Integral definido

1. Calcule os seguintes integrais:

(a) $\displaystyle\int_1^2 e^{\pi x}\ dx = \left[\frac{e^{\pi x}}{\pi}\right]_1^2 = \frac{1}{\pi}(e^{2\pi} - e^{\pi})$

(b) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\operatorname{sen}(x)|\ dx;$

Notamos que, para $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$,

$$|\operatorname{sen}(x)| = \begin{cases} \operatorname{sen}(x) & \text{se} \quad x \in]0, \pi/2], \\ -\operatorname{sen}(x) & \text{se} \quad x \in [-\pi/2, 0]. \end{cases}$$

Assim,

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\operatorname{sen}(x)|\ dx &= \int_{-\frac{\pi}{2}}^{0} -\operatorname{sen}(x)dx + \int_0^{\frac{\pi}{2}} \operatorname{sen}(x)dx \\ &= \Big[\cos(x)\Big]_{-\frac{\pi}{2}}^{0} + \Big[-\cos(x)\Big]_0^{\frac{\pi}{2}} \\ &= \cos(0) - \cos(\pi/2) + [-\cos(\pi/2) - (-\cos(0))] \\ &= 2\end{aligned}$$

Outra forma: verifica-se que $|\operatorname{sen}(x)|$ é uma função par. Logo

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\operatorname{sen}(x)|\ dx = 2\int_0^{\frac{\pi}{2}} |\operatorname{sen}(x)|\ dx = 2\int_0^{\frac{\pi}{2}} \operatorname{sen}(x)\ dx$$

(c) $\displaystyle\int_{-3}^{5} |x-1|\ dx;$

Notamos que, para $x \in [-3, 5]$,

$$|x-1| = \begin{cases} x-1 & \text{se} \quad x \in]1, 5], \\ -x+1 & \text{se} \quad x \in [-3, 1]. \end{cases}$$

Assim,

$$\begin{aligned}\int_{-3}^{5} |x-1|\ dx &= \int_{-3}^{1} -x+1\ dx + \int_1^5 x-1\ dx \\ &= \left[-\frac{x^2}{2} + x\right]_{-3}^{1} + \left[\frac{x^2}{2} - x\right]_1^5 \\ &= \Big((-\frac{1}{2}+1) - (-\frac{9}{2}-3)\Big) + \Big((\frac{25}{2}-5) - (\frac{1}{2}-1)\Big) \\ &= 16\end{aligned}$$

(d) $\displaystyle\int_0^2 |(x-1)(3x-2)|\ dx$.
Analisando a função $(x-1)(3x-2)$, verificamos que tem $x=1$ e $x=2/3$ como zeros e

		$2/3$		1	
$x-1$	$-$	$-$	$-$	0	$+$
$3x-2$	$-$	0	$+$	$+$	$+$
f	$+$	0	$-$	0	$+$

donde concluímos que, para $x \in [0,2]$,

$$|(x-1)(3x-2)| = |3x^2-5x+2| = \begin{cases} 3x^2-5x+2 & \text{se} \quad x \in [0,2/3[\cup]1,2], \\ -3x^2+5x-2 & \text{se} \quad x \in [2/3,1]. \end{cases}$$

Então,

$$\begin{aligned}\int_0^2 |(x-1)(3x-2)|\ dx &= \int_0^{\frac{2}{3}} 3x^2-5x+2\ dx + \int_{\frac{2}{3}}^1 -3x^2+5x-2\ dx + \int_1^2 3x^2-5x+2\ dx \\ &= \Big[x^3-5\frac{x^2}{2}+2x\Big]_0^{\frac{2}{3}} + \Big[-x^3+5\frac{x^2}{2}-2x\Big]_{\frac{2}{3}}^1 + \Big[x^3-5\frac{x^2}{2}+2x\Big]_1^2 \\ &= \frac{55}{27}\end{aligned}$$

2. Calcule os seguintes integrais:

(a) $\displaystyle\int_0^2 f(x)\ dx$, com $f(x) = \begin{cases} x^2 & \text{se} \quad 0 \le x \le 1, \\ 3-x & \text{se} \quad 1 < x \le 2. \end{cases}$
A função f é integrável pois apenas possui um ponto de descontinuidade em $x=1$. Assim,

$$\begin{aligned}\int_0^2 f(x)\ dx &= \int_0^1 x^2\ dx + \int_1^2 3-x\ dx \\ &= \Big[\frac{x^3}{3}\Big]_0^1 + \Big[3x-\frac{x^2}{2}\Big]_1^2 \\ &= (\frac{1}{3}-0)+(6-2)-(3-\frac{1}{2}) \\ &= \frac{11}{6}\end{aligned}$$

(b) $\displaystyle\int_{-5}^0 2x\sqrt{4-x}\ dx$.
Usamos integração por partes, escolhendo $u' = \sqrt{4-x}$ e $v=2x$, donde $u = -\frac{2}{3}(4-x)^{3/2}$ e $v'=2$. Assim,

$$\begin{aligned}\int_{-5}^0 2x\sqrt{4-x}\ dx &= \Big[-\frac{2}{3}(4-x)^{3/2}.2x\Big]_{-5}^0 - \int_{-5}^0 \frac{-2}{3}(4-x)^{3/2}.2\ dx \\ &= \Big(0+\frac{2}{3}9^{3/2}(-10)\Big) + \frac{4}{3}\int_{-5}^0 (4-x)^{3/2}\ dx \\ &= -180 + \frac{4}{3}\left[-\frac{(4-x)^{5/2}}{5/2}\right]_{-5}^0 \\ &= -180 - \frac{8}{15}(4^{5/2}-9^{5/2}) \\ &= -\frac{1012}{15}\end{aligned}$$

(c) $\displaystyle\int_0^2 x^3\,\mathrm{e}^{x^2}\ dx.$

Notando que $\displaystyle\int_0^2 x^3\,\mathrm{e}^{x^2}\ dx = \int_0^2 x^2.x\,\mathrm{e}^{x^2}\ dx$, usamos integração por partes, escolhendo $u' = x\,\mathrm{e}^{x^2}$ e $v = x^2$, donde $u = \frac{1}{2}\,\mathrm{e}^{x^2}$ e $v' = 2x$. Assim,

$$\begin{aligned}
\int_0^2 x^3\,\mathrm{e}^{x^2}\ dx &= \left[\frac{1}{2}\,\mathrm{e}^{x^2}.x^2\right]_0^2 - \int_{-5}^0 \frac{1}{2}\,\mathrm{e}^{x^2}.2x\ dx \\
&= \left(\frac{1}{2}\,\mathrm{e}^4.4 - 0\right) - \left[\frac{1}{2}\,\mathrm{e}^{x^2}\right]_0^2 \\
&= 2\,\mathrm{e}^4 - \left(\frac{1}{2}\,\mathrm{e}^4 - \frac{1}{2}\right) \\
&= \frac{1}{2} + \frac{3}{2}\,\mathrm{e}^4
\end{aligned}$$