The Drude Theory of Metals (1900)

1. Basic assumptions:

a) A collision is the scattering of an electron by an ion core

b) Between collisions, electrons do not interact with each other or with ions or anything else.

c) Collisions are instantaneous events

d) The scattering rate (an electron suffers a collision with a probability per unit time) is constant : $\frac{1}{\tau}$

e) Thermal equilibrium is achieved only through collisions.

(in 1900, electrons and ion cores are classical objects)

Let z be the number of free electrons per atom, ρ the density of the metal, and A its atomic number. Then:

$A \equiv$ mass of one mole (in grams)

$z \cdot \mathcal{N} \equiv$ number of "free" electrons in one mole

Volume per mole : $\frac{A}{\rho}$ (in grams/cm^3, ρ)

Hence

$$\frac{N}{V} = n = z \mathcal{N} \frac{\rho}{A} \sim z \cdot 6{,}022 \times 10^{23} \cdot \left(\frac{\rho}{A}\right) \sim 10^{22} \frac{\text{electron}}{\text{cm}^3}$$

The volume per electron $\left(\frac{1}{n}\right) = \frac{4}{3}\pi r_s^3 \rightsquigarrow 1\ \mathring{A} \sim r_s \sim 2-3\ a_0$

These densities are $\sim 10^3$ greater than those of a classical gas !

2. DC-electrical conductivity

$t \to t + dt$, an electron is acted upon by a force $\vec{f}(t)$

Its momentum changes $\delta \vec{p} = \vec{f}(t)\, \delta t + \Theta(\delta t^2)$ if it does not collide.

$$\vec{p}(t+\delta t) = \left(1 - \frac{\delta t}{\tau}\right) \left[\vec{p}(t) + \delta \vec{p}(t)\right]$$

$$= \left(1 - \frac{\delta t}{\tau}\right) \left[\vec{p}(t) + \vec{f}(t)\, \delta t + \Theta(\delta t^2)\right]$$

$\left(1 - \frac{\delta t}{\tau}\right) \equiv$ probability of surviving without colliding

The electrons that collide between t and $t + \delta t$ are a fraction $\frac{\delta t}{\tau}$ of the number of electrons. Since they have immediately after the collision zero momentum (on average), they can only acquire a momentum $\propto \vec{f}(t)\, \delta t$. Therefore, their contribution to the total momentum will be $\sim (\delta t)^2$, and therefore negligible.

$$\vec{p}(t+\delta t) - \vec{p}(t) \simeq -\frac{\delta t}{\tau} \vec{p}(t) + \vec{f}(t)\, \delta t + \dots$$

$$\frac{d\vec{p}}{dt} = -\frac{1}{\tau} \vec{p}(t) + \vec{f}(t)$$

Collisions produce a frictional damping term.

Electric force $\vec{f} = -e\vec{E}$; steady state :

$$\frac{1}{\tau} p(t) = -e\vec{E} \quad \to \quad \vec{v} = -\frac{e\vec{E}\tau}{m}$$

$$\vec{J} = -ne\vec{v} = \frac{ne^2\tau}{m} \vec{E} \implies \boxed{\sigma = \frac{ne^2\tau}{m}}$$

Since $n = z N_A \frac{\rho}{A}$, then $\sigma \sim \frac{z N_A \rho}{A} \frac{e^2 \tau}{m} \implies \underline{\tau \sim 10^{-14}\, s}$!

What is the average distance a electron travels between collisions?

Using the equipartition theorem, $\frac{1}{2} m v^2 = \frac{3}{2} k_B T$ (assuming that the electron is a point-like particle), then:

$$v = \sqrt{\frac{3 k_B T}{m}} \sim 10^7 \ cm\ s^{-1} \ \text{at RT}$$

$$l = \sqrt{\frac{3 k_B T}{m}} \, \tau \sim 1 - 10 \ \mathring{A} \quad \text{(mean free path)}$$

3. **Thermal Conductivity:**

$$\vec{j}^q = -K \nabla T$$

Consider that $\nabla T = \frac{\partial T}{\partial x} \hat{x}$; $\vec{j}_q = -K \frac{dT}{dx}$ (one-dim. model)

$$\vec{j}_q = n v \, \mathcal{E}(x)$$ $\mathcal{E}(x) \equiv$ average energy of an electron at x.

Hence:

$$\vec{j}_{\vec{q}} = \frac{n}{2} v \left[\mathcal{E}(T(x - v\tau)) - \mathcal{E}(T(x + v\tau)) \right]$$

If the variation of T over the length $l = v\tau$ is small:

$$\vec{j}_{\vec{q}} = \frac{n}{2} v_x \left[\mathcal{E}(T(x)) - \left(\frac{\partial \mathcal{E}}{\partial T}\right)_x v\tau \left(\frac{\partial T}{\partial x}\right) - \mathcal{E}(T(x)) - \left(\frac{\partial \mathcal{E}}{\partial T}\right)_x \frac{\partial T}{\partial x} v\tau + \cdots \right]$$

$$= n v_x^2 \tau \left(\frac{\partial \mathcal{E}}{\partial T}\right) \cdot \left(-\frac{\partial T}{\partial x}\right)$$

Now: $\langle v_x^2 \rangle = \langle v_y^2 \rangle + \langle v_z^2 \rangle = \frac{1}{3} \langle v^2 \rangle$

$$n \frac{\partial \mathcal{E}}{\partial T} = \frac{N}{V} \frac{\partial \mathcal{E}}{\partial T} = \frac{dE}{dt} \frac{1}{V} = c_v$$

$$\vec{j}_q = \frac{1}{3} c_v \tau v^2 \left(-\frac{dT}{dx}\right) \Rightarrow \boxed{K = \frac{1}{3} c_v \tau v^2}$$

4. Wiedemann - Franz law

$$\frac{K}{\sigma} = \frac{\frac{1}{3} C_V m v^2}{n e^2} \qquad C_V = \frac{3}{2} n K_B$$

$$\frac{1}{2} m v^2 = \frac{3}{2} K_B T$$

$$\frac{K}{\sigma} = \frac{\frac{1}{3}\left(\frac{3}{2} n K_B\right) \cdot m \, \frac{3 K_B T}{v^2} \cdot v^2}{n e^2} = \frac{3}{2}\left(\frac{K_B}{e}\right)^2 T$$

Hence

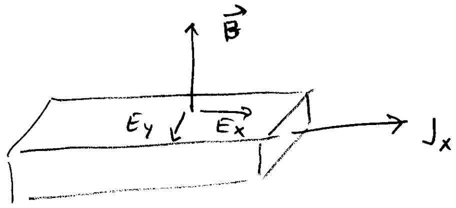
$$\frac{K}{\sigma T} = \frac{3}{2}\left(\frac{K_B}{e}\right)^2 \text{ is independent of the metal!}$$

$$\downarrow$$

$$\sim 1{,}11 \times 10^{-8} \frac{W \, \Omega}{K^2} \qquad \left(\tfrac{1}{2} \text{ of the exp. value}\right)$$

5. Hall effect and magnetoresistance

(Hall, 1879)



$$\frac{d\vec{p}}{dt} = -e\left[\vec{E} + \frac{\vec{p}}{m} \wedge \vec{B}\right] - \frac{\vec{p}}{\tau}$$

Steady state:

$$\begin{cases} 0 = -eE_x - \dfrac{eB}{m} p_y - \dfrac{p_x}{\tau} \\[2ex] 0 = -eE_y + \dfrac{eB}{m} p_x - \dfrac{p_y}{\tau} \end{cases}$$

$$\begin{cases} \frac{ne^2\tau}{m} E_x = \frac{eB}{m} \frac{ne\tau}{m} p_y - \frac{ne}{m} p_x \\ (\ldots) \end{cases}$$

$$\left(\omega_c = \frac{eB}{m} \right)$$

$$\begin{cases} \sigma_0 E_x = \omega_c \dot{j}_y \tau + \dot{j}_x \\ \sigma_0 E_y = -\omega_c \tau \dot{j}_x + \dot{j}_y \end{cases}$$

$j_y = 0 \Rightarrow \quad \dot{j}_x = \sigma_0 E_x \qquad$ (<u>No magnetoresistance</u>)

$$E_y = -\frac{\omega_c \tau}{\sigma_0} \dot{j}_x = -\frac{eB}{m} \frac{m}{ne^2\tau} \cdot \tau \, \dot{j}_x = -\frac{1}{ne} B$$

$$R_H = -\frac{1}{ne} \qquad \left[E_y = R_H B \right]$$

<u>The Hall coefficient depends only on the density of carriers!</u>

Remark: Notice that $\omega_c \tau$ is a dimensionless quantity that measures the strength of a magnetic field.

Remark: Lord Kelvin established around 1880 that the resistance does depend on the magnetic field. But this cannot be accounted for by the present model.

In the case of an electromagnetic wave, there is, in addition a magnetic force $-e(\vec{v} \wedge \vec{B})$. This force can be ignored provided that the mean electronic drift velocity is small (which is indeed the case: $v_{drif} \sim 0{,}1 \text{ cm s}^{-1}$ check this)

Consider the Maxwell's equations:

$$\nabla \cdot \vec{E} = 0 = \nabla \cdot \vec{B} \qquad\qquad \vec{B} = \mu \vec{H}$$

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \wedge \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \wedge \nabla \wedge \vec{E} = -\nabla^2 \vec{E} = -\frac{\partial}{\partial t}\left(\mu \nabla \wedge \vec{H}\right) = -\frac{\partial}{\partial t}\left(\mu \vec{J} + \mu \frac{\partial \vec{D}}{\partial t}\right)$$

$$\vec{J} = \sigma(\omega) \vec{E} \qquad\qquad = -\sigma(\omega)\mu \frac{\partial \vec{E}}{\partial t} - \mu \varepsilon \frac{\partial^2 E}{\partial t^2}$$

$$\frac{\partial \vec{E}}{\partial t} = -i\omega \vec{E}(\omega) \qquad \frac{\partial^2 \vec{E}}{\partial t^2} = -\omega^2 \vec{E}(\omega) \qquad = +i\omega\, \sigma(\omega)\mu E(\omega) + \mu_0 \varepsilon \omega^2 E(\omega)$$

$$= \omega^2 \mu_0 \underbrace{\left[i \frac{\sigma(\omega)}{\omega} + \varepsilon \right]}_{\tilde{\varepsilon}} E(\omega)$$

$$\boxed{\tilde{\varepsilon} = \varepsilon + i \frac{\sigma(\omega)}{\omega}}$$

The dielectric function becomes complex

$$\tilde{\varepsilon} = \varepsilon + i \frac{\sigma_0}{\omega(1 - i\omega\tau)}$$

Real number.

- If $\omega\tau \gg 1 \Rightarrow \tilde{\varepsilon} = \varepsilon + i \dfrac{\sigma_0}{-i\omega^2\tau} = \varepsilon - \dfrac{\frac{ne^2}{m}}{\omega^2} = \varepsilon - \dfrac{\Omega_p^2}{\omega^2}$

$$\Omega_p = \frac{ne^2}{m} \quad \text{(Plasma frequency)}$$

6. Electronic specific heat

As we have already ~~stated~~ seen, $C_V = \frac{3}{2} n K_B$ = const.

(which is not true)

7. Ac Electric conductivity of a metal:

$$\frac{d\vec{p}}{dt} = -\frac{\vec{p}}{\tau} - e\vec{E} \qquad (*)$$

$$\vec{E}(t) = \mathbb{R}\left[\vec{E}(\omega)\, e^{-i\omega t}\right]$$

Steady-state solution: $\vec{p} = \mathbb{R}\left[\vec{p}(\omega)\, e^{-i\omega t}\right]$

Then $(*)$ becomes:

$$-i\omega\, \vec{p}(\omega) = -\frac{\vec{p}(\omega)}{\tau} - e\,\vec{E}(\omega)$$

$\vec{J} = -ne\dfrac{\vec{p}}{m}$

$$+i\omega \frac{m\vec{J}(\omega)}{ne} = \frac{m\,\vec{J}(\omega)}{ne\tau} - e\,\vec{E}(\omega)$$

$$\vec{J}(\omega)\, \frac{m}{ne\tau}\left(1 - i\omega\tau\right) = e\,\vec{E}(\omega)$$

$$\vec{j}(\omega) = \frac{ne^2\tau}{m} \frac{\vec{E}(\omega)}{(1 - i\omega\tau)} = \frac{\sigma_0}{1 - i\omega\tau}\vec{E}(\omega)$$

Notice that we have assumed the force (hence, the electric field) to be spatially homogeneous. This means that the electric field is assumed to be nearly constant over a mean free path.