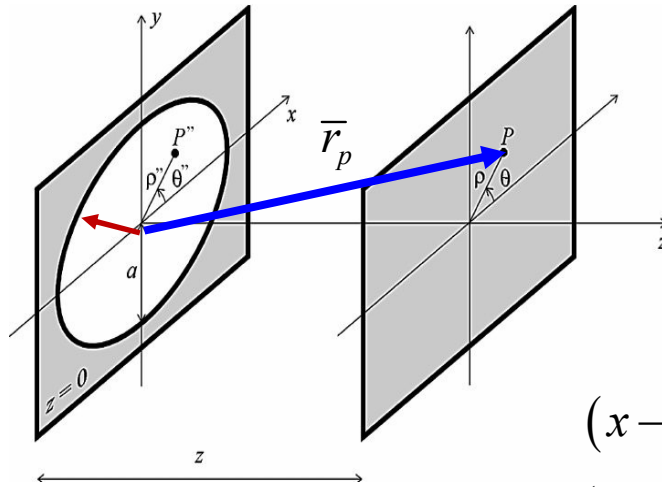


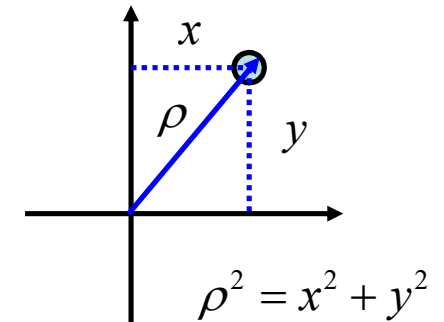
Integral Fresnel-Huygens com Simetria circular

$$\mathcal{E}(x, y, z) = \frac{\mathcal{E}_0}{i\lambda z} \int_{-\infty}^{\infty} \int f(x', y') e^{ikz} e^{ik[(x-x')^2 + (y-y')^2]/2z} dx' dy'$$

Com simetria circular convêm usar coordenados polares



$$\begin{aligned} x' &= \rho' \cos \theta' & y' &= \rho' \sin \theta' \\ x &= \rho \cos \theta & y &= \rho \sin \theta \end{aligned}$$



$$\bar{r}_p = z + \rho^2 / 2z$$

$$(x-x')^2 = \rho^2 \cos^2 \theta - 2\rho\rho' \cos \theta \cos \theta' + \rho'^2 \cos^2 \theta'$$

$$+ (y-y')^2 = \rho^2 \sin^2 \theta - 2\rho\rho' \sin \theta \sin \theta' + \rho'^2 \sin^2 \theta'$$

$$(x-x')^2 + (y-y')^2 = \rho^2 - 2\rho\rho' \cos(\theta - \theta') + \rho'^2$$

$$f(\rho') = \begin{cases} 1 & \rho' < R_a \\ 0 & \rho' \geq R_a \end{cases}$$

$$\mathcal{E}(\rho, \theta, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ik\bar{r}_p} \int_0^{2\pi} d\theta' \int_0^{R_a} \rho' d\rho' \exp \left[-ik \frac{\rho\rho'}{z} \cos(\theta - \theta') + ik \frac{\rho'^2}{2z} \right]$$

Abertura circular irradiância no eixo ótico

$$\mathcal{E}(\rho, \theta, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ik\bar{r}_p} \int_0^{2\pi} d\theta' \int_0^{R_A} \rho' d\rho' \exp \left[-ik \frac{\rho\rho'}{z} \cos(\theta - \theta') + ik \frac{\rho'^2}{2z} \right]$$

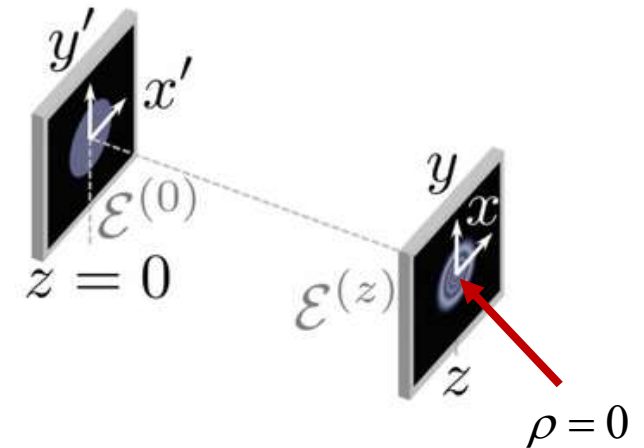
Esta integral é difícil em geral e tem ser resolvida numericamente. No entanto no eixo ótico $\rho = 0$

$$\mathcal{E}(\rho = 0, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} 2\pi \int_0^{R_A} \rho' d\rho' \exp \left[ik \frac{\rho'^2}{2z} \right]$$

$$u' = \frac{ik\rho'^2}{2z} = \frac{i\pi\rho'^2}{\lambda z} \quad \frac{\rho'^2}{2} = \frac{\lambda z}{i2\pi} u' \quad \rho' d\rho' = \frac{\lambda z}{i2\pi} du'$$

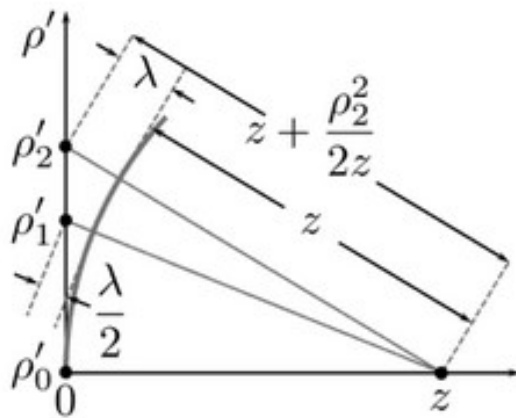
$$\begin{aligned} \mathcal{E}(\rho = 0, z) &= \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} 2\pi \frac{\lambda z}{i2\pi} \int_0^{i\pi R_A^2/\lambda z} e^{u'} du' = -\mathcal{E}_0 e^{ikz} \left[e^{i\pi R_A^2/\lambda z} - 1 \right] \\ &= -2i\mathcal{E}_0 e^{ikz} e^{i\pi R_A^2/2\lambda z} \left[\frac{e^{i\pi R_A^2/2\lambda z} - e^{-i\pi R_A^2/2\lambda z}}{2i} \right] \end{aligned}$$

$$\mathcal{I}(\rho = 0, z) = 4\mathcal{I}_0 \sin^2 \left(\frac{\pi R_a^2}{2\lambda z} \right)$$



Intensidade no eixo ótico – as Zonas de Fresnel

No eixo ótico ($r = 0$) $\mathcal{E}(\rho = 0, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} 2\pi \int_0^{R_a} \exp\left[i \frac{k\rho'^2}{2z}\right] \rho' d\rho'$



Considere um ponto fixo, z , no eixo ótica

A maneira que ρ' aumenta a fase $e^{ik\rho'^2/2z}$ vai oscilar. Cada vez que a diferença em fase é igual meio ciclo, o sinal inverte.

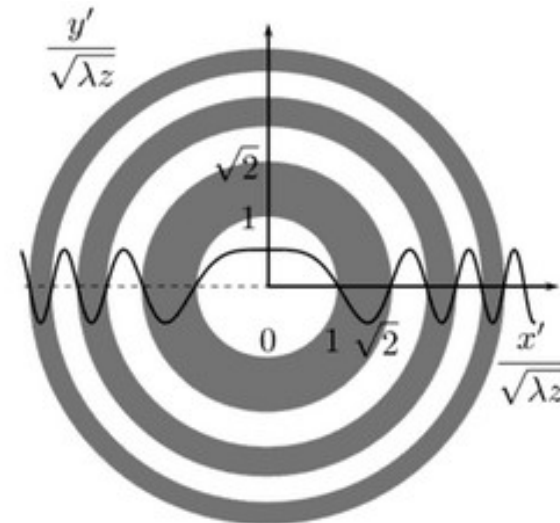
Fresnel usou este efeito para definir zonas sobre qual o sinal da fase é positiva ou negativa

Zonas brancas (fase positiva)

Zonas cinzentas (fase negativa)

Raios limitadores $\frac{\rho_m'^2}{2z} = m \frac{\lambda}{2}$ $k \frac{\rho_m'^2}{2z} = m\pi$

$\rho_m' = \sqrt{m\lambda z}$



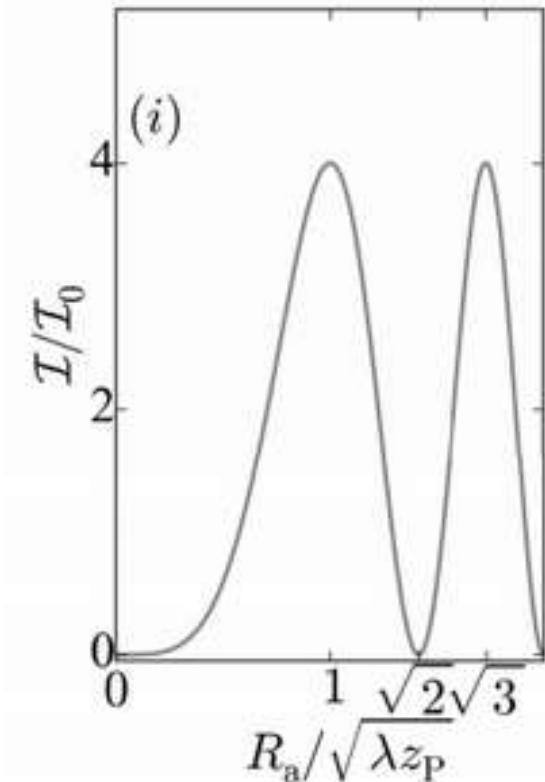
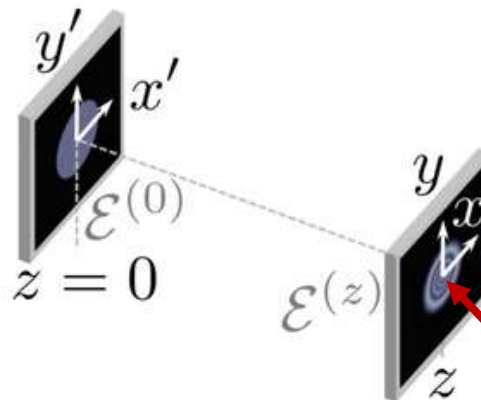
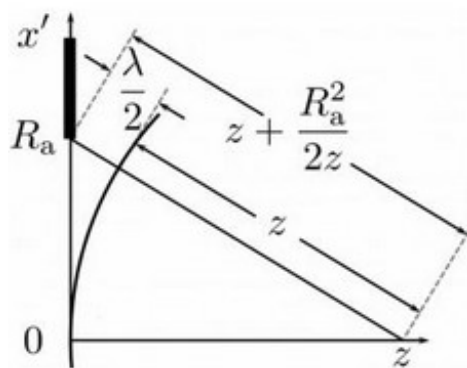
Abertura circular

Imagine que temos uma abertura circular com um raio R_a que podemos variar. Qual é a intensidade no eixo ótico ($\rho = 0$) a uma distância z da abertura?

$$\mathcal{E}(\rho = 0, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} \int_0^{R_a} e^{ik\rho'^2/2z} 2\pi\rho' d\rho' \quad \rho'_m = \sqrt{m\lambda z}$$

$0 \leq R_a \leq \rho'_1$ dentro a primeira zona Fresnel
interferência é construtiva e a
irradiância aumenta com R_a

$\rho_1 \leq R_a \leq \rho_2$ Começa haver interferência
destrutiva e a irradiância no eixo vai
diminuir até atingir 0 quando $R_a = \rho_2$
⋮

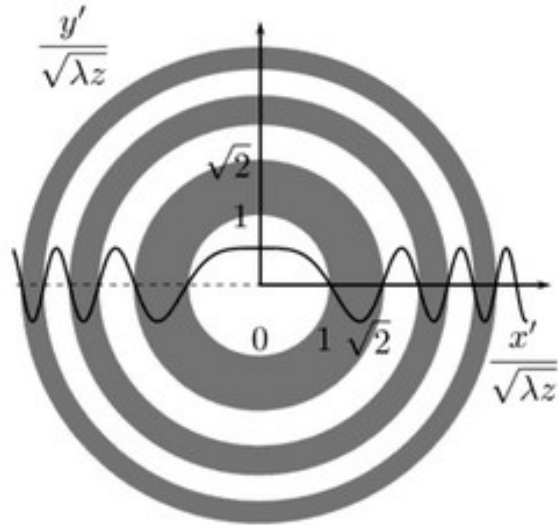


$z = z_p$ fixo, variar
 R_a

$\rho = 0$

$$\mathcal{I}(\rho = 0, z) = 4\mathcal{I}_0 \sin^2 \left(\frac{\pi R_a^2}{2\lambda z} \right)$$

Zonas de Fresnel



$$\rho'_m = \sqrt{m\lambda z}$$

Note que depende da distância entre a abertura e o plano de observação (z)

A área entre ρ'_{m-1} e ρ'_m é a $m^{\text{íssima}}$ zona Fresnel

Todas as zonas têm a mesma área

$$\pi(\rho'^2_{m+1} - \rho'^2_m) = \pi\lambda z(m+1 - m) = \pi\lambda z$$

Placa de Fresnel

Se eliminamos todas as zonas cinzentas (ou em alternativa todas as zona brancas) a luz que passa pelos os restantes zonas vai interfere construtivamente (no eixo).

É equivalente á uma lente

$$f = \frac{\rho'^2_1}{\lambda}$$

i.e. foca uma onda plana na posição $z = f$

Útil nas situações em que refração não é uma opção (Raios X, focagem dum feixe de átomos neutrais,...)



Uma placa Fresnel para átomos de He

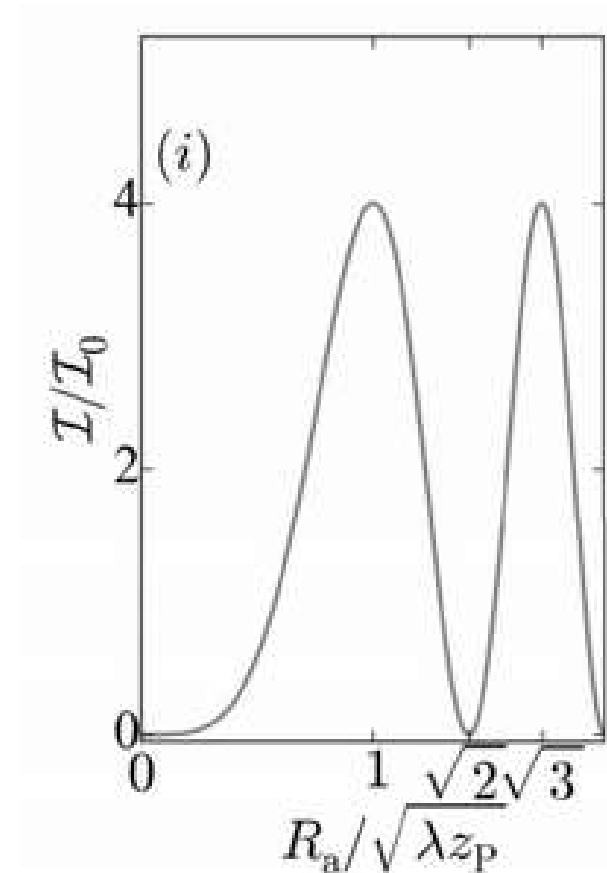
Lentes Fresnel



Abertura circular irradiância no eixo ótico

$$\begin{aligned}\mathcal{E}(\rho=0, z) &= \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} \int_0^{R_a} e^{ik\rho'^2/2z} 2\pi\rho' d\rho' \\ &= \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} \frac{2\pi z}{ik} \left[e^{ikR_a^2/2z} - 1 \right] \\ &= -2i\mathcal{E}_0 e^{ikz} e^{ikR_a^2/4z} \left[\frac{e^{ikR_a^2/4z} - e^{-ikR_a^2/4z}}{2i} \right] \\ &= -2i\mathcal{E}_0 e^{ikz} e^{ikR_a^2/4z} \sin\left(\frac{kR_a^2}{4z}\right)\end{aligned}$$

$$\mathcal{I}(\rho=0, z) = 4\mathcal{I}_0 \sin^2\left(\frac{\pi R_a^2}{2\lambda z}\right)$$



$z = z_p$ fixo, variar R_a

Variação da intensidade ao longo do eixo ótico

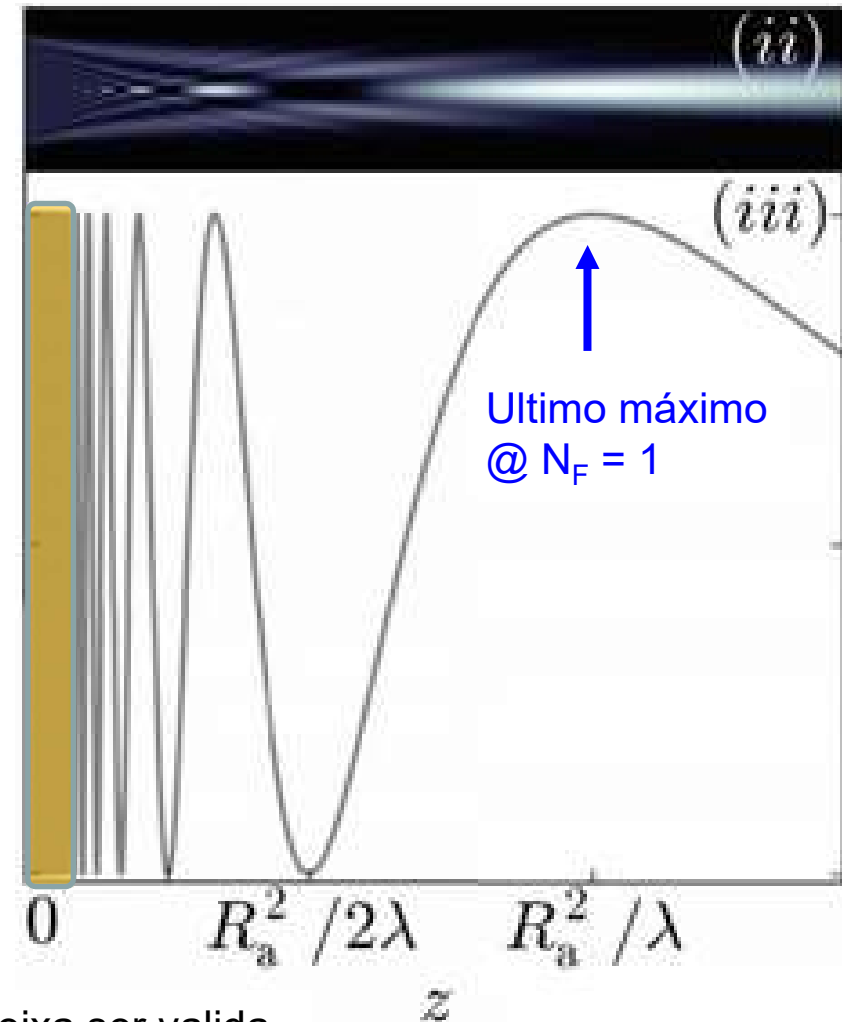
$$\mathcal{I}(\rho = 0, z) = 4\mathcal{I}_0 \sin^2\left(\frac{\pi R_a^2}{2\lambda z}\right)$$

$$N_F = \frac{R_a^2}{\lambda z} \quad \text{Número Fresnel}$$

Quando $N_F \ll 1$
no limite paraxial
o padrão de difração fica mais estável
e mais simples, tal como aconteceu
no caso duma fenda simples.



Quando $z < R_a^2$ a aproximação de Fresnel deixa de ser válida



Variação da padrão de difração com z

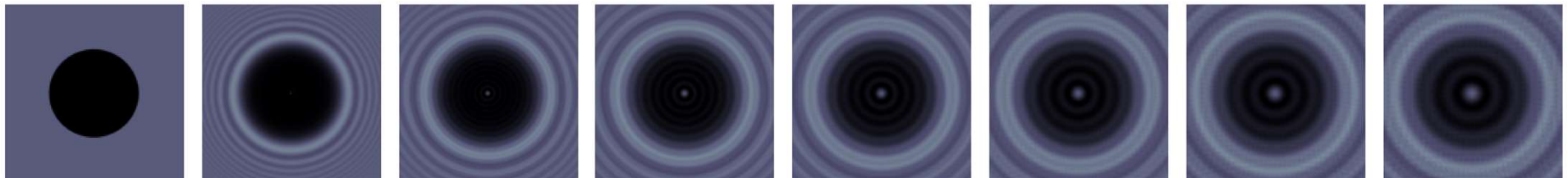
$$\mathcal{I}(\rho, \theta, z) = \mathcal{I}_0 \left| \int_0^{2\pi} d\theta' \int_0^{R_A} \rho' d\rho' \exp \left[-ik \frac{\rho\rho'}{z} \cos(\theta - \theta') + ik \frac{\rho'^2}{2z} \right] \right|^2$$

Simulação numérica da integral

$z = 0$

Abertura circular

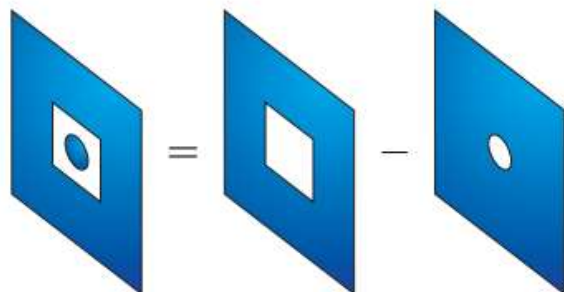
$z = \frac{R_a^2}{\lambda}$



Disco circular

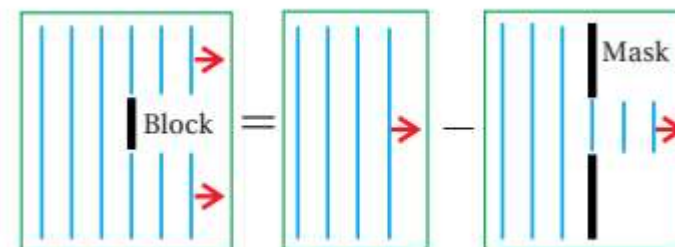
Princípio de Babinet

O princípio de Babinet é efetivamente o princípio de sobreposição



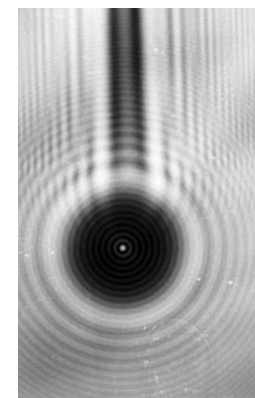
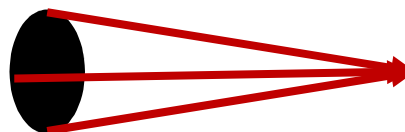
Jaques Babinet
(1794-1872)

Em particular considere um disco circular
O campo no eixo ótico é



$$\begin{aligned}\mathcal{E}_{disco}(\rho=0, z) &= \mathcal{E}_{plana}(0, z) - \mathcal{E}_{abertura}(0, z) \\ &= \mathcal{E}_0 e^{ikz} + \mathcal{E}_0 e^{ikz} \left[e^{i\pi R_a^2 / \lambda z} - 1 \right] \\ &= \mathcal{E}_0 e^{ikz} e^{i\pi R_a^2 / \lambda z}\end{aligned}$$

$$\mathcal{I}_{disco}(\rho=0, z) = \mathcal{I}_0$$



Interferência
construtiva no
centro do disco

O spot de
Poisson/Arago

Complementaridade

No limite Fraunhofer a integral Fresnel de difração é uma transformada Fourier

$$N_F \ll 1$$

Desprezar termos $\frac{x'^2}{z\lambda}, \frac{y'^2}{z\lambda}$

$$N_F \sim \frac{(\text{tamnaho da abertura})^2}{\lambda z}$$

$$\mathcal{E}(x, y, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') e^{\left\{ ik \left[(x-x')^2 + (y-y')^2 \right] / 2z \right\}} dx' dy'$$

$$\Rightarrow \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} e^{ik(x^2+y^2)/2z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x', y') e^{-ik[xx'+yy']/z} dx' dy'$$

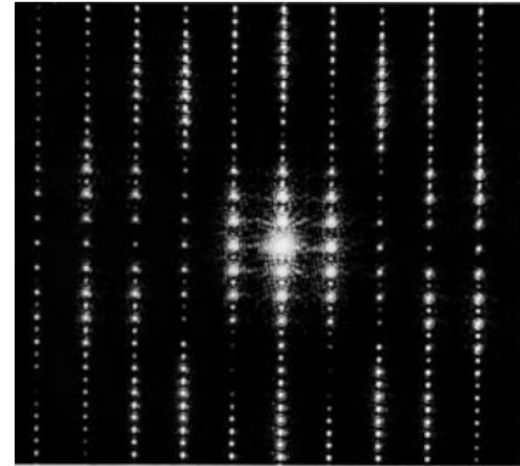
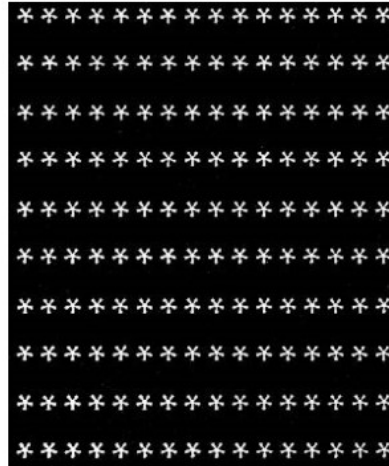
Uma abertura complementar (uma que é o inverso da abertura original) da a mesma padrão (exceto no eixo ótico)

$$\begin{aligned} \mathcal{E}_{comp}(x, y, z) &= \frac{\mathcal{E}_0}{i\lambda z} e^{ikz} e^{ik(x^2+y^2)/2z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [1 - f(x', y')] e^{-ik[xx'+yy']/z} dx' dy' \\ &= -\mathcal{E}_{original}(x, y, z) + -i\mathcal{E}_0 \lambda z e^{ikz} 2\pi \delta(x) \delta(y) \end{aligned}$$

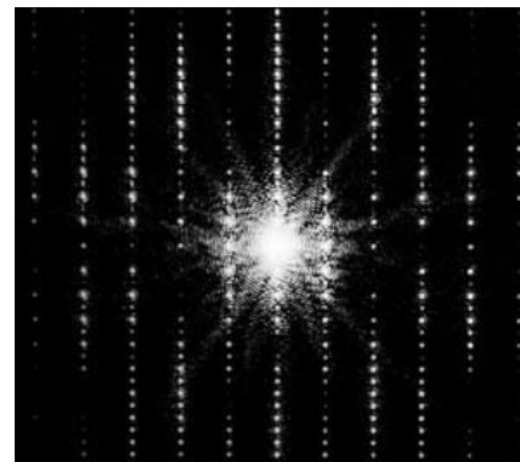
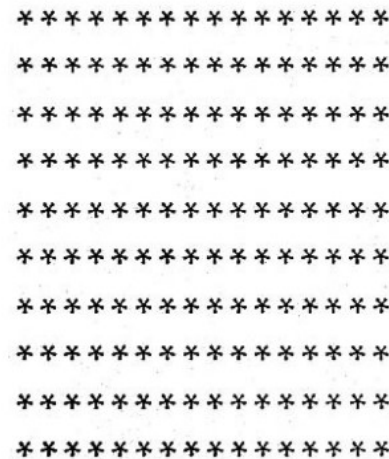
$$\text{Exceto na origem} \quad \mathcal{E}_{comp}(x, y, z) = -\mathcal{E}_{original}(x, y, z) \quad \mathcal{I}_{comp}(x, y, z) = \mathcal{I}_{original}(x, y, z)$$

Exemplo do principio de Babinet

Uma rede
de buracos



Uma rede
de anti-buracos



Difração de abertura circulares

Integral da difração de Fresnel

$$\mathcal{E}(\rho, \theta, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ik\bar{r}_p} \int_0^{2\pi} d\theta' \int_0^{R_A} \rho' d\rho' \exp \left[-ik \frac{\rho\rho'}{z} \cos(\theta - \theta') + ik \frac{\rho'^2}{2z} \right]$$

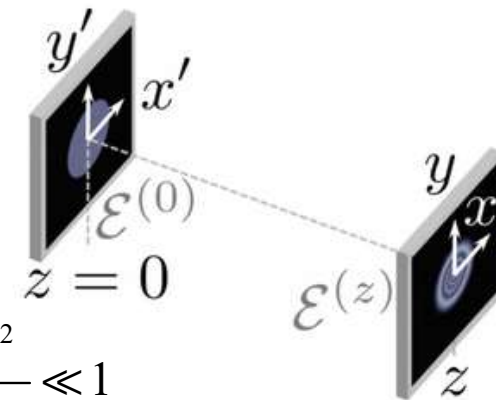
Para o caso de uma abertura circular com diâmetro D ,
existe 2 casos particulares onde a dependência nos
termo $\rho'^2 / 2z$

é anulada / desprezável

Caso I : Onda plana incidente, no limite Fraunhofer $N_F = \frac{D^2}{4\lambda z} \ll 1$

$$\frac{k\rho'^2}{2z} \leq \pi \frac{D^2}{\lambda z} = 4\pi N_F \ll 1$$

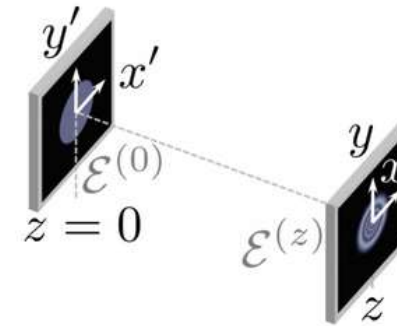
$$\mathcal{E}(\rho, \theta, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ik\bar{r}_p} \int_0^{2\pi} d\theta' \int_0^{R_A} \rho' d\rho' \exp \left[-ik \frac{\rho\rho'}{z} \cos(\theta - \theta') + ik \frac{\rho'^2}{2z} \right]$$



Difração de abertura circulares

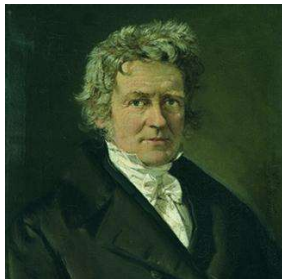
Integral Fresnel de difração

$$\mathcal{E}(\rho, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz - ik\rho^2/2z} \underbrace{\int_0^{2\pi} d\phi' \int_0^{D/2} \rho' d\rho' e^{[-ik\rho\rho' \cos(\phi - \phi')/z]}}_{\text{Transformada Fourier}}$$



Transformada Fourier numa abertura circular do diâmetro D

ajuda dum amigo matemático



Friedrich Bessel
(1784-1846)

$$\int_0^{2\pi} d\phi' e^{[-ik\rho\rho' \cos(\phi - \phi')/z]} = 2\pi J_0\left(\frac{k\rho\rho'}{z}\right)$$

$$\int_0^{D/2} \rho' d\rho' J_0\left(\frac{k\rho\rho'}{z}\right) = \frac{Dz}{2k\rho} J_1\left(\frac{kD\rho}{2z}\right)$$

$$\mathcal{E}(\rho, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ikz - ik\rho^2/2z} \frac{\pi Dz}{k\rho} J_1\left(\frac{kD\rho}{2z}\right)$$

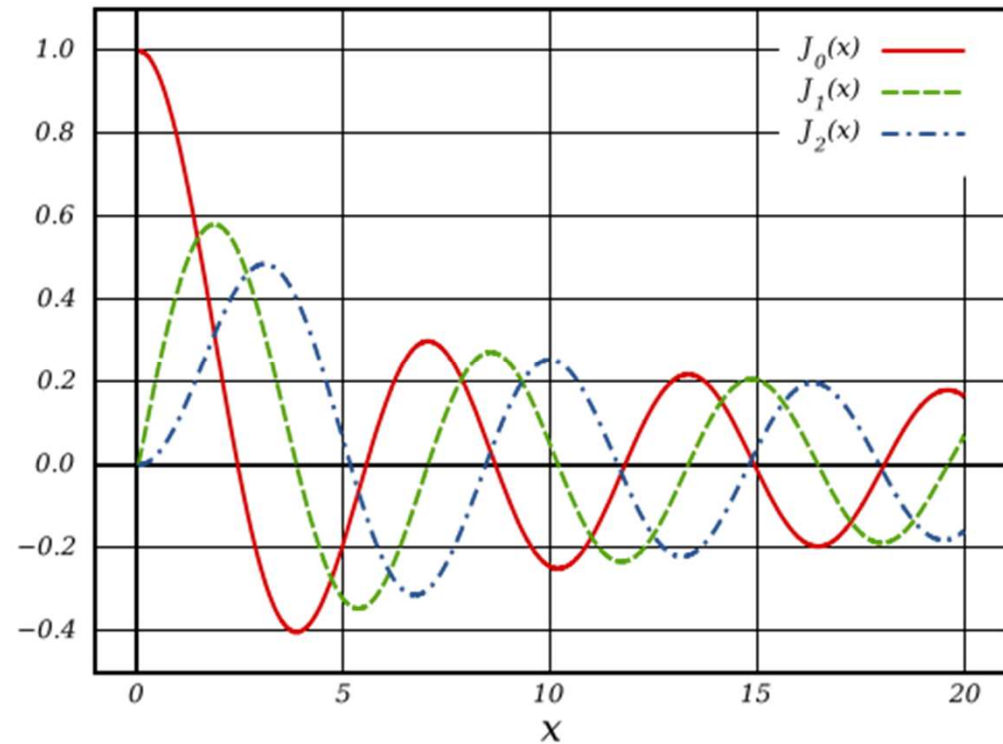
Difração de abertura circulares

Integral Fresnel de difração limite Fraunhofer

$$\begin{aligned}\mathcal{E}(\rho, z) &= \frac{\mathcal{E}_0}{i\lambda z} e^{ikz - ik\rho^2/2z} \frac{\pi D z}{k\rho} J_1\left(\frac{kD\rho}{2z}\right) \\ &= -i\mathcal{E}_0 e^{ikz - ik\rho^2/2z} \frac{D}{2\rho} J_1(kD\rho/2z)\end{aligned}$$

$$\mathcal{I}(\rho, z) = \mathcal{I}_0 \left(\frac{\pi D^2}{4\lambda z} \right)^2 \underbrace{\left[2 \frac{J_1(kD\rho/2z)}{(kD\rho/2z)} \right]^2}_{\text{jinc}^2(kD\rho/2z)}$$

$$\lim_{\xi \rightarrow 0} \underbrace{\left[2 \frac{J_1(\xi)}{\xi} \right]^2}_{\text{jinc}^2(\xi)} = 1$$



Zeros de J_1 @ 3.8317, 7.0156, 10.1735,...

O padrão de Airy

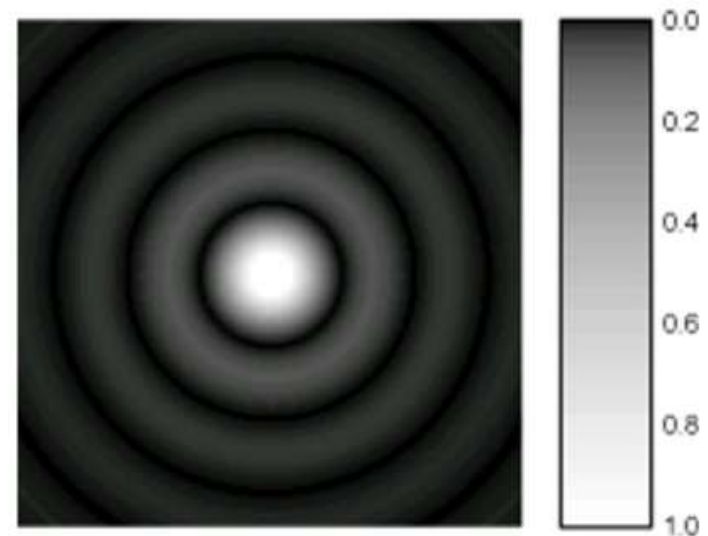
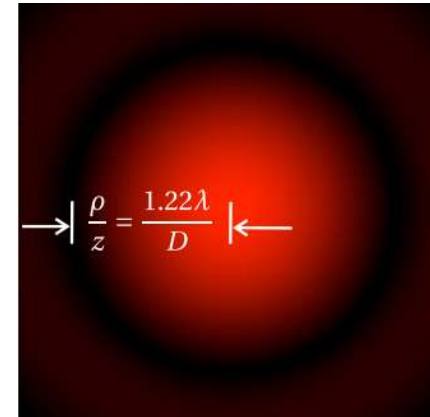
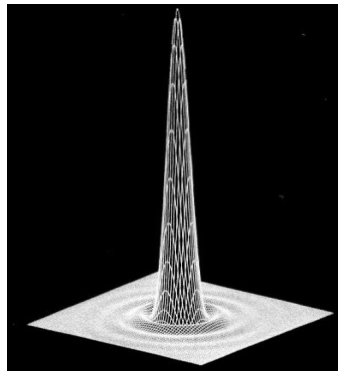
$$\mathcal{I}(\rho, z) = \mathcal{I}_0 \left(\frac{\pi D^2}{4\lambda z} \right)^2 \underbrace{\left[2 \frac{J_1(kD\rho/2z)}{(kD\rho/2z)} \right]^2}_{\text{jinc}^2(kD\rho/2z)}$$
$$= \mathcal{I}_0 (\pi N_F)^2 \text{jinc}^2(4\pi N_F \rho / D)$$



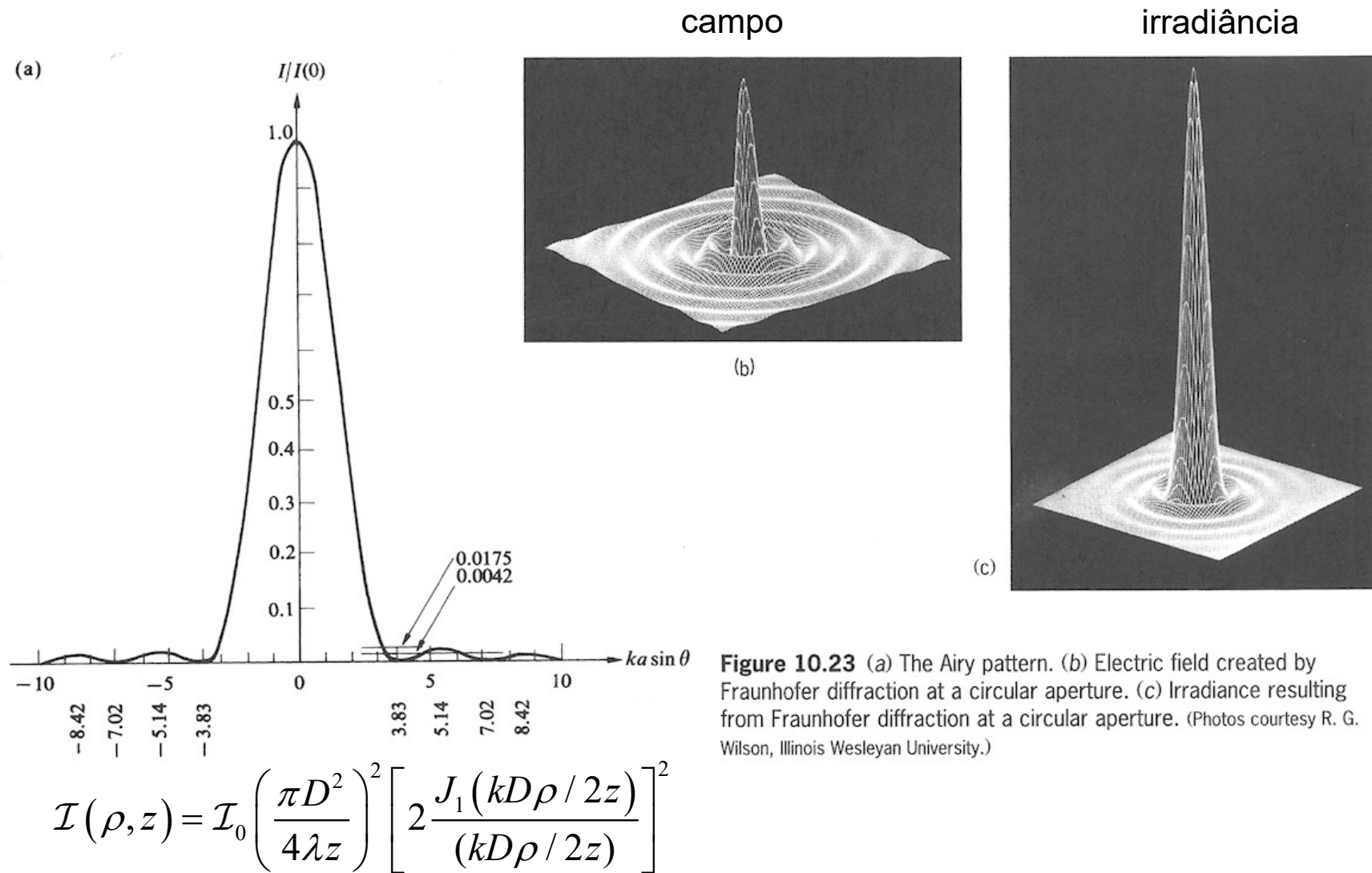
Sir George Biddell Airy
1801-1892

Esta padrão é conhecida como o “padrão de Airy”

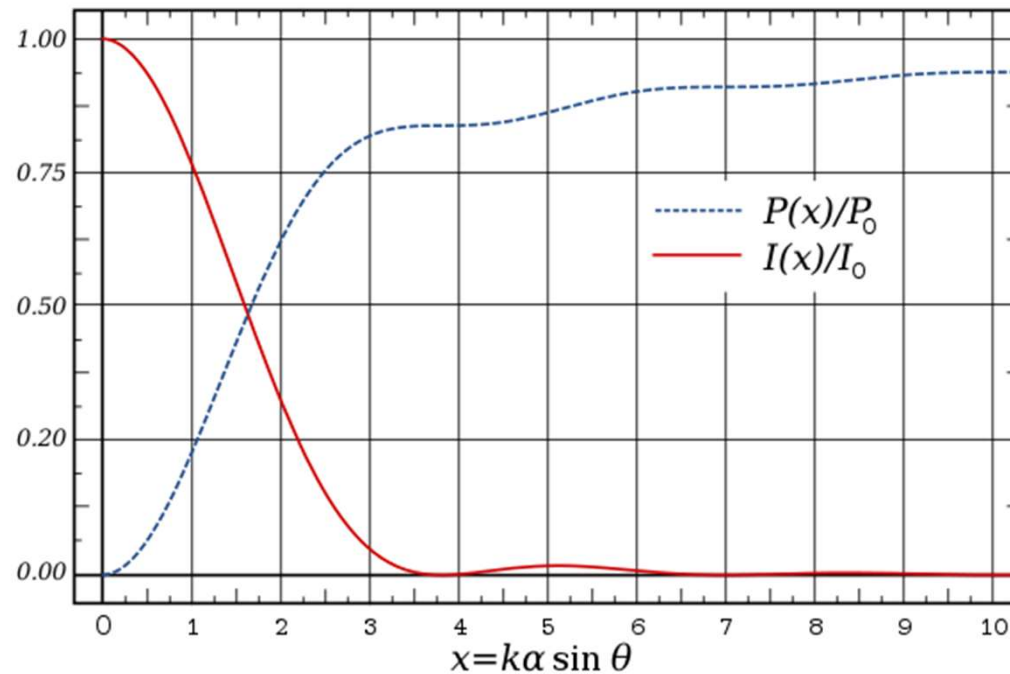
A maior parte da intensidade está no pico central



Padrão de Airy



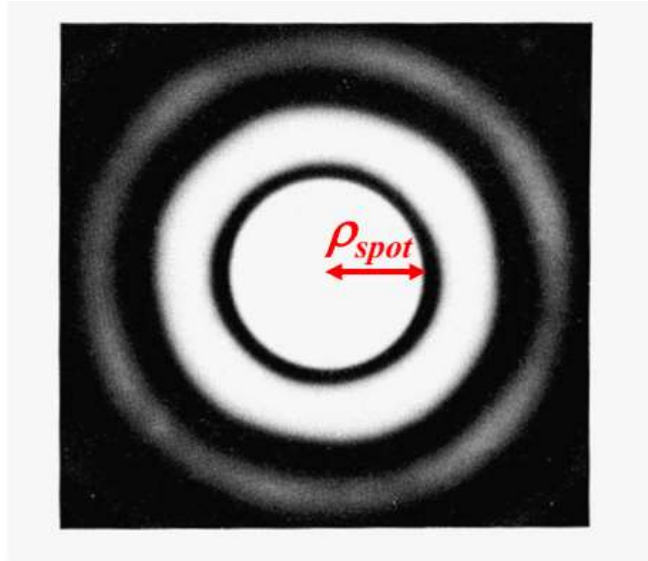
A maior parte da potencia está no pico central



$$\frac{\int_0^{3.8317} \rho d\rho \left[\frac{J_1(\rho)}{\rho} \right]^2}{\int_0^{\infty} \rho d\rho \left[\frac{J_1(\rho)}{\rho} \right]^2} \approx 0,838$$

Cerca de 83,8% da potência total está dentro no pico central

Tamanho do spot central



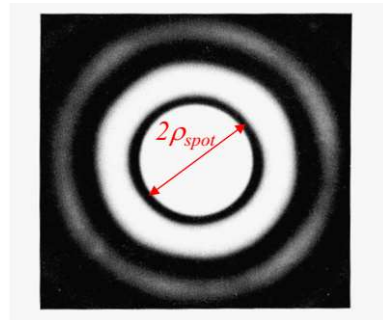
Definir o tamanho do spot centro como a distância radial do centro até o primeiro mínimo

$$\mathcal{I}(\rho, z) = \mathcal{I}_0 \left(\frac{\pi D^2}{4\lambda z} \right)^2 \left[2 \frac{J_1(kD\rho/2z)}{(kD\rho/2z)} \right]^2$$

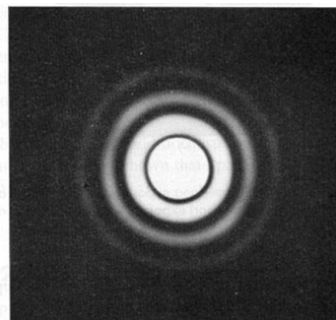
$$J_1(\xi) = 0 \quad \xi \approx 3.83$$

$$\frac{kD\rho_{spot}}{2z} = \frac{\pi D\rho_{spot}}{\lambda z} \approx 3.83$$

$$\rho_{spot} \approx \frac{3.83}{\pi} \frac{\lambda z}{D} = 1.22 \frac{\lambda z}{D}$$



Abertura menor



Abertura maior

$$\rho_{spot} \sim \frac{\lambda z}{D}$$

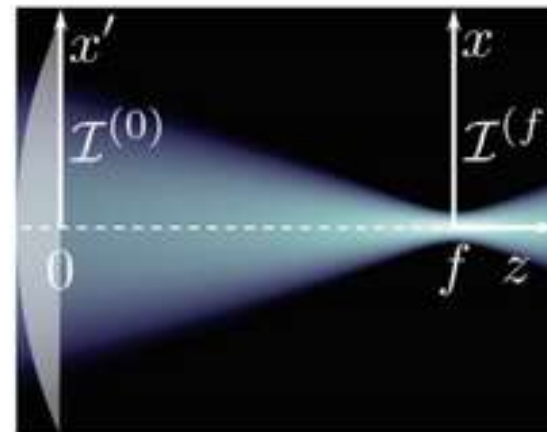
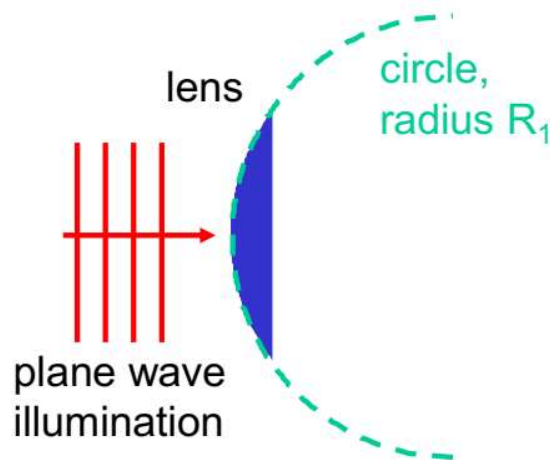
Característica de difração

Simetria circular - Difração Fraunhofer

Integral da difração de Fresnel em coordenados polares

$$\mathcal{E}(\rho, \theta, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ik\bar{r}_p} \int_0^{2\pi} d\theta' \int_0^{R_A} \rho' d\rho' \exp \left[-ik \frac{\rho\rho'}{z} \cos(\theta - \theta') + ik \frac{\rho'^2}{2z} \right]$$

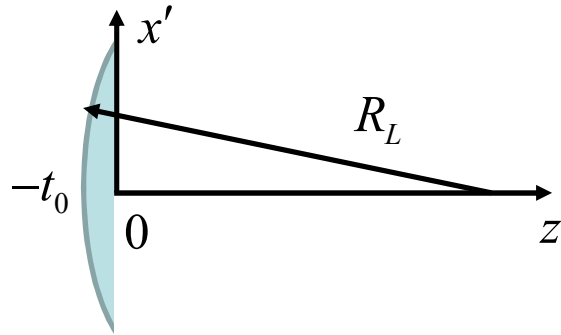
Caso II : Onda plana incidente numa lente fina, observação no plano focal duma lente



Qual a função da abertura duma lente?

$$f_{lente}(\rho') = ?$$

Função da abertura duma lente plana-convexa



A espessura da lente
nos coordenados (x', y')

$$t(x', y') = \sqrt{R_L^2 - (x'^2 + y'^2)} - R_L + t_0$$

$$\approx R_L - \rho'^2 / 2R_L - R_L + t_0 = t_0 - \rho'^2 / 2R_L$$

Ao atravessar da lente a fase que uma onda plana adquire é

$$\mathcal{E}(x', y') = \mathcal{E}_0 e^{ik(t_0 - t)} e^{ikn_l t} = \mathcal{E}_0 e^{ikt_0} e^{ik(n_l - 1)t}$$

$$\approx \mathcal{E}_0 e^{ikt_0} \exp\left[-ik(n_l - 1)\rho'^2 / 2R_L\right]$$

$$= \mathcal{E}_0 e^{ikt_0} \exp\left[-ik\rho'^2 / 2f\right]$$

Lente delgada plano-convexa

$$\frac{1}{f} = \frac{(n_l - 1)}{R_L}$$

Desprezando a fase global

$$f(\rho') = \begin{cases} \exp\left[-ik\rho'^2 / 2f\right] & \rho' < D_L / 2 \\ 0 & \rho' > D_L / 2 \end{cases}$$

No plano focal duma lente

$$\mathcal{E}(\rho, \theta, z) = \frac{\mathcal{E}_0}{i\lambda z} e^{ik\bar{r}_p} \int_0^{2\pi} d\theta' \int_0^{D_L/2} f(\rho') \rho' d\rho' \exp \left[-ik \frac{\rho\rho'}{z} \cos(\theta - \theta') + ik \frac{\rho'^2}{2z} \right]$$

$$f(\rho') = \exp \left[-ik \frac{\rho'^2}{2f} \right]$$

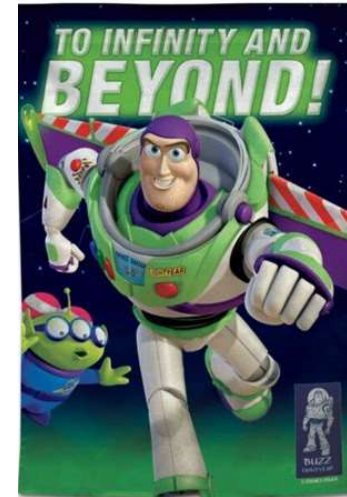
No plano $z = f$ as fases proporcionais $k\rho'^2 / 2f$ cortam

Efetivamente a lente transporta o limite $z \rightarrow \infty$ ao plano focal

$$\mathcal{E}_{lente}(\rho, \theta, z = f) = \frac{\mathcal{E}_0}{i\lambda f} e^{ikz - ik\rho^2/2f} \int_0^{2\pi} d\theta' \int_0^{D_L/2} \rho' d\rho' e^{[-ik\rho\rho' \cos(\theta - \theta')/f]}$$

Igual ao caso I: Limite Fraunhofer $N_F = \frac{D^2}{4z\lambda} \ll 1$

$$\mathcal{I}_{lente}(\rho, z = f) = \mathcal{I}_0 \left(\frac{\pi D^2}{4\lambda f} \right)^2 \left[2 \frac{J_1(kD\rho/2f)}{(kD\rho/2f)} \right]^2$$



Ótica Fourier

Logo o plano focal duma lente é o plano Fourier (Limite Fraunhofer)

$$\mathcal{E}_{lente}(\rho, \theta, z = f) = \frac{\mathcal{E}_0}{i\lambda f} e^{ikz - ik\rho^2/2f} \int_0^{2\pi} d\theta' \int_0^{D_L/2} \rho' d\rho' e^{[-ik\rho\rho'\cos(\theta-\theta')/f]}$$

$$x = \rho \cos \theta$$

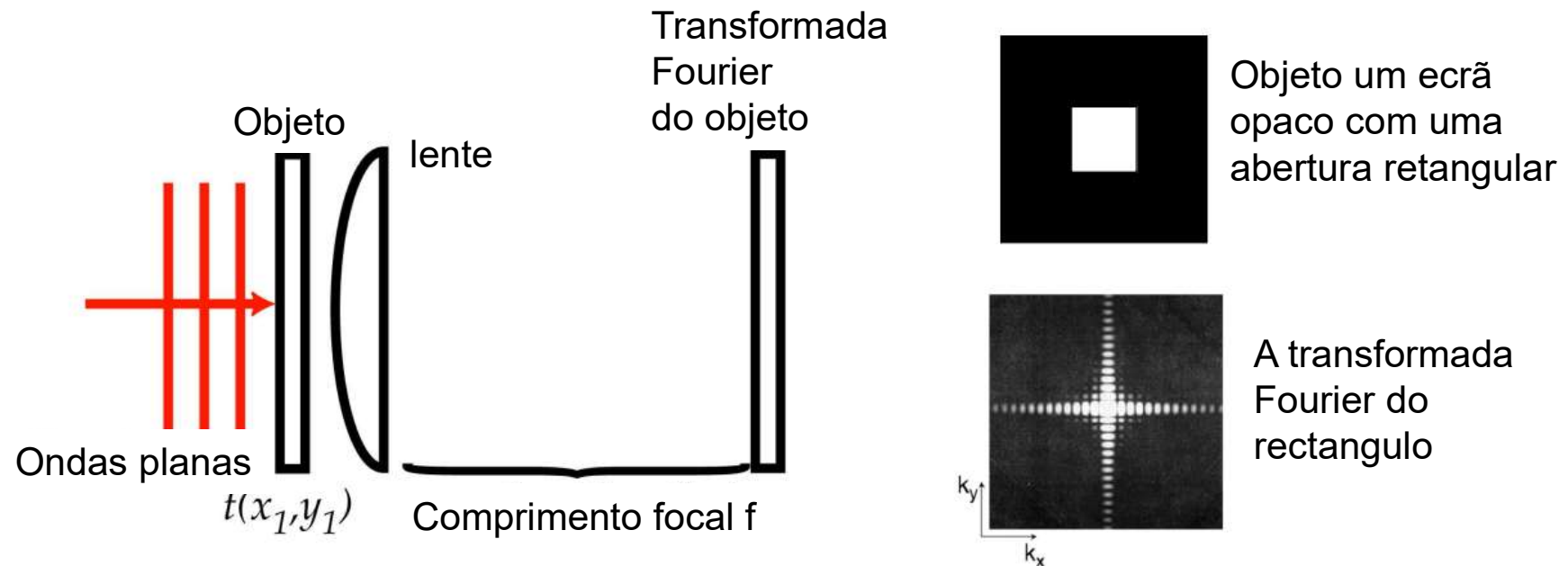
$$y = \rho \sin \theta$$

$$\mathcal{E}_{lente}(x, y, z = f) = \frac{\mathcal{E}_0}{i\lambda f} e^{ikz - ik(x^2+y^2)/2f} \iint_{lente} dx' dy' e^{-i2\pi(xx'+yy')/\lambda f}$$

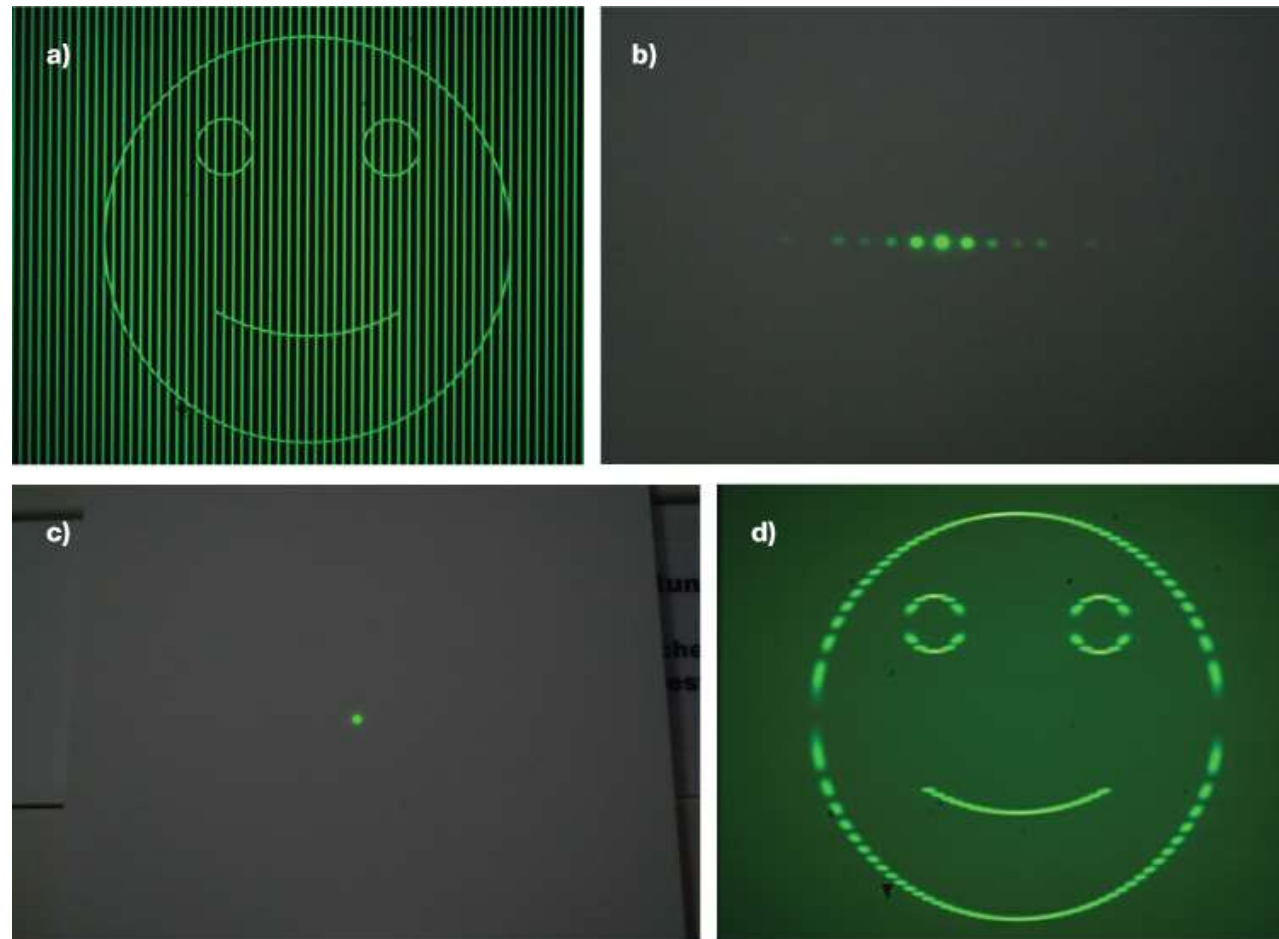
Frequências espaciais

$$k_x = 2\pi x / \lambda f$$

$$k_y = 2\pi y / \lambda f$$



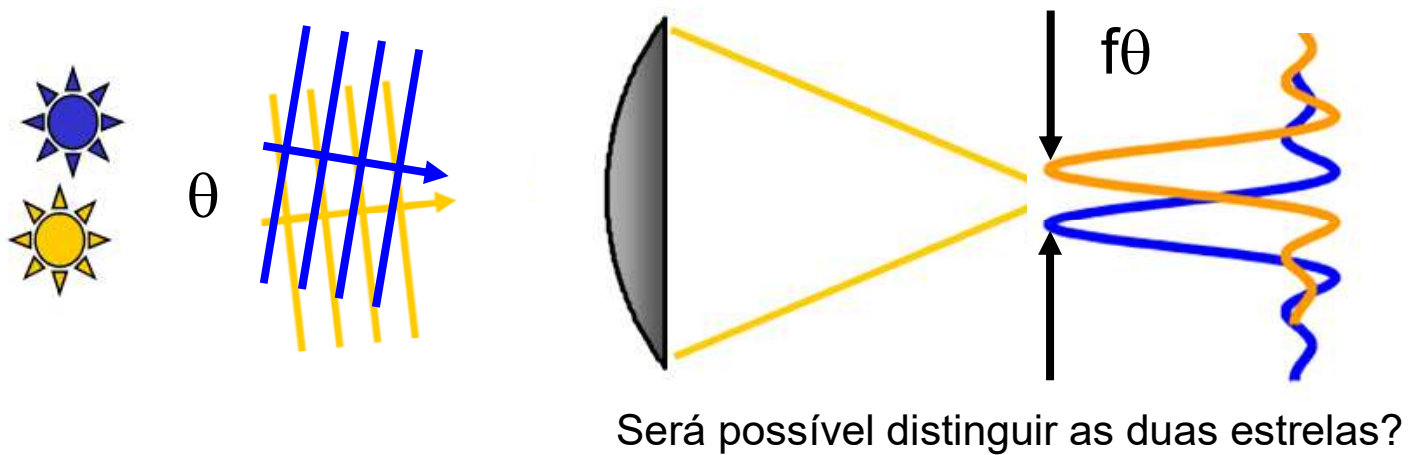
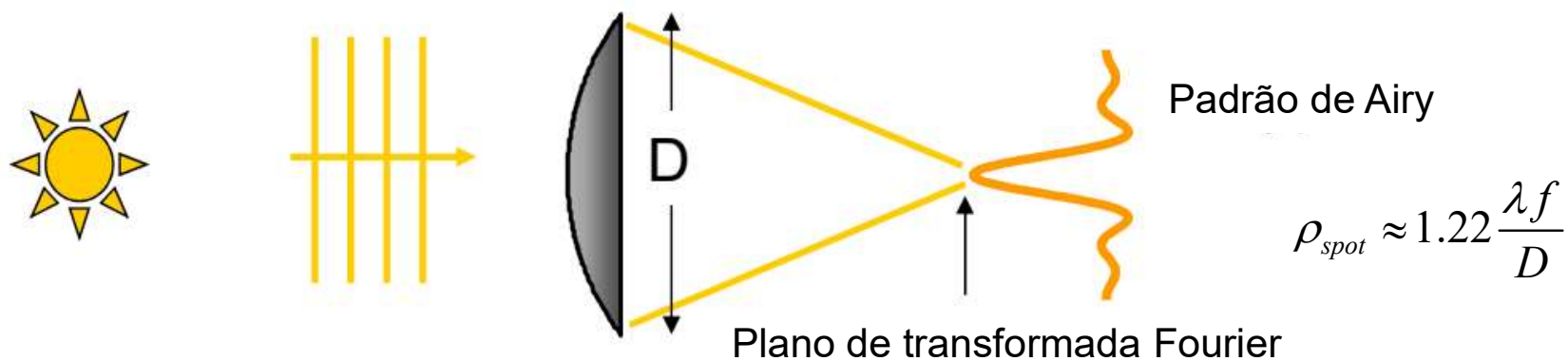
Exemplo de Processamento Ótico



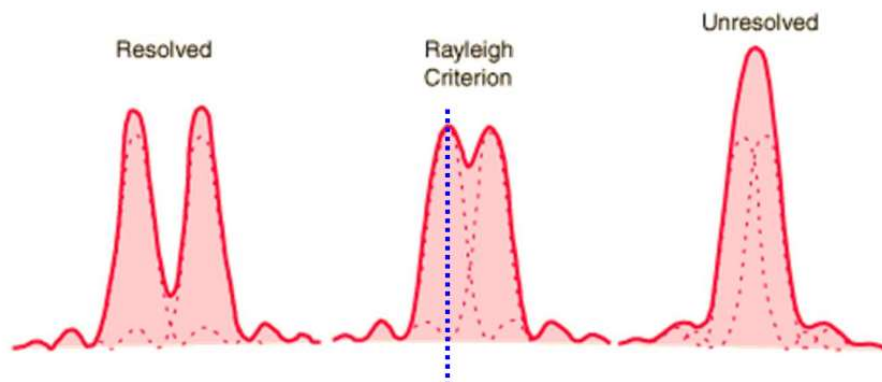
A smiley face behind bars (a) generates a Fourier pattern of dots (b). When a slit is used to block all but the zeroth order of the pattern (c), the bars disappear and the smiley is free (d).

Limite da difração

Imagine que utilize um telescópio para observar uma estrela distante



Critério de Rayleigh



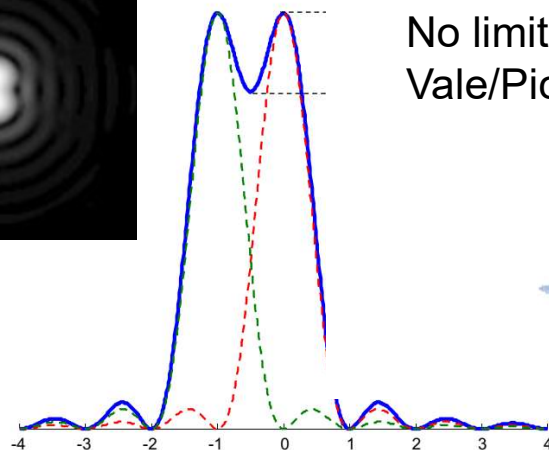
$$f \theta_{\min} = \rho_{spot} \approx 1.22 \frac{\lambda f}{D}$$

$$\theta_{\min} \approx 1.22 \frac{\lambda}{D}$$

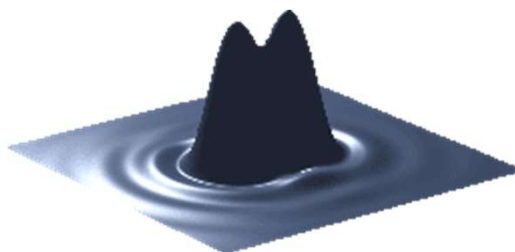


John William Strutt,
3rd Baron Rayleigh
1842 - 1919

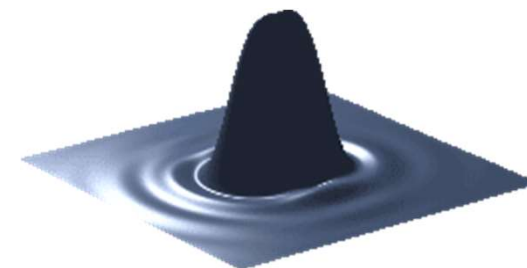
Limite de resolução: quando o máximo de uma padrão é sobreposto no primeiro mínimo da outra padrão. (É algo arbitrário)



No limite de Rayleigh a razão
Vale/Pico é ~ 73%

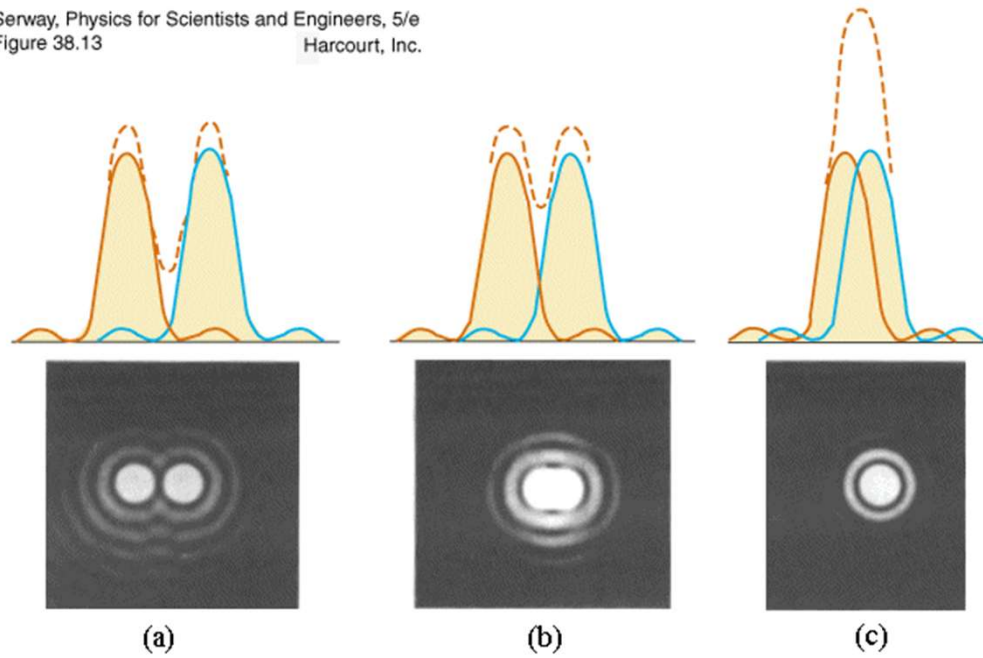


No limite



Não resolvidos

Serway, Physics for Scientists and Engineers, 5/e
Figure 38.13
Harcourt, Inc.



$$\theta_{\min} \approx 1.22 \frac{\lambda}{D}$$

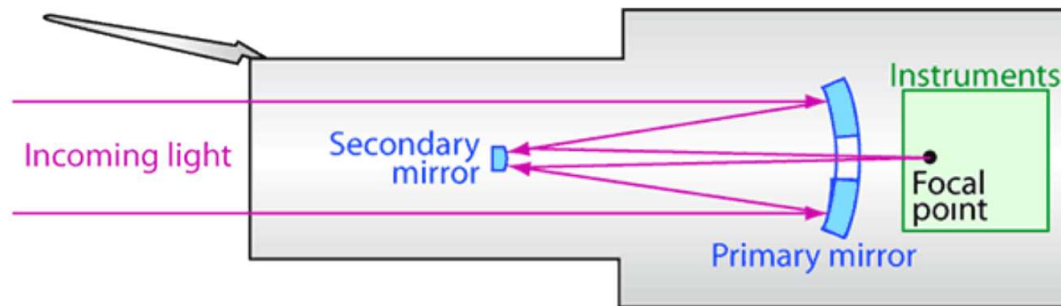


Telescópio de Hubble



Espelho principal tem um diâmetro $D = 2,4\text{m}$

Luz @ $\lambda = 500\text{ nm}$ luz verde



$$\begin{aligned}\theta_{\min} &\approx 1.22 \frac{\lambda}{D} \\ &\approx 1.22 \frac{5 \times 10^{-7} \text{ m}}{2.4 \text{ m}} \\ &\approx 2.5 \times 10^{-7} \text{ radianos} \\ &0.05 \text{ arco segundos}\end{aligned}$$

Na prática o limite da resolução é cerca de 0.1 arco segundos devida aberração esférica
~2x o limite de difração



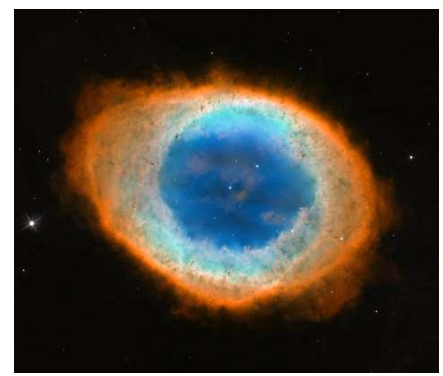
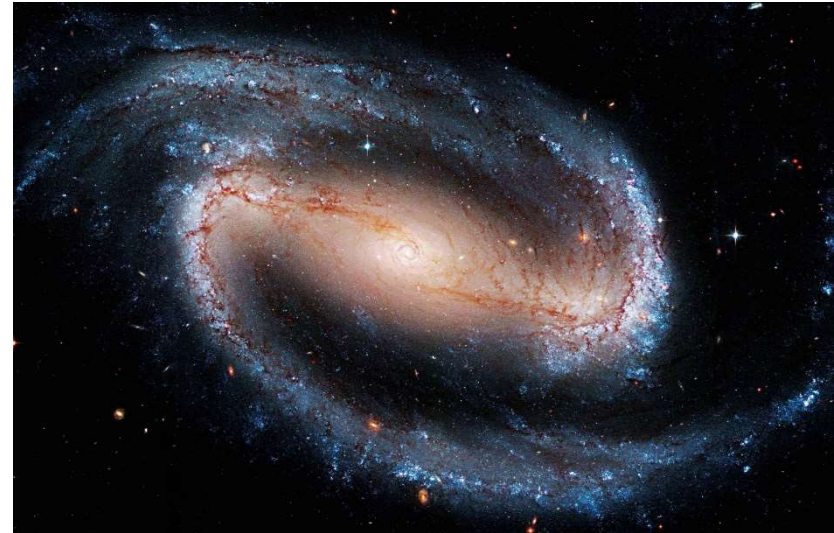
This photograph shows the Hubble Space Telescope's primary mirror being ground at the *Perkin-Elmer Corporation's* large optics fabrication facility in 1979, more than a decade before its very small but very significant flaw was discovered.

Antes

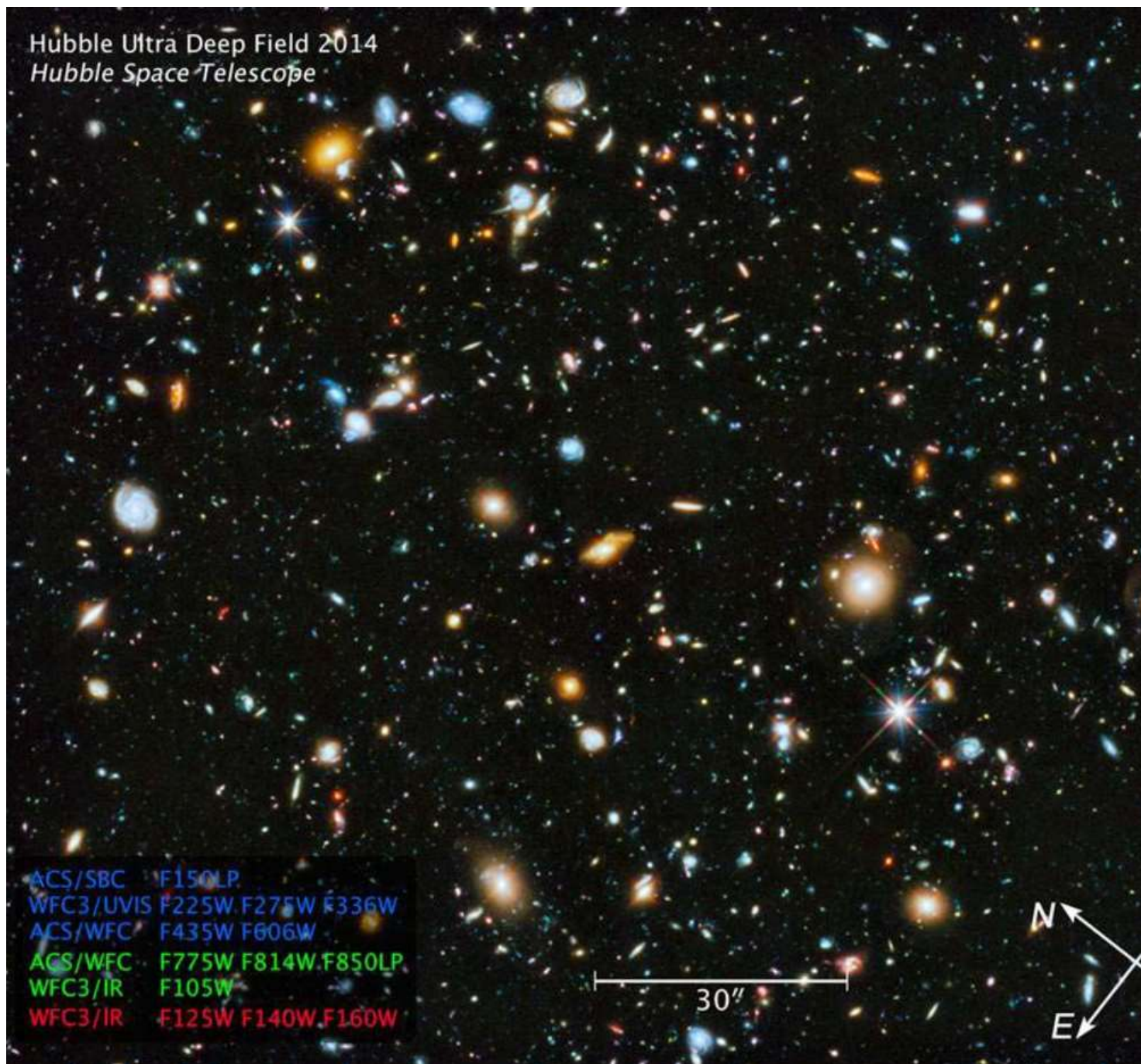


Depois a correção





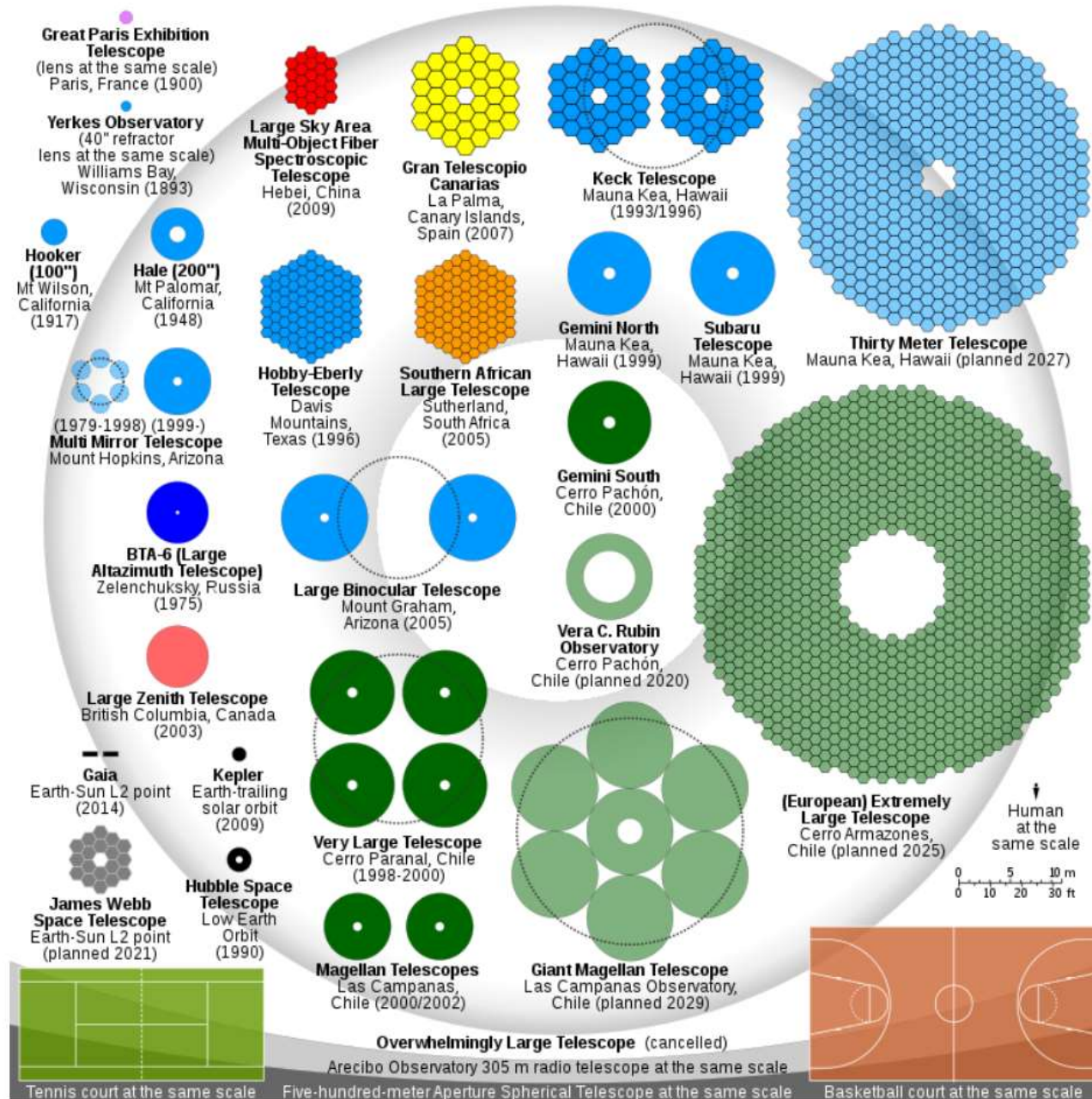
Hubble Ultra Deep Field 2014
Hubble Space Telescope



ACS/SBC F150LP
WFC3/UVIS F225W F275W F336W
ACS/WFC F435W F606W
ACS/WFC F775W F814W F850LP
WFC3/IR F105W
WFC3/IR F125W F140W F160W

30"

N
E



https://en.wikipedia.org/wiki/List_of_largest_optical_reflecting_telescopes