Guias de ondas

Considere um tubo com uma condutividade eléctrica perfeita.

Isto significa que, dentro deste condutor perfeito $\vec{E} = \vec{B} = 0$.

As condições de fronteira impostas pelas equações de Maxwell são (como vimos):

$$B_1^{\perp} = B_2^{\perp} \qquad \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = \sigma_f$$

$$E_1^{\parallel} = E_2^{\parallel} \qquad \frac{1}{\mu_1} B_1^{\parallel} - \frac{1}{\mu_2} B_2^{\parallel} = \vec{K}_f \wedge \hat{n} \qquad (*)$$

$$\downarrow$$

No tubo: (nas paredes interiores do tubo)
$$\begin{cases} B_1^{\perp} = 0 \\ E_1^{\parallel} = 0 \end{cases}$$

Que ondas monocromáticas se podem propagar no tubo? (orientado segundo zz'). Vejamos: Procuramos soluções do tipo:

$$\tilde{\vec{E}}(x,y,z,t) = \tilde{\vec{E}}_0(x,y)\, e^{i(kz-\omega t)}$$
$$\tilde{\vec{B}}(x,y,z,t) = \tilde{\vec{B}}_0(x,y)\, e^{i(kz-\omega t)} \qquad (**)$$

Estas funções devem ser constrangidas de forma a obedecerem simultaneamente às condições de fronteira $(*)$ e às equações de Maxwell:

$$\nabla \cdot \vec{E} = 0 \qquad \nabla \wedge \vec{E} = -\dot{\vec{B}}$$
$$\nabla \cdot \vec{B} = 0 \qquad \nabla \wedge \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

Admitimos que $\vec{E}_0$ $(\vec{B}_0)$ possa ter uma componente longitudinal $(\vec{E}_0 = E_{0x}\hat{x} + E_{0y}\hat{y} + E_{0z}\hat{z})$.

Consideremos então a eq. $\nabla\wedge\vec{E} = -\dot{\vec{B}}$ c/ $\vec{E}$ e $\vec{B}$ dados por (xx)

$$(\nabla\wedge\vec{E})_x = \frac{\partial\tilde{E}_z}{\partial y} - \frac{\partial\tilde{E}_y}{\partial z} = \left(\frac{\partial\tilde{E}_{0z}}{\partial y} - ik\tilde{E}_{0y}\right)e^{i(kz-\omega t)} = -\dot{B}_x = i\omega B_x \quad (ii)$$

$$(\nabla\wedge\vec{E})_y = \left(ik\tilde{E}_{0x} - \frac{\partial E_{0z}}{\partial x}\right)e^{i(kz-\omega t)} \Rightarrow ik\tilde{E}_x - \frac{\partial\tilde{E}_z}{\partial x} = i\omega B_y \quad (iii)$$

$$(\nabla\wedge\vec{E})_z = \quad \Rightarrow \quad \frac{\partial\tilde{E}_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z \quad (i)$$

De forma semelhante, a eq. $\nabla\wedge\vec{B} = \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$ impõe:

$$\frac{\partial\tilde{B}_z}{\partial y} - ik\tilde{B}_y = -\frac{i\omega}{c^2}\tilde{E}_x \qquad (v)$$

$$ik\tilde{B}_x - \frac{\partial\tilde{B}_z}{\partial x} = -\frac{i\omega}{c^2}E_y \qquad (vi)$$

$$\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i\frac{\omega}{c^2}E_z \qquad (iv)$$

Podemos resolver ii) iii) v) e vi) para exprimir B_x, B_y, E_x e E_y à custa das componentes z dos campos:

Por exemplo: Multiplique iii) por k e v) por ω:

$$k \times iii) \Rightarrow ik^2E_x - k\frac{\partial E_z}{\partial x} = i\omega k B_y$$

$$\omega \times v) \Rightarrow \omega\frac{\partial B_z}{\partial y} - ik\omega B_y = -\frac{i\omega^2}{c^2}E_x$$

Subtraindo ordenadamente:

$$i\left(k^2 - \frac{\omega^2}{c^2}\right)E_x - k\frac{\partial E_z}{\partial x} - \omega\frac{\partial B_z}{\partial y} = 0 \quad \Rightarrow$$

$$\Rightarrow \quad E_x = \frac{-i}{k^2 - \frac{\omega^2}{c^2}} \cdot \left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right)$$

De forma semelhante:

$$E_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right]$$

$$B_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right)$$

$$B_y = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right)$$

As componentes transversais do campo podem ser expressas à custa das componentes longitudinais E_z e B_z.

Consideremos agora $\nabla\cdot\vec{E} = 0$ e $\nabla\cdot\vec{B} = 0$

$$\nabla\cdot\vec{E} = 0 \quad \Rightarrow \quad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \left(\frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} + ikE_{0z}\right)e^{i(kz-\omega t)} = 0$$

$$\Rightarrow \quad \frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} + ikE_{0z} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{\partial}{\partial x}\left[\frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial E_{0z}}{\partial x} + \omega\cancel{\frac{\partial B_{0z}}{\partial y}}\right)\right] +$$

$$+ \frac{\partial}{\partial y}\left[\frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left(k\frac{\partial E_{0z}}{\partial y} - \omega\cancel{\frac{\partial B_{0z}}{\partial x}}\right)\right] + ikE_{0z} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{\cancel{i}}{\left(\frac{\omega}{k}\right)^2 - k^2}\left[\cancel{k}\frac{\partial^2 E_{0z}}{\partial x^2} + \cancel{k}\frac{\partial^2 E_{0z}}{\partial y^2}\right] + \cancel{ik}E_{0z} = 0 \quad \Rightarrow$$

$$\frac{\partial^2 E_{0z}}{\partial x^2} + \frac{\partial^2 E_{0z}}{\partial y^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] E_{0z} = 0$$

De forma semelhante: $\nabla \cdot \vec{B} = 0 \Rightarrow$

$$\frac{\partial^2 B_{0z}}{\partial x^2} + \frac{\partial^2 B_{0z}}{\partial y^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] B_{0z} = 0$$

Temos que resolver estas duas equações para encontrarmos as componentes longitudinais dos campos. Se $E_z = 0 \Rightarrow$ $\Rightarrow$ o campo eléctrico é puramente transversal (TE-mode). Se $B_z = 0$, TM-mode.; se $E_z = B_z = 0 \Rightarrow$ TEM-mode.

Num tubo oco, modos TEM não podem ocorrer:

$$E_z = 0 \rightarrow \nabla \cdot \vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0$$

$$B_z = 0 \rightarrow (\nabla \wedge \vec{E})_z = 0 \Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0$$

$\vec{E}_0$ é tal que o seu rotacional e a sua divergência são nulos. Logo, $\vec{E}_0 = \nabla \phi$, onde ϕ é uma função harmonica ($\nabla^2 \phi = 0$)

A condição de fronteira $\vec{E}^{\parallel} = 0 \Rightarrow$ que as paredes do tubo são equipotenciais. Então $\nabla^2 \phi = 0 \Rightarrow \phi = \text{const.}$ (é a única solução possível) $\Rightarrow \vec{E} = 0$.

Caso de um tubo rectangular

1. · Modos TE $(E_z = 0)$

$$B_z(x,y) = X(x)\, Y(y)$$

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial B_z}{\partial y^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] B_z = 0 \iff$$

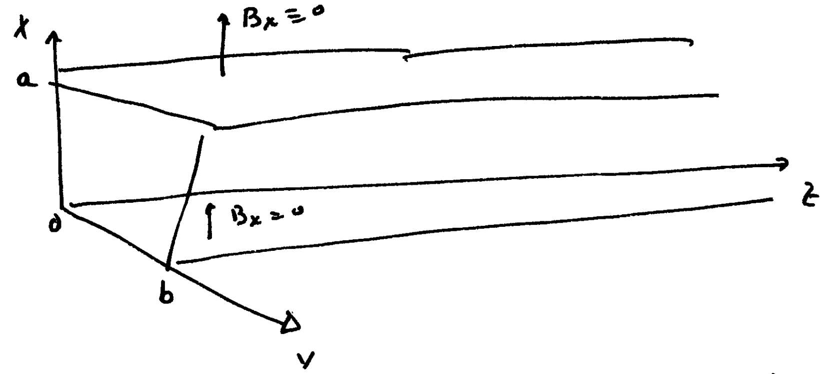
$$\Rightarrow Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] XY = 0 \Rightarrow$$

$$\Rightarrow \underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-k_x^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}}_{-k_y^2} + \left(\frac{\omega}{c}\right)^2 - k^2 = 0$$

se $\boxed{-k_x^2 - k_y^2 - k^2 + \left(\frac{\omega}{c}\right)^2 = 0}$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \Rightarrow X(x) = A \sin(k_x x) + B \cos(k_x x)$$

Mas



$x = 0$ e $x = a$ $\quad B_\perp = 0 \Rightarrow B_x(x=0;\ \text{ou}\ x = a) = 0 \Rightarrow$

$\Rightarrow$ (dado que $B_x = \dfrac{i}{\left(\frac{\omega}{c}\right)^2 - k^2}\left[k \dfrac{\partial B_z}{\partial x} - \dfrac{\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right]$ e $E_z = 0$)

$\Rightarrow \dfrac{\partial B_z}{\partial x} = 0$ nas 2 superfícies; ou seja $\dfrac{dX}{dx} = 0$ para

$x = 0$ ou $x = a$; Isto impõe que

$$\frac{dX}{dx} = A\, k_x \cos(k_x x) - B\, k_x \sin(k_x x)$$

$$x = 0 \Rightarrow A = 0$$

$$x = a \Rightarrow k_x a = m\pi$$

De forma semelhante para $\dfrac{dY}{dy}$ em $y=0$ e $y=b$:$\Rightarrow$

$$\Rightarrow k_y = n\frac{\pi}{b}$$

Então: $$B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cdot\cos\left(\frac{n\pi y}{b}\right)$$

(Modo TE_{mn})

Como $-k_x^2 - k_y^2 + \left(\frac{\omega}{c}\right)^2 = k^2 \Rightarrow$

$$\Rightarrow \quad k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Soluções com k - real impõem que:

$$\left(\frac{\omega}{c}\right)^2 > \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \quad \text{ou}$$

$$\omega > c\pi\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \quad (\text{cut-off})$$

<u>Observações</u>: qual a menor frequência ω que se pode propagar no tubo? (Modo TE)

$$m = n = 0 \Rightarrow \omega = 0 \,! \quad \text{Mas: então } \frac{\omega}{c} = k$$

<u>Se $m=n=0$</u>: como

$$\underbrace{\frac{\partial E_z}{\partial y}}_{=0} - ik E_y = i\omega B_x \Rightarrow E_y = -c B_x$$

$$ik E_x - \underbrace{\frac{\partial E_z}{\partial x}}_{=0} = i\omega B_y \Rightarrow E_x = c B_y$$

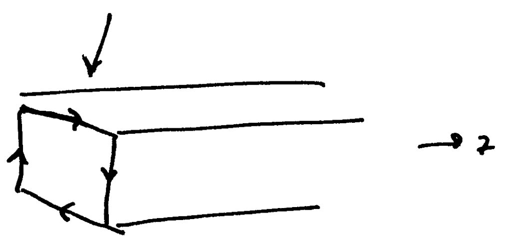
$$\frac{\partial B_z}{\partial y} - ik B_y = -\frac{i\omega}{c^2} E_x \Rightarrow$$

$$\Rightarrow \frac{\partial B_z}{\partial y} = i\left(k B_y - \frac{\omega}{c^2} E_x\right) = i\left(k B_y - \frac{\omega}{c} B_y\right) = 0$$

$$\frac{\partial B_z}{\partial x} = i\left(k B_x + \frac{\omega}{c^2} E_y\right) = i\left(k B_x - \frac{\omega}{c} B_x\right) \equiv 0$$

Logo $B_z = \text{const}$. É fácil mostrar que, então, $B_z \equiv 0$:

$$\oint \vec{E}\cdot d\vec{l} = -\int \dot{\vec{B}}\cdot d\vec{\Sigma} = +i\omega\int \vec{B}\cdot d\vec{\Sigma} = i\omega B_z ab \equiv$$



$$\equiv \oint_c \vec{E}\cdot d\vec{l} \equiv 0 \Rightarrow B_z = 0.$$

($c \to$ no metal)

i.e. TEM-modos são proibidos aqui.

Logo, a frequência mínima que se pode propagar é

$$\boxed{\omega_{10} = c\,\frac{\pi}{a}}$$

$$k = \sqrt{\left(\frac{\omega}{c}\right)^2 - \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

$$\downarrow$$

Vel. fase: $v = \frac{\omega}{k} = ?$

$$\frac{\omega}{c}\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}$$

$$v = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}} > c$$

Mas a velocidade de grupo: $v_g = \frac{\partial \omega}{\partial k} = \frac{1}{\partial k / \partial \omega} =$

$$= \frac{1}{\frac{1}{c}\cancel{\frac{1}{2}}\left(\omega^2 - \omega_{mn}^2\right)^{-1/2} \cdot \cancel{2}\omega} =$$

$$= \frac{c\sqrt{\omega^2 - \omega_{mn}^2}}{\omega} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2} < c .$$

$\square$

- **Modos TM**

Neste caso $E_z \neq 0$ e $B_z = 0$; entas temos que resolver

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \left(\frac{\omega}{c}\right)^2 - k^2\right] E_z = 0$$

com $\vec{E}'' = 0$ na fronteira.

$E_z(x,y) = X(x)\, Y(y) \rightarrow \quad Y X'' + X Y'' + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] X Y = 0$

$$\underbrace{\frac{1}{X} X''}_{-k_x^2} + \underbrace{\frac{1}{Y} Y''}_{-k_y^2} + \left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = 0$$

$$X'' + k_x^2 X = 0 \quad \rightarrow \quad X(x) = A \sin(k_x x) + B \cos(k_x x)$$

$x = 0$, $x = a$ $\rightarrow$ $X(x) = 0$ (visto que $E_z = 0$ na fronteira)

Então: $B = 0$ e $k_x a = m\pi \rightarrow k_x = \frac{m\pi}{a}$; $X = A \sin\left(\frac{m\pi x}{a}\right)$

$Y(y) = A_2 \sin\left(\frac{n\pi y}{b}\right)$; $k_y = \frac{n\pi}{b}$

Logo

$$E_z = E_0 \sin\left(\frac{m\pi x}{a}\right) . \sin\left(\frac{n\pi y}{b}\right) \quad , \quad n = 1, 2, 3, \ldots$$
$$m = 1, 2, 3, \ldots$$

Nota: Se $n = m = 0 \Rightarrow E_z = 0 \Rightarrow$ modo TEM (inexistente).

$\omega_{mn} = c\pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$ $\rightarrow$ frequência de corte para o modo mn.

Neste caso a frequência mínima que se pode propagar no tubo é $\omega_{11} = c\pi \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$ (TM)

[Como vimos, a menor frequência EM é $\omega_{10} = \frac{c\pi}{a}$

Então: $\frac{\omega_{11}^{TM}}{\omega_{10}^{TE}} = \sqrt{1 + \left(\frac{a}{b}\right)^2}$]

• $v = \frac{\omega}{k} = ?$ * $-\left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 + \frac{\omega^2}{c^2} - k^2 = 0 \Rightarrow$

$$\Rightarrow k = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} = \sqrt{\frac{\omega^2}{c^2} - \frac{\omega_{mn}^2}{c^2}} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}$$

$$V = \frac{\omega}{k} = \frac{c}{\sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}} \quad ; \quad V_g = \left(\frac{\partial k}{\partial \omega}\right)^{-1} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}$$

Problema 9.28 (Griffiths):

Um tubo rectangular com $a = 2{,}28$ cm e $b = 1{,}01$ cm. é excitado com uma frequencia $1{,}7 \times 10^{10}$ Hz. Que modos TE se podem propagar?

$$\nu_{10} = \frac{\omega_{10}}{2\pi} = \frac{c\pi}{a} \cdot \frac{1}{2\pi} = \frac{c}{2a} \cong 0{,}66 \times 10^{10} \text{ Hz}.$$

$$\nu_{20} = \frac{\omega_{20}}{2\pi} = \frac{c\,2\pi}{a\,2\pi} = \frac{c}{a} = 1{,}32 \times 10^{10}$$

$$\nu_{30} = \frac{c\,3\pi}{a\,2\pi} = \frac{3}{2}\frac{c}{a} = 1{,}97 \times 10^{10} \text{ Hz} \quad \times$$

(não se propaga)

$$\nu_{01} = \frac{c}{2b} = 1{,}49 \times 10^{10} \text{ Hz}$$

$$\nu_{02} = \frac{c}{b} = 2{,}97 \times 10^{10} \text{ Hz} \quad \times \text{ Não se propaga}$$

$$\nu_{11} = \frac{c}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}} = 1{,}62 \times 10^{10} \text{ Hz}$$

Propagam-se os modos 10, 20, 01, 11

Se quisermos que apenas um modo se propague devemos excitar com uma frequencia. $\nu_{10} < \nu < \nu_{20}$

Problema 9.29 (Griffiths)

Mostre que a energia de um modo TE_{mn} se propaga com a velocidade de grupo desse modo.

Solução:

Como vimos $B_z = B_0 \cos\left(\frac{m\pi x}{a}\right) \cdot \cos\left(\frac{n\pi y}{b}\right)$

$$E_z = 0$$

Substituindo nas expressões da página-38 obtém-se:

$$E_x = \frac{+i\omega}{\left(\frac{\omega}{c}\right)^2 - k^2} \frac{\partial B_z}{\partial y} = B_0 \frac{i\omega}{\left(\frac{\omega}{c}\right)^2 - k^2} \left(-\frac{n\pi}{b}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

De forma semelhante:

$$E_y = \frac{-i\omega}{\left(\frac{\omega}{c}\right)^2 - k^2} \left(-\frac{m\pi}{a}\right) B_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$B_x = \frac{i}{\left(\frac{\omega}{c}\right)^2 - k^2} k \frac{\partial B_z}{\partial x} = \frac{ik B_0}{\left(\frac{\omega}{c}\right)^2 - k^2} \left(-\frac{m\pi}{a}\right) \sin\left(\frac{m\pi}{a}x\right) \cos\left(\frac{n\pi}{b}y\right)$$

$$B_y = \frac{ik B_0}{\left(\frac{\omega}{c}\right)^2 - k^2} \left(\frac{n\pi}{b}\right) \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi}{b}y\right)$$

Podemos mostrar (ver adiante) que

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} (\vec{E} \times \vec{B}^*)$$

$$\tilde{\vec{E}} = \tilde{\vec{E}}_0(x,y)\, e^{i(kz-\omega t)} \qquad \tilde{\vec{B}} = \tilde{\vec{B}}_0(x,y)\, e^{i(kz-\omega t)}$$

$$\langle \vec{S} \rangle = \frac{1}{2\mu_0} \left\{ \left(E_{0x}\,\hat{x} + E_{0y}\,\hat{y} \right) \wedge \left(B^*_{0x}\,\hat{x} + B^*_{0y}\,\hat{y} + B^*_{0z}\,\hat{z} \right) \right\}$$

$$= \frac{1}{2\mu_0} \left\{ E_{0x} B^*_{0y}\,\hat{z} - E_{0x} B^*_{0z}\,\hat{y} - E_{0y} B^*_{0y}\,\hat{z} + E_{0y} B^*_{0z}\,\hat{x} \right\}$$

$$= \frac{1}{2\mu_0} \left\{ \left(E_{0x} B^*_{0y} - E_{0y} B^*_{0y} \right) \hat{z} + E_{0y} B^*_{0z}\,\hat{x} - E_{0x} B^*_{0z}\,\hat{y} \right\}$$

$$= \frac{1}{2\mu_0} \left\{ \frac{i\omega\pi B_0^2}{\left(\frac{\omega}{c}\right)^2 - k^2} \left(\frac{m}{a}\right) \cos\left(\frac{m\pi x}{a}\right) \cos^2\left(\frac{n\pi y}{b}\right) \sin\left(\frac{m\pi x}{a}\right) \hat{x} + \right.$$

$$+ \frac{i\omega\pi B_0^2}{\left(\frac{\omega}{c}\right)^2 - k^2} \left(\frac{n}{b}\right) \cos^2\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \cos\left(\frac{n\pi y}{b}\right) \hat{y} +$$

$$+ \frac{\omega k \pi^2 B_0^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2} \left[\left(\frac{n}{b}\right)^2 \cos^2\left(\frac{m\pi x}{a}\right) \sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2 \sin^2\left(\frac{m\pi x}{a}\right) \cos^2\left(\frac{n\pi y}{b}\right) \right] \hat{z}$$

Integrando sobre a seccion transversal

$$\int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx = \int_0^a \cos^2\left(\frac{m\pi x}{a}\right) dx = \frac{a}{2}$$

$$\int_0^b \sin^2\left(\frac{n\pi y}{b}\right) dy = \int_0^b \cos^2\left(\frac{n\pi y}{b}\right) dy = \frac{b}{2} \longrightarrow$$

$$\rightarrow \int \langle \vec{S} \rangle \cdot d\vec{\Sigma} = \frac{1}{8\mu_0} \frac{\omega k \pi^2 B_0^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2}\, ab \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right]$$

$$= \frac{1}{8\mu_0} \frac{\omega k \pi^2 B_0^2}{\left(\omega_{mn}/c\right)^4}\, ab \left(\frac{\omega_{mn}}{\pi c}\right)^2 = \frac{\omega k a b c^2}{8\mu_0\, \omega_{mn}^2} B_0^2$$

$$\langle u \rangle = \frac{1}{4}\left(\varepsilon_0 \tilde{\vec{E}} \cdot \tilde{\vec{E}}^* + \frac{1}{\mu_0} \tilde{\vec{B}} \tilde{\vec{B}}^* \right)$$

$$= \frac{\varepsilon_0}{4} \frac{\omega^2 \pi^2 B_0^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2} \left[\left(\frac{n}{b}\right)^2 \cos^2\left(\frac{m\pi x}{a}\right) \cdot \sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2 \sin^2\left(\frac{m\pi x}{a}\right) \cdot \cos^2\left(\frac{n\pi y}{b}\right) \right]$$

$$+ \frac{1}{4\mu_0} \left\{ B_0^2 \cos^2\left(\frac{m\pi x}{a}\right) \cos^2\left(\frac{n\pi y}{b}\right) + \frac{k^2\pi^2 B_0^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2} \left[\left(\frac{n}{b}\right)^2 \cos^2\left(\frac{m\pi x}{a}\right) \cdot \sin^2\left(\frac{n\pi y}{b}\right) + \left(\frac{m}{a}\right)^2 \sin^2\left(\frac{m\pi x}{a}\right) \cos^2\left(\frac{n\pi y}{b}\right) \right] \right\}$$

~~Integrando Sobre~~

Integrando sobre a secção transversal temos uma energia por unidade de comprimento:

$$\int_0^a\int_0^b \langle u \rangle \, dx\, dy = \frac{ab}{4} \left\{ \frac{\varepsilon_0}{4} \frac{\omega^2 \pi^2 B_0^2}{\left[\left(\frac{\omega}{c}\right)^2 - k^2\right]^2} \left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2 \right] + \frac{B_0^2}{4\mu_0} + \frac{1}{4\mu_0} \frac{k^2\pi^2 B_0^2}{\left[\left(\frac{\omega}{c}\right)^2 + k^2\right]^2} \left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2 \right] \right\} =$$

$$\left[\left(\frac{\omega}{c}\right)^2 - k^2\right] = \left(\frac{\omega_{mn}}{c}\right)^2$$

$$\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 = \left(\frac{\omega_{mn}}{\pi c}\right)^2 \qquad \varepsilon_0 \mu_0 c^2 = 1 \rightarrow \varepsilon_0 = \frac{1}{\mu_0 c^2}$$

$$= \frac{ab}{4} \left\{ \frac{1}{4\mu_0 \cancel{c^2}} \frac{\omega^2 \cancel{\pi^2} B_0^2}{\omega_{mn}^{\cancel{4}\,2}/\cancel{c^4}} \left(\frac{\omega_{mn}}{\cancel{\pi c}}\right)^2 + \frac{B_0^2}{4\mu_0} + \frac{B_0^2}{4\mu_0} \frac{k^2 \cancel{\pi^2}}{\left(\frac{\omega_{mn}}{c}\right)^{\cancel{4}\,2}} \left(\frac{\omega_{mn}}{\cancel{\pi c}}\right)^2 \right\}$$

$$= \cancel{\frac{ab B_0^2}{8} \left\{ \frac{1}{2\mu_0} \right.}$$

k

$$= \frac{ab}{4}\left\{ \frac{B_0^2}{4\mu_0}\frac{\omega^2}{\omega_{mn}^2} + \frac{B_0^2}{4\mu_0} + \frac{B_0}{4\mu_0}\frac{k^2c^2}{\omega_{mn}^2} \right\}$$

Mas $\quad c^2k^2 = \omega^2 - \omega_{mn}^2$

$$= \frac{ab\,B_0^2}{8\mu_0}\frac{\omega^2}{\omega_{mn}^2} \;\Big\downarrow$$

densidade linear de energia:

$$\frac{\text{energia por unidade de tempo através da secção transv.}}{\text{energia por unidade de comprimento}} \equiv \text{vel. prop. energia} =$$

$$= \frac{\cancel{\omega}\, k\, \cancel{ab}\, c^2}{\cancel{8\mu_0}\, \cancel{\omega_{mn}^2}} \cancel{B_0^2} \Bigg/ \frac{\cancel{ab}\, \cancel{B_0^2}}{\cancel{8\mu_0}} \frac{\omega^{\cancel{2}}}{\cancel{\omega_{mn}^2}} = \frac{k\,c^2}{\omega} =$$

$$= \frac{c^2}{\omega/k} = \frac{c^{\cancel{2}}}{\omega}\frac{1}{\cancel{c}}\sqrt{\omega^2-\omega_{mn}^2} = c\sqrt{1-\frac{\omega_{mn}^2}{\omega^2}} = V_g .$$

$\square$

Problema 9.11

$$\langle f g \rangle = \frac{1}{T}\int_0^T a\cos(\bar{k}\cdot\bar{r} - \omega t + \delta_a)\cdot b\cos(\bar{k}\cdot\bar{r} - \omega t + \delta_b)\, dt =$$

$$= \frac{ab}{2T}\int_0^T \left[\cos\left(2\bar{k}\cdot\bar{r} - 2\omega t + \delta_a + \delta_b\right) + \cos(\delta_a - \delta_b)\right] dt =$$

$$= \frac{ab}{2T}\cos(\delta_a-\delta_b)\,T = \frac{1}{2}ab\cos(\delta_a-\delta_b)$$

$\langle \tilde{f} \cdot \tilde{g} \rangle$;

$$\tilde{f} = \tilde{a}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

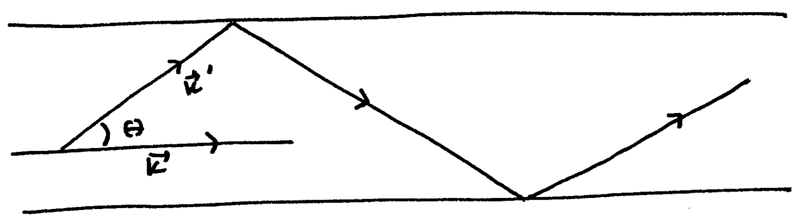
$$\tilde{g} = \tilde{b}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\frac{1}{2}\left(\tilde{f}\,\tilde{g}^{*}\right) = \frac{1}{2}\ \tilde{a}\,\tilde{b}^{*} = \frac{1}{2}\, a b\, e^{i(\delta_a - \delta_b)}$$

$$\mathbb{R}\left(\frac{1}{2}\tilde{f}\tilde{g}^{*}\right) = \frac{1}{2}\, ab \cos(\delta_a - \delta_b) = \langle f g \rangle$$

$\square$

<u>Observação:</u>



$$\vec{k}' = k\,\hat{z} + k_x\,\hat{x} + k_y\,\hat{y}$$

interferência construtiva : $k_x = \dfrac{m\pi}{a}$; $k_y = \dfrac{n\pi}{b}$

$$\vec{k}' = k\,\hat{z} + \frac{m\pi}{a}\,\hat{x} + \frac{n\pi}{b}\,\hat{y}$$

$$\omega = c\,|\vec{k}'| = c\sqrt{k^2 + \pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} =$$

$$= \sqrt{c^2 k^2 + \omega_{mn}^2} \quad \rightarrow \cancel{|\vec{k}'| = \frac{1}{c}\sqrt{c^2 k^2 + \omega_{mn}^2}}$$

$$\cancel{\cos\theta = \frac{k}{|\vec{k}'|} = \frac{kc}{\omega}}$$

$$c^2k^2 + \omega_{mn}^2 = \omega^2 = c^2k'^2$$

$$\frac{c^2k^2}{\omega^2} + \frac{\omega_{mn}^2}{\omega^2} = 1$$

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{mn}^2}{\omega^2} = \frac{k^2}{k'^2} = \cos^2\theta$$

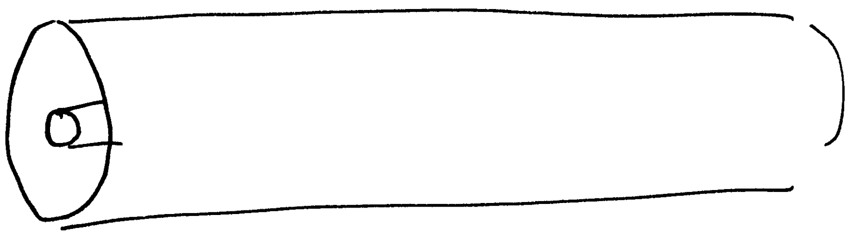
Logo: $\cos\theta = \sqrt{1 - \frac{\omega_{mn}^2}{\omega^2}}$ e $v_g = c\cos\theta < c$

A onda propaga-se com velocidade c mas a propagação do campo ao longo de z é $c\cos\theta$.

Por outro lado: $\frac{\omega^2}{k^2c^2} = \frac{1}{\cos^2\theta} \rightarrow \frac{\omega}{k} = v_f = \frac{c}{\cos\theta} > c.$

□

Modos TEM : o caso de um guia de ondas coaxial



As equações de Maxwell para campos do tipo

$$\vec{E}(\vec{r},t) = \vec{E}_0(x,y)\, e^{i(kz-\omega t)}$$

$$\vec{B}(\vec{r},t) = \vec{B}_0(x,y)\, e^{i(kz-\omega t)} \quad \text{c/ } \tilde{B}_{0z} = \tilde{E}_{0z} = 0$$

impõem que

$$\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y} = 0 \qquad \frac{\partial B_{0y}}{\partial x} - \frac{\partial B_{0x}}{\partial y} = 0$$

$$E_{0y} = -\frac{\omega}{k} B_{0x} = -\frac{B_{0x}}{c} \quad ; \quad B_{0y} = \frac{E_{0x}}{c} \qquad \left(\frac{\omega}{k} = c\right)$$

$$\left(\nabla \wedge \vec{E}_0 = 0 \ ; \ \nabla \wedge \vec{B}_0 = 0\right)$$

e que: $\left(\nabla \cdot \vec{E}_0 = \nabla \cdot \vec{B}_0 = 0\right)$

$$\frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} = 0 = \frac{\partial B_{0x}}{\partial x} + \frac{\partial B_{0y}}{\partial y}$$

Estas equações a amplitudes $\vec{E}_0$ e $\vec{B}_0$: $\nabla \wedge \vec{E}_0 = 0$, $\nabla \cdot \vec{E}_0 = 0$

$(\nabla \wedge \vec{B}_0) = 0$; $\nabla \cdot \vec{B} = 0$. Estas equações têm soluções para campos $\vec{E}_0 = \frac{A}{s}\hat{s}$ e $\vec{B}_0 = \frac{A}{cs}\hat{\phi}$ (ver magneto- e electro-estática).

O modo TEM é assim possível

$$\vec{E}(s,\phi,z,t) = \frac{A}{s}\cos(kz-\omega t)\ \hat{s}$$

$$\vec{B}(s,\phi,z,t) = \frac{A}{cs}\cos(kz-\omega t)\ \hat{\phi}$$

com $\frac{\omega}{k} = c$ (Modo TEM).