Quantum Physics II: Angular Momentum in Real Space

Lecture notes 2020-21

REVISÕES MOMENTO ANGULAR NO ESPAÇO REAL

$$\hat{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

$$\hat{L}^2 |l,m\rangle = \hbar^2 l(l+1) |l,m\rangle$$

Fazendo o produto interno com $\langle\theta,\varphi|$

$$\langle\theta,\varphi|\hat{L}^2|l,m\rangle = \hat{L}^2 \langle\theta,\varphi|l,m\rangle$$

$$= \hat{L}^2 Y_{l,m}(\theta,\varphi) = \hbar^2 l(l+1) Y_{l,m}$$

harmónicos esféricos

Qual a forma de $\hat{\vec{L}}$?

$$\hat{\vec{L}} = \vec{r} \times \hat{\vec{p}} \quad ; \quad \hat{\vec{p}} = -i\hbar \vec{\nabla}$$

$$\vec{r} = r\,\hat{u}_r$$ (coordenadas esféricas)

$$\hat{\vec{p}} = -i\hbar\left(\hat{u}_r \frac{\partial}{\partial r} + \hat{u}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{u}_\varphi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)$$

(coordenadas esféricas)

$\hat{u}_r, \hat{u}_\theta, \hat{u}_\varphi \longrightarrow$ versores

$$\hat{\vec{L}} = \vec{r} \times \hat{\vec{p}} =$$

$$= -i\hbar\left(\hat{u}_\varphi \frac{\partial}{\partial \theta} - \hat{u}_\theta \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}\right)$$

$$\hat{L}_z = \hat{u}_z \cdot \hat{\vec{L}} = -i\hbar \frac{\partial}{\partial \varphi}$$

$$\hat{L}_x = -i\hbar \left(-\sin\varphi \frac{\partial}{\partial \theta} - \cot g\theta \cos\varphi \frac{\partial}{\partial \varphi} \right)$$

$$\hat{L}_y = -i\hbar \left(\cos\varphi \frac{\partial}{\partial \theta} - \cot g\theta \sin\varphi \frac{\partial}{\partial \varphi} \right)$$

Eq. valores próprios

$$\hat{L}_z Y_{lm}(\theta,\varphi) = -i\hbar \frac{\partial}{\partial \varphi} Y_{lm}(\theta,\varphi)$$

Separação de variáveis:

$$Y_{lm} = \Theta_{lm}(\theta)\, \Phi_m(\varphi)$$

$$\hat{L}_z Y_{lm} = \cancel{\Theta_{lm}}(\theta)(-i\hbar) \frac{\partial}{\partial \varphi} \Phi_m(\varphi)$$

$$= \hbar m\, \cancel{\Theta_{lm}}(\theta)\, \Phi_m(\varphi)$$

$$-i\hbar \frac{\partial}{\partial \varphi} \Phi_m(\varphi) = \hbar m \, \Phi_m(\varphi)$$

a solução é: $\Phi_m(\varphi) = e^{i\varphi m}$

$\varphi \to \varphi + 2\pi$: $\Phi_m(\varphi) = \Phi_m(\varphi + 2\pi)$

$$\Leftrightarrow e^{i\varphi m} = e^{i(\varphi + 2\pi) m}$$

$\Rightarrow \quad m = 0, \pm 1, \pm 2, \ldots$

$\Rightarrow \ell =$ inteiros.

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Encontrar $\Theta_{\ell m}(\theta)$

$$L_+ = L_x + i L_y$$

$$= -i\hbar \left(i \frac{\partial}{\partial \theta} - \cot g\,\theta \frac{\partial}{\partial \varphi} \right)$$

$$Y_{lm}(\theta,\varphi) = \Theta_{lm}(\theta)\, e^{im\varphi}$$

$$L_+ Y_{ll}(\theta,\varphi) = 0 \qquad \left(L_+ |l,l\rangle = 0 \right)$$

$$-i\hbar\left(i\frac{\partial}{\partial\theta} - \cot g\,\theta \frac{\partial}{\partial\varphi} \right)\Theta_{ll}(\theta)\, e^{il\varphi} = 0$$

$$\left(i\frac{\partial}{\partial\theta} - \cot g\,\theta \frac{\partial}{\partial\varphi} \right)\Theta_{ll}(\theta)\, e^{il\varphi}$$

$$= \left(\frac{\partial}{\partial\theta} - \frac{l\cos\theta}{\sin\theta} \right)\Theta_{ll}(\theta) = 0$$

solução é $\Theta_{l,l}(\theta) = \sin^l\theta$

$$\boxed{Y_{ll}(\theta,\varphi) = A \sin^l\theta \cdot e^{il\varphi}}$$

O próximo estado da torre:

$$Y_{l,l-1}(\theta,\varphi) = \Theta_{l,l-1}(\theta)\, e^{i(l-1)\varphi}$$

$$Y_{l,l-1}(\theta,\varphi) = L_- \, Y_{l,l}(\theta,\varphi)$$

$$L_- = L_x - i L_y$$

$$= -i\hbar\left(-\sin\varphi \frac{\partial}{\partial\theta} - \cot g\,\theta \cos\phi \frac{\partial}{\partial\varphi}\right)$$

$$-i(-i\hbar)\left(\cos\varphi \frac{\partial}{\partial\theta} - \cot g\,\theta \sin\varphi \frac{\partial}{\partial\varphi}\right)$$

$$= -i\hbar\left(-i(\cos\varphi - i\sin\varphi)\frac{\partial}{\partial\theta}\right.$$

$$\left. - (\cos\varphi - i\sin\varphi)\cot g\,\theta \frac{\partial}{\partial\varphi}\right)$$

$$L_- = -i\hbar\left(-i e^{-i\varphi}\frac{\partial}{\partial\theta} - \cot g\theta\, e^{-i\varphi}\frac{\partial}{\partial\varphi}\right)$$

$$= -i\hbar\, e^{-i\varphi}\left(-i\frac{\partial}{\partial\theta} - \cot g\theta\frac{\partial}{\partial\varphi}\right)$$

$$L_- Y_{ll}(\theta,\varphi) = L_- \sin^l\theta\, e^{il\varphi}$$

$$= -i\hbar\, e^{i(l-1)\varphi}\left(-il\sin^{l-1}\theta - \cot g\theta \cdot \sin^l\theta \cdot il\right)$$

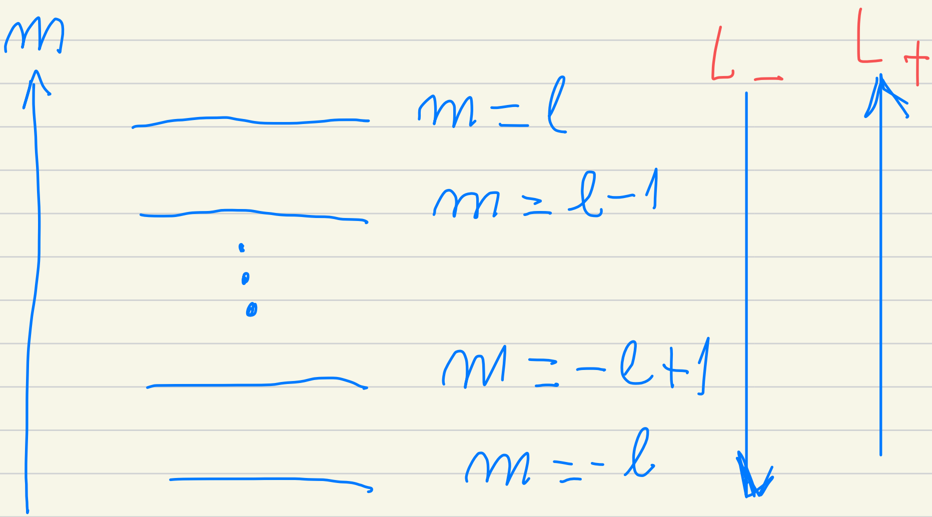
$$= -i\hbar\, e^{i(l-1)\varphi}(-il) \times \left(\sin^{l-1}\theta - \cos\theta\sin^{l-1}\theta\right)$$

$$= l\hbar\sin^{l-1}\theta\,(1-\cos\theta)\, e^{i(l-1)\varphi} \quad (*)$$

(**) $$L_- Y_{l,l}(\theta,\varphi) = \hbar\sqrt{l(l+1) - l(l-1)} \times$$
$$\times\, Y_{l,l-1}(\theta,\varphi)$$

(*) = (**

$$\boxed{Y_{l,l-1} = \frac{l}{\sqrt{2l}} \cdot \sin^{l-1}\theta\,(1-\cos\theta) \times e^{i(l-1)\varphi}}$$



$L_+ Y_{ll} = 0$ e $L_- Y_{l,-l} = 0$.

Dado um harmônico esférico

$$Y_{lm}(\theta,\varphi)$$

A condição de normalização,

$$\int d\Omega \, Y^*_{lm}(\theta,\varphi) \, Y_{lm}(\theta,\varphi)$$

$$\boxed{= \int_0^{2\pi} d\varphi \cdot \int_0^{\pi} \sin d\theta \cdot |Y_{lm}(\theta,\varphi)|^2 = 1.}$$

Outro harmônico esférico $Y_{l0}(\theta,\varphi)$

$$\hat{L}^2 \, Y_{l0}(\theta,\varphi) = \hbar^2 l(l+1) \, Y_{l0}(\theta,\varphi)$$

$$\Leftrightarrow \hat{L}^2 \, \Theta_{l0}(\theta) = \hbar \, l(l+1) \, \Theta_{l0}(\theta)$$

$$\boxed{\cos\theta \frac{\partial \Theta_{l0}(\theta)}{\partial\theta} + \sin\theta \frac{\partial^2 \Theta_{l0}(\theta)}{\partial\theta^2} + \sin\theta \; l(l+1)\, \Theta_{l0}(\theta) = 0}$$

↳ Eq. Legendre

Soluções: funções Legendre

se l = inteiros

funções Legendre = polinómios

$P_{l0}(\cos\theta)$

$$\boxed{Y_{l0} = \sqrt{\frac{2l+1}{4\pi}} \cdot P_l(\cos\theta)}$$

Os 4 primeiros $Y_{l,0}(\theta)$

$$Y_{00} = \sqrt{\frac{1}{4\pi}}.$$

$$Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cdot \cos\theta.$$

usando o L_+ em $Y_{1,0}$

$$Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \cdot e^{+i\varphi} \cdot \sin\theta.$$

usando o L_- em $Y_{1,0}$

$$Y_{1,-1} = +\sqrt{\frac{3}{8\pi}} \cdot e^{-i\varphi} \sin\theta.$$

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1. ADIÇÃO DE MOMENTOS ANGULARES

1.1) Lista de resultados importantes:

exemplo possível: átomo He que tem 2 electrões: $s_1 = 1/2$ e $s_2 = 1/2$ para o momento angular de spin.

$$[L^2, L_z] = 0 \quad , \quad [S_i^2, S_{zi}] = 0$$

$$[L_x, L_y] = i\hbar L_z \quad , \quad [S_{ix}, S_{iy}] = i\hbar S_{iz}$$

$$L^2 |l, m\rangle = \hbar^2 l(l+1) |l, m\rangle$$

$$L_z |l, m\rangle = \hbar m |l, m\rangle$$

$$S_i^2 |s_i, s_{zi}\rangle = \hbar^2 s_i(s_i+1) |s_i, s_{zi}\rangle$$

$$S_{zi} |s_i, s_{zi}\rangle = \hbar s_{zi} |s_i, s_{zi}\rangle$$

$$L_+ |l, m\rangle = \hbar \sqrt{l(l+1) - m(m+1)}\, |l, m+1\rangle$$

$$L_- |l, m\rangle = \hbar \sqrt{l(l+1) - m(m-1)}\, |l, m-1\rangle$$

$$S_{+i} |s_i, s_{zi}\rangle = \hbar \sqrt{s_i(s_i+1) - s_{zi}(s_{zi}+1)} \times |s_i, s_{zi}+1\rangle$$

$$S_{-i} |s_i, s_{zi}\rangle = \hbar \sqrt{s_i(s_i+1) - s_{zi}(s_{zi}-1)} \times |s_i, s_{zi}-1\rangle$$

$$s_i = 1/2 \,,\; s_{zi} = \pm \frac{1}{2} .$$

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1.2 ADIÇÃO DOIS SPINS 1/2

$\vec{S}_1$ e $\vec{S}_2$

$$\vec{S}_1 = \frac{\hbar}{2}\vec{\sigma}_1 \;,\; \vec{S}_2 = \frac{\hbar}{2}\vec{\sigma}_2$$

Momento angular total:

$$\vec{S} = \vec{S}_1 + \vec{S}_2 \;,\; S_z = S_{1z} + S_{2z}$$

$$\vec{S}^2 = (\vec{S}_1 + \vec{S}_2)^2$$

$$= \vec{S}_1^2 + \vec{S}_2 + 2\vec{S}_1\cdot\vec{S}_2$$

Conjunto de resultados:

$$[\vec{S}^2, S_z] = 0 \;,\; [\vec{S}^2, \vec{S}_i^2] = 0 \;,$$

$$[S_z, \vec{S}_i^2] = 0$$

Existe uma base de estados:

$$|s_1, s_2; s, s_z\rangle \quad (2)$$

Existe uma base de estados:

$$|s_1, s_2; s_{z1}, s_{z2}\rangle \quad (1)$$

O número de estados nas duas bases tem o mesmo:

Exemplo: base (1)

$$s_1 = 1/2 \Rightarrow s_{z1} = -1/2 \text{ ou } +1/2$$

$$s_2 = 1/2 \Rightarrow s_{z2} = -1/2 \text{ ou } +1/2$$

$$\#\text{estados} = 2 \times 2 = 4$$
$$= (2s_1+1)(2s_2+1).$$

Exprimir a base (2) à custa da base (1)

$$|s, s_z; s_1, s_2\rangle =$$

$$\sum_{\substack{s_{z1} \\ s_{z2}}} |s_1, s_2; s_{z1}, s_{z2}\rangle \cdot$$

$$\cdot \underbrace{\langle s_1, s_2; s_{z1}, s_{z2} | s, s_z; s_1, s_2\rangle}$$

coeficientes de Clebsch-Gordon

Objectivo: encontrar os C.G.

$$S_z |s_1, s_2; s_{z1}, s_{z2}\rangle = (S_{z1} + S_{z2}) \times |s_1, s_2; s_{z1}, s_{z2}\rangle$$

$$= S_z |s_1, s_2; s_{z1}, s_{z2}\rangle = (s_{z1} + s_{z2}) \times |s_1, s_2; s_{z1}, s_{z2}\rangle$$

$$\langle s_1 s_z ; s_1, s_2 | S_z | s_1, s_2 ; s_{z1}, s_{z2} \rangle$$

$$= \langle s_1 s_z ; s_1, s_2 | s_z | s_1, s_2 ; s_{z1}, s_{z2} \rangle$$

$$= s_z \langle s_1 s_z ; s_1, s_2 | s_1, s_2 ; s_{z1}, s_{z2} \rangle$$

$$= (s_{z1} + s_{z2}) \langle s_1 s_z ; s_1 s_2 | s_1, s_2 ; s_{z1}, s_{z2} \rangle$$

$$\Leftrightarrow (s_z - s_{z1} - s_{z2}) \cdot \underbrace{\langle s_1 s_z ; s_1, s_2 | s_1, s_2 ; s_{z1}, s_{z2} \rangle}_{\text{coeficiente de C.-G.}} = 0$$

Para que o coeficiente de C.-G. seja diferente de zero:

$$\boxed{s_{z1} + s_{z2} = s_z}$$

$$|s, s_z; s_1, s_2\rangle =$$

$$\sum_{\substack{s_{z1} \\ s_{z2}}} |s_1, s_2; s_{z1}, s_{z2}\rangle \cdot$$

$$\cdot \underbrace{\langle s_1, s_2; s_{z1}, s_{z2} | s, s_z; s_1, s_2\rangle}_{\text{coeficientes de Clebsch-Gordan}}$$

A norma na eq. atrás:

$$\sum_{\substack{s_{z1} \\ s_{z2}}} = \sum_{\substack{s_{z1}, s_{z2} \\ s_z = s_{z1} + s_{z2}}}$$

Valores esperados para s_z

$$s_z = 0 \quad , \quad s_z = 0, \pm 1$$

Podemos mostrar que os valores possíveis de S estão compreendidos entre:

$$S = |s_1 - s_2| \cdots (s_1 + s_2)$$

$$s_1 = s_2 = 1/2$$

$$S = 0, \; 1$$

$$S_z = 0 \qquad\qquad S_z = 0, \pm 1$$

$$\#\text{estados} = 1 + 3 = 4$$

As duas bases são:

base de partida
$$\left\{\begin{array}{l} |1/2,1/2;1/2,1/2\rangle = |\uparrow,\uparrow\rangle \\ |1/2,1/2;1/2,-1/2\rangle = |\uparrow,\downarrow\rangle \\ |1/2,1/2;-1/2,1/2\rangle = |\downarrow,\uparrow\rangle \\ |1/2,1/2;-1/2,-1/2\rangle = |\downarrow\downarrow\rangle \end{array}\right.$$

na base s, s_z, s_1, s_2

$$|0,0;1/2,1/2\rangle = |0,0\rangle$$
$$|1,1;1/2,1/2\rangle = |1,1\rangle$$
$$|1,0;1/2,1/2\rangle = |1,0\rangle$$
$$|1,-1;1/2,1/2\rangle = |1,-1\rangle$$

O estado mais alto da torre

$$\boxed{|1,1\rangle = |\uparrow,\uparrow\rangle}$$

$$S_- = S_{1-} + S_{2-}$$

$$\underbrace{S_-|1,1\rangle} = \underbrace{(S_{1-} + S_{2-})|\uparrow,\uparrow\rangle}$$

$$\hbar\sqrt{1(1+1)-1(1-1)}\,|1,0\rangle =$$

$$= \hbar|\downarrow,\uparrow\rangle + \hbar|\uparrow,\downarrow\rangle$$

$$\Leftrightarrow \boxed{|1,0\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow,\uparrow\rangle + |\uparrow,\downarrow\rangle\right)}$$

O próximo estado

$$S_-|1,0\rangle = (S_{1-} + S_{2-})\frac{1}{\sqrt{2}}\left(|\downarrow,\uparrow\rangle + |\uparrow,\downarrow\rangle\right)$$

$$\Leftrightarrow \hbar\sqrt{1(1+1)-0}\,|1,-1\rangle =$$

$$= \frac{1}{\sqrt{2}}\hbar\left(|\downarrow,\downarrow\rangle + |\downarrow,\downarrow\rangle\right) = \hbar\sqrt{2}\,|\downarrow,\downarrow\rangle$$

$$\boxed{|1,-1\rangle = |\downarrow,\downarrow\rangle}$$

$s=1$ três estados

$$\text{tripleto}\begin{cases} |1,1\rangle = |\uparrow,\uparrow\rangle \\ |1,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle\right) \\ |1,-1\rangle = |\downarrow,\downarrow\rangle \end{cases}$$

ortogonais

$s=0$, $s_z=0$

$$\boxed{|0,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\right)}$$

TABELA DOS coeficientes de C.-G.

	$\|\uparrow\uparrow\rangle$	$\|\uparrow\downarrow\rangle$	$\|\downarrow\uparrow\rangle$	$\|\downarrow\downarrow\rangle$
$\|0,0\rangle$	0	$1/\sqrt{2}$	$-1/\sqrt{2}$	0
$\|1,1\rangle$	1	0	0	0
$\|1,0\rangle$	0	$1/\sqrt{2}$	$1/\sqrt{2}$	0
$\|1,-1\rangle$	0	0	0	1

As linhas e as colunas são ortogonais entre si:

A tabela de C.-G. define uma transformação unitária.

1.3 ADIÇÃO MOMENTO ANGULAR ORBITAL $l=1$ e $s=1/2$

Original basis: $|l, m; s, s_z\rangle$

$\#_s = (2l+1)(2s+1) = (2+1)(1+1) = 6$ estados

New bases: $|j, j_z, l, s\rangle$

$\#_s = (2j_1+1)+(2j_2+1)$; $\begin{cases} j_2 = 1-1/2 = 1/2 & \text{torre 2} \\ j_1 = 1+1/2 = 3/2 & \text{torre 1} \end{cases}$

$= (2\cdot\frac{1}{2}+1)+(2\,\frac{3}{2}+1)$

$= 2+4 = 6$ estados

$$\boxed{|j, j_z; l, s\rangle = \sum_{\substack{m, s_z \\ m+s_z=j_z}} C^{j l s}_{m, s_z} |l, m; s, s_z\rangle}$$

Torre 1:

$$|j, j_z; l, s\rangle = \begin{cases} |3/2, 3/2; 1, 1/2\rangle \equiv |3/2, 3/2\rangle \\ |3/2, 1/2; 1, 1/2\rangle \equiv |3/2, 1/2\rangle \\ |3/2, -1/2; 1, 1/2\rangle \equiv |3/2, -1/2\rangle \\ |3/2, -3/2; 1, 1/2\rangle \equiv |3/2, -3/2\rangle \end{cases}$$

$$\overset{j}{|3/2}, \overset{j_z}{3/2}\rangle = \sum_{\substack{m, s_z \\ m+s_z=3/2}} C^{3/2,1,1/2}_{m,s_z} |1, m; 1/2, s_z\rangle \quad (*)$$

$m = 0, \pm 1$; $s_z = \pm 1/2$: $\frac{3}{2} = \underset{m}{1} + \underset{s_z}{1/2}$

$(*) \Rightarrow |\frac{3}{2}, \frac{3}{2}\rangle = C^{3/2,1,1/2}_{1,1/2} |\underset{l}{1}, \underset{m}{1}; \underset{s}{1/2}, \underset{s_z}{1/2}\rangle$

$\Downarrow$

1 (por normalização)

logo: $|3/2, 3/2\rangle = |1, 1; 1/2, 1/2\rangle \equiv |\underset{m}{1}, \underset{s_z}{\uparrow}\rangle$

Aplicando J_- constrói-se a torre de momento angular:

$$J_- |3/2, 3/2\rangle = J_- |1, \uparrow\rangle = (L_- + S_-)|1, \uparrow\rangle$$

$$J_- |j, j_z\rangle = \hbar \sqrt{j(j+1) - j_z(j_z - 1)}\ |j, j_z - 1\rangle$$

$$\hbar \sqrt{3/2(3/2+1) - \frac{3}{2}\left(\frac{3}{2} - 1\right)}\ |3/2, 1/2\rangle$$

$$= \hbar \sqrt{\frac{15}{4} - \frac{3}{4}}\ |3/2, 1/2\rangle = \hbar\sqrt{3}\ |3/2, 1/2\rangle \quad (o)$$

Lado direito:

$$(L_- + S_-)|1, \uparrow\rangle = L_-|1, \uparrow\rangle + S_-|1, \uparrow\rangle =$$

$$\hbar \sqrt{1(1+1)-1(1-1)}\ |0,\uparrow\rangle + \hbar|1,\downarrow\rangle =$$

$$= \hbar\sqrt{2}\ |0,\uparrow\rangle + \hbar\ |1,\downarrow\rangle \quad (00)$$

$(0) = (00)$

$$\cancel{\hbar}\sqrt{3}\,|3/2,1/2\rangle = \cancel{\hbar}\sqrt{2}\ |0,\uparrow\rangle + \cancel{\hbar}\ |1,\downarrow\rangle$$

$$\boxed{|3/2,1/2\rangle = \sqrt{\frac{2}{3}}\ |0,\uparrow\rangle + \frac{1}{\sqrt{3}}\ |1,\downarrow\rangle}$$

Próximo estado:

$$J_-|3/2,1/2\rangle = (L_- + S_-)\left[\sqrt{\frac{2}{3}}\ |0,\uparrow\rangle + \frac{1}{\sqrt{3}}\ |1,\downarrow\rangle\right]$$

$$\Leftrightarrow \hbar\sqrt{\frac{3}{2}\left(\frac{3}{2}+1\right) - \frac{1}{2}\left(\frac{1}{2}-1\right)}\ |3/2,-1/2\rangle =$$

$$= \sqrt{\frac{2}{3}}\ L_-|0,\uparrow\rangle + \frac{1}{\sqrt{3}}\ L_-|1,\downarrow\rangle$$

$$\sqrt{\frac{2}{3}}\ S_-|0,\uparrow\rangle + \frac{1}{\sqrt{3}}\ S_-|1,\downarrow\rangle \quad \Leftrightarrow$$

$$\Leftrightarrow \hbar\sqrt{\frac{15}{4} + \frac{1}{4}}\ |3/2,-1/2\rangle = \hbar\sqrt{\frac{2}{3}}\sqrt{1(1+1)-0}\ |-1,\uparrow\rangle$$

$$\hbar\frac{1}{\sqrt{3}}\sqrt{1(1+1)-0}\ |0,\downarrow\rangle + \hbar\sqrt{\frac{2}{3}}\ |0,\downarrow\rangle + 0$$

$$\hbar\, 2|3/2,-1/2\rangle = \hbar\sqrt{\frac{2}{3}}\sqrt{2}\,|-1,\uparrow\rangle + \hbar\sqrt{\frac{2}{3}}\,|0,\downarrow\rangle$$
$$+\,\hbar\sqrt{\frac{2}{3}}\,|0,\downarrow\rangle$$

$$\Leftrightarrow \quad \boxed{|3/2,-1/2\rangle = \frac{1}{\sqrt{3}}|-1,\uparrow\rangle + \sqrt{\frac{2}{3}}\,|0,\downarrow\rangle}$$

↳ próximo estado:

$$J_-|3/2,-1/2\rangle = \frac{1}{\sqrt{3}}(L_- + S_-)|-1,\uparrow\rangle$$
$$+\sqrt{\frac{2}{3}}(L_- + S_-)|0,\downarrow\rangle$$

$$\Leftrightarrow \quad \hbar\sqrt{\frac{15}{4}+\frac{1}{2}\left(-\frac{1}{2}-1\right)}\,|3/2,-3/2\rangle =$$

$$= \frac{1}{\sqrt{3}}\hbar|-1,\downarrow\rangle + \hbar\sqrt{\frac{2}{3}}\sqrt{2}\,|-1,\downarrow\rangle$$

$$\Leftrightarrow \quad \hbar\sqrt{\frac{12}{4}}\,|3/2-3/2\rangle = \frac{3}{\sqrt{3}}\hbar\,|-1,\downarrow\rangle$$

$$\Leftrightarrow \quad \boxed{|3/2,-3/2\rangle = |-1,\downarrow\rangle}$$

estado mais baixo da torre.

Torre 1:

$$|3/2, 3/2\rangle = |1, \uparrow\rangle$$

$$|3/2, 1/2\rangle = \sqrt{\frac{2}{3}}\,|0, \uparrow\rangle + \frac{1}{\sqrt{3}}\,|1, \downarrow\rangle$$

$$|3/2, -1/2\rangle = \frac{1}{\sqrt{3}}\,|-1, \uparrow\rangle + \sqrt{\frac{2}{3}}\,|0, \downarrow\rangle$$

$$|3/2, -1/2\rangle = |-1, \downarrow\rangle$$

Torre 2:

$$|j, j_z\rangle = \sum_{\substack{m, s_z \\ m + s_z = j_z}} C^{j\,l\,s}_{m, s_z}\,|m, s_z\rangle$$

torre 1: $j = \frac{3}{2}$, $j_z = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}$

torre 2: $j = \frac{1}{2}$, $j_z = \frac{1}{2}, -\frac{1}{2}$

A soma é satisfeita para estados das duas torres:

$$|\tfrac{3}{2}, \tfrac{1}{2}\rangle = C^{\frac{3}{2}, 1, 1/2}_{0, 1/2}\,|0, 1/2\rangle + C^{\frac{3}{2}, 1, 1/2}_{1, -\frac{1}{2}}\,|1, -1/2\rangle$$

$$|\tfrac{1}{2}, \tfrac{1}{2}\rangle = C^{1/2, 1, 1/2}_{0, 1/2}\,|0, 1/2\rangle + C^{1/2, 1, 1/2}_{1, -1/2}\,|1, -1/2\rangle$$

$$|3/2, 1/2\rangle = \sqrt{\frac{2}{3}}\,|0, 1/2\rangle + \frac{1}{\sqrt{3}}\,|1, -1/2\rangle$$

$$\boxed{|1/2, 1/2\rangle = \frac{1}{\sqrt{3}}\,|0, 1/2\rangle - \sqrt{\frac{2}{3}}\,|1, -1/2\rangle}$$

$$\langle 3/2, 1/2 | 1/2, 1/2\rangle = 0$$

1º estado da torre 2

2º estado da torre 2:

$$\boxed{|1/2, -1/2\rangle = \sqrt{\frac{2}{3}}\,|-1, \uparrow\rangle - \frac{1}{\sqrt{3}}\,|0, \downarrow\rangle}$$

$$\langle 3/2, -1/2 | 1/2, -1/2\rangle$$

	$\|\frac{3}{2}, \frac{3}{2}\rangle$	$\|\frac{3}{2}, \frac{1}{2}\rangle$	$\|\frac{3}{2}, -\frac{1}{2}\rangle$	$\|\frac{3}{2}, -\frac{3}{2}\rangle$	$\|\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{1}{2}, -\frac{1}{2}\rangle$
$\|1, 1/2\rangle$	1	0		0	0	0
$\|0, 1/2\rangle$	0	$\sqrt{2/3}$		0	$\sqrt{1/3}$	0
$\|-1, 1/2\rangle$	0	0	$\sqrt{1/3}$	0	0	$\sqrt{2/3}$
$\|1, -1/2\rangle$	0	$\sqrt{1/3}$		0	$-\sqrt{2/3}$	
$\|0, -1/2\rangle$	0	0	$\sqrt{2/3}$	0	0	$-1/\sqrt{3}$
$\|-1, -1/2$	0	0		1	0	0

- Notar que as linhas e as colunas são ortogonais
- A tabela pode ser lida nos dois sentidos