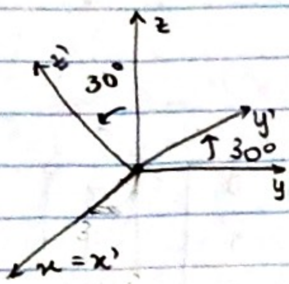
MECÂNICA ANALÍTICA E ONDAS

SÉRIE 4 - TRANSFORMAÇÕES ORTOGONAIS

①



$S' \equiv S$ em $t=0$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(x',x) & \cos(x',y) & \cos(x',z) \\ \cos(y',x) & \cos(y',y) & \cos(y',z) \\ \cos(z',x) & \cos(z',y) & \cos(z',z) \end{bmatrix}}_{R_x(30^\circ)} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow$$

y, y', z, z' : plano $\perp$ a $x' \equiv x$

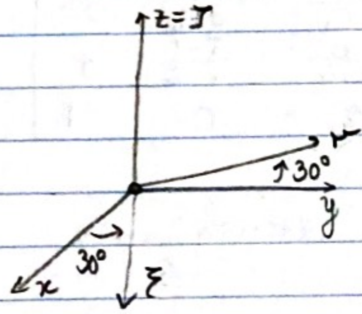
$$\Leftrightarrow \underbrace{\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}}_{\text{em } S'} = \begin{bmatrix} \cos(0^\circ) & \cos(90^\circ) & \cos(90^\circ) \\ \cos(90^\circ) & \cos(30^\circ) & \cos(90-30) \\ \cos(90^\circ) & \cos(90+30) & \cos(30^\circ) \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\text{em } S}$$

$$R_x(30^\circ) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(30^\circ) & \sin(30^\circ) \\ 0 & -\sin(30^\circ) & \cos(30^\circ) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

②

a) $S' \equiv S$ em $t=0$

ROTAÇÃO 1

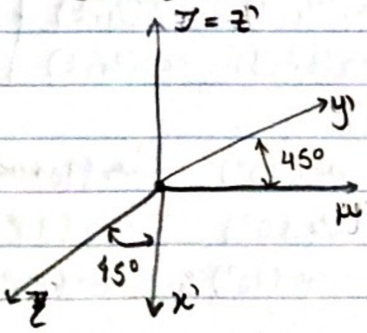


$$\begin{bmatrix} \xi \\ \mu \\ \zeta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\xi,x) & \cos(\xi,y) & \cos(\xi,z) \\ \cos(\mu,x) & \cos(\mu,y) & \cos(\mu,z) \\ \cos(\zeta,x) & \cos(\zeta,z) & \cos(\zeta,z) \end{bmatrix}}_{R_z(30^\circ)} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} \xi \\ \mu \\ \zeta \end{bmatrix} = \begin{bmatrix} \cos(+30^\circ) & \cos(90-30) & \cos(90^\circ) \\ \cos(+(90+30)) & \cos(+30^\circ) & \cos(90^\circ) \\ \cos(90^\circ) & \cos(90^\circ) & \cos(0^\circ) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$R_z(30^\circ) = \begin{bmatrix} \cos(30^\circ) & \sin(30^\circ) & 0 \\ -\sin(30^\circ) & \cos(30^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ROTAÇÃO 2



$$R_z(45^\circ) = \begin{bmatrix} \cos(x',\xi) & \cos(x',\mu) & \cos(x',\zeta) \\ \cos(y',\xi) & \cos(y',\mu) & \cos(y',\zeta) \\ \cos(z',\xi) & \cos(z',\mu) & \cos(z',\zeta) \end{bmatrix} =$$

$$= \begin{bmatrix} \cos(45^\circ) & \cos(90-45) & \cos(90^\circ) \\ \cos(90+45) & \cos(45^\circ) & \cos(90^\circ) \\ \cos(90^\circ) & \cos(90^\circ) & \cos(0^\circ) \end{bmatrix} = \rightarrow$$

$$= \begin{bmatrix} \cos(45°) & \sin(45°) & 0 \\ -\sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ROTAÇÃO TOTAL

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{1+\sqrt{3}}{2\sqrt{2}} & 0 \\ \frac{-1-\sqrt{3}}{2\sqrt{2}} & \frac{-1+\sqrt{3}}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{matriz transformação pedida.}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

b)

$$R'_{TOTAL} = R_z(45°)\, R_z(30°) = R_z(30°)\, R_z(45°) = R_{TOTAL}$$

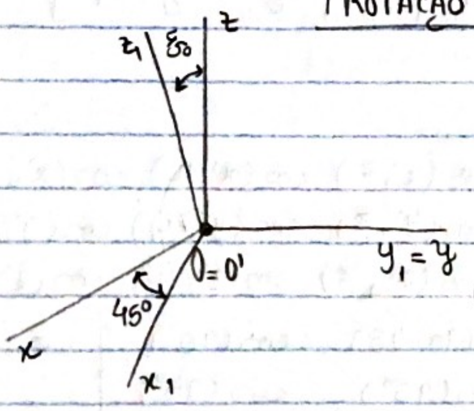
$$\begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} & 0 \\ \frac{-1-\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

↳ As matrizes comutam porque as rotação são ambas em torno do mesmo eixo.

③ $S = S'$ em $t = 0$

a)

ROTAÇÃO 1



$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos(x_1,x) & \cos(x_1,y) & \cos(x_1,z) \\ \cos(y_1,x) & \cos(y_1,y) & \cos(y_1,z) \\ \cos(z_1,x) & \cos(z_1,y) & \cos(z_1,z) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow$$

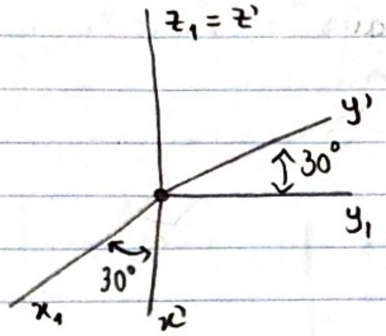
$$\Leftrightarrow \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \cos(45°) & \cos(90°) & \cos(90+45) \\ \cos(90°) & \cos(0°) & \cos(90°) \\ \cos(90-45) & \cos(90°) & \cos(45°) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Leftrightarrow$$

→

$$\Leftrightarrow \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$R_y\,(45^\circ) = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

ROTAÇÃO 2



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(x', x_1) & \cos(x', y_1) & \cos(x', z_1) \\ \cos(y', x_1) & \cos(y', y_1) & \cos(y', z_1) \\ \cos(z', x_1) & \cos(z', y_1) & \cos(z', z_1) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(30^\circ) & \cos(90^\circ - 30^\circ) & \cos(90^\circ) \\ \cos(90+30) & \cos(30^\circ) & \cos(90^\circ) \\ \cos(90^\circ) & \cos(90^\circ) & \cos(0^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \Leftrightarrow$$

$$\Leftrightarrow \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos(30^\circ) & \sin(30^\circ) & 0 \\ -\sin(30^\circ) & \cos(30^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$$R_z\,(30^\circ) = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ROTAÇÃO TOTAL

$$R_{TOTAL} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2} & -\frac{\sqrt{3}}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2} & \frac{1}{2\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

b)

$$R'_{TOTAL} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} =$$

↳ As matrizes ñ comutam porque correspondem a rotações em torno de eixos distintos.