Ondas - Apontamentos

Simão Cardoso

Janeiro 2022

1 Periodic Motions

The description of simple harmonic motion [1]

If the body is of mass m and the mass of the spring is negligible, the equation of motion of the body becomes:

$$m\frac{d^2x}{dt^2} = -kx \tag{1}$$

with solution:

$$x = A\sin(\omega t + \phi_0) \tag{2}$$

where $\omega = \sqrt{\frac{k}{m}}$ and $T = \frac{2\pi}{\omega}$.

For the particular time $t = 0$, we have these following identities:

$$A = \left[x_0^2 + \left(\frac{v_0}{\omega}\right)^2\right]^{\frac{1}{2}}$$

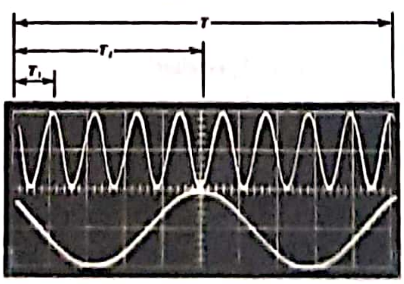
$$\phi_0 = \tan^{-1}\left(\frac{\omega x_0}{v_0}\right)$$

2 The Superposition of Periodic Motions

Superposed vibrations of different frequency; beats

The condition for any sort of true periodicity in the combined motion of two waves with different frequencies is that periods of the component motions be commensurable - there exist two integers n_1 and n_2 such that:

$$T = n_1T_1 = n_2T_2 \tag{3}$$



If two waves are quite close in frequency, the combined disturbance exhibits what are called *beats.*

$$x_1 = A\cos(\omega_1 t)$$
$$x_2 = A\cos(\omega_2 t)$$

Then by addition we get:

$$\boxed{x = 2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)} \tag{4}$$

Many superposed vibrations of the same frequency

We suppose there are N combining vibrations, each with amplitude A_0 and differing in phase from the next one by angle δ.
Let the first of the component vibrations be described by the equation:

$$x = A_0\cos(\omega t)$$

And the resultant disturbance will be given by:

$$X = A\cos(\omega t + \alpha)$$

We can then write the following geometrical statements:

$$\begin{cases} A = A_0\frac{\sin(N\delta/2)}{\sin(\delta/2)} \\ \alpha = \frac{(N-1)\delta}{2} \end{cases} \tag{5}$$

Hence the resulting vibration along the x axis is described by the followig equation:

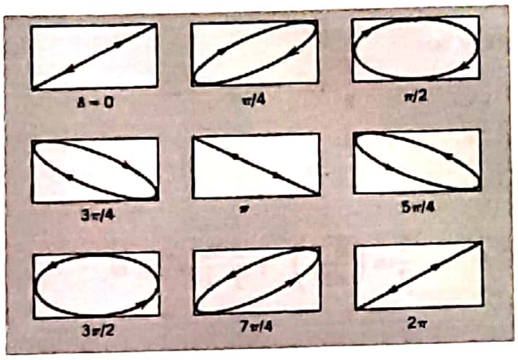
$$\boxed{X = A_0\frac{\sin(N\delta/2)}{\sin(\delta/2)}\cos\left[\omega t + \frac{(N-1)\delta}{2}\right]} \tag{6}$$

Perpendicular motions with equal frequencies

We can write the combining vibrations in the following simple form where δ is the inital phase difference between the motions:

$$\begin{cases} x = A_1\cos(\omega t) \\ y = A_2\cos(\omega t + \delta) \end{cases} \tag{7}$$

By specializing still further, to particular values of δ, we can quickly build up a qualitative picture of all possible motions for which the combining frequencies are equal.



3 The Free Vibrations of Physical Systems

The basic mass-spring problem

If a system can be regarded as being effecively a concentrated mass at the end of a spring, then we can write its equation of motion either of two ways:

1- By Newton's law $F = ma$,

$$-kx = ma$$

2- By conservation of total mechanincal energy (E),

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$$

In explicit differential form, they may be written as follows:

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{8}$$

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2 = E \tag{9}$$

Floating objects

If a floating object is slightly depressed or raised from its normal position of equilibrium, there is called into play a restoring force equal to the increase or decrease in the weight of liquid displaced by the object, and periodic motion ensues.

$$m\frac{d^2x}{dt^2} = -g\rho Ax \tag{10}$$

Where A is the diameter, ρ the density of the liquid, $k = g\rho A$, $\omega = \sqrt{\frac{g\rho A}{m}}$ and $T = 2\pi\sqrt{\frac{m}{g\rho A}}$.

Pendulums

The statement of conservation of energy is:

$$\frac{1}{2}mv^2 + mgy = E$$

Given the approximations, it is correct to put:

$$\frac{1}{2}m\left(\frac{d^2x}{dt^2}\right)^2 + \frac{1}{2}\frac{mg}{l}x^2 = E$$

Using angular displacement θ, we have $v = l\left(\frac{d\theta}{dt}\right)$ and $y \approx \frac{1}{2}l\theta^2$:

$$\frac{1}{2}ml^2\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{2}mgl\theta^2 = E \tag{11}$$

The decay of free vibrations

The resistive force is exerted oppositely to the direction of v itself. In this case the statement of Newton's law for the moving mass can be written:

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{12}$$

Where $\gamma = \frac{b}{m}$ and $\omega_0^2 = \frac{k}{m}$.

The solution is:

$$x = Ae^{-\gamma t/2}\cos(\omega t + \alpha) \tag{13}$$

Where $\omega^2 = \omega_0^2 - \frac{\gamma^2}{4}$ and $A(t) = A_0 e^{-\gamma t/2}$

Now the total mechanical energy of a simple harmonic oscillator, using the value of A, is given by:

$$E(t) = E_0 e^{-\gamma t} \tag{14}$$

For convenience, we definee a parameter called the Q (quality) value of the oscillatory system, given by the ration of these two quantities:

$$Q = \frac{\omega_0}{\gamma} \tag{15}$$

In terms of the Q value, ω^2 becomes:

$$\omega^2 = \omega_0^2\left(1 - \frac{1}{4Q^2}\right) \tag{16}$$

If Q is large compared to unity and $\omega \approx \omega_0$ then the motion is given nearly by:

$$x = A_0 e^{-\omega_0 t/2Q} \cos(\omega_0 t + \alpha) \tag{17}$$

Let us measure the time t in terms of the number of complete cycles of oscillation, n. Then given the approximation that $\omega \approx \omega_0$, we can put $t \approx 2\pi n/\omega_0$. In terms of the number of cycles elapsed, therefore, we can put:

$$A(n) \approx A_0 e^{-n\pi/Q} \tag{18}$$

The effects of very large damping

If $\gamma > 2\omega_0$, a rigorous analysis shows that both exponentials are in general necessary, and that the complete variation of x with t is given by the following equation:

$$\boxed{x = A_1 e^{-(\gamma/2+\beta)t} + A_2 e^{-(\gamma/2-\beta)t}} \tag{19}$$

Where $\beta = \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{\frac{1}{2}}$

If $\gamma = 2\omega_0$ the appropriate form of solution for this case is:

$$\boxed{x = (A + Bt)e^{-\gamma t/2}} \tag{20}$$

4 Forced vibrations and resonance

Undamped oscillator with harmonic forcing

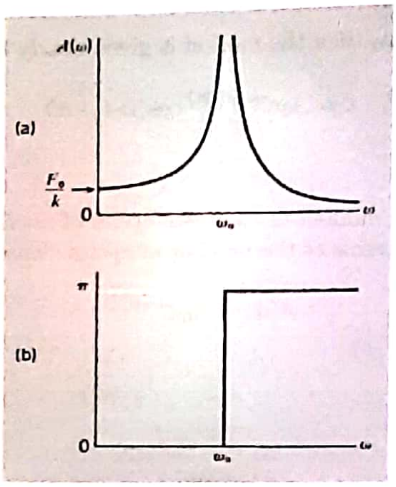
We shall imagine the application of a sinusoidal driving force $F = F_0 \cos(\omega t)$. Then the statement of the equation of motion in the form of $ma =$ net force, is:

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos(\omega t) \tag{21}$$

The solution is:

$$x = A\cos(\omega t + \alpha) \tag{22}$$

Where $\boxed{A = |C| = \left|\frac{F_0/m}{\omega_0^2 - \omega^2}\right|}$. The infinite value A at $\omega = \omega_0$, and the discontinuous jump from zero to π in the value α as one passes through ω_0, must be unphysical, but as we'll see, they represent a mathematically limiting case of what actually occurs in systems with nonzero damping.



Forced oscillations with damping

We shall now consider the result of acting on a system with a force just like that considered in the previous section:

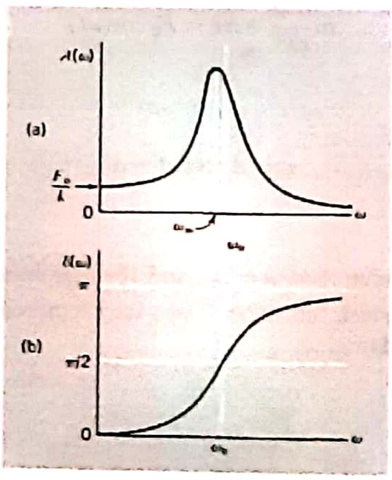
$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt}+\omega_0^2x=F_0\cos(\omega t) \tag{23}$$

One simply assumes a solution of the form:

$$x=A\cos(\omega t+\delta) \tag{24}$$

Where $A(\omega)$ (resonance) and $\delta(\omega)$ are:

$$\begin{cases} A(\omega)=\frac{F_0/m}{[(\omega_0^2-\omega^2)^2+(\gamma\omega)^2]^{\frac{1}{2}}} \\ \delta(\omega)=\tan^{-1}\left(\frac{\gamma\omega}{\omega_0^2-\omega^2}\right) \end{cases} \tag{25}$$



We shall put Eq.(25) into more convenient form. Substituting $\gamma = \frac{\omega_0}{Q}$:

$$\begin{cases} A = \frac{F_0}{k} \frac{\omega_0/\omega}{\left[\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}\right]^{\frac{1}{2}}} \\ \delta = \tan^{-1}\left(\frac{1/Q}{\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}}\right) \end{cases} \tag{26}$$

Transient phenomena

Turning now to the more realistic case in which damping is assumed to be present, we can postulate the following combination of free and steady-state motions:

$$\boxed{x = Be^{-\gamma t/2}\cos(\omega_1 t + \beta) + A\cos(\omega t + \alpha)} \tag{27}$$

δ

Where $\omega_1 = \sqrt{\omega_0^2 - \gamma^2/4}$ and $\delta = \tan^{-1}(\gamma\omega/(\omega_0^2 - \omega^2))$. The first part of the solution goes to 0 while the second one is the steady-state motion and A is the resonance.

Power absorbed by a driven oscillator

Power with decay

$$P = -(F_0 v_0 \cos(\delta))\sin(\omega t)\cos(\omega t) + (F_0 v_0 \sin(\delta))\cos^2(\omega t) \tag{28}$$

Which becomes:

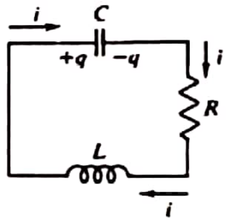
$$\overline{P}(\omega) = \frac{F_0^2 \omega_0}{2kQ} \frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{1}{Q^2}} \tag{29}$$

We see that this power input passes through a maximum at precisely $\omega = \omega_0$ for any Q. The maximum power is given by:

$$\boxed{\overline{P}(\omega_0) = \frac{QF_0^2}{2m\omega_0}} \tag{30}$$

X

Electrical resonance



The statement of zero net voltage drop in one complete tour of the circuit is as follow:

$$\frac{q}{C} + IR + L\frac{dI}{dt} = 0$$
$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = 0$$

In this equation, R/L plays the exact role of the damping constant γ, and in such a circuit the charge on the capacitor plates will undergo exponentially damped harmonic oscillations.
Finally, if the circuit is driven by an altering applied voltage, we have a typical forced-oscillator equation:

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{V_0}{L}\cos(\omega t) \tag{31}$$

5 Normal modes of continuous systems

The free vibrations of streched strings

The equation for the vibrating string is as follows:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \tag{32}$$

Where $v = \left(\frac{T}{\mu}\right)^{\frac{1}{2}}$.

We shall now look for solutions of this equation corresponding to the kind of situation physically represented by a stationary vibration. This means that every point on the string is moving with a time dependence of the form of $\cos(\omega t)$, but that the amplitude of this motion is a function of distance x of that point from the end of the string.

$$y_n(x,t) = A_n \sin\left(\frac{2\pi x}{x_n}\right)\cos(\omega_n t) \tag{33}$$

Where $n = 1, 2, 3, \ldots$, $\omega_n = \frac{n\pi}{L}v$ and $x_n = \frac{2L}{n}$.

It will be convenient to introduce the number of cycles per unit time, ϑ, equal to $\omega/2\pi$. The frequencies of the permitted stationary vibrations are thus given by:

$$\vartheta = \frac{nv}{2L} = \frac{n}{2L}\left(\frac{T}{\mu}\right)^{\frac{1}{2}} \tag{34}$$

We can define a wavelength x_n, associated with the mode n, such that:

$$x_n = \frac{2L}{n} \tag{35}$$

Forced harmonic vibration of a stretched string

We shall suppose a steady-state solution of the form:

$$y(x,t) = f(x)\cos(\omega t)$$

But now subject to the following conditions:

$$\begin{cases} y(0,t) = B\cos(\omega t) \\ y(L,t) = 0 \end{cases}$$

And:

$$\begin{cases} f(x) = A\sin\left(\frac{\omega x}{v} + \alpha\right) \\ f(L) = 0 \end{cases}$$

Therefore:

$$y(0,t) = B\cos(\omega t) = f(0)\cos(\omega t) = A\sin(\alpha)\cos(\omega t) \tag{36}$$

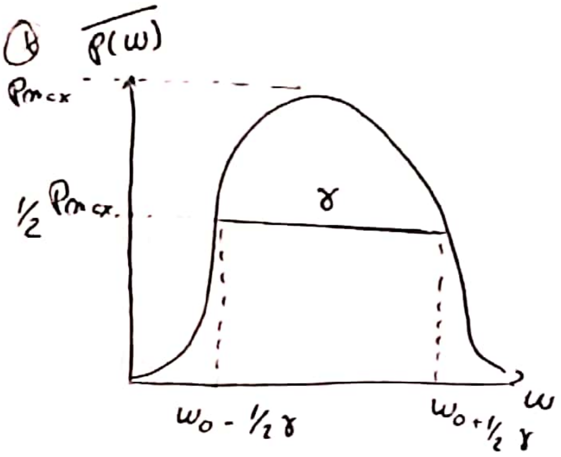
Where:

$$\boxed{\begin{cases} A = \frac{B}{\sin(n\pi - \omega L/v)} \\ \alpha = n\pi - \omega L/v \end{cases}} \tag{37}$$

References

[1] Anthony Philip French. Vibrations and waves, 2001.

①



$E = E_0\, e^{-\gamma t}$