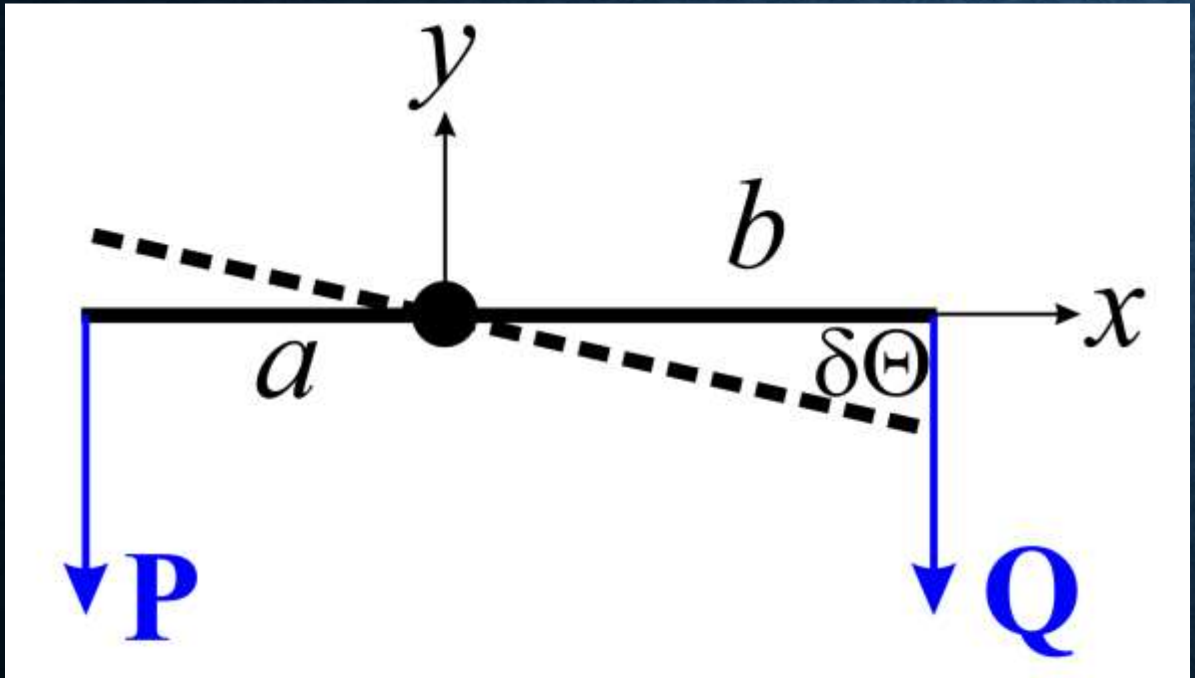
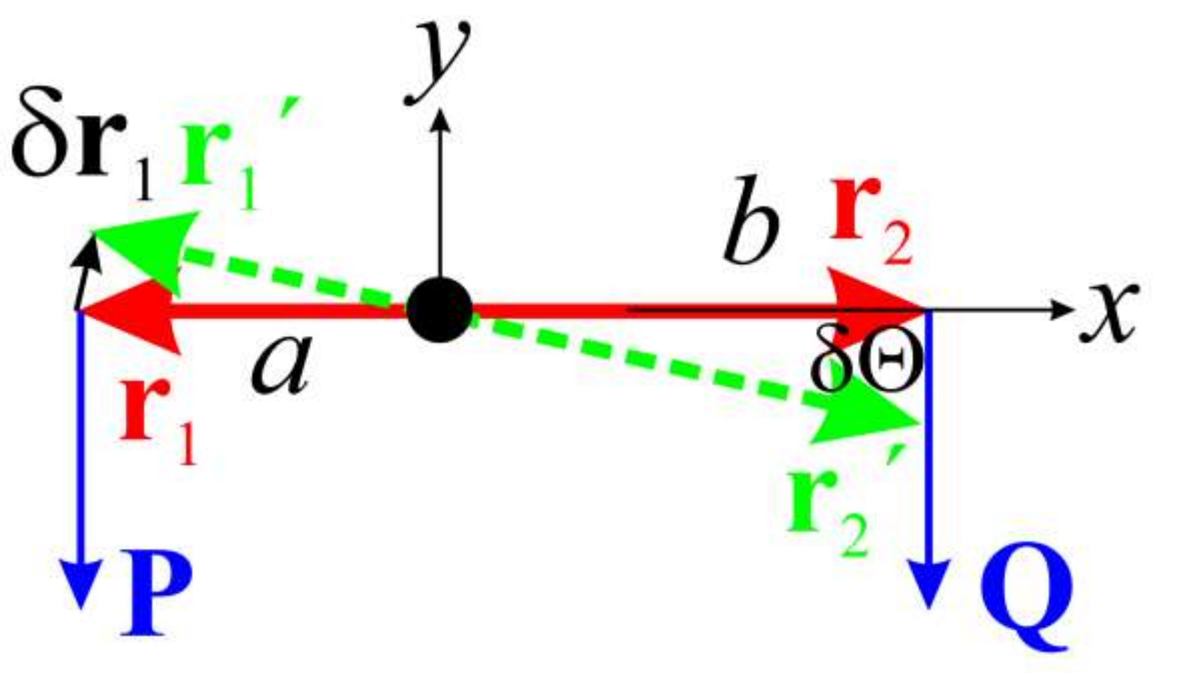
1- Deduza, utilizando o princípio dos trabalhos virtuais, as condições de equilíbrio de uma alavanca interfixa com distâncias do ponto de aplicação aos extremos da barra a e b, respectivamente, tais que $a < b$. Supõe-se desprezável o peso da barra.



$$T_a = Pa$$
$$T_b = Qb$$

$$T_a = T_b$$
$$Pa = Qb$$

$$\sum_j \mathbf{F}_j \cdot \delta \mathbf{r}_j = 0$$



$$\mathbf{F}_1 = -P\mathbf{e}_y = \begin{pmatrix} 0 \\ -P \end{pmatrix}$$

$$\mathbf{r}_1 = -a\mathbf{e}_x = \begin{pmatrix} -a \\ 0 \end{pmatrix}$$

$$\mathbf{F}_1 \cdot \delta\mathbf{r}_1 + \mathbf{F}_2 \cdot \delta\mathbf{r}_2 = 0$$

$$\mathbf{r}_1' = -a\cos\delta\Theta\mathbf{e}_x + a\sin\delta\Theta\mathbf{e}_y = \begin{pmatrix} -a\cos\delta\Theta \\ a\sin\delta\Theta \end{pmatrix}$$

$$\delta\mathbf{r}_1 = \mathbf{r}_1' - \mathbf{r}_1 = a\left[1 - \cos\delta\Theta\right]\mathbf{e}_x + a\sin\delta\Theta\mathbf{e}_y \approx a\,\delta\Theta\mathbf{e}_y = \begin{pmatrix} 0 \\ a\,\delta\Theta \end{pmatrix}$$

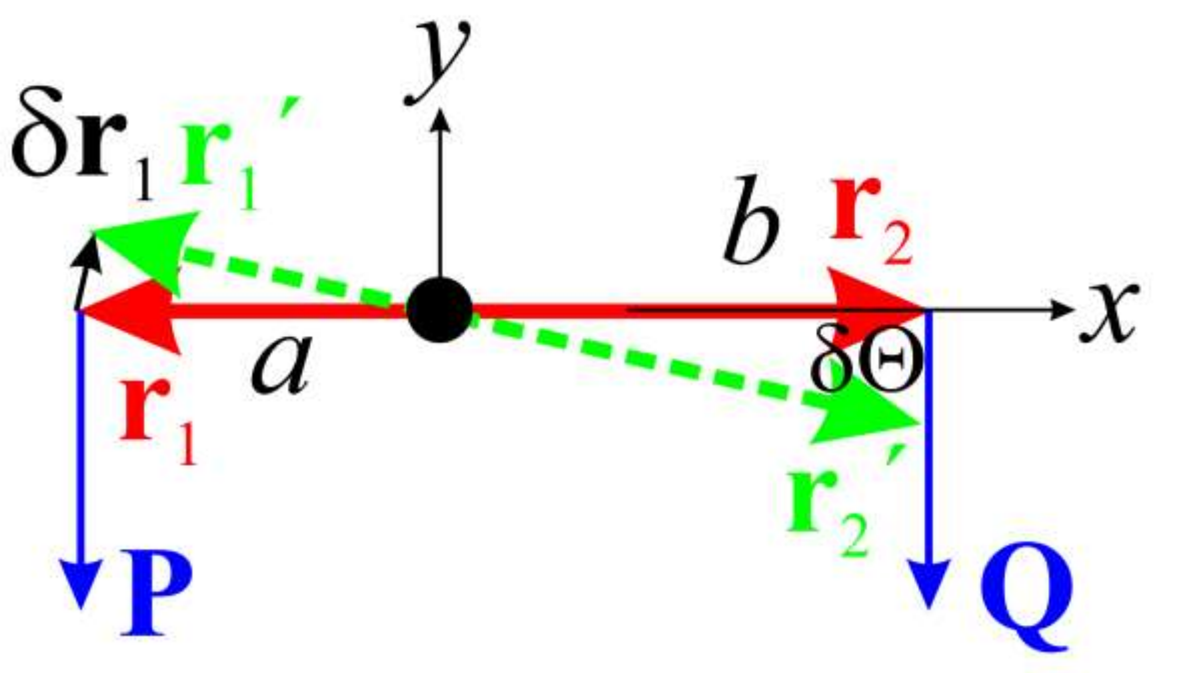
$$\cos\delta\Theta \approx 1$$

$$\sin\delta\Theta \approx \delta\Theta$$

$$\mathbf{A} = A_x\mathbf{e}_x + A_y\mathbf{e}_y = \begin{pmatrix} A_x \\ A_y \end{pmatrix}$$

$$\mathbf{B} = B_x\mathbf{e}_x + B_y\mathbf{e}_y = \begin{pmatrix} B_x \\ B_y \end{pmatrix}$$

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y$$



$$\mathbf{F}_2 = -Q\mathbf{e}_y = \begin{pmatrix} 0 \\ -Q \end{pmatrix}$$

$$\mathbf{r}_2 = b\mathbf{e}_x = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

$$\mathbf{r}'_2 = b\cos\delta\Theta\mathbf{e}_x - b\sin\delta\Theta\mathbf{e}_y = \begin{pmatrix} b\cos\delta\Theta \\ -b\sin\delta\Theta \end{pmatrix}$$

$$\delta\mathbf{r}_2 = \mathbf{r}'_2 - \mathbf{r}_2 = b\left[\cos\delta\Theta - 1\right]\mathbf{e}_x - b\sin\delta\Theta\mathbf{e}_y \approx -b\,\delta\Theta\mathbf{e}_y = \begin{pmatrix} 0 \\ -b\,\delta\Theta \end{pmatrix}$$

$$\mathbf{F}_1\cdot\delta\mathbf{r}_1 + \mathbf{F}_2\cdot\delta\mathbf{r}_2 \approx -Pa\,\delta\Theta + Qb\,\delta\Theta = 0$$

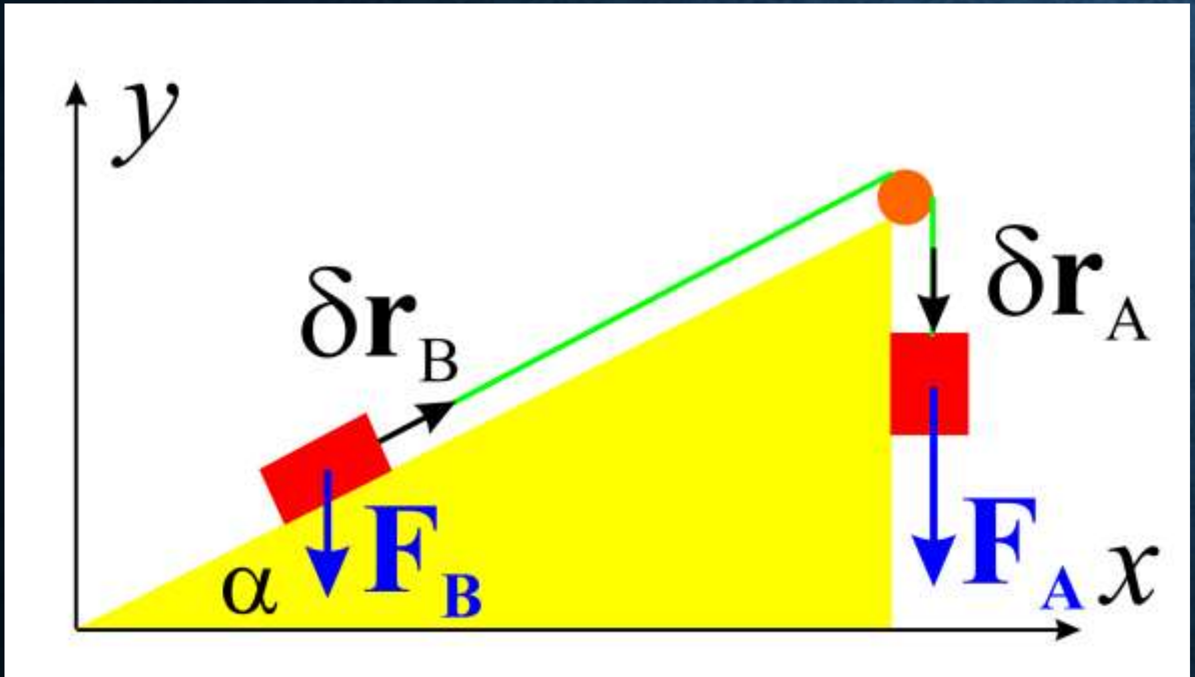
$$Pa = Qb$$

$$\cos\delta\Theta \approx 1$$
$$\sin\delta\Theta \approx \delta\Theta$$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{2\cdot 3}x^3 + \ldots + \frac{f^{\overbrace{\prime\prime\ldots\prime}^{n}}(0)}{n!}x^n + \ldots$$

$$\sin(x) = \sin(0) + \cos(0)\,x + \frac{-\sin(0)}{2}x^2 + \frac{-\cos(0)}{2\cdot 3}x^3 + \ldots = x - \frac{x^3}{6} + \ldots$$
$$\cos(x) = \cos(0) - \sin(0)\,x + \frac{-\cos(0)}{2}x^2 + \frac{\sin(0)}{2\cdot 3}x^3 = 1 - \frac{x^2}{2} + \ldots$$

2- Numa das extremidades de um fio que passa por uma roldana, está suspenso verticalmente um corpo A de peso $\vec{P}_1$ e na outra extremiade um corpo B, assente num plano inclinado de ângulo α relativamente à direcção horizontal. Sabendo que se pode desprezar o atrito na roldana e o atrito entre o corpo B e o plano inclinado sobre o qual está assente, determine o peso do corpo B para que o sistema esteja em equilíbrio.

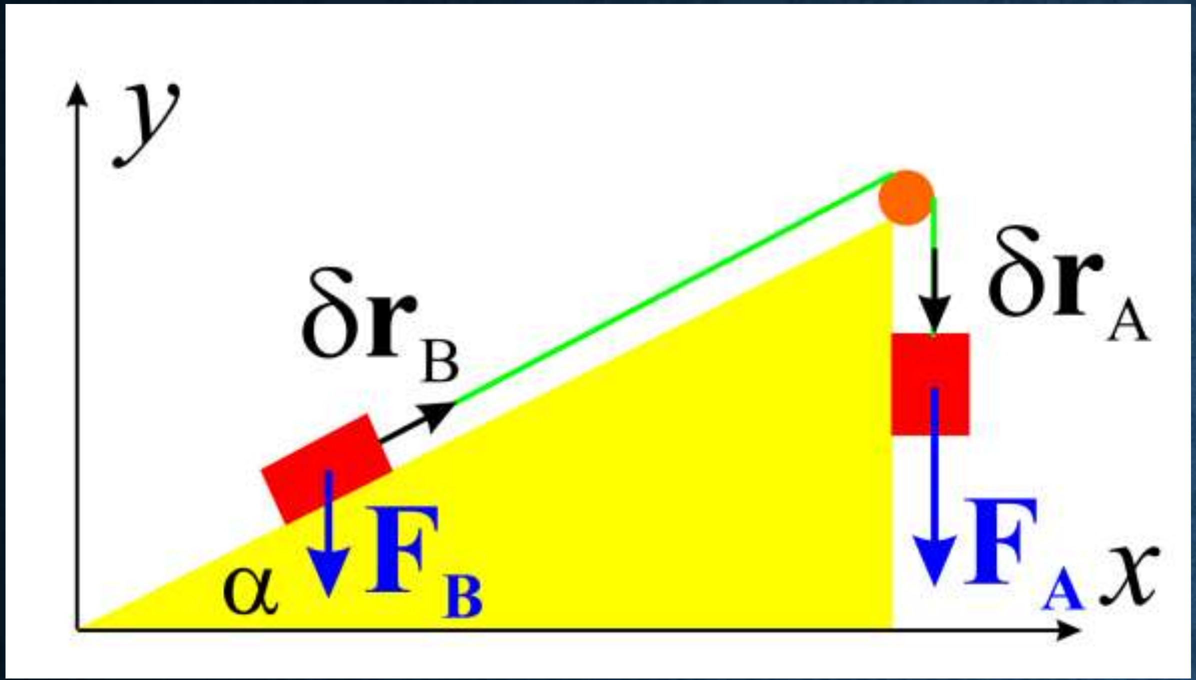


$$\mathbf{F}_A = -P_1 \mathbf{e}_y = \begin{pmatrix} 0 \\ -P_1 \end{pmatrix}$$

$$\mathbf{F}_B = -P_2 \mathbf{e}_y = \begin{pmatrix} 0 \\ -P_2 \end{pmatrix}$$

$$|\delta \mathbf{r}_A| = |\delta \mathbf{r}_B| = \Delta$$

$$\delta \mathbf{r}_A = -\Delta \mathbf{e}_y = \begin{pmatrix} 0 \\ -\Delta \end{pmatrix}$$



$$\mathbf{F}_A = -P_1\mathbf{e}_y = \begin{pmatrix} 0 \\ -P_1 \end{pmatrix}$$

$$\mathbf{F}_B = -P_2\mathbf{e}_y = \begin{pmatrix} 0 \\ -P_2 \end{pmatrix}$$

$$\delta\mathbf{r}_A = -\Delta\mathbf{e}_y = \begin{pmatrix} 0 \\ -\Delta \end{pmatrix}$$

$$\delta\mathbf{r}_B = \Delta\cos\alpha\mathbf{e}_x + \Delta\sin\alpha\mathbf{e}_y = \begin{pmatrix} \Delta\cos\alpha \\ \Delta\sin\alpha \end{pmatrix}$$

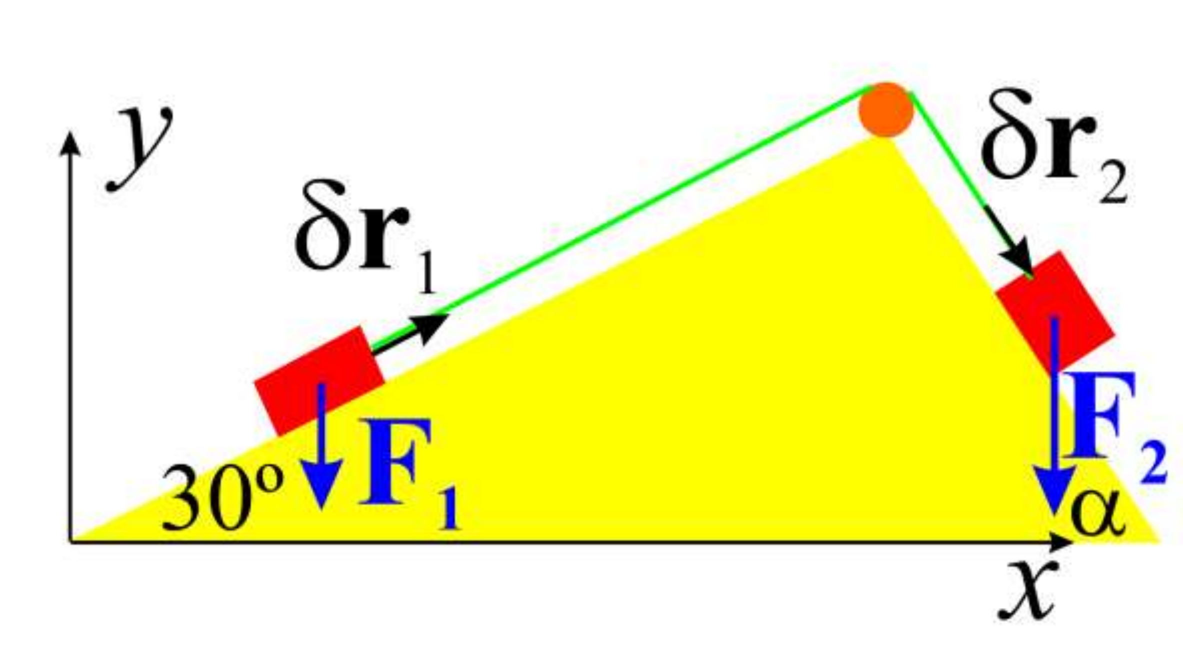
$$\mathbf{F}_A\cdot\delta\mathbf{r}_A + \mathbf{F}_B\cdot\delta\mathbf{r}_B = P_1\Delta - P_2\Delta\sin\alpha = 0$$

$$P_2 = \frac{P_1}{\sin\alpha}$$

$$\begin{aligned} m_1 a &= P_1 - T = 0 \\ m_2 a &= T - P_2\sin\alpha = 0 \end{aligned}$$

$$P_1 = P_2\sin\alpha$$

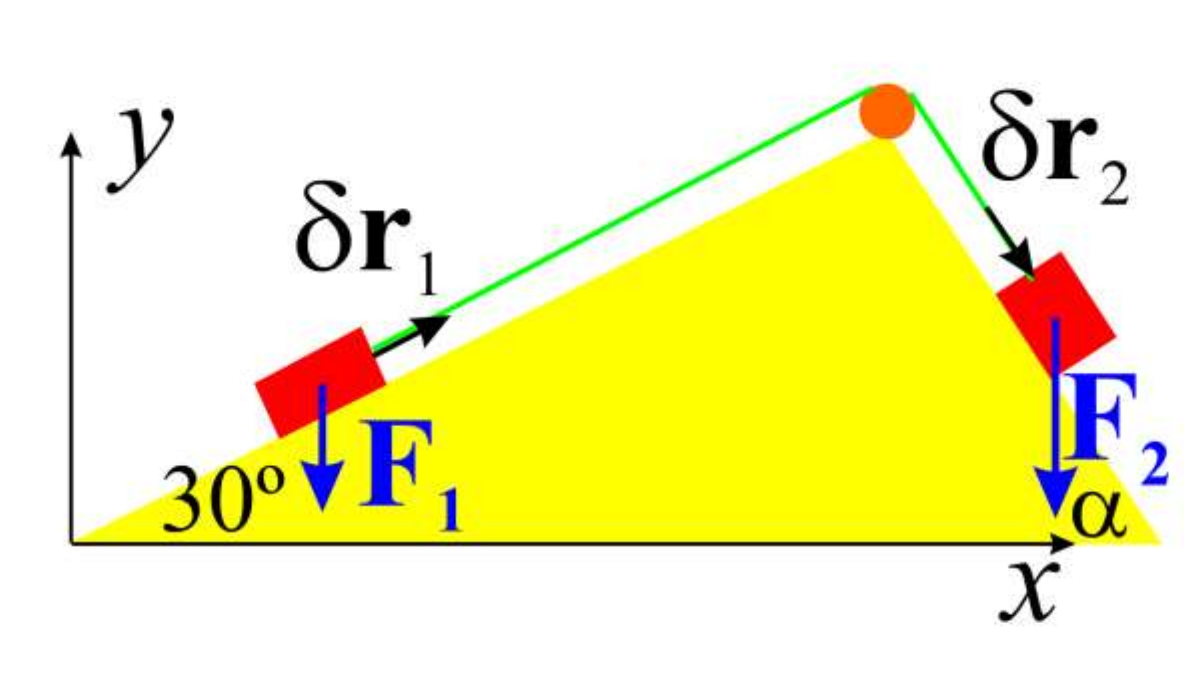
3- Duas massas m_1 e m_2, sujeitas à força da gravidade e deslocando-se sem atrito sobre um duplo plano inclinado, estão ligadas entre si por um fio inextensível e de massa desprezável, que passa igualmente sem atrito por uma roldana. O valor do ângulo do plano inclinado sobre o qual está assente o corpo de massa m_1 relativamente à direcção horizontal é de $30^o = \pi/6$ radianos. Determine, a partir do princípio dos trabalhos virtuais, o valor do correspondente ângulo α relativamente à direcção horizontal do plano inclinado sobre o qual está assente o corpo de massa m_2 para o qual o sistema está em equilíbrio, quando $m_2 = 2m_1$.



$$\mathbf{F}_1 \cdot \delta\mathbf{r}_1 + \mathbf{F}_2 \cdot \delta\mathbf{r}_2 = 0$$

$$\mathbf{F}_1 = -m_1 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix}$$

$$\delta\mathbf{r}_1 = \Delta\cos\left(\frac{\pi}{6}\right)\mathbf{e}_x + \Delta\sin\left(\frac{\pi}{6}\right)\mathbf{e}_y = \begin{pmatrix} \Delta\cos\left(\frac{\pi}{6}\right) \\ \Delta\sin\left(\frac{\pi}{6}\right) \end{pmatrix}$$



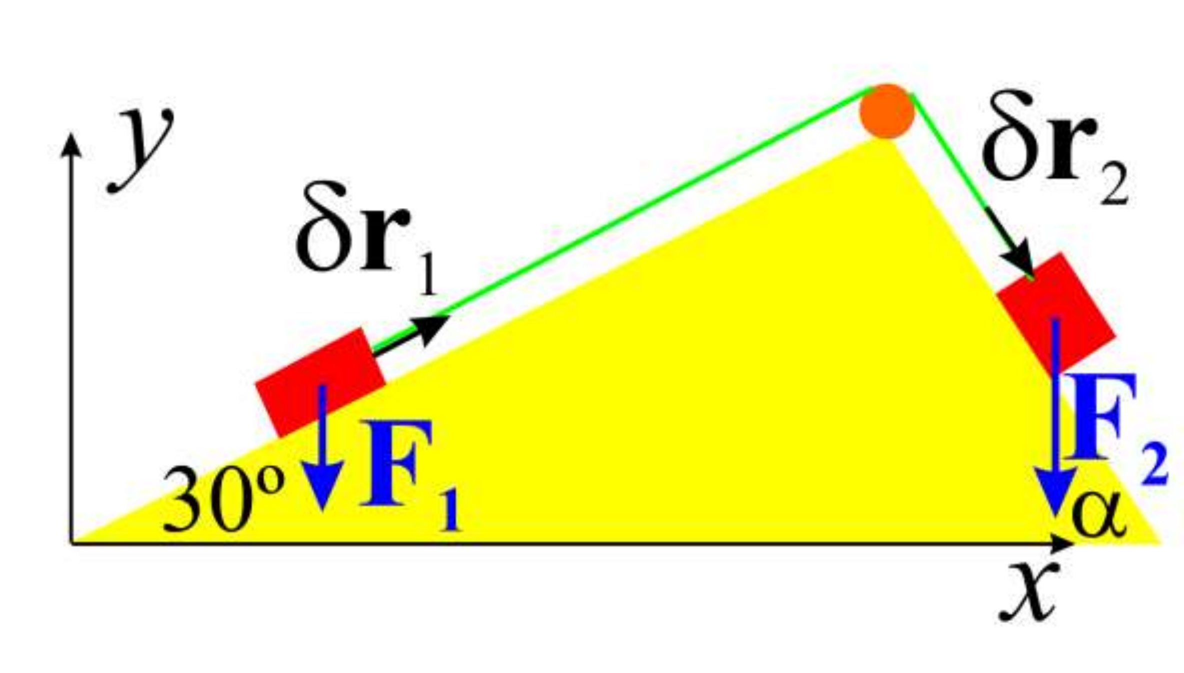
$$\mathbf{F}_1 = -m_1 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix}$$

$$\mathbf{F}_2 = -m_2 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix}$$

$$\delta\mathbf{r}_1 = \Delta\cos\left(\frac{\pi}{6}\right)\mathbf{e}_x + \Delta\sin\left(\frac{\pi}{6}\right)\mathbf{e}_y = \begin{pmatrix} \Delta\cos\left(\frac{\pi}{6}\right) \\ \Delta\sin\left(\frac{\pi}{6}\right) \end{pmatrix}$$

$$\delta\mathbf{r}_2 = \Delta\cos\alpha\mathbf{e}_x - \Delta\sin\alpha\mathbf{e}_y = \begin{pmatrix} \Delta\cos\alpha \\ -\Delta\sin\alpha \end{pmatrix}$$

$$\mathbf{F}_1\cdot\delta\mathbf{r}_1 + \mathbf{F}_2\cdot\delta\mathbf{r}_2 = -m_1 g\Delta\sin\left(\frac{\pi}{6}\right) + m_2 g\Delta\sin\alpha = 0$$



$$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

$$\mathbf{F}_1 \cdot \delta\mathbf{r}_1 + \mathbf{F}_2 \cdot \delta\mathbf{r}_2 = -m_1 g \Delta \sin\left(\frac{\pi}{6}\right) + m_2 g \Delta \sin\alpha = 0$$

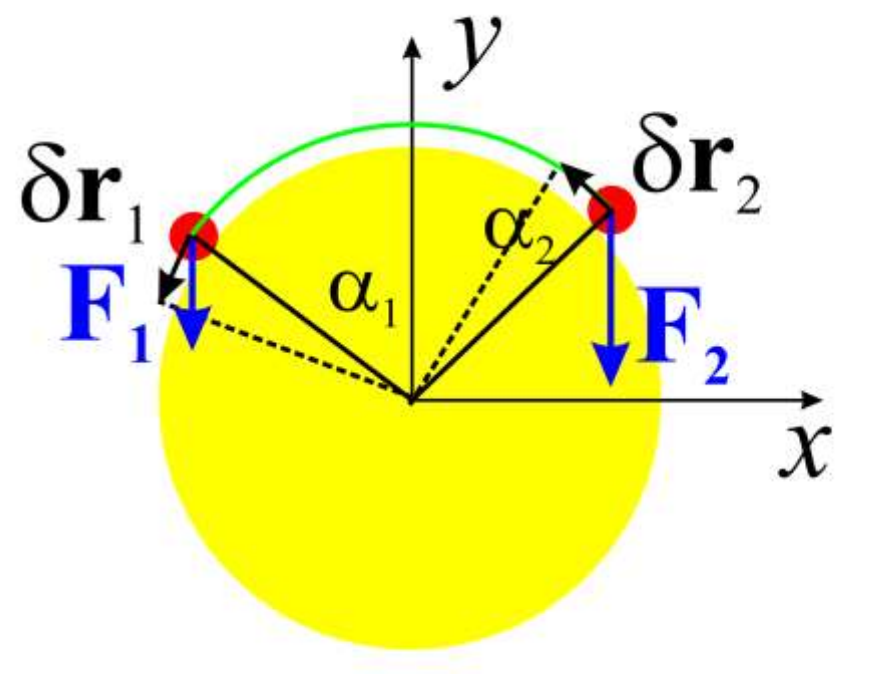
$$-\frac{m_1}{2} + m_2 \sin\alpha = 0$$

$$m_2 = 2m_1$$

$$\sin\alpha = \frac{1}{4}$$

$$\alpha = \arcsin\left(\frac{1}{4}\right) \approx 0.2527 \approx 14.76^{\text{º}}$$

4- Duas massas pontuais m_1 e m_2, ligadas por uma barra rígida de massa desprezável, podem deslocar-se sem atrito sobre uma circunferência vertical. Os raios da circunferência que ligam o seu centro às massas pontuais m_1 e m_2 fazem ângulos α_1 e α_2, respectivamente, com a direção vertical. Determine, usando o princípio dos trabalhos virtuais, a que relação devem obedecer as grandezas m_1 e m_2 e os ângulos α_1 e α_2 para que o sistema esteja em equilíbrio.

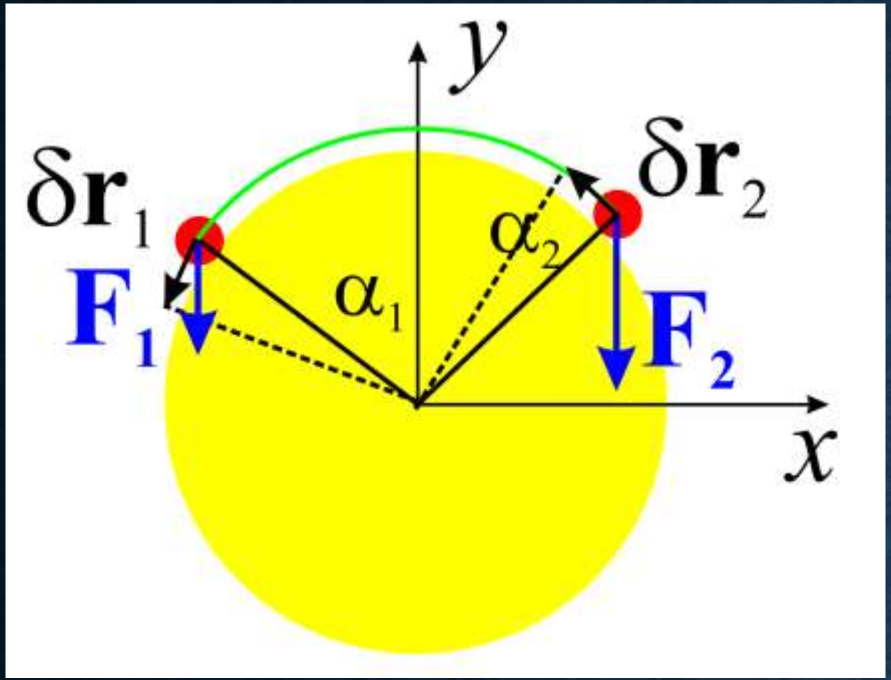


$$\mathbf{F}_1 = -m_1 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix}$$

$$\mathbf{F}_2 = -m_2 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix}$$

$$\mathbf{r}_1 = -R\sin\alpha_1 \mathbf{e}_x + R\cos\alpha_1 \mathbf{e}_y = \begin{pmatrix} -R\sin\alpha_1 \\ R\cos\alpha_1 \end{pmatrix}$$

$$\mathbf{r}_1' = -R\sin\alpha_1' \mathbf{e}_x + R\cos\alpha_1' \mathbf{e}_y = \begin{pmatrix} -R\sin\alpha_1' \\ R\cos\alpha_1' \end{pmatrix}$$



$$\mathbf{r}_1 = -R\sin\alpha_1\mathbf{e}_x + R\cos\alpha_1\mathbf{e}_y = \begin{pmatrix} -R\sin\alpha_1 \\ R\cos\alpha_1 \end{pmatrix}$$

$$\mathbf{r}'_1 = -R\sin\alpha'_1\mathbf{e}_x + R\cos\alpha'_1\mathbf{e}_y = \begin{pmatrix} -R\sin\alpha'_1 \\ R\cos\alpha'_1 \end{pmatrix}$$

$$\sin(\alpha'_1) = \sin(\alpha_1) + \cos(\alpha_1)(\alpha'_1 - \alpha_1) + \ldots$$

$$= \sin(\alpha_1) + \cos(\alpha_1)\,\delta\alpha + \ldots$$

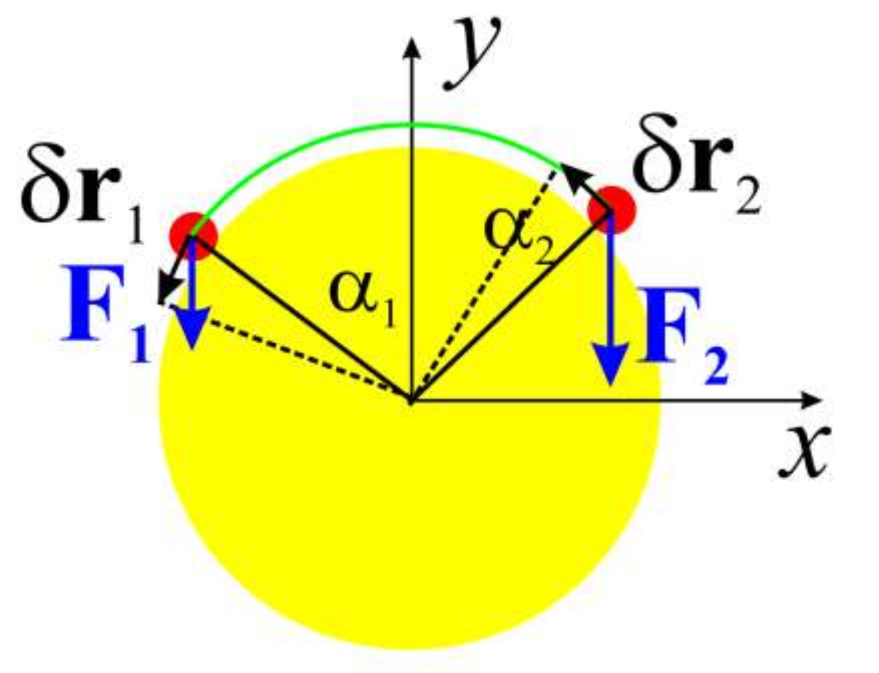
$$\alpha'_1 = \alpha_1 + \delta\alpha$$

$$\cos(\alpha'_1) = \cos(\alpha_1) - \sin(\alpha_1)(\alpha'_1 - \alpha_1) + \ldots = \cos(\alpha_1) - \sin(\alpha_1)\,\delta\alpha$$

$$\mathbf{r}'_1 \approx -R[\sin(\alpha_1) + \cos(\alpha_1)\,\delta\alpha]\,\mathbf{e}_x + R[\cos(\alpha_1) - \sin(\alpha'_1)\,\delta\alpha]\,\mathbf{e}_y = \begin{pmatrix} -R[\sin(\alpha_1) + \cos(\alpha_1)\,\delta\alpha] \\ R[\cos(\alpha_1) - \sin(\alpha_1)\,\delta\alpha] \end{pmatrix}$$

$$\delta\mathbf{r}_1 = \mathbf{r}'_1 - \mathbf{r}_1$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f'''(x_0)}{2\cdot 3}(x - x_0)^3 + \ldots + \frac{f^{\overbrace{\prime\prime\ldots\prime}^{n}}(x_0)}{n!}(x - x_0)^n + \ldots$$



$$\mathbf{r}_1 = -R\sin\alpha_1 \mathbf{e}_x + R\cos\alpha_1 \mathbf{e}_y = \begin{pmatrix} -R\sin\alpha_1 \\ R\cos\alpha_1 \end{pmatrix}$$

$$\mathbf{r}_2 = R\sin\alpha_2 \mathbf{e}_x + R\cos\alpha_2 \mathbf{e}_y = \begin{pmatrix} R\sin\alpha_2 \\ R\cos\alpha_2 \end{pmatrix}$$

$$\mathbf{r}_2' = R\sin\alpha_2' \mathbf{e}_x + R\cos\alpha_2' \mathbf{e}_y = \begin{pmatrix} R\sin\alpha_2' \\ R\cos\alpha_2' \end{pmatrix}$$

$$\mathbf{r}_1' \approx -R\left[\sin(\alpha_1) + \cos(\alpha_1)\,\delta\alpha\right]\mathbf{e}_x + R\left[\cos(\alpha_1) - \sin(\alpha_1')\,\delta\alpha\right]\mathbf{e}_y = \begin{pmatrix} -R\left[\sin(\alpha_1) + \cos(\alpha_1)\,\delta\alpha\right] \\ R\left[\cos(\alpha_1) - \sin(\alpha_1)\,\delta\alpha\right] \end{pmatrix}$$

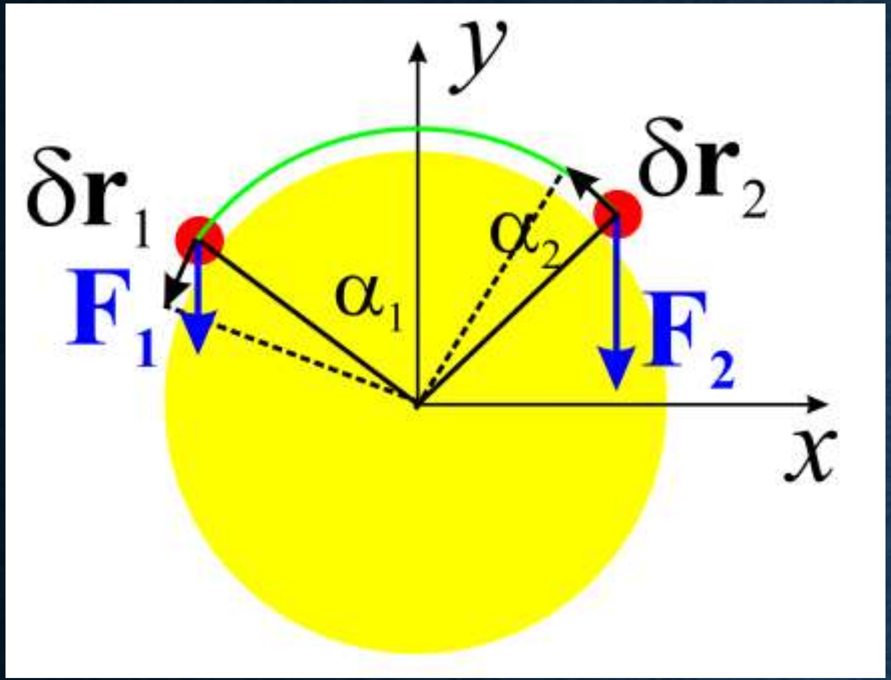
$$\delta\mathbf{r}_1 = \mathbf{r}_1' - \mathbf{r}_1 \approx -R\cos(\alpha_1)\,\delta\alpha\mathbf{e}_x - R\sin(\alpha_1)\,\delta\alpha\mathbf{e}_y = \begin{pmatrix} -R\cos(\alpha_1)\,\delta\alpha \\ -R\sin(\alpha_1)\,\delta\alpha \end{pmatrix}$$

$$\sin(\alpha_2') = \sin(\alpha_2) + \cos(\alpha_2)(\alpha_2' - \alpha_2) + \ldots = \sin(\alpha_2) - \cos(\alpha_2)\,\delta\alpha + \ldots$$

$$\cos(\alpha_2') = \cos(\alpha_2) - \sin(\alpha_2)(\alpha_2' - \alpha_2) + \ldots = \cos(\alpha_2) + \sin(\alpha_2)\,\delta\alpha$$

$$\alpha_2' = \alpha_2 - \delta\alpha$$

$$\alpha_1' = \alpha_1 + \delta\alpha$$



$$\mathbf{r}_2 = R\sin\alpha_2 \mathbf{e}_x + R\cos\alpha_2 \mathbf{e}_y = \begin{pmatrix} R\sin\alpha_2 \\ R\cos\alpha_2 \end{pmatrix}$$

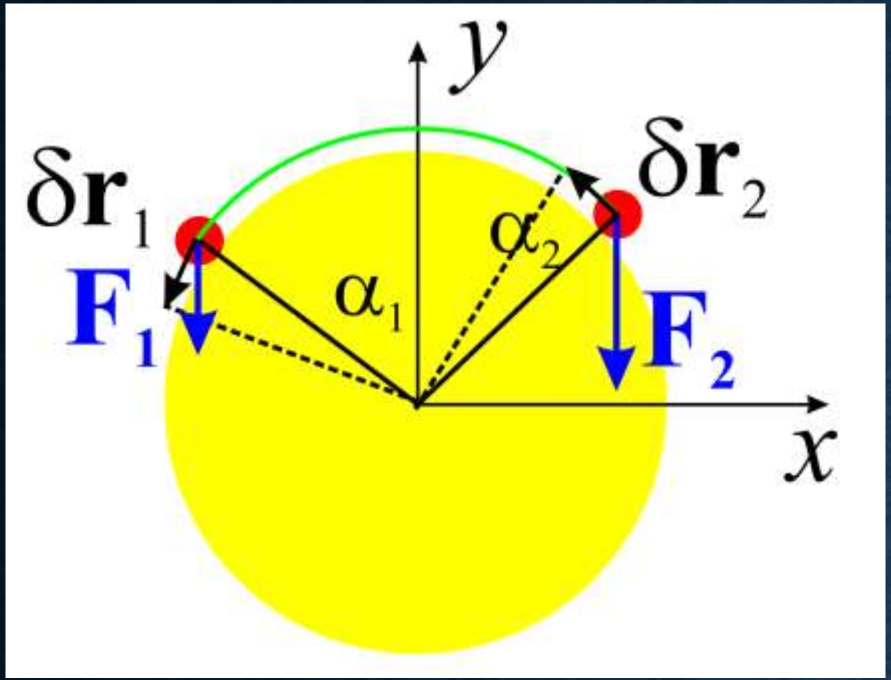
$$\mathbf{r}_2' = R\sin\alpha_2' \mathbf{e}_x + R\cos\alpha_2' \mathbf{e}_y = \begin{pmatrix} R\sin\alpha_2' \\ R\cos\alpha_2' \end{pmatrix}$$

$$\sin(\alpha_2') = \sin(\alpha_2) + \cos(\alpha_2)(\alpha_2' - \alpha_2) + \ldots = \sin(\alpha_2) - \cos(\alpha_2)\,\delta\alpha + \ldots$$

$$\cos(\alpha_2') = \cos(\alpha_2) - \sin(\alpha_2)(\alpha_2' - \alpha_2) + \ldots = \cos(\alpha_2) + \sin(\alpha_2)\,\delta\alpha$$

$$\mathbf{r}_2' = R\left[\sin(\alpha_2) - \cos(\alpha_2)\,\delta\alpha\right]\mathbf{e}_x + R\left[\cos(\alpha_2') + \sin(\alpha_2')\,\delta\alpha\right]\mathbf{e}_y = \begin{pmatrix} R\left[\sin(\alpha_2) - \cos(\alpha_2)\right]\delta\alpha \\ R\left[\cos(\alpha_2) + \sin(\alpha_2)\,\delta\alpha\right] \end{pmatrix}$$

$$\delta\mathbf{r}_2 = \mathbf{r}_2' - \mathbf{r}_2 = -R\cos(\alpha_2)\,\delta\alpha\mathbf{e}_x + R\sin(\alpha_2')\,\delta\alpha\mathbf{e}_y = \begin{pmatrix} -R\cos(\alpha_2)\,\delta\alpha \\ R\sin(\alpha_2)\,\delta\alpha \end{pmatrix}$$



$$\mathbf{F}_1 = -m_1 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix}$$

$$\mathbf{F}_2 = -m_2 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix}$$

$$T = m_2 g \sin(\alpha_2)$$

$$T = m_1 g \sin(\alpha_1)$$

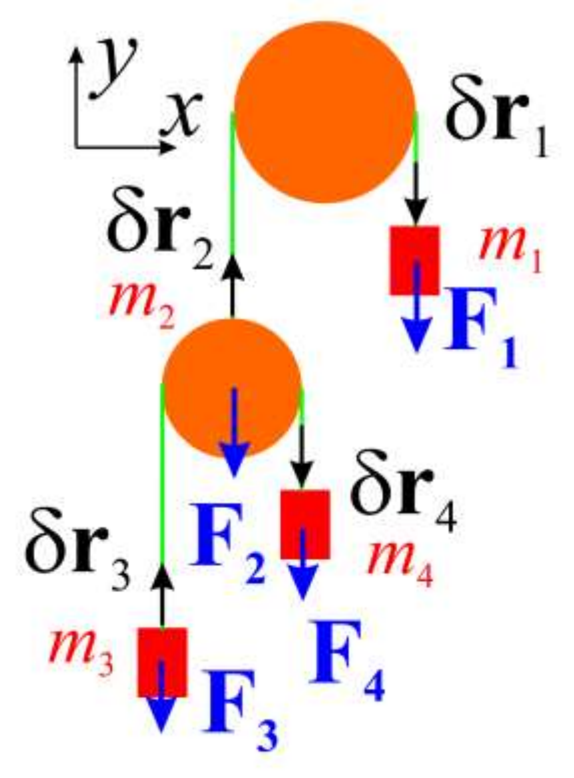
$$\delta\mathbf{r}_1 = \mathbf{r}_1' - \mathbf{r}_1 \approx -R\cos(\alpha_1)\,\delta\alpha\mathbf{e}_x - R\sin(\alpha_1)\,\delta\alpha\mathbf{e}_y = \begin{pmatrix} -R\cos(\alpha_1)\,\delta\alpha \\ -R\sin(\alpha_1)\,\delta\alpha \end{pmatrix}$$

$$\delta\mathbf{r}_2 = \mathbf{r}_2' - \mathbf{r}_2 = -R\cos(\alpha_2)\,\delta\alpha\mathbf{e}_x + R\sin(\alpha_2')\,\delta\alpha\mathbf{e}_y = \begin{pmatrix} -R\cos(\alpha_2)\,\delta\alpha \\ R\sin(\alpha_2)\,\delta\alpha \end{pmatrix}$$

$$\mathbf{F}_1 \cdot \delta\mathbf{r}_1 + \mathbf{F}_2 \cdot \delta\mathbf{r}_2 = m_1 g R\sin(\alpha_1)\,\delta\alpha - m_2 g R\sin(\alpha_2)\,\delta\alpha = 0$$

$$m_1 \sin(\alpha_1) - m_2 \sin(\alpha_2) = 0$$

5- Uma massa m_1 está suspensa por um fio inextensível, que passa sem atrito numa roldana fixa. Na outra extremidade do fio, também inextensível, encontra-se uma outra roldana de massa m_2, na qual passa um segundo fio, também inextensível, em cujas extremidades estão suspensas as massas m_3 e m_4. Esta última roldana não roda em torno da vertical. Determine, utilizando o princípio dos trabalhos virtuais, as relações que as massas m_1, m_2, m_3 e m_4 devem verificar para que o sitema esteja em equilíbrio.



$$\mathbf{F}_1 \cdot \delta\mathbf{r}_1 + \mathbf{F}_2 \cdot \delta\mathbf{r}_2 + \mathbf{F}_3 \cdot \delta\mathbf{r}_3 + \mathbf{F}_4 \cdot \delta\mathbf{r}_4 = 0$$

$$\mathbf{F}_1 = -m_1 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix}$$

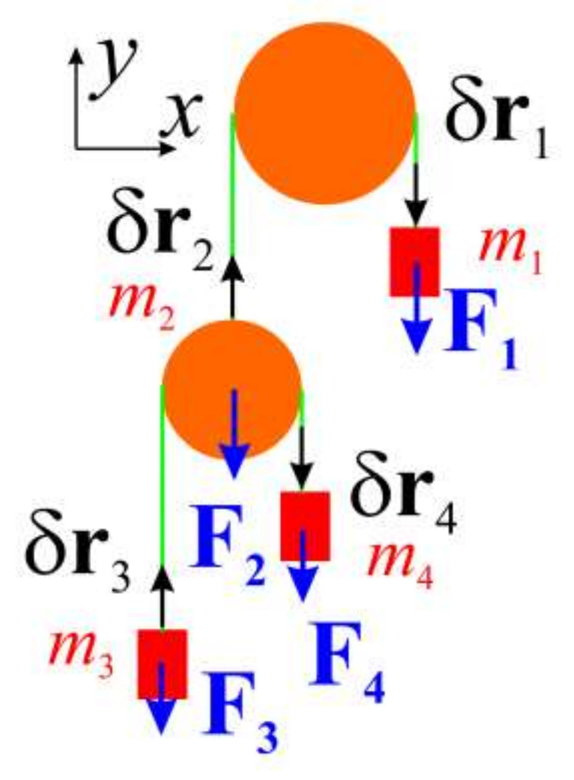
$$\mathbf{F}_2 = -m_2 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix}$$

$$\mathbf{F}_3 = -m_3 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_3 g \end{pmatrix}$$

$$\mathbf{F}_4 = -m_4 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_4 g \end{pmatrix}$$

$$\delta\mathbf{r}_2 = \Delta \mathbf{e}_y = \begin{pmatrix} 0 \\ \Delta \end{pmatrix}$$

$$\delta\mathbf{r}_3 = (\Delta' + \Delta)\, \mathbf{e}_y = \begin{pmatrix} 0 \\ \Delta' + \Delta \end{pmatrix}$$



$$\mathbf{F}_1 \cdot \delta\mathbf{r}_1 + \mathbf{F}_2 \cdot \delta\mathbf{r}_2 + \mathbf{F}_3 \cdot \delta\mathbf{r}_3 + \mathbf{F}_4 \cdot \delta\mathbf{r}_4 = 0$$

$$\mathbf{F}_1 = -m_1 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_1 g \end{pmatrix}$$

$$\mathbf{F}_2 = -m_2 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_2 g \end{pmatrix}$$

$$\mathbf{F}_3 = -m_3 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_3 g \end{pmatrix}$$

$$\mathbf{F}_4 = -m_4 g \mathbf{e}_y = \begin{pmatrix} 0 \\ -m_4 g \end{pmatrix}$$

$$\delta\mathbf{r}_2 = \Delta \mathbf{e}_y = \begin{pmatrix} 0 \\ \Delta \end{pmatrix}$$

$$\delta\mathbf{r}_3 = (\Delta' + \Delta)\, \mathbf{e}_y = \begin{pmatrix} 0 \\ \Delta' + \Delta \end{pmatrix}$$

$$\delta\mathbf{r}_1 = -\Delta \mathbf{e}_y = \begin{pmatrix} 0 \\ -\Delta \end{pmatrix}$$

$$\delta\mathbf{r}_4 = (-\Delta' + \Delta)\, \mathbf{e}_y = \begin{pmatrix} 0 \\ -\Delta' + \Delta \end{pmatrix}$$

$$\mathbf{F}_1 \cdot \delta\mathbf{r}_1 + \mathbf{F}_2 \cdot \delta\mathbf{r}_2 + \mathbf{F}_3 \cdot \delta\mathbf{r}_3 + \mathbf{F}_4 \cdot \delta\mathbf{r}_4 =$$
$$= m_1 g \Delta - m_2 g \Delta - m_3 g (\Delta' + \Delta) - m_4 g (-\Delta' + \Delta) = 0$$

$$m_1 = m_2 + m_3 + m_4$$

$$(m_1 - m_2 - m_3 - m_4)\, \Delta + \Delta' (m_4 - m_3) = 0$$

$$m_4 = m_3$$