SÉRIE 5

①

$$\vec{r_1} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{r_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{r_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{R}_{CM} = \frac{m_1 \vec{r_1} + m_2 \vec{r_2} + m_3 \vec{r_3}}{m_1 + m_2 + m_3} =$$

$$= \frac{-m_1 \vec{e_y} + m_3 \vec{e_x} + m_3 \vec{e_y} + m_3 \vec{e_z}}{M} =$$

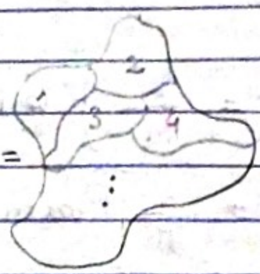
$$= A \vec{e_x} + A \vec{e_z} + \left(\frac{-m_1 + m_3}{M} \right) \vec{e_y} =$$

$$= A \vec{e_x} + A \vec{e_z} + \left(A - AB \right) \vec{e_y}$$

SÉRIE 5

②

$$\vec{R}_{CM} = \frac{1}{M}\int_V d\vec{r}\,\rho(\vec{r})\,\vec{r} =$$



$$= \frac{1}{M}\left(\int_{V_1}\dots + \int_{V_2}\dots + \dots + \int_{V_m}\dots\right) =$$

$$= \frac{1}{M}\left(m_1 \underbrace{\int_{V_1}\frac{\dots}{m_1}}_{\vec{R_1}} + m_2 \underbrace{\int_{V_2}\dots}_{\vec{R_2}} + \dots + m_m \underbrace{\int_{V_m}\dots}_{\vec{R_m}}\right) =$$

$$= \frac{m_1\vec{R_1} + m_2\vec{R_2} + \dots + m_m\vec{R_m}}{M}$$

// c.q.m.

③

$$\vec{R_{CM}} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + m_3\vec{r_3}}{m_1+m_2+m_3}$$

$m_1 = \rho\, a\,(b-c-e)$

$\vec{r_1} = \left(\frac{b-c-e}{2}\right)\vec{e_x} + \left(d+\frac{f}{2}\right)\vec{e_y}$

$m_2 = \rho\, f\,(c+e)$

$\vec{r_2} = \left(b-\frac{c+e}{2}\right)\vec{e_x} + \left(d+\frac{f}{2}\right)\vec{e_y}$

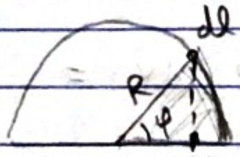
$m_3 = \rho\, e\,\frac{d}{2}$

$\vec{r_3} = \left(b-\frac{e}{2}\right)\vec{e_x} + \frac{d}{2}\vec{e_y}$

... } agora fazer as contas

④

a)



$$\vec{R_{CM}} = \frac{1}{M}\int_V d\vec{r}\,\rho(\vec{r})\,\vec{r}$$

$$\rho = \frac{M}{R\pi}$$

$$R_{CM_x} = \frac{1}{M}\int_0^\pi \rho\, R\, d\varphi\, R\cos(\varphi) =$$

$$= \frac{1}{\cancel{M}}\int_0^\pi \frac{\cancel{M}}{\cancel{R}\pi}\cancel{R}\, d\varphi\, R\cos(\varphi) = \frac{R}{\pi}\int_0^\pi \cos(\varphi)\,d\varphi =$$

→

$$= \frac{R}{\pi}\left[+\sin(\varphi)\right]_0^{\pi} = \frac{R}{\pi}\left(\sin(\pi) - \sin(0)\right) = 0$$

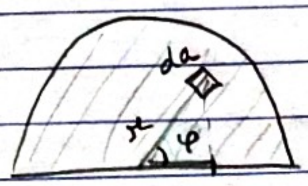
$$R_{CM_y} = \frac{1}{M}\int_0^{\pi} \rho R\, d\varphi\, R \sin(\varphi) = \frac{1}{\cancel{M}}\int_0^{\pi} \frac{\cancel{M}}{R\pi} \cancel{R}\, d\varphi\, R \sin(\varphi) =$$

$$= \frac{R}{\pi}\int_0^{\pi} \sin(\varphi)\, d\varphi = \frac{R}{\pi}\left[-\cos(\varphi)\right]_0^{\pi} =$$

$$= \frac{R}{\pi}\left(-\cos(\pi) + \cos(0)\right) = \frac{R}{\pi}(1+1) = \frac{2R}{\pi}$$

$$\vec{R_{CM}} = \frac{2R}{M}\vec{e_x}$$

b)



$$\vec{R_{CM}} = \frac{1}{M}\int_V d\vec{x}\, \rho(\vec{x})\, \vec{x}$$

$$\rho = \frac{2M}{\pi R^2}$$

$$R_{CM_x} = \frac{1}{M}\int_0^R\int_0^{\pi} \rho\, r\, d\varphi\, dr\; r \cos(\varphi) =$$

$$= \frac{2}{\pi R^2}\int_0^R\int_0^{\pi} r^2 \cos(\varphi)\, d\varphi\, dr =$$

$$= \frac{2}{\pi R^2}\int_0^R r^2\, dr \int_0^{\pi} \cos(\varphi)\, d\varphi =$$

$$= \frac{2}{\pi R^2}\left[\frac{r^3}{3}\right]_0^R \left[\sin(\varphi)\right]_0^{\pi} =$$

$$= \frac{2R^3}{3\pi R^2}\left(\sin(\pi) - \sin(0)\right) = 0$$

$$R_{CM_y} = \frac{1}{M}\int_0^R\int_0^{\pi} \rho\, r\, d\varphi\, dr\; r \sin(\varphi) =$$

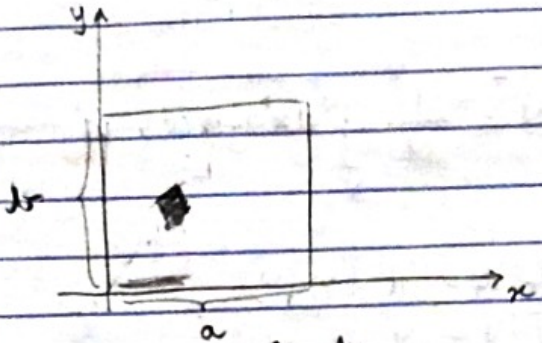
$$= \frac{2}{\pi R^2}\int_0^R r^2\, dr \int_0^{\pi} \sin(\varphi)\, d\varphi =$$

$$= \frac{2}{\pi R^2}\left[\frac{r^3}{3}\right]_0^R \left[-\cos(\varphi)\right]_0^{\pi} =$$

$$= \frac{2R^3}{3\pi R^2}\left(-\cos(\pi) + \cos(0)\right) = \frac{2R}{3\pi}(1+1) = \frac{4R}{3\pi}$$

⑤

a) $\rho = Ky$



$$\vec{R}_{CM} = \frac{1}{M}\int_V d\vec{x}\,\rho(\vec{x})\,\vec{x}$$

$$R_{CM_x} = \frac{1}{M}\int_0^a\int_0^b \rho\, dx\, dy\; x = \frac{1}{M}\int_0^a\int_0^b K\, y\, x\, dy\, dx =$$

$$= \frac{K}{M}\int_0^a x\, dx \int_0^b y\, dy =$$

$$= \frac{K}{M}\left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b = \frac{K}{M}\,\frac{a^2}{2}\,\frac{b^2}{2} =$$

$$= \frac{K}{M}\,\frac{a^2 b^2}{4}$$

$$R_{CM_y} = \frac{1}{M}\int_0^a\int_0^b \rho\, dx\, dy\; y = \frac{1}{M}\int_0^a\int_0^b K y^2\, dy\, dx =$$

$$= \frac{K}{M}\int_0^a dx \int_0^b y^2\, dy =$$

$$= \frac{K}{M}\, a \left[\frac{y^3}{3}\right]_0^b = \frac{K}{M}\, a\, \frac{b^3}{3}$$

$$\vec{R}_{CM} = \left(\frac{K}{4M}\, a^2 b^2\right)\vec{e_x} + \left(\frac{K a b^3}{3M}\right)$$

b)

$$M = \int_0^a\int_0^b \rho\, dx\, dy = \int_0^a\int_0^b Ky\, dx\, dy =$$

$$= K\int_0^a dx \int_0^b y\, dy =$$

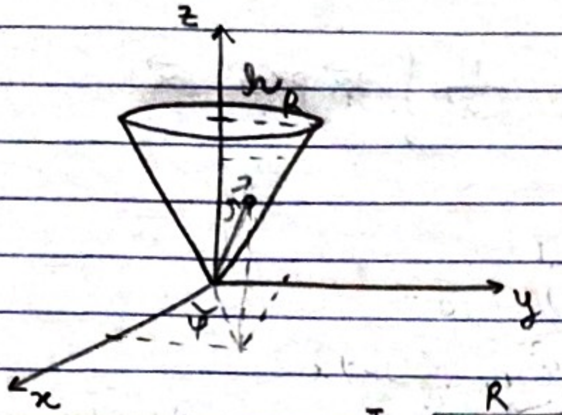
$$= K a \left[\frac{y^2}{2}\right]_0^b = K a \frac{b^2}{2}$$

Logo,

$$\vec{R}_{CM} = \left(\frac{a}{2}\,;\ \frac{2b}{3}\right)$$

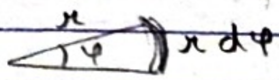
⑥

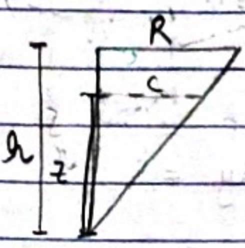
a) homogéneo $\rightarrow \overset{\text{constante}}{\rho} = \frac{M}{V}$



$$\vec{R_{CM}} = \frac{1}{M}\int_V d\vec{r}\, \rho(\vec{r})\, \vec{r}$$

$$\begin{cases} x = r\cos(\varphi) \\ y = r\sin(\varphi) \\ z = z \end{cases}$$





$$dr = r\, d\varphi\, dr\, dz$$

$$\frac{z}{h} = \frac{c}{R} \Leftrightarrow c = \frac{zR}{h}$$

$$R_{CM_x} = \frac{1}{M}\int_0^{2\pi}\int_0^{h}\int_0^{\frac{zR}{h}} \frac{M}{V}\, r\, d\varphi\, dr\, dz\, r\cos(\varphi) =$$

$$= \frac{M}{VM}\int_0^{2\pi}\cos(\varphi)\, d\varphi \int_0^{h} dz \int_0^{\frac{zR}{h}} r^2\, dr =$$

$$= \frac{M}{VM}\left[\cancel{\sin(\varphi)}\right]_0^{2\pi}\left[z\right]_0^{h}\left[\frac{r^3}{3}\right]_0^{\frac{zR}{h}} =$$

$$= 0$$

$$R_{CM_y} = \frac{1}{M}\int_0^{2\pi}\int_0^{h}\int_0^{\frac{zR}{h}} \frac{M}{V}\, r\, d\varphi\, dr\, dz\, r\sin(\varphi) =$$

$$= \frac{M}{VM}\int_0^{2\pi}\sin(\varphi)\, d\varphi \int_0^{h} dz \int_0^{\frac{zR}{h}} r^2\, dr =$$

$$= \frac{M}{VM}\left[-\cancel{\cos(\varphi)}\right]_0^{2\pi}\left[z\right]_0^{h}\left[\frac{r^3}{3}\right]_0^{\frac{zR}{h}} =$$

$$= 0$$

$$R_{CM_y} = \frac{1}{M}\int_0^{2\pi}\int_0^{h}\int_0^{\frac{zR}{h}} \frac{M}{\pi R^2 h}\, r\, d\varphi\, dr\, dz\; z =$$

$$= \frac{M}{VM}\int_0^{2\pi} d\varphi \int_0^{h} z\, dz \int_0^{\frac{zR}{h}} r\, dr =$$

$$= \frac{M}{VM}\, 2\pi \int_0^{h} z\, dz \left[\frac{r^2}{2}\right]_0^{\frac{zR}{h}} =$$

$$= \frac{M}{VM}\, 2\pi \int_0^{h} z\, dz\, \frac{z^2 R^2}{2h^2} =$$

$$= \frac{\cancel{2}\pi}{V}\frac{R^2}{\cancel{2}h^2}\int_0^{h} z^3\, dz =$$

$$= \frac{R^2\pi}{V h^2}\left[\frac{z^4}{4}\right]_0^{h} = \frac{R^2\pi}{V \cancel{h^2}}\frac{h^4}{4} = \frac{R^2\pi h^2}{4V}$$

$$\overrightarrow{R_{CM}} = \frac{R^2\pi h^2}{4V}\,\vec{e_z}$$

b)

$$V = \int_0^{2\pi}\int_0^{h}\int_0^{\frac{zR}{h}} r\, dr\, d\varphi\, dz =$$

$$= \int_0^{2\pi} d\varphi \int_0^{h} dz \int_0^{\frac{zR}{h}} r\, dr =$$

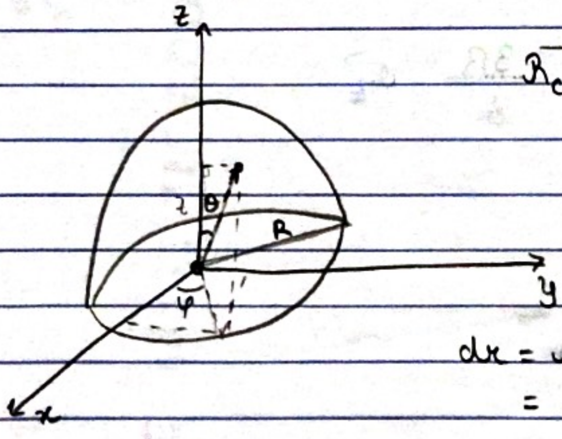
$$= 2\pi \int_0^{h} dz \left[\frac{r^2}{2}\right]_0^{\frac{zR}{h}} = 2\pi \int_0^{h} dz\, \frac{z^2 R^2}{2h^2} =$$

$$= \frac{\pi}{h^2}R^2 \int_0^{h} z^2\, dz = \frac{\pi R^2}{h^2}\left[\frac{z^3}{3}\right]_0^{h} =$$

$$= \frac{\pi R^2}{\cancel{h^2}}\frac{h^3}{3} = \frac{\pi R^2 h}{3}$$

$$\overrightarrow{R_{CM}} = \frac{3h}{4}\,\vec{e_z}$$

7) homogénea $\rightarrow$ $\rho = \frac{M}{V}$ constante!!



$$\overrightarrow{R_{CM}} = \frac{1}{M}\int_V d\vec{r}\,\rho(\vec{x})\,\vec{x}$$

$$\begin{cases} x = r\sin(\theta)\cos(\varphi) \\ y = r\sin(\theta)\sin(\varphi) \\ z = r\cos(\theta) \end{cases}$$

$$dr = \cancel{r}\, dr\; d\theta\; r\sin(\theta)\, d\varphi =$$

$$= r^2\sin(\theta)\, dr\, d\theta\, d\varphi$$

$$V = \frac{4}{3}\pi R^3 \qquad \rho = \frac{3M}{4\pi R^3}$$

$\rightarrow$

$$R_{CM_x} = \frac{1}{M}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R r^2 \sin(\theta)\, dr\, d\theta\, d\varphi \, \frac{M}{V} \, r \sin(\theta)\cos(\varphi) =$$

$$= \frac{1}{V}\underbrace{\int_0^{2\pi}\cos(\varphi)\, d\varphi}_{=0} \int_0^{\pi/2} \sin^2(\theta)\, d\theta \int_0^R r^3\, dr = 0$$

$$R_{CM_y} = \frac{1}{\cancel{M}}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R r^2 \sin(\theta)\, dr\, d\theta\, d\varphi \, \frac{\cancel{M}}{V} \, r \sin(\theta)\sin(\varphi) =$$

$$= \frac{1}{V}\int_0^{2\pi}\sin(\varphi)\, d\varphi \underbrace{\int_0^{\pi/2} \sin^2(\theta)\, d\theta}_{=0} \int_0^R r^3\, dr = 0$$

$$R_{CM_z} = \frac{1}{\cancel{M}}\int_0^{2\pi}\int_0^{\pi/2}\int_0^R r^2 \sin(\theta)\, dr\, d\theta\, d\varphi \, \frac{\cancel{M}}{V} \, r \cos(\theta) =$$

$$= \frac{1}{V}\int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin(\theta)\cos(\theta)\, d\theta \int_0^R r^3\, dr =$$

$$= \frac{1}{V} \times 2\pi \int_0^{\pi/2} \frac{\sin(2\theta)}{2}\, d\theta \left[\frac{r^4}{4}\right]_0^R =$$

$$= \frac{\pi}{V}\left[-\frac{\cos(2\theta)}{2}\right]_0^{\pi/2} \frac{R^4}{4} =$$

$$= \frac{\pi}{V}\frac{R^4}{4}$$

$$V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^R r^2 \sin(\theta)\, dr\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/2} \sin(\theta)\, d\theta \int_0^R r^2\, dr =$$

$$= 2\pi \left[-\cos(\theta)\right]_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^R =$$

$$= 2\pi (+1) \frac{R^3}{3} = \frac{2\pi R^3}{3}$$

$$\vec{R_{CM}} = \frac{\cancel{\pi} R^{\cancel{4}} \times 3}{2\cancel{\pi} \cancel{R^3} \times 4} \vec{e_z} = \frac{3R}{8} \vec{e_z}$$