Física Quântica I / Mecânica Quântica (2021/22)

Folha de Problemas 8 (Trem de ondas da partícula livre, 1D)

Vitor M. Pereira | Universidade do Minho | 19-Abr-2022

— Soluções —

Problema 1 | Trem de ondas (*wavepacket*) de uma partícula livre

Mostrou-se nas aulas que a dependência temporal da função de onda de uma partícula livre em 1 dimensão é dada por

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{i\,[px/\hbar - \omega(p)\,t]}\, \psi(p,0)\, dp, \qquad \text{onde} \qquad \omega(p) \equiv \frac{p^2}{2m\hbar},$$

e $\psi(p,0)$ é a função de onda no instante $t=0$ na base de momento. Para as questões abaixo, considere a seguinte função de onda inicial:

$$\psi(x,0) = \mathcal{N}\, e^{ik_0 x - \frac{x^2}{4\sigma^2}} \qquad \text{(trem de ondas Gaussiano).} \qquad (\star)$$

1. Determine a constante de normalização $\mathcal{N}$. Esquematize graficamente a distribuição de probabilidade descrita por esta função de onda, identificando o seu centro e largura em termos dos parâmetros k_0 e σ.

2. Calcule os valores esperados $\langle \hat{X} \rangle$ e $\langle \hat{X}^2 \rangle$.

3. Determine a função de onda na base de momento em $t=0$: $\psi(p,0)$.

4. Calcule os valores esperados $\langle \hat{P} \rangle$ e $\langle \hat{P}^2 \rangle$. (*É mais fácil fazê-lo na base de momento.*)

5. Verifique que a função de onda inicial ($\star$) minimiza a relação de incerteza entre posição e momento; ou seja, que neste estado $\delta X \delta P = \frac{\hbar}{2}$.

6. Calcule $\psi(x,t)$ e a distribuição de probabilidade para a posição no instante t.

7. Calcule a posição média da partícula, $\langle \hat{X} \rangle$, e a incerteza δX em função do tempo. *Sugestão: é conveniente escrever $\psi(x,t)$ na mesma forma da eq. ($\star$), onde σ passa a ser uma função do tempo, $\sigma(t)$.*

8. O que significam os resultados da questão anterior quanto à dinâmica da partícula? Qual a sua velocidade média?

9. Se ($\star$) representar a função de onda de uma partícula com massa 1 g e cuja incerteza na posição é da ordem do raio do protão ($\sim 10^{-15}$m), quanto tempo é necessário para esta incerteza aumentar até ~ 1 mm?

Nota: Integrais Gaussianos podem ser obtidos através das relações seguintes, válidas quando $\mathrm{Re}\,\alpha > 0$:

$$I_0(\alpha) \equiv \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}, \qquad I_{2n}(\alpha) \equiv \int_{-\infty}^{+\infty} x^{2n} e^{-\alpha x^2} dx = (-1)^n \frac{\partial^n}{\partial \alpha^n} I_0(\alpha),$$

$$I_2(\alpha) \equiv \int_{-\infty}^{+\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha^3}}, \qquad I_{2n+1}(\alpha) \equiv \int_{-\infty}^{+\infty} x^{2n+1} e^{-\alpha x^2} dx = 0.$$

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2 + \beta x} dx = e^{\beta^2/(4\alpha)} \sqrt{\frac{\pi}{\alpha}}.$$

Solution

To solve this problem it is useful to recall that a Gaussian probability distribution (also known as “normal distribution”), which is defined by the probability density

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}, \tag{a}$$

has the following properties:

$$\begin{aligned} 1 &= \int_{-\infty}^{+\infty} p(x)\, dx, && \text{(normalization)} \\ \langle x \rangle &= \int_{-\infty}^{+\infty} p(x)\, x\, dx = \mu, && \text{(average value)} \\ \langle x^2 \rangle &= \int_{-\infty}^{+\infty} p(x)\, x^2\, dx = \sigma^2 + \mu^2. && \text{(b)} \end{aligned}$$

These results follow by directly applying the formulas given at the end of the problem text to evaluate the integrals above. They will be useful for various of the questions below.

1. The normalization is found by equating the integrated squared modulus of the wavefunction to one. Since

$$\psi(x) = \mathcal{N}\, e^{ik_0 x - \frac{x^2}{4\sigma^2}} \quad \longrightarrow \quad |\psi(x)|^2 = |\mathcal{N}|^2 \cancelto{1}{\left|e^{ik_0 x}\right|^2} \left|e^{-\frac{x^2}{4\sigma^2}}\right|^2 = |\mathcal{N}|^2\, e^{-\frac{x^2}{2\sigma^2}},$$

we get the normalization constant (which we choose to be a real number) by solving

$$1 = \int |\psi(x)|^2\, dx = \mathcal{N}^2 \int e^{-\frac{x^2}{2\sigma^2}} dx = \mathcal{N}^2 \sqrt{2\pi\sigma^2} \qquad \Rightarrow \qquad \mathcal{N} = \left(2\pi\sigma^2\right)^{-1/4}.$$

The probability density function is therefore given by

$$|\psi(x)|^2 = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}.$$

This is a Gaussian probability distribution centred at $x = 0$ with standard deviation σ, as we can easily see by comparing with the results discussed in eq. (b) above. The standard deviation, σ, provides the "width" of this probability distribution in the sense that most of the probability density is contained between $-\sigma$ and σ; more precisely, for this wavefunction,

$$\int_{-\sigma}^{+\sigma} |\psi(x)|^2 \, dx \simeq 0.68.$$

This tells us that $68\,\%$ of the probability is contained in the positions $-\sigma < x < +\sigma$. Another way to see the role of the parameter σ is to note that, for $x = \sigma$,

$$|\psi(x=\sigma)|^2 = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}\ |\psi(x=0)|^2 \simeq 0.6\ |\psi(x=0)|^2 \,,$$

which means that σ is the distance at which the probability $|\psi(x)|^2$ has decayed to $\simeq 60\,\%$ of its value at the origin. Because of the exponential decay, σ defines the characteristic distances from the origin where the probability is of the order of 1; at distances to the origin much larger than σ ($x \gg \sigma$ or $x \ll -\sigma$), the probability is very small, because

$$e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \xrightarrow[x \gg \sigma]{} 0.$$

Note that the parameter k_0 does not appear in the probability distribution for the particle's position.

2. The average position is given by

$$\langle \hat{X} \rangle_\psi = \int_{-\infty}^{+\infty} |\psi(x)|^2 \, x \, dx = 0.$$

This is zero because $|\psi(x)|^2 x$ is an odd function (it is the product of an even and an odd function). The other expectation value is

$$\langle \hat{X}^2 \rangle_\psi = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2\sigma^2}} \, dx = \frac{1}{\sqrt{2\pi\sigma^2}} \left[\frac{1}{2} \pi^{\frac{1}{2}} \left(2\sigma^2\right)^{\frac{3}{2}} \right] = \sigma^2,$$

where, to evaluate the integral, we used the formulas listed at the end of the Problem text.

3. Recall that the position and momentum wavefunctions are related by the following integral relations (Fourier transform):

$$\psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \psi(x) \, dx, \qquad \psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{ipx/\hbar} \psi(p) \, dp.$$

These relations ensure that, if $\psi(x)$ is normalized, then $\psi(p)$ is also normalized.

For this question we want to find $\psi(p)$ from the given $\psi(x)$, so we use the first of these relations.

$$\begin{aligned}
\psi(p) &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \psi(x)\, dx \\
&= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar} \left[\frac{1}{(2\pi\sigma^2)^{1/4}}\, e^{ik_0 x - \frac{x^2}{4\sigma^2}} \right] dx \\
&= \frac{1}{(2\pi\hbar)^{\frac{1}{2}}\,(2\pi\sigma^2)^{\frac{1}{4}}} \int_{-\infty}^{+\infty} e^{i(k_0 x - px/\hbar) - \frac{x^2}{4\sigma^2}}\, dx \\
&\Big\downarrow \text{ use the last integral formula provided at the end} \\
&= \frac{1}{(2\pi\hbar)^{\frac{1}{2}}\,(2\pi\sigma^2)^{\frac{1}{4}}}\, e^{-(k_0 - p/\hbar)^2\sigma^2} \sqrt{4\sigma^2\pi} \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2} \right)^{\frac{1}{4}} e^{-\frac{\sigma^2}{\hbar^2}(p - \hbar k_0)^2}.
\end{aligned}$$

We can see that $\psi(p)$ is also a Gaussian function, but now in the variable p, with a centre at the value $p = \hbar k_0$.

4. In the momentum basis the momentum expectation value is calculated as

$$\langle \hat{P} \rangle_\psi = \int p\, |\psi(p)|^2\, dp = \left(\frac{2\sigma^2}{\pi\hbar^2} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} p e^{-\frac{2\sigma^2}{\hbar^2}(p - \hbar k_0)^2}\, dp.$$

To perform this integral, we changing the variable to $y = p - \hbar k_0$. The integral becomes

$$\langle \hat{P} \rangle = \left(\frac{2\sigma^2}{\pi\hbar^2} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} (y + \hbar k_0)\, e^{-\frac{2\sigma^2}{\hbar^2} y^2}\, dy = \hbar k_0 \left(\frac{2\sigma^2}{\pi\hbar^2} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} e^{-\frac{2\sigma^2}{\hbar^2} y^2}\, dy = \hbar k_0.$$

This is what we should expect because, as we have seen from the result of the previous question, $|\psi(p)|^2$ is a normalized Gaussian distribution centred at $\hbar k_0$.

The average of the squared momentum is calculated in a similar way:

$$\begin{aligned}
\langle \hat{P}^2 \rangle &= \left(\frac{2\sigma^2}{\pi\hbar^2} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} p^2 e^{-\frac{2\sigma^2}{\hbar^2}(p - \hbar k_0)^2}\, dp \\
&\Big\downarrow \text{ change variable: } y = p - \hbar k_0 \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} (y + \hbar k_0)^2\, e^{-\frac{2\sigma^2}{\hbar^2} y^2}\, dy \\
&\Big\downarrow \text{ split the two terms and integrate the 1st one} \\
&= (\hbar k_0)^2 + \left(\frac{2\sigma^2}{\pi\hbar^2} \right)^{\frac{1}{2}} \int_{-\infty}^{+\infty} y^2 e^{-\frac{2\sigma^2}{\hbar^2} y^2}\, dy \\
&\Big\downarrow \text{ use the integral formulas provided} \\
&= (\hbar k_0)^2 + \left(\frac{2\sigma^2}{\pi\hbar^2} \right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{2} \left(\frac{\hbar^2}{2\sigma^2} \right)^{\frac{3}{2}} \\
&= (\hbar k_0)^2 + \frac{\hbar^2}{4\sigma^2}.
\end{aligned}$$

5. Recalling that the uncertainty of an observable $\hat{A}$ is defined as

$$\delta A = \sqrt{\langle \hat{A}^2 \rangle_\psi - \langle \hat{A} \rangle_\psi^2},$$

we need only to gather the results of the questions 2 and 4 above:

$$\delta X = \sigma, \qquad \delta P = \sqrt{\left(\hbar^2 k^2 + \frac{\hbar^2}{4\sigma^2}\right) - \hbar^2 k^2} = \frac{\hbar}{2\sigma}.$$

Therefore,

$$\delta X \delta P = \frac{\hbar}{2} \qquad \boxdot$$

6. Using the result derived above for the momentum wavefunction at $t = 0$,

$$\psi(p,0) = \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} e^{-\frac{\sigma^2}{\hbar^2}(p-\hbar k_0)^2},$$

we have

$$\psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} \underbrace{\int_{-\infty}^{+\infty} dp\, e^{i[px/\hbar - \omega(p)t]} e^{-\frac{\sigma^2}{\hbar^2}(p-\hbar k_0)^2}}_{\text{Int}}, \qquad \omega(p) \equiv \frac{p^2}{2m\hbar}. \tag{c}$$

In order to perform this integral over p it is convenient to expand and rearrange the argument of the exponential in to identify the coefficients of the various powers of p. The argument of the exponential can be rearranged as

$$\begin{aligned}
\frac{ipx}{\hbar} - i\omega(p)t - \frac{\sigma^2}{\hbar^2}(p-\hbar k_0)^2 &= \frac{ipx}{\hbar} - \frac{ip^2 t}{2m\hbar} - \frac{p^2\sigma^2}{\hbar^2} + \frac{2pk_0\sigma^2}{\hbar} - k_0^2\sigma^2 \\
&= -p^2 \underbrace{\left(\frac{it}{2m\hbar} + \frac{\sigma^2}{\hbar^2}\right)}_{\alpha} + p \underbrace{\left(\frac{ix + 2\sigma^2 k_0}{\hbar}\right)}_{\beta} - \sigma^2 k_0^2.
\end{aligned}$$

Therefore, we can write the integral "Int" that appears in eq. (c) as

$$\begin{aligned}
\text{Int} &= e^{-\sigma^2 k_0^2} \int_{-\infty}^{+\infty} e^{-\alpha p^2 + \beta p}\, dp \\
&\Big\downarrow \text{ use the last of the integral formulas provided} \\
&= e^{-\sigma^2 k_0^2} e^{\beta^2/(4\alpha)} \sqrt{\frac{\pi}{\alpha}}.
\end{aligned}$$

Replacing back the α and β, and going back to eq. (c),

$$\begin{aligned}
\psi(x,t) &= \frac{1}{\sqrt{2\hbar}} \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} \left(\frac{it\hbar + 2m\sigma^2}{2m\hbar^2}\right)^{-\frac{1}{2}} \exp\left[\frac{(ix + 2\sigma^2 k_0)^2}{\frac{2it\hbar}{m} + 4\sigma^2} - \sigma^2 k_0^2\right] \\
&= \left(\frac{1}{2\pi}\right)^{\frac{1}{4}} \left(\frac{it\hbar}{2m\sigma} + \sigma\right)^{-\frac{1}{2}} \exp\left[\frac{-x^2 + 4i\sigma^2 k_0 x - \frac{2it\hbar\sigma^2 k_0^2}{m}}{\frac{2it\hbar}{m} + 4\sigma^2}\right].
\end{aligned}$$

We see that the wavefunction still has a Gaussian shape because it has a factor decaying as e^{-x^2}. In order to expose this more clearly, we re-write the argument of the exponential as follows

$$\begin{aligned}
&\frac{-x^2+4i\sigma^2k_0x-\frac{2it\hbar\sigma^2k_0^2}{m}}{\frac{2it\hbar}{m}+4\sigma^2} \\
&=-\frac{\left(x-\frac{\hbar k_0 t}{m}\right)^2}{\frac{2it\hbar}{m}+4\sigma^2}+\frac{\hbar^2k_0^2t^2-2x\hbar k_0tm+4i\sigma^2k_0xm^2-2it\hbar\sigma^2k_0^2m}{2it\hbar m+4\sigma^2m^2} \\
&=-\frac{\left(x-\frac{\hbar k_0 t}{m}\right)^2}{\frac{2it\hbar}{m}+4\sigma^2}+ik_0\left(x-\frac{\hbar k_0 t}{2m}\right)
\end{aligned}$$

Putting this back in the result obtained for the time-evolved wavefunction leads to

$$\psi(x,t)=\left(\frac{1}{2\pi}\right)^{\frac{1}{4}}\left(\frac{it\hbar}{2m\sigma}+\sigma\right)^{-\frac{1}{2}}\exp\left[-\frac{\left(x-\frac{\hbar k_0 t}{m}\right)^2}{\frac{2it\hbar}{m}+4\sigma^2}\right]\exp\left[ik\left(x-\frac{\hbar k_0 t}{2m}\right)\right].$$

From here the probability density follows immediately:

$$|\psi(x,t)|^2=\frac{1}{\sqrt{2\pi}\sqrt{\frac{t^2\hbar^2}{4m^2\sigma^2}+\sigma^2}}\exp\left[-\frac{\left(x-\frac{\hbar k_0 t}{m}\right)^2}{\frac{t^2\hbar^2}{2\sigma^2m^2}+2\sigma^2}\right].$$

To understand better what this probability distribution looks like, let us write it as

$$|\psi(x,t)|^2=\frac{1}{\sqrt{2\pi\left[\sigma(t)\right]^2}}\exp\left[-\frac{\left(x-\frac{\hbar k_0 t}{m}\right)^2}{2\left[\sigma(t)\right]^2}\right], \qquad \sigma(t)\equiv\sigma\sqrt{1+\frac{t^2\hbar^2}{4\sigma^4m^2}}.$$

Comparing with eq. (a) we can see this is a Gaussian probability distribution. It describes a Gaussian wavepacket with the following characteristics:

- Its centre (which is defined by the condition $x-\frac{\hbar k_0 t}{m}=0$) is located at $x_M=\frac{\hbar k_0 t}{m}$. Consequently, the wavepacket moves with a constant velocity given by $\hbar k_0/m$.
- Its spread (or width), which is defined by the parameter $\sigma(t)$, increases because $\sigma(t)$ increases monotonically with time.
- The wavepacket remains Gaussian at all times. It moves in space with velocity $\hbar k_0/m$ and gets progressively wider with time.

7. Since $|\psi(x,t)|^2$ obtained above is still a Gaussian centred at $\frac{\hbar k_0 t}{m}$, and it is (necessarily) normalized, the average position as a function of time is easy to calculate in the same way we did before from the integrals listed in the appendix, with the result

$$\langle\hat{X}\rangle=\int_{-\infty}^{+\infty}x\,|\psi(x,t)|^2\,dx=\frac{\hbar k_0 t}{m}.$$

Likewise, the spread, or position uncertainty δX, can be obtained similarly to question 2 above, and it is given by

$$\delta X=\sigma(t)=\sigma\sqrt{1+\frac{t^2\hbar^2}{4\sigma^4m^2}}.$$

8. From all that we have seen so far in the above questions, at $t = 0$ the wavefunction of the particle is a Gaussian wavepacket centred at $x = 0$ (question 1) with average momentum $\langle \hat{P} \rangle = \hbar k_0$ (question 4). The probability density at $t = 0$ has a spread/uncertainty of $\delta X = \sigma$ (question 2). This initial wavefunction represents a particle localized within a distance of the order of 2σ, with a starting group velocity given by $v_g = \hbar k_0/m$. Note, however, that neither the position, nor the momentum are precisely defined, since the uncertainties δX and δP are both finite (question 5).

   As time passes, the wavepacket remains Gaussian but its centre changes to $\hbar k_0 t/m$ (question 6). This is what we expected because the initial wavefunction represents a particle with an average velocity of $\langle \hat{P} \rangle/m = \hbar k_0/m$. Therefore, since the particle's most likely position was $x = 0$ at $t = 0$, after a time t the most likely position for this particle should be $(\langle \hat{P} \rangle/m) \times t = \hbar k_0 t/m$, which is precisely where the centre of the wavepacket lies at time t.

   In addition to the motion of the centre of the wavepacket, its width is also changing with time — more specifically, δX increases with time. This is expected because there was an uncertainty in the momentum of the particle at $t = 0$, which means that the uncertainty in the position can only increase with time (see Comment III below).

9. At $t = 0$ the position uncertainty is given by $\delta X = \sigma$ (question 5), which means that $\sigma \sim 10^{-15}$ m initially. Some time t later, the uncertainty should be (result of question 7 above)
$$\delta X = \sigma \sqrt{1 + \frac{t^2 \hbar^2}{4\sigma^4 m^2}}.$$
   We want to find the time T at which we get $\delta X \sim 1$ mm. Solving the last equation for t,
$$t = \frac{2\sigma^2 m}{\hbar} \sqrt{\left(\frac{\delta X}{\sigma}\right)^2 - 1}.$$
   We now replace $\sigma \to 10^{-15}$ (initial uncertainty) and $\delta X \to 10^{-3}$ (uncertainty at $t = T$) to obtain
$$T \simeq \frac{2 \times 10^{-30} \times 10^{-3}}{10^{-34}} \times \frac{10^{-3}}{10^{-15}} \simeq 2 \times 10^{13}\,\text{s},$$
   which is well above 600 000 years!

Comment I. This problem shows us that a well localized particle is described by a wavepacket whose initial spread corresponds to the uncertainty in the position of the particle. The uncertainty in the position will increase in time in a way that, for asymptotically large times, approaches
$$\delta X \sim t \left(\frac{\hbar}{2\sigma m}\right) \qquad \text{(at large times).}$$
But if the particle is macroscopic (here reflected by the fact that its mass is about 1g), the rate of increase in δX is very small (because $\hbar/m$ is very small).

This means that, within any reasonable time-span, the uncertainty in the position of this classical particle will not grow enough to become of the same order of its size (say, a few mm, for example, or even a few nm!). Consequently, for all time-spans of interest in any experiment or daily life, the particle's probability distribution remains sharply peaked at a position that varies as

$$\langle \hat{X} \rangle = \frac{\hbar k_0}{m} t = \frac{\langle \hat{P} \rangle}{m} t.$$

Since the probability distribution remains sharp about this point $\langle \hat{X} \rangle$, this equation (which is the classical equation of motion for the free particle) predicts with *nearly* complete certainty the particle's position at all times of interest (the sharply peaked probability distribution implies this). Therefore, the classical equations of motion (in this case simply $x = v_g t$ because we are dealing with a free particle) are enough to fully describe the motion in time of this particle, and the uncertainty in its position will remain extremely small for all meaningful times. This is precisely what we expect for a macroscopic particle.

Comment II. We showed above that the product of position and momentum uncertainties at $t = 0$ corresponds to the absolute minimum allowed by the Heisenberg uncertainty relation

$$\delta X \delta P = \frac{\hbar}{2} \tag{1.1}$$

At any other time $t \neq 0$ the product of the uncertainties will change (it can only be higher, of course). The reason for this is that, as we have seen, at time t the uncertainty in the position is given by

$$\delta X = \sigma \sqrt{1 + \frac{t^2 \hbar^2}{4\sigma^4 m^2}}.$$

But what is then the uncertainty in the momentum at this same time? We can anticipate that the uncertainty in momentum will not change, because $\hat{P}$ commutes with $\hat{H}$ for a free particle, and hence both $\langle \hat{P} \rangle$ and $\langle \hat{P}^2 \rangle$ should be constants of motion.

If we want to see that explicitly we just need to consider, for example, the wavefunction $\psi(p,t)$, which is given by

$$\psi(p,t) = \psi(p,0)\, e^{-ip^2 t/2m\hbar}. \tag{1.2}$$

Therefore it is obvious that the expectation of any power of $\hat{P}$ is constant in time, because $|\psi(p,t)|^2 = |\psi(p,0)|^2$ and so

$$\langle \hat{P}^n \rangle(t) = \int dp\, |\psi(p,t)|^2\, p^n = \int dp\, |\psi(p,0)|^2\, p^n = \langle \hat{P}^n \rangle \Big|_{t=0}.$$

Consequently the product of the uncertainties changes with time as

$$\delta X \delta P = \sigma \sqrt{1 + \frac{t^2 \hbar^2}{4\sigma^4 m^2}} \times \frac{\hbar}{2\sigma} = \frac{\hbar}{2} \sqrt{1 + \frac{t^2 \hbar^2}{4\sigma^4 m^2}}$$

and, as expected, it is always $> \hbar/2$, except at $t = 0$ where it coincides with the minimum allowed by the uncertainty principle.

Comment III. As we stated above, for long times the position uncertainty becomes

$$\delta X \sim t \left(\frac{\hbar}{2\sigma m} \right) \qquad \text{at large times.}$$

Let us think classically for a moment: if the initial uncertainty in the momentum is $\delta P = \frac{\hbar}{2\sigma}$, after a time t the uncertainty in position should be $\delta X_{\text{classical}} = \delta v \times t = (\delta P/m) \times t$, (to see this, think of two free particles starting at $x = 0$, $t = 0$ with velocity p/m and $(p + \delta P)/m$, and work out their separation after a time t) or

$$\delta X_{\text{classical}} = \frac{\delta P}{m} t.$$

If we plug here the uncertainty in momentum associated with this wavepacket,$\delta P = \frac{\hbar}{2\sigma}$, we obtain

$$\delta X_{\text{classical}} = \frac{\hbar}{2\sigma m} t$$

In other words, for times $t \gg 2\sigma m/\hbar$, the quantum mechanical solution for the spread of the quantum wavepacket recovers the uncertainty in the position that we would expect simply from the classical equations of motion of a free particle.
