Problem → $\boxed{TP2}$

Maxwell equation in vacuum and the Heaviside duality

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \wedge \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

These equations are interchanged under the replacements

$E \to -c\vec{B}$ and $c\vec{B} \to \vec{E}$ (check this)

Larmor generalization to a continuous transformation:

$$E' = E\cos\eta - cB\sin\eta$$

$$cB' = E\sin\eta + cB\cos\eta$$

$$\begin{bmatrix} E' \\ cB' \end{bmatrix} = \begin{bmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} E \\ cB \end{bmatrix}$$

- Show that the Maxwell's equations in vacuum are invariant under such a transformation.

- Can this symmetry be maintained with sources if one considers the existence of magnetic charges?

ii

a)

$$\nabla \wedge \vec{E}' = \nabla \wedge \left[\vec{E} \cos\eta - cB \sin\eta \right] =$$

$$= \left(\nabla \wedge \vec{E}\right) \cos\eta - c \sin\eta \left(\nabla \wedge \vec{B}\right)$$

$$= -\dot{\vec{B}} \cos\eta - c \sin\theta \frac{1}{c^2} \dot{\vec{E}} = -\dot{\vec{B}}'$$ (Maxwell's in the transf. fields.)

$$-\dot{\vec{B}} = -\frac{\dot{\vec{E}}}{c} \sin\eta + \dot{B} \cos\eta$$

$$\nabla \wedge \vec{B}' = \nabla \wedge \left[\frac{E}{c} \sin\eta + B \cos\eta \right]$$

$$= \frac{1}{c} \sin\eta \left(\nabla \wedge \vec{E}\right) + (\nabla \wedge B) \cos\eta \overset{?}{\equiv} \frac{1}{c^2} \frac{\partial \vec{E}'}{\partial t}$$

$$\frac{1}{c^2} \dot{\vec{E}}' = \frac{\dot{E}}{c^2} \cos\eta - \sin\eta \frac{\dot{B}}{c}$$

$$-\frac{1}{c} \sin\eta \, \dot{B} + \cos\eta \frac{1}{c^2} \dot{\vec{E}}$$ O.K.

b) Magnetic charges?

$$\nabla \cdot \vec{B}' = \mu_0 \rho'_{mag.}$$

$$\nabla \cdot \vec{B}' = \nabla \cdot \left] \frac{1}{c} E \sin\eta + \cos\eta\, B \right] =$$

$$= \frac{1}{c} \sin\eta \left(\nabla \cdot \vec{E}\right) + \cos\theta \left(\nabla \cdot \vec{B}\right) = \frac{1}{c} \frac{\rho_e}{\varepsilon_0} \sin\eta + \cos\eta\, \mu_0 \rho_{mag.}$$

$$\stackrel{?}{=} \mu_0 \rho'_{mag}$$

$$\rho'_{mag} = \frac{1}{c} \frac{1}{\varepsilon_0 \mu_0} \rho_e \sin\eta + \rho_{mag} \cos\eta$$

$$\boxed{\rho'_{mag} = c \rho_e \sin\eta + \rho_{mag} \cos\eta}$$

$$\nabla \cdot \vec{E}' = \nabla \cdot \left[\bar{E} \cos\eta - \sin\eta\, c \vec{B} \right] = \cos\theta \left(\nabla \cdot \vec{E}\right) - \sin\eta\, c \left(\nabla \cdot \vec{B}\right)$$

$$= \cos\theta \frac{\rho_e}{\varepsilon} - \sin\eta\, c\, \mu_0 \rho_{mag.}$$

$\downarrow$

$$\frac{\rho'_e}{\varepsilon_0} \stackrel{!}{=} \frac{\rho_e}{\varepsilon} \cos\eta - \sin\eta\, c\, \mu_0 \rho_{mag.}$$

$$\boxed{\rho'_e = \rho_e \cos\eta - \frac{1}{c} \rho_{mag.} \sin\eta}$$

$$\nabla \wedge \vec{E}' = -\dot{\vec{B}}' - \frac{1}{\varepsilon_0} \vec{J}'_{mag}.$$

$$\downarrow$$

$$\nabla \wedge \left[\cos\eta \, \vec{E} - c \sin\eta \, \vec{B} \right] \overset{?}{\equiv} -\left[\frac{\dot{E}}{c} \sin\eta + \cos\eta \, \dot{B} \right] - \frac{1}{\varepsilon_0} \vec{J}'_{mag}.$$

$$\cos\eta \, (\nabla \wedge \vec{E}) - c \sin\eta \, (\nabla \wedge \vec{B}) \equiv -\frac{\dot{\vec{E}}}{c} \sin\eta - \cos\eta \, \dot{\vec{B}} - \frac{1}{\varepsilon_0} \vec{J}'_{mag}$$

$$\cos\eta \left[\cancel{-\dot{\vec{B}}} - \frac{1}{\varepsilon_0} \vec{J}_{mag} \right] - c \sin\eta \left[\mu_0 \vec{J}_e + \cancel{\frac{1}{c^2} \dot{\vec{E}}} \right] \equiv \cancel{-\frac{\dot{\vec{E}}}{c} \sin\eta} \cancel{- \dot{\vec{B}} \cos\eta} - \frac{1}{\varepsilon_0} \vec{J}'_{mag}.$$

$$-\frac{1}{\varepsilon_0} \cos\eta \, \vec{J}_{mag} - \mu_0 c \sin\eta \, \vec{J}_e = -\frac{1}{\varepsilon_0} \vec{J}'_{mag}. \Rightarrow$$

$$\Rightarrow \boxed{\vec{J}'_{mag} \equiv \vec{J}_{mag} \cos\eta + \frac{1}{c} \sin\eta \, \vec{J}_e}$$

$$\nabla \wedge \vec{B}' = \mu_0 \vec{J}'_e + \frac{1}{c^2} \dot{\vec{E}}'$$

$$\downarrow$$

$$\nabla \wedge \left[\frac{\vec{E}}{c} \sin\eta + B \cos\eta \right] \equiv \mu_0 \vec{J}'_e + \frac{1}{c^2} \left[\cos\eta \, \dot{\vec{E}} - \sin\eta \, c \, \dot{\vec{B}} \right]$$

$$\Longleftrightarrow \cos\eta \, (\nabla \wedge \vec{B}) + \frac{1}{c} \sin\eta (\nabla \wedge \vec{E}) \equiv \mu_0 \vec{J}'_e + \frac{1}{c^2} \left[\cos\eta \, \dot{\vec{E}} - \sin\eta \, c \, \dot{\vec{B}} \right]$$

$$\ldots\eta\left[\mu_0\vec{J}_e + \cancel{\mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}}\right] + \frac{1}{c}\sin\eta\left(\cancel{-\dot{\vec{B}}} - \frac{1}{\varepsilon_0}\vec{J}_{mag}\right) \equiv$$

$$\equiv \mu_0\vec{J}_e' + \frac{1}{c^2}\left[\cancel{\cos\eta\,\dot{\vec{E}}} - \cancel{\sin\eta\, c\,\dot{\vec{B}}}\right] \implies$$

$$\implies \boxed{\vec{J}_e' \equiv \cos\eta\; j_e + c\sin\eta\; j_{mag}.}$$

- The Heaviside-Larmor duality is consistent with the generalised Maxwell's-equations if

$$\begin{bmatrix}\rho_e'\\ \rho_m'\end{bmatrix} = \begin{bmatrix}\cos\eta & -\frac{1}{c}\sin\eta\\ c\sin\eta & \cos\eta\end{bmatrix}\begin{bmatrix}\rho_e\\ \rho_m\end{bmatrix}$$

check this.

-

$$\begin{bmatrix}\vec{J}_e'\\ \vec{J}_m'\end{bmatrix} = \begin{bmatrix}\cos\eta & -c\sin\eta\\ \frac{1}{c}\sin\eta & \cos\eta\end{bmatrix}\begin{bmatrix}\vec{J}_e\\ \vec{J}_m\end{bmatrix}$$

$$\begin{bmatrix}\vec{E}'\\ c\vec{B}'\end{bmatrix} = \begin{bmatrix}\cos\eta & -c\sin\eta\\ \frac{1}{c}\sin\eta & \cos\eta\end{bmatrix}\begin{bmatrix}E\\ cB\end{bmatrix}$$

[redacted]

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para $\nu = 4 \times 10^{8}$ Hz

1. A permitividade da água do mar é $\varepsilon \sim 81\,\varepsilon_0$ e $\mu \sim \mu_0$ e a resistividade é $\rho \sim 0,23\ \Omega \cdot m$. Obtenha a relação entre a corrente óhmica e a de deslocamento num condensador plano imerso em água do mar e ligado a um gerador de sinal operando àquela frequência:

$$E = \frac{V_0 \cos(\omega t)}{d} \qquad \vec{J}_c = \sigma \vec{E} \Rightarrow \frac{1}{\rho}\frac{V}{d}$$

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t} = \frac{\partial(\varepsilon E)}{\partial t} = \frac{\varepsilon V_0}{d}\left(-\omega \sin \omega t\right)$$

$$\frac{j_c}{j_D} = \left(\frac{1}{\rho d} V_0 \cos(\omega t)\right) \Big/ \left(\frac{\varepsilon V_0}{d}\, \omega \sin(\omega t)\right)$$

A razão das amplitudes é:

$$\frac{\frac{1}{\rho \cancel{d}} V_0}{\omega \varepsilon \cancel{V_0/d}} = \frac{1}{\omega \varepsilon \rho} \rightarrow \left[2\pi\,(4 \times 10^{8})\,(81)\,8,85 \times 10^{-12} \cdot 0,23\right]^{-1}$$

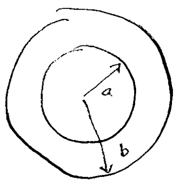
$$\| $$

$$\underline{\underline{2,41}}$$

[redacted] TP2 + TP3 [redacted]

1. Problema (Ex. 4.5 Griffiths)

Uma esfera metálica de raio a transporta uma carga Q. É revestida por uma coroa esférica de espessura $(b-a)$, e que tem uma permitividade ε. Calcule o potencial no centro da esfera (tomando como referência $\vec{r} \to \infty$).



$-\nabla V = \vec{E}$; para calcular $\vec{E}$ precisaríamos de conhecer ρ_b e σ_b (ou $\vec{P}$), mas para isso precisaríamos de conhecer $\vec{E}$ (!). Mas, como conhecemos σ_ℓ podemos calcular $\vec{D}$!

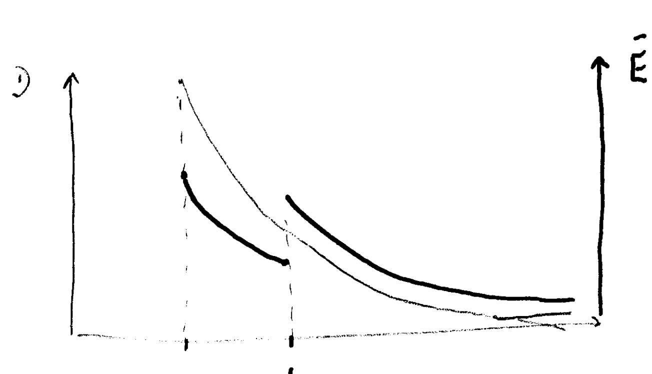
$\underline{r < a}$, $\rho_\ell = 0 \Rightarrow |\vec{D}| = 0$ ($\vec{D}$ tem sempre simetria radial)

$\left.\begin{array}{l} a < r < b \\ r > b \end{array}\right\}$ $|\vec{D}| = \dfrac{Q}{4\pi r^2}\,\vec{r}$

Admitindo que o meio dieléctrico é linear e isotrópico

$$a < r < b \Rightarrow \vec{E} = \frac{Q}{4\pi r^2 \varepsilon}\,\vec{r}$$

$$r > b \qquad E = \frac{Q}{4\pi r^2 \varepsilon_0}$$



Potencial no centro da esfera:

$$V = -\int_{-\infty}^{0} \vec{E}\cdot d\vec{\ell} = -\int_{-\infty}^{b} \frac{Q}{4\pi\varepsilon_0 r^2}\, dr - \int_{b}^{a} \frac{Q}{4\pi\varepsilon r^2}\, dr =$$

$$= +\frac{Q}{4\pi}\left[\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right]$$

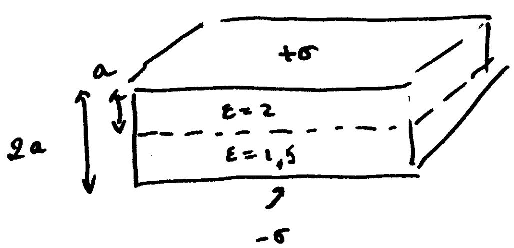
Podemos também calcular a polarização na casca dielétrica

$$\vec{P} = \varepsilon_0 \chi_e \vec{E} = \frac{\varepsilon_0 \chi_e Q}{4\pi\varepsilon r^2}\,\hat{r}$$

$$\rho_b = -\nabla\cdot\vec{P} = 0 \quad \left[\left[\frac{1}{r^2}\partial_r(r^2 P_r)\right] = 0\right]$$

$$\sigma_b = \vec{P}\cdot\hat{n} = \begin{cases} -\dfrac{\varepsilon_0\chi_e Q}{4\pi\varepsilon a^2} & r = a \\[2ex] +\dfrac{\varepsilon_0\chi_e Q}{4\pi\varepsilon b^2} & r = b \end{cases}$$

Problema 4.18 (Griffiths)



a) $\vec{D}$ em cada dielétrico

b) $\vec{E}$ em cada dielétrico

c) $\vec{P}$ em cada dielétrico

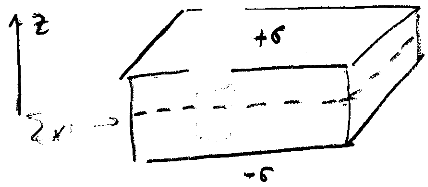
d) ΔV entre as armaduras

e) Cargas ligadas

f) Confirme que a) e b) são compatíveis.

$$\boxed{\nabla\cdot\vec{D} = \rho}$$

a)



$\vec{D} = -\sigma\,\hat{z}$ (diel. 1)

também: $\vec{D} = -\sigma\,\hat{z}$ (diel. 2)

b) Dielectrico 1 : $E_1 = \frac{D}{\varepsilon_1} = -\frac{\sigma}{\varepsilon_1} = -\frac{\sigma}{2\varepsilon_0}$ $\quad(\varepsilon_1 = 2\varepsilon_0)$

$E_2 = \frac{D}{\varepsilon_2} = -\frac{\sigma}{\varepsilon_2} = -\frac{2\sigma}{3\varepsilon_0}$ $\quad(\varepsilon_2 = \frac{3}{2}\varepsilon_0)$

c) $\vec{P} = \varepsilon_0\chi_e\vec{E} \rightarrow P_1 = \varepsilon_0\chi_{e_1}E_1 = -\varepsilon_0\chi_{e_1}\frac{\sigma}{2\varepsilon_0} = -\frac{\chi_{e_1}\sigma}{2}$

$\left(\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}\right)$ $\qquad \chi_e = \varepsilon_r - 1 \Rightarrow \chi_{e_1} = 1 \longrightarrow -\frac{\sigma}{2}$

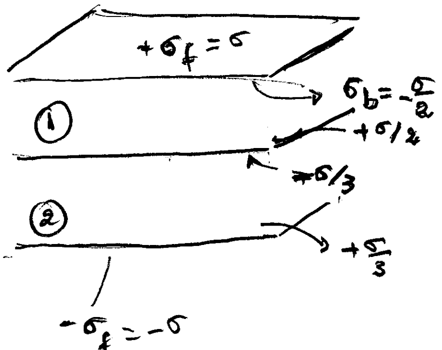
$P_2 = \varepsilon_0(\varepsilon_r - 1)E_2 = -\frac{\varepsilon_0}{2}\frac{2\sigma}{3\varepsilon_0} = -\frac{\sigma}{3}$

d) $-\int\vec{E}\cdot d\vec{l} = -(E_1 a + E_2 a) = \frac{\sigma}{\varepsilon_0}a\left[\frac{1}{2} + \frac{2}{3}\right] = \frac{7\sigma a}{\varepsilon_0 6}$

e) $\nabla\cdot\vec{P} = 0 \rightarrow \rho_b = 0$;

$\sigma_b = P_1\cdot\hat{m} = -\frac{\sigma}{2}$; topo do dielectrico 1 ($\frac{\sigma}{2}$ em baixo)

topo slab 2 : $+P_2\hat{m} = -\frac{\sigma}{3}$; (baixo: $\frac{\sigma}{3}$)



$\longrightarrow E_1 = -\frac{\sigma}{2\varepsilon_0}$

$E_2 = \left(\sigma - \frac{\sigma}{2} + \frac{\sigma}{2} - \frac{\sigma}{3}\right)\frac{1}{\varepsilon_0} = \frac{2\sigma}{3\varepsilon_0}$

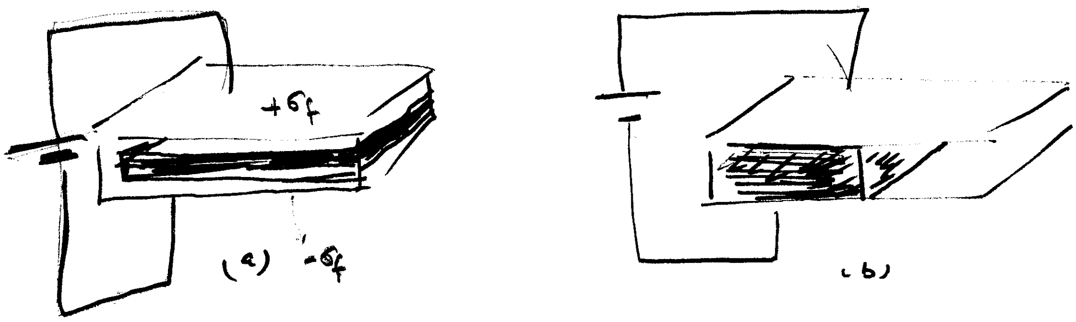
$\left(-\sigma + \frac{\sigma}{3} = -\frac{2\sigma}{3}\right)$ o.k.

em baixo

4.19 (Griffiths)

Dispõe de ½ do material dieléctrico necessário para encher o entre-ferro de um condensador plano. Qual é o maior aumento de capacidade? (para a configuração (a) ou (b)?

Calcule $\bar{E}$, $\bar{D}$, $\bar{P}$, σ_b e ρ_b em cada caso



sem dieléctrico, $C_0 = \varepsilon_0 \dfrac{A}{d}$

em (a) $+\sigma_f \rightarrow D = \sigma_f$ entre armaduras.

$E_1 = \dfrac{\sigma}{\varepsilon_0}$ no vazio

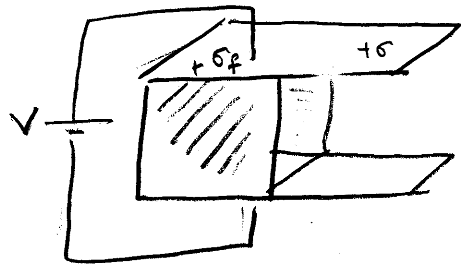
$E_2 = \dfrac{\sigma}{\varepsilon}$ no dieléctrico

$$V = \frac{\sigma d}{\varepsilon_0 2} + \frac{\sigma d}{\varepsilon 2} = \frac{\sigma d}{2\varepsilon_0}\left[1 + \frac{1}{\varepsilon_r}\right]$$

$$C_a = \frac{Q}{V} = \frac{\cancel{\sigma} A}{\frac{\cancel{\sigma} d}{2\varepsilon_0}\left[1 + \frac{1}{\varepsilon_r}\right]} = \frac{2\,\varepsilon_0\, A}{d\left[1 + \frac{1}{\varepsilon_r}\right]} = \frac{\varepsilon_0 A}{d}\,\frac{2\varepsilon_r}{\varepsilon_r + 1}$$

$$\frac{C_a}{C_0} = \frac{2}{1 + \frac{1}{\varepsilon_r}} = \frac{2\varepsilon_r}{\varepsilon_r + 1}$$ = constante dieléctrica relativa.

em b)



$D = \sigma$

Na metade ar: $E = \frac{\sigma_c}{\varepsilon_0} = \frac{V}{d}$

No metade diel. $E = \frac{\sigma'}{\varepsilon} = \frac{V}{d}$

$$P = \varepsilon_0 \chi_e E = \varepsilon_0 \chi_e \frac{V}{d} \Longrightarrow \underset{\hat{n}\cdot\vec{P}}{\underset{\shortparallel}{\sigma_b}} = -\varepsilon_0 \chi_e \frac{V}{d}$$

(carga sup. no topo do mat. dielétrico)

(Gauss)

$$E = \frac{\sigma_{TOT}}{\varepsilon_0} \longleftarrow \sigma_{TOT} = \sigma_f + \sigma_b = \sigma_f - \varepsilon_0 \chi_e \frac{V}{d}$$

$$E d = V = \frac{\sigma_{TOT} d}{\varepsilon_0} \longrightarrow \varepsilon_0 \frac{V}{d} = \sigma_f - \varepsilon_0 \chi_e \frac{V}{d} \rightarrow \sigma_f = \varepsilon_0 \frac{V}{d}(1+\chi_e) = \varepsilon_0 \varepsilon_r \frac{V}{d}$$

$$\Longrightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma \frac{A}{2} + \sigma_f \frac{A}{2}\right) = \frac{A}{2V}\left(\varepsilon_0 \frac{V}{d} + \varepsilon_0 \varepsilon_r \frac{V}{d}\right) =$$

$$= \frac{A \varepsilon_0}{2d}(1+\varepsilon_r)$$

$$\boxed{C_b = C_0 \frac{1+\varepsilon_r}{2}}$$

Recorde que $\boxed{C_a = C_0 \frac{2\varepsilon_r}{\varepsilon_r + 1}}$

$\frac{C_a}{C_0} = \frac{2\varepsilon_r}{\varepsilon_r+1}$

Resumo

a)

	E (1)	D	P	σ_b (top)	σ_f (top plate) (2)
Vácuo	$\frac{2\varepsilon_r}{\varepsilon_r+1}\frac{V}{d}$	$\frac{2\varepsilon_r}{\varepsilon_r+1}\varepsilon_0\frac{V}{d}$	0	0	$\frac{2\varepsilon_r}{\varepsilon_r+1}\frac{\varepsilon_0 V}{d}$
dieléctrico	$\frac{2}{\varepsilon_r+1}\frac{V}{d}$ (3)	$\frac{2\varepsilon_r}{\varepsilon_r+1}\varepsilon_0\frac{V}{d}$	$\frac{2(\varepsilon_r-1)}{\varepsilon_r+1}\frac{\varepsilon_0 V}{d}$ (4)	$-\frac{2(\varepsilon_r-1)}{\varepsilon_r+1}\frac{\varepsilon_0 V}{d}$ (5)	—

b)

	E	D	P	$\sigma_b = \vec{P}\cdot\hat{n}$	σ
Vácuo	$\frac{V}{d}$	$\varepsilon_0\frac{V}{d}$	0	0	$\varepsilon_0\frac{V}{d}$
dielectrico	$\frac{V}{d}$	$\varepsilon_r\varepsilon_0\frac{V}{d}$	$(\varepsilon_r-1)\frac{\varepsilon_0 V}{d}$	$-(\varepsilon_r-1)\frac{\varepsilon_0 V}{d}$	$\varepsilon_r\varepsilon_0\frac{V}{d}$

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(1) $E_1 = \frac{\sigma}{\varepsilon_0}$; como $\sigma = \frac{V\,2}{d}\varepsilon_0\frac{\varepsilon_r}{\varepsilon_r+1} \Rightarrow E_1 = \frac{2V}{d}\frac{\varepsilon_r}{\varepsilon_r+1}$

(2) $E_1 = \frac{\sigma}{\varepsilon_0} \rightarrow \sigma = \varepsilon_0 E_1 = D$

(3) $E_2 = \frac{\sigma}{\varepsilon_r\varepsilon_0} = E_1\frac{1}{\varepsilon_r} = \frac{2}{\varepsilon_r+1}\frac{V}{d}$

(4) $D = \varepsilon_0 E + P \Rightarrow P = D - \varepsilon_0 E = \frac{2\varepsilon_0 V}{(\varepsilon_r+1)\,d}[\varepsilon_r-1]$

(5) $\hat{n}\cdot\vec{P} = \sigma_b \Rightarrow$