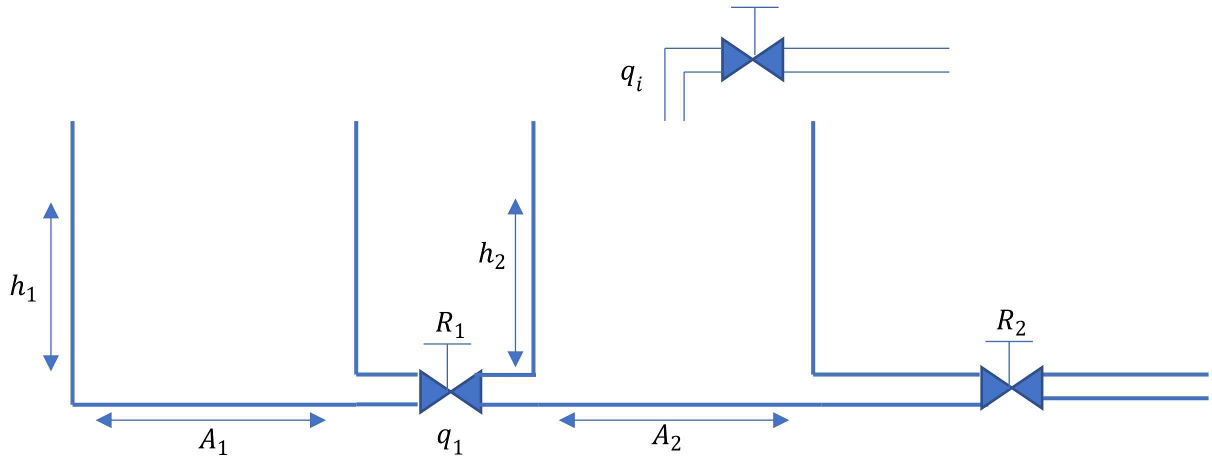
Exercise 1 (2 tanks in series):



a) q_i is the input and h_1 is the output:

Tank 1: $out = accumulates$ (because the tank 1 doesn't have an input (a tap))

$q_1(t) = \frac{h_2(t)-h_1(t)}{R_1}$; $\quad volume\ do\ tanque\ 1 = A_1\frac{dh_1(t)}{dt}$

$$\frac{h_2(t)-h_1(t)}{R_1} = A_1\frac{dh_1(t)}{dt} (=) h_2(t) = h_1(t) + R_1A_1\frac{dh_1(t)}{dt}$$

Tank 2: $in = out + acumulates$

In this case we have: 1 input -> q_i; 2 outs -> q_1 e q_2; accumulates -> volume on tank 2

$$q_i(t) = \frac{h_2(t)}{R_2} + \frac{h_2(t)-h_1(t)}{R_1} + A_2\frac{dh_2(t)}{dt}$$

So:

$$\begin{cases} h_2(t) = h_1(t) + R_1A_1\dfrac{dh_1(t)}{dt} & (eq.1) \\ q_i(t) = \dfrac{h_2(t)}{R_2} + \dfrac{h_2(t)-h_1(t)}{R_1} + A_2\dfrac{dh_2(t)}{dt} & (eq.2) \end{cases}$$

Replacing the equation 1 on equation 2 we have:

$$q_i(t) = \frac{1}{R_2}\left(h_1(t) + A_1R_1\frac{dh_1(t)}{dt}\right) + \frac{1}{R_1}\left(h_1(t) + A_1R_1\frac{dh_1(t)}{dt}\right) - \frac{h_1(t)}{R_1} + A_2\frac{dh_1(t)}{dt} + A_2A_1R_1\frac{dh_1^2(t)}{dt} (=)$$

$$(=) q_i(t) = \frac{h_1(t)}{R_2} + \frac{A_1R_1}{R_2}\frac{dh_1(t)}{dt} + \frac{h_1(t)}{R_1} + A_1\frac{dh_1(t)}{dt} - \frac{h_1(t)}{R_1} + A_2\frac{dh_1(t)}{dt} + A_2A_1R_1\frac{dh_1^2(t)}{dt} (=)$$

$$(=) q_i(t) = \frac{h_1(t)}{R_2} + \frac{A_1R_1}{R_2}\frac{dh_1(t)}{dt} + A_1\frac{dh_1(t)}{dt} + A_2\frac{dh_1(t)}{dt} + A_2A_1R_1\frac{dh_1^2(t)}{dt} (=)(\times R_2)$$

$$(=) R_2q_i(t) = h_1(t) + A_1R_1\frac{dh_1(t)}{dt} + A_1R_2\frac{dh_1(t)}{dt} + A_2R_2\frac{dh_1(t)}{dt} + A_2A_1R_1R_2\frac{dh_1^2(t)}{dt}$$

Laplace:

$$(=) R_2Q_i(s) = H_1(s) + A_1R_1sH_1(s) + A_1R_2sH_1(s) + A_2R_2sH_1(s) + A_2A_1R_1R_2s^2H_1(s)$$

$$H_1(s) = \frac{R_2}{(A_2A_1R_1R_2s^2 + (A_1R_1 + A_1R_2 + A_2R_2)s + 1)}Q_i(s)$$

b) q_i is the input and h_2 is the output:

$$\begin{cases} h_2(t) = h_1(t) + R_1A_1\dfrac{dh_1(t)}{dt} & (eq. 1) \\ q_i(t) = \dfrac{h_2(t)}{R_2} + \dfrac{h_2(t) - h_1(t)}{R_1} + A_2\dfrac{dh_2(t)}{dt} & (eq. 2) \end{cases}$$

Rearranging the equation 2:

$$R_2q_i(t) = h_2(t) + \frac{R_2}{R_1}h_2(t) - \frac{R_2}{R_1}h_1(t) + A_2R_2\frac{dh_2(t)}{dt} (=)$$

$$(=)h_1(t) = \frac{R_1}{R_2}\left(A_2R_2\frac{dh_2(t)}{dt} + \frac{R_2}{R_1}h_2(t) + h_2(t) - R_2q_i(t)\right)(=)$$

$$(=)h_1(t) = R_1A_2\frac{dh_2(t)}{dt} + \frac{R_1}{R_2}h_2(t) + h_2(t) - R_1q_i(t)$$

Replacing equation 2 on equation 1:

$$A_1A_2R_1R_1\frac{dh_2^2(t)}{dt} + \frac{A_1R_1R_1}{R_2}\frac{dh_2(t)}{dt} + A_1R_1\frac{dh_2(t)}{dt} - A_1R_1R_1\frac{dq_i(t)}{dt} + A_2R_1\frac{dh_2(t)}{dt} + \frac{R_1}{R_2}h_2(t) + h_2(t) - R_1q_i(t) = h_2(t)$$

Multiplying the previous equation by $\frac{R_2}{R_1}$:

$$A_1A_2R_1R_2\frac{dh_2^2(t)}{dt} + A_1R_1\frac{dh_2(t)}{dt} + A_1R_2\frac{dh_2(t)}{dt} - A_1R_1R_2\frac{dq_i(t)}{dt} + A_2R_2\frac{dh_2(t)}{dt} + h_2(t) - R_2q_i(t) = 0$$

Laplace:

$$A_1A_2R_1R_2s^2H_2(s) + A_1R_1sH_2(s) + A_1R_2sH_2(s) - A_1R_1R_2sQ_i(s) + A_2R_2sH_2(s) + H_2(s) - R_2Q_i(s) = 0$$

$$A_1A_2R_1R_2s^2H_2(s) + A_1R_1sH_2(s) + A_1R_2sH_2(s) + A_2R_2sH_2(s) + H_2(s) = A_1R_1R_2sQ_i(s) + R_2Q_i(s)$$

$$H_2(s)(A_1A_2R_1R_2s^2 + (A_1R_1 + A_1R_2 + A_2R_2)s + 1) = (A_1R_1R_2s + R_2)Q_i(s)$$

$$H_2(s) = \frac{(A_1R_1R_2s + R_2)}{(A_1A_2R_1R_2s^2 + (A_1R_1 + A_1R_2 + A_2R_2)s + 1)}Q_i(s)$$

c) Representation Space of States:

$$\begin{cases} h_2(t) = h_1(t) + R_1A_1\dfrac{dh_1(t)}{dt} & (eq. 1) \\ q_i(t) = \dfrac{h_2(t)}{R_2} + \dfrac{h_2(t) - h_1(t)}{R_1} + A_2\dfrac{dh_2(t)}{dt} & (eq. 2) \end{cases}$$

Rearrange each equations in order of $\frac{dh_1(t)}{dt}$ and $\frac{dh_2(t)}{dt}$:

$$\begin{cases} \dfrac{dh_1(t)}{dt} = \dfrac{h_2(t)}{R_1A_1} - \dfrac{h_1(t)}{R_1A_1} \\ \dfrac{dh_2(t)}{dt} = \dfrac{q_i(t)}{A_2} - \dfrac{h_2(t)}{A_2R_2} - \dfrac{h_2(t)}{A_2R_1} + \dfrac{h_1(t)}{A_2R_1} \end{cases}$$

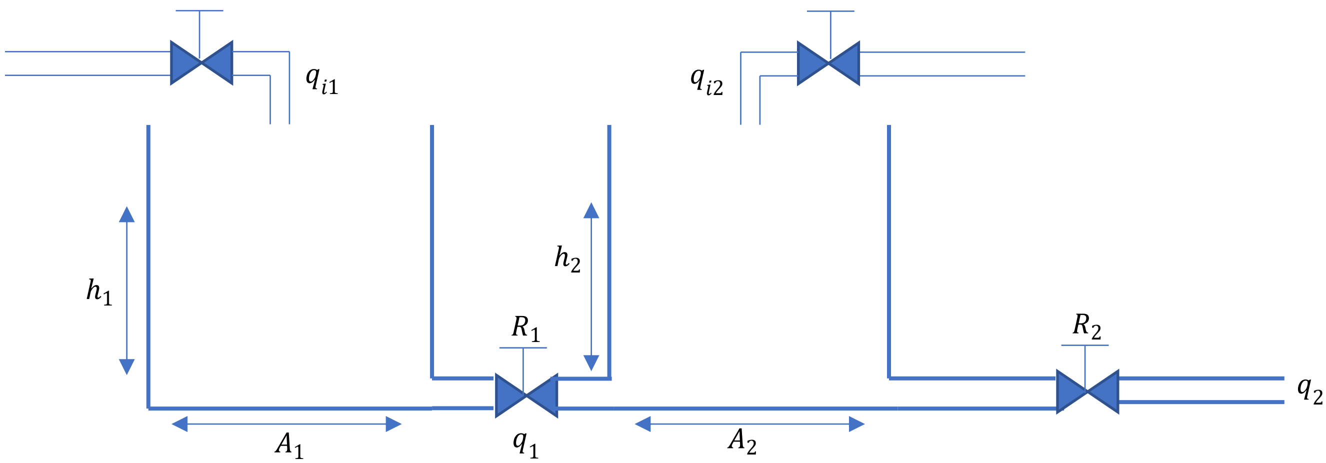
$$\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1A_1} & \dfrac{1}{R_1A_1} \\ \dfrac{1}{A_2R_1} & -\left(\dfrac{1}{A_2R_2} + \dfrac{1}{A_2R_1}\right) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{A_2} \end{bmatrix} [q_i]$$

If we want to observe the evolution of water in both tanks, then our output is:

$$y = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$

<u>Exercise 2:</u>

Model the following system using state space notation:



Tank 1 -> $in = out + accumulates$:

$$q_{i1}(t) = \frac{h_1(t) - h_2(t)}{R_1} + A_1 \frac{dh_1(t)}{dt}$$

Tank 2 -> $in = out + accumulates$:

$$q_{i2}(t) + q_1(t) = q_2(t) + A_2 \frac{dh_2(t)}{dt} \text{ (=) } q_{i2}(t) + \frac{h_1(t) - h_2(t)}{R_1} = \frac{h_2(t)}{R_2} + A_2 \frac{dh_2(t)}{dt}$$

Rearrange each equations in order of $\frac{dh_1(t)}{dt}$ and $\frac{dh_2(t)}{dt}$:

$$\begin{cases} \dfrac{dh_1(t)}{dt} = \dfrac{q_{i1}(t)}{A_1} - \dfrac{h_1(t)}{R_1A_1} + \dfrac{h_2(t)}{R_1A_1} \\ \dfrac{dh_2(t)}{dt} = \dfrac{q_i(t)}{A_2} + \dfrac{h_1(t)}{A_2R_1} - \dfrac{h_2(t)}{A_2R_2} - \dfrac{h_2(t)}{A_2R_1} \end{cases}$$

Space of states:

$$\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{1}{R_1A_1} & \dfrac{1}{R_1A_1} \\ \dfrac{1}{A_2R_1} & -\left(\dfrac{1}{A_2R_2} + \dfrac{1}{A_2R_1}\right) \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{A_1} & 0 \\ 0 & \dfrac{1}{A_2} \end{bmatrix} \begin{bmatrix} q_{i1} \\ q_{i2} \end{bmatrix}$$

Considering h_1 and h_2 as outputs:

$$y = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$$