1- Calcule o centro de massa de um sistema constituído por três pontos materiais de massas m_1, m_2 e m_3 e colocados, respetivamente, nos pontos de coordenadas Cartesianas,

$$\vec{r}_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \qquad \vec{r}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{r}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

As coordenadas Cartesianas do centro de massa devem ser expressas apenas em termos dos dois quocientes $A = (m_3/M)$ e $B = (m_1/m_3)$ onde $M = m_1+m_2+m_3$.

$$\mathbf{r}_1 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = -\mathbf{e}_y \qquad \mathbf{r}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \qquad \mathbf{r}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z$$

$$\mathbf{R}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3} = \frac{-m_1\mathbf{e}_y + m_3\left(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z\right)}{m_1 + m_2 + m_3} =$$

$$\mathbf{R}_{CM} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3} = \frac{-m_1\mathbf{e}_y + m_3\left(\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z\right)}{m_1 + m_2 + m_3} =$$

$$= \frac{m_3}{m_1 + m_2 + m_3}\mathbf{e}_x + \frac{m_3 - m_1}{m_1 + m_2 + m_3}\mathbf{e}_y + \frac{m_3}{m_1 + m_2 + m_3}\mathbf{e}_z =$$

$$= \frac{m_3}{m_1 + m_2 + m_3}\mathbf{e}_x + \frac{m_3}{m_1 + m_2 + m_3}\left(1 - \frac{m_1}{m_3}\right)\mathbf{e}_y + \frac{m_3}{m_1 + m_2 + m_3}\mathbf{e}_z =$$

$$= A\mathbf{e}_x + A\left(1 - B\right)\mathbf{e}_y + A\mathbf{e}_z$$

$A = (m_3/M)$ e $B = (m_1/m_3)$ onde $M = m_1 + m_2 + m_3$.

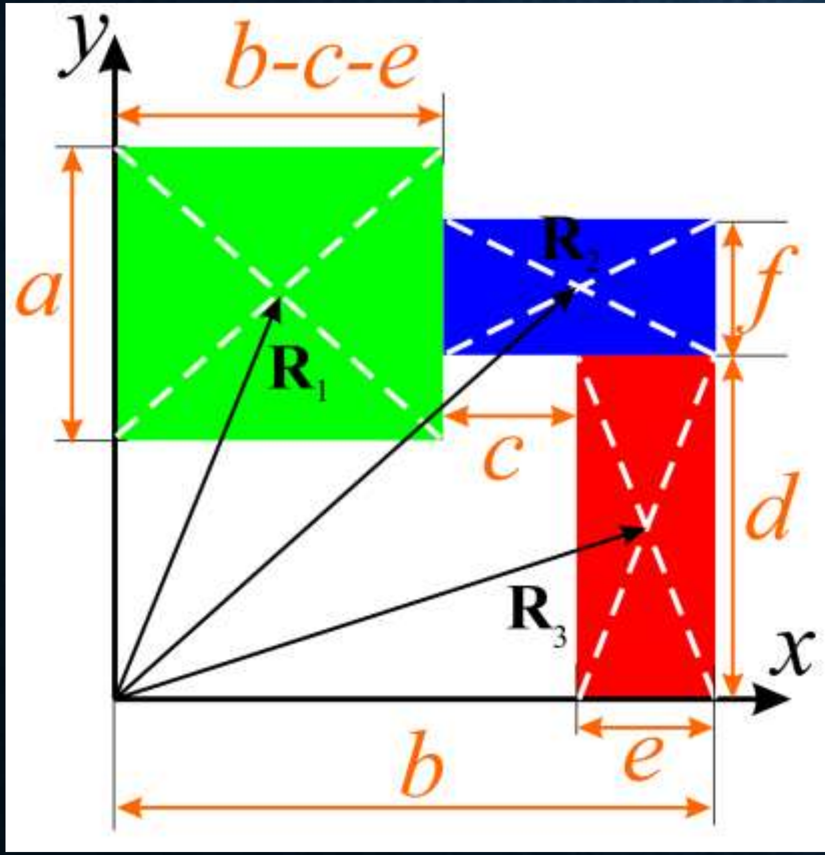
2- Considere um sólido homogéneo de massa m e decomponha-o em n porções disjuntas de massas m_1, m_2,...,m_n e centros de massa $\vec{R}_1$, $\vec{R}_2$,...,$\vec{R}_n$, respetivamente. Mostre que o centro de massa do sólido dado coincide com o centro de massa de um sistema constituído por n partículas de massas m_1, m_2,...,m_n cujos vetores posição são, respetivamente, $\vec{r}_1 = \vec{R}_1$, $\vec{r}_2 = \vec{R}_2$,...,$\vec{r}_n = \vec{R}_n$.

$$\mathbf{R}_{CM} = \frac{1}{M}\int_V d\mathbf{r}\rho\left(\mathbf{r}\right)\mathbf{r} = \frac{1}{M}\left\{\int_{V_1} d\mathbf{r}\rho\left(\mathbf{r}\right)\mathbf{r} + \int_{V_2} d\mathbf{r}\rho\left(\mathbf{r}\right)\mathbf{r} + \ldots + \int_{V_n} d\mathbf{r}\rho\left(\mathbf{r}\right)\mathbf{r}\right\} =$$

$$= \frac{1}{M}\left\{M_1\frac{\int_{V_1} d\mathbf{r}\rho\left(\mathbf{r}\right)\mathbf{r}}{M_1} + M_2\frac{\int_{V_2} d\mathbf{r}\rho\left(\mathbf{r}\right)\mathbf{r}}{M_2} + \ldots + M_n\frac{\int_{V_n} d\mathbf{r}\rho\left(\mathbf{r}\right)\mathbf{r}}{M_n}\right\} =$$

$$= \frac{1}{M}\left\{M_1\mathbf{R}_1 + M_2\mathbf{R}_2 + \ldots + M_n\mathbf{R}_n\right\}$$

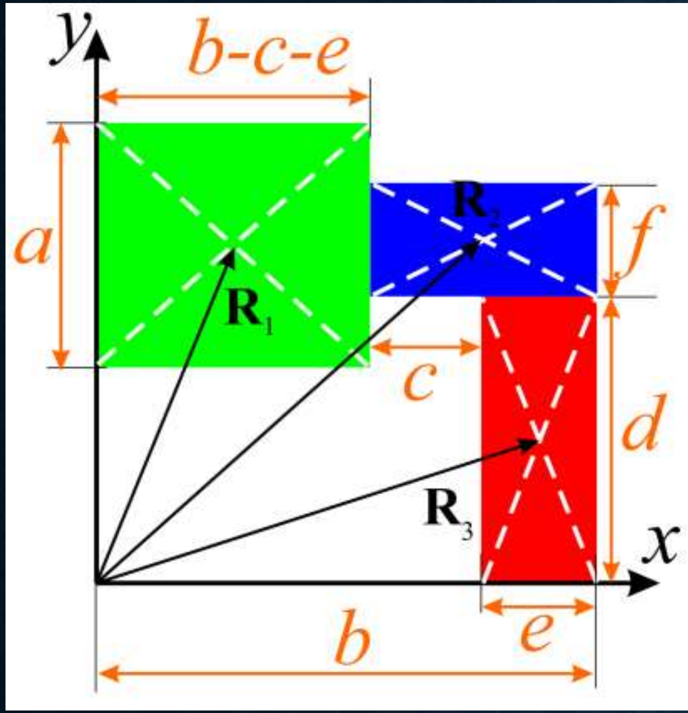
3- Determine o centro de massa do sistema plano de massa total M e superfície S desenhado no quadro constituído por três retângulos supondo que é homogéneo e expresse as suas componentes Cartersianas apenas e termos das distâncias a, b, c, d, e, f da figura. Faz-se notar que a massa de cada retângulo é igual à densidade superficial constante ρ vezes a correspondente área.



$$\mathbf{R}_{CM} = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2 + m_3\mathbf{R}_3}{m_1 + m_2 + m_3}$$

$$\mathbf{R}_3 = \left(b - \frac{e}{2}\right)\mathbf{e}_x + \frac{d}{2}\mathbf{e}_y$$

$$m_1 = \rho a\left(b - c - e\right)$$



$$\mathbf{R}_3 = \left(b - \frac{e}{2}\right)\mathbf{e}_x + \frac{d}{2}\mathbf{e}_y$$

$$\mathbf{R}_2 = \left(b - \frac{c+e}{2}\right)\mathbf{e}_x + \left(d + \frac{f}{2}\right)\mathbf{e}_y$$

$$\mathbf{R}_1 = \frac{b-c-e}{2}\mathbf{e}_x + \left(d + \frac{f}{2}\right)\mathbf{e}_y$$

$$m_1 = \rho a\left(b - c - e\right)$$

$$m_2 = \rho f\left(c + e\right)$$

$$m_3 = \rho d e$$

$$\mathbf{R}_{CM} = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2 + m_3\mathbf{R}_3}{m_1 + m_2 + m_3} =$$

$$= \frac{1}{\rho a\left(b-c-e\right) + \rho f\left(c+e\right) + \rho d e}\left\{\rho a\left(b-c-e\right)\left[\frac{b-c-e}{2}\mathbf{e}_x + \left(d+\frac{f}{2}\right)\mathbf{e}_y\right] + \right.$$

$$\left. + \rho f\left(c+e\right)\left[\left(b - \frac{c+e}{2}\right)\mathbf{e}_x + \left(d+\frac{f}{2}\right)\mathbf{e}_y\right] + \rho d e\left[\left(b-\frac{e}{2}\right)\mathbf{e}_x + \frac{d}{2}\mathbf{e}_y\right]\right\} =$$

$$\mathbf{R}_{CM} = \frac{m_1\mathbf{R}_1 + m_2\mathbf{R}_2 + m_3\mathbf{R}_3}{m_1 + m_2 + m_3} =$$

$$= \frac{1}{\rho a\left(b-c-e\right)+\rho f\left(c+e\right)+\rho de}\left\{\rho a\left(b-c-e\right)\left[\frac{b-c-e}{2}\mathbf{e}_x + \left(d+\frac{f}{2}\right)\mathbf{e}_y\right] + \right.$$

$$\left. +\rho f\left(c+e\right)\left[\left(b-\frac{c+e}{2}\right)\mathbf{e}_x + \left(d+\frac{f}{2}\right)\mathbf{e}_y\right] + \rho de\left[\left(b-\frac{e}{2}\right)\mathbf{e}_x + \frac{d}{2}\mathbf{e}_y\right]\right\} =$$

$$= \frac{a\frac{\left(b-c-e\right)^2}{2} + f\left(c+e\right)\left(b-\frac{c+e}{2}\right) + de\left(b-\frac{e}{2}\right)}{a\left(b-c-e\right)+f\left(c+e\right)+de}\mathbf{e}_x +$$

$$+\frac{a\left(b-c-e\right)\left(d+\frac{f}{2}\right) + f\left(c+e\right)\left(d+\frac{f}{2}\right) + e\frac{d^2}{2}}{a\left(b-c-e\right)+f\left(c+e\right)+de}\mathbf{e}_y$$

4- Determine o centro de massa dos seguintes sistemas:

(a) Um fio homogéneo semi-circular de massa M e raio R.

$$dm = \rho_1 \, dl$$

$$\rho_1 = \frac{M_1}{R\pi}$$

$$dl = R d\varphi$$

$$\mathbf{R}_{CM} = \frac{1}{M} \int_V d\mathbf{r} \rho(\mathbf{r}) \, \mathbf{r} = \frac{1}{M} \int dm \, \mathbf{r}$$

$$\mathbf{r} = R\cos\varphi \mathbf{e}_x + R\sin\varphi \mathbf{e}_y$$

$$\mathbf{R}_{CM1} = \frac{1}{\cancel{M_1}} \int_0^\pi d\varphi \, \cancel{R} \frac{\cancel{M_1}}{\cancel{R}\pi} \left[R\cos\varphi \mathbf{e}_x + R\sin\varphi \mathbf{e}_y\right] =$$

$$= \frac{R}{\pi} \left[\mathbf{e}_x \int_0^\pi d\varphi \, \cos\varphi + \mathbf{e}_y \int_0^\pi d\varphi \, \sin\varphi \right] =$$

$$= \frac{R}{\pi} \left[\mathbf{e}_x \, \sin\varphi|_0^\pi - \mathbf{e}_y \, \cos\varphi|_0^\pi \right] = \frac{R}{\pi} \left[\mathbf{e}_x \, (0-0) - \mathbf{e}_y \, (-1-1) \right] = \frac{2R}{\pi} \mathbf{e}_y$$

$$\mathbf{r} = r\cos\varphi\,\mathbf{e}_x + r\sin\varphi\,\mathbf{e}_y$$

$$dm = \rho_2\, dS$$

$$\rho_2 = \frac{M_2}{0.5\,\pi R^2} = \frac{2M_2}{\pi R^2}$$

$$dS = r d\varphi dr$$

4- Determine o centro de massa dos seguintes sistemas:

(b) Uma placa homogénea semi-circular de massa M e raio R.

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r}$$

$$\mathbf{R}_{CM2} = \frac{1}{M_2}\int_0^R r dr \int_0^\pi d\varphi\,\frac{2M_2}{\pi R^2}\left[r\cos\varphi\,\mathbf{e}_x + r\sin\varphi\,\mathbf{e}_y\right] =$$

$$= \frac{2}{\pi R^2}\int_0^R r^2 dr \int_0^\pi d\varphi\,\left[\cos\varphi\,\mathbf{e}_x + \sin\varphi\,\mathbf{e}_y\right] =$$

$$= \frac{2}{\pi R^2}\int_0^R r^2 dr\, 2\mathbf{e}_y = \frac{4}{\pi R^2}\frac{R^3}{3}\mathbf{e}_y = \frac{4R}{3\pi}\mathbf{e}_y$$

5- Considere uma placa retangular de massa M e lados a e b cuja densidade é proporcional à distância de cada ponto ao lado de comprimento a, sendo pois dada por $\rho = K\,y$ onde K é uma constante.

(a) Determine o centro de massa da placa.

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r} = \frac{1}{M}\int_0^a dx \int_0^b dy\,\rho\,(x,y)\,[x\mathbf{e}_x + y\mathbf{e}_y] =$$

$$dm = \rho\,(x,y)\,dx\,dy \qquad \mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y \qquad \rho\,(x,y) = K\,y$$

(b) Expresse as componentes Cartesianas do vetor centro de massa apenas em termos de a e b.

$$M = \int dm = \int_0^a dx \int_0^b dy\,\rho\,(x,y) = K\int_0^a dx \int_0^b y\,dy$$

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r} = \frac{1}{M}\int_0^a dx \int_0^b dy\,\rho(x,y)\left[x\mathbf{e}_x + y\mathbf{e}_y\right] =$$

$$= \mathbf{e}_x \frac{1}{M}\int_0^a dx \int_0^b dy\,Kxy + \mathbf{e}_y \frac{1}{M}\int_0^a dx \int_0^b dy\,Ky^2 = \mathbf{e}_x \frac{K}{M}\int_0^a x\,dx \int_0^b y\,dy + \mathbf{e}_y \frac{K}{M}\int_0^a dx \int_0^b y^2\,dy =$$

$$= \mathbf{e}_x \frac{K}{M}\left[\left.\frac{x^2}{2}\right|_0^a\right]\left[\left.\frac{y^2}{2}\right|_0^b\right] + \mathbf{e}_y \frac{K}{M}\left[x|_0^a\right]\left[\left.\frac{y^3}{3}\right|_0^b\right] = \mathbf{e}_x \frac{K}{M}\frac{a^2b^2}{4} + \mathbf{e}_y \frac{K}{M}\frac{ab^3}{3}$$

$$M = \int dm = \int_0^a dx \int_0^b dy\,\rho(x,y) = K\int_0^a dx \int_0^b y\,dy = K\left[x|_0^a\right]\left[\left.\frac{y^2}{2}\right|_0^b\right] = K\frac{ab^2}{2}$$

$$\mathbf{R}_{CM} = \mathbf{e}_x K \frac{2}{K\cancel{a}\cancel{b^2}}\frac{a^{\cancel{2}}\cancel{b^2}}{4} + \mathbf{e}_y K \frac{2}{K\cancel{a}\cancel{b^2}}\frac{\cancel{a}b^{\cancel{3}}}{3} = \mathbf{e}_x \frac{a}{2} + \mathbf{e}_y \frac{2}{3}b$$

6- Considere um cone homogéneo invertido de massa M, volume V e altura h cujo eixo coincide com o eixo OZ e cujo vértice corresponde à origem do sistema de referência.

(a) Determine o centro de massa do cone.

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r} = \frac{\rho}{\rho V}\int_V r\,dr\,d\varphi\,dz\,\left[r\cos\varphi\,\mathbf{e}_x + r\sin\varphi\,\mathbf{e}_y + z\mathbf{e}_z\right]$$

$$dm = \rho\,dV = \rho r\,dr\,d\varphi\,dz$$

$$\mathbf{r} = r\cos\varphi\,\mathbf{e}_x + r\sin\varphi\,\mathbf{e}_y + z\mathbf{e}_z$$

$$M = \rho V$$

$$0 \le z \le h$$

$$0 \le \varphi \le 2\pi$$

$$0 \le r \le \frac{R}{h}z$$

(b) Expresse as componentes Cartesianas do vetor centro de massa apenas em termos da altura do cone h.

$$V = \int_V r\,dr\,d\varphi\,dz$$

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r} = \frac{\rho}{\rho V}\int_V r\,dr\,d\varphi\,dz\,\left[r\cos\varphi\mathbf{e}_x + r\sin\varphi\mathbf{e}_y + z\mathbf{e}_z\right] =$$

$$= \mathbf{e}_x\left[\frac{1}{V}\int_0^h dz\int_0^{Rz/h} r^2 dr\int_0^{2\pi}\cos\varphi\,d\varphi\right] + \mathbf{e}_y\left[\frac{1}{V}\int_0^h dz\int_0^{Rz/h} r^2 dr\int_0^{2\pi}\sin\varphi\,d\varphi\right] +$$

$$+\mathbf{e}_z\left[\frac{1}{V}\int_0^h z\,dz\int_0^{Rz/h} r dr\int_0^{2\pi} d\varphi\right] =$$

$$= \mathbf{e}_x\frac{1}{V}\int_0^h dz\int_0^{Rz/h} r^2 dr\,\cancel{\left[\sin\varphi\Big|_0^{2\pi}\right]} + \mathbf{e}_y\frac{1}{V}\int_0^h dz\int_0^{Rz/h} r^2 dr\,\cancel{\left[-\cos\varphi\Big|_0^{2\pi}\right]} +$$

$$+\mathbf{e}_z\frac{2\pi}{V}\int_0^h z\,dz\left[\frac{r^2}{2}\Bigg|_0^{Rz/h}\right] = \mathbf{e}_z\frac{2\pi}{V}\int_0^h \frac{1}{2}\left(\frac{Rz}{h}\right)^2 z\,dz =$$

$$\mathbf{R}_{CM} = \mathbf{e}_z \frac{2\pi}{V} \int\limits_0^h \frac{1}{2} \left(\frac{Rz}{h} \right)^2 z\, dz =$$

$$= \mathbf{e}_z \frac{\pi}{V} \left(\frac{R}{h} \right)^2 \int\limits_0^h z^3\, dz = \mathbf{e}_z \frac{\pi}{V} \left(\frac{R}{h} \right)^2 \left[\frac{z^4}{4} \bigg|_0^h \right] = \mathbf{e}_z \frac{\pi}{4V} \left(\frac{R}{h} \right)^2 h^4 = \mathbf{e}_z \frac{\pi}{4V} R^2 h^2$$

$$V = \int_V r\, dr\, d\varphi\, dz = \int\limits_0^h dz \int\limits_0^{Rz/h} r dr \int\limits_0^{2\pi} d\varphi = 2\pi \int\limits_0^h dz \left[\frac{r^2}{2} \bigg|_0^{Rz/h} \right] = \pi \int\limits_0^h \left(\frac{Rz}{h} \right)^2 dz =$$

$$= \pi \left(\frac{R}{h} \right)^2 \int\limits_0^h z^2\, dz = \pi \left(\frac{R}{h} \right)^2 \left[\frac{z^3}{3} \bigg|_0^h \right] = \pi \left(\frac{R}{h} \right)^2 \frac{h^3}{3} = \frac{\pi}{3} R^2 h$$

$$\mathbf{R}_{CM} = \mathbf{e}_z \frac{\pi}{4V} R^2 h^2 = \mathbf{e}_z \frac{\cancel{\pi}}{4} \frac{3}{\cancel{\pi} \cancel{R^2} \cancel{h}} \cancel{R^2} h^{\cancel{2}} = \mathbf{e}_z \frac{3}{4} h$$

7- Determine as coordenadas do centro de massa de uma semi-esfera homogénea de massa M e raio R e expresse as mesmas apenas em termos de R.

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r} =$$

$$= \frac{\rho}{\rho V}\int_V r^2 \sin\theta\, dr\, d\varphi\, d\theta\, \left[r\sin\theta\cos\varphi\mathbf{e}_x + r\sin\theta\sin\varphi\mathbf{e}_y + r\cos\theta\mathbf{e}_z\right]$$

$$dm = \rho\, dV = \rho r^2 \sin\theta\, dr\, d\varphi\, d\theta$$

$$M = \rho V$$

$$\mathbf{r} = r\sin\theta\cos\varphi\mathbf{e}_x + r\sin\theta\sin\varphi\mathbf{e}_y + r\cos\theta\mathbf{e}_z$$

$$V = \frac{2}{3}\pi R^3$$

$$0 \le r \le R$$

$$0 \le \theta \le \pi/2$$

$$0 \le \varphi \le 2\pi$$

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r} =$$

$$= \frac{\rho}{\rho V}\int_V r^2 \sin\theta\, dr\, d\varphi\, d\theta\ \left[r\sin\theta\cos\varphi\mathbf{e}_x + r\sin\theta\sin\varphi\mathbf{e}_y + r\cos\theta\mathbf{e}_z\right]$$

$$= \mathbf{e}_x\left[\frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^{2\pi} \cos\varphi\, d\varphi\right] + \mathbf{e}_y\left[\frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^{2\pi} \sin\varphi\, d\varphi\right] +$$

$$+\mathbf{e}_z\left[\frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\varphi\right] =$$

$0 \leq r \leq R$ $0 \leq \theta \leq \pi/2$ $0 \leq \varphi \leq 2\pi$

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r} =$$

$$= \mathbf{e}_x \left[\frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^{2\pi} \cos\varphi\, d\varphi\right] + \mathbf{e}_y \left[\frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin^2\theta\, d\theta \int_0^{2\pi} \sin\varphi\, d\varphi\right] +$$

$$+\mathbf{e}_z \left[\frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin\theta\cos\theta\, d\theta \int_0^{2\pi} d\varphi\right] =$$

$$= \mathbf{e}_x \frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin^2\theta\, d\theta \cancel{\left[\sin\varphi|_0^{2\pi}\right]} + \mathbf{e}_y \frac{1}{V}\int_0^R r^3 dr \int_0^{\pi/2} \sin^2\theta\, d\theta \cancel{\left[-\cos\varphi|_0^{2\pi}\right]} +$$

$$+\mathbf{e}_z \frac{2\pi}{V}\int_0^R r^3 dr \int_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta =$$

$$\mathbf{R}_{CM} = \frac{1}{M}\int dm\,\mathbf{r} =$$

$$V = \frac{2}{3}\pi R^3$$

$$= \mathbf{e}_x \frac{1}{V}\int\limits_0^R r^3 dr \int\limits_0^{\pi/2} \sin^2\theta\, d\theta \left[\sin\varphi|_0^{2\pi}\right] + \mathbf{e}_y \frac{1}{V}\int\limits_0^R r^3 dr \int\limits_0^{\pi/2} \sin^2\theta\, d\theta \left[-\cos\varphi|_0^{2\pi}\right] +$$

$$+\mathbf{e}_z \frac{2\pi}{V}\int\limits_0^R r^3 dr \int\limits_0^{\pi/2} \frac{\sin 2\theta}{2}\, d\theta =$$

$$= \mathbf{e}_z \frac{2\pi}{V}\left[\frac{r^4}{4}\bigg|_0^R\right]\left[-\frac{\cos 2\theta}{4}\bigg|_0^{\pi/2}\right] = -\mathbf{e}_z \frac{\pi}{V}\frac{R^4}{8}\left[\cos 2\theta|_0^{\pi/2}\right] =$$

$$= -\mathbf{e}_z \frac{\pi}{V}\frac{R^4}{8}[-1-1] = \mathbf{e}_z \frac{\pi}{V}\frac{R^4}{4} = \mathbf{e}_z \cancel{\pi}\frac{3}{2\cancel{\pi}\cancel{R^3}}\frac{R^{\cancel{4}}}{4} = \mathbf{e}_z \frac{3}{8}R$$