Análise Complexa
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Segundo Teste
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Proposta de Resolução

1. $f(z) = \dfrac{z+1}{z^4 - 3z^3 + 2z^2}$, $A = \{ z \in \mathbb{C} : 1 < |z| < 2 \}$

$$f(z) = \frac{1}{z^2} \frac{z+1}{z^2 - 3z + 2} = \frac{1}{z^2} \frac{z+1}{(z-2)(z-1)}$$

$$\frac{z+1}{(z-2)(z-1)} = \frac{A}{z-2} + \frac{B}{z-1} \Rightarrow z+1 = A(z-1) + B(z-2)$$

$\underline{z=1}$: $2 = -B \Rightarrow B = -2$

$\underline{z=2}$: $3 = A \Rightarrow A = 3$

$$f(z) = \frac{1}{z^2} \left(\frac{3}{z-2} + \frac{-2}{z-1} \right) = \frac{1}{z^2} \left(\frac{-3}{2-z} - \frac{2}{z} \cdot \frac{1}{1 - 1/z} \right)$$

$$= \frac{1}{z^2} \left(-\frac{3}{2} \frac{1}{1 - z/2} - \frac{2}{z} \frac{1}{1 - 1/z} \right)$$

No anel A, temos $|z| > 1 \Leftrightarrow |1/z| < 1$

$|z| < 2 \Leftrightarrow |z/2| < 1$

Logo podemos aplicar a fórmula da série geométrica:

$$f(z) = \frac{1}{z^2} \left(-\frac{3}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n - \frac{2}{z} \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n \right) =$$

$$= -3 \sum_{n=0}^{\infty} \frac{z^{n-2}}{2^{n+1}} - 2 \sum_{n=0}^{\infty} z^{-n-3}$$

2. $f(z) = \dfrac{\cos z - 1}{z^5}$

a. Temos $\cos z = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + O(z^6)$

Logo $\dfrac{\cos(z) - 1}{z^5} = \displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{z^{2n-5}}{(2n)!} = -\frac{1}{2z^3} + \frac{1}{4!\, z} + O(z)$

Portanto $z_0 = 0$ é um pólo de ordem 3 de $f(z)$.

b Pelo teorema dos resíduos $\displaystyle\int_\gamma f(z)\, dz = 2\pi i \operatorname{Res}_{z=0} f(z)$

Da alínea anterior temos que $\operatorname{Res}_{z=0} f(z) = c_{-1} = \dfrac{1}{4!} = \dfrac{1}{24}$

Logo $\displaystyle\int_\gamma f(z)\, dz = \frac{2\pi i}{24} = \frac{\pi i}{12}$.

3

a $\displaystyle\int_0^{+\infty} \frac{x^2}{(x^2+4)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)^2}\,dx$

uma vez que a função integranda é par.

Consideremos $f(z) = \dfrac{z^2}{(z^2+4)^2}$

As singularidades de $f(z)$ são $\pm 2i$, pelo que $2i$ é a única singularidade de $f(z)$ no semiplano superior.

Sabemos então que $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \operatorname{Res}_{z=2i} f(z)$

$f(z) = \dfrac{z^2}{(z-2i)^2(z+2i)^2}$ pelo que $2i$ é um pólo de ordem 2.

Portanto $\operatorname{Res}_{z=2i} f(z) = g'(2i)$ onde $g(z) = (z-2i)^2 f(z)$

$g(z) = \dfrac{z^2}{(z+2i)^2} \Rightarrow g'(z) = \dfrac{2z\,(z+2i)^2 - z^2\, 2\,(z+2i)}{(z+2i)^4}$

$\Rightarrow g'(2i) = \dfrac{4i\,(4i)^2 - 2(2i)^2(4i)}{(4i)^4} = \dfrac{-4^3 i + 8\,(4i)}{(4i)^4} = \dfrac{4^2}{4^2}\,\dfrac{-4i+2i}{4^2}$

$= \dfrac{-2i}{16} = -\dfrac{i}{8}$

Logo $\displaystyle\int_0^{+\infty} f(x)\,dx = \pi i \operatorname{Res}_{z=2i} f(z) = \frac{\pi}{8}$

b $\displaystyle\int_0^{+\infty} \frac{\cos(2x)}{x^2+1}\,dx = \underbrace{\frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}\,dx}_{\text{função par}} =$

$\displaystyle = \frac{1}{2}\left[\int_{-\infty}^{\infty} \frac{\cos(2x)}{x^2+1}\,dx + i\int_{-\infty}^{+\infty} \frac{\operatorname{sen}(2x)}{x^2+1}\,dx\right] =$

$\displaystyle = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{i2x}}{x^2+1}\,dx$ $\rightarrow$ função ímpar

Consideremos $f(z) = \dfrac{e^{i2z}}{z^2+1}$. Singularidades de $f(z)$: $\pm i$

Singularidaridades de $f(z)$ no semiplano superior: i

Logo $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i \operatorname{Res}_{z=i} f(z)$

$f(z) = \dfrac{\varphi(z)}{\psi(z)}$ onde $\varphi(z) = e^{i2z}$

$\psi(z) = z^2+1 \Rightarrow \psi'(z) = 2z$

$\operatorname{Res}_{z=i} f(z) = \dfrac{\varphi(i)}{\psi'(i)} = \dfrac{e^{-2}}{2i}$

Portanto: $\displaystyle\int_0^{+\infty} \frac{\cos(2x)}{x^2+1}\,dx = \pi i \frac{e^{-2}}{2i} = \pi \frac{e^{-2}}{2} = \frac{\pi}{2e^2}$.

4 $f(x) = |x|$, $x \in [-\pi, \pi]$

a A série de Fourier pretendida é dada por

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \operatorname{sen}(nx) \quad \text{onde:}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad , \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$\text{e} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\operatorname{sen}(nx)\,dx \quad , \quad n \in \mathbb{N}.$$

De notar que $f(x) = |x|$ é par logo $f(x)\operatorname{sen}(nx)$ é ímpar, pelo que $b_n = 0$, $\forall n \in \mathbb{N}$

Temos então:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left.\frac{x^2}{2}\right|_0^{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos(nx)\,dx = \frac{2}{\pi}\int_0^{\pi} \underset{u}{x}\,\underset{v'}{\cos(nx)}\,dx =$$

$$= \frac{2}{\pi}\left[\left. x\frac{\operatorname{sen}(nx)}{n}\right|_0^{\pi} - \int_0^{\pi} \frac{\operatorname{sen}(nx)}{n}\,dx \right]$$

$$= \frac{2}{\pi n}\left.\frac{\cos(nx)}{n}\right|_0^{\pi} = \frac{2}{\pi n^2}\left(\cos(n\pi) - \cos(0)\right) =$$

$$= \frac{2}{\pi n^2}\left((-1)^n - 1\right) = \begin{cases} -\dfrac{4}{\pi n^2} & , \text{ se } n \text{ é ímpar} \\ 0 & , \text{ se } n \text{ é par.} \end{cases}$$

Logo:

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos((2n+1)x)$$

b Tomando $x=0$ temos que $F(0) = \dfrac{f(0^-) + f(0^+)}{2} = f(0) = 0$

Logo:

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \Rightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$