Cálculo

Resolução da Ficha 3 — Outubro de 2008

Primitivas imediatas

1. Determine a primitiva das seguintes funções:

   (a) $a(x) = x^2 \cosh\left(x^3\right) + x.4^{x^2}$

$$
\begin{aligned}
P(x^2\cosh\left(x^3\right) + x.4^{x^2}) &= P(x^2\cosh\left(x^3\right)) + P(x.4^{x^2}) && P(f+g) = P(f)+P(g)\\
&= \frac{1}{3}P(\underbrace{3x^2}_{u'}\underbrace{\cosh(x^3)}_{\cosh u}) + \frac{1}{2}P(\underbrace{2x}_{u'}\underbrace{4^{x^2}}_{a^u}) && P(u'\cosh(u)) = \sinh(u)+C\\
&= \frac{1}{3}\sinh(x^3) + \frac{1}{2}\frac{4^{x^2}}{\ln 4} + C, C\in\mathbb{R} && P(a^u u') = \frac{a^u}{\ln a} + C
\end{aligned}
$$

   (b) $b(x) = \frac{\sinh(5x)}{\sqrt[3]{\cosh^4(5x)}}$

$$
\begin{aligned}
P\left(\frac{\sinh(5x)}{\sqrt[3]{\cosh^4(5x)}}\right) &= P(\sinh(5x)\cosh^{-4/3}(5x))\\
&= \frac{1}{5}P(\underbrace{5\sinh(5x)}_{u'}\underbrace{\cosh^{-4/3}(5x)}_{u^\alpha}) && P(u'u^\alpha) = \frac{u^{\alpha+1}}{\alpha+1} + C,\ \alpha\neq -1\\
&= \frac{1}{5}\frac{\cosh^{-1/3}(5x)}{-1/3} + C\\
&= -\frac{3}{5}\frac{1}{\sqrt[3]{\cosh(5x)}} + C, \quad C\in\mathbb{R}
\end{aligned}
$$

   (c) $c(x) = \frac{1}{\sqrt{4-9x^2}}$

$$
\begin{aligned}
P\left(\frac{1}{\sqrt{4-9x^2}}\right) &= P\left(\frac{1}{\sqrt{4(1-\frac{9}{4}x^2)}}\right)\\
&= P\left(\frac{1}{2\sqrt{1-(\frac{3}{2}x)^2}}\right)\\
&= \frac{2}{3}\times\frac{1}{2}P\left(\frac{\overbrace{3/2}^{u'}}{\sqrt{1-\left(\underbrace{\frac{3}{2}x}_{u}\right)^2}}\right) && P\left(\frac{u'}{\sqrt{1-u^2}}\right) = \arcsin(u)+C\\
&= \frac{1}{3}\arcsin\left(\frac{3}{2}x\right) + C, \quad C\in\mathbb{R}
\end{aligned}
$$

(d) $d(x) = \frac{(\ln(x)+e)^4}{x}$

$$\begin{aligned} P\left(\frac{(\ln(x)+e)^4}{x}\right) &= P\left(\underbrace{\frac{1}{x}}_{u'} \underbrace{(\ln(x)+e)^4}_{u^\alpha}\right) \\ &= \frac{(\ln(x)+e)^5}{5} + C, \quad C \in \mathbb{R} \end{aligned}$$

(e) $e(x) = \tan(x)$

$$\begin{aligned} P(\tan x) &= P\left(\frac{\sin(x)}{\cos(x)}\right) \\ &= -P\left(\frac{\overbrace{-\sin(x)}^{u'}}{\underbrace{\cos(x)}_{u}}\right) \qquad\qquad P\left(\frac{u'}{u}\right) = \ln|u| + C \\ &= \ln|\cos(x)| + C, \quad C \in \mathbb{R} \end{aligned}$$

(f) $f(x) = \frac{5x}{4+4x^2}$

$$\begin{aligned} P\left(\frac{5x}{4+4x^2}\right) &= 5P\left(\frac{x}{4(1+x^2)}\right) \\ &= \frac{5}{4}P\left(\frac{x}{1+x^2}\right) \\ &= \frac{1}{2} \times \frac{5}{4}P\left(\frac{\overbrace{2x}^{u'}}{\underbrace{1+x^2}_{u}}\right) \\ &= \frac{5}{8}\ln(1+x^2) + C, \quad C \in \mathbb{R} \end{aligned}$$

(g) $g(x) = \frac{3x}{\sqrt{1+5x^2}}$

$$\begin{aligned} P\left(\frac{3x}{\sqrt{1+5x^2}}\right) &= 3P\left(x(1+5x^2)^{-1/2}\right) \\ &= \frac{3}{10}P\left(\underbrace{10x}_{u'} \underbrace{(1+5x^2)^{-1/2}}_{u^\alpha}\right) \\ &= \frac{3}{10}\frac{(1+5x^2)^{1/2}}{1/2} + C \\ &= \frac{3}{5}\sqrt{1+5x^2} + C, \quad C \in \mathbb{R} \end{aligned}$$

(h) $h(x)=\frac{3}{\sqrt{4-3x^2}}$

$$\begin{aligned}
P\left(\frac{3}{\sqrt{4-3x^2}}\right) &= 3P\left(\frac{1}{\sqrt{4(1-\frac{3}{4}x^2)}}\right)\\
&= \frac{3}{2}P\left(\frac{1}{\sqrt{1-(\frac{\sqrt{3}}{2}x)^2}}\right)\\
&= \frac{2}{\sqrt{3}}\times\frac{3}{2}P\left(\frac{\overbrace{\frac{\sqrt{3}}{2}}^{u'}}{\sqrt{1-\left(\underbrace{\frac{\sqrt{3}}{2}x}_{u}\right)^2}}\right)\\
&= \sqrt{3}\arcsin\left(\frac{\sqrt{3}}{2}x\right)+C,\quad C\in\mathbb{R}
\end{aligned}$$

(i) $i(x)=\frac{x+5}{\sqrt{1+x^2}}$

$$\begin{aligned}
P\left(\frac{x+5}{\sqrt{1+x^2}}\right) &= P\left(\frac{x}{\sqrt{1+x^2}}\right)+P\left(\frac{5}{\sqrt{1+x^2}}\right)\\
&= P\left(x(1+x^2)^{-1/2}\right)+5P\left(\frac{1}{\sqrt{1+x^2}}\right)\\
&= \frac{1}{2}P\left(\underbrace{2x}_{u'}\underbrace{(1+x^2)^{-1/2}}_{u^\alpha}\right)+5P\left(\frac{\overbrace{1}^{u'}}{\sqrt{1+\underbrace{x}_{u}{}^2}}\right)\quad P\left(\frac{u'}{\sqrt{1+u^2}}\right)=\arg\sinh(u)+C\\
&= \frac{1}{2}\frac{(1+x^2)^{1/2}}{1/2}+5\arg\sinh(x)+C\\
&= \sqrt{1+x^2}+5\arg\sinh(x)+C,\quad C\in\mathbb{R}
\end{aligned}$$

(j) $j(x) = \frac{2x-1}{x^2-2x+10}$

$$
\begin{aligned}
P\left(\frac{2x-1}{x^2-2x+10}\right) &= P\left(\frac{2x-2+1}{x^2-2x+10}\right) \\
&= P\left(\frac{\overbrace{2x-2}^{u'}}{\underbrace{x^2-2x+10}_{u}}\right) + P\left(\frac{1}{(x-1)^2+9)}\right) \\
&= \ln(x^2-2x+10) + P\left(\frac{1}{9\left(\frac{(x-1)^2}{9}+1\right)}\right) \\
&= \ln(x^2-2x+10) + \frac{1}{9}P\left(\frac{1}{\left(\frac{x-1}{3}\right)^2+1}\right) \\
&= \ln(x^2-2x+10) + 3\times\frac{1}{9}P\left(\frac{\overbrace{1/3}^{u'}}{\left(\underbrace{\frac{x-1}{3}}_{u}\right)^2+1}\right) \qquad P\left(\frac{u'}{1+u^2}\right) = \arctan(u)+C \\
&= \ln(x^2-2x+10) + \frac{1}{3}\arctan\left(\frac{x-1}{3}\right) + C, \quad C\in\mathbb{R}
\end{aligned}
$$

2. Determine a função f que verifica a condição

$$f'(x) = \frac{x}{(1+x^2)^2}$$

e tal que $f(0) = 2$.

$$
\begin{aligned}
P\left(\frac{x}{(1+x^2)^2}\right) &= P\left(x(1+x^2)^{-2}\right) \\
&= \frac{1}{2}P\left(2x(1+x^2)^{-2}\right) \\
&= \frac{1}{2}\frac{(1+x)^{-1}}{-1} + C \\
&= -\frac{1}{2+2x^2} + C, \quad C\in\mathbb{R}
\end{aligned}
$$

Então,

$$f(x) = -\frac{1}{2}\frac{1}{x^2+1} + C \wedge f(0) = 2$$

Determinamos C fazendo

$$
\begin{aligned}
f(0) = 2 &\Leftrightarrow -\frac{1}{2}\frac{1}{0^2+1} + C = 2 \\
&\Leftrightarrow -\frac{1}{2}\times 1 + C = 2 \\
&\Leftrightarrow C = \frac{5}{2}
\end{aligned}
$$

Logo,

$$f(x) = -\frac{1}{2}\frac{1}{x^2+1} + \frac{5}{2}$$