Problem

Maxwell equation in vacuum and the Heaviside duality

$$\nabla \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \nabla \wedge \vec{B} = \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}$$

These equations are interchanged under the replacements

$E \to -c\vec{B}$ and $c\vec{B} \to \vec{E}$ (check this)

Larmor generalization to a continuous transformation:

$$E' = E\cos\eta - cB\sin\eta$$

$$cB' = E\sin\eta + cB\cos\eta$$

$$\begin{bmatrix} E' \\ cB' \end{bmatrix} = \begin{bmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} E \\ cB \end{bmatrix}$$

- Show that the Maxwell's equations in vacuum are invariant under such a transformation.

- Can this symmetry be maintained with sources if one considers the existence of magnetic charges?

a)

$$\nabla \wedge \vec{E}' = \nabla \wedge \left[\vec{E} \cos\eta - c B \sin\eta \right] =$$

$$= (\nabla \wedge \vec{E}) \cos\eta - c \sin\eta \, (\nabla \wedge \vec{B})$$

$$= -\dot{\vec{B}} \cos\eta - c \sin\theta \, \frac{1}{c^2} \dot{\vec{E}} = -\dot{\vec{B}}'$$

(Maxwell's in the transf. fields).

$$-\dot{\vec{B}} = -\frac{\dot{\vec{E}}}{c} \sin\eta \; \dot{B} \cos\eta$$

$$\nabla \wedge \vec{B}' = \nabla \wedge \left[\frac{E}{c} \sin\eta + B \cos\eta \right]$$

$$= \frac{1}{c} \sin\eta \, (\nabla \wedge \vec{E}) + (\nabla \wedge B) \cos\eta \overset{?}{\equiv} \frac{1}{c^2} \frac{\partial \vec{E}'}{\partial t}$$

$$\frac{1}{c^2} \dot{\vec{E}}' = \frac{\dot{E}}{c^2} \cos\eta - \sin\eta \frac{\dot{B}}{c}$$

$$-\frac{1}{c} \sin\eta \, \dot{B} + \cos\eta \, \frac{1}{c^2} \dot{\vec{E}}$$

O.K.

b) Magnetic charges?

$$\nabla \cdot \vec{B}' = \mu_0 \rho'_{mag.}$$

$$\nabla \cdot \vec{B}' = \nabla \cdot \left[\frac{1}{c} E \sin\eta + \cos\eta \, B \right] =$$

$$= \frac{1}{c} \sin\eta \left(\nabla \cdot \vec{E} \right) + \cos\theta \left(\nabla \cdot \vec{B} \right) = \frac{1}{c} \frac{\rho_e}{\varepsilon_0} \sin\eta + \cos\eta \, \mu_0 \rho_{mag.}$$

$$\stackrel{?}{=} \mu_0 \rho'_{mag}$$

$$\rho'_{mag} = \frac{1}{c} \frac{1}{\varepsilon_0 \mu_0} \rho_e \sin\eta + \rho_{mag} \cos\eta$$

$$\boxed{\rho'_{mag} = c \rho_e \sin\eta + \rho_{mag} \cos\eta}$$

$$\nabla \cdot \vec{E}' = \nabla \cdot \left[\bar{E} \cos\eta - \sin\eta \, c \vec{B} \right] = \cos\theta \left(\nabla \cdot \vec{E} \right) - \sin\eta \, c \left(\nabla \cdot \vec{B} \right)$$

$$= \cos\theta \frac{\rho_e}{\varepsilon} - \sin\eta \, c \, \mu_0 \rho_{mag.}$$

$$\downarrow$$

$$\frac{\rho'_e}{\varepsilon_0} \stackrel{!}{\equiv} \frac{\rho_e}{\varepsilon} \cos\eta - \sin\eta \, c \, \mu_0 \rho_{mag}.$$

$$\boxed{\rho'_e = \rho_e \cos\eta - \frac{1}{c} \rho_{mag.} \sin\eta}$$

$$\nabla\wedge\vec{E}' = -\dot{\vec{B}}' - \frac{1}{\varepsilon_0}\vec{J}'_{mag}. \quad \cancel{= -\mu_0\dot{\vec{H}}' - \frac{1}{\varepsilon_0}\dot{\vec{J}}_{mag}}.$$

$$\downarrow$$

$$\nabla\wedge\left[\cos\eta\,\vec{E} - c\sin\eta\,\vec{B}\right] \overset{?}{\equiv} -\left[\frac{\dot{E}}{c}\sin\eta + \cos\eta\,\dot{B}\right] - \frac{1}{\varepsilon_0}\vec{J}'_{mag}.$$

$$\cos\eta\,(\nabla\wedge\vec{E}) - c\sin\eta\,(\nabla\wedge\vec{B}) \equiv -\frac{\dot{\vec{E}}}{c}\sin\eta - \cos\eta\,\dot{\vec{B}} - \frac{1}{\varepsilon_0}\vec{J}'_{mag}$$

$$\cos\eta\left[-\cancel{\dot{\vec{B}}} - \frac{1}{\varepsilon_0}\vec{J}_{mag}\right] - c\sin\eta\left[\mu_0 j_e + \cancel{\frac{1}{c^2}\dot{\vec{E}}}\right] \equiv -\cancel{\frac{\dot{E}}{c}\sin\eta} - \cancel{\dot{B}\cos\eta} - \frac{1}{\varepsilon_0}\vec{J}'_{mag}.$$

$$-\frac{1}{\varepsilon_0}\cos\eta\,\vec{J}_{mag} - \mu_0 c\sin\eta\,\vec{J}_e = -\frac{1}{\varepsilon_0}\vec{J}'_{mag}. \Rightarrow$$

$$\Rightarrow \boxed{\vec{J}'_{mag} \equiv \vec{J}_{mag}\cos\eta + \frac{1}{c}\sin\eta\,\vec{J}_e}$$

$$\nabla\wedge\vec{B}' = \mu_0\vec{J}'_e + \frac{1}{c^2}\dot{\vec{E}}'$$

$$\downarrow$$

$$\nabla\wedge\left[\frac{\vec{E}}{c}\sin\eta + B\cos\eta\right] \equiv \mu_0\vec{J}'_e + \frac{1}{c^2}\left[\cos\eta\,\dot{\vec{E}} - \sin\eta\,c\,\dot{\vec{B}}\right]$$

$$\Leftrightarrow \cos\eta\,(\nabla\wedge\vec{B}) + \frac{1}{c}\sin\eta\,(\nabla\wedge\vec{E}) \equiv \mu_0\vec{J}'_e + \frac{1}{c^2}\left[\cos\eta\,\dot{E} - \sin\eta\,c\,\dot{\vec{B}}\right]$$

$$\cos\eta \left[\mu_0 \vec{J}_e + \mu_0 \varepsilon_0 \cancel{\frac{\partial E}{\partial t}} \right] + \frac{1}{c} \sin\eta \left(\cancel{-\dot{\vec{B}}} - \frac{1}{\varepsilon_0} \vec{J}_{mag} \right) \equiv$$

$$\equiv \mu_0 \vec{J}_e' + \frac{1}{c^2}\left[\cancel{\cos\eta \, \dot{\vec{E}}} - \cancel{\sin\eta \, c \, \dot{\vec{B}}} \right] \Rightarrow$$

$$\Rightarrow \boxed{\vec{J}_e' = \cos\eta \, j_e + c \sin\eta \, j_{mag}.}$$

The Heaviside - Larmor duality is consistent with the generalised Maxwell's - equations if

$$\begin{bmatrix} \rho_e' \\ \rho_m' \end{bmatrix} = \begin{bmatrix} \cos\eta & -\frac{1}{c}\sin\eta \\ c\sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} \rho_e \\ \rho_m \end{bmatrix}$$ check this.

$$\begin{bmatrix} \vec{J}_e' \\ \vec{J}_m' \end{bmatrix} = \begin{bmatrix} \cos\eta & -c\sin\eta \\ \frac{1}{c}\sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} \vec{J}_e \\ \vec{J}_m \end{bmatrix}$$

$$\begin{bmatrix} \vec{E}' \\ c\vec{B}' \end{bmatrix} = \begin{bmatrix} \cos\eta & -c\sin\eta \\ \frac{1}{c}\sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} E \\ cB \end{bmatrix}$$