Cálculo EE

Ficha 7 — Dezembro

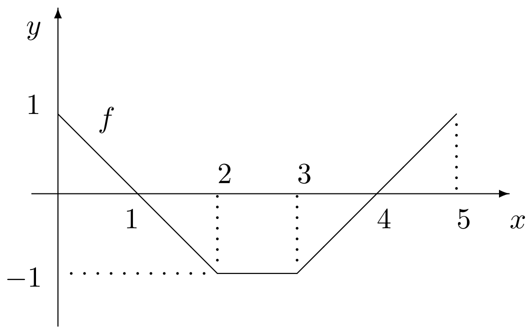
Integral definido

1. Sabendo que $\int_1^4 f(x)\,dx = 3$ e que $\int_2^4 f(x)\,dx = 5$, determine:

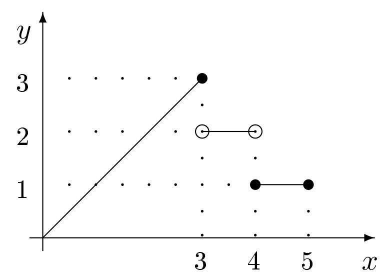
   (a) $\int_1^4 f(t)\,dt$; (b) $\int_4^2 f(t)\,dt$; (c) $\int_1^2 f(x)\,dx$.

2. Seja $f\colon [0,5] \longrightarrow \mathbb{R}$ a função representada na figura ao lado. Recorrendo ao significado geométrico do integral em termos de área, calcule

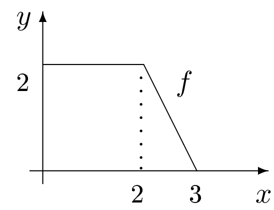
   $$\int_0^1 f(x)dx,\ \int_1^2 f(x)dx,\ \int_0^5 f(x)dx.$$



3. Sem recorrer ao Teorema Fundamental do Cálculo, determine $\int_0^5 f(x)\,dx$, sendo f a função representada na figura. Justifique convenientemente a sua resposta.



4. Sejam $f : [0,3] \longrightarrow \mathbb{R}$ a função representada na figura e $F\colon [0,3] \longrightarrow \mathbb{R}$ uma sua primitiva. Sem calcular qualquer integral, determine $F(3) - F(0)$.



5. Apresente um exemplo de:

   (a) uma função $f\colon [0,2] \longrightarrow \mathbb{R}$ tal que $\int_0^2 f(x)\,dx = 0$ e $f(x) \neq 0, \forall x \in [0,2]$;

   (b) duas funções $f, g\colon [0,2] \longrightarrow \mathbb{R}$ tais que $\int_0^2 f(x)\,dx = \int_0^2 g(x)\,dx$ e $f(x) \neq g(x)$, $\forall x \in [0,2]$.

6. Calcule os seguintes integrais definidos:

(a) $\displaystyle\int_0^2 (x+1)^2 \ dx;$ (b) $\displaystyle\int_{-1}^1 \frac{1}{1+x^2} \ dx;$

(e) $\displaystyle\int_{-1}^2 x|x| \ dx;$ (f) $\displaystyle\int_0^\pi x \ \sin x \ dx;$

(g) $\displaystyle\int_0^1 x \arctan x^2 \ dx;$ (h) $\displaystyle\int_0^{\sqrt{2}/2} \arcsin x \ dx;$

(i) $\displaystyle\int_0^1 \frac{2x^3+x^2-3x-1}{x^2+1} dx;$ (j) $\displaystyle\int_0^2 \frac{2x-1}{(x-3)(x+1)} \ dx;$

(k) $\displaystyle\int_0^1 \sqrt{4-4x^2} \ dx$ com $x = \cos t$ (l) $\displaystyle\int_{-5}^0 2x\sqrt{4-x}dx$ com $\sqrt{4-x} = y;$

(m) $\displaystyle\int_0^1 \frac{\sqrt{t}}{1+t\sqrt{t} \ dt}$ com $s = \sqrt{t};$ (n) $\displaystyle\int_{3/4}^{4/3} \frac{1}{x^2\sqrt{1+x^2}} \ dx$ com $x = \sinh t;$

(o) $\displaystyle\int_0^1 g(x) \ dx,$ com $g(x) = \begin{cases} 2x & \text{se} \quad 0 \le x \le \frac{1}{2}, \\ -3x & \text{se} \quad \frac{1}{2} < x \le 1 \end{cases}$

7. Sem calcular os integrais $I = \displaystyle\int_0^1 \sqrt{1-x^2}\, dx$ e $J = \displaystyle\int_{2\pi}^{3\pi/2} \sin^2 x \, dx$, justifique que $I > 0$ e $J < 0$.

8. Comparando o integral dado com um integral mais fácil de calcular, verifique as seguintes estimativas:

(a) $\displaystyle\frac{\pi}{4} < \int_0^1 \frac{dx}{1+x^3} < 1;$ (b) $\displaystyle 0 < \int_0^\pi \sin^2 x \ dx < 2;$

9. Seja f uma função contínua tal que se verifica a igualdade seguinte para todo o númro real x:

$$\int_0^x f(t) \ dt = \frac{4}{3} + 3x^2 + \sin(2x).$$

Calcule $f(\frac{\pi}{2})$ e $f'(\frac{\pi}{4})$.

Aplicações de integrais

1. Usando integrais, escreva uma expressão que permita calcular a área da região plana limitada pelas curvas seguintes.

(a) $x = 0, \quad x = 1, \quad y = 3x, \quad y = -x^2 + 4$
(b) $x = 0, \quad x = \frac{\pi}{2}, \quad y = \sin x, \quad y = \cos x;$
(c) $x^2 = 12(y-1), \quad x^2 + y^2 = 16;$
(d) $\{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 \le 4 \ \text{ e } \ 0 \le y \le x\};$

2. Usando integrais, escreva uma expressão que permita calcular o volume do sólido de revolução gerado pela rotação da região plana limitada pelas curvas seguintes, em torno do eixo OX.

   (a) $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : x^2 \leq y \leq \sqrt{x}\}$;
   (b) $\mathcal{R} = \{(x,y) \in \mathbb{R}^2 : |x-2|+1 \leq y \leq 3\}$;

Integrais impróprios

1. Determine, se existir, o integral

   (a) $I = \displaystyle\int_0^1 \frac{1}{2t}\,dt, \quad I = \int_{-1}^1 \frac{1}{1+t}\,dt, \quad I = \int_{-1}^1 \frac{1}{\sqrt{1+t}}\,dt, \quad I = \int_1^3 \frac{1}{\sqrt[5]{t-1}}\,dt.$
   (b) $I = \displaystyle\int_0^5 \frac{t^2-1}{2t}\,dt, \quad I = \int_1^e \frac{1}{t\ln(t)}\,dt, \quad I = \int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)}\,dx.$
   (c) $I = \displaystyle\int_0^1 \frac{e^t}{1-e^t}\,dt, \quad I = \int_0^{\pi/4} \frac{\cos(x)}{\sqrt{\sin(x)}}\,dx, \quad I = \int_0^{\frac{\pi}{2}} \frac{\sin(x)}{(\cos(x))^2}\,dx.$

2. Determine, se existir, o integral

   (a) $I = \displaystyle\int_1^{+\infty} \frac{1}{2t}\,dt, \quad I = \int_0^{+\infty} \frac{1}{(1+t)^2}\,dt, \quad I = \int_0^{+\infty} \frac{1}{\sqrt{1+t}}\,dt, \quad I = \int_0^{+\infty} \frac{e^{t/2}}{e^t+1}\,dt.$
   (b) $I = \displaystyle\int_1^{+\infty} \frac{\ln(t)}{t^2}\,dt$ por partes, $\quad I = \displaystyle\int_1^{+\infty} \frac{1}{1+3t^2}\,dt, \quad I = \int_{-\infty}^{-1} \frac{7t}{1-4t^2}\,dt.$
   (c) $I = \displaystyle\int_1^{+\infty} \frac{1}{\sqrt{t}(1+\sqrt{t})}\,dt$ e $u = \sqrt{t}$, $\quad I = \displaystyle\int_0^{+\infty} \frac{t}{(1+t^2)^{5/2}}\,dt$ e $t = \sinh(x)$.

3. Indique se o integral é convergente ou divergente (sem calcular explicitamente o valor)

   (a) $I = \displaystyle\int_1^{+\infty} \frac{1}{2te^t}\,dt, \; I = \int_1^{+\infty} \frac{\sqrt{t}+t}{(1+t)^2}\,dt, \quad I = \int_0^1 \frac{1}{\sqrt{1+t}+\sqrt{1-t}}\,dt.$
   (b) $I = \displaystyle\int_0^{+\infty} \ln\left(\frac{e^t}{e^t+1}\right)dt, \quad I = \int_0^{1/4} \frac{\tan(\pi\sqrt{x})}{x}\,dt, \quad I = \int_1^{+\infty} (1-\tanh(x))dx.$

Soluções de alguns exercícios

1. (a) 3 (b) -5 (c) -2

2. $\int_0^1 f(x)dx = \frac{1}{2}$, $\int_1^2 f(x)dx = -\frac{1}{2}$ $\int_0^5 f(x)dx = -1$. 3. $\frac{15}{2}$ 4. 5

5. (a) $f(x) = \begin{cases} k & \text{se } 0 \leq x \leq 1, \\ -k & \text{se } 1 < x \leq 2 \end{cases}$ para qualquer valor real de k. (b) f da alínea anterior e $g(x) = \begin{cases} -k & \text{se } 0 \leq x \leq 1, \\ k & \text{se } 1 < x \leq 2 \end{cases}$ para qualquer valor real de k.

6.

(a) $\frac{26}{3}$; (b) $\frac{\pi}{2}$; (e) $\frac{7}{3}$; (f) π;

(g) $\frac{\pi}{8} - \frac{\ln 2}{4}$; (h) $\frac{\sqrt{2}}{2}\left(\frac{\pi}{4}+1\right) - 1$; (i) $2 - \frac{5}{2}\ln 2$; (j) $\ln\frac{1}{\sqrt{3}}$;

(k) $\dfrac{\pi}{4}$ (l) (m) $\frac{2}{3}\ln 2$ (n) $\frac{5}{12}$

(o) $-\frac{7}{8}$

8. (a) A partir de $\frac{1}{1+x^2} \leq \frac{1}{1+x^3} \leq 1$ para todo $x \in [0,1]$ chega-se ao resultado pretendido; (b) A partir de $0 \leq \sin^2 x \leq \sin x$ para todo $x \in [0,\pi]$ chega-se ao resultado pretendido;

9. $f(\pi/2) = 3\pi - 2$; $f'(\pi/4) = 2$.

Aplicações de integrais

1.
(a) $\int_0^1 -x^2 - 3x + 4 \; dx$ (b) $\int_0^{\pi/4} \cos x - \sin x \; dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \; dx$

(c) $\int_{-\sqrt{12}}^{\sqrt{12}} \sqrt{16 - x^2} - (1 + x^2/2) \; dx$ (d) $\int_0^2 x \; dx + \int_2^4 \sqrt{4 - (x-2)^2} \; dx$

2.(a) $\pi \int_0^1 x - x^4 \; dx$ (b) $2\pi \int_0^2 9 - (3-x)^2 \; dx$

Integrais impróprios

1.

(a) i) $\displaystyle\int_0^1 \frac{1}{2t}\, dt = +\infty$, ii) $\displaystyle\int_{-1}^1 \frac{1}{1+t}\, dt = +\infty$, iii) $\displaystyle\int_{-1}^1 \frac{1}{\sqrt{1+t}}\, dt = 2\sqrt{2}$,
iv) $\displaystyle\int_1^3 \frac{1}{\sqrt[5]{t-1}}\, dt = \frac{5\sqrt[5]{2}}{4}$.

(b) i) $\displaystyle\int_0^5 \frac{t^2-1}{2t}\, dt = +\infty$, ii) $\displaystyle\int_1^e \frac{1}{t\ln(t)}\, dt = +\infty$, iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos(x)}{\sin(x)}\, dx = +\infty$.

(c) i) $\displaystyle\int_0^1 \frac{e^t}{1-e^t}\, dt = +\infty$, ii) $\displaystyle\int_0^{\pi/4} \frac{\cos(x)}{\sqrt{\sin(x)}}\, dx = \sqrt[4]{8}$, iii) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin(x)}{(\cos(x))^2}\, dx = +\infty$.

2.

(a) i) $\displaystyle\int_1^{+\infty} \frac{1}{2t}\, dt = +\infty$, ii) $\displaystyle\int_0^{+\infty} \frac{1}{(1+t)^2}\, dt = 1$, iii) $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{1+t}}\, dt = +\infty$,
iv) $\displaystyle\int_0^{+\infty} \frac{e^{t/2}}{e^t+1}\, dt = \frac{\pi}{2}$.

(b) i) $\displaystyle\int_1^{+\infty} \frac{\ln(t)}{t^2}\, dt = 1$, ii) $\displaystyle\int_1^{+\infty} \frac{1}{1+3t^2}\, dt = \frac{\pi}{6\sqrt{3}}$, iii) $\displaystyle\int_{-\infty}^{-1} \frac{7t}{1-4t^2}\, dt = +\infty$.

(c) i) $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{t}(1+\sqrt{t})}\, dt = +\infty$, ii) $\displaystyle\int_0^{+\infty} \frac{t}{(1+t^2)^{5/2}}\, dt = \frac{1}{3}$.