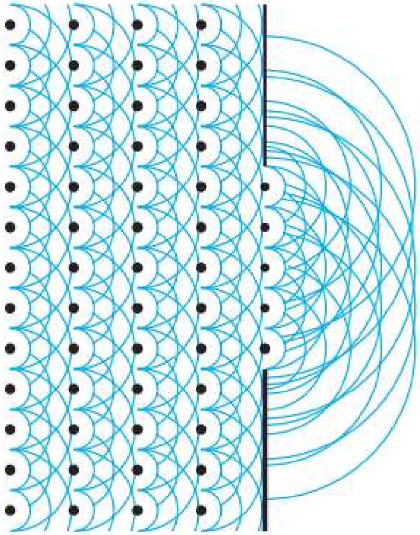
Integral da propagação Fresnel-Huygens



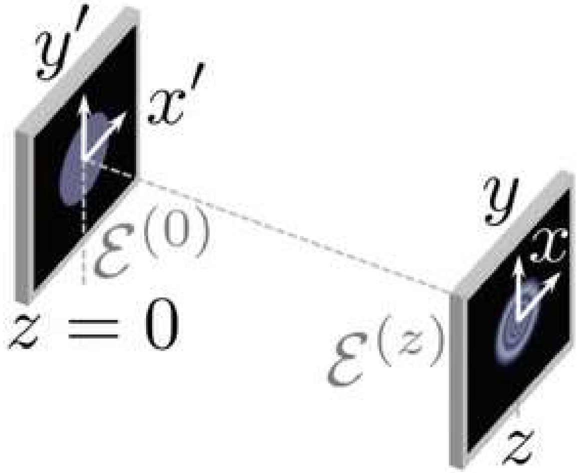
Cada ponto na abertura [f(x´, y´)] é tratado como uma fonte de onda esférica
Dentro da aproximação Fresnel o integral representa a sobreposição destas fontes.

$$\mathcal{E}(x,y,z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} \int_{-\infty}^{\infty}\int f(x',y') \mathrm{e}^{\left\{ik\left[(x-x')^2+(y-y')^2\right]/2z\right\}} dx'dy'$$

Efetivamente uma soma infinita de fasores

$$r_p^{\ 2} = (x-x')^2 + (y-y') + z^2$$

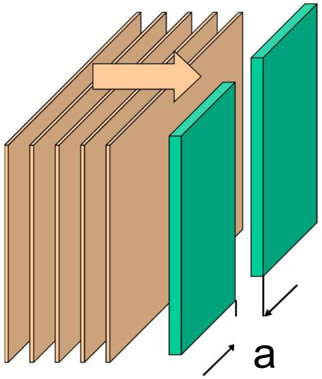
$$r_p \approx z + \frac{(x-x')^2 + (y-y')^2}{2z}$$



Aproximação Fresnel
(paraxial)

$$z \gg x, y, x', y'$$

Integral sobre y´



$$\mathcal{E}(x,y,z)=\frac{\mathcal{E}_0}{i\lambda z}e^{ikz}\int_{-a/2}^{a/2}\mathrm{e}^{\left\{ik(x-x')^2/2z\right\}}\,dx'\int_{-\infty}^{\infty}\mathrm{e}^{\left\{ik(y-y')^2/2z\right\}}dy'$$

$$\int_{-\infty}^{\infty}\mathrm{e}^{ik(y-y')^2/2z}dy'=\int_{-\infty}^{\infty}\mathrm{e}^{-\pi(y-y')^2/i\lambda z}dy'=\sqrt{i\lambda z}$$

Integral Gaussiano $\displaystyle\int_{-\infty}^{\infty}du\,\mathrm{e}^{-\beta(u-u_0)^2}=\sqrt{\frac{\pi}{\beta}}$ Re(β) ≥0

$$\mathcal{E}(x,y,z)=\frac{\mathcal{E}_0}{\sqrt{i\lambda z}}e^{ikz}\int_{-a/2}^{a/2}\mathrm{e}^{\left\{ik(x-x')^2/2z\right\}}\,dx'$$

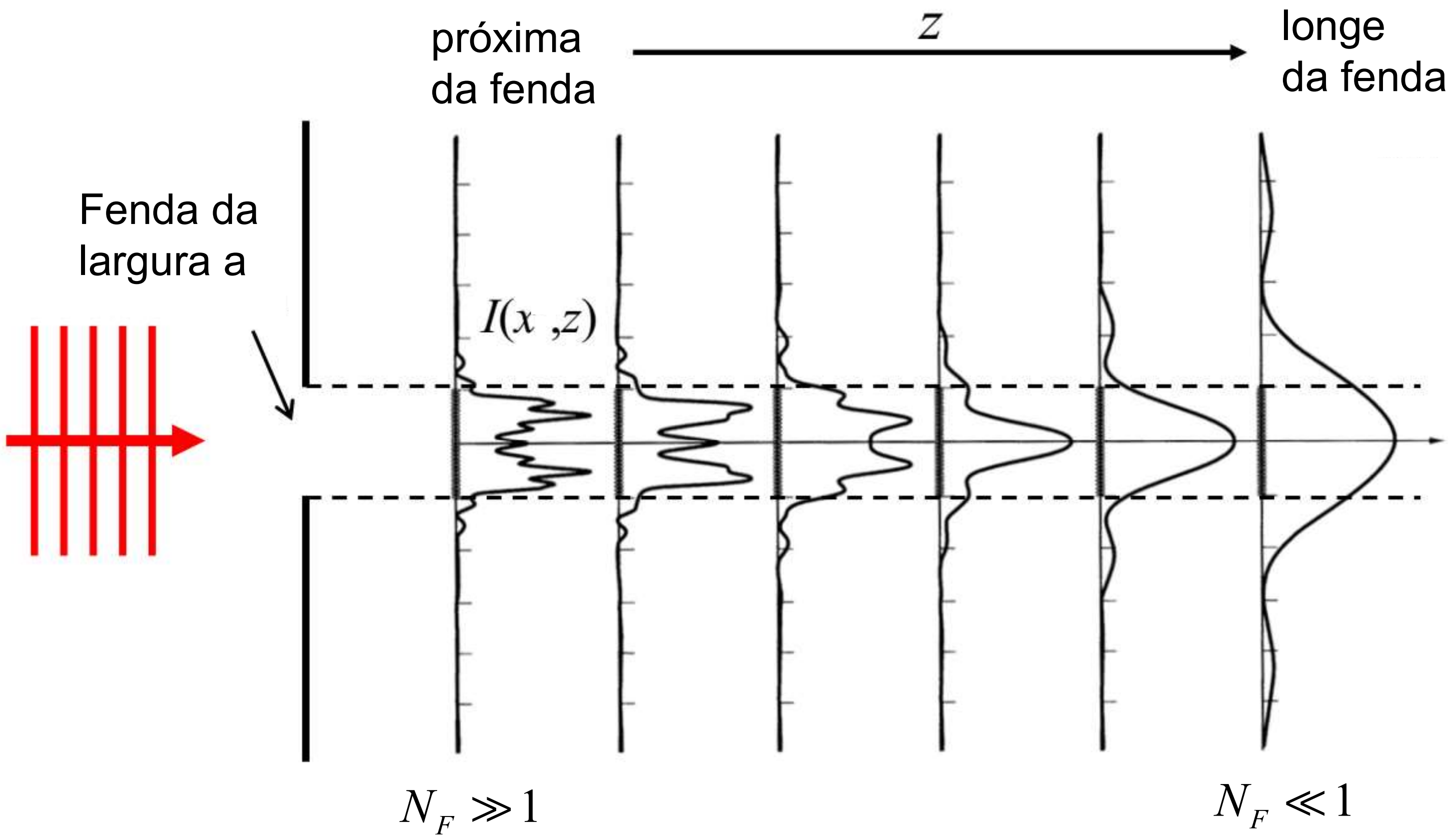
Soma infinita de ondas cilíndricas

O amplitude das ondas cilíndricas $\sim\frac{1}{\sqrt{z}}$

O amplitude das ondas esféricas $\sim\frac{1}{z}$

Difração de uma fenda simples

$$\mathcal{I}(x,z) = N_F \mathcal{I}_0 \left| \int_{-1}^{1} \exp\left[i\pi N_F (u-u')^2 \right] du' \right|^2 \qquad u = \frac{2x}{a} \qquad N_F \equiv \frac{(a/2)^2}{z\lambda}$$



Importância do número de Fresnel

$$N_F \equiv \frac{(a/2)^2}{z\lambda}$$

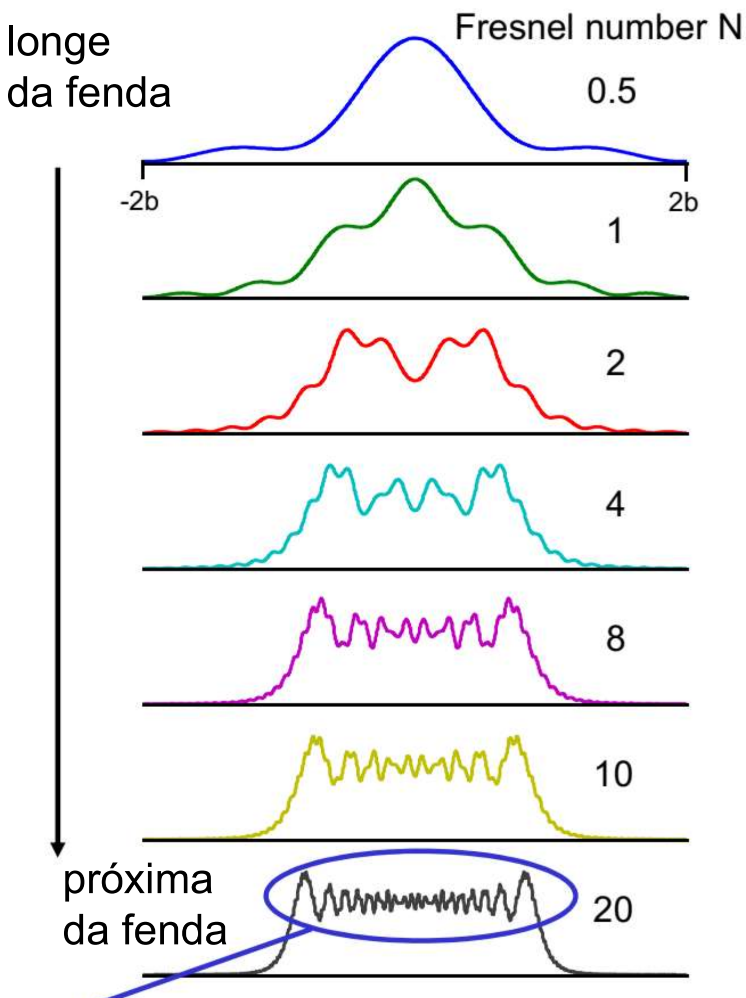
exemplo $\lambda = 500\ nm$ luz verde

$$a/2 = 10\ \mu m = 20\lambda$$

$$N = 1 \quad @ \quad z = 400\lambda = 0,2mm$$

$$a/2 = 1\ mm = 2000\lambda$$

$$N = 1 \quad @ \quad z = 4x10^6\lambda = 2m$$

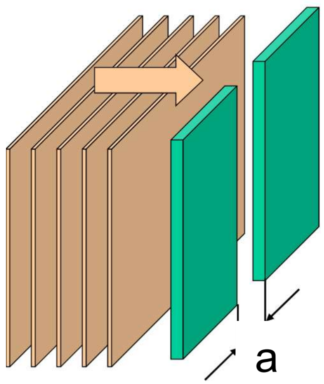


Nº de oscilações = N_F

Aproximação Fraunhofer N_F<<1 (campo distante)

$$\mathcal{E}(x,z)=\frac{\mathcal{E}_0}{\sqrt{i\lambda z}}e^{ikz}\int_{-a/2}^{a/2}\exp\left[i\frac{\pi}{\lambda z}(x-x')^2\right]dx' \qquad N_F=\frac{(a/2)^2}{\lambda z}\ll 1$$

Aproximação Fraunhofer (depende de λ)



$$e^{i\pi x^2/\lambda z}\int_{-a2}^{a/2}\exp\left[i\frac{\pi}{\lambda z}\left(-2xx'+x'^2\right)\right]du' \qquad \left(\frac{x'^2}{\lambda z}\right)\le\left(\frac{(a/2)^2}{\lambda z}\right)=N_F$$

$$e^{i\pi x^2/\lambda z}\int_{-a/2}^{a/2}\exp\left[-i2\pi\frac{xx'}{\lambda z}\right]dx'=e^{i\pi x^2/\lambda z}\left[\frac{e^{-i\pi ax/\lambda z}-e^{i\pi ax/\lambda z}}{-i2\pi x/\lambda z}\right]$$

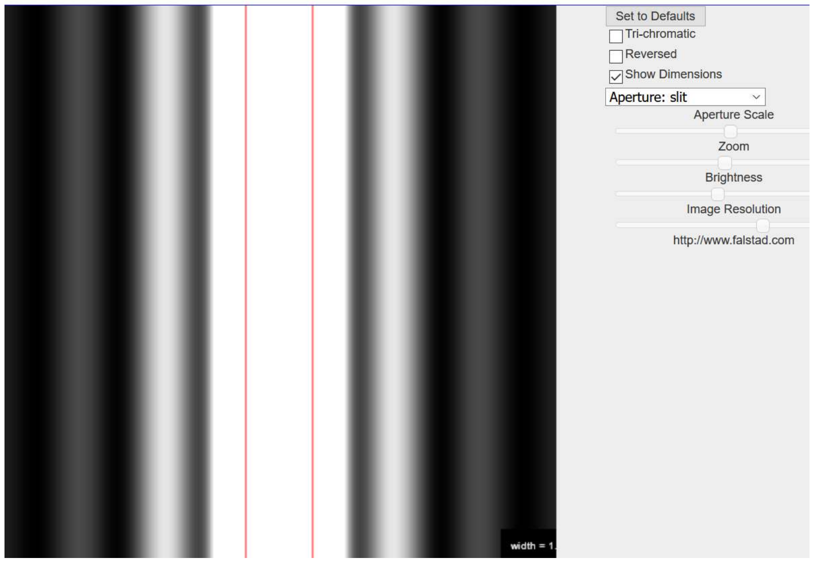
$$=e^{i\pi x^2/\lambda z}a\frac{\sin(\pi ax/\lambda z)}{\pi ax/\lambda z}$$

$$\mathcal{E}(x,z)=\frac{\mathcal{E}_0}{\sqrt{i\lambda z}}e^{ikz}e^{i\pi x^2/\lambda z}a\left[\frac{\sin(\pi ax/\lambda z)}{\pi ax/\lambda z}\right]$$

$$\mathcal{I}(x,z)=\mathcal{I}_0\left(\frac{a^2}{\lambda z}\right)\left[\frac{\sin(\pi ax/\lambda z)}{\pi ax/\lambda z}\right]^2$$

$$=4\mathcal{I}_0N_F\left[\frac{\sin(4\pi N_F x/a)}{(4\pi N_F x/a)}\right]^2$$

Difração Fenda simples (aproximação Fraunhofer)



https://www.falstad.com/diffraction/

$$\mathcal{I}(x,z) = \mathcal{I}_0 \left(\frac{a^2}{\lambda z}\right)\left[\frac{\sin(\pi a x/\lambda z)}{\pi a x/\lambda z}\right]^2$$

$$\mathrm{sinc}(\xi) = \left[\frac{\sin(\xi)}{\xi}\right]^2 \qquad \lim_{\xi\to 0}\mathrm{sinc}(\xi) = 1$$

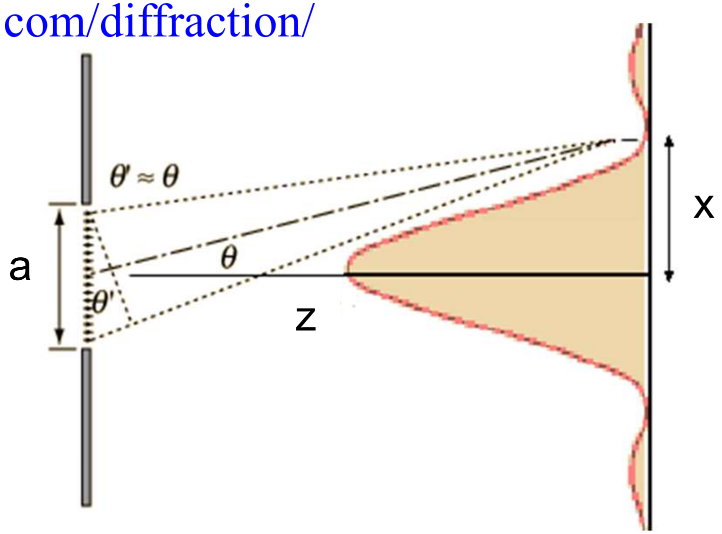
mínimos

$$\mathcal{I}(x,z) = 0 \quad @\frac{\pi a x_{\min}}{\lambda z} = m\pi$$

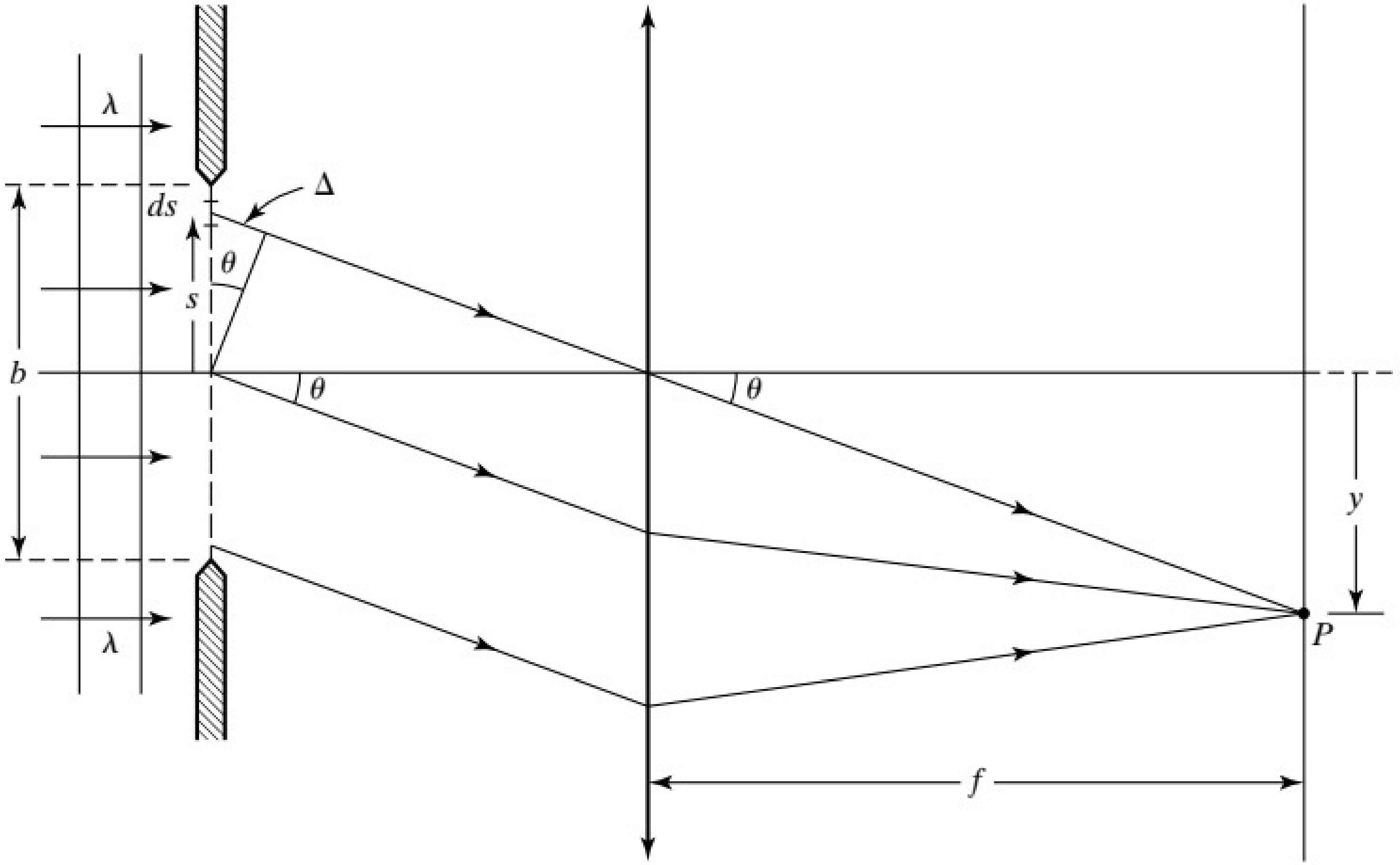
exceto m=0

$$a\frac{x_{\min}}{z} \approx a\sin\theta_{\min} = m\lambda$$

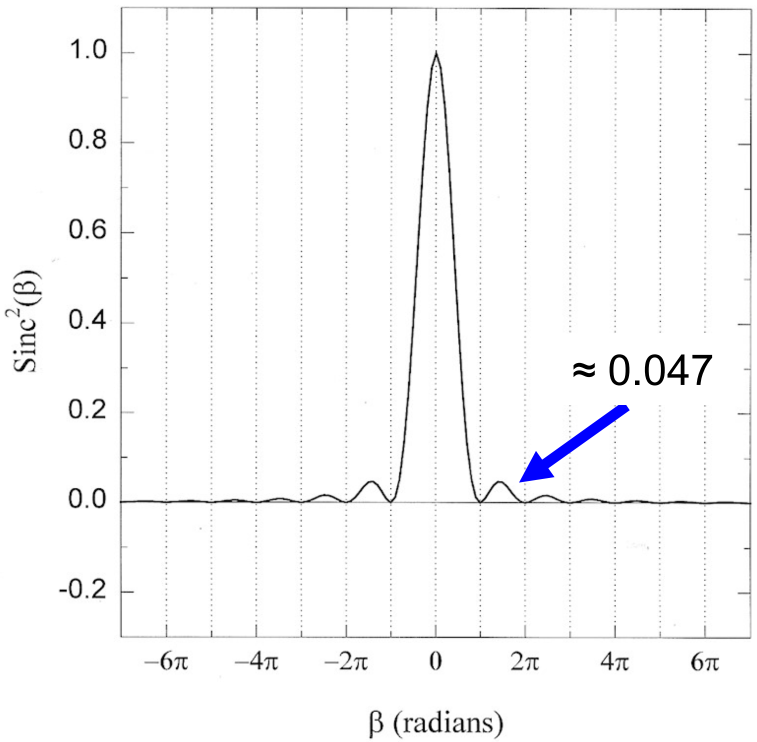
$$\sin\theta_{\min} = \frac{m\lambda}{a}$$



Uso duma lente para aproximar o infinito…

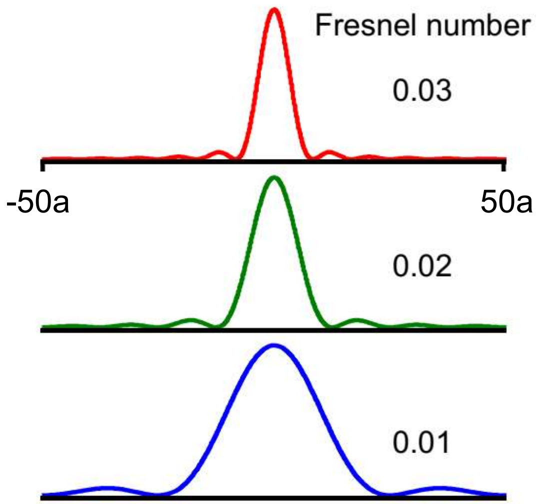


A função sinc



$$\mathrm{sinc}^2(\beta)=\left[\frac{\sin(\beta)}{\beta}\right]^2$$

$$N_F \equiv \frac{(a/2)^2}{z\lambda}$$



$$N_F \sim \frac{1}{z}$$

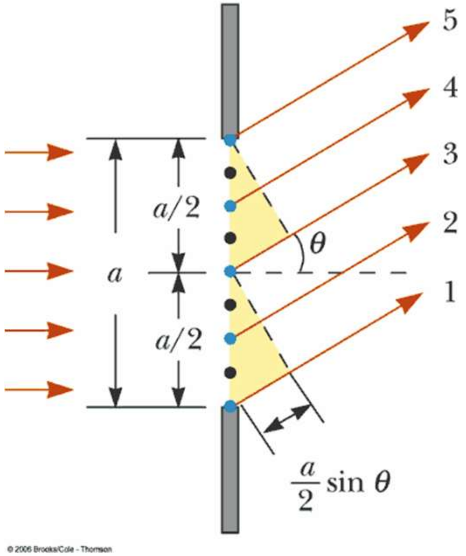
$$\mathcal{I}(x,z)=\mathcal{I}_0\left(\frac{a^2}{\lambda z}\right)\left[\frac{\sin(\pi a x/\lambda z)}{\pi a x/\lambda z}\right]^2=\mathcal{I}_0 4N_F \mathrm{sinc}^2\left(\frac{4\pi N_F x}{a}\right)$$

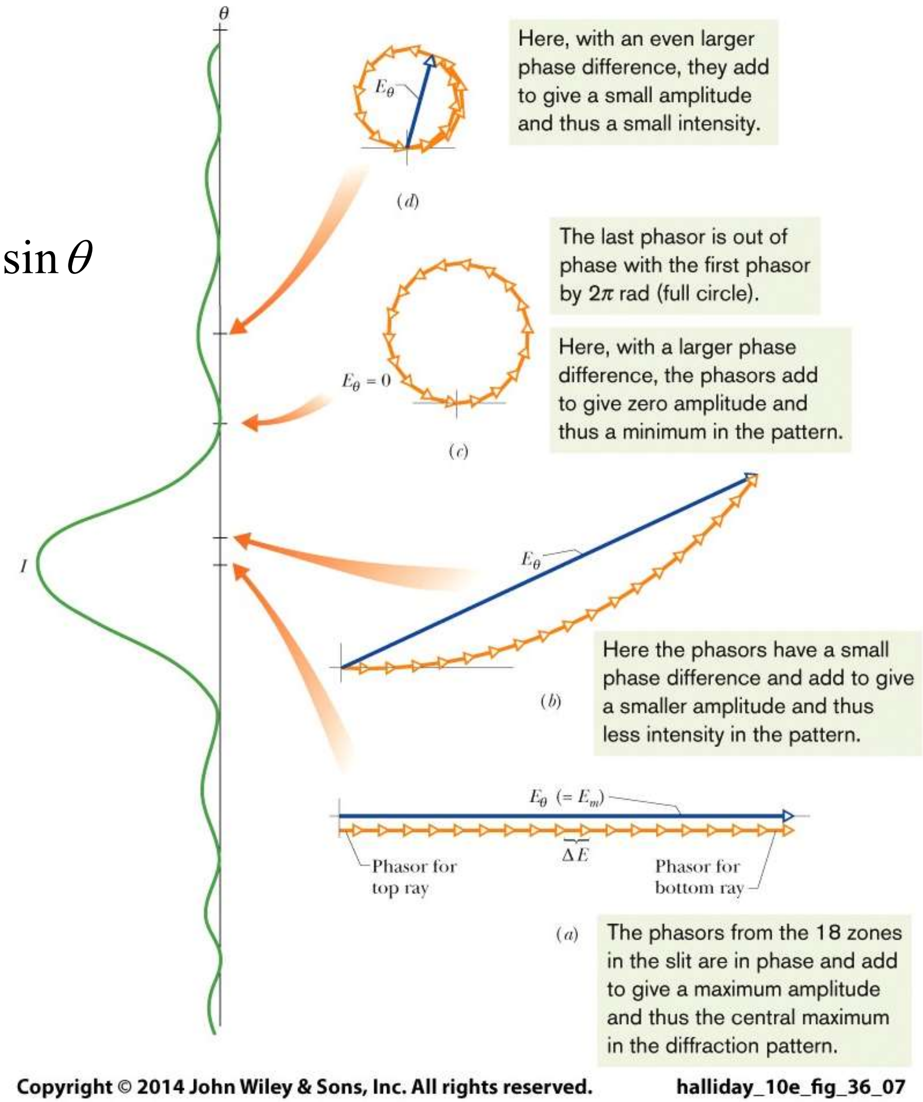
Quando z aumenta a padrão alarga mas mantêm a mesma forma

Interpretação em termos de fasores

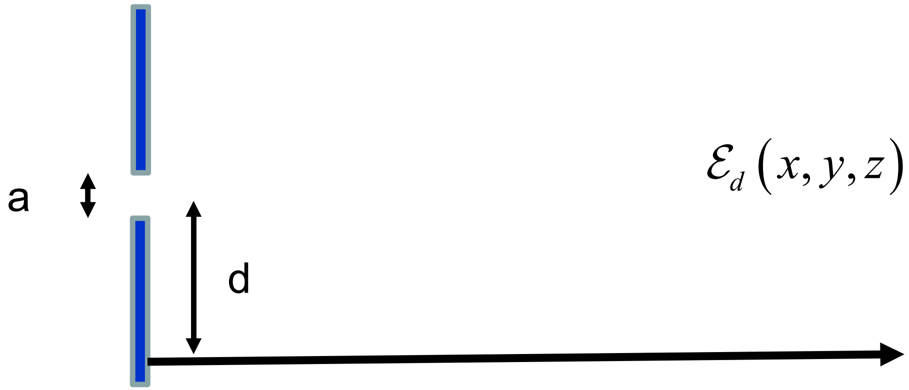
$$\mathcal{I}(x,z) = \mathcal{I}_0\left(\frac{a^2}{\lambda z}\right)\left[\frac{\sin(\pi a x/\lambda z)}{\pi a x/\lambda z}\right]^2$$

$$\frac{\pi a x}{\lambda z} \approx k\frac{a}{2}\sin\theta$$





Padrão duma fenda simples deslocada



$$\mathcal{E}_d(x,y,z) = \frac{\mathcal{E}_0}{\sqrt{i\lambda z}} e^{ikz} \int_{-a/2+d}^{a/2+d} \exp\left[ik(x-x')^2/2z\right] dx'$$

$$x'' = x' - d$$

Integral na aproximação Fraunhofer

$$\int_{-a/2+d}^{a/2+d} \exp\left[-i2\pi \frac{xx'}{\lambda z}\right] dx' = e^{-i2\pi dx/\lambda z} \int_{-a/2}^{a/2} \exp\left[-i2\pi \frac{xx''}{\lambda z}\right] dx''$$

igual exceto duma fase adicional

$$\mathcal{E}_d(x,z) = e^{-i2\pi dx/\lambda z} \mathcal{E}(x,z)$$

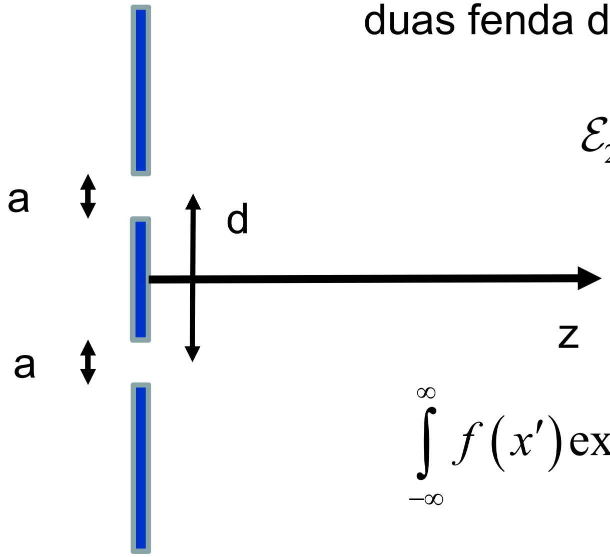
$$\mathcal{I}(x,z) = \mathcal{I}_0 \left(\frac{a^2}{\lambda z}\right) \left[\frac{\sin(\pi a x/\lambda z)}{\pi a x/\lambda z}\right]^2$$

Padrão de difração igual ao padrão não deslocada!

Aproximação Fraunhofer $\max\left(\frac{x'^2}{\lambda z}\right) \ll 1$

implica que a distância de deslocação $d \ll \sqrt{\lambda z}$

Dupla Fenda



duas fenda de largura a, separadas por uma distância d

$$\mathcal{E}_{2\,fendas}(x,y,z) = \frac{\mathcal{E}_0}{\sqrt{i\lambda z}} e^{ikz} \int_{-\infty}^{\infty} f(x') \exp\left[ik(x-x')^2/2z\right] dx'$$

Duas fendas deslocada por ±d/2

$$\int_{-\infty}^{\infty} f(x') \exp\left[-i2\pi \frac{xx'}{\lambda z}\right] dx' = \left[e^{-i\pi dx/\lambda z} + e^{i\pi dx/\lambda z}\right] \int_{-a/2}^{a/2} \exp\left[-i2\pi \frac{xx''}{\lambda z}\right] dx''$$

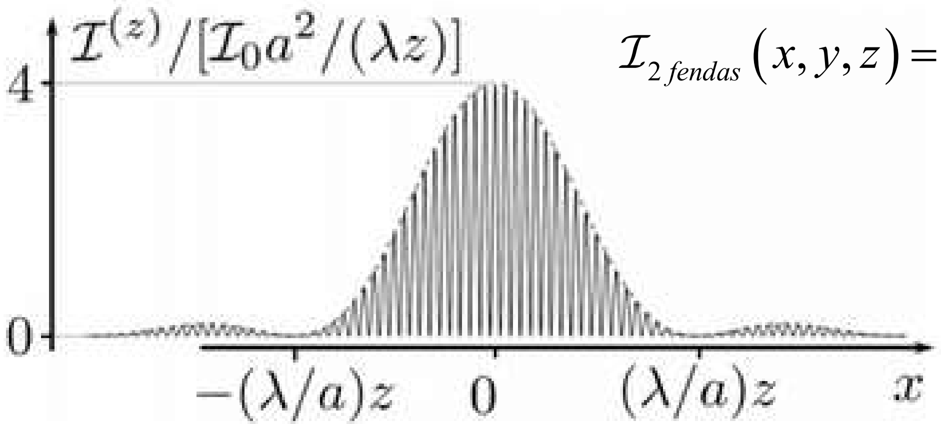
$$= 2\cos\left(\frac{\pi dx}{\lambda z}\right) e^{i\pi x^2/\lambda z} a \frac{\sin(\pi a x/\lambda z)}{\pi a x/\lambda z}$$

$$\mathcal{E}_{2\,fendas}(x,y,z) = \frac{\mathcal{E}_0}{\sqrt{i\lambda z}} e^{ikz} e^{i\pi x^2/\lambda z} 2\cos\left(\frac{\pi dx}{\lambda z}\right) a \frac{\sin(\pi a x/\lambda z)}{\pi a x/\lambda z}$$

$$\mathcal{I}_{2\,fendas}(x,y,z) = 4\left(\frac{a^2}{\lambda z}\right) \mathcal{I}_0 \cos^2\left(\frac{\pi dx}{\lambda z}\right) \left[\frac{\sin(\pi a x/\lambda z)}{\pi a x/\lambda z}\right]^2$$

Padrão de 2 fendas multiplicada pela padrão de uma fenda (teorema de convolução)

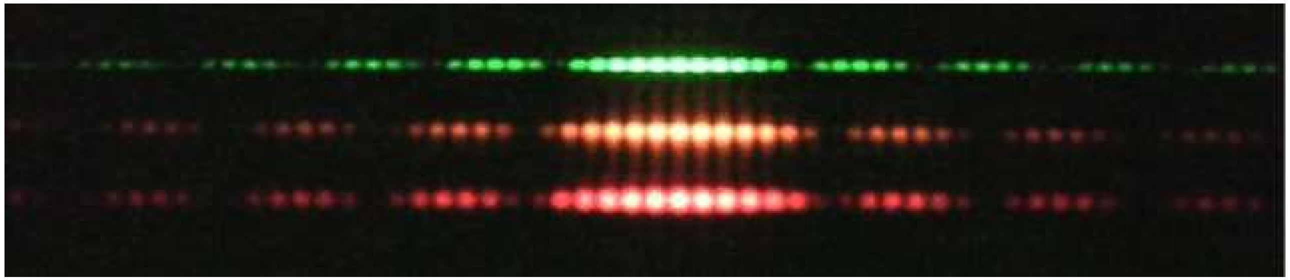
Padrão difração dupla fendas



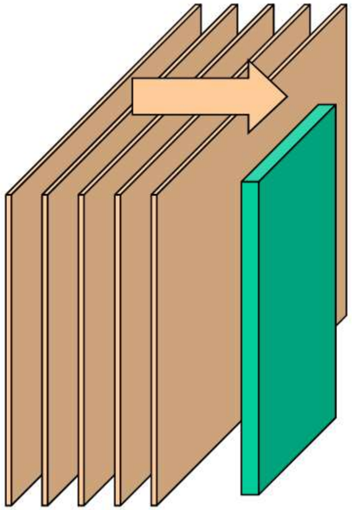
$$\mathcal{I}_{2\,fendas}(x,y,z)=\mathcal{I}_0\boxed{4\cos^2\left(\frac{\pi d x}{\lambda d}\right)}\boxed{\left(\frac{a^2}{\lambda z}\right)\left[\frac{\sin(\pi a x/\lambda z)}{\pi a x/\lambda z}\right]^2}$$

Oscilação rápida devida a interferência das duas fendas

Modulação do amplitude devida a padrão da difração de uma fenda com largura a

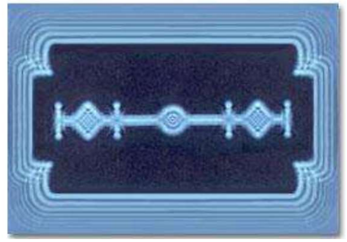


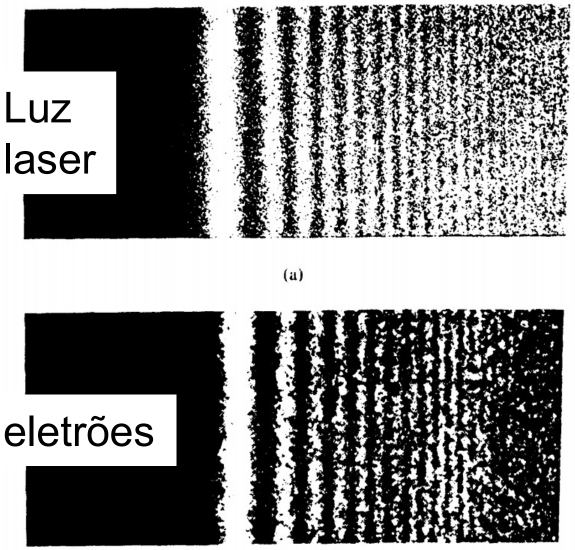
Difração duma aresta



$$\mathcal{E}(x,z)=\frac{\mathcal{E}_0}{\sqrt{i\lambda z}}e^{ikz}\int_{-\infty}^{\infty} f(x')\exp\left[ik(x-x')^2/2z\right]dx'$$

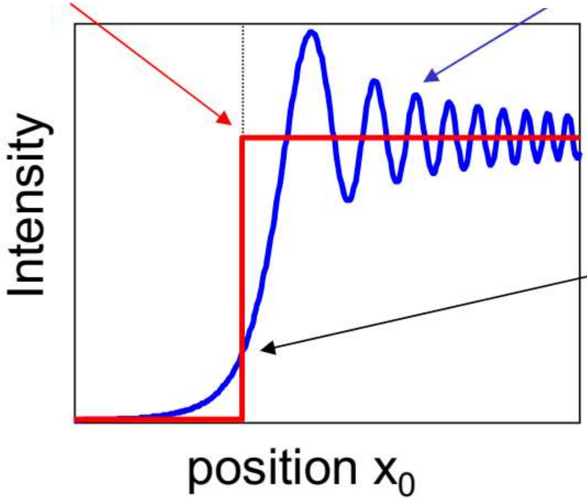
$$=\frac{\mathcal{E}_0}{\sqrt{i\lambda z}}e^{ikz}\int_{0}^{\infty}\exp\left[ik(x-x')^2/2z\right]dx'$$





Sombra
geométrica

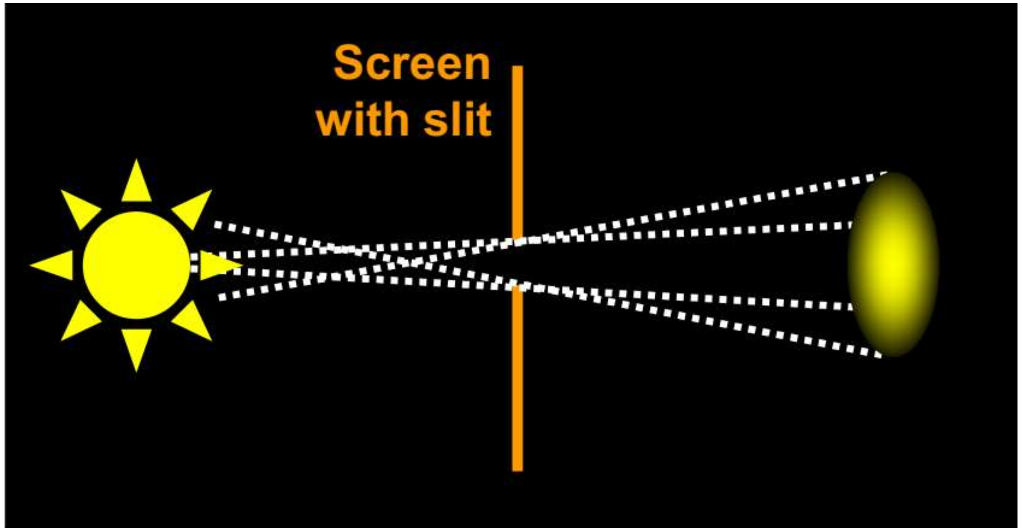
Resultado do integral Fresnel-Huygens



Irradiância
na aresta é
¼ do valor
no x $\rightarrow\infty$

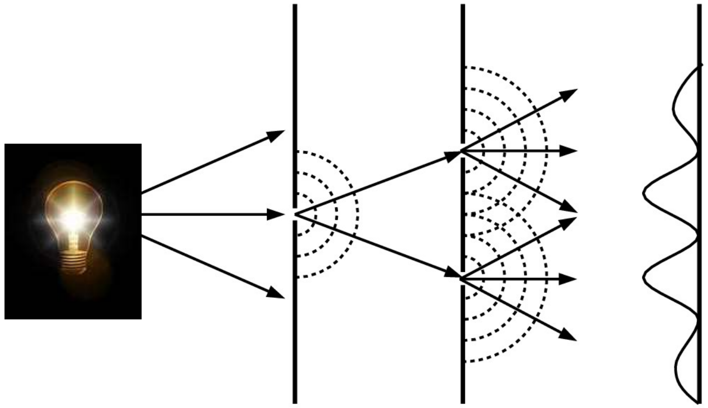
Porque é raro ver os efeitos de difração com luz ambiente?

Basicamente é um problema com a coerência de fontes extensas



Em principio, raios esféricos duma fonte pontual produzam efeitos de difração, mas os raios provenientes de outras zonas ofuscam a padrão

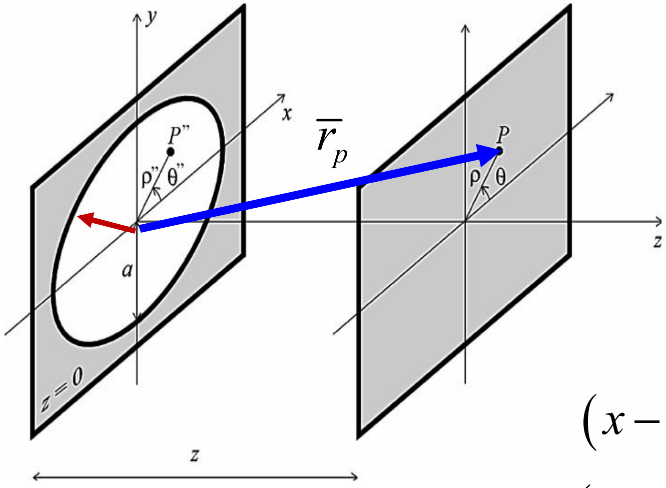
Em geral uma fonte extensa (sol, lâmpada) cria sombras borratadas



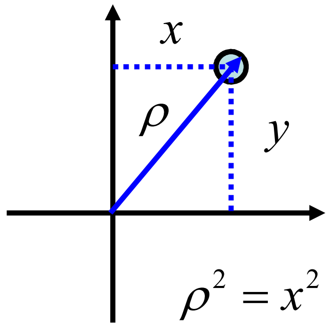
Integral Fresnel-Huygens com Simetria circular

$$\mathcal{E}(x,y,z)=\frac{\mathcal{E}_0}{i\lambda z}\int_{-\infty}^{\infty}\int f(x',y')e^{ikz}\,\mathrm{e}^{ik\left[(x-x')^2+(y-y')^2\right]/2z}dx'dy'$$

Com simetria circular convêm usar coordenados polares



$$x'=\rho'\cos\theta' \qquad y'=\rho'\sin\theta'$$
$$x=\rho\cos\theta \qquad y=\rho\sin\theta$$



$$\rho^2=x^2+y^2$$

$$(x-x')^2=\rho^2\cos^2\theta-2\rho\rho'\cos\theta\cos\theta'+\rho'^2\cos^2\theta'$$
$$+(y-y')^2=\rho^2\sin^2\theta-2\rho\rho'\sin\theta\sin\theta'+\rho'^2\sin^2\theta'$$

$$\overline{r}_p=z+\rho^2/2z$$

$$(x-x')^2+(y-y')^2=\rho^2-2\rho\rho'\cos(\theta-\theta')+\rho'^2$$

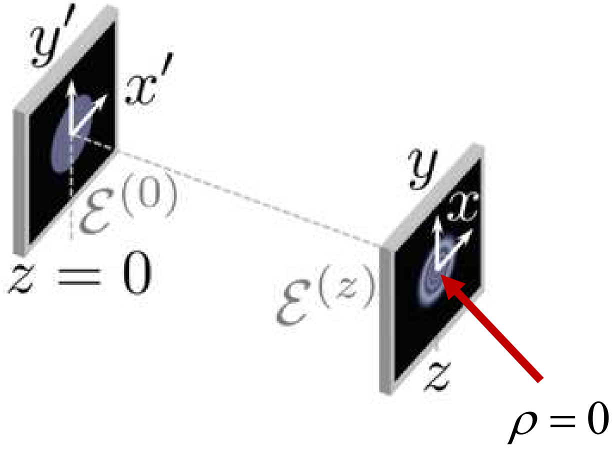
$$f(\rho')=\begin{cases}1 & \rho'<R_a\\ 0 & \rho'\geq R_a\end{cases}$$

$$\mathcal{E}(\rho,\theta,z)=\frac{\mathcal{E}_0}{i\lambda z}e^{ik\overline{r}_p}\int_0^{2\pi}d\theta'\int_0^{R_a}\rho'd\rho'\exp\left[-ik\frac{\rho\rho'}{z}\cos(\theta-\theta')+ik\frac{\rho'^2}{2z}\right]$$

Abertura circular irradiância no eixo ótico

$$\mathcal{E}(\rho,\theta,z)=\frac{\mathcal{E}_0}{i\lambda z}e^{ik\bar{r}_p}\int_0^{2\pi}d\theta'\int_0^{R_A}\rho' d\rho' \exp\left[-ik\frac{\rho\rho'}{z}\cos(\theta-\theta')+ik\frac{\rho'^2}{2z}\right]$$

Esta integral é difícil em geral e tem ser resolvida numericamente. No entanto no eixo ótico $\rho=0$



$$\mathcal{E}(\rho=0,z)=\frac{\mathcal{E}_0}{i\lambda z}e^{ikz}2\pi\int_0^{R_A}\rho' d\rho' \exp\left[ik\frac{\rho'^2}{2z}\right]$$

$$u'=\frac{ik\rho'^2}{2z}=\frac{i\pi\rho'^2}{\lambda z}\qquad \frac{\rho'^2}{2}=\frac{\lambda z}{i2\pi}u\qquad \rho' d\rho'=\frac{\lambda z}{i2\pi}du'$$

$$\mathcal{E}(\rho=0,z)=\frac{\mathcal{E}_0}{i\lambda z}e^{ikz}2\pi\frac{\lambda z}{i2\pi}\int_0^{i\pi R_A^2/\lambda z}\mathrm{e}^{u'}du'=-\mathcal{E}_0 e^{ikz}\left[e^{i\pi R_A^2/\lambda z}-1\right]$$

$$=-2i\mathcal{E}_0 e^{ikz}e^{i\pi R_A^2/2\lambda z}\left[\frac{e^{i\pi R_A^2/2\lambda z}-e^{-i\pi R_A^2/2\lambda z}}{2i}\right]$$

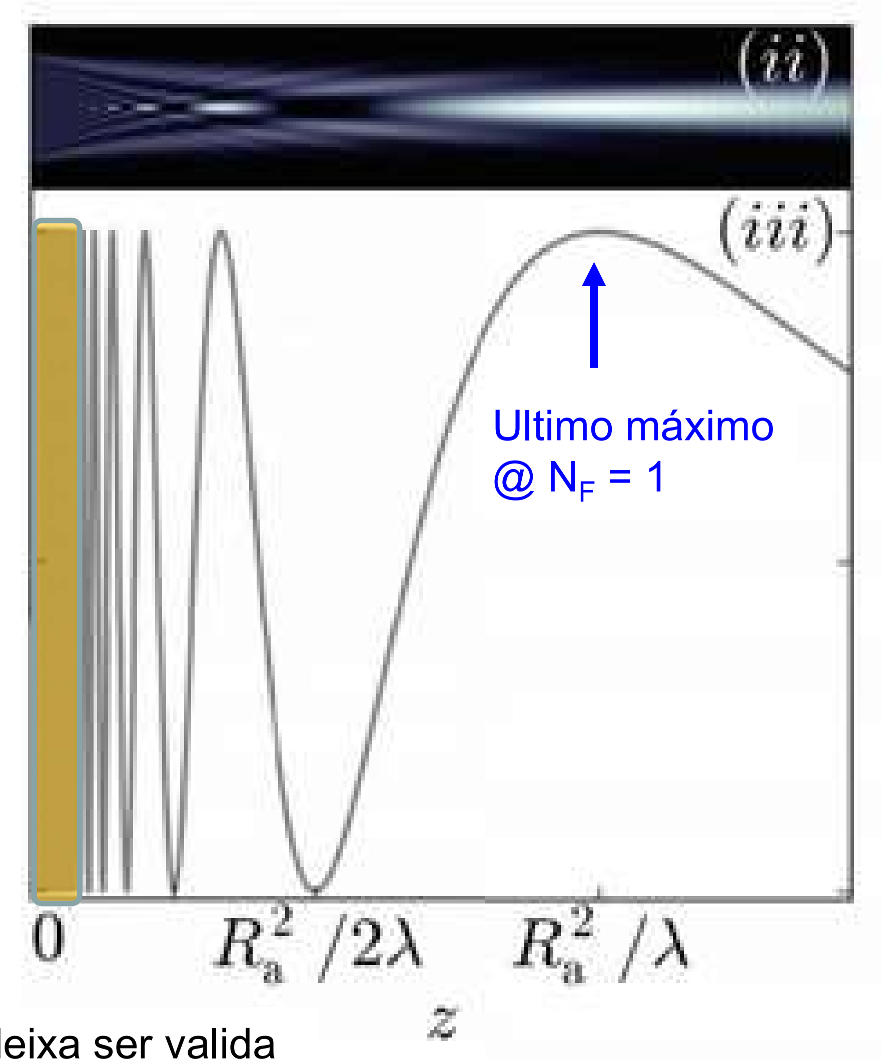
$$\mathcal{I}(\rho=0,z)=4\mathcal{I}_0\sin^2\left(\frac{\pi R_a^2}{2\lambda z}\right)$$

Variação da intensidade no eixo ótico

$$\mathcal{I}\left(\rho=0,z\right)=4\mathcal{I}_0\sin^2\left(\frac{\pi R_a^2}{2\lambda z}\right)$$

$$N_F=\frac{R_a^2}{\lambda z}$$ Número Fresnel

Quando $N_F \ll 1$
no limite paraxial
a padrão de difração fica mais estável
e mais simples, tal como aconteceu
no caso duma fenda simples.



Quando $\boxed{z<R_a}$ a aproximação de Fresnel deixa ser valida

Variação da padrão de difração com z

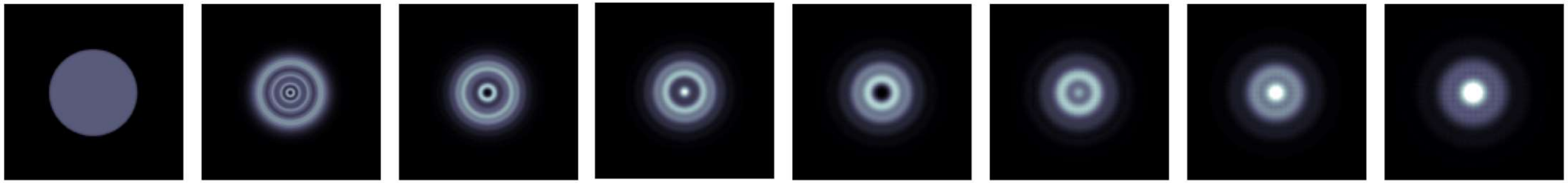
$$\mathcal{I}(\rho,\theta,z)=\mathcal{I}_0\left|\int_0^{2\pi}d\theta'\int_0^{R_A}\rho'd\rho'\exp\left[-ik\frac{\rho\rho'}{z}\cos(\theta-\theta')+ik\frac{\rho'^2}{2z}\right]\right|^2$$

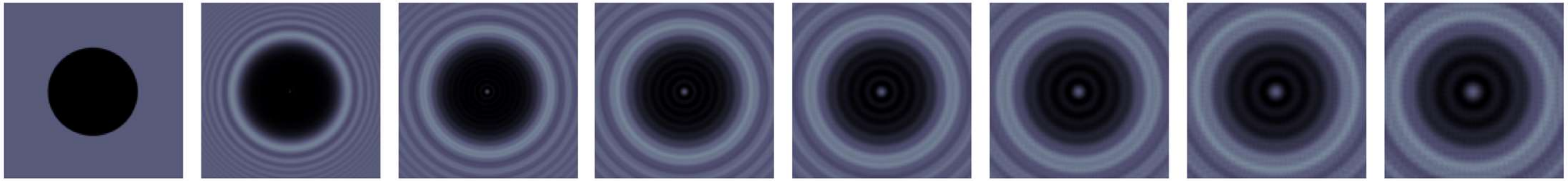
Simulação numérica da integral

$z=0$

Abertura circular

$z=\dfrac{R_a^2}{\lambda}$

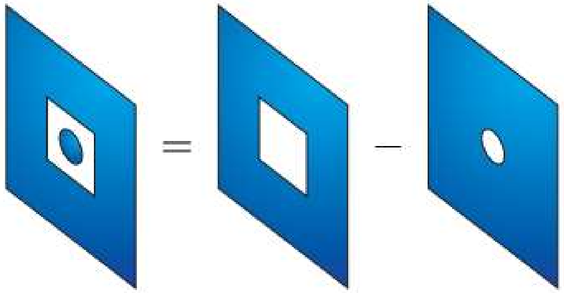




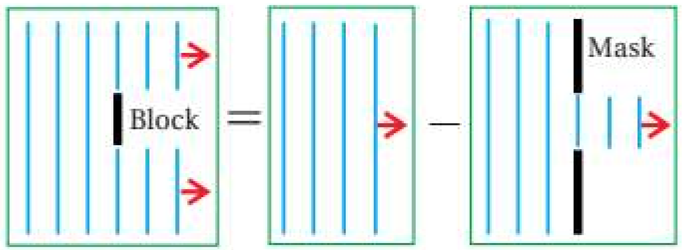
Disco circular

Principio de Babinet

O principio de Babinet é efetivamente o principio de sobreposição



Jaques Babinet
(1794-1872)



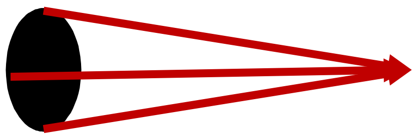
Em particular considere um disco circular
O campo no eixo ótico é

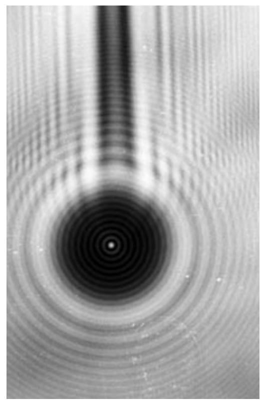
$$\mathcal{E}_{disco}(\rho = 0, z) = \mathcal{E}_{plana}(0, z) - \mathcal{E}_{abertura}(0, z)$$

$$= \mathcal{E}_0 e^{ikz} + \mathcal{E}_0 e^{ikz}\left[e^{i\pi R_a^2/\lambda z} - 1\right]$$

$$= \mathcal{E}_0 e^{ikz} e^{i\pi R_a^2/\lambda z}$$

$$\mathcal{I}_{disco}(\rho = 0, z) = \mathcal{I}_0$$





Interferência construtiva no centro do disco

O spot de Poisson/Arago

Complementaridade

No limite Fraunhofer a integral Fresnel de difração é uma transformada Fourier

$$N_F \ll 1$$

$$\mathcal{E}(x,y,z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} \int_{-\infty}^{\infty}\int f(x',y') \mathrm{e}^{\left\{ik\left[(x-x')^2+(y-y')^2\right]/2z\right\}} dx'dy'$$

$$\Rightarrow \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} e^{ik(x^2+y^2)/2z} \int_{-\infty}^{\infty}\int f(x',y') \mathrm{e}^{-ik[xx'+yy']/z} dx'dy'$$

Uma abertura complementaria (uma que é o inverso da abertura original) da a mesma padrão (exceto no eixo ótico)

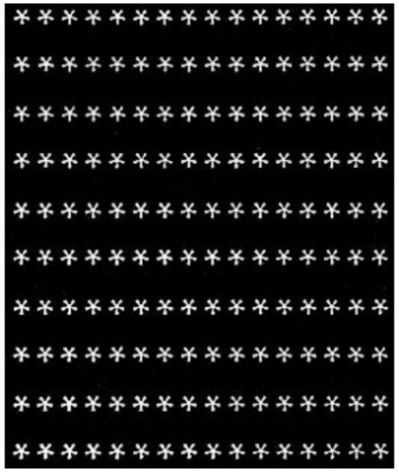
$$\mathcal{E}_{comp}(x,y,z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} e^{ik(x^2+y^2)/2z} \int_{-\infty}^{\infty}\int \left[1 - f(x',y')\right] \mathrm{e}^{-ik[xx'+yy']/z} dx'dy'$$

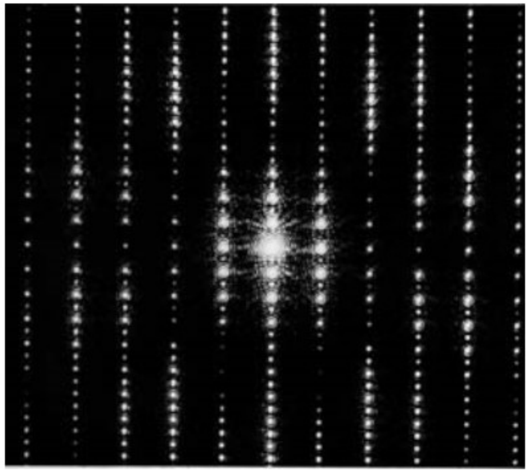
$$= -\mathcal{E}_{original}(x,y,z) + -i\mathcal{E}_0 \lambda z e^{ikz} 2\pi \delta(x)\delta(y)$$

Exceto na origem $\mathcal{E}_{comp}(x,y,z) = -\mathcal{E}_{original}(x,y,z)$ $\quad \mathcal{I}_{comp}(x,y,z) = \mathcal{I}_{original}(x,y,z)$

Exemplo do principio de Babinet

Uma rede
de buracos





Uma rede
de anti-buracos