Equações Diferenciais Parciais

A partial differential equation (**PDE**) is an equation that involves an unknown function and its partial derivatives.

Example :

$$\frac{\partial^2 \ u(x,t)}{\partial x^2} = \frac{\partial \ u(x,t)}{\partial t}$$

PDE involves two or more independent variable (in the example x and t are independent variables).

Order of the PDE = order of the Highest order derivative.

Equações Diferenciais Parciais

A second order linear PDE (2 - independent variables)

$A\, u_{xx} + B\, u_{xy} + C\, u_{yy} + D = 0,$

A, B, and C are functions of x and y

D is a function of $x, y, u, u_x,$ and u_y

is classified based on $(B^2 - 4AC)$ as follows:

$$\boxed{\begin{array}{ll} B^2 - 4AC < 0 & \text{Elliptic} \\ B^2 - 4AC = 0 & \text{Parabolic} \\ B^2 - 4AC > 0 & \text{Hyperbolic} \end{array}}$$

$$u_{xx} = \frac{\partial^2\ u(x,t)}{\partial x^2}$$

$$u_{xt} = \frac{\partial^2\ u(x,t)}{\partial x\ \partial t}$$

Equações Diferenciais Parciais. Classificação

A PDE is linear if it is linear in the unknown function and its derivatives

Example of linear PDE :

$$2\ u_{xx} + 1\ u_{xt} + 3\ u_{tt} + 4\ u_x + \cos(2t) = 0$$

$$2\ u_{xx} - 3\ u_t + 4\ u_x = 0$$

Examples of Nonlinear PDE

$$2\ u_{xx} + (u_{xt})^2 + 3\ u_{tt} = 0$$

$$\sqrt{u_{xx}} + 2\ u_{xt} + 3u_t = 0$$

$$2\ u_{xx} + 2\ u_{xt}\, u_t + 3u_t = 0$$

Exemplos de PDEs

Laplace Equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

$$A = 1, B = 0, C = 1 \Rightarrow B^2 - 4AC < 0$$

$\Rightarrow$ Laplace Equation *is Elliptic*

Heat Equation $\quad \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$

$$A = \alpha,\ B = 0,\ C = 0 \Rightarrow B^2 - 4AC = 0$$

$\Rightarrow$ Heat Equation *is Parabolic*

Wave Equation $\quad c^2 \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(x,t)}{\partial t^2} = 0$

$$A = c^2 > 0,\ B = 0,\ C = -1 \Rightarrow B^2 - 4AC > 0$$

$\Rightarrow$ Wave Equation *is Hyperbolic*

Condições fronteira para PDE

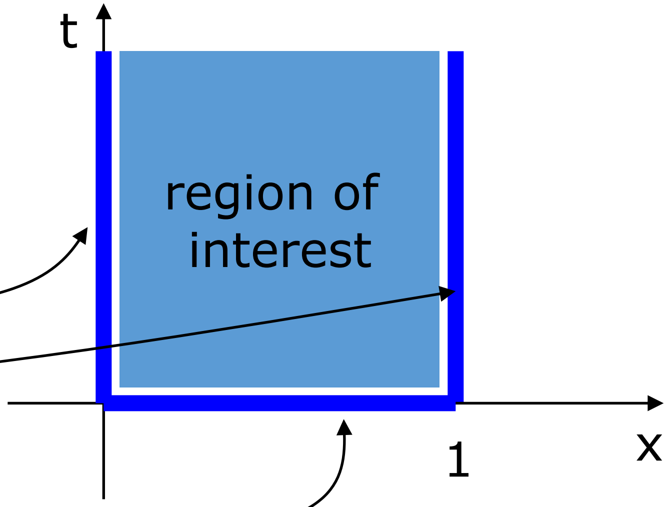
To uniquely specify a solution to the PDE, a set of boundary conditions are needed.

$$\text{Heat Equation}: \alpha \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial u(x,t)}{\partial t} = 0$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

$$u(x,0) = \sin(\pi x)$$



Several types boundary conditions can be specified along the boundaries:

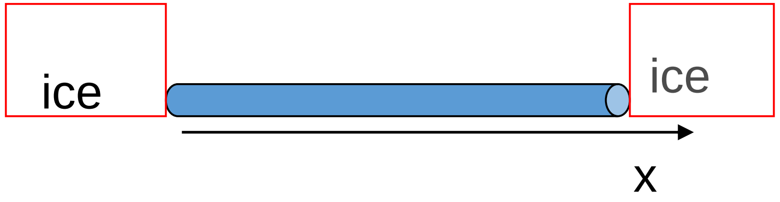
Dirichlet: u=c **Neumann:** $\frac{\partial u}{\partial \eta} = c$ **Mixed:** $\frac{\partial u}{\partial \eta} + b.u$=c

PDE Parabólicas.

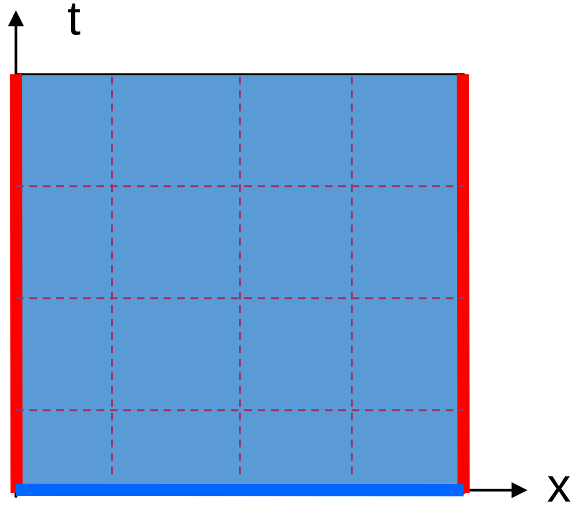
Heat Equation : $\dfrac{\partial\, T(x,t)}{\partial t} = \dfrac{\partial^2\, T(x,t)}{\partial x^2}$

$\boxed{T(0,t) = T(1,t) = 0}$

$\boxed{T(x,0) = \sin(\pi\, x)}$



- Divide the interval *x* into sub-intervals, each of width *h*
- Divide the interval *t* into sub-intervals, each of width *k*
- A grid of points is used for the finite difference solution
- $T_{i,j}$ represents $T(x_i, t_j)$
- Replace the derivates by finite-difference formulas



PDE Parabólicas.

Replace the derivatives by finite difference formulas

Central Difference Formula for $\dfrac{\partial^2 T}{\partial x^2}$:

$$\frac{\partial^2 T(x,t)}{\partial x^2} \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{(\Delta x)^2} = \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h^2}$$

Forward Difference Formula for $\dfrac{\partial T}{\partial t}$:

$$\frac{\partial T(x,t)}{\partial t} \approx \frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \frac{T_{i,j+1} - T_{i,j}}{k}$$

- **Two types of solution to the Heat Equation are possible:**

1. Explicit Method: Simple, Stability Problems.

2. Implicit Methods: Involves the solution of a Tridiagonal system of equations, Stable.

PDE Parabólicas. Método explicito.

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}$$

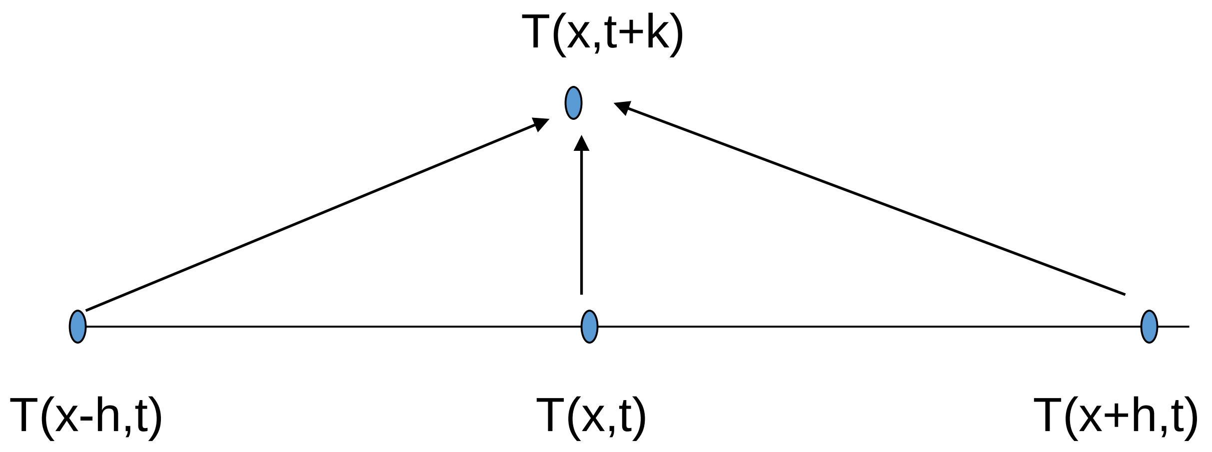
$$\frac{T(x,t+k) - T(x,t)}{k} = \alpha \frac{T(x-h,t) - 2T(x,t) + T(x+h,t)}{h^2}$$

Definindo:

$$\lambda = \frac{\alpha\, k}{h^2}$$

$$\boxed{T(x,t+k) = \lambda\, T(x-h,t) + (1-2\lambda)\, T(x,t) + \lambda\, T(x+h,t)}$$

Explicit method is first order in time (Δt) and second order in space (Δx^2).

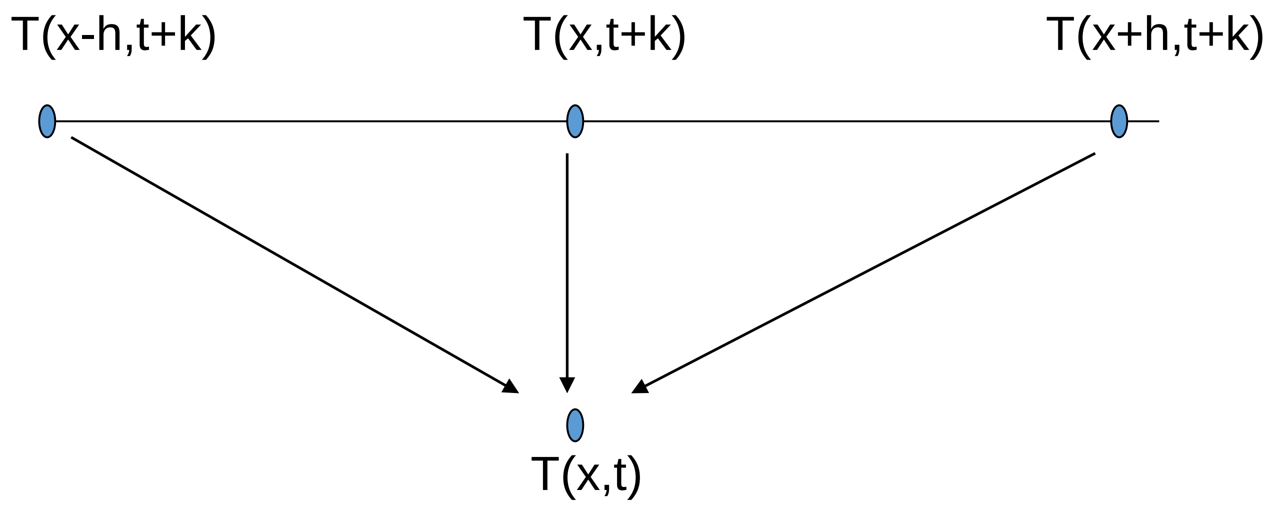


PDE Parabólicas. Método Implicito.

$$\frac{T(x,t+k)-T(x,t)}{k}=\frac{T(x-h,t+k)-2T(x,t+k)+T(x+h,t+k)}{h^2} \qquad Define \quad \lambda=\frac{k}{h^2}$$

$$\boxed{-\lambda T(x-h,t+k)+(1+2\lambda)T(x,t+k)-\lambda T(x+h,t+k)=T(x,t)}$$

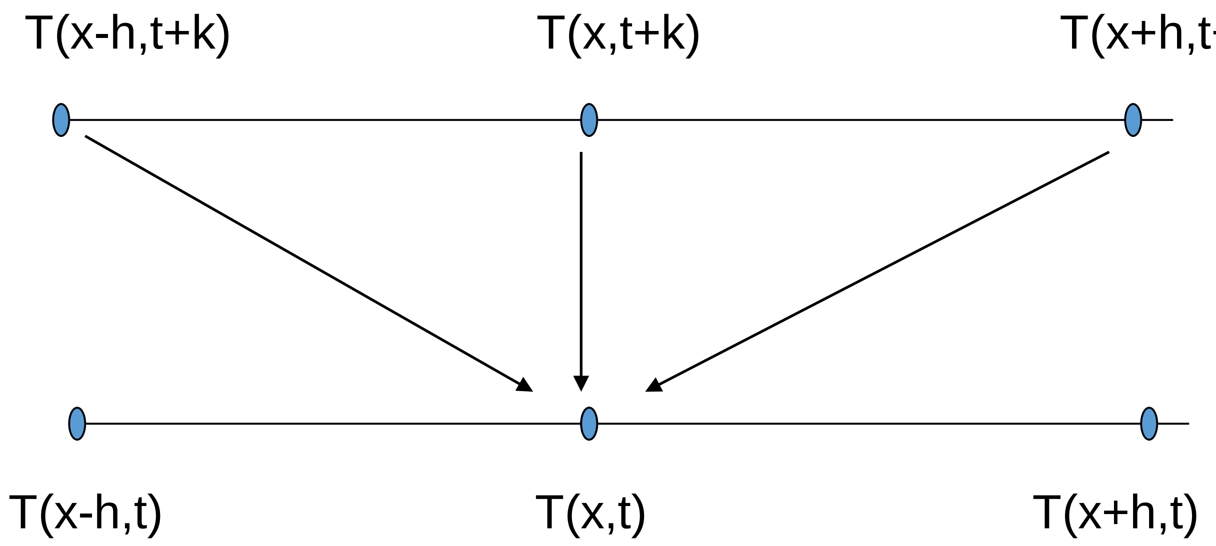
Implicit method is first order in time (Δt) and second order in space (Δx^2).



PDE Parabólicas. Método Crank-Nicholson.

$$-\lambda T(x-h,t+k)+2(1+\lambda)T(x,t+k)-\lambda T(x+h,t+k)=$$
$$\lambda T(x-h,t)+2(1-\lambda)T(x,t)+\lambda T(x+h,t)$$

Cranck – Nicholson is second order in time (Δt^2) and space (Δx^2).



PDE Parabólicas. Forma Matricial.

In vector notation, the explicit scheme can be written as

$$w^{n+1} = A\,w^n + b^n\,,$$

where $w^n = (w_1^n, \ldots, w_{N-1}^n)^T \in \mathbb{R}^{N-1}$ and

$$A = \begin{pmatrix} 1-2\lambda & \lambda & & \\ \lambda & 1-2\lambda & \lambda & \\ & \ddots & \ddots & \\ & & \lambda & 1-2\lambda \end{pmatrix} \in \mathbb{R}^{(N-1)\times(N-1)}, \qquad b^n = \begin{pmatrix} \lambda\,w_0^n \\ 0 \\ \vdots \\ 0 \\ \lambda\,w_N^n \end{pmatrix} \in \mathbb{R}^{N-1}\,.$$

For the implicit method we get

$$B\,w^{n+1} = w^n + b^{n+1}\,, \qquad \text{where} \quad B = \begin{pmatrix} 1+2\lambda & -\lambda & & \\ -\lambda & 1+2\lambda & -\lambda & \\ & \ddots & \ddots & \\ & & -\lambda & 1+2\lambda \end{pmatrix}.$$

Estabilidade numérica .

The *Fourier method* can be used to check if a scheme is stable.

Assume that a numerical scheme admits a solution of the form

$$T_j^n = a^{(n)}(\omega)\, e^{i\, j\, \omega\, \Delta x}\,,$$

Define

$$G(\omega) = \frac{a^{(n+1)}(\omega)}{a^{(n)}(\omega)}\,,$$

where $G(\omega)$ is an amplification factor, which governs the growth of the Fourier component $a(\omega)$.

The *von Neumann stability condition* is given by

$$|G(\omega)| \leq 1$$

for $0 \leq \omega\Delta x \leq \pi$.

Estabilidade numérica. Método Explicito.

For the explicit scheme we get on substituting (9) into (6) that

$$a^{(n+1)}(\omega)\, e^{i\,j\,\omega\,\Delta x} = \lambda\, a^{(n)}(\omega)\, e^{i\,(j+1)\,\omega\,\Delta x} + (1-2\,\lambda)\, a^{(n)}(\omega)\, e^{i\,j\,\omega\,\Delta x} + \lambda\, a^{(n)}(\omega)\, e^{i\,(j-1)\,\omega\,\Delta x}$$

$$\Rightarrow \quad G(\omega) = \frac{a^{(n+1)}(\omega)}{a^{(n)}(\omega)} = \lambda\, e^{i\,\omega\,\Delta x} + (1-2\,\lambda) + \lambda\, e^{-i\,\omega\,\Delta x}.$$

Type equation here.

The von Neumann stability condition then is

$$|G(\omega)| \leq 1 \quad \Longleftrightarrow \quad |\lambda\, e^{i\,\omega\,\Delta x} + (1-2\,\lambda) + \lambda\, e^{-i\,\omega\,\Delta x}| \leq 1$$

$$\Longleftrightarrow \quad |(1-2\,\lambda) + 2\,\lambda\, \cos(\omega\,\Delta x)| \leq 1 \qquad [\cos 2\,\alpha = 1 - 2\,\sin^2\alpha]$$

$$\Longleftrightarrow \quad 0 \leq \lambda \leq \frac{1}{2\,\sin^2\left(\frac{\omega\,\Delta x}{2}\right)} \qquad \text{for all } 0 \leq \omega\,\Delta x \leq \pi.$$

Explicit method is stable if and only if :

(Condição de Courant-Friedrichs-Lewy)

$$\boxed{\Delta t \leq \frac{(\Delta x)^2}{2\alpha}}$$

Estabilidade numérica. Métodos Implicitos.

Implicit method is unconditionally:

$$G(\omega) = \frac{1}{1 + 4\lambda \sin^2\left(\frac{\omega \Delta x}{2}\right)} \qquad |G(\omega)| \leq 1 \quad , \forall \lambda$$

Cranck-Nicholson method is unconditionally:

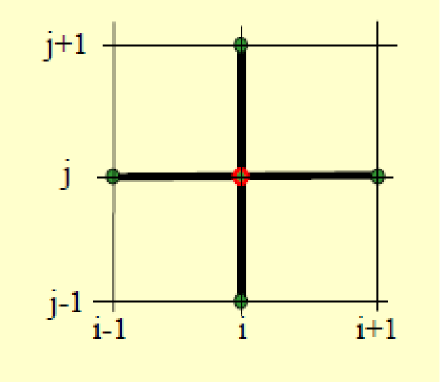
$$G(\omega) = \frac{1 - 2\lambda \sin^2\left(\frac{\omega \Delta x}{2}\right)}{1 + 2\lambda \sin^2\left(\frac{\omega \Delta x}{2}\right)} \qquad |G(\omega)| \leq 1 \quad , \forall \lambda$$

PDE Elípticas.

$$\frac{\partial^2 T(x,y)}{\partial x^2} + \frac{\partial^2 T(x,y)}{\partial y^2} = f(x,y)$$

T : steady state temperature at point (x, y)

$f(x,y)$: heat source (or heat sink)



A finite difference approximation is obtained at each grid point.

$$\frac{\partial^2 T(x,y)}{\partial x^2} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}, \qquad \frac{\partial^2 T(x,y)}{\partial y^2} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}$$

PDE Elípticas.

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = h^2 f_{i,j}$$

Assuming Δx= Δy=h:

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = h^2 f_{i,j}$$