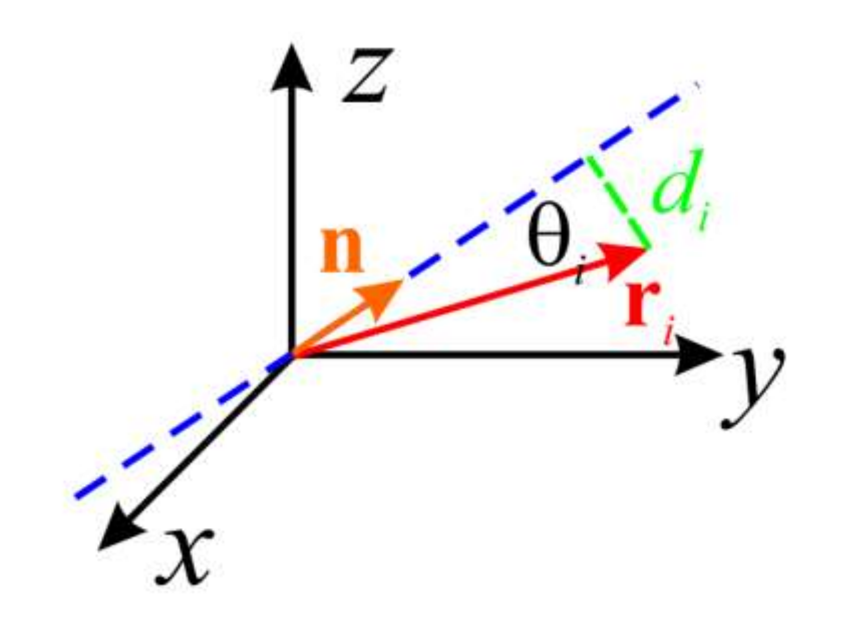
1- Considere uma reta que passa pela origem de um referencial OXYZ e cujo versor, neste referencial, é,

$$\vec{n} = \alpha\,\vec{e}_x + \beta\,\vec{e}_y + \gamma\,\vec{e}_z$$

onde α, β e γ são cosenos diretores da direção da reta. Calcule o momento de inércia de um sistema de N pontos materiais de massas m_1, m_2,...,m_N relativamente à reta dada, em termos dos momentos e produtos de inércia do sistema relativamente ao referencial considerado.



$$I = \sum_j m_j d_j^2$$

$$d_j = r_j \sin(\theta_j)$$

$$\mathbf{d}_j = \mathbf{n} \times \mathbf{r}_j = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \alpha & \beta & \gamma \\ x_j & y_j & z_j \end{vmatrix} =$$

$$= \mathbf{e}_x\left(\beta z_j - \gamma y_j\right) + \mathbf{e}_y\left(\gamma x_j - \alpha z_j\right) + \mathbf{e}_z\left(\alpha y_j - \beta x_j\right)$$

$$\mathbf{d}_j = \mathbf{n} \times \mathbf{r}_j = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \alpha & \beta & \gamma \\ x_j & y_j & z_j \end{vmatrix} =$$

$$= \mathbf{e}_x \left(\beta z_j - \gamma y_j\right) + \mathbf{e}_y \left(\gamma x_j - \alpha z_j\right) + \mathbf{e}_z \left(\alpha y_j - \beta x_j\right)$$

$$I = \sum_j m_j d_j^2$$

$$d_j = r_j \sin\left(\theta_j\right)$$

$$d_j^2 = \left|\mathbf{d}_j\right|^2 = \left(\beta z_j - \gamma y_j\right)^2 + \left(\gamma x_j - \alpha z_j\right)^2 + \left(\alpha y_j - \beta x_j\right)^2 =$$

$$= \beta^2 z_j^2 - 2\beta\gamma y_j z_j + \gamma^2 y_j^2 + \gamma^2 x_j^2 - 2\alpha\gamma x_j z_j + \alpha^2 z_j^2 + \alpha^2 y_j^2 - 2\alpha\beta x_j y_j + \beta^2 x_j^2 =$$

$$= \alpha^2 \left(y_j^2 + z_j^2\right) + \beta^2 \left(x_j^2 + z_j^2\right) + \gamma^2 \left(x_j^2 + y_j^2\right) - 2\alpha\beta x_j y_j - 2\alpha\gamma x_j z_j - 2\beta\gamma y_j z_j$$

$$I = \sum_j m_j d_j^2 = \alpha^2 \sum_j m_j \left(y_j^2 + z_j^2\right) + \beta^2 \sum_j m_j \left(x_j^2 + z_j^2\right) + \gamma^2 \sum_j m_j \left(x_j^2 + y_j^2\right) -$$

$$-2\alpha\beta \sum_j m_j x_j y_j - 2\alpha\gamma \sum_j m_j x_j z_j - 2\beta\gamma \sum_j m_j y_j z_j$$

$$I = \sum_j m_j d_j^2 = \alpha^2 \sum_j m_j \left(y_j^2 + z_j^2\right) + \beta^2 \sum_j m_j \left(x_j^2 + z_j^2\right) + \gamma^2 \sum_j m_j \left(x_j^2 + y_j^2\right) -$$

$$-2\alpha\beta \sum_j m_j x_j y_j - 2\alpha\gamma \sum_j m_j x_j z_j - 2\beta\gamma \sum_j m_j y_j z_j$$

$$I = \mathbf{n} \cdot \left(\hat{I}\mathbf{n}\right) = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \begin{pmatrix} I_{xx}\alpha + I_{xy}\beta + I_{xz}\gamma \\ I_{yx}\alpha + I_{yy}\beta + I_{yz}\gamma \\ I_{zx}\alpha + I_{zy}\beta + I_{zz}\gamma \end{pmatrix} =$$

$$= \alpha\left(I_{xx}\alpha + I_{xy}\beta + I_{xz}\gamma\right) + \beta\left(I_{yx}\alpha + I_{yy}\beta + I_{yz}\gamma\right) + \gamma\left(I_{zx}\alpha + I_{zy}\beta + I_{zz}\gamma\right) =$$

$$= \alpha^2 I_{xx} + \beta^2 I_{yy} + \gamma^2 I_{zz} + \alpha\beta\left(I_{xy} + I_{yx}\right) + \alpha\gamma\left(I_{xz} + I_{zx}\right) + \beta\gamma\left(I_{zy} + I_{yz}\right)$$

$$I = \sum_j m_j d_j^2 = \alpha^2 \sum_j m_j \left(y_j^2 + z_j^2\right) + \beta^2 \sum_j m_j \left(x_j^2 + z_j^2\right) + \gamma^2 \sum_j m_j \left(x_j^2 + y_j^2\right) -$$

$$-2\alpha\beta \sum_j m_j x_j y_j - 2\alpha\gamma \sum_j m_j x_j z_j - 2\beta\gamma \sum_j m_j y_j z_j$$

$$= \alpha \left(I_{xx}\alpha + I_{xy}\beta + I_{xz}\gamma\right) + \beta \left(I_{yx}\alpha + I_{yy}\beta + I_{yz}\gamma\right) + \gamma \left(I_{zx}\alpha + I_{zy}\beta + I_{zz}\gamma\right) =$$

$$= \alpha^2 I_{xx} + \beta^2 I_{yy} + \gamma^2 I_{zz} + \alpha\beta \left(I_{xy} + I_{yx}\right) + \alpha\gamma \left(I_{xz} + I_{zx}\right) + \beta\gamma \left(I_{zy} + I_{yz}\right)$$

$$I_{xx} = \sum_j m_j \left(y_j^2 + z_j^2\right)$$

$$I_{yy} = \sum_j m_j \left(x_j^2 + z_j^2\right)$$

$$I_{zz} = \sum_j m_j \left(x_j^2 + y_j^2\right)$$

$$I_{xy} = I_{yx} = -\sum_j m_j x_j y_j$$

$$I_{xz} = I_{zx} = -\sum_j m_j x_j z_j$$

$$I_{yz} = I_{zy} = -\sum_j m_j y_j z_j$$

2- Determine os momentos e produtos de inércia do sistema constituido por dois pontos materiais P_1 de massa m e P_2 de massa $2m$, no referencial em que as coordenadas destes pontos são, respetivamente,

$$\vec{r}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ e } \qquad \vec{r}_2 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

$$m_1 = m$$

$$m_2 = 2m$$

$$I_{xx} = \sum_j m_j \left(y_j^2 + z_j^2\right)$$

$$I_{yy} = \sum_j m_j \left(x_j^2 + z_j^2\right)$$

$$I_{zz} = \sum_j m_j \left(x_j^2 + y_j^2\right)$$

$$I_{xy} = I_{yx} = -\sum_j m_j x_j y_j$$

$$I_{xz} = I_{zx} = -\sum_j m_j x_j z_j$$

$$I_{yz} = I_{zy} = -\sum_j m_j y_j z_j$$

$$m_1 = m$$

$$m_2 = 2m$$

$$\mathbf{r}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\mathbf{r}_2 = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$I_{xx} = \sum_{j=1}^{2} m_j \left(y_j^2 + z_j^2\right) = m\left(0^2 + 1^2\right) + 2m\left(2^2 + 1^2\right) = 11m$$

$$I_{yy} = \sum_{j=1}^{2} m_j \left(x_j^2 + z_j^2\right) = m\left(1^2 + 1^2\right) + 2m\left(2^2 + 1^2\right) = 12m$$

$$I_{zz} = \sum_{j=1}^{2} m_j \left(x_j^2 + y_j^2\right) = m\left(1^2 + 0^2\right) + 2m\left(2^2 + 2^2\right) = 17m$$

$$I_{xy} = I_{yx} = -\sum_{j=1}^{2} m_j x_j y_j = -m\,(1 \cdot 0) - 2m\,(2 \cdot 2) = -8m$$

$$I_{xz} = I_{zx} = -\sum_{j=1}^{2} m_j x_j z_j = -m\,(1 \cdot 1) - 2m\,(2 \cdot 1) = -5m$$

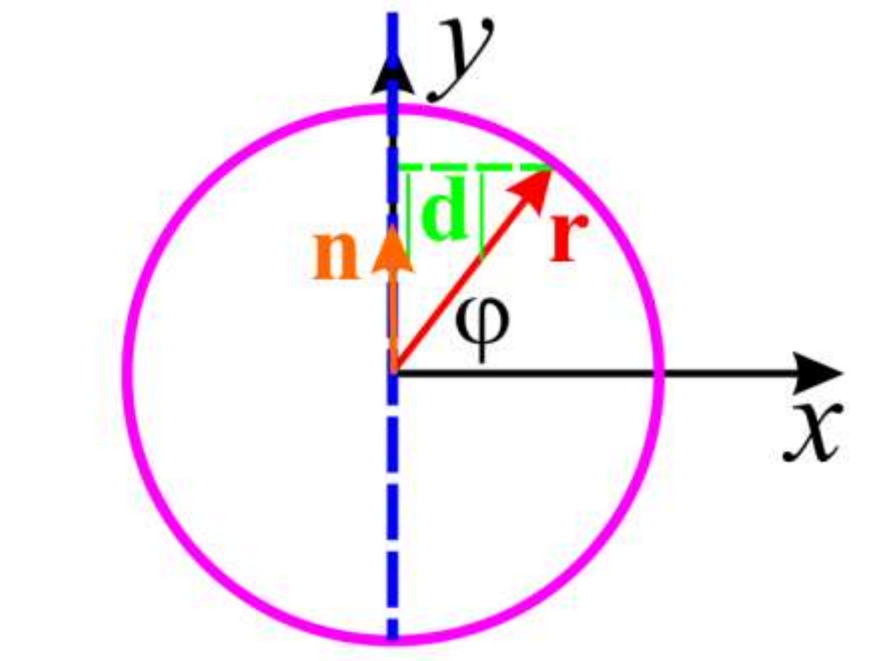
$$I_{yz} = I_{zy} = -\sum_{j=1}^{2} m_j y_j z_j = -m\,(0 \cdot 1) - 2m\,(2 \cdot 1) = -4m$$

$$\hat{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} =$$

$$= \begin{pmatrix} 11m & -8m & -5m \\ -8m & 12m & -4m \\ -5m & -4m & 17m \end{pmatrix}$$

3- Determine o momento de inércia de um fio circular homogéneo expresso em termos da sua massa M e raio R em relação a:

(a) Um diâmetro (por exemplo, coincidente com o eixo OY);



$$I = \int dm \, |\mathbf{d}|^2$$

$$dm = \rho_1 \, dl$$
$$\rho_1 = \frac{M}{R2\pi}$$
$$dl = Rd\varphi$$

$$\mathbf{n} = \mathbf{e}_y$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ x & y & 0 \end{vmatrix} = -x\mathbf{e}_z = -R\cos\varphi\mathbf{e}_x$$

$$I = \int dm \, |\mathbf{d}|^2$$

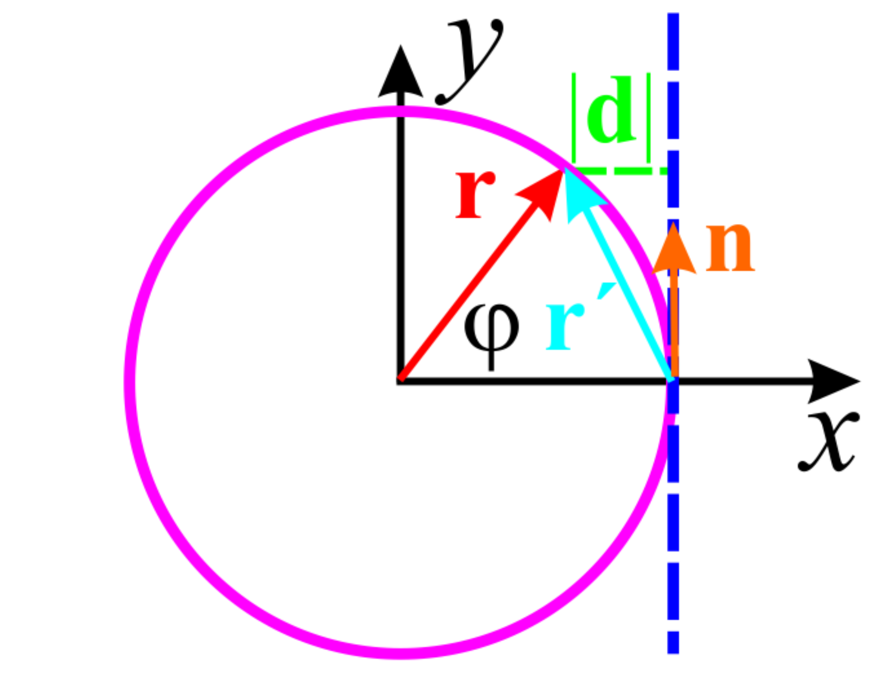
$$dm = \rho_1 \, dl$$
$$\rho_1 = \frac{M}{R2\pi}$$
$$dl = R d\varphi$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ x & y & 0 \end{vmatrix} = -x\mathbf{e}_z = -R\cos\varphi\mathbf{e}_x$$

$$I = \int_0^{2\pi} d\varphi \, R \frac{M}{2R\pi} \left[R\cos\varphi\right]^2 = R^2 \frac{M}{2\pi} \int_0^{2\pi} d\varphi \, \cos^2\varphi =$$

$$= R^2 \frac{M}{2\pi} \int_0^{2\pi} d\varphi \, \frac{1}{2} \left[1 + \cos 2\varphi\right] = R^2 \frac{M}{4\pi} \left[\varphi\Big|_0^{2\pi} + \frac{1}{2} \sin 2\varphi\Big|_0^{2\pi} \right] = \frac{MR^2}{2}$$

(b) Uma tangente ao fio, contida no plano deste (por exemplo, paralelo ao eixo OY).



$$I = \int dm\, |\mathbf{d}|^2$$

$$\mathbf{n} = \mathbf{e}_y$$

$$\mathbf{r}' = -R\mathbf{e}_x + \mathbf{r}$$

$$dm = \rho_1\, dl$$

$$\rho_1 = \frac{M}{R2\pi}$$

$$dl = Rd\varphi$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ x - R & y & 0 \end{vmatrix} = (R - x)\,\mathbf{e}_z = R\,[1 - \cos\varphi]\,\mathbf{e}_z$$

$$I = \int dm \, |\mathbf{d}|^2$$

$$dm = \rho_1 \, dl$$
$$\rho_1 = \frac{M}{R2\pi}$$
$$dl = R d\varphi$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ x - R & y & 0 \end{vmatrix} = (R - x)\,\mathbf{e}_z = R\,[1 - \cos\varphi]\,\mathbf{e}_z$$

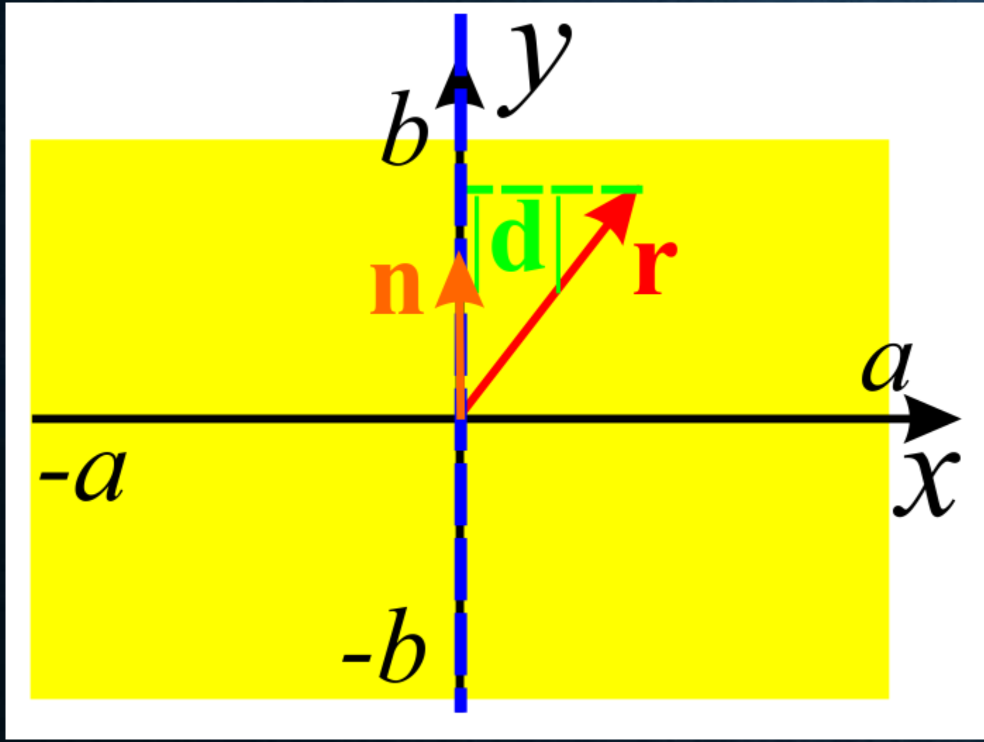
$$I = \int_0^{2\pi} d\varphi \, R \frac{M}{2R\pi} R^2 \, [1 - \cos\varphi]^2 = R^2 \frac{M}{2\pi} \int_0^{2\pi} d\varphi \, [1 - \cos\varphi]^2 =$$

$$= R^2 \frac{M}{2\pi} \int_0^{2\pi} d\varphi \, \left[1 - 2\cos\varphi + \cos^2\varphi\right] = R^2 \frac{M}{2\pi} \int_0^{2\pi} d\varphi \, \left[1 - 2\cos\varphi + \frac{1}{2} + \frac{1}{2}\cos 2\varphi\right] =$$

$$= R^2 \frac{M}{2\pi} \left[\frac{3}{2}\,\varphi\Big|_0^{2\pi} - 2\,\sin\varphi\Big|_0^{2\pi} + \frac{1}{4}\,\sin 2\varphi\Big|_0^{2\pi}\right] = \frac{3MR^2}{2} = \frac{MR^2}{2} + MR^2$$

4- Determine os momentos de inércia de uma placa retangular homogénea de lados $2a$ e $2b$, expressos em termos de a, b e da sua massa M, em relação a:

(a) Uma reta situada no plano da placa, passando pelo centro e perpendicular ao lado $2a$;



$$I = \int dm\, |\mathbf{d}|^2$$

$$I = \int_{-a}^{a} dx \int_{-b}^{b} dy \rho_2 |\mathbf{d}|^2$$

$$dm = \rho_2\, dS$$
$$\rho_2 = \frac{M}{4ab}$$
$$dS = dx\, dy$$

$$\mathbf{n} = \mathbf{e}_y$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ x & y & 0 \end{vmatrix} = -x\mathbf{e}_z$$

$$I = \int dm \, |\mathbf{d}|^2$$

$$dm = \rho_2 \, dS$$

$$\rho_2 = \frac{M}{4ab}$$

$$dS = dx \, dy$$

$$\mathbf{n} = \mathbf{e}_y$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & 1 & 0 \\ x & y & 0 \end{vmatrix} = -x\mathbf{e}_z$$

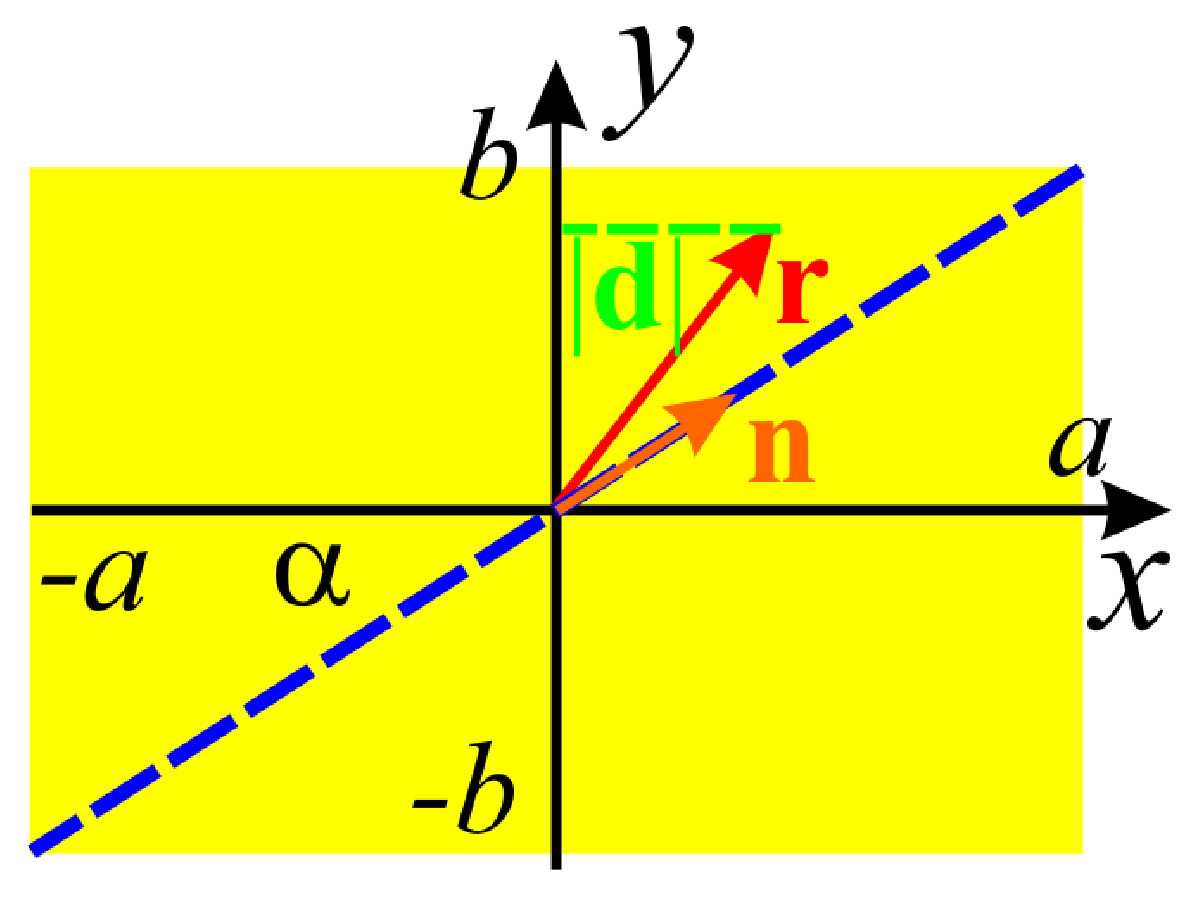
$$I = \int_{-a}^{a} dx \int_{-b}^{b} dy \rho_2 \, |\mathbf{d}|^2 = \frac{M}{4ab} \int_{-a}^{a} x^2 \, dx \int_{-b}^{b} dy =$$

$$= \frac{M}{4ab} \left. \frac{x^3}{3} \right|_{-a}^{a} \left. y \right|_{-b}^{b} = \frac{M}{4ab} \frac{2a^3}{3} 2b = \frac{Ma^2}{3}$$

(b) Uma diagonal da placa.

$$I = \int dm \, |\mathbf{d}|^2$$

$$dm = \rho_2 \, dS$$
$$\rho_2 = \frac{M}{4ab}$$
$$dS = dx \, dy$$



$$I = \int_{-a}^{a} dx \int_{-b}^{b} dy \rho_2 \, |\mathbf{d}|^2$$

$$\mathbf{n} = \cos\alpha \, \mathbf{e}_x + \sin\alpha \, \mathbf{e}_y =$$

$$= \frac{a}{\sqrt{a^2+b^2}}\mathbf{e}_x + \frac{b}{\sqrt{a^2+b^2}}\mathbf{e}_y$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \cos\alpha & \sin\alpha & 0 \\ x & y & 0 \end{vmatrix} =$$

$$= \mathbf{e}_z \left[y\cos\alpha - x\sin\alpha \right]$$

$$I = \int dm\, |\mathbf{d}|^2$$

$$dm = \rho_2\, dS$$
$$\rho_2 = \frac{M}{4ab}$$
$$dS = dx\, dy$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \cos\alpha & \sin\alpha & 0 \\ x & y & 0 \end{vmatrix} =$$

$$= \mathbf{e}_z \left[y\cos\alpha - x\sin\alpha \right]$$

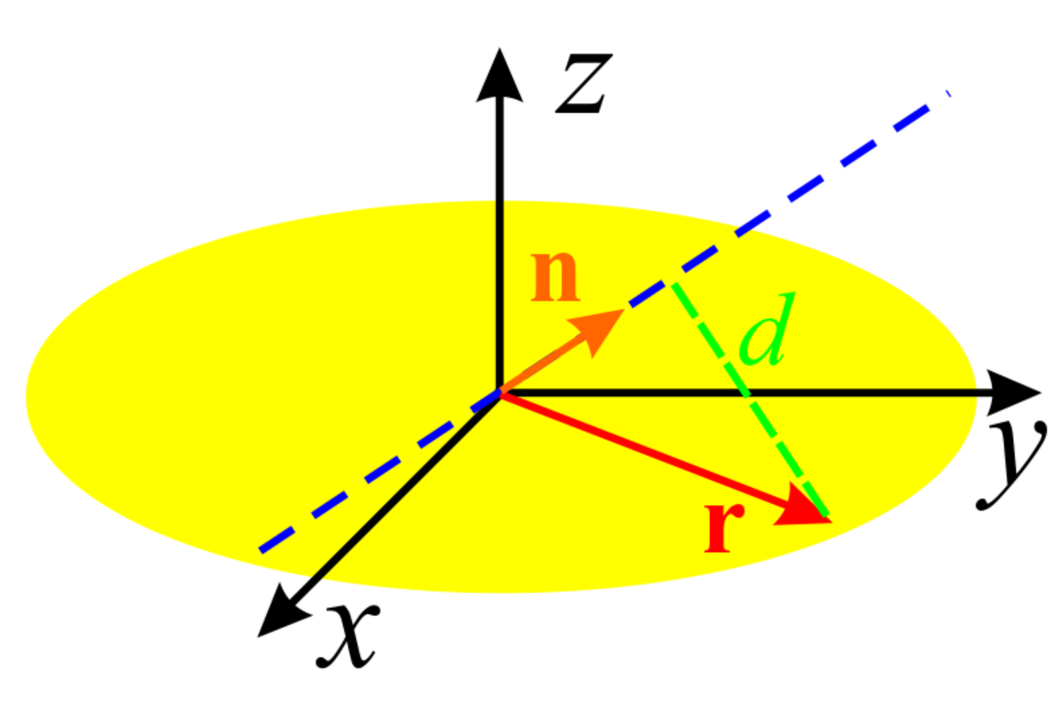
$$I = \int_{-a}^{a} dx \int_{-b}^{b} dy \rho_2 \,|\mathbf{d}|^2 = \frac{M}{4ab} \int_{-a}^{a} dx \int_{-b}^{b} \left[y\cos\alpha - x\sin\alpha \right]^2 dy$$

$$= \frac{M}{4ab} \int_{-a}^{a} dx \int_{-b}^{b} \left[y^2 \cos^2\alpha - xy\cos\alpha\sin\alpha + x^2\sin^2\alpha \right] dy$$

$$= \frac{M}{4ab} \left[x\big|_{-a}^{a} \left.\frac{y^3}{3}\right|_{-b}^{b} \cos^2\alpha - \left.\frac{x^2}{2}\right|_{-a}^{a} \left.\frac{y^2}{2}\right|_{-b}^{b} \cos\alpha\sin\alpha + \left.\frac{x^3}{3}\right|_{-a}^{a} y\big|_{-b}^{b} \sin^2\alpha \right] =$$

$$= \frac{M}{4ab} \left[2a\frac{2b^3}{3}\cos^2\alpha + \frac{2a^3}{3} 2b \sin^2\alpha \right] = \frac{M}{3} \left[b^2\cos^2\alpha + a^2\sin^2\alpha \right] = \frac{2Ma^2b^2}{3\left(a^2+b^2\right)}$$

5- Calcule o momento de inércia de um disco na forma de um círculo homogéneo de massa M e raio R contido no plano XY em relação a uma reta formando um ângulo α com o plano do disco e que passa pelo centro deste.



$$I = \int dm \, |\mathbf{d}|^2$$

$$\mathbf{n} = \cos\alpha \mathbf{e}_y + \sin\alpha \mathbf{e}_z$$

$$dm = \rho_2 \, dS$$
$$\rho_2 = \frac{M}{\pi R^2}$$
$$dS = r dr \, d\varphi$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & \cos\alpha & \sin\alpha \\ x & y & 0 \end{vmatrix} = -\mathbf{e}_x y \sin\alpha + \mathbf{e}_y x \sin\alpha - \mathbf{e}_z x \cos\alpha$$

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

$$I = \int dm\ |\mathbf{d}|^2$$

$$dm = \rho_2\, dS$$
$$\rho_2 = \frac{M}{\pi R^2}$$
$$dS = r dr\, d\varphi$$

$$\mathbf{d} = \mathbf{n} \times \mathbf{r} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 0 & \cos\alpha & \sin\alpha \\ x & y & 0 \end{vmatrix} = -\mathbf{e}_x y \sin\alpha + \mathbf{e}_y x \sin\alpha - \mathbf{e}_z x \cos\alpha$$

$$|\mathbf{d}|^2 = y^2 \sin^2\alpha + x^2 \left[\sin^2\alpha + \cos^2\alpha\right] = r^2 \sin^2\varphi \sin^2\alpha + r^2 \cos^2\varphi$$

$$I = \int\limits_0^R r dr \int\limits_0^{2\pi} d\varphi \rho_2 \, |\mathbf{d}|^2 = \frac{M}{\pi R^2} \int\limits_0^R r^3 dr \int\limits_0^{2\pi} d\varphi \left[\sin^2\varphi \sin^2\alpha + \cos^2\varphi\right] =$$

$$= \frac{M}{\pi R^2} \int\limits_0^R r^3 dr \int\limits_0^{2\pi} d\varphi \frac{1}{2} \left[(1 - \cos 2\varphi)\sin^2\alpha + (1 + \cos 2\varphi)\right] =$$

$$= \frac{M}{\pi R^2} \int\limits_0^R r^3 dr \int\limits_0^{2\pi} d\varphi \frac{1}{2} \left[(1 + \sin^2\alpha) + \cos 2\varphi \left(1 - \sin^2\alpha\right)\right] =$$

$$I = \int_0^R r dr \int_0^{2\pi} d\varphi \rho_2 \left|\mathbf{d}\right|^2 = \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\varphi \left[\sin^2\varphi \sin^2\alpha + \cos^2\varphi\right] =$$

$$= \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\varphi \frac{1}{2} \left[(1 - \cos 2\varphi)\sin^2\alpha + (1 + \cos 2\varphi)\right] =$$

$$= \frac{M}{\pi R^2} \int_0^R r^3 dr \int_0^{2\pi} d\varphi \frac{1}{2} \left[\left(1 + \sin^2\alpha\right) + \cos 2\varphi \left(1 - \sin^2\alpha\right)\right] =$$

$$= \frac{M}{2\pi R^2} \left[\left.\frac{r^4}{4}\right|_0^R\right] \left[\left(1 + \sin^2\alpha\right) \left.\varphi\right|_0^{2\pi} + \left(1 - \sin^2\alpha\right) \left.\frac{\sin 2\varphi}{2}\right|_0^{2\pi}\right] =$$

$$= \frac{M}{2\pi R^2} \frac{R^4}{4} \left(1 + \sin^2\alpha\right) 2\pi = \frac{MR^2}{4} \left(1 + \sin^2\alpha\right)$$