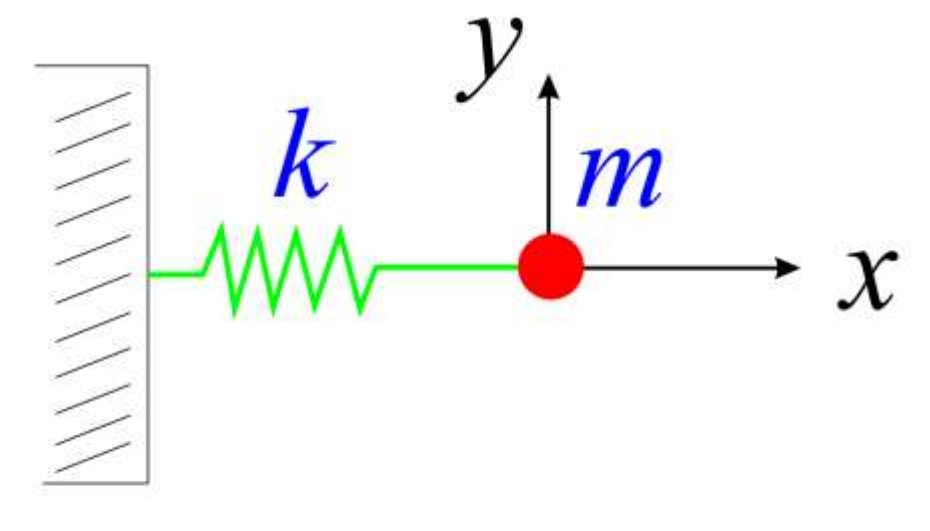
1- Um objeto de massa 1 g suspenso de uma mola de massa desprezável é induzido em movimento oscilatório. No instante $t = 0$ o desvio é de 43.785 cm e a aceleração de -1.7514 cm s^{-2}. Calcule o valor da constante elástica da mola.



Problema 1 de série 2

$$\boxed{m\ddot{x} + kx = 0}$$

$$k = -\frac{ma}{x}$$

$$\ddot{x} = a$$

$$a = -1.7514 \cdot 10^{-2}\,\mathrm{m/s}$$
$$x = 43.785 \cdot 10^{-2}\,\mathrm{m/s}$$
$$m = 10^{-3}\,\mathrm{kg}$$

$$k = \frac{10^{-3}\,\mathrm{kg}\,1.7514 \cdot 10^{-2}\,\mathrm{m/s}}{43.785 \cdot 10^{-2}\,\mathrm{m/s}} = 4 \cdot 10^{-5}\,\mathrm{N/m}$$

2- Considere-se um paralelepípedo de densidade uniforme e massa m.

(a) Qual o período das oscilações do sistema se o objeto estiver suspenso de uma mola uniforme de massa desprezável e constante elástica k?

Problema 2 de série 2

$$\boxed{m\ddot{x} + kx = 0}$$

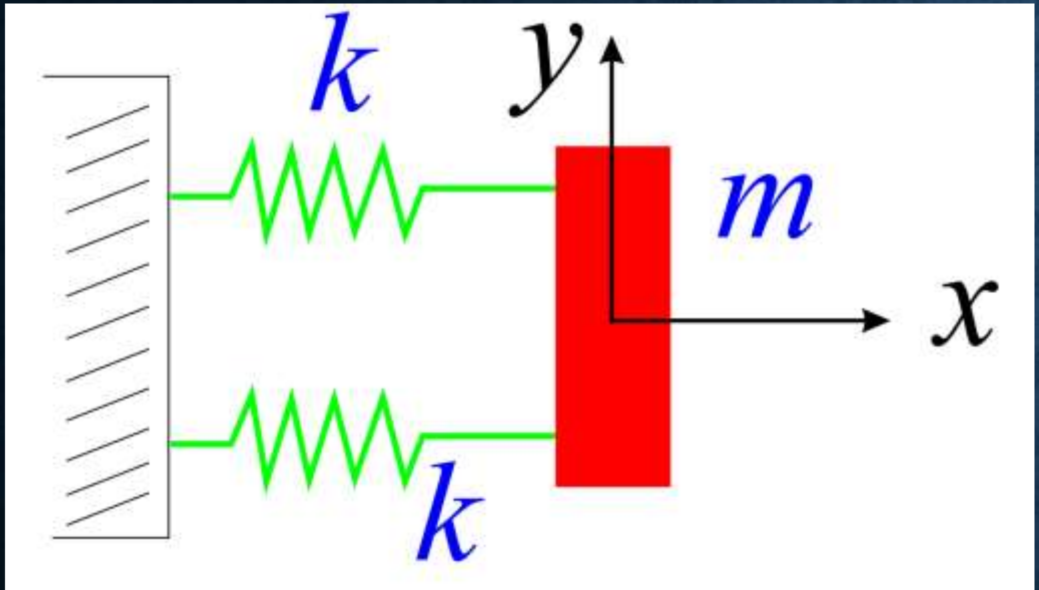
$$x(t) = \exp(i\omega t) \qquad \dot{x}(t) = i\omega \exp(i\omega t)$$

$$\ddot{x}(t) = -\omega^2 \exp(i\omega t) \qquad -m\omega^2 \exp(i\omega t) + k\exp(i\omega t) = 0$$

$$-m\omega^2 + k = 0 \qquad \omega = \pm\sqrt{\frac{k}{m}}$$

$$T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

(b) Qual o período das oscilações do sistema se o objeto estiver suspenso de duas molas idênticas de constante elástica k, dispostas em paralelo, cujos pontos de suspensão são equidistantes relativamente ao centro de uma das faces do paralelepípedo?



$$\{q_1\} = \{x\}$$

$$T = \frac{1}{2}m\dot{x}^2$$

$$V(x) = \frac{k}{2}x^2 + \frac{k}{2}x^2 = kx^2$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - kx^2$$

$$\omega = \sqrt{\frac{2k}{m}}$$

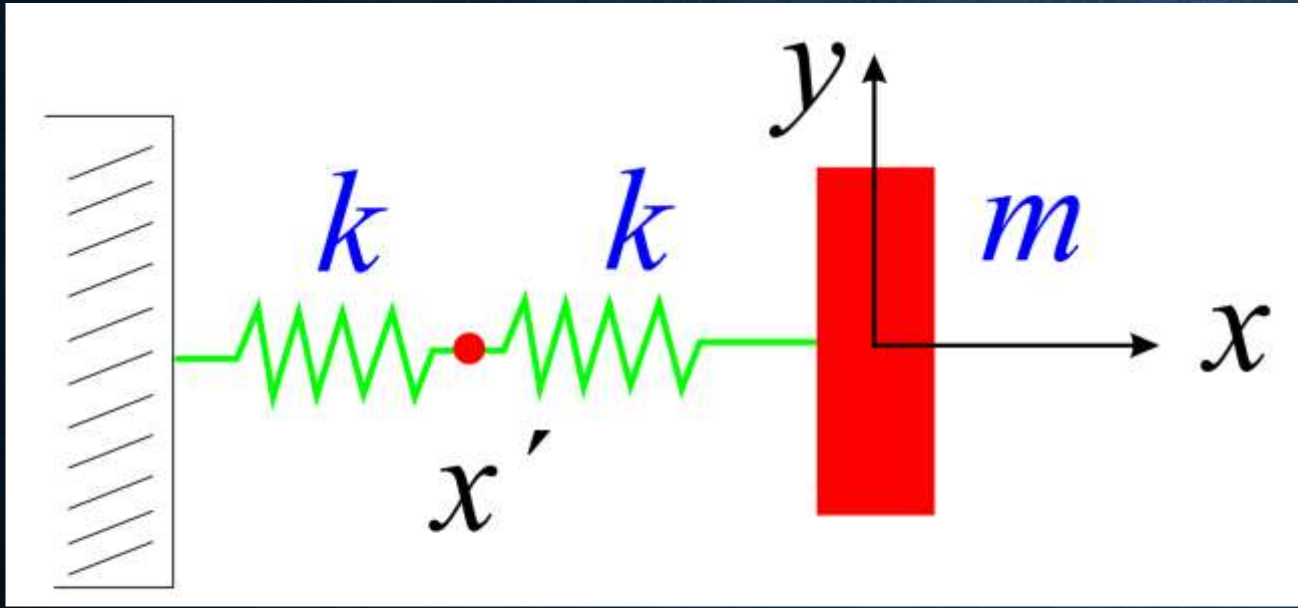
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$m\ddot{x} + 2kx = 0$$

$$T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{2k}} = \frac{T_0}{\sqrt{2}}$$

(c) Qual o período das oscilações do sistema se o objeto estiver suspenso da mola inferior de um conjunto de duas molas idênticas de constante elástica k, dispostas em série, ligadas uma à outra por um fio vertical de massa e raio desprezáveis?



$$\{q_1, q_2\} = \{x, x'\}$$

$$T = \frac{1}{2}m\dot{x}^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \left(\frac{\partial L}{\partial x'}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$V(x) = \frac{k}{2}\left(x' - x'_0\right)^2 + \frac{k}{2}\left(x - x' + x'_0\right)^2$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{k}{2}\left(x' - x'_0\right)^2 - \frac{k}{2}\left(x - x' + x'_0\right)^2$$

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{k}{2}\left(x' - x'_0\right)^2 - \frac{k}{2}\left(x - x' + x'_0\right)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) - \left(\frac{\partial L}{\partial x'}\right) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0$$

$$-\frac{\partial L}{\partial x'} = 0$$

$$k\left(x' - x'_0\right) + k\left(-x + x' - x'_0\right) = 0$$

$$2x' - x - 2x'_0 = 0$$

$$x' = \frac{x}{2} + x'_0$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}\left(m\dot{x}\right) = m\ddot{x}$$

$$\omega = \sqrt{\frac{k}{2m}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2m}{k}} = \sqrt{2}T_0$$

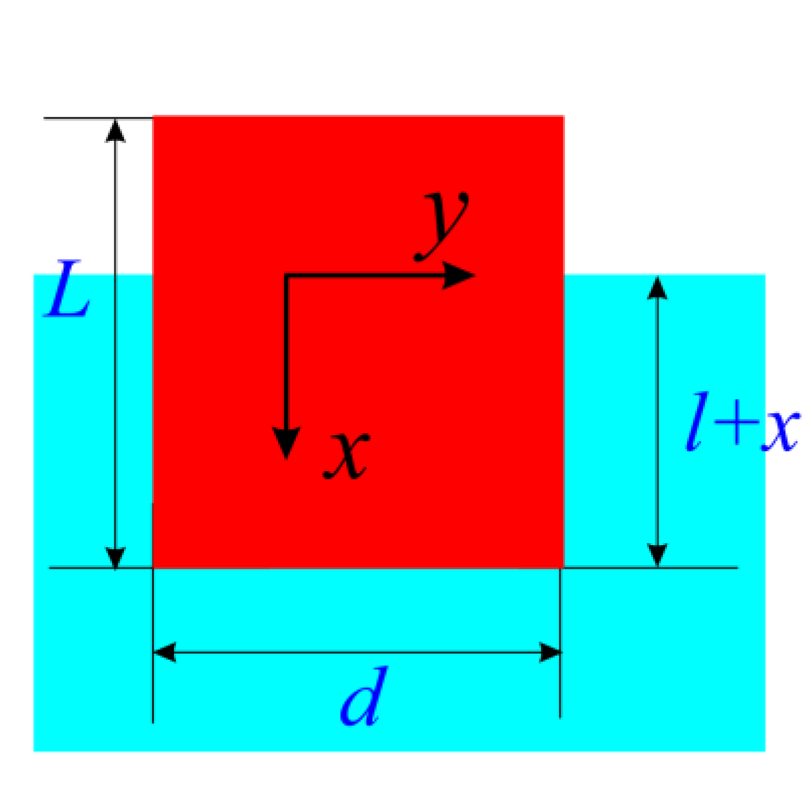
$$\frac{\partial L}{\partial x} = -k\left(x - x' + x'_0\right) = -k\frac{x}{2}$$

$$T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

$$m\ddot{x} + \frac{k}{2}x = 0$$

3- Um cilindro de diâmetro d e comprimento L flutua num líquido, sendo $l < L$ o comprimento da sua parte mergulhada no mesmo. Considera-se que não há amortecimento e que no instante $t = 0$ o cilindro é empurrado para baixo, de uma distância A, e em seguida libertado.

(a) Qual o valor da frequência angular da oscilação induzida?



$$F(x) = mg - \rho g \pi \left(\frac{d}{2}\right)^2 (l + x) = -\frac{dV}{dx}$$

$$V(x) = -mgx + \rho g \pi \left(\frac{d}{2}\right)^2 \frac{(l + x)^2}{2}$$

$$T = \frac{1}{2} m \dot{x}^2$$

$$L = \frac{1}{2} m \dot{x}^2 + mgx - \rho g \pi \left(\frac{d}{2}\right)^2 \frac{(l + x)^2}{2}$$

$$L=\frac{1}{2}m\dot{x}^2+mgx-\rho g\pi\left(\frac{d}{2}\right)^2\frac{(l+x)^2}{2}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=0$$

$$m\ddot{x}-mg+\rho g\pi\left(\frac{d}{2}\right)^2(l+x)=0$$

$$\ddot{x}=x=0$$

$$-mg+\rho g\pi\left(\frac{d}{2}\right)^2 l=0$$

$$\rho\cancel{\pi\left(\frac{d}{2}\right)^2} l\ddot{x}-\rho\cancel{\pi\left(\frac{d}{2}\right)^2} lg+\rho g\cancel{\pi\left(\frac{d}{2}\right)^2}(\cancel{l}+x)=0$$

$$m=\rho\pi\left(\frac{d}{2}\right)^2 l$$

$$l\ddot{x}+gx=0$$

$$-Bl\omega^2\cos(\omega t+\varphi)+gB\cos(\omega t+\varphi)=0$$

$$x(t)=B\cos(\omega t+\varphi)$$

$$\omega=\sqrt{\frac{g}{l}}$$

(b) Produza um gráfico representativo da velocidade em função do tempo, desde $t = 0$ até $t = T$, onde T designa o período da oscilação. Os valores da amplitude e fase devem ser indicados.

$$x(t) = B\cos(\omega t + \varphi)$$

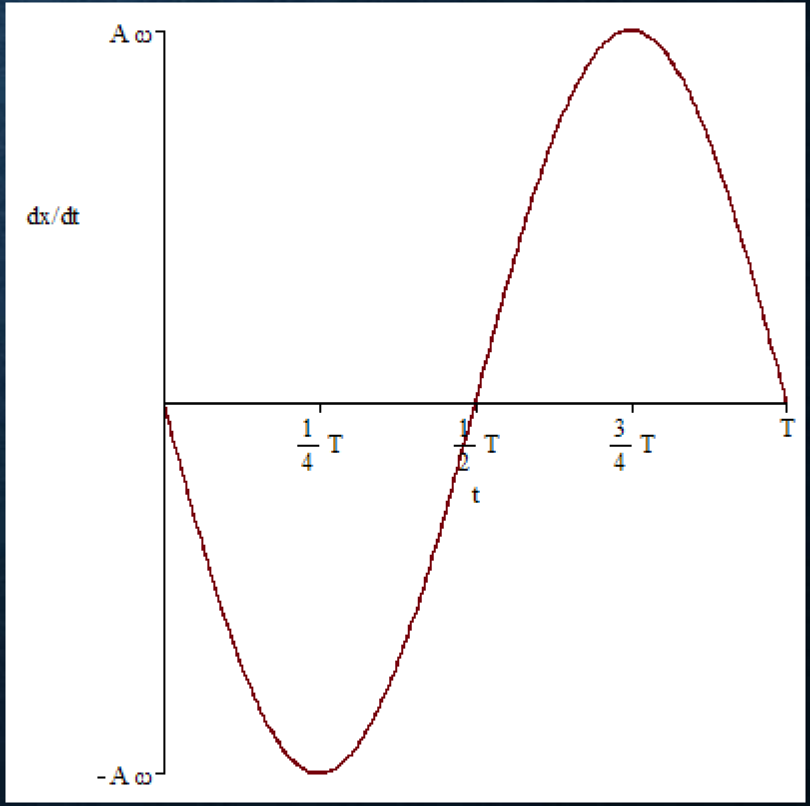
$$\dot{x}(t) = -B\omega\sin(\omega t + \varphi)$$

$$x(0) = A$$
$$\dot{x}(0) = 0$$

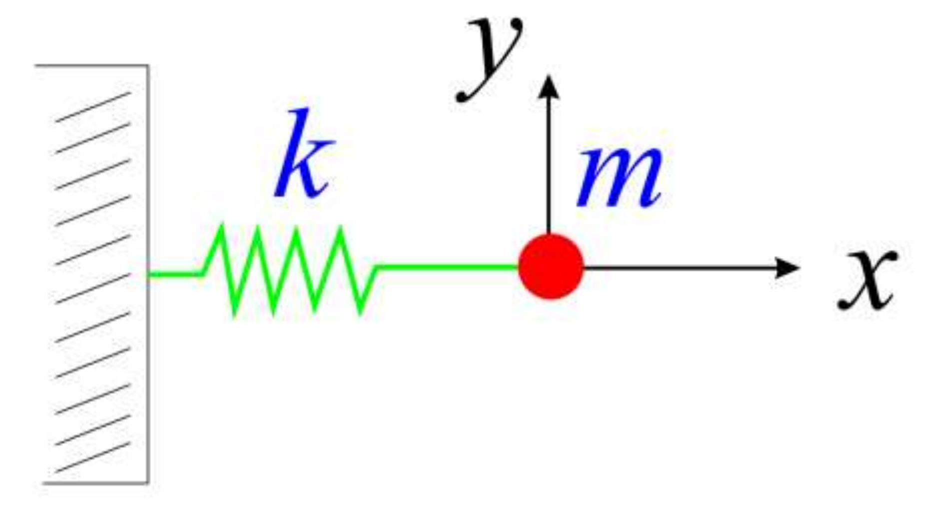
$$x(0) = B\cos(\varphi) = A$$
$$\dot{x}(0) = -B\omega\sin(\varphi) = 0$$

$$\varphi = 0$$
$$B = A$$

$$x(t) = A\cos\left(\sqrt{\frac{g}{l}}t\right)$$
$$\dot{x}(t) = -A\sqrt{\frac{g}{l}}\sin\left(\sqrt{\frac{g}{l}}t\right)$$



4 - Um objeto de massa 0.2 Kg é suspenso de uma mola de massa desprezável e constante elástica 80 N m^{-1}. O objeto está sujeito a uma força resistiva de amortecimento dada por $-b\,v$, onde v é a sua velocidade em metros por segundo.



$$\{q_1\} = \{x\}$$

$$T = \frac{1}{2} m\dot{x}^2$$

$$V(x) = \frac{k}{2} x^2$$

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = Q_x$$

$$Q_x = -b\dot{x}$$

$$m\ddot{x} + kx = -b\dot{x}$$

$$m\ddot{x} + kx = -b\dot{x}$$

$$x(t) = \exp(i\omega t)$$

$$\dot{x}(t) = i\omega \exp(i\omega t)$$
$$\ddot{x}(t) = -\omega^2 \exp(i\omega t)$$
$$-m\omega^2 \exp(i\omega t) + ib\omega \exp(i\omega t) + k \exp(i\omega t) = 0$$

$$-m\omega^2 + ib\omega + k = 0$$
$$\omega = \frac{ib}{2m} \pm \sqrt{\left(\frac{k}{m}\right)^2 - \left(\frac{b}{2m}\right)^2}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\gamma = \frac{b}{m}$$

$$\Omega = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

$$\omega = i\frac{\gamma}{2} \pm \Omega$$

$$x_1(t) = \exp(-\gamma t/2)\exp(i\Omega t)$$
$$x_2(t) = \exp(-\gamma t/2)\exp(-i\Omega t)$$
$$x(t) = A_1 x_1(t) + A_2 x_2(t)$$

(a) Sabendo que a razão da frequência angular amortecida e frequência angular natural é dada por $\sqrt{3}/2$, determine o valor da constante b.

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\Omega = \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

$$\gamma = \frac{b}{m}$$

$$\frac{\Omega}{\omega_0} = \frac{\sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}}{\omega_0} = \frac{\sqrt{3}}{2}$$

$$1 - \frac{1}{4}\left(\frac{\gamma}{\omega_0}\right)^2 = \frac{3}{4}$$

$$\left(\frac{\gamma}{\omega_0}\right)^2 = 1$$

$$\frac{b^2}{mk} = 1$$

$$b = \sqrt{mk} = \sqrt{0.2\,\mathrm{kg} \cdot 80\,\mathrm{Nm}^{-1}} = 4\,\mathrm{kgs}^{-1}$$

(b) Calcule ainda o valor da qualidade do sistema Q.

$$Q = \frac{\omega_0}{\gamma}$$

$$\boxed{Q = 1}$$

$$1 - \frac{1}{4}\left(\frac{\gamma}{\omega_0}\right)^2 = \frac{3}{4}$$

$$\left(\frac{\gamma}{\omega_0}\right)^2 = 1$$

5- Derive a solução de estado estacionário da equação de movimento,

$$m\,\frac{d^2x}{dt^2} = -k\,x + F_0\,\sin(\omega t)\,,$$

representativa de um oscilador forçado constituido por um objeto de massa m suspenso de uma mola de massa desprezável e constante elástica k.

$$m\ddot{x} + kx = F_0 \sin(\omega t) = F_0 \cos\left(\omega t - \frac{\pi}{2}\right)$$

$$\Phi_0 = F_0 \exp(-i\pi/2)$$
$$x(t) = \mathrm{Re}\left[X(t)\right]$$

$$m\ddot{X} + kX = F_0 \cos\left(\omega t - \frac{\pi}{2}\right) + iF_0 \sin\left(\omega t - \frac{\pi}{2}\right) =$$
$$= F_0 \exp(i\omega t)\exp(-i\pi/2) = \Phi_0 \exp(i\omega t)$$

$$X(t) = A\exp(i\omega' t)$$

$$-\omega'^2 A\exp(i\omega' t) + \frac{k}{m}A\exp(i\omega' t) = \frac{\Phi_0}{m}\exp(i\omega t)$$

$$\dot{X}(t) = i\omega' A\exp(i\omega' t)$$
$$\ddot{X}(t) = -(\omega')^2 A\exp(i\omega' t)$$

$$X(t) = A\exp(i\omega' t)$$

$$-\omega'^2 A\exp(i\omega' t) + \frac{k}{m}A\exp(i\omega' t) = \frac{\Phi_0}{m}\exp(i\omega t)$$

$$\Phi_0 = F_0\exp(-i\pi/2)$$

$$\omega' = \omega$$

$$\frac{k}{m} = \omega_0^2$$

$$A = \frac{\Phi_0}{m}\frac{1}{\omega_0^2 - \omega^2}$$

$$\boxed{\omega > \omega_0}$$

$$\boxed{\omega < \omega_0}$$

$$A = \frac{\Phi_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}$$

$$A = -\frac{\Phi_0}{m}\frac{1}{|\omega_0^2 - \omega^2|} = \frac{\Phi_0}{m}\frac{\exp(i\pi)}{|\omega_0^2 - \omega^2|}$$

$$X(t) = \frac{F_0}{m}\frac{\exp(-i\pi/2)}{|\omega_0^2 - \omega^2|}\exp(i\omega t)$$

$$X(t) = \frac{F_0}{m}\frac{\exp(i\pi/2)}{|\omega_0^2 - \omega^2|}\exp(i\omega t)$$

$$x(t) = \mathrm{Re}[X(t)] = \frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\cos\left(\omega t - \frac{\pi}{2}\right) =$$

$$x(t) = \mathrm{Re}[X(t)] = \frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\cos\left(\omega t + \frac{\pi}{2}\right) =$$

$$= \frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\sin(\omega t)$$

$$= -\frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\sin(\omega t)$$

$\omega < \omega_0$

$$x(t) = \mathrm{Re}\left[X(t)\right] = \frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\cos\left(\omega t - \frac{\pi}{2}\right) =$$

$$= \frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\sin(\omega t)$$

$\omega > \omega_0$

$$x(t) = \mathrm{Re}\left[X(t)\right] = \frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\cos\left(\omega t + \frac{\pi}{2}\right) =$$

$$= -\frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\sin(\omega t)$$

$$X(t) = \frac{F_0}{m}\frac{\exp\left(-\mathrm{sign}\left(\omega_0^2 - \omega^2\right) i\pi/2\right)}{|\omega_0^2 - \omega^2|}\exp(i\omega t)$$

$$x(t) = \mathrm{Re}\left[X(t)\right] = \frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\cos\left(\omega t - \mathrm{sign}\left(\omega_0^2 - \omega^2\right)\frac{\pi}{2}\right) =$$

$$= \mathrm{sign}\left(\omega_0^2 - \omega^2\right)\frac{F_0}{m}\frac{1}{|\omega_0^2 - \omega^2|}\sin(\omega t) = \frac{F_0}{m}\frac{1}{\omega_0^2 - \omega^2}\sin(\omega t)$$

6 - Considere um oscilador amortecido de massa $m = 0.2$ Kg, constante de amortecimento $b = 4$ Nm^{-1}s e constante elástica $k = 80$ Nm^{-1}. Considere que ao oscilador é aplicada uma força $F = F_0 \cos(\omega t)$, onde $F_0 = 2$ N e a frequência angular é dada por $\omega = 30$ radianos s^{-1}.

(a) Quais são os valores de A e δ da resposta do estado estacionário descrita por $x = A\cos(\omega t - \delta)$?

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

$$m\ddot{X} + b\dot{X} + kX = F_0 \cos(\omega t) + iF_0 \sin(\omega t) = F_0 \exp(i\omega t)$$

$$x(t) = \mathrm{Re}[X(t)]$$

$$X(t) = A\exp(i\omega t)$$

$$\dot{X}(t) = i\omega A \exp(i\omega t)$$

$$\ddot{X}(t) = -\omega^2 A \exp(i\omega t)$$

$$-\omega^2 A \exp(i\omega t) + i\frac{b}{m}\omega A \exp(i\omega t) + \frac{k}{m} A \exp(i\omega t) = \frac{F_0}{m}\exp(i\omega t)$$

$$\frac{b}{m} = \gamma$$

$$\frac{k}{m} = \omega_0^2$$

$$A = \frac{F_0}{m}\frac{1}{\omega_0^2 - \omega^2 + i\gamma\omega} = \frac{F_0}{m}\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}\exp(i\delta)} =$$

$$A=\frac{F_0}{m}\frac{1}{\omega_0^2-\omega^2+i\gamma\omega}=\frac{F_0}{m}\frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}\exp\left(i\delta\right)}=$$

$$\frac{b}{m}=\gamma$$

$$\frac{k}{m}=\omega_0^2$$

$$=\frac{F_0}{m}\frac{\exp\left(-i\delta\right)}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}$$

$$\tan\left(\delta\right)=\frac{\gamma\omega}{\omega_0^2-\omega^2}$$

$$a(\omega)=\frac{F_0}{m}\frac{1}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}=\frac{F_0}{m}\frac{1}{\sqrt{\left(\frac{k}{m}-\omega^2\right)^2+\frac{b^2}{m^2}\omega^2}}=\frac{F_0}{\sqrt{\left(k-\omega^2 m\right)^2+b^2\omega^2}}$$

$$a(\omega)=\frac{2\,\mathrm{N}}{\sqrt{\left(80\,\mathrm{Nm}^{-1}-(30\,\mathrm{s}^{-1})^2 0.2\,\mathrm{kg}\right)^2+(4\,\mathrm{Nsm}^{-1}\,30\,\mathrm{s}^{-1})^2}}\approx 1.28\cdot 10^{-2}\,\mathrm{m}$$

$$\tan\left(\delta\right)=\frac{\frac{b}{m}\omega}{\frac{k}{m}-\omega^2}=\frac{b\omega}{k-\omega^2 m}=\frac{4\,\mathrm{Nsm}^{-1}\,30\,\mathrm{s}^{-1}}{80\,\mathrm{Nm}^{-1}-(30\,\mathrm{s}^{-1})^2 0.2\,\mathrm{kg}}=-1.2$$

$$\delta\approx -50^{\text{o}}$$

(b) Qual é a potência média fornecida durante um ciclo?

$$P = \frac{1}{2}\mathrm{Re}\left[F_0\overline{V}\right] = \frac{F_0}{2}\mathrm{Re}\left[-i\omega\overline{A}\right]$$

$$\dot{X}(t) = i\omega A \exp(i\omega t) = V \exp(i\omega t)$$

$$V = i\omega A$$

$$A = \frac{F_0}{m}\frac{\exp(-i\delta)}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}$$

$$\tan(\delta) = \frac{\gamma\omega}{\omega_0^2-\omega^2}$$

$$P = \frac{F_0^2}{2m}\mathrm{Re}\left[\frac{-i\omega\exp(i\delta)}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}\right] =$$

$$= \frac{F_0^2}{2m}\frac{\omega}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}\mathrm{Re}\left[-i\cos(\delta)+\sin(\delta)\right] = \frac{F_0^2}{2m}\frac{\omega\sin(\delta)}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2}}$$

$$\frac{1}{\sin^2(\delta)} = \frac{\sin^2(\delta)+\cos^2(\delta)}{\sin^2(\delta)} = 1 + \frac{1}{\tan^2(\delta)}$$

$$P = \frac{F_0^2}{2m} \frac{\omega \sin(\delta)}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2 \omega^2}}$$

$$\frac{1}{\sin^2(\delta)} = \frac{\sin^2(\delta) + \cos^2(\delta)}{\sin^2(\delta)} = 1 + \frac{1}{\tan^2(\delta)}$$

$$\tan(\delta) = \frac{\gamma \omega}{\omega_0^2 - \omega^2}$$

$$\sin(\delta) = \left[1 + \frac{1}{\tan^2(\delta)}\right]^{-1/2} = \left[1 + \frac{\left(\omega_0^2 - \omega^2\right)^2}{\gamma^2 \omega^2}\right]^{-1/2} = \frac{\gamma \omega}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2 \omega^2}}$$

$$P = \frac{F_0^2}{2m} \frac{\gamma \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma^2 \omega^2} = \frac{F_0^2}{2m} \frac{\gamma}{\left(\frac{\omega_0^2}{\omega} - \omega\right)^2 + \gamma^2} =$$

$$= \frac{F_0^2 \omega_0}{2m\omega_0^2} \frac{\frac{\gamma}{\omega_0}}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \left(\frac{\gamma}{\omega_0}\right)^2} = \frac{F_0^2 \omega_0}{2kQ} \frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\gamma = \frac{b}{m}$$

$$Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{km}}{b}$$

$$P = \frac{F_0^2}{2m}\frac{\gamma\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2} = \frac{F_0^2}{2m}\frac{\gamma}{\left(\frac{\omega_0^2}{\omega}-\omega\right)^2+\gamma^2} =$$

$$= \frac{F_0^2\omega_0}{2m\omega_0^2}\frac{\frac{\gamma}{\omega_0}}{\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\left(\frac{\gamma}{\omega_0}\right)^2} = \frac{F_0^2\omega_0}{2kQ}\frac{1}{\left(\frac{\omega_0}{\omega}-\frac{\omega}{\omega_0}\right)^2+\frac{1}{Q^2}}$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$\gamma = \frac{b}{m}$$

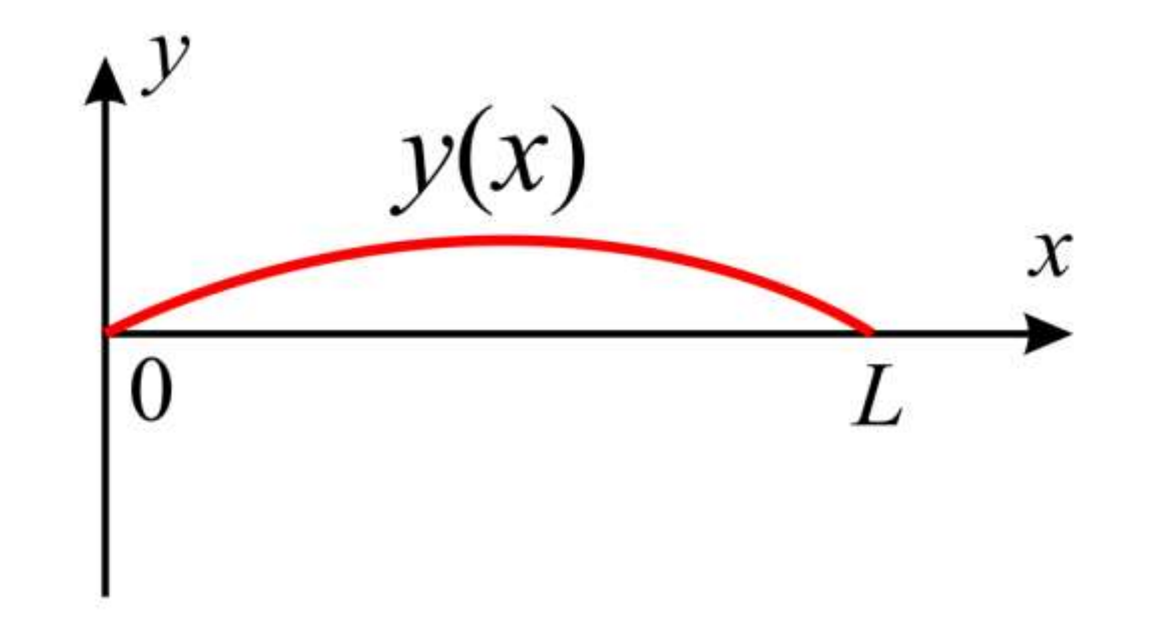
$$Q = \frac{\omega_0}{\gamma} = \frac{\sqrt{km}}{b}$$

$$P = \frac{F_0^2}{2m}\frac{\gamma\omega^2}{\left(\omega_0^2-\omega^2\right)^2+\gamma^2\omega^2} = \frac{F_0^2}{2m}\frac{\frac{b}{m}\omega^2}{\left(\frac{k}{m}-\omega^2\right)^2+\frac{b^2}{m^2}\omega^2} = \frac{F_0^2}{2}\frac{b\omega^2}{\left(k-\omega^2 m\right)^2+b^2\omega^2}$$

$$P \approx \frac{\left(2\,\mathrm{N}\right)^2}{2}\frac{4\,\mathrm{Nsm}^{-1}\left(30\,\mathrm{s}^{-1}\right)^2}{\left(80\,\mathrm{Nm}^{-1}-(30\,\mathrm{s}^{-1})^2 0.2\,\mathrm{kg}\right)^2+\left(4\,\mathrm{Nsm}^{-1}\,30\,\mathrm{s}^{-1}\right)^2} \approx 0.295\,W$$

7 - Uma corda uniforme de comprimento $L = 2.5$ m e massa $M = 0.01$ Kg é sujeita a uma tensão de $T = 10$ N.

(a) Qual é a frequência do seu modo fundamental?

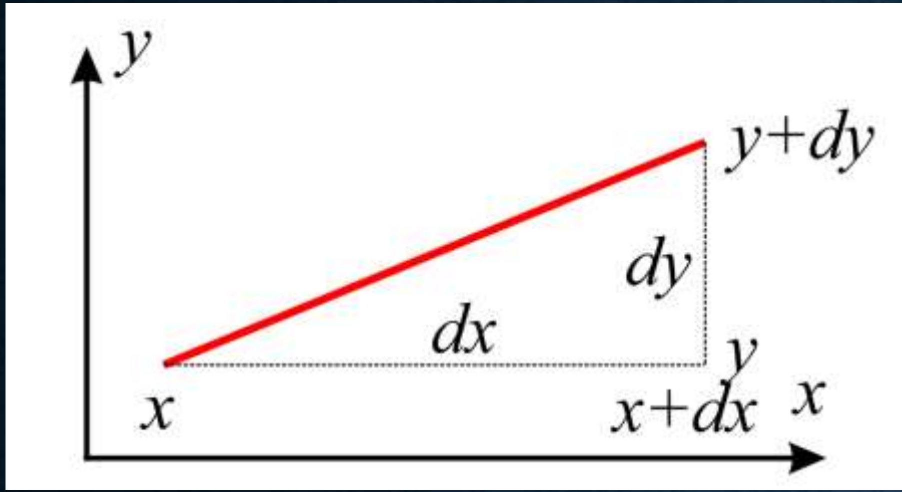


$$\{q_1, q_2, ..., q_N\} = \{y_1,\ y_2, ..., y_N\}$$

$$y\,(x)$$

$$dm = \rho_1 dx \qquad \rho_1 = \frac{M}{L}$$

$$d\mathcal{T} = \frac{1}{2}\left[\dot{y}\,(x)\right]^2 dm = \frac{1}{2}\left[\dot{y}\,(x)\right]^2 \rho_1 dx$$



$$d\mathcal{T} = \frac{1}{2}\left[\dot{y}(x)\right]^2 dm = \frac{1}{2}\left[\dot{y}(x)\right]^2 \rho_1 dx$$

$$d\mathcal{L} = d\mathcal{T} - d\mathcal{V} = \frac{1}{2}\left\{\rho_1\left[\dot{y}(x)\right]^2 - T\left(\frac{dy}{dx}\right)^2\right\} dx$$

$$d\mathcal{V} = T\left(\sqrt{dx^2 + dy^2} - dx\right) = T\left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - 1\right) dx \approx \frac{T}{2}\left(\frac{dy}{dx}\right)^2 dx$$

$$\mathcal{L} = \frac{1}{2}\int_0^L \left\{\rho_1\left[\dot{y}(x)\right]^2 - T\left(\frac{dy}{dx}\right)^2\right\} dx$$

$$\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{y}}\right) - \frac{\delta\mathcal{L}}{\delta y} = 0$$

$$\frac{\delta\mathcal{L}}{\delta\dot{y}} = \rho_1\dot{y}(x)$$

$$\mathcal{L} = \frac{\rho_1}{2}\int_0^L \left[\dot{y}(x)\right]^2 dx - \frac{T}{2}\int_0^L \left(\frac{dy}{dx}\right)^2 dx = \frac{\rho_1}{2}\int_0^L \left[\dot{y}(x)\right]^2 dx - \frac{T}{2}\int_0^L \frac{dy}{dx}\frac{dy}{dx} dx =$$

$$\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{y}}\right)-\frac{\delta\mathcal{L}}{\delta y}=0$$

$$\frac{\delta\mathcal{L}}{\delta\dot{y}}=\rho_1\dot{y}\left(x\right)$$

$$\boxed{\rho_1\ddot{y}\left(x\right)-T\frac{d^2y}{dx^2}=0}$$

$$\mathcal{L}=\frac{\rho_1}{2}\int\limits_0^L\left[\dot{y}\left(x\right)\right]^2dx-\frac{T}{2}\int\limits_0^L\left(\frac{dy}{dx}\right)^2dx=\frac{\rho_1}{2}\int\limits_0^L\left[\dot{y}\left(x\right)\right]^2dx-\frac{T}{2}\int\limits_0^L\frac{dy}{dx}\frac{dy}{dx}dx=$$

$$=\frac{\rho_1}{2}\int\limits_0^L\left[\dot{y}\left(x\right)\right]^2dx-\frac{T}{2}\int\limits_0^L\frac{dy}{dx}dy=\frac{\rho_1}{2}\int\limits_0^L\left[\dot{y}\left(x\right)\right]^2dx-\frac{T}{2}\left\{\left(\frac{dy}{dx}y\right)\Bigg|_0^L-\int\limits_0^L yd\left(\frac{dy}{dx}\right)\right\}=$$

$$y(0)=y(L)=0$$

$$=\frac{\rho_1}{2}\int\limits_0^L\left[\dot{y}\left(x\right)\right]^2dx+\frac{T}{2}\int\limits_0^L y\frac{d^2y}{dx^2}dx$$

$$\frac{\delta\mathcal{L}}{\delta y}=\frac{T}{2}\int\limits_0^L y\frac{d^2y}{dx^2}dx=\frac{T}{2}\left(\frac{d^2y}{dx^2}+y\frac{d^2}{dx^2}\right)=\frac{T}{2}\left(\frac{d^2y}{dx^2}+\frac{d^2}{dx^2}y\right)=T\frac{d^2y}{dx^2}$$

$$\rho_1 \frac{d^2 y}{dt^2} - T\frac{d^2 y}{dx^2} = 0$$

$$y(x,t) = \exp(ikx - i\omega t)$$

$$-\rho_1\omega^2 + Tk^2 = 0$$

$$k = \pm\omega\sqrt{\frac{\rho_1}{T}}$$

$$y_1(x,t) = \exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x - i\omega t\right)$$

$$y_2(x,t) = \exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x - i\omega t\right)$$

$$y_3(x,t) = \exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x + i\omega t\right)$$

$$y_4(x,t) = \exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x + i\omega t\right)$$

$$y(x,t) = A_1 y_1(x,t) + A_2 y_2(x,t) + A_3 y_3(x,t) + A_4 y_4(x,t) =$$
$$= A_1 \exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x - i\omega t\right) + A_2 \exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x - i\omega t\right) +$$
$$+ A_3 \exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x + i\omega t\right) + A_4 \exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x + i\omega t\right)$$

$$y(x,t) = A_1 y_1(x,t) + A_2 y_2(x,t) + A_3 y_3(x,t) + A_4 y_4(x,t) =$$
$$= A_1 \exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x - i\omega t\right) + A_2 \exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x - i\omega t\right) +$$
$$+A_3 \exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x + i\omega t\right) + A_4 \exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x + i\omega t\right)$$

$$A_2 = -A_1$$
$$A_4 = -A_3$$

$$y(0,t) = A_1 \exp(-i\omega t) + A_2 \exp(-i\omega t) +$$
$$+A_3 \exp(i\omega t) + A_4 \exp(i\omega t) = 0$$

$$y(0) = y(L) = 0$$

$$y(x,t) = A_1 \left\{\exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x\right) - \exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x\right)\right\}\exp(-i\omega t) +$$
$$+A_3 \left\{\exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x\right) - \exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x\right)\right\}\exp(i\omega t) =$$

$$y\left(x,t\right)=A_1\left\{\exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x\right)-\exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x\right)\right\}\exp\left(-i\omega t\right)+$$

$$+A_3\left\{\exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x\right)-\exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x\right)\right\}\exp\left(i\omega t\right)=$$

$$=\left\{\exp\left(i\omega\sqrt{\frac{\rho_1}{T}}x\right)-\exp\left(-i\omega\sqrt{\frac{\rho_1}{T}}x\right)\right\}\left\{A_1\exp\left(-i\omega t\right)+A_3\exp\left(i\omega t\right)\right\}=$$

$$=2i\sin\left(\omega\sqrt{\frac{\rho_1}{T}}x\right)\left\{A_1\exp\left(-i\omega t\right)+A_3\exp\left(i\omega t\right)\right\}$$

$$y(0)=y(L)=0$$

$$y\left(L,t\right)=2i\sin\left(\omega\sqrt{\frac{\rho_1}{T}}L\right)\left\{A_1\exp\left(-i\omega t\right)+A_3\exp\left(i\omega t\right)\right\}=0$$

$$\sin\left(\omega\sqrt{\frac{\rho_1}{T}}L\right)=0$$

$$\sin\left(\omega\sqrt{\frac{\rho_1}{T}}L\right) = 0$$

$$\omega\sqrt{\frac{\rho_1}{T}}L = n\pi$$

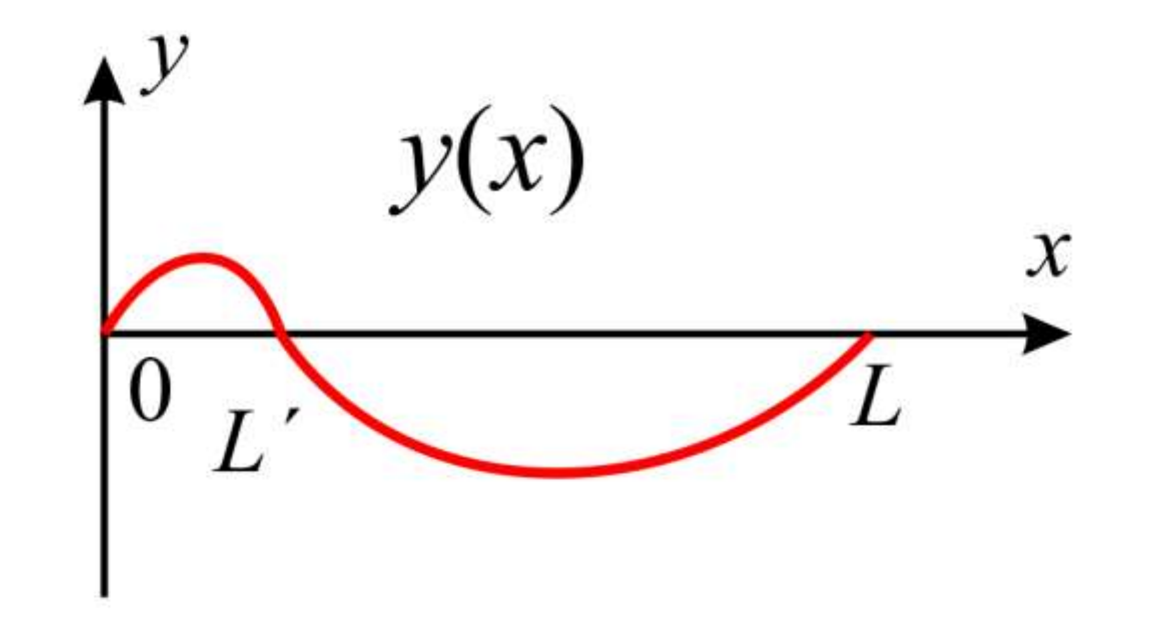
$$\omega_n = \frac{n\pi}{L}\sqrt{\frac{T}{\rho_1}}$$

$$\rho_1 = \frac{M}{L}$$

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{n}{2L}\sqrt{\frac{TL}{M}} = \frac{n}{2}\sqrt{\frac{T}{LM}}$$

$$\nu_1 = \frac{1}{2}\sqrt{\frac{10\,\mathrm{N}}{2.5\,\mathrm{m}\,0.01\,\mathrm{kg}}} = 10\,\mathrm{Hz}$$

(b) Se a corda for puxada transversalmente, de modo a vibrar, e em seguida um dos seus pontos a 0.5 m de um dos extremos for fixado, quais as frequências que persistirão para as vibrações do segmento da corda entre esse ponto e o correspondente extremo?



$$\rho_1 = \frac{M}{L} = \frac{M'}{L'}$$

$$M' = M\frac{L'}{L}$$

$$\nu_n = \frac{\omega_n}{2\pi} = \frac{n}{2L}\sqrt{\frac{TL}{M}} = \frac{n}{2}\sqrt{\frac{T}{LM}}$$

$$\nu'_n = \frac{n}{2}\sqrt{\frac{T}{L'M'}} = \frac{n}{2}\sqrt{\frac{TL}{(L')^2 M}} = \frac{n}{2}\frac{L}{L'}\sqrt{\frac{T}{LM}} = \frac{L}{L'}\nu_n$$

$$\nu'_1 = \frac{2.5\ \mathrm{m}}{0.5\ \mathrm{m}}\,10\ \mathrm{Hz} = 50\ \mathrm{Hz}$$