Análise Complexa

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D'Alembert solução (Onda)

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \rightarrow u(x,t) = \frac{1}{2}\left(\varphi(x-ct) + \varphi(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy$$

1)
$$\begin{cases} u(x,0) = 0 \\ \frac{\partial u}{\partial t}(x,0) = \cos(2\pi x) \end{cases}$$

$$u(x,t) = \frac{1}{2}(0+0) + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(2\pi x)\,dx$$

$$= \frac{1}{2c}\left[\frac{\operatorname{sen}(2\pi x)}{2\pi}\right]_{x-ct}^{x+ct}$$

$$= \frac{1}{2c}\left(\frac{\operatorname{sen}(2\pi(x+ct))}{2\pi} - \frac{\operatorname{sen}(2\pi(x-ct))}{2\pi}\right)$$

$$u(x,t) = \frac{1}{4c\pi}\left(\operatorname{sen}(2\pi x + 2\pi c t) - \operatorname{sen}(2\pi x - 2\pi c t)\right)$$

2)
$$\begin{cases} u(x,0) = e^{-x^2} \\ \frac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

$$u(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,dx$$

$$u(x,t) = \frac{1}{2}\left(e^{-(x-ct)^2} + e^{-(x+ct)^2}\right)$$

Solução corda vibrante

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \longrightarrow u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos(nct) + \frac{b_n}{nc}\operatorname{sen}(nct)\right)\operatorname{sen}(nx)$$

1)
$$\begin{cases} u(x,0) = \operatorname{sen}(3x) \\ \frac{\partial u}{\partial t}(x,0) = 2\operatorname{sen}(4x) \end{cases}$$

$$u(x,t) = \cos(3ct)\operatorname{sen}(3x) + \frac{2}{4c}\operatorname{sen}(4ct)\operatorname{sen}(4x)$$

2)
$$\begin{cases} u(x,0) = 3\operatorname{sen}(x) - \operatorname{sen}(2x) \\ \frac{\partial u}{\partial t}(x,0) = \operatorname{sen}(3x) \end{cases}$$

$$u(x,t) = 3\cos(ct)\operatorname{sen}(x) - \cos(2ct)\operatorname{sen}(2x) + \frac{1}{3c}\operatorname{sen}(3ct)\operatorname{sen}(3x)$$

Solução eq. calor fronteira nula

$$\frac{\partial u}{\partial t} - \beta\frac{\partial^2 u}{\partial x^2} = 0 \rightarrow u(x,t) = \sum b_n e^{-\beta n^2 t}\operatorname{sen}(nx)$$

1)
$$\begin{cases} u(0,t) = 0 = u(\pi,t) \\ u(x,0) = \operatorname{sen}(x) + 3\operatorname{sen}(2x) \end{cases}$$

$$u(x,t) = e^{-\beta t}\operatorname{sen}(x) + 3e^{-\beta 4 t}\operatorname{sen}(2x)$$

2)
$$\begin{cases} u(0,t) = 0 = u(\pi,t) \\ u(x,0) = \pi\operatorname{sen}(7x) - \operatorname{sen}(5x) \end{cases}$$

$$u(x,t) = \pi e^{-\beta 7^2 t}\operatorname{sen}(7x) - e^{-\beta 5^2 t}\operatorname{sen}(5x)$$

Solução eq. calor condutor isolado

$$\frac{\partial u}{\partial t} - \beta\frac{\partial^2 u}{\partial x^2} = 0 \rightarrow u(x,t) = a_0 + \sum a_m e^{-\beta m^2 t}\cos(mx)$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} u(x,0)\,dx$$

Se $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0$

$u(x,t) = \text{constante}$

1)

$u(x,0) = \cos(x) + 3\cos(2x)$

- $a_0 = \frac{1}{\pi}\int_0^{\pi}(\cos(x)+3\cos(2x))\,dx =$

$= \frac{1}{\pi}\left[\operatorname{sen}(x) + \frac{3}{2}\operatorname{sen}(2x)\right]_0^{\pi}$

$= 0$

- $u(x,t) = e^{-\beta t}\cos(x) + 3e^{-\beta 4t}\cos(2x)$

2)

$u(x,0) = 3 - \cos(5x)$

- $a_0 = \frac{1}{\pi}\int_0^{\pi}(3-\cos(5x))\,dx = 3$

- $u(x,t) = 3 - e^{-\beta 25t}\cos(5x)$

Soluções separáveis (calor)

$$\frac{\partial u}{\partial t} - \beta\frac{\partial^2 u}{\partial x^2} = 0$$

$X(x)T(t)$

$X(x)T'(t) - \beta X''(x)T(t) = 0$

$$\frac{1}{\beta}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$$

$$\begin{cases} \frac{T'(t)}{T(t)} = \beta\lambda \\ \frac{X''(x)}{X(x)} = \lambda \end{cases} \quad \begin{cases} T(t) = e^{\lambda\beta t} \\ X(x) = A\operatorname{sen}(mx) \end{cases}$$

$u(x,t) = X(x)T(t)$

$= Ae^{-m^2\beta t}\operatorname{sen}(mx)$

Sobreposição

$$u(x,t) = \sum b_m e^{-m^2\beta t}\operatorname{sen}(mx)$$

Separação Variáveis

1) $t\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial t} = 0$

$X(x)T(t)$

$t\cdot X''(x)T(t) + X(x)T'(t) = 0$

$$-\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}t = \lambda$$

$$\begin{cases} X''(x) = -\lambda X(x) \\ T'(t) = \lambda t\,T(t) \end{cases} \quad \begin{cases} X(x) = a\cos(\sqrt{\lambda}x) + b\operatorname{sen}(\sqrt{\lambda}x) \\ T(t) = e^{\lambda t^2/2} + \beta \end{cases}$$

$$u(x,t) = \sum\left(a_m\cos(\sqrt{-m^2}x) + b_m\operatorname{sen}(\sqrt{-m^2}x)\right)e^{-m^2\frac{t^2}{2}} + \beta$$

de Fourier

$$\simeq c_0 + \sum c_m e^{imx} \simeq c_0 + \sum \left(a_m \cos(mx) + b_m \,\text{sen}(mx)\right)$$

$$c_0 \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad c_m \to \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx \qquad a_m \to \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \qquad b_m \to \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\text{sen}(mx)\,dx$$

1)

$\Theta(\theta)$

ou

- $$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Theta(\theta)\,d\theta = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\,d\theta + \int_{0}^{\pi} 1\,d\theta\right)$$

$$= \frac{1}{2}$$

- $$c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Theta(\theta)e^{-in\theta}\,d\theta = \frac{1}{2\pi}\left(\int_{-\pi}^{0} 0\cdot e^{-in\theta}\,d\theta + \int_{0}^{\pi} e^{-in\theta}\,d\theta\right)$$

$$= \frac{1}{2\pi}\int_{0}^{\pi} e^{-in\theta}\,d\theta = \frac{i}{2\pi m}\left(-e^{-in\pi}+1\right) \to \begin{cases} \frac{i}{\pi m}, & m \text{ ímpar} \\ 0, & m \text{ par} \end{cases}$$

$$\Theta(\theta) \simeq \frac{1}{2} + \sum_{m \text{ ímpar}}^{\infty} \frac{i}{m\pi} e^{im\theta}$$

- $c_0 = \frac{1}{2}$

- $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta(\theta)\cos(m\theta)\,d\theta = 0$$

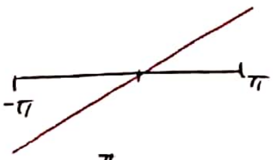
- $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\Theta(\theta)\,\text{sen}(m\theta)\,d\theta = \frac{1}{\pi}\left[-\frac{\cos(m\theta)}{m}\right]_0^{\pi}$$

$$= \frac{1}{\pi m}\left(-\cos(m\pi)+1\right) = \begin{cases} \frac{2}{\pi m}, & m \text{ ímpar} \\ 0, & m \text{ par} \end{cases}$$

$$\Theta(\theta) \simeq \frac{1}{2} + \frac{2}{\pi}\sum_{m \text{ ímpar}}^{\infty} \frac{1}{m}\,\text{sen}(m\theta)$$

2)

$f(x) = x$



- $$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\cdot dx = 0$$

- $$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\cos(mx)\,dx = 0$$

- $$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\text{sen}(mx)\,dx = \frac{1}{\pi}\left(\left[-x\cdot\frac{\cos(mx)}{m}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{\cos(mx)}{m}\,dx\right)$$

$$= \frac{1}{\pi}\left[\left(\frac{-\pi\cos(m\pi) - \pi\cos(m\pi)}{m} + 0\right)\right]$$

$$= -\frac{2}{m}\cos(m\pi) = \begin{cases} \frac{2}{m}, & m \text{ ímpar} \\ -\frac{2}{m}, & m \text{ par} \end{cases}$$

$$f(x) = \sum \frac{2}{m}(-1)^{m+1}\,\text{sen}(mx)$$

Aplicações série Fourier às EDPs

Calor
(Temp. const. front.)

$$\frac{\partial u}{\partial t} - \beta \frac{\partial^2 u}{\partial x^2} = 0 \longrightarrow u(x,t) = \sum \underbrace{b_n}_{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}(nx)\,dx} e^{-n^2\beta t}\,\mathrm{sen}(nx)$$

1)
$$\underbrace{u(x,0) = x}_{f(x)}$$

$$\bullet\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \mathrm{sen}(nx)\,dx$$

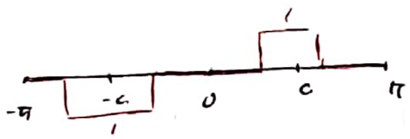
$$= \frac{1}{\pi}\left[\left[-\frac{x\cos(nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} -\frac{\cos(nx)}{n}\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(n\pi)}{n} - \frac{\pi\cos(n\pi)}{n} + \frac{1}{n}\cdot 0\right]$$

$$= -\frac{2}{n}\cos(n\pi)$$

$$u(x,t) = -\sum \frac{2}{n}\cos(n\pi)\, e^{-n^2\beta t}\,\mathrm{sen}(nx)$$

2)
$$u(x,0) = \begin{cases} \lambda & \text{se } |x-c| \le \varepsilon \\ 0 & \text{se } |x-c| > \varepsilon \end{cases}$$



$$\bullet\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{sen}(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{sen}(nx)\,dx$$

$$= \frac{2\lambda}{\pi}\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \mathrm{sen}(nx)\,dx = \frac{2\lambda}{\pi}\left[-\frac{\cos(nx)}{n}\right]_{\alpha-\varepsilon}^{\alpha+\varepsilon}$$

$$= \frac{2\lambda}{\pi}\left(\cos(n\alpha - n\varepsilon) - \cos(n\alpha + n\varepsilon)\right)$$

$$= \frac{4\lambda}{\pi n}\,\mathrm{sen}(n\alpha)\,\mathrm{sen}(n\varepsilon)$$

$$u(x,t) = \frac{4\lambda}{\pi}\sum \frac{e^{-n^2\beta t}}{n}\,\mathrm{sen}(n\alpha)\,\mathrm{sen}(n\varepsilon)\,\mathrm{sen}(nx)$$

Calor
(condutor isolado)

$$\frac{\partial u}{\partial t} - \beta\frac{\partial^2 u}{\partial t^2} = 0 \rightarrow u(x,t) = \underbrace{c_0}_{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx} + \sum \underbrace{a_n}_{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx} e^{-n^2\beta t}\cos(nx)$$

1)
$$u(x,0) = x^2$$

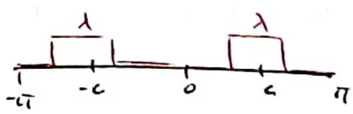
$$\bullet\ c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$$

$$\bullet\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx$$

$$= \frac{4\cos(n\pi)}{n^2}$$

$$u(x,t) = \frac{\pi^2}{3} + \sum \frac{4\cos(n\pi)}{n^2} e^{-n^2\beta t}\cos(nx)$$

2)
$$u(x,0) = \begin{cases} \lambda & \text{se } |x-c| \le \varepsilon \\ 0 & \text{se } |x-c| > \varepsilon \end{cases}$$



$$\bullet\ c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{\alpha-\varepsilon}^{\alpha+\varepsilon} \lambda\,dx = \frac{2\lambda\varepsilon}{\pi}$$

$$\bullet\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx) = \frac{2\lambda}{\pi}\int_{\alpha-\varepsilon}^{\alpha+\varepsilon}\cos(nx)\,dx$$

$$= \frac{4\lambda}{\pi n}\,\mathrm{sen}(n\varepsilon)\cos(n\alpha)$$

$$u(x,t) = \frac{2\lambda\varepsilon}{\pi} + \sum \frac{4\lambda}{\pi n} e^{-n^2\beta t}\cos(nx)\,\mathrm{sen}(n\varepsilon)\cos(n\alpha)$$

Vibrante $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \rightarrow u(x,t) = \sum \left(a_m \cos(cmt) + \frac{b_m}{cm} \operatorname{sen}(cmt) \right) \operatorname{sen}(mx)$

$a_m \downarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx$, $f \downarrow u(x,0)$

b_m: $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen}(mx)\,dx$, f: $\frac{\partial u}{\partial x}(x,0)$

$$\begin{cases} u(x,0) = \begin{cases} x & \text{se } 0 < x < \pi/2 \\ \pi - x & \text{se } \pi/2 < x < \pi \end{cases} \\ \frac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

- $a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} u(x,0)\cdot\cos(mx)\,dx = \frac{2}{\pi}\int_0^{\pi} u(x,0)\cos(nx)\,dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos(mx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos(mx)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[\frac{x\operatorname{sen}(mx)}{m}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\operatorname{sen}(mx)}{m}\,dx + \left[\frac{(\pi - x)\operatorname{sen}(mx)}{m}\right]_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi}\frac{\operatorname{sen}(mx)}{m}\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{2\cos\left(\frac{m\pi}{2}\right)}{m^2} - \frac{1}{n^2} - \frac{\cos(m\pi)}{m^2}\right] = \frac{2}{\pi m^2}\left(2\cos\left(\frac{m\pi}{2}\right) - 1 - \cos(m\pi)\right)$$

- $b_m = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial u}{\partial t}(x,0)\cdot\operatorname{sen}(mx)\,dx = 0$

$$u(x,t) = \sum\left(\frac{2}{\pi m^2}\left(2\cos\left(\frac{m\pi}{2}\right) - 1 - \cos(m\pi)\right)\cos(cmt)\right)\operatorname{sen}(mx)$$

Transformada de Fourier

Linear $\Rightarrow$ $\mathcal{F}(f+g) = \mathcal{F}f + \mathcal{F}g$ e $\mathcal{F}(\lambda f) = \lambda\mathcal{F}f$

Translação $\Rightarrow$ $\mathcal{F}(T_c f)(\xi) = e^{-2\pi i\xi c}\mathcal{F}f$

Modulação $\Rightarrow$ $\mathcal{F}(M_b f)(\xi) = \mathcal{F}f(\xi - b)$

Homotetia $\Rightarrow$ $\mathcal{F}(H_\lambda f)(\xi) = \frac{1}{\lambda}\mathcal{F}f(\xi/\lambda)$

Conjugação $\Rightarrow$ $\mathcal{F}\bar{f} = \overline{\mathcal{F}f}$

$$\mathcal{F}f = \int_{-\infty}^{\infty} f\cdot e^{-2\pi i\xi x}\,dx$$

1) $f(x) = \dfrac{1}{x^2+a^2}$

$$\mathcal{F}f = \int_{-\infty}^{\infty} \frac{1}{x^2+c^2}\, e^{-2\pi i \xi x}\, dx$$

$$= \int_\gamma \frac{e^{-2\pi i \xi z}}{z^2+a^2}\, dz$$

$$= \int_\gamma \frac{e^{-2\pi i \xi z}}{(z-ci)(z+ci)}\, dz$$

$$= \frac{\pi}{a} e^{2\pi \xi a}$$

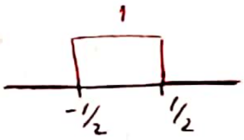
2) $f(x) = e^{-2\pi b|x|}$

$$\mathcal{F}f = \int_{-\infty}^{\infty} e^{-2\pi b|x|} e^{-i2\pi\xi x}\, dx$$

$$= \int_{-\infty}^{0} e^{2\pi(b-i\xi)x}\, dx + \int_0^{+\infty} e^{-2\pi(b+i\xi)x}\, dx$$

$$= \frac{1}{2\pi}\left(\frac{1}{b-i\xi} - \frac{1}{b+i\xi}\right)$$

3) $f(x) = \begin{cases} 1 & |x| < 1/2 \\ 0 & |x| > 1/2 \end{cases}$



$$\mathcal{F}f = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x}\, dx = \int_{-1/2}^{1/2} e^{-2\pi i \xi x}\, dx$$

$$= \frac{\operatorname{sen}(\pi\xi)}{\pi\xi} \to \boxed{\operatorname{sinc}}$$

4) $f(x) = e^{-\pi x^2}$

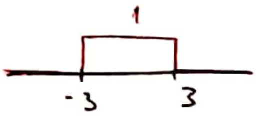
$$\mathcal{F}f = \int_{-\infty}^{\infty} e^{-\pi x^2} \cdot e^{-2\pi i \xi x}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i \xi x}\, dx$$

$$= e^{-\pi\xi^2}$$

Propiedades

1)



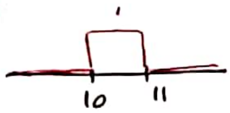
$f\left(\frac{x}{6}\right)$

$$\mathcal{F}f = 6\,\mathcal{F}f(6\xi)$$

$$= 6\int_{-1/2}^{1/2} 1 \cdot e^{-2\pi i 6 \xi x}\, dx$$

$$= \frac{\operatorname{sen}(6\pi\xi)}{\pi\xi}$$

2)

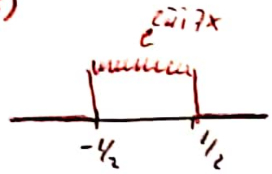


$f(x - 10{,}5)$

$$\mathcal{F}(T_{10,5} f)(\xi) = e^{-2\pi i \xi (10,5)}\, \mathcal{F}f$$

$$= e^{-2\pi i \xi (10,5)} \cdot \frac{\operatorname{sen}(\pi\xi)}{\pi\xi}$$

3)



$$\mathcal{F}(M_7 f)(\xi) = \mathcal{F}f(\xi-7)$$

$$= \int_{-1/2}^{1/2} 1 \cdot e^{-2\pi i(\xi-7)x}\, dx$$

$$= \frac{\operatorname{sen}(\pi(\xi-7))}{\pi(\xi-7)}$$

4) $G_\tau(x) = \dfrac{1}{\sqrt{\tau}} e^{-\pi x^2/\tau}$

$$\mathcal{F}G_\tau = \frac{1}{\sqrt{\tau}} \mathcal{F}(H_{1/\tau} G_\tau)(\xi)$$

$$= e^{-\pi\tau\xi^2}$$

5) $f(x) = e^{-\pi(x-c)^2}$

$$\mathcal{F}(T_c f)(\xi) = e^{-2\pi i \xi c}\, \mathcal{F}f$$

$$= e^{-\pi\xi(2ic-\xi)}$$

...formada de Fourier às EDPs

$$(f * g)(t) = \int_{-\infty}^{\infty} f(s)\, g(t-s)\, ds$$

• $\mathcal{F}(f * g)(\xi) = \mathcal{F}f \cdot \mathcal{F}g$

• $H_t(x) = \dfrac{1}{\sqrt{4\pi t}} e^{-x^2/4t}$

$\mathcal{F}H_t = e^{-4\pi^2 \xi^2 t}$

• $P_y(x) = \dfrac{1}{\pi} \dfrac{y}{x^2+y^2}$

$\mathcal{F}P_y = e^{-2\pi|\xi|y}$

2) $$\mathcal{F}\left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}\right) = 0$$

$$\frac{\partial^2 \mathcal{F}u}{\partial t^2} - c^2(-2\pi i \xi)^2 \mathcal{F}u = 0$$

$$\frac{\partial^2 \mathcal{F}u}{\partial t^2} = -4\pi^2 c^2 \xi^2 \mathcal{F}u$$

$$\mathcal{F}u(\xi,t) = \mathcal{F}f\, e^{2\pi i c \xi t} + \mathcal{F}g\, e^{-2\pi i c\xi t}$$

$$u(x,t) = f(x+ct) + g(x-ct)$$

1) $$\mathcal{F}\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -u\right)$$

$$\frac{\partial \mathcal{F}u}{\partial t} - (-2\pi i\xi)^2 \mathcal{F}u = -\mathcal{F}u$$

$$\frac{\partial \mathcal{F}u}{\partial t} + 4\pi^2\xi^2 \mathcal{F}u = -\mathcal{F}u$$

$$\frac{\partial \mathcal{F}u}{\partial t} = (-1-4\pi^2\xi^2)\mathcal{F}u$$

$$\mathcal{F}u = u(\xi,0)\, e^{-(-1-4\pi^2\xi^2)t}$$

$$\mathcal{F}u(\xi,t) = u(\xi,0)\, e^{(1+4\pi^2\xi^2)t}$$

$$\mathcal{F}u(\xi,t) = \left(\varphi * e^{t} \cdot e^{4\pi^2\xi^2 t}\right)$$

$$\mathcal{F}u(\xi,t) = e^t (\varphi * H_t)(x)$$

$$u(x,t) = e^t \int_{-\infty}^{\infty} \varphi(y) \frac{1}{\sqrt{4\pi t}} e^{\frac{(x-y)^2}{4t}} dy$$

3) $$\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\frac{\partial^2 \mathcal{F}u}{\partial y^2} = -(2\pi i \xi)^2 \mathcal{F}u$$

$$\frac{\partial^2 \mathcal{F}u}{\partial y^2} = 4\pi\xi^2 \mathcal{F}u$$

$$\mathcal{F}u = \mathcal{F}f\, e^{2\pi\xi y} + \mathcal{F}g\, e^{-2\pi\xi y}$$

$$\mathcal{F}u = \mathcal{F}h\, e^{-2\pi|\xi|y}$$

$$\mathcal{F}u = \mathcal{F}h\, \mathcal{F}P_y$$

$$u(x,y) = (\varphi * P_y)(x)$$

$$u(x,y) = \int_{-\infty}^{\infty} \varphi(z) \frac{1}{\pi} \frac{y}{(x-z)^2+y^2} dz$$