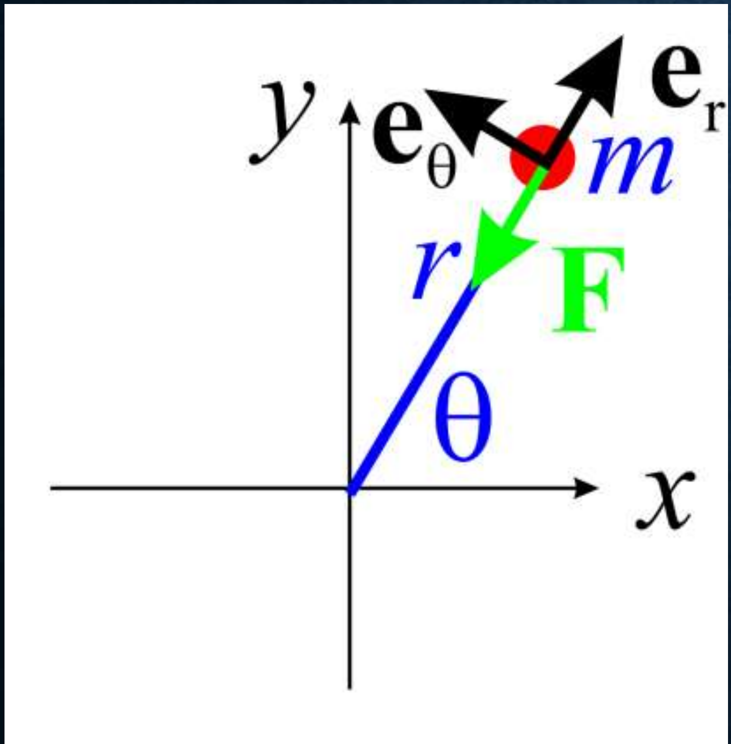
1- Considere o caso do movimento no plano de uma partícula de massa m sob um campo central de força central, $V(r)$. Para simplificar, considera-se que o movimento da partícula ocorre apenas no plano XY.

(a) Escreva as equações de Hamilton do sistema.



Problema 3 de série 2

$$\{q_1, q_2\} = \{r, \theta\}$$

$$L = \frac{1}{2}m\left[(\dot{r})^2 + r^2\left(\dot{\theta}\right)^2\right] - \int F(r)\, dr$$

$$p_r = \frac{\partial L}{\partial \dot{r}} \qquad p_\theta = \frac{\partial L}{\partial \dot{\theta}} \qquad \dot{p}_r = -\frac{\partial H}{\partial r} \qquad \dot{r} = \frac{\partial H}{\partial p_r}$$

$$H = \sum_j p_{q_j}\dot{q}_j - L = p_r\dot{r} + p_\theta\dot{\theta} - L$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$L = \frac{1}{2}m\left[(\dot{r})^2 + r^2\left(\dot{\theta}\right)^2\right] - \int F(r)\,dr$$

$$\dot{r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = \sum_j p_{q_j}\dot{q}_j - L = p_r\dot{r} + p_\theta\dot{\theta} - L =$$

$$= m(\dot{r})^2 + mr^2\left(\dot{\theta}\right)^2 - \frac{1}{2}m\left[(\dot{r})^2 + r^2\left(\dot{\theta}\right)^2\right] + \int F(r)\,dr =$$

$$= \frac{1}{2}m\left[(\dot{r})^2 + r^2\left(\dot{\theta}\right)^2\right] + \int F(r)\,dr$$

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\theta^2}{r^2}\right] + \int F(r)\,dr$$

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{2m}\frac{2}{r^3} + F(r)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}$$

(b) O que pode dizer sobre a componente L_z segundo o eixo OZ do momento angular da partícula a partir do comportamento do momento generalizado p_θ?

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$L_z = mr^2\dot{\theta} = p_\theta$$

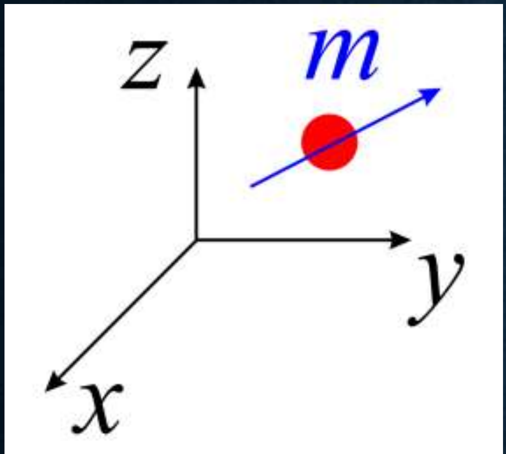
$$\dot{r} = \frac{p_r}{m}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2}$$

$$p_\theta = const$$

2- Escreva as equações de Hamilton para uma partícula livre de massa m, isto é, uma partícula que não é actuada por nenhuma força.

Problema 1(a) de série 2



$$\{q_j\} = \{x, y, z\}$$

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

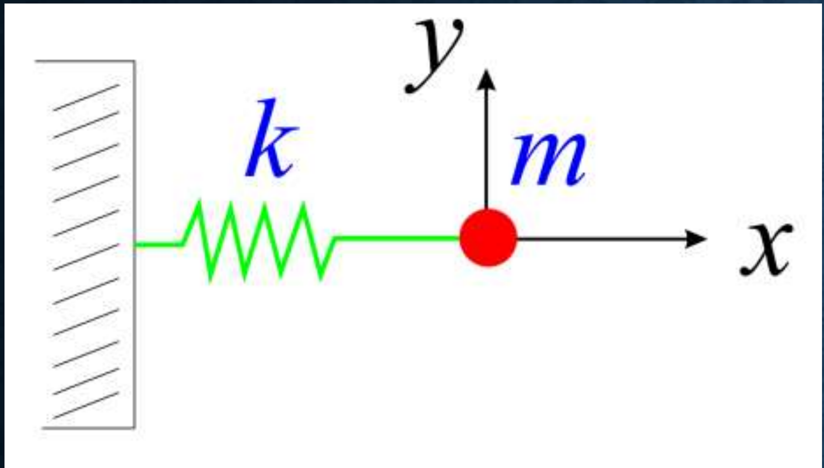
$$H = p_x\dot{x} + p_y\dot{y} + p_z\dot{z} - L = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m}$$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \dot{p}_x = -\frac{\partial H}{\partial x} = 0 \qquad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \qquad \dot{p}_y = -\frac{\partial H}{\partial y} = 0 \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m}$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \qquad \dot{p}_z = -\frac{\partial H}{\partial z} = 0 \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

3- Escreva as equações de Hamilton para o oscilador harmónico linear.



Problema 1(b) de série 2

$$\{q_1\} = \{x\}$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{k}{2}x^2$$

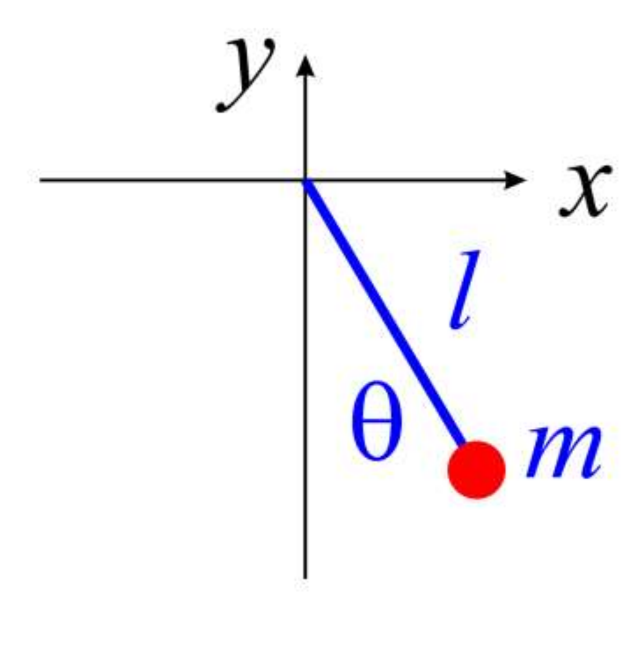
$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$H = p_x\dot{x} - L = \frac{p_x^2}{2m} + \frac{k}{2}x^2$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = -kx$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m}$$

4- Escreva as equações de Hamilton para o pêndulo simples.



Problema 1(c) de série 2

$$\{q_1\} = \{\theta\}$$

$$L = \frac{1}{2}m\left(l\dot{\theta}\right)^2 - mgl\left[1 - \cos\left(\theta\right)\right]$$

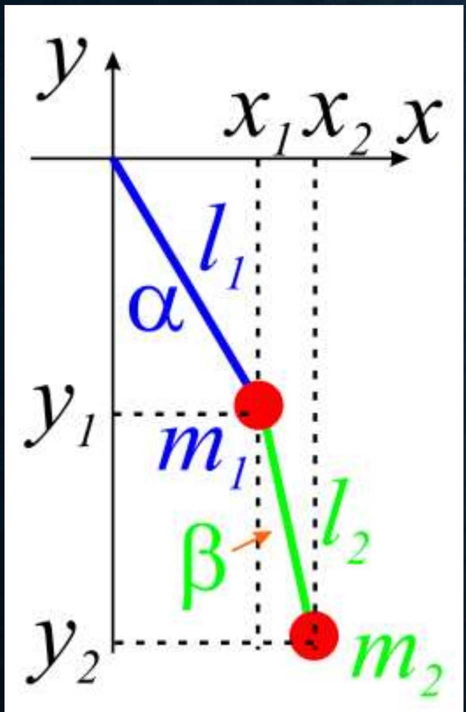
$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

$$H = p_\theta\dot{\theta} - L = \frac{p_\theta^2}{2ml^2} + mgl\left[1 - \cos\left(\theta\right)\right]$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\left(\theta\right)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}$$

5- Expresse o Hamiltoniano para o pêndulo duplo coplanar em termos das coordenadas generalizadas e dos momentos generalizados e indique como se chegaria às equações de Hamilton desse sistema.



Problema 1(d) de série 2

$$\{q_j\} = \{\alpha, \beta\}$$

$$L = \frac{1}{2}(m_1+m_2)(l_1\dot{\alpha})^2 + \frac{1}{2}m_2\left\{\left(l_2\dot{\beta}\right)^2 + 2l_1l_2\dot{\alpha}\dot{\beta}\cos(\alpha-\beta)\right\} - -(m_1+m_2)gl_1[1-\cos(\alpha)] - m_2gl_2[1-\cos(\beta)]$$

$$p_\alpha = \frac{\partial L}{\partial\dot{\alpha}} \qquad p_\beta = \frac{\partial L}{\partial\dot{\beta}}$$

$$\begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}$$

$$H = p_\alpha\dot{\alpha} + p_\beta\dot{\beta} - L$$

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}^{-1}\begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix} = \frac{1}{ab-c^2}\begin{pmatrix} b & -c \\ -c & a \end{pmatrix}\begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix}$$

$$L = \frac{1}{2}(m_1 + m_2)(l_1\dot{\alpha})^2 + \frac{1}{2}m_2\left\{\left(l_2\dot{\beta}\right)^2 + 2l_1l_2\dot{\alpha}\dot{\beta}\cos(\alpha - \beta)\right\} -$$
$$-(m_1 + m_2)gl_1[1 - \cos(\alpha)] - m_2gl_2[1 - \cos(\beta)]$$

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = (m_1 + m_2)l_1^2\dot{\alpha} + m_2l_1l_2\dot{\beta}\cos(\alpha - \beta)$$
$$p_\beta = \frac{\partial L}{\partial \dot{\beta}} = m_2l_2^2\dot{\beta} + m_2l_1l_2\dot{\alpha}\cos(\alpha - \beta)$$

$$a = (m_1 + m_2)l_1^2$$
$$b = m_2l_2^2$$
$$c = m_2l_1l_2\cos(\alpha - \beta)$$

$$\begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}$$

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} = \begin{pmatrix} a & c \\ c & b \end{pmatrix}^{-1}\begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix} = \frac{1}{ab - c^2}\begin{pmatrix} b & -c \\ -c & a \end{pmatrix}\begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix}$$

$$L=\frac{1}{2}\left(m_1+m_2\right)\left(l_1\dot{\alpha}\right)^2+\frac{1}{2}m_2\left\{\left(l_2\dot{\beta}\right)^2+2l_1l_2\dot{\alpha}\dot{\beta}\cos\left(\alpha-\beta\right)\right\}-$$
$$-\left(m_1+m_2\right)gl_1\left[1-\cos\left(\alpha\right)\right]-m_2gl_2\left[1-\cos\left(\beta\right)\right]$$

$$a=\left(m_1+m_2\right)l_1^2$$
$$b=m_2l_2^2$$
$$c=m_2l_1l_2\cos\left(\alpha-\beta\right)$$

$$\begin{pmatrix}\dot{\alpha}\\ \dot{\beta}\end{pmatrix}=\begin{pmatrix}a & c\\ c & b\end{pmatrix}^{-1}\begin{pmatrix}p_\alpha\\ p_\beta\end{pmatrix}=\frac{1}{ab-c^2}\begin{pmatrix}b & -c\\ -c & a\end{pmatrix}\begin{pmatrix}p_\alpha\\ p_\beta\end{pmatrix}$$

$$\dot{\alpha}=\frac{1}{ab-c^2}\left(bp_\alpha-cp_\beta\right)$$

$$\dot{\beta}=\frac{1}{ab-c^2}\left(-cp_\alpha+ap_\beta\right)$$

$$L=\frac{1}{2}a\dot{\alpha}^2+\frac{1}{2}b\dot{\beta}^2+c\dot{\alpha}\dot{\beta}-$$
$$-\left(m_1+m_2\right)gl_1\left[1-\cos\left(\alpha\right)\right]-m_2gl_2\left[1-\cos\left(\beta\right)\right]=$$
$$=\frac{1}{2}\left(\frac{1}{ab-c^2}\right)^2\left[a\left(b^2p_\alpha^2-2bcp_\alpha p_\beta+c^2p_\beta^2\right)+\right.$$
$$\left.+2c\left(bp_\alpha-cp_\beta\right)\left(-cp_\alpha+ap_\beta\right)+b\left(c^2p_\alpha^2-2acp_\alpha p_\beta+a^2p_\beta^2\right)\right]-$$
$$-\left(m_1+m_2\right)gl_1\left[1-\cos\left(\alpha\right)\right]-m_2gl_2\left[1-\cos\left(\beta\right)\right]=$$

$$
\begin{aligned}
L = \frac{1}{2} a\dot{\alpha}^2 + \frac{1}{2} b\dot{\beta}^2 + c\dot{\alpha}\dot{\beta} - \\
- (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] - m_2 g l_2 \left[1 - \cos(\beta)\right] = \\
= \frac{1}{2}\left(\frac{1}{ab - c^2}\right)^2 \left[a\left(b^2 p_\alpha^2 - 2bc p_\alpha p_\beta + c^2 p_\beta^2\right) + \right. \\
\left. +2c\left(b p_\alpha - c p_\beta\right)\left(-c p_\alpha + a p_\beta\right) + b\left(c^2 p_\alpha^2 - 2ac p_\alpha p_\beta + a^2 p_\beta^2\right)\right] - \\
- (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] - m_2 g l_2 \left[1 - \cos(\beta)\right] =
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{2}\left(\frac{1}{ab - c^2}\right)^2 \left[p_\alpha^2\left(ab^2 - \cancel{2}bc^2 + \cancel{bc^2}\right) + \right. \\
\left. +p_\alpha p_\beta\left(\cancel{-2abc} + \cancel{2cab} + 2c^3 - 2abc\right) + p_\beta^2\left(\cancel{ac^2} - \cancel{2}ac^2 + ba^2\right)\right] - \\
- (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] - m_2 g l_2 \left[1 - \cos(\beta)\right] =
\end{aligned}
$$

$$
\begin{aligned}
L = \frac{1}{2}\left(\frac{1}{ab-c^2}\right)^2 \left[p_\alpha^2\left(ab^2 - \cancel{2}bc^2 + \cancel{bc^2}\right) + \right. \\
+ p_\alpha p_\beta\left(\cancel{-2abc} + \cancel{2cab} + 2c^3 - 2abc\right) + p_\beta^2\left(\cancel{ac^2} - \cancel{2}ac^2 + ba^2\right)\Big] - \\
- (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] - m_2 g l_2 \left[1 - \cos(\beta)\right] =
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{2}\left(\frac{1}{ab-c^2}\right)^{\cancel{2}} \left[p_\alpha^2 b\, \cancel{\left(ab - c^2\right)} + \right. \\
-2c p_\alpha p_\beta \cancel{\left(-c^2 + ab\right)} + a p_\beta^2 \cancel{\left(-c^2 + ba\right)}\Big] - \\
- (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] - m_2 g l_2 \left[1 - \cos(\beta)\right] =
\end{aligned}
$$

$$
\begin{aligned}
= \frac{1}{2}\left(\frac{1}{ab-c^2}\right) \left[p_\alpha^2 b - 2c p_\alpha p_\beta + a p_\beta^2\right] - \\
- (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] - m_2 g l_2 \left[1 - \cos(\beta)\right]
\end{aligned}
$$

$$L = \frac{1}{2}\left(\frac{1}{ab-c^2}\right)\left[p_\alpha^2 b - 2cp_\alpha p_\beta + ap_\beta^2\right] -$$
$$- (m_1+m_2)\, gl_1 \left[1-\cos(\alpha)\right] - m_2 g l_2 \left[1-\cos(\beta)\right]$$

$$\begin{aligned}
H &= p_\alpha \dot{\alpha} + p_\beta \dot{\beta} - L = \\
&= \frac{1}{ab-c^2}\left[p_\alpha\left(bp_\alpha - cp_\beta\right) + p_\beta\left(-cp_\alpha + ap_\beta\right)\right] - L = \\
&= \frac{1}{ab-c^2}\left[bp_\alpha^2 - 2cp_\alpha p_\beta + ap_\beta^2\right] - \frac{1}{2}\left(\frac{1}{ab-c^2}\right)\left[p_\alpha^2 b - 2cp_\alpha p_\beta + ap_\beta^2\right] + \\
&+ (m_1+m_2)\, gl_1 \left[1-\cos(\alpha)\right] + m_2 g l_2 \left[1-\cos(\beta)\right] = \\
&= \frac{1}{2}\left(\frac{1}{ab-c^2}\right)\left[p_\alpha^2 b - 2cp_\alpha p_\beta + ap_\beta^2\right] + \\
&+ (m_1+m_2)\, gl_1 \left[1-\cos(\alpha)\right] + m_2 g l_2 \left[1-\cos(\beta)\right]
\end{aligned}$$

$$H = \frac{1}{2}\left(\frac{1}{ab - c^2}\right)\left[p_\alpha^2 b - 2cp_\alpha p_\beta + ap_\beta^2\right] +$$
$$+ (m_1 + m_2)\, gl_1 \left[1 - \cos(\alpha)\right] + m_2 gl_2 \left[1 - \cos(\beta)\right]$$

$$\dot{\alpha} = \frac{\partial H}{\partial p_\alpha} = \left(\frac{1}{ab - c^2}\right)\left[p_\alpha b - cp_\beta\right]$$

$$\dot{\beta} = \frac{\partial H}{\partial p_\beta} = \left(\frac{1}{ab - c^2}\right)\left[p_\beta a - cp_\alpha\right]$$

$$\dot{p}_\alpha = -\frac{\partial H}{\partial \alpha} = -\frac{1}{2}\left(\frac{1}{ab - c^2}\right)^2 \frac{\partial c}{\partial \alpha}\left[-2p_\alpha p_\beta\left(ab - c^2\right) - \left(p_\alpha^2 b - 2cp_\alpha p_\beta + ap_\beta^2\right)(-2c)\right] -$$
$$- (m_1 + m_2)\, gl_1 \sin(\alpha) =$$
$$= \left(\frac{1}{ab - c^2}\right)^2 \frac{\partial c}{\partial \alpha}\left[p_\alpha p_\beta\left(ab + c^2\right) - c\left(p_\alpha^2 b + ap_\beta^2\right)\right] - (m_1 + m_2)\, gl_1 \sin(\alpha)$$

$$\dot{p}_\beta = -\frac{\partial H}{\partial \beta} = -\frac{1}{2}\left(\frac{1}{ab - c^2}\right)^2 \frac{\partial c}{\partial \beta}\left[-2p_\alpha p_\beta\left(ab - c^2\right) - \left(p_\alpha^2 b - 2cp_\alpha p_\beta + ap_\beta^2\right)(-2c)\right] -$$
$$-m_2 gl_2 \sin(\beta) =$$
$$= \left(\frac{1}{ab - c^2}\right)^2 \frac{\partial c}{\partial \beta}\left[p_\alpha p_\beta\left(ab + c^2\right) - c\left(p_\alpha^2 b + ap_\beta^2\right)\right] - m_2 gl_2 \sin(\beta)$$

$$\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix}^T = \frac{1}{ab-c^2} \begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix}^T \begin{pmatrix} b & -c \\ -c & a \end{pmatrix}^T = \frac{1}{ab-c^2} \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \begin{pmatrix} b & -c \\ -c & a \end{pmatrix}$$

$$\begin{aligned} H = p_\alpha \dot{\alpha} + p_\beta \dot{\beta} - L = \\ = p_\alpha \dot{\alpha} + p_\beta \dot{\beta} - \frac{1}{2} a \dot{\alpha}^2 - \frac{1}{2} b \dot{\beta}^2 - c \dot{\alpha} \dot{\beta} + \\ + (m_1 + m_2) g l_1 [1 - \cos(\alpha)] + m_2 g l_2 [1 - \cos(\beta)] = \\ = \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \dot{\alpha} & \dot{\beta} \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + \\ + (m_1 + m_2) g l_1 [1 - \cos(\alpha)] + m_2 g l_2 [1 - \cos(\beta)] = \end{aligned}$$

$$\begin{aligned} = \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \left[\hat{I} - \frac{1}{2} \frac{1}{ab-c^2} \begin{pmatrix} b & -c \\ -c & a \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \right] \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} + \\ + (m_1 + m_2) g l_1 [1 - \cos(\alpha)] + m_2 g l_2 [1 - \cos(\beta)] = \end{aligned}$$

$$H = \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \left[\hat{I} - \frac{1}{2} \frac{1}{ab - c^2} \begin{pmatrix} b & -c \\ -c & a \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix} \right] \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} +$$

$$+ (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] + m_2 g l_2 \left[1 - \cos(\beta)\right] =$$

$$= \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \left[\hat{I} - \frac{1}{2} \frac{1}{ab - c^2} \begin{pmatrix} ab - c^2 & 0 \\ 0 & ab - c^2 \end{pmatrix} \right] \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} +$$

$$+ (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] + m_2 g l_2 \left[1 - \cos(\beta)\right] =$$

$$= \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \left[\hat{I} - \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} +$$

$$+ (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] + m_2 g l_2 \left[1 - \cos(\beta)\right] =$$

$$H = \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \left[\hat{I} - \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} +$$

$$+ (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] + m_2 g l_2 \left[1 - \cos(\beta)\right] =$$

$$= \frac{1}{2} \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \end{pmatrix} +$$

$$+ (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] + m_2 g l_2 \left[1 - \cos(\beta)\right] =$$

$$= \frac{1}{2} \frac{1}{ab - c^2} \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \begin{pmatrix} b & -c \\ -c & a \end{pmatrix} \begin{pmatrix} p_\alpha \\ p_\beta \end{pmatrix} +$$

$$+ (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] + m_2 g l_2 \left[1 - \cos(\beta)\right] =$$

$$= \frac{1}{2} \frac{1}{ab - c^2} \begin{pmatrix} p_\alpha & p_\beta \end{pmatrix} \begin{pmatrix} b p_\alpha - c p_\beta \\ -c p_\alpha + a p_\beta \end{pmatrix} +$$

$$+ (m_1 + m_2)\, g l_1 \left[1 - \cos(\alpha)\right] + m_2 g l_2 \left[1 - \cos(\beta)\right] =$$

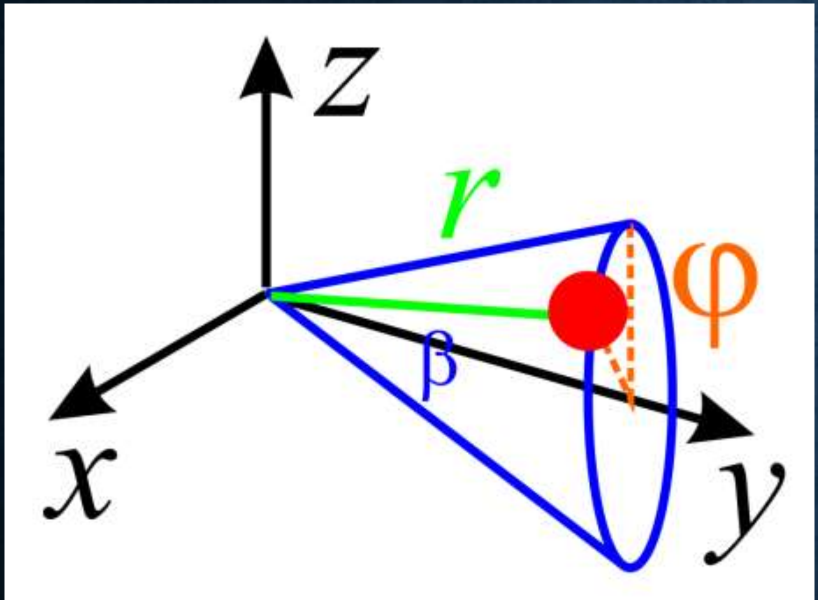
$$H = \frac{1}{2}\frac{1}{ab-c^2}\left(p_\alpha \qquad p_\beta\right)\begin{pmatrix} bp_\alpha - cp_\beta \\ -cp_\alpha + ap_\beta \end{pmatrix} +$$

$$+\left(m_1+m_2\right) gl_1\left[1-\cos\left(\alpha\right)\right] + m_2 gl_2\left[1-\cos\left(\beta\right)\right] =$$

$$= \frac{1}{2}\frac{1}{ab-c^2}\left(bp_\alpha^2 - 2cp_\alpha p_\beta + ap_\beta^2\right) +$$

$$+\left(m_1+m_2\right) gl_1\left[1-\cos\left(\alpha\right)\right] + m_2 gl_2\left[1-\cos\left(\beta\right)\right]$$

6- Um ponto material de massa m, sujeito à acção da gravidade, é obrigado a permanecer sobre a superfície de um cone de eixo horizontal. Determine as equações de Hamilton do movimento deste sistema.



Problema 5 de série 2

$$\{q_1, q_2\} = \{r, \varphi\}$$

$$L = \frac{1}{2}m\left[\dot{r}^2 + r^2\sin^2(\beta)\,\dot{\varphi}^2 - 2gr\sin(\beta)\cos(\varphi)\right]$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\sin^2(\beta)\,\dot{\varphi}$$

$$\dot{r} = \frac{p_r}{m} \qquad \dot{\varphi} = \frac{p_\varphi}{mr^2\sin^2(\beta)}$$

$$H = p_r\dot{r} + p_\varphi\dot{\varphi} - L = m\dot{r}^2 + mr^2\sin^2(\beta)\,\dot{\varphi}^2 - \frac{1}{2}m\left[\dot{r}^2 + r^2\sin^2(\beta)\,\dot{\varphi}^2 - 2gr\sin(\beta)\cos(\varphi)\right] =$$

$$= \frac{1}{2}m\left[\dot{r}^2 + r^2\sin^2(\beta)\,\dot{\varphi}^2 + 2gr\sin(\beta)\cos(\varphi)\right] = \frac{1}{2m}\left[p_r^2 + \frac{p_\varphi^2}{r^2\sin^2(\beta)}\right] + mgr\sin(\beta)\cos(\varphi)$$

$$H = \frac{1}{2m}\left[p_r^2 + \frac{p_\varphi^2}{r^2 \sin^2(\beta)}\right] + mgr\sin(\beta)\cos(\varphi)$$

$$\dot{p_r} = -\frac{\partial H}{\partial r} = \frac{p_\varphi^2}{mr^3 \sin^2(\beta)} - mg\sin(\beta)\cos(\varphi)$$

$$\dot{p_\varphi} = -\frac{\partial H}{\partial \varphi} = mg\sin(\beta)\sin(\varphi)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}$$

$$\dot{\varphi} = \frac{p_\varphi}{mr^2 \sin^2(\beta)}$$

7- O Lagrangeano de uma partícula de massa m e de carga eléctrica e num campo electromagnético é dado por:

$$L = \frac{m\,v^2}{2} + \frac{e}{c}\,\vec{A}\cdot\vec{v} - e\,\phi$$

onde c é a velocidade da luz no vácuo, $\vec{v}$ a velocidade da partícula, $\vec{A}$ o vector potencial magnético e ϕ o potencial eléctrico. Para simplificar, omite-se a dependência explícita de $\vec{A}$ e ϕ nas coordenadas Cartesianas, o que fisicamente corresponde ao caso em que essas quantidades têm o mesmo valor em todos os pontos do espaço.

Usando a transformação de Legendre,

$$H = \sum_i p_i\,\dot{q}_i - L$$

obtenha:

(a) O Hamiltoniano.

(b) As equações de Hamilton.

$$L=\frac{mv^2}{2}+e\left(\mathbf{A}\cdot\mathbf{v}\right)-e\phi=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)+e\left(A_x\dot{x}+A_y\dot{y}+A_z\dot{z}\right)-e\phi$$

$$p_x=\frac{\partial L}{\partial \dot{x}}=m\dot{x}+eA_x$$

$$p_y=\frac{\partial L}{\partial \dot{y}}=m\dot{y}+eA_y$$

$$p_z=\frac{\partial L}{\partial \dot{z}}=m\dot{z}+eA_z$$

$$\mathbf{p}=m\dot{\mathbf{r}}+e\mathbf{A}$$

$$\mathbf{v}=\dot{\mathbf{r}}$$

$$H=p_x\dot{x}+p_y\dot{y}+p_z\dot{z}-L=\left(\mathbf{p}\cdot\dot{\mathbf{r}}\right)-L=$$

$$=m\left(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}\right)\cancel{+e\left(\mathbf{A}\cdot\dot{\mathbf{r}}\right)}-\frac{m\dot{r}^2}{2}\cancel{-e\left(\mathbf{A}\cdot\mathbf{v}\right)}+e\phi=$$

$$=\frac{m\dot{r}^2}{2}+e\phi=\frac{1}{2m}\left(\mathbf{p}-e\mathbf{A}\right)^2+e\phi=$$

$$=\frac{1}{2m}\left[\left(p_x-eA_x\right)^2+\left(p_y-eA_y\right)^2+\left(p_z-eA_z\right)^2\right]+e\phi$$

$$H = \frac{1}{2m}\left[(p_x - eA_x)^2 + (p_y - eA_y)^2 + (p_z - eA_z)^2\right] + e\phi$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{e}{m}\left[(p_x - eA_x)\frac{\partial A_x}{\partial x} + (p_y - eA_y)\frac{\partial A_y}{\partial x} + (p_z - eA_z)\frac{\partial A_z}{\partial x}\right] - e\frac{\partial \phi}{\partial x}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = \frac{e}{m}\left[(p_x - eA_x)\frac{\partial A_x}{\partial y} + (p_y - eA_y)\frac{\partial A_y}{\partial y} + (p_z - eA_z)\frac{\partial A_z}{\partial y}\right] - e\frac{\partial \phi}{\partial y}$$

$$\dot{p}_y = -\frac{\partial H}{\partial z} = \frac{e}{m}\left[(p_x - eA_x)\frac{\partial A_x}{\partial z} + (p_y - eA_y)\frac{\partial A_y}{\partial z} + (p_z - eA_z)\frac{\partial A_z}{\partial z}\right] - e\frac{\partial \phi}{\partial z}$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}(p_x - eA_x)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{m}(p_y - eA_y)$$

$$\dot{\mathbf{r}} = \boldsymbol{\nabla}_{\mathbf{p}} H = \frac{1}{m}(\mathbf{p} - e\mathbf{A})$$

$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{1}{m}(p_z - eA_z)$$

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{e}{m}\left[(p_x - eA_x)\frac{\partial A_x}{\partial x} + (p_y - eA_y)\frac{\partial A_y}{\partial x} + (p_z - eA_z)\frac{\partial A_z}{\partial x}\right] - e\frac{\partial \phi}{\partial x}$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = \frac{e}{m}\left[(p_x - eA_x)\frac{\partial A_x}{\partial y} + (p_y - eA_y)\frac{\partial A_y}{\partial y} + (p_z - eA_z)\frac{\partial A_z}{\partial y}\right] - e\frac{\partial \phi}{\partial y}$$

$$\dot{p}_y = -\frac{\partial H}{\partial z} = \frac{e}{m}\left[(p_x - eA_x)\frac{\partial A_x}{\partial z} + (p_y - eA_y)\frac{\partial A_y}{\partial z} + (p_z - eA_z)\frac{\partial A_z}{\partial z}\right] - e\frac{\partial \phi}{\partial z}$$

$$\begin{aligned}\dot{\mathbf{p}} = -\boldsymbol{\nabla} H = \frac{e}{m}\mathbf{e}_x &\left[(p_x - eA_x)\frac{\partial A_x}{\partial x} + (p_y - eA_y)\frac{\partial A_y}{\partial x} + (p_z - eA_z)\frac{\partial A_z}{\partial x}\right] + \\ + \frac{e}{m}\mathbf{e}_y &\left[(p_x - eA_x)\frac{\partial A_x}{\partial y} + (p_y - eA_y)\frac{\partial A_y}{\partial y} + (p_z - eA_z)\frac{\partial A_z}{\partial y}\right] + \\ + \frac{e}{m}\mathbf{e}_z &\left[(p_x - eA_x)\frac{\partial A_x}{\partial z} + (p_y - eA_y)\frac{\partial A_y}{\partial z} + (p_z - eA_z)\frac{\partial A_z}{\partial z}\right] - e\boldsymbol{\nabla}\phi\end{aligned}$$

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \mathbf{e}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{e}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{e}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$(\mathbf{p} - e\mathbf{A}) \times \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ p_x - eA_x & p_y - eA_y & p_z - eA_z \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} & \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{vmatrix} =$$

$$= \mathbf{e}_x \left[(p_y - eA_y) \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - (p_z - eA_z) \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] +$$

$$+\mathbf{e}_y \left[(p_z - eA_z) \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - (p_x - eA_x) \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] +$$

$$+\mathbf{e}_z \left[(p_x - eA_x) \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) - (p_y - eA_y) \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \right]$$

$$
(\mathbf{p} - e\mathbf{A}) \times \boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ p_x - eA_x & p_y - eA_y & p_z - eA_z \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} & \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{vmatrix} =
$$

$$
= \mathbf{e}_x \left[(p_y - eA_y) \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) - (p_z - eA_z) \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \right] +
$$

$$
- (p_x - eA_x) \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \Bigg] +
$$

$$
\Bigg) - (p_y - eA_y) \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \Bigg]
$$

$$
\begin{aligned}
&= \mathbf{e}_x \left[(p_x - eA_x) \frac{\partial A_x}{\partial x} + (p_y - eA_y) \frac{\partial A_y}{\partial x} + (p_z - eA_z) \frac{\partial A_z}{\partial x} - \right. \\
&\quad \left. - (p_x - eA_x) \frac{\partial A_x}{\partial x} - (p_y - eA_y) \frac{\partial A_x}{\partial y} - (p_z - eA_z) \frac{\partial A_x}{\partial z} \right] + \\
&+ \mathbf{e}_y \left[(p_x - eA_x) \frac{\partial A_x}{\partial y} + (p_y - eA_y) \frac{\partial A_y}{\partial y} + (p_z - eA_z) \frac{\partial A_z}{\partial y} - \right. \\
&\quad \left. - (p_x - eA_x) \frac{\partial A_y}{\partial x} - (p_y - eA_y) \frac{\partial A_y}{\partial y} - (p_z - eA_z) \frac{\partial A_y}{\partial z} \right] + \\
&+ \mathbf{e}_z \left[(p_x - eA_x) \frac{\partial A_x}{\partial z} + (p_y - eA_y) \frac{\partial A_y}{\partial z} + (p_z - eA_z) \frac{\partial A_z}{\partial z} - \right. \\
&\quad \left. - (p_x - eA_x) \frac{\partial A_z}{\partial x} - (p_y - eA_y) \frac{\partial A_z}{\partial y} - (p_z - eA_z) \frac{\partial A_z}{\partial z} \right]
\end{aligned}
$$

$$= \mathbf{e}_x \left[(p_x - eA_x)\frac{\partial A_x}{\partial x} + (p_y - eA_y)\frac{\partial A_y}{\partial x} + (p_z - eA_z)\frac{\partial A_z}{\partial x} - \right.$$

$$\left. - (p_x - eA_x)\frac{\partial A_x}{\partial x} - (p_y - eA_y)\frac{\partial A_x}{\partial y} - (p_z - eA_z)\frac{\partial A_x}{\partial z} \right] +$$

$$\boxed{\left([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla} \right) A_x}$$

$$+\mathbf{e}_y \left[(p_x - eA_x)\frac{\partial A_x}{\partial y} + (p_y - eA_y)\frac{\partial A_y}{\partial y} + (p_z - eA_z)\frac{\partial A_z}{\partial y} - \right.$$

$$\left. - (p_x - eA_x)\frac{\partial A_y}{\partial x} - (p_y - eA_y)\frac{\partial A_y}{\partial y} - (p_z - eA_z)\frac{\partial A_y}{\partial z} \right] +$$

$$\boxed{\left([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla} \right) A_y}$$

$$+\mathbf{e}_z \left[(p_x - eA_x)\frac{\partial A_x}{\partial z} + (p_y - eA_y)\frac{\partial A_y}{\partial z} + (p_z - eA_z)\frac{\partial A_z}{\partial z} - \right.$$

$$\left. - (p_x - eA_x)\frac{\partial A_z}{\partial x} - (p_y - eA_y)\frac{\partial A_z}{\partial y} - (p_z - eA_z)\frac{\partial A_z}{\partial z} \right]$$

$$\boxed{\left([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla} \right) A_z}$$

$$
\begin{aligned}
&= \mathbf{e}_x \left[(p_x - eA_x) \frac{\partial A_x}{\partial x} + (p_y - eA_y) \frac{\partial A_y}{\partial x} + (p_z - eA_z) \frac{\partial A_z}{\partial x} \right] + \\
&+ \mathbf{e}_y \left[(p_x - eA_x) \frac{\partial A_x}{\partial y} + (p_y - eA_y) \frac{\partial A_y}{\partial y} + (p_z - eA_z) \frac{\partial A_z}{\partial y} \right] + \\
&+ \mathbf{e}_z \left[(p_x - eA_x) \frac{\partial A_x}{\partial z} + (p_y - eA_y) \frac{\partial A_y}{\partial z} + (p_z - eA_z) \frac{\partial A_z}{\partial z} \right] - \\
&- \mathbf{e}_x \left([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla} \right) A_x - \mathbf{e}_y \left([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla} \right) A_y - \mathbf{e}_z \left([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla} \right) A_z
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{e}_x \left[(p_x - eA_x) \frac{\partial A_x}{\partial x} + (p_y - eA_y) \frac{\partial A_y}{\partial x} + (p_z - eA_z) \frac{\partial A_z}{\partial x} \right] + \\
&+ \mathbf{e}_y \left[(p_x - eA_x) \frac{\partial A_x}{\partial y} + (p_y - eA_y) \frac{\partial A_y}{\partial y} + (p_z - eA_z) \frac{\partial A_z}{\partial y} \right] + \\
&+ \mathbf{e}_z \left[(p_x - eA_x) \frac{\partial A_x}{\partial z} + (p_y - eA_y) \frac{\partial A_y}{\partial z} + (p_z - eA_z) \frac{\partial A_z}{\partial z} \right] = \\
&= (\mathbf{p} - e\mathbf{A}) \times \boldsymbol{\nabla} \times \mathbf{A} + \left([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla} \right) \mathbf{A}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{e}_x\left[(p_x - eA_x)\frac{\partial A_x}{\partial x} + (p_y - eA_y)\frac{\partial A_y}{\partial x} + (p_z - eA_z)\frac{\partial A_z}{\partial x}\right] + \\
&+\mathbf{e}_y\left[(p_x - eA_x)\frac{\partial A_x}{\partial y} + (p_y - eA_y)\frac{\partial A_y}{\partial y} + (p_z - eA_z)\frac{\partial A_z}{\partial y}\right] + \\
&+\mathbf{e}_z\left[(p_x - eA_x)\frac{\partial A_x}{\partial z} + (p_y - eA_y)\frac{\partial A_y}{\partial z} + (p_z - eA_z)\frac{\partial A_z}{\partial z}\right] = \\
&= (\mathbf{p} - e\mathbf{A}) \times \boldsymbol{\nabla} \times \mathbf{A} + ([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla})\,\mathbf{A}
\end{aligned}
$$

$$
\dot{\mathbf{r}} = \boldsymbol{\nabla}_{\mathbf{p}} H = \frac{1}{m}(\mathbf{p} - e\mathbf{A})
$$

$$
\begin{aligned}
\dot{\mathbf{p}} = -\boldsymbol{\nabla} H = &\frac{e}{m}\mathbf{e}_x\left[(p_x - eA_x)\frac{\partial A_x}{\partial x} + (p_y - eA_y)\frac{\partial A_y}{\partial x} + (p_z - eA_z)\frac{\partial A_z}{\partial x}\right] + \\
&+\frac{e}{m}\mathbf{e}_y\left[(p_x - eA_x)\frac{\partial A_x}{\partial y} + (p_y - eA_y)\frac{\partial A_y}{\partial y} + (p_z - eA_z)\frac{\partial A_z}{\partial y}\right] + \\
&+\frac{e}{m}\mathbf{e}_z\left[(p_x - eA_x)\frac{\partial A_x}{\partial z} + (p_y - eA_y)\frac{\partial A_y}{\partial z} + (p_z - eA_z)\frac{\partial A_z}{\partial z}\right] - e\boldsymbol{\nabla}\phi
\end{aligned}
$$

$$
\begin{aligned}
\dot{\mathbf{p}} = -\boldsymbol{\nabla} H &= \frac{e}{m}\left\{[\mathbf{p} - e\mathbf{A}] \times \boldsymbol{\nabla} \times \mathbf{A} + ([\mathbf{p} - e\mathbf{A}] \cdot \boldsymbol{\nabla})\,\mathbf{A}\right\} - e\boldsymbol{\nabla}\phi \\
&= e\left\{\dot{\mathbf{r}} \times \boldsymbol{\nabla} \times \mathbf{A} + (\dot{\mathbf{r}} \cdot \boldsymbol{\nabla})\,\mathbf{A}\right\} - e\boldsymbol{\nabla}\phi
\end{aligned}
$$

$$\dot{\mathbf{r}} = \boldsymbol{\nabla}_{\mathbf{p}} H = \frac{1}{m}\left(\mathbf{p} - e\mathbf{A}\right)$$

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{A}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{A}}{\partial z}\frac{\partial z}{\partial t} =$$

$$= \frac{\partial \mathbf{A}}{\partial t} + \dot{x}\frac{\partial \mathbf{A}}{\partial x} + \dot{y}\frac{\partial \mathbf{A}}{\partial y} + \dot{z}\frac{\partial \mathbf{A}}{\partial z} = \frac{\partial \mathbf{A}}{\partial t} + \left(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\right)\mathbf{A}$$

$$\ddot{\mathbf{r}} = \frac{1}{m}\left(\dot{\mathbf{p}} - e\frac{d\mathbf{A}}{dt}\right)$$

$$\dot{\mathbf{p}} = -\boldsymbol{\nabla} H = \frac{e}{m}\left\{\left[\mathbf{p} - e\mathbf{A}\right] \times \boldsymbol{\nabla} \times \mathbf{A} + \left(\left[\mathbf{p} - e\mathbf{A}\right] \cdot \boldsymbol{\nabla}\right)\mathbf{A}\right\} - e\boldsymbol{\nabla}\phi$$

$$= e\left\{\dot{\mathbf{r}} \times \boldsymbol{\nabla} \times \mathbf{A} + \left(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\right)\mathbf{A}\right\} - e\boldsymbol{\nabla}\phi$$

$$\ddot{\mathbf{r}} = \frac{1}{m}\left(e\left\{\dot{\mathbf{r}} \times \boldsymbol{\nabla} \times \mathbf{A} + \cancel{\left(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\right)\mathbf{A}}\right\} - e\boldsymbol{\nabla}\phi - e\frac{\partial \mathbf{A}}{\partial t} - \cancel{e\left(\dot{\mathbf{r}} \cdot \boldsymbol{\nabla}\right)}\mathbf{A}\right)$$

$$m\ddot{\mathbf{r}} = e\left(-\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t} + \dot{\mathbf{r}} \times \boldsymbol{\nabla} \times \mathbf{A}\right)$$

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

$$m\ddot{\mathbf{r}} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$