.
2. Substract all the cost element c_{ij} of the cost matrix from the highest cost element in the cost matrix.

1. A company has a team of four salesman from 4 districts where the company wants to start its business. After taking into the account the capabilities of the salesman and nature of the districts the company estimates that the profit per day in rupees for each salesman in given below

$$\text{districts}\qquad \begin{array}{c|c} & \begin{array}{cccc}1 & 2 & 3 & 4\end{array} \\ \text{salesman}\begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix} 16 & 10 & 14 & 11 \\ 14 & 11 & 15 & 15 \\ 15 & 15 & 13 & 12 \\ 13 & 12 & 14 & 15 \end{bmatrix}\end{array}$$

Find the assignment of salesman to various district which will yield max profit?

Sol. Since this is a maximization problem it can be converted into equivalent minimization problem by substracting all cost elements in the cost matrix from the highest element '16' in the cost matrix

$$\begin{bmatrix} 0 & 6 & 2 & 5 \\ 2 & 5 & 1 & 1 \\ 1 & 1 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix}$$

Substract smallest element of each row from every element of corresponding row we get the reduced matrix

$$\begin{bmatrix} 0 & 6 & 2 & 5 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 2 & 3 & 1 & 0 \end{bmatrix}$$

Substracting smallest element from each column from every element of corresponding column we get reduced matrix

$$\begin{bmatrix} \boxed{0} & 6 & 2 & 5 \\ 1 & 4 & \boxed{0} & \cancel{0} \\ \cancel{0} & \boxed{0} & 2 & 3 \\ 2 & 3 & 1 & \boxed{0} \end{bmatrix}$$

∴ No. of assignment = 4 = n = order of matrix.

∴ The assignments which are optimal are $A \to 1$; $B \to 3$; $C \to 2$; $D \to 4$

And the optimum profit is : $16+15+15+15 = $ Rs. 61/-

2. Solve the following assignment problem for maximization given profit matrix.

$$\text{jobs}\qquad \begin{array}{c|c} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \text{Persons}\begin{array}{c}A\\B\\C\\D\end{array} & \begin{bmatrix} 62 & 78 & 50 & 101 & 82 \\ 71 & 84 & 61 & 73 & 59 \\ 87 & 92 & 111 & 71 & 81 \\ 48 & 64 & 87 & 77 & 80 \end{bmatrix}\end{array}$$

Sol. Since the given maximization problem is an unbalanced assignment problem. First we convert this into Balanced AP by adding a dummy row E with cost as zero

$$\begin{array}{c|c} & \begin{array}{ccccc}1 & 2 & 3 & 4 & 5\end{array} \\ \begin{array}{c}A\\B\\C\\D\\E\end{array} & \begin{bmatrix} 62 & 78 & 50 & 101 & 82 \\ 71 & 84 & 61 & 73 & 59 \\ 87 & 92 & 111 & 71 & 81 \\ 48 & 64 & 87 & 77 & 80 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\end{array}$$

Since the given problem is maximization we convert it into a minimization. Substracting all the element from the highest cost elements '111' in cost matrix.

$$\begin{bmatrix} 49 & 33 & 61 & 10 & 29 \\ 40 & 27 & 50 & 38 & 52 \\ 24 & 19 & 0 & 40 & 30 \\ 63 & 47 & 24 & 34 & 31 \\ 111 & 111 & 111 & 111 & 111 \end{bmatrix}$$

substract smallest ele of each row from every ele of correspoing row we get the reduced matrix

$$\begin{bmatrix} 39 & 23 & 51 & 0 & 9 \\ 13 & 0 & 23 & 11 & 25 \\ 24 & 19 & 0 & 40 & 30 \\ 39 & 23 & 0 & 10 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Sub smallest ele of each column from every ele of corresponding column.

$$\begin{bmatrix} 39 & 23 & 51 & \boxed{0} & 19 \\ 13 & \boxed{0} & 23 & 11 & 25 \\ 24 & 19 & \boxed{0} & 40 & 30 \\ 39 & 23 & \boxed{0} & 10 & 7 \\ \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Smallest ele not covered by line is i.e, '7' sub this element from all the uncovered ele and add the same through all the ele lying at the intersection of the line we get,

$$\begin{bmatrix} 39 & 23 & 58 & \boxed{0} & 19 \\ 13 & \boxed{0} & 30 & 11 & 25 \\ 17 & 12 & \boxed{0} & 33 & 23 \\ 32 & 16 & \cancel{0} & 3 & \boxed{0} \\ \boxed{0} & \cancel{0} & 7 & \cancel{0} & \cancel{0} \end{bmatrix}$$

∴ no. of assignments = 5 = n = order of matrix

∴ N = n

∴ The assignments are $A \to 4$, $B \to 2$, $C \to 3$, $D \to 5$, $E \to 1$

and the optimum profit is, $= 101+84+111+80+0 = $ Rs. 376/-

3. Solve the following Assignment Problem for maximization given the profit matrix.

Machines

$$\begin{array}{c|ccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline J_1 & 32 & 38 & 40 & 28 & 40 \\ J_2 & 40 & 24 & 28 & 21 & 36 \\ J_3 & 41 & 27 & 33 & 30 & 37 \\ J_4 & 22 & 38 & 41 & 36 & 36 \\ J_5 & 29 & 33 & 40 & 35 & 39 \end{array}$$

Jobs

Sol. Since this is a maximization problem it can be converted into equivalents minimization problem by substracting all cost elements in the cost matrix from the highest element '41' in the cost matrix.

$$\begin{array}{c|ccccc} & M_1 & M_2 & M_3 & M_4 & M_5 \\ \hline J_1 & 9 & 3 & 1 & 13 & 1 \\ J_2 & 1 & 17 & 13 & 20 & 5 \\ J_3 & 0 & 14 & 8 & 11 & 4 \\ J_4 & 19 & 3 & 0 & 5 & 5 \\ J_5 & 12 & 8 & 1 & 6 & 2 \end{array}$$

Rows :-

$$\begin{bmatrix} 8 & 2 & 0 & 12 & 0 \\ 0 & 16 & 12 & 19 & 4 \\ 0 & 14 & 8 & 11 & 4 \\ 19 & 3 & 0 & 5 & 5 \\ 11 & 7 & 0 & 5 & 1 \end{bmatrix}$$

Columns :-

$$\begin{bmatrix} 8 & \boxed{0} & \cancel{0} & 7 & \cancel{0} \\ \boxed{0} & 14 & 12 & 14 & 4 \\ \cancel{0} & 12 & 8 & 6 & 4 \\ 19 & 1 & \boxed{0} & \cancel{0} & 5 \\ 11 & 5 & \cancel{0} & \boxed{0} & 1 \end{bmatrix} \begin{matrix} \\ \checkmark \\ \checkmark \\ \\ \\ \end{matrix}$$

$N<n$ we go to next step

$$\begin{bmatrix} 12 & \boxed{0} & 0 & 7 & \cancel{0} \\ \boxed{0} & 10 & 8 & 10 & \cancel{0} \\ \cancel{0} & 8 & 4 & 2 & \boxed{0} \\ 23 & 5 & \boxed{0} & \cancel{0} & 9 \\ 15 & 9 & \cancel{0} & \boxed{0} & 5 \end{bmatrix}$$

Since the no. of assignments $= N = n =$ order of matrix

The assignments which are optimal are,

$J_1 \rightarrow M_2$, $J_2 \rightarrow M_1$, $J_3 \rightarrow M_5$, $J_4 \rightarrow M_3$, $J_5 \rightarrow M_4$

And optimum profit is $= 38+40+37+41+35 =$ Rs. 191/-

4. Solve the following Assignment Problem

Jobs

$$\begin{array}{c|cccc} & P & Q & R & S \\ \hline A & 18 & 26 & 17 & 11 \\ B & 13 & 28 & 14 & 26 \\ C & 38 & 19 & 18 & 15 \\ D & 19 & 26 & 24 & 10 \end{array}$$

machines

rows :-

$$\begin{bmatrix} 7 & 15 & 6 & 0 \\ 0 & 15 & 1 & 13 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

columns :-

$$\begin{bmatrix} 7 & 11 & 5 & \boxed{0} \\ \boxed{0} & 11 & \cancel{0} & 13 \\ 23 & \boxed{0} & 2 & \cancel{0} \\ 9 & 12 & 13 & \cancel{0} \end{bmatrix} \begin{matrix} \checkmark \\ \\ \\ \checkmark \end{matrix}$$

$$\begin{bmatrix} 2 & 6 & \boxed{0} & 5 \\ \boxed{0} & 11 & \cancel{0} & 18 \\ 23 & \boxed{0} & 2 & \cancel{0} \\ 4 & 7 & 8 & \boxed{0} \end{bmatrix}$$

$A \rightarrow 3 = 17$
$B \rightarrow 1 = 13$
$C \rightarrow 2 = 19$
$D \rightarrow 4 = 10$

Rs. 59/-

* Travelling Salesman Problem :-

A salesman normally must visit a no. of cities starting from his head quarters. The distance (or) time (or) cost. Between every pair of cities are assumed to be known. The problem of finding the shortest distance (or) minimum time (or) minimum cost. If the sales man starts from headquarters and crosses through each city; exactly once and return to the head quarters is called the Travelling salesman Problem.

A travelling salesman problem is very similar to the assignment problem with the additional constraints.

1. The salesman should go through every city exactly once except the starting city.

2. The salesman starts from one city (head quarters) and comes back to that city (head quarters).

3. Obviously going from any city to the same city directly is not allowed. i.e, no assignment should be made along the diagonal line.

Solve the following travelling salesman problem

To

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & - & 46 & 16 & 40 \\ B & 41 & - & 50 & 40 \\ C & 82 & 32 & - & 60 \\ D & 40 & 40 & 36 & - \end{array}$$

From

Sol. The cost matrix of the travelling salesman problem is

$$\begin{array}{c|cccc} & & \text{To} & & \\ & A & B & C & D \\ \hline A & \infty & 46 & 16 & 40 \\ B & 41 & \infty & 50 & 40 \\ C & 82 & 32 & \infty & 60 \\ D & 40 & 40 & 36 & \infty \end{array}$$

(From: rows)

Substracting the smallest cost ele in each row from every ele of the corresponding row we get the reduced matrix

$$\begin{bmatrix} \infty & 30 & 0 & 24 \\ 1 & \infty & 10 & 0 \\ 50 & 0 & \infty & 28 \\ 4 & 4 & 0 & \infty \end{bmatrix}$$

sub the smallest ele in each column from every element of the corresponding column we get the reduced matrix

$$\begin{bmatrix} \infty & 30 & \boxed{0} & 24 \\ \boxed{0} & \infty & 10 & \cancel{0} \\ 49 & \boxed{0} & \infty & 28 \\ 3 & 4 & \cancel{0} & \infty \end{bmatrix}$$

$\therefore N<n$

$$\begin{bmatrix} \infty & 27 & \boxed{0} & 21 \\ \cancel{0} & \infty & 13 & \boxed{0} \\ 49 & \boxed{0} & \infty & 28 \\ \boxed{0} & 1 & \cancel{0} & \infty \end{bmatrix}$$

$N=4=n$

From	To	Cost
A	C	16
B	D	40
C	B	32
D	A	40
		Rs. 128/-

$\therefore$ The shortest path is $A \to C \to B \to D \to A$

$\therefore$ The required assignment schedule is given by $A \to C \to B \to D \to A$

Hence required distance is $16+40+32+40=128$

2. Solve the following salesman problem to find least cost route.

$$\begin{array}{c|ccccc} & & & \text{To} & & \\ & A & B & C & D & E \\ \hline A & \infty & 4 & 10 & 14 & 2 \\ B & 12 & \infty & 6 & 10 & 4 \\ C & 16 & 14 & \infty & 8 & 14 \\ D & 24 & 8 & 12 & \infty & 10 \\ E & 2 & 6 & 4 & 16 & \infty \end{array}$$

(From: rows)

rows:-

$$\begin{bmatrix} \infty & 2 & 8 & 12 & 0 \\ 8 & \infty & 2 & 6 & 0 \\ 8 & 6 & \infty & 0 & 6 \\ 16 & 0 & 4 & \infty & 2 \\ 0 & 4 & 2 & 14 & \infty \end{bmatrix}$$

columns:-

$$\begin{bmatrix} \infty & 2 & 6 & 12 & \boxed{0} \\ 8 & \infty & \boxed{0} & 6 & \cancel{0} \\ 8 & 6 & \infty & \boxed{0} & 6 \\ 16 & \boxed{0} & 2 & \infty & 2 \\ \boxed{0} & 4 & \cancel{0} & 4 & \infty \end{bmatrix}$$

$A \to E, B \to C, C \to D, D \to B, B \to E$ i.e, $A \to E \to A$ $\quad N=n=5$

As the salesman should go from $A \to E$ and come back to A without covering B, C, D hence we obtain the next back solution by abstracting with next minimum non-zero element.

$$\begin{bmatrix} \infty & \boxed{0} & 4 & 10 & \cancel{0} \\ 6 & \infty & \boxed{0} & 4 & \cancel{0} \\ 6 & 4 & \infty & \boxed{0} & 4 \\ 14 & \cancel{0} & \cancel{0} & \infty & \boxed{0} \\ \boxed{0} & 2 & \cancel{0} & 12 & \infty \end{bmatrix}$$

$\therefore$ The requirment assignment schedule is given by $A \to B$, $B \to C$, $C \to D$, $D \to E$, $E \to A$

$\therefore A \to B \to C \to D \to E$

$\therefore$ Hence required distance is $4+6+8+10+2=30$

3. Solve the following TSP to find shortest distance.

$$\begin{array}{c|ccccc} & A & B & C & D & E \\ \hline A & \infty & 4 & 7 & 3 & 4 \\ B & 4 & \infty & 6 & 3 & 4 \\ C & 7 & 6 & \infty & 7 & 5 \\ D & 3 & 3 & 7 & \infty & 7 \\ E & 4 & 4 & 5 & 7 & \infty \end{array}$$

rows:-

$$\begin{bmatrix} \infty & 1 & 4 & 0 & 1 \\ 1 & \infty & 2 & 0 & 1 \\ 2 & 1 & \infty & 2 & 0 \\ 0 & 0 & 4 & \infty & 4 \\ 0 & 0 & 1 & 3 & \infty \end{bmatrix}$$

columns:-

$$\begin{bmatrix} \infty & 1 & 3 & \boxed{0} & 1 \\ 1 & \infty & 1 & 0 & 1 \\ 2 & 1 & \infty & 2 & \boxed{0} \\ \boxed{0} & \cancel{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & 3 & \infty \end{bmatrix}$$

$$\begin{bmatrix} \infty & \boxed{0} & 2 & \cancel{0} & \cancel{0} \\ \cancel{0} & \infty & 1 & \boxed{0} & \cancel{0} \\ 2 & 1 & \infty & 3 & \boxed{0} \\ \boxed{0} & \cancel{0} & 3 & \infty & 4 \\ \cancel{0} & \cancel{0} & \boxed{0} & 4 & \infty \end{bmatrix}$$

$A \to B, B \to D, C \to E, D \to A, E \to C$

$A \to B \to D \to A$ × no paths

$$\begin{bmatrix} \infty & \cancel{0} & 1 & \boxed{0} & \cancel{0} \\ \cancel{0} & \infty & \boxed{0} & \cancel{0} & \cancel{0} \\ 1 & \cancel{0} & \infty & 2 & \boxed{0} \\ \cancel{0} & \boxed{0} & 2 & \infty & 3 \\ \boxed{0} & \cancel{0} & \cancel{0} & 3 & \infty \end{bmatrix}$$

$A \to D, B \to C, C \to E, D \to B, E \to A$

$A \to D \to B \to C \to E \to A$

Required distance: $3+6+5+3+4=21$

4. Solve the following TSP:-

$$\begin{array}{c|c} & \text{To city} \\ & \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \text{From City} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} \infty & 10 & 25 & 25 & 10 \\ 1 & \infty & 10 & 15 & 2 \\ 8 & 9 & \infty & 20 & 10 \\ 14 & 10 & 24 & \infty & 15 \\ 10 & 8 & 25 & 27 & \infty \end{bmatrix} \end{array}$$

rows:-

$$\begin{bmatrix} \infty & 0 & 15 & 15 & 0 \\ 0 & \infty & 9 & 14 & 1 \\ 0 & 1 & \infty & 12 & 2 \\ 4 & 0 & 14 & \infty & 5 \\ 2 & 0 & 17 & 19 & \infty \end{bmatrix}$$

columns:-

$$\begin{bmatrix} \infty & \cancel{0} & 6 & 3 & \boxed{0} \\ 0 & \infty & \boxed{0} & 2 & 1 \\ 0 & 1 & \infty & \boxed{0} & 2 \\ 4 & \boxed{0} & 5 & \infty & 5 \\ 2 & \cancel{0} & 8 & 7 & \infty \end{bmatrix} \begin{array}{c} \\ \\ \\ \checkmark \\ \checkmark \end{array}$$

$$\begin{bmatrix} \infty & 2 & 6 & 3 & \boxed{0} \\ \cancel{0} & \infty & \boxed{0} & 2 & 1 \\ \cancel{0} & 3 & \infty & \boxed{0} & 2 \\ 2 & \boxed{0} & 3 & \infty & 3 \\ \boxed{0} & \cancel{0} & 6 & 5 & \infty \end{bmatrix}$$

$1 \to 5, 2 \to 3, 3 \to 4, 4 \to 2, 5 \to 1$

$1 \to 5 \to 1$ ×

$$\begin{bmatrix} \infty & 1 & 5 & 2 & \boxed{0} \\ \cancel{0} & \infty & \boxed{0} & 1 & \cancel{0} \\ \cancel{0} & 2 & \infty & \boxed{0} & 1 \\ 1 & \boxed{0} & 2 & \infty & 2 \\ \boxed{0} & \cancel{0} & 5 & 4 & \infty \end{bmatrix}$$

$1 \to 5, 2 \to 3, 3 \to 4, 4 \to 2, 5 \to 1$

$1 \to 5 \to 1$ ×

$$\begin{bmatrix} \infty & \cancel{0} & 4 & 1 & \boxed{0} \\ \cancel{0} & \infty & \boxed{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & 1 & \infty & \boxed{0} & \cancel{0} \\ \boxed{0} & \cancel{0} & 1 & \infty & 1 \\ \cancel{0} & \boxed{0} & 4 & 3 & \infty \end{bmatrix}$$

$1 \to 5, 2 \to 3, 3 \to 4, 4 \to 1, 5 \to 2$

$1 \to 5 \to 2 \to 3 \to 4 \to 1$

$\therefore$ required distance $= 10 + 10 + 20 + 14 + 8$

$= 62$/-