Análise complexa

Números complexos

Simão Cardoso

- $z = x + iy \Rightarrow x = \dfrac{z+\bar{z}}{2}$, $\quad y = \dfrac{z-\bar{z}}{2}$
- $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2}$
- $\bar{z} = x - iy$
- $|z| = \sqrt{x^2+y^2}$ • $|zw| = |z||w|$
- $\overline{zw} = \bar{z}\,\bar{w}$ • $|z+w| \leq |z|+|w|$
- $\overline{z+w} = \bar{z}+\bar{w}$ • $||z|-|w|| \leq |z-w|$

Forma Polar

- $z = |z|e^{i\varphi}$
- $zw = |z|e^{i\varphi}|w|e^{i\phi} = |z||w|\,e^{i(\varphi+\phi)}$

Fórmula de Euler $\Rightarrow e^{i\varphi} = \cos\varphi + i\sin\varphi$

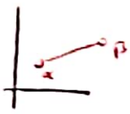
- $\dfrac{1}{z} = \dfrac{1}{|z|}e^{-i\varphi}$
- $e^{z} = e^{x}e^{iy} = e^{x}(\cos y + i \sin y)$
- $e^{z+w} = e^{z}e^{w}$

Retas

$\mathrm{Im}(\alpha z+\beta) = 0$

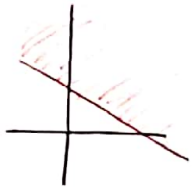
Parametrização

- $z(t) = \alpha + (\beta-\alpha)t$, $\quad t \in [0,1]$



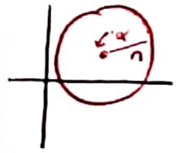
- $z(t) = \alpha + vt$

Semiplanos



$\mathrm{Im}(\alpha z+\beta) > 0$

Circunferências



$|z-\alpha| = r$

Parametrização

- $z(t) = re^{it}+\alpha$

Cauchy-Riemann

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}$$

ou

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

onde $z = u + iv$

$\Rightarrow$ Uma função é holomorfa se é derivável em todo o seu domínio

$\Rightarrow$ é inteira se é holomorfa e domínio $= \mathbb{C}$

$\Rightarrow$ é holomorfa se $\dfrac{\partial}{\partial \bar{z}} f = 0$

- $\bar{\partial} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$
- $\partial = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$

$\Rightarrow \bar{\partial}\partial = \partial\bar{\partial} = \dfrac{1}{4}\nabla^2$

$\Rightarrow$ Se $f(z) = u+iv$ é holomorfa então u e v são harmónicas se $\nabla^2 u = 0$, $\nabla^2 v = 0$.

Jacobiano

$$J = \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

Séries convergentes $\Rightarrow \sum \frac{1}{n^2}$, $\sum \frac{(-1)^n}{n}$, $\sum \frac{1}{a^n}$, $\sum \frac{1}{(-a)^n}$

Séries divergentes $\Rightarrow \sum \frac{1}{n}$, $\sum 1^n$, $\sum (-a)^n$, $\sum a^n$

Séries

$$\sum_{n=0}^{\infty} z_n = z_0 + z_1 + z_2 + \dots$$

converge se existe $\lim_{n\to\infty} \sum z_n = S$

$\Rightarrow \sum_{n=0}^{\infty} z_n$ converge absolutamente se $\sum_{n=0}^{\infty} |z_n|$ converge

Série geométrica

Se $\lambda < 1$

$$\sum_{n=0}^{\infty} \lambda^n = 1 + \lambda^2 + \lambda^3 + \dots = \frac{1}{1-\lambda}$$

Série de potências

$$\sum_{n=0}^{\infty} c_n (z-f)^n$$

$\Rightarrow$ se $|c_n| \leq \lambda^n$ com $0 \leq \lambda \leq 1$

$\Rightarrow \sum c_n$ é convergente

$\Rightarrow$ Se existe $\lim_{n\to\infty} |c_n|^{1/n} = \lambda$ e $\lambda < 1$ então $\sum c_n$ é convergente

Raio de convergência

$$R = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}$$

Derivada

$$f(z) = \sum_{n=0}^{\infty} c_n z^n = c_n \frac{1}{1-z}$$

$$f'(z) = \sum_{n=0}^{\infty} c_n n z^{n-1} = c_n \frac{1}{(1-z)^2}$$

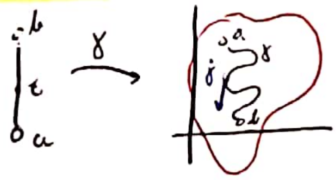
Série exponencial

$$\sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^z$$

Exponencial

- $(e^z)' = e^z$
- $e^{\bar{z}} = \overline{(e^z)}$
- $\cos(z) = \dfrac{e^{iz}+e^{-iz}}{2}$
- $\operatorname{sen}(z) = \dfrac{e^{iz}-e^{-iz}}{2i}$

Contorno



- $\gamma(t) = \alpha + (\beta-\alpha)t \to$ reta

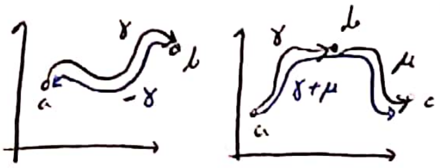
$$\frac{d\gamma}{dt} = \beta - \alpha$$

- $\gamma(t) = p + re^{it} \to$ circunferência

$$\frac{d\gamma}{dt} = ire^{it}$$

$\Rightarrow$ comprimento de γ é

$$|\gamma| = \int_a^b \left|\frac{d\gamma}{dt}\right| dt$$



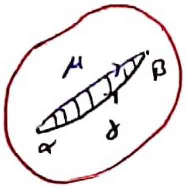
Integral de contorno

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\frac{d\gamma}{dt}\,dt$$

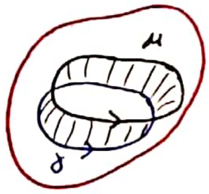
Propriedades

- $\displaystyle\int_{-\gamma} f(z)\,dz = -\int_\gamma f(z)\,dz$
- $\displaystyle\int_{\gamma+\mu} f(z)\,dz = \int_\gamma f(z)\,dz + \int_\mu f(z)\,dz$
- se $|f(z)| < M \to \left|\displaystyle\int_\gamma f(z)\,dz\right| \leq M\cdot|\gamma|$

Teorema $\Rightarrow$ se $\oint_\gamma f(z)\,dz = 0$ para todo o caminho fechado γ em $\Omega \subset \mathbb{C}$ então existe uma primitiva $F(z)$ de $f(z)$.



$$\int_\gamma f(z)\,dz = \int_\mu f(z)\,dz$$



$$\oint_\gamma f(z)\,dz = \oint_\mu f(z)\,dz$$

$\Rightarrow$ Se γ é contratível

$\Rightarrow \oint_\gamma f(z)\,dz = 0$

Integral de Cauchy

Teorema $\Rightarrow$ se $f(z)$ em Ω e $\overline{D_\rho(A)} \subset \Omega$, então

$$\boxed{f(A) = \frac{1}{2\pi i}\oint_{|z-A|=\rho} \frac{f(z)}{z-A}\,dz}$$

- $$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_{|w-A|=\rho} \frac{f(w)}{(w-z)^{n+1}}\,dw$$

Em geral

$$f(z) = \sum_{n=0}^{\infty} c_n (z-A)^n$$

onde

$$c_n = \frac{1}{2\pi i}\oint_{|w-A|=\rho} \frac{f(w)}{|w-z|^{n+1}}\,dw$$

$\Rightarrow$ holomorfa $\equiv$ analítica

$\Rightarrow f$ holomorfa em $\Omega \Rightarrow \frac{df}{dz}$ holomorfa em Ω $\therefore$ $f^{(n)}$ são holomorfas em Ω

Série de Taylor

- $\operatorname{sen}(z) = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \dots$
- $\cos(z) = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - \dfrac{z^6}{6!} + \dots$
- $\log(z) = (z-1) - \frac{1}{2}(z-1)^2 + \frac{1}{3}(z-1)^3 - \dots$

Série de Laurent

Teorema $\Rightarrow$ se $f(z)$ é holomorfa no anel $A_{r,R}(A) = \{r < |z-A| < R\}$ então

$$\boxed{f(z) = \sum_{-\infty}^{\infty} a_n (z-A)^n}$$

onde

$$a_n = \frac{1}{2\pi i}\oint_{|z-A|=\rho} \frac{f(z)}{(z-A)^{n+1}}\,dz$$

Singularidades Isoladas

$\Rightarrow$ O ponto A é uma singularidade isolada da função $f(z)$ se $f(z)$ é holomorfa num disco perfurado $\{0 < |z-A| < \varepsilon\}$

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-A)^n$$

Essencial $\Rightarrow$ se existem infinitos $a_n \neq 0$ com n negativo

Removível $\Rightarrow$ se todos os coeficientes negativos são nulos

• cálculos resíduos

• Se p é um polo simples, logo

$$f(z) = \frac{c_{-1}}{z-p} + c_0 + c_1(z-p) + c_2(z-p)^2 + \dots$$

então

$$\operatorname{res}(f,p) = \lim_{z\to p} (z-p) f(z)$$

• Se p é um polo duplo, logo

$$f(z) = \frac{c_{-2}}{(z-p)^2} + \frac{c_{-1}}{(z-p)} + c_0 + c_1(z-p) + c_2(z-p)^2 + \dots$$

então

$$\operatorname{res}(f,p) = \lim_{z\to p} \frac{d}{dz}(z-p)^2 f(z)$$

• Se p é um polo de ordem m, logo

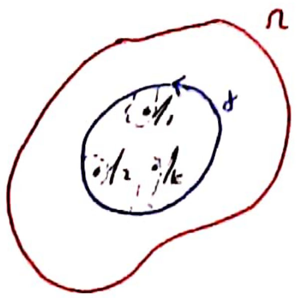
$$f(z) = \frac{c_{-m}}{(z-p)^m} + \dots + c_0 + \dots + c_m(z-p)^m$$

então

$$\operatorname{res}(f,p) = \frac{1}{(m-1)!} \lim_{z\to p} \frac{d^{m-1}}{dz^{m-1}} (z-p)^m f(z)$$

⇒ $f(z)$ holomorfa em Ω exceto a umas singularidades isoladas no interior da curva

$$\boxed{\oint_\gamma f(z)\,dz = 2\pi i \sum_k \operatorname{res}(f, p_k)}$$



⇒ $\cos(\alpha x) = e^{i\alpha x}$

$\operatorname{sen}(\alpha x) = e^{i\alpha x}$

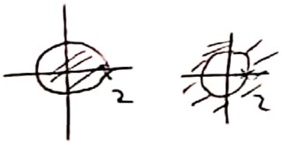
⇒ $\cos(\varphi) = \dfrac{z + 1/z}{2}$

$\operatorname{sen}(\varphi) = \dfrac{z - 1/z}{2i}$

$\dfrac{dz}{iz} = d\varphi$

Laurent Series

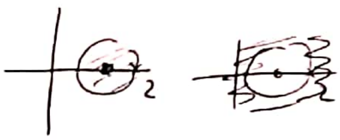
1) $f(z) = \dfrac{1}{z-2}$, around $z=0$



$|z|<2 \rightarrow \dfrac{1}{z-2} = -\dfrac{1}{2}\dfrac{1}{1-z/2} = -\dfrac{1}{2}\sum\left(\dfrac{z}{2}\right)^n = -\sum \dfrac{z^n}{2^{n+1}}$

$|z|>2 \rightarrow \dfrac{1}{z-2} = \dfrac{1}{z}\dfrac{1}{1-2/z} = \sum \dfrac{2^n}{z^{n+1}}$

2) $f(z) = \dfrac{1}{z-2}$, around $z=1$

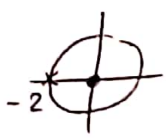


$|z-1|<1 \rightarrow \dfrac{1}{z-2} = \dfrac{1}{(z-1)-1} = -\dfrac{1}{1-(z-1)} = -\sum (z-1)^n$

$|z-1|>1 \rightarrow \dfrac{1}{z-2} = \dfrac{1}{(z-1)-1} = \dfrac{1}{z-1} \cdot \dfrac{1}{1-\frac{1}{z-1}} = \sum \dfrac{1}{(z-1)^{n+1}}$

Taylor Series

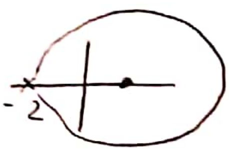
1) $f(z) = \dfrac{1}{2+z}$, around $z=0$



$R = 2$

$\dfrac{1}{2+z} = \dfrac{1}{2}\dfrac{1}{1-(-z/2)} = \sum (-1)^n \dfrac{z^n}{2^{n+1}}$

2) $f(z) = \dfrac{1}{2+z}$, around $z=1$



$R = 3$

$\dfrac{1}{2+z} = \dfrac{1}{2+1+(z-1)} = \dfrac{1}{3}\dfrac{1}{1-\left(-\frac{(z-1)}{3}\right)} = \sum \dfrac{(-1)^n (z-1)^n}{3^{n+1}}$