Física Quântica I / Mecânica Quântica (2021/22)

Folha de Problemas 4 (spin 1/2)

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— Soluções —

Problema 1 | Representação matricial das projeções de spin 1/2

Nota: Neste problema, assuma que nenhum operador tem espectro degenerado.

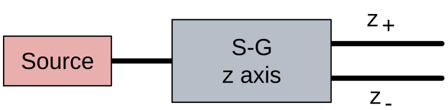
Nas aulas teóricas foi revelada, sem derivação, a representação matricial dos operadores $\hat{\mathrm{S}}_{x,y,z}$ na base própria de $\hat{\mathrm{S}}_z$. Neste problema vamos dar um passo atrás e, incrementalmente, *derivar* essas expressões através de resultados experimentais do tipo Stern-Gerlach (SG) conhecidos para uma partícula com spin $S = 1/2$. As respostas às questões seguintes requerem cálculos mínimos e simples, sendo principalmente necessário aplicar os postulados da mecânica quântica no contexto de cada um dos resultados descritos.

1. Antes de tudo, *mostre* que, se $|a_n\rangle$ for um auto-estado normalizado da observável genérica $\hat{\mathrm{A}}$ (com espectro não degenerado) associado ao autovalor a_n, então esse operador pode ser expresso como

$$\hat{\mathrm{A}} = \sum_{k=1}^{D} a_k \, |a_k\rangle\langle a_k|, \qquad \text{(decomposição espectral)}, \tag{1.1}$$

   onde D é a dimensão do espaço de estados/Hilbert em questão.

2. Comecemos por medir $\mathcal{S}_z$ através da experiência esquematizada abaixo. Esta experiência revela apenas dois resultados para $\mathcal{S}_z$: $\pm\frac{\hbar}{2}$. Revela também que a fração de partículas que emerge em cada um dos feixes defletidos z_+ e z_- é idêntica.



   Usando o resultado (1.1) derivado acima, *mostre* que podemos escrever

$$\hat{\mathrm{S}}_z = s_+ \, |+\rangle_{zz}\langle +| + s_- \, |-\rangle_{zz}\langle -|,$$

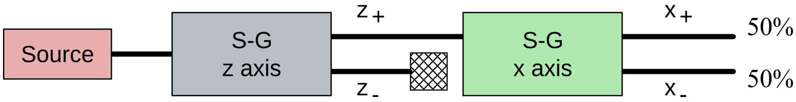
   onde $s_\pm$ são números relacionados com o resultado desta experiência. *Escreva* a matriz que representa $\hat{\mathrm{S}}_z$ na sua base própria $\{|+\rangle_z, \, |-\rangle_z\}$.

3. As matrizes que representam $\hat{\mathrm{S}}_x$ e $\hat{\mathrm{S}}_y$ na base $\{|+\rangle_z, \, |-\rangle_z\}$ são para já desconhecidas. No entanto, como o espaço de estados tem dimensão 2, terão de ser matrizes 2×2 com a forma

$$\hat{\mathrm{S}}_x \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \hat{\mathrm{S}}_y \mapsto \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}. \tag{1.2}$$

   Considerando que estas matrizes representam observáveis (quantidades físicas), que relações e propriedades deverão existir entre os elementos a, b, c, d? Use essas relações para *simplificar* a forma geral dada na eq. (1.2).

4. Combinando um seletor SG segundo z com um dispositivo SG orientado segundo x (figura abaixo) descobrimos que a fração de partículas detetadas à saída do último dispositivo SG em cada um dos feixes é igual: 50%. Essas partículas emergem do segundo SG no estado $|+\rangle_x$ ou $|-\rangle_x$.

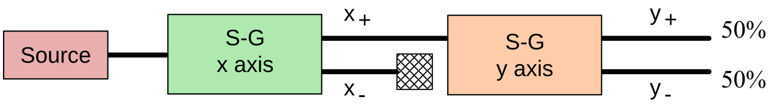


Com base neste resultado, qual é o valor esperado de $\hat{S}_x$ no estado $|+\rangle_z$?

5. Se na questão anterior tivéssemos utilizado um dispositivo SG orientado segundo a direção y (em vez de segundo x), as frações finais seriam igualmente 50% em cada feixe de saída. Tendo isso em conta, *mostre* que as matrizes (1.2) devem ter a forma

$$\hat{S}_x \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}, \qquad \hat{S}_y \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{pmatrix}, \qquad \alpha, \beta \in [0,\, 2\pi[. \tag{1.3}$$

6. Numa outra experiência, enviamos partículas selecionadas por um SG segundo x através de um dispositivo SG orientado segundo y. Descobrimos que a fração de partículas que emerge deste último nos estados $|+\rangle_y$ e $|-\rangle_y$ é igual.



Use esta informação para mostrar que as constantes α e β em (1.3) estão relacionadas segundo

$$\alpha - \beta = \pm\frac{\pi}{2}.$$

7. Escolhendo $\alpha = 0$ no resultado acima, *determine* o valor de β de modo a obter a seguinte relação de comutação

$$[\hat{S}_x,\, \hat{S}_y] = i\hbar\, \hat{S}_z.$$

8. Reunindo todos os resultados anteriores, *escreva* as matrizes que representam cada uma das projeções de spin $\hat{S}_{x,y,z}$ na base própria de $\hat{S}_z$.

Solution

1. The simplest way is to show what happens when the operator defined in this way acts on an arbitrary eigenstate $|a_m\rangle$. If the definition is correct, it should give us

$$\hat{A}|a_m\rangle = a_m|a_m\rangle$$

for all eigenstates $|a_m\rangle$. Using the definition given,

$$\hat{A}|a_m\rangle = \left(\sum_{k=1}^{D} a_k\, |a_k\rangle\langle a_k|\right)|a_m\rangle = \sum_{k=1}^{D} a_k\, |a_k\rangle\langle a_k|a_m\rangle = \sum_{k=1}^{D} a_k\, |a_k\rangle\delta_{km} = a_m|a_m\rangle,$$

where we used the fact that $\langle a_k|a_m\rangle = \delta_{km}$ for eigenstates associated with different eigenvalues of an Hermitian operator. Since this holds for the action of $\hat{A}$ in *any* eigenstate and, by assumption, the operator has no degenerate eigenvalues, it means that the definition (1.1) completely specifies the action of $\hat{A}$ on any vector of the state space.

2. The expression
$$\hat{S}_z = s_+\,|+\rangle_{zz}\langle+| + s_-\,|-\rangle_{zz}\langle-|$$
is simply the general result derived above applied to this spin projection. Hence, the numbers $s_\pm$ correspond to the eigenvalues of $\hat{S}_z$. Since we know from the quoted experiment that the outcome of measuring $\mathcal{S}_z$ is either $\frac{\hbar}{2}$ or $-\frac{\hbar}{2}$, the eigenvalues must be $s_\pm = \pm\frac{\hbar}{2}$ (postulate P3). Therefore, in this basis $\hat{S}_z$ has the matrix representation
$$\hat{S}_z \mapsto \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

3. The fact that $\hat{S}_{x,y}$ are observables means that they are *Hermitian* operators and, consequently, must be represented by Hermitian matrices. Therefore it must be true that
$$\hat{S}_x^\dagger = \hat{S}_x \quad \text{(Hermitian)} \qquad \Leftrightarrow \qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a^* & c^*\\ b^* & d^*\end{pmatrix},$$
or, more explicitly, we must have
$$a = a^*, \quad d = d^*, \quad b = c^*.$$
This can only be true if
$$a, d \in \mathbb{R}, \quad \text{and} \quad b = c^*.$$
Moreover, these operators represent spin projections. The first experiment shows that measuring one projection has always one of the two outcomes $\pm\frac{\hbar}{2}$. Since any direction of space must be equivalent, the outcome of the first experiment would have been the same if the SG apparatus was oriented along x or y (or any other direction), which means that the eigenvalues of any spin projection $\hat{S}_{x,y,z}$ are $\pm\frac{\hbar}{2}$. If we recall that the trace of a matrix is equal to the sum of its diagonal elements in *any* basis (and, hence, equal to the sum of its eigenvalues), it must be true that
$$\mathrm{Tr}\,\hat{S}_x = a + d = \frac{\hbar}{2} - \frac{\hbar}{2} = 0 \qquad \Rightarrow \qquad a = -d.$$
With this information, the matrix representing $\hat{S}_x$ must have the form
$$\hat{S}_x \mapsto \begin{pmatrix}a & b\\ b^* & -a\end{pmatrix}, \quad \text{with } a \in \mathbb{R}.$$
The situation for $\hat{S}_y$ is entirely analogous, yielding
$$\hat{S}_y \mapsto \begin{pmatrix}a' & b'\\ b'^* & -a'\end{pmatrix}, \quad \text{with } a' \in \mathbb{R}.$$

4. We must calculate ${}_z\langle +|\hat{S}_x|+\rangle_z$, but we don't know the matrix representation of $\hat{S}_z$ at this point. On the other hand, since the experiment prepares atoms in the state $|+\rangle_z$ and measures $\hat{S}_x$, we can use its results to directly compute the expectation value that is asked from the probability of obtaining either of the eigenvalues of $\hat{S}_x$:

$$ {}_z\langle +|\hat{S}_x|+\rangle_z = \tfrac{\hbar}{2}\,\mathcal{P}(x_+) - \tfrac{\hbar}{2}\,\mathcal{P}(x_-) = \frac{\hbar}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = 0, $$

because the experiment tells us that the probabilities are 0.5 for each outcome.

5. The results of the experiment in the previous question tell us that

$$ {}_z\langle +|\hat{S}_x|+\rangle_z = 0. $$

Since $|+\rangle_z$ is the first element of our basis $\{|+\rangle_z,\, |-\rangle_z\}$, this expectation value corresponds to the matrix element $[S_x]_{11}$. We saw above that the matrix representing $\hat{S}_x$ in this basis must have the form

$$ \hat{S}_x \mapsto \begin{pmatrix} a & b \\ b^* & -a \end{pmatrix}, \quad \text{with } a \in \mathbb{R}, $$

from where it follows that ${}_z\langle +|\hat{S}_x|+\rangle_z = a = 0$, and we can narrow down the matrix further to

$$ \hat{S}_x \mapsto \begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}. $$

A direct computation shows that the eigenvalues of this matrix are $\pm\,|b|$. But we know that the eigenvalues of $\hat{S}_x$ *must* be $\pm\hbar/2$ and, consequently, we must have

$$ |b| = \frac{\hbar}{2} \quad \Leftrightarrow \quad b = |b|\,e^{i\arg b} = \frac{\hbar}{2}\,e^{i\alpha}, \quad \text{where } 0 \le (\alpha = \arg b) < 2\pi. $$

In conclusion, the matrix S_x must have the form

$$ \hat{S}_x \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}, $$

where the phase α is the only thing left undetermined.

Since the question tells us that replacing the second SG box by one along y leads to the same fractions exiting as $|\pm\rangle_y$, the same analysis and consequences apply for S_y and its matrix representation is thus of the same form:

$$ \hat{S}_y \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{pmatrix}. $$

6. In this experiment the first selector prepares particles in the state $|+\rangle_x$. The subsequent measurement tells us that, in such a state, we have 50% probability of obtaining the positive eigenvalue of $\hat{S}_y$, and another 50% of obtaining the negative eigenvalue. Therefore, similarly to what we did in one of the questions above, this tells us that

$$ {}_x\langle +|\hat{S}_y|+\rangle_x = \tfrac{\hbar}{2}\,\mathcal{P}(y_+) - \tfrac{\hbar}{2}\,\mathcal{P}(y_-) = 0. $$

The state $|+\rangle_x$ is the eigenstate associated with the positive eigenvalue of $\hat{S}_x$, which we can determine by computing the corresponding eigenvector of the matrix we found in the previous question:

$$|+\rangle_x \mapsto \begin{pmatrix} u \\ v \end{pmatrix}: \qquad \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} u \\ v \end{pmatrix} \quad \Leftrightarrow \quad u = e^{i\alpha} v.$$

From here we conclude that this eigenstate is, after normalization, given by

$$|+\rangle_x \mapsto \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{-i\alpha} \end{pmatrix}.$$

It now becomes straightforward to compute the expectation value:

$${}_x\langle +|\hat{S}_y|+\rangle_x = \frac{\hbar}{2}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}e^{i\alpha} \end{pmatrix}\begin{pmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}e^{-i\alpha} \end{pmatrix} = \frac{\hbar}{2}\cos(\beta - \alpha).$$

Therefore, since the experiment is telling us that ${}_x\langle +|\hat{S}_y|+\rangle_x = 0$, this imposes the condition

$$\cos(\beta - \alpha) = 0 \quad \Leftrightarrow \quad \beta - \alpha = \pm\frac{\pi}{2}.$$

7. If we put $\alpha = 0$, as suggested, the matrices representing $\hat{S}_x$ and $\hat{S}_y$ obtained so far become

$$\hat{S}_x \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{S}_y \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{pmatrix}, \qquad \beta = \pm\tfrac{\pi}{2}.$$

So, the last thing left is to determine which sign should β have so that the commutation relation is verified. Replacing these matrices in the commutator,

$$[S_x, S_y] = \left(\frac{\hbar}{2}\right)^2\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{pmatrix} - \begin{pmatrix} 0 & e^{i\beta} \\ e^{-i\beta} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{i\hbar^2}{2}\begin{pmatrix} -\sin\beta & 0 \\ 0 & \sin\beta \end{pmatrix}.$$

In order to satisfy the commutation relation this must be equal to $i\hbar S_z$, i.e.,

$$\frac{i\hbar^2}{2}\begin{pmatrix} -\sin\beta & 0 \\ 0 & \sin\beta \end{pmatrix} = i\hbar S_z = i\frac{\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow \quad \sin\beta = -1 \quad \beta = -\frac{\pi}{2}.$$

8. With $\alpha = 0$ and $\beta = -\frac{\pi}{2}$, plus what we found in question 5 above, the matrix representations of the three spin components are as follows:

$$\hat{S}_x \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{S}_y \mapsto \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{S}_z \mapsto \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

in the eigenbasis of $\hat{S}_z$.

Problema 2

1. Mostre que o vetor de estado de uma partícula cujo spin aponta na direção definida pelo vetor unitário

$$\boldsymbol{n} = \sin\theta\cos\varphi\,\boldsymbol{u}_x + \sin\theta\sin\varphi\,\boldsymbol{u}_y + \cos\theta\,\boldsymbol{u}_z,$$

(θ, φ são os ângulos polar e azimutal) pode ser expresso na base própria de $\hat{\mathrm{S}}_z$ como

$$|\psi\rangle = \cos\frac{\theta}{2}|+\rangle_z + \sin\frac{\theta}{2}e^{i\varphi}|-\rangle_z. \tag{2.1}$$

2. Um conjunto de partículas, preparadas exatamente no mesmo estado inicial, são analisadas com recurso a 3 experiências de Stern-Gerlach *independentes*, tendo-se obtido os resultados tabelados abaixo:

	SG segundo x		SG segundo y		SG segundo z	
Percentagens em cada feixe	$+x$	$-x$	$+y$	$-y$	$+z$	$-z$
	50%	50%	10%	90%	80%	20%

Determine o vetor de estado inicial dessas partículas, $|\psi\rangle$, e os ângulos (θ, φ) que definem a direção $\boldsymbol{n}$ em que o seu spin aponta.

Solution

1. If the spin has been determined to be pointing along $\boldsymbol{n}$ it means that its state is the eigenstate of $\hat{S}_{\boldsymbol{n}} \equiv \hat{\boldsymbol{S}}\cdot\boldsymbol{n}$ associated with the eigenvalue $+\hbar/2$. We know that the operator $\hat{\boldsymbol{S}}\cdot\boldsymbol{n}$ is represented in the basis $\{|+\rangle_z, |-\rangle_z\}$ by the matrix

$$\hat{\boldsymbol{S}}\cdot\boldsymbol{n} \quad\mapsto\quad \frac{\hbar}{2}\boldsymbol{\sigma}\cdot\boldsymbol{n},$$

so we need only to calculate the positive eigenstate of $\boldsymbol{\sigma}\cdot\boldsymbol{n}$. We will not calculate this explicitly now since we saw already (Folha de Problemas 1) that such eigenstate is (up to an irrelevant global phase factor):

$$|+\rangle_{\boldsymbol{n}} = \cos\frac{\theta}{2}|+\rangle_z + \sin\frac{\theta}{2}e^{i\varphi}|-\rangle_z.$$

2. Since we don't know the state of the atoms, we assume the general state pointing along an arbitrary direction defined by some unit vector $\boldsymbol{n}$:

$$|\psi\rangle = |+\rangle_{\boldsymbol{n}} = \cos\frac{\theta}{2}|+\rangle_z + \sin\frac{\theta}{2}e^{i\varphi}|-\rangle_z.$$

Each of the 3 separate experiments consists of a measurement of one particular projection of spin. The values in the table allow us to compute the outcome probabilities for each of those measurements. We can relate those values with the expected probabilities for the state $|\psi\rangle$ and use that to obtain the values of θ and φ.

- *Measurement of $\mathcal{S}_z$* — The probabilities for each outcome are read directly from the values on the table:

$$\mathcal{P}(z_+) = 0.8 = \tfrac{4}{5} \qquad \text{(from the 80\% in the tabled results for } \mathcal{S}_z\text{)},$$
$$\mathcal{P}(z_-) = 0.2 = \tfrac{1}{5} \qquad \text{(from the 20\% in the tabled results for } \mathcal{S}_z\text{)}.$$

  On the other hand, for a general spin state given by eq. (2.1), these probabilities are expected to be (postulate P4)

$$\mathcal{P}(z_+) = |{}_z\langle + | \psi\rangle|^2 = \cos^2\frac{\theta}{2}, \qquad \mathcal{P}(z_-) = |{}_z\langle - | \psi\rangle|^2 = \sin^2\frac{\theta}{2}.$$

  So, equating these to the measured probabilities we get

$$\begin{cases} \cos^2\frac{\theta}{2} = \frac{4}{5} \\ \sin^2\frac{\theta}{2} = \frac{1}{5} \end{cases} \quad \Leftrightarrow \quad \begin{cases} \cos\frac{\theta}{2} = \frac{2}{\sqrt{5}} \\ \sin\frac{\theta}{2} = \frac{1}{\sqrt{5}} \end{cases} \quad \Leftrightarrow \quad \theta = 2\arctan\frac{1}{2},$$

  where we used the fact that, because $0 \leq \theta \leq \pi$, both the $\cos$ and $\sin$ have to be positive.

- *Measurement of $\mathcal{S}_x$* — We proceed analogously to the previous case:

$$\mathcal{P}(x_+) = \frac{1}{2} \quad \Rightarrow \quad |{}_x\langle + | \psi\rangle|^2 = \frac{1}{2}.$$

  Expanding the lhs of the last equation, the condition becomes

$$\begin{aligned} &\left|\frac{\cos\frac{\theta}{2}}{\sqrt{2}} + \frac{\sin\frac{\theta}{2}e^{i\varphi}}{\sqrt{2}}\right|^2 = \frac{1}{2}\left|\cos\frac{\theta}{2} + \sin\frac{\theta}{2}e^{i\varphi}\right|^2 = \frac{1}{2} \\ \Leftrightarrow &\left|\cos\frac{\theta}{2} + \sin\frac{\theta}{2}e^{i\varphi}\right|^2 = 1 + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} = 1 \end{aligned}$$

  Putting this together with the knowledge gathered from the z measurements we have

$$\begin{aligned} \text{(result above implies)} \qquad & \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} = 0 \\ \Leftrightarrow \quad & 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\varphi = 0 \\ \Leftrightarrow \quad & \cos\varphi = 0 \\ \Leftrightarrow \quad & \varphi = \pm\frac{\pi}{2} \end{aligned}$$

  ...because we concluded above that both $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ are non-zero. We still don't know which sign to choose for φ. So we'll need the remaining measurement to establish that.

- *Measurement of $\mathcal{S}_y$* — Just as above,

$$\mathcal{P}(y_+) = \frac{1}{10} \quad \Rightarrow \quad |{}_y\langle + | \psi\rangle|^2 = \frac{1}{10},$$

which translates into

$$\left|\frac{\cos\frac{\theta}{2}}{\sqrt{2}} - \frac{i\sin\frac{\theta}{2}e^{i\varphi}}{\sqrt{2}}\right|^2 = \frac{1}{2}\left|\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}e^{i\varphi}\right|^2 = \frac{1}{10}$$

$$\Leftrightarrow \quad \left|\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}e^{i\varphi}\right|^2 = 1 + i\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi} - i\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} = \frac{1}{5}.$$

This implies

$$\begin{aligned} & i\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\varphi} - i\cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\varphi} = 1 - \frac{1}{5} = \frac{4}{5} \\ \Leftrightarrow \quad & -2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\varphi = \frac{4}{5} \\ \text{(replacing } \theta \text{ found above)} \Leftrightarrow \quad & \cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\varphi = -\frac{2}{5} \\ \Leftrightarrow \quad & \sin\varphi = -1 \end{aligned}$$

and, together with the previous result, this fixes the azimuthal angle

$$\varphi = -\frac{\pi}{2}.$$

In conclusion, since the experimental data tell us that $\theta = 2\arctan\frac{1}{2}$ and $\varphi = -\frac{\pi}{2}$, the spin state of the atoms is

$$|\psi\rangle = \cos\frac{\theta}{2}|+\rangle_z + \sin\frac{\theta}{2}e^{i\varphi}|-\rangle_z = \frac{2}{\sqrt{5}}|+\rangle_z - \frac{i}{\sqrt{5}}|-\rangle_z,$$

and the direction of $\boldsymbol{n}$ is determined by the following polar and azimuthal angles

$$\theta = 2\ \arctan\frac{1}{2}, \qquad \varphi = -\frac{\pi}{2}.$$

Note: We could have solved this question using a different, but entirely equivalent strategy, based on the expectation values of each spin projection. To do that we just need to recall that the measurement of *any* spin component on *each individual particle* always yields either the value $+\hbar/2$ or $-\hbar/2$. This means that we can extract the expectation value of $\mathcal{S}_{x,y,z}$ in the state ψ directly from the table of results provided:

$$\begin{aligned} \langle\psi|\hat{\mathrm{S}}_x|\psi\rangle &= +\frac{\hbar}{2}\mathcal{P}(x_+) - \frac{\hbar}{2}\mathcal{P}(x_-) = \frac{\hbar}{2}\left[\mathcal{P}(x_+) - \mathcal{P}(x_-)\right] = \frac{\hbar}{2}\left(\frac{1}{2} - \frac{1}{2}\right) = 0, \\ \langle\psi|\hat{\mathrm{S}}_y|\psi\rangle &= \frac{\hbar}{2}\left[\mathcal{P}(y_+) - \mathcal{P}(y_-)\right] = \frac{\hbar}{2}\left(\frac{1}{10} - \frac{9}{10}\right) = -\frac{2\hbar}{5}, \\ \langle\psi|\hat{\mathrm{S}}_z|\psi\rangle &= \frac{\hbar}{2}\left[\mathcal{P}(z_+) - \mathcal{P}(z_-)\right] = \frac{\hbar}{2}\left(\frac{8}{10} - \frac{2}{10}\right) = \frac{3\hbar}{10}. \end{aligned}$$

On the other hand, for the spin state $|\psi\rangle = |+\rangle_{\boldsymbol{n}}$, these expectation values cab be computed directly in terms of the Pauli matrices associated with each spin projection:

$$\langle\psi|\hat{\mathrm{S}}_x|\psi\rangle = {}_{\boldsymbol{n}}\langle+|\hat{\mathrm{S}}_x|+\rangle_{\boldsymbol{n}} = \frac{\hbar}{2}\begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi}\end{bmatrix}\overbrace{\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}}^{\sigma_x}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{bmatrix} = \frac{\hbar}{2}\sin\theta\cos\varphi,$$

$$\langle\psi|\hat{S}_y|\psi\rangle = {}_{\boldsymbol{n}}\langle+|\hat{S}_y|+\rangle_{\boldsymbol{n}} = \frac{\hbar}{2}\begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi}\end{bmatrix}\overbrace{\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}}^{\sigma_y}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{bmatrix} = \frac{\hbar}{2}\sin\theta\sin\varphi,$$

$$\langle\psi|\hat{S}_z|\psi\rangle = {}_{\boldsymbol{n}}\langle+|\hat{S}_z|+\rangle_{\boldsymbol{n}} = \frac{\hbar}{2}\begin{bmatrix}\cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\varphi}\end{bmatrix}\overbrace{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}^{\sigma_z}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\varphi}\end{bmatrix} = \frac{\hbar}{2}\cos\theta.$$

(We used the fact that, in terms of Pauli matrices, the matrices representing $\hat{S}_{x,y,z}$ are, respectively, $\hbar/2\,\sigma_{x,y,z}$ in the eigenbasis of $\hat{S}_z$.) So, equating each of these 3 expressions to the expectation values obtained from the table of results we obtain

$$\begin{cases}\sin\theta\cos\varphi = 0\\ \sin\theta\sin\varphi = -\frac{4}{5}\\ \cos\theta = \frac{3}{5}\end{cases} \quad\Leftrightarrow\quad \begin{cases}\theta = 2\arctan\frac{1}{2}\\ \varphi = -\frac{\pi}{2}\end{cases},$$

which is precisely the result obtained earlier (as it should be) using the probabilities.

Problema 3

Usando o facto de que os auto-estados de $\hat{S}_y$ são

$$|\pm\rangle_y = \frac{1}{\sqrt{2}}|+\rangle_z \pm \frac{i}{\sqrt{2}}|-\rangle_z$$

e a representação já conhecida dos operadores $\hat{S}_{x,y,z}$ na base $|\pm\rangle_z$, *determine* as matrizes que representam os operadores $\hat{S}_x$, $\hat{S}_y$ e $\hat{S}_z$ na base própria de $\hat{S}_y$. Por exemplo,

$$\hat{S}_y \mapsto \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \text{na base } \{|+\rangle_y,\, |-\rangle_y\}.$$

Obtenha as restantes duas matrizes nesta base.

Solution

We know from the discussion in the lectures (and from Problem 1 above) that, in the eigenbasis of $\hat{S}_z$,

$$\hat{S}_x \mapsto \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \hat{S}_y \mapsto \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \quad \hat{S}_z \mapsto \frac{\hbar}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad \text{in the basis } \{|+\rangle_z,\, |-\rangle_z\}.$$

The normalized eigenstates of $\hat{S}_y$ are given as linear combinations of $|\pm\rangle_z$ by computing the eigenvectors of the second of these matrices; they are

$$|+\rangle_y = \tfrac{1}{\sqrt{2}}|+\rangle_z + \tfrac{i}{\sqrt{2}}|-\rangle_z \mapsto \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ i\end{bmatrix}, \qquad |-\rangle_y = \tfrac{1}{\sqrt{2}}|+\rangle_z - \tfrac{i}{\sqrt{2}}|-\rangle_z \mapsto \frac{1}{\sqrt{2}}\begin{bmatrix}1\\ -i\end{bmatrix}.$$

With this we can calculate the matrix elements of any operator in the basis defined by the two orthonormal states $|\pm\rangle_y$, namely,

$$
\begin{aligned}
{}_y\langle +|\hat{S}_x|+\rangle_y &= \frac{\hbar}{2}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{bmatrix} = 0,\\
{}_y\langle +|\hat{S}_x|-\rangle_y &= \frac{\hbar}{2}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{bmatrix} = \frac{\hbar}{2}\left(-i\right).
\end{aligned}
$$

We can use the Hermiticity and traceless character of these matrices ($\mathrm{Tr}\,\hat{S}_x = \mathrm{Tr}\,\hat{S}_y = \mathrm{Tr}\,\hat{S}_z = 0$) to avoid the explicit computation of the other two matrix elements. We then obtain

$$
\hat{S}_x \mapsto \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \quad \text{in the basis } \{|+\rangle_y,\, |-\rangle_y\}.
$$

Proceeding analogously, we can obtain that

$$
\hat{S}_z \mapsto \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \text{in the basis } \{|+\rangle_y,\, |-\rangle_y\}.
$$
