Unit-5

Complex Numbers

5.1 Introduction :

We have studied the number sets N, Z, Q and R in the earlier classes. We also checked the closure property, commutative law, associative law, ... etc. for addition and multiplication in these sets. We also discussed the solutions of quadratic equation $ax^2 + bx + c = 0$, $a, b, c \in R$, $a \neq 0$ in one variable. We observed that if the discriminant $D = b^2 - 4ac < 0$ then this quadratic equation has a solution in R. For example $x^2 + 1 = 0$ has no solution in R, as there does not exist $x \in R$ such that $x^2 = -1$. Hence we need a new set in which such equations can be solved. Thus the set of real numbers R needed to be expanded to a larger number system.

Greeks were the first to recognize the fact that square root of a negative number does not exist in the set of real number R. The Indian mathematician Mahavira (850 A.D.) too mentions this difficulty in his work 'Ganitasara Sangraha'. The extension of real number system should be in such a way that the algebraic operations such as addition, subtraction, multiplication and division can be defined properly. This new set is called the set of complex numbers and is denoted by C.

5.2 Set R × R and Complex Numbers :

We begin with the set of real numbers to obtain the set C of complex numbers. R × R is the set of all ordered pairs of real numbers.

$R \times R = \{(a, b) \mid a \in R, b \in R\}$

We shall defined the equality addition and multiplication of two elements of R × R.

(1) Equality : Two elements (a, b) and (c, d) of R × R are defined to be equal if $a = c$ and $b = d$.

Thus $(a, b) = (c, d) \Leftrightarrow a = c, b = d$.

For Example $\left(\sqrt{4}, \sqrt[3]{27}\right) = (2, 3)$ but $(3, 5) \neq (5, 3)$, $(1, 2) \neq (1, -2)$.

(2) Addition : The sum of two elements (a, b) and (c, d) of R × R is defined as follows :

$(a, b) + (c, d) = (a + c, b + d)$

For example $(2, 3) + (5, 1) = (2 + 5, 3 + 1) = (7, 4)$

(3) Multiplication : The product of two elements (a, b) and (c, d) of R × R is defined as follows :

$(a, b)(c, d) = (ac - bd, ad + bc)$

For example

$(1, 2)(5, -3) = ((1)(5) - (2)(-3), (1)(-3) + (2)(5))$

$= (5 + 6, -3 + 10)$

$= (11, 7)$

The set R × R with these rules is called the set of complex numbers and it is denoted by C and it element (a, b) is called a complex number. Generally we denote a complex number by $z, w, ...$ etc. For example complex number $z = (2, -3)$, $w = (1, 2)$.

5.3 Properties of Addition and Multiplication of Complex Numbers :

We have discussed the properties of closure, commutativity, associativity and distributivity with respect to operations of addition and multiplication on R. We shall see that these properties holds good in C too.

The operation of addition satisfies the following peroperties :

(1) The closure property : The sum of two complex numbers is a complex number. i.e. For all $z_1, z_2 \in C$, $z_1 + z_2 \in C$.

(2) The commutative property : For all $z_1, z_2 \in C$, $z_1 + z_2 = z_2 + z_1$.

(3) The associative property : For all $z_1, z_2, z_3 \in C$, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

(4) The existence of additive identity : There exist a complex number $O = (0, 0)$, such that for all $z \in C$, $z + O = O + z = z$.

Here $O = (0, 0)$ is called the additive identity or zero complex number and it is unique.

(5) The existence of additive inverse : To every complex number $z = (a, b)$, there corresponds a complex number $(-a, -b)$, such that $(a, b) + (-a, -b) = (0, 0) = 0$. The complex number $(-a, -b)$ is called the additive inverse of (a, b) and it is denoted by $-z$.

Thus, $z = (a, b)$, then $-z = (-a, -b)$ and $z + (-z) = (-z) + z = 0$

The operation of multiplication satisfies following properties :

(1) The closure property : The product of two complex numbers is a complex number. i.e. For all $z_1, z_2 \in C$, $z_1 z_2 \in C$

(2) **The commutative Property :** For all $z_1, z_2, \in C$, $z_1 z_2 = z_2 z_1$.

(3) **The associative property :** For all $z_1, z_2, z_3 \in C$, $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.

(4) **The existence of multiplicative identity :** There exist a complex number (1, 0), such that for all $z = (a, b) \in C$,

$$\begin{aligned} z \cdot (1, 0) &= (a, b)(1, 0) \\ &= ((a)(1) - (b)(0), (a)(0) + (b)(1)) \\ &= (a - 0, 0 + b) \\ &= (a, b) = z \end{aligned}$$

Thus, $z(1, 0) = (1, 0) z = z$

$\therefore$ (1, 0) is called the multiplicative identity and it is unique.

(5) **The existence of multiplicative inverse :** To each non-zero complex number

$z = (a, b)$, there corresponds a complex number such that $\left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

$$z \cdot \left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right) = (a, b)\left(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2}\right)$$

$$= \left(\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}, \frac{-ab}{a^2+b^2} + \frac{ab}{a^2+b^2}\right)$$

$= (1, 0)$

Thus, the inverse of complex number $z = (a, b)$ is

$\left(\dfrac{a^2}{a^2+b^2}, \dfrac{-b^2}{a^2+b^2}\right)$, which is denoted by z^{-1}.

For each non-zero complex number z, its multiplication inverse is unique and it is denoted by z^{-1} or $\frac{1}{z}$.

$\therefore \quad z \cdot z^{-1} = z^{-1} \cdot z = (1, 0)$

(6) **The distributive law :** For any three complex numbers, $z_1, z_2, z_3 \in C$,

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

5.4 R as a subset of C :

By definition, every complex number is an order pair of real numbers. Let us denote by R^1 the set of those complex numbers (a, b) in which $b = 0$,

So $R^1 = \{(a, 0) \mid a \in R\}$. Obviously $R^1 \subset C$.

Now, for any two elements $(a, 0)$ and $(b, 0)$ of R'

(i) $(a, 0) = (b, 0) \Leftrightarrow a = b$

(ii) $(a, 0) + (b, 0) = (a + b, 0) \in R^1$

(iii) $(a, 0) \cdot (b, 0) = ((a)(b) - (0)(0), (a)(0) + (0)(b))$
$= (ab, 0) \in R^1$

Thus, the sum as well as the product of two elements of R^1 is again an element of R^1. Moreover the first component of the sum or product of two numbers $(a, 0)$ and $(b, 0)$ is obtained merely by adding or multiplying respectively the first components a and b, while the second component remains zero. So as far as equality, sum and multiplication are concerned, the complex numbers of the form $(a, 0)$ behave exactly like real number a. Hence we identify complex numbers of the form $(a, 0)$ with a and write $(a, 0) = a$.

For example $(5, 0) = 5$, $(-3, 0) = -3$

$(0, 0) = 0$, $(1, 0) = 1$

Thus, every real number a is also a complex number $(a, 0)$ so $R^1 = R \subset C$.

5.5 Representation of a complex number (a, b) in the form of a + bi :

By writing (a, 0) = a we are able to represent a complex number (a, b) in another form. But first let us get familiar with a special complex number (0, 1). We use symbol i for this complex number.

Thus, $i = (0, 1)$

$$\begin{aligned} \therefore \quad i^2 = i \cdot i &= (0, 1) \cdot (0, 1) \\ &= ((0)(0) - (1)(1), (0)(1) + (1)(0)) \\ &= (0 - 1, 0 + 0) \\ &= (-1, 0) \\ &= -1 \end{aligned}$$

Thus $i^2 = -1$. This $i = (0, 1)$ is called an imaginary number. In the year 1737 Euler was the first person to introduce the symbol i.

Now, for any complex number (a, b)

$(a, b) = (a, 0) + (0, b)$

$= (a, 0) + (0, 1) \cdot (b, 0)$

$(\because (0, 1)(b, 0) = (0 - 0, 0 + b))$

$= (0, b)$

$= a + ib$

$\therefore \quad (a, b) = a + ib$ or $a + bi$

Hence, every complex number (a, b) can be expressed in the form $a + ib$, where $a, b \in R$ and $i^2 = -1$.

$\therefore$ The set of complex number

$C = \{a + ib \mid a, b \in R\}$

For example, $(2, 3) = 2 + 3i$, $(3, -1) = 3 - i$

$(0, 2) = 0 + 2i = 2i$, $(7, 0) = 7 + 0i = 7$

For the complex number $z = a + bi$, a is called the Real part of z and is denoted by $Re(z)$ and b is called the Imaginary part of z and is denoted by $Im(z)$.

Thus, $z = a + bi = Re(z) + i\,Im(z)$

For example in $z = 3 + 2i$, $Re(z) = 3$, $Im(z) = 2$

If in a complex number $z = a + bi$, $a = 0$ and $b \neq 0$, then it is called purely imaginary number.

For example $5i = 0 + 5i$, $-3i = 0 - 3i$

5.6 Properties of complex numbers in the form $a + bi$:

For two complex numbers $z_1 = a + bi$ and $z_2 = c + di$

(1) Equality of two complex numbers :

$z_1 = z_2 \Leftrightarrow a + bi = c + di$

$\Leftrightarrow a = c$ and $b = d$

(2) Addition of two complex numbers :

$z_1 + z_2 = (a + bi) + (c + di)$

$= a + bi + c + di$

$= (a + c) + (b + d)i$

For example,

$(3 + 2i) + (5 + i) = 3 + 2i + 5 + i$

$= (3 + 5) + (2 + 1)i$

$= 8 + 3i$

(3) Difference of two complex numbers :

$z_1 - z_2 = z_1 + (-z_2)$

$= (a + bi) + (-c - di)$

$= a + bi - c - di$

$= (a - c) + (b - d)i$

For example,

$(3 + 2i) - (5 + i) = 3 + 2i - 5 - i$

$= (3 - 5) + (2 - 1)i$

$= -2 + i$

(4) Multiplication of two complex numbers :

$z_1 z_2 = (a + bi)(c + di)$

$= a(c + di) + bi(c + di)$

$= ac + adi + bci + bdi^2$

$= ac + adi + bci - bd \quad (\because i^2 = -1)$

$= (ac - bd) + (ad + bc)i$

For example,

$(3 + 2i) . (5 + i) = 15 + 3i + 10i + 2i^2$

$= 15 + 13i - 2 (\because i^2 = -1)$

$= 13 + 13i$

(5) Quotient of two complex numbers :

Let z_1 and z_2 be two complex numbers, where $z_2 \neq 0$.

The quotient $\frac{z_1}{z_2}$ is defined as

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = z_1 \cdot \frac{1}{z_2},$$

where $z_2^{-1} = \frac{1}{z_2} = \frac{c}{c^2 + d^2} - \frac{d}{c^2 + d^2} i$

Observe that the above properties are compatible with the properties shown in 5.3.

5.7 Power of i :

We know that $i^2 = -1$

$\therefore \quad i^3 = i^2 \cdot i = (-1)i = -i$

$i^4 = (i^2)^2 = (-1)^2 = i$

$i^5 = i^4 \cdot i = (1)i = i$

Also, $i^{-1} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$

$$i^{-2} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

$$i^{-3} = \frac{1}{i^3} = \frac{1}{i^3} \times \frac{i}{i} = \frac{i}{i^4} = \frac{i}{1} = i$$

$$i^{-4} = \frac{1}{i^4} = \frac{1}{1} = 1 ....$$

In general, for any integer n,

$i^{4n} = 1$

$i^{4n+1} = i$

$i^{4n+2} = -1$

$i^{4n+3} = -i$

We will accept that the rules of exponents are true for integer exponents of complex number.

5.8 Square roots of a negative real number :

We know that $i^2 = -1$ and $(-i)^2 = i^2 = -1$

Thus square roots of -1 are $\pm i$ but the notation $\sqrt{-1}$ indicates only i.

Similarly, $(\sqrt{5}\, i)^2 = (\sqrt{5})^2\, i^2 = 5(-1) = -5$ and

$(-\sqrt{5}\, i)^2 = (-\sqrt{5})^2\, i^2 = 5(-1) = -5$

$\therefore$ Square roots of -5 are $\pm\sqrt{5}\, i$ but the notation $\sqrt{-5}$ indicates only $\sqrt{5}\, i$.

i.e. $\sqrt{-5} = \sqrt{5}\, i$

In general, if a is any positive real number then $\sqrt{-a} = \sqrt{a}\,\sqrt{-1} = \sqrt{a}\, i$

Ex. 1 : Evaluate :

(1) $\sqrt{-9}$ **(GTU : June 2014, Dec. 2015, Jan. 2018)**

(2) i^9 **(GTU : May 2015, 2018, Nov. 2020)**

(3) i^{11} **(GTU : March 2021)**

(4) $i + i^2 + i^3 + i^4$ **(GTU : June 2013, 2017, 2019)**

(5) $(2 + 3i)(3 - 2i)$ **(GTU : June 2014)**

(6) $(1 + i)^2$ **(GTU : June 2018)**

(7) $(1 - i)^4$

(8) $\left[i^{13} - \left(\frac{1}{i}\right)^{19}\right]^4$

Solution :

(1) $\sqrt{-9} = \sqrt{9}\cdot\sqrt{-1} = 3i$

(2) $i^9 = i^{4(2)+1} = i^{4(2)}\cdot i = (i^4)^2\cdot i = (1)^2\cdot i = -i$

(3) $i^{11} = i^{2(5)+1} = i^{2(5)}\cdot i = (i^2)^5\cdot i = (-1)^5 i = i$

OR

$i^{11} = i^{4(2)+3} = -i$ $\quad (\because i^{4n+3} = -i)$

(4) $i + i^2 + i^3 + i^4 = i + (-1) + i^2\cdot i + (i^2)^2$

$= i - 1 + (-1)\, i + (-1)^2$

$= i - 1 - i + 1 = 0$

(5) $(2 + 3i)(3 - 2i) = 2(3 - 2i) + 3i\,(3 - 2i)$

$= 6 - 4i + 9i - 6i^2$

$= 6 + 5i - 6(-1)$

$= 6 + 5i + 6$

$= 12 + 5i$

(6) $(1 + i)^2 = 1 + 2i + i^2$

$= 1 + 2i + (-1) = 2i$

(7) $(1 - i)^4 = [(1 - i)^2]^2$

$= (1 - 2i + i^2)^2$

$= (1 - 2i - 1)^2 = (-2i)^2 = 4i^2 = -4$

(8) $$\left[i^{13} - \left(\frac{1}{i}\right)^{19}\right]^4 = \left[\left(i^2\right)^6\cdot i - \frac{1}{(i^2)^9\cdot i}\right]^4$$

$$= \left[(-1)^6\cdot i - \frac{1}{(-1)^9\cdot i}\right]^4$$

$$= \left(i + \frac{1}{i}\right)^4 = \left(i + \frac{i}{i^2}\right)^4 = \left(i + \frac{i}{-1}\right)^4$$

$= (i - i)^4 = 0$

Ex. 2 : Find $x, y \in$ R :

(1) $3x + 2yi = 6 + 4i$ **(GTU : June 2019)**

(2) $x + 4iy = xi + y + 3$ **(GTU : Jan. 2017)**

(3) $(3x - 7) + 2iy = 5y + (5 + x)\, i$ **(GTU : June 2017)**

(4) $(2x - y) + 2yi = 6 + 4i$ **(GTU : Dec. 2019)**

(5) $(3 - 4i)(x + iy) = 1$ **(GTU : Nov. 2020)**

(6) $(x - yi) + (3 + 5i) = -6 + 24i$

(7) $\dfrac{x}{1 - i} + \dfrac{y}{1 + i} = 1 + 3i$

Solution :

To solve these equations, we will use the equality of complex numbers.

i.e. $a + bi = c + di \Leftrightarrow a = c$ and $b = d$

(1) $3x + 2yi = 6 + 4i$

Comparing real and imaginary parts of both the sides,

$3x = 6$ and $2y = 4$

$\therefore$ $x = 2$ and $y = 2$

(2) $x + 4iy = xi + y + 3$

$\therefore$ $x + 4yi = (y + 3) + xi$

Comparing real and imaginary parts of both the sides,

$x = y + 3$ and $4y = x$

$\therefore$ $4y = y + 3$

$\therefore$ $3y = 3$

$\therefore$ $y = 1$

$\therefore$ $x = y + 3 = 1 + 3 = 4$

Thus $x = 4$ and $y = 1$

(3) $(3x - 7) + 2iy = 5y + (5 + x)\, i$

Comparing real and imaginary parts of both the sides,

$3x - 7 = 5y$ and $2y = 5 + x$

$\therefore$ $3x - 5y = 7$ and $x - 2y = -5$

Multiplying on both the sides of the equation $x - 2y = -5$ and subtracting from the equation $3x - 5y = 7$,

$3x - 5y = 7$

$3x - 6y = -15$

$- \quad + \quad +$

$y = 22$

$\therefore$ Substituting $y = 22$ in $x - 2y = -5$,

$x - 2(22) = -5$ $\therefore x - 44 = -5$ $\therefore x = 39$

Thus, $x = 39$ and $y = 22$

(4) $(2x - y) + 2yi = 6 + 4i$

Comparing real and imaginary parts of both the sides,

$2x - y = 6$ and $2y = 4$

$\therefore$ $y = 2$

Substituting $y = 2$ in $2x - y = 6$,

$2x - 2 = 6$ $\therefore 2x = 8$ $\therefore x = 4$

Thus, $x = 4$ and $y = 2$

(5) $(3 - 4i)(x + iy) = 1$

$\therefore$ $3x + 3yi - 4xi - 4yi^2 = 1$

$\therefore$ $3x + 3yi - 4xi + 4y = 1$ $(\because i^2 = -1)$

$\therefore$ $(3x + 4y) + (3y - 4x)\, i = 1 + 0i$

Comparing real and imaginary parts of both the sides,

$3x + 4y = 1$ and $3y - 4x = 0$

$\therefore$ $y = \frac{4x}{3}$

Now, taking $y = \frac{4x}{3}$ in $3x + 4y = 1$,

$3x + 4\left(\frac{4x}{3}\right) = 1$

$\therefore$ $9x + 16x = 3$ $\therefore 25x = 3$ $\therefore x = \frac{3}{25}$

$\therefore$ $y = \frac{4x}{3} = \frac{4\left(\frac{3}{25}\right)}{3} = \frac{4}{25}$

Thus, $x = \frac{3}{25}$ and $y = \frac{4}{25}$

(**Note :** Observe that $(x + iy) = \frac{1}{3 - 4i} = (3 - 4i)^{-1}$)

(6) $(x - yi) + (3 + 5i) = -6 + 24i$

$\therefore$ $x - yi + 3 + 5i = -6 + 24i$

$\therefore$ $(x + 3) + (5 - y)\, i = -6 + 24i$

Comparing real and imaginary parts of both the sides,

$x + 3 = -6$ and $5 - y = 24$

$\therefore$ $x = -9$ and $y = -19$

(7) $\frac{x}{1 - i} + \frac{y}{1 + i} = 1 + 3i$

Multiplying both sides of the equation by $(1 - i)(1 + i)$,

$x(1 + i) + y(1 - i) = (1 - i)(1 + i)(1 + 3i)$

$\therefore$ $x + xi + y - yi = (1 - i^2)(1 + 3i)$

$\therefore$ $(x + y) + (x - y)\, i = (1 + 1)(1 + 3i)$

$\therefore$ $(x + y) + (x - y)\, i = 2 + 6i$

Comparing real and imaginary parts of both the sides,

$x + y = 2$ and $x - y = 6$

$\therefore$ $(x + y) + (x - y) = 2 + 6$

$\therefore$ $2x = 8$ $\therefore x = 4$

Now, taking $x = 4$ in $x + y = 2$,

$4 + y = 2$ $\therefore y = -2$

Thus, $x = 4$ and $y = -2$

Ex. 3 : Do as directed :

(1) **If $z = 3 - 2i$, then prove that $z^2 - 6z + 13 = 0$, hence obtain the value of $z^4 - 4z^3 + 6z^2 - 4z + 17$.**

(2) **If $z = -3 + \sqrt{2}\,i$, then prove that $z^4 + 5z^3 + 8z^2 + 7z + 4$.** **(GTU : May 2016)**

(3) **If $z = \dfrac{3}{1 + \sqrt{2}\,i}$, then prove that $z^2 - 2z + 3 = 0$, and hence obtain the value of $z^3 + z^2 - 3z + 10$.** **(GTU : Dec. 2013)**

Solution :

(1) $z = 3 - 2i$

$\therefore \quad z - 3 = -2i$

$\therefore \quad (z-3)^2 = (-2i)^2$

$\therefore \quad z^2 - 6z + 9 = 4i^2 = -4$

$\therefore \quad z^2 - 6z + 13 = 0 \quad \text{.... (a)}$

Now,

$$
\begin{array}{r|l}
 & z^2 + 2z + 5 \\
\hline
z^2 - 6z + 13 & z^4 - 4z^3 + 6z^2 - 4z + 17 \\
 & z^4 - 6z^3 + 13z^2 \\
 & - \quad + \quad - \\
\hline
 & 2z^3 - 7z^2 - 4z + 17 \\
 & 2z^3 - 12z^2 + 26z \\
 & - \quad + \quad - \\
\hline
 & 5z^2 - 30z + 17 \\
 & 5z^2 - 30z + 65 \\
 & - \quad + \quad - \\
\hline
 & -48
\end{array}
$$

$\therefore \quad z^4 - 4z^3 + 6z^2 - 4z + 13$

$= (z^2 - 6z + 13)(z^2 + 2z + 5) - 48$

$= (0)(z^2 + 2z + 5) - 48 \quad (\because \text{From (a)})$

$= -48$

(2) $z = -3 + \sqrt{2}\,i$

$\therefore \quad z + 3 = \sqrt{2}\,i$

$\therefore \quad (z+3)^2 = (\sqrt{2}\,i)^2$

$\therefore \quad z^2 + 6z + 9 = 2i^2 = -2$

$\therefore \quad z^2 + 6z + 11 = 0 \quad \text{.... (a)}$

Now,

$$
\begin{array}{r|l}
 & z^2 - z + 3 \\
\hline
z^2 + 6z + 11 & z^4 + 5z^3 + 8z^2 + 7z + 4 \\
 & z^4 + 6z^3 + 11z^2 \\
 & - \quad - \quad - \\
\hline
 & -z^3 - 3z^2 + 7z + 4 \\
 & -z^3 - 6z^2 - 11z \\
 & + \quad + \quad + \\
\hline
 & 3z^2 + 18z + 4 \\
 & 3z^2 + 18z + 33 \\
 & - \quad - \quad - \\
\hline
 & -29
\end{array}
$$

$\therefore \quad z^4 + 5z^3 + 8z^2 + 7z + 4$

$= (z^2 + 6z + 11)(z^2 - z + 3) - 29$

$= (0)(z^2 - z + 3) - 29 \quad (\because \text{From (a)})$

$= -29$

(3) $z = \dfrac{3}{1 + \sqrt{2}\,i}$

$= \dfrac{3}{1 + \sqrt{2}\,i} \times \dfrac{1 - \sqrt{2}\,i}{1 - \sqrt{2}\,i}$

$= \dfrac{3(1 - \sqrt{2}\,i)}{1 - 2i^2} = \dfrac{3(1 - \sqrt{2}\,i)}{1 + 2} = 1 - \sqrt{2}\,i$

$\therefore \quad z - 1 = -\sqrt{2}\,i$

$\therefore \quad (z-1)^2 = (-\sqrt{2}\,i)^2$

$\therefore \quad z^2 - 2z + 1 = 2i^2 = -2$

$\therefore \quad z^2 - 2z + 3 = 0 \quad \text{.... (a)}$

Now,

$$
\begin{array}{r|l}
 & z + 3 \\
\hline
z^2 - 2z + 3 & z^3 + z^2 - 3z + 10 \\
 & z^3 - 2z^2 + 3z \\
 & - \quad + \quad - \\
\hline
 & 3z^2 - 6z + 10 \\
 & 3z^2 - 6z + 9 \\
 & - \quad + \quad - \\
\hline
 & 1
\end{array}
$$

$\therefore \quad z^3 + z^2 - 3z + 10 = (z^2 - 2z + 3)(z + 3) + 1$

$= (0)(z + 3) + 1$

$(\because$ From (a))

$= 1$

EXERCISE-5.1

1. **Evaluate :**

(1) $\sqrt{-25}$

(2) $\sqrt{-12}$

(3) i^{12}

(4) i^{26}

(5) $i + i^2 + i^3 + i^{2020}$

(6) $i^{2019} + i^{2020} + i^{2021} + i^{2022}$

(7) $(5 + 3i)(3 + 5i)$

(8) $(1 - i)^2$

(9) $(1 + i)^4$

(10) $\left[i^{15} + \left(\frac{1}{i}\right)^{29}\right]^2$

2. **Find $x, y \in R$:**

(1) $2x + 3yi = 4 + 9i$

(2) $(3y - 2) + (5 - 4x)\ i = 0$

(3) $(4 + 5i)x + (3 - zi)y + 10i^2 - i = 0$

(4) $(3x - 2yi)(3 + 4i) = 10(1 + i)$

(5) $(2x + 3yi) + (5 - 4i) = 11 + 5i$

(6) $\frac{x}{2 - i} + \frac{y}{2 + i} = 2 + 3i$

(7) $(1 + i)x + (1 - i)y - 2i^2 - 8i = 0$

3. **Do as directed :**

(1) If $z = 1 + 2i$, then find the value of $z^3 + z^2 - z + 22$.

(2) If $z = 2 + \sqrt{3}\, i$, then find the value of $z^3 - z^2 - 5z + 26$.

(3) If $z = \frac{10}{2 + \sqrt{6}\, i}$, then find the value of $z^2 - 4z + 10 = 0$, hence find the value of $z^4 - 4z^3 + 11z^2 - 4z + 20$.

(4) If $z = -5 + 4i$, then prove that $z^4 + 9z^3 + 35z^2 - z + 164 = 0$.

ANSWERS

1. (1) $5i$ (2) $2\sqrt{3}\, i$ (3) 1 (4) -1 (5) 0 (6) 0 (7) $34\ i$ (8) $-2\ i$ (9) -4 (10) -4

2. (1) $x = 2,\ y = 3$ (2) $x = \frac{5}{4},\ y = \frac{2}{3}$ (3) $x = 1,\ y = 2$ (4) $x = \frac{14}{15},\ y = \frac{1}{5}$ (5) $x = 3,\ y = 3$ (6) $x = 10,\ y = -5$ (7) $x = 3,\ y = -5$

3. (1) 7 (2) 5 (3) 10

5.9 Conjugate Complex number :

If $z = a + bi$ is any complex number then the complex number $a - bi$ is called the conjugate complex number of z. It is denoted by $\bar{z}$.

Thus, $z = a + bi$, then $\bar{z} = a - bi$

(**Note :** We can get $\bar{z}$ by changing the sing of only imaginary part of the complex number z.)

For example,

$z = 2 + 3i$, then $\bar{z} = 2 - 3i$

$z = 1 - 4i$, then $\bar{z} = 1 + 4i$

$z = 5i$, then $\bar{z} = -5i$

$z = 3 = 3 + 0i$, then $\bar{z} = 3 - 0i = 3$

Also,

$z \cdot \bar{z} = (a + bi)(a - bi) = a^2 - b^2 i^2 = a^2 + b^2$

Thus, $z\bar{z}$ is a real number. i.e. $\bar{z}$ acts like a rationalising factor of z.

- **Properties of Complex Conjugate :**

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\mathrm{Re}(z)$

(iii) $z - \bar{z} = 2i\ \mathrm{Im}(z)$

(iv) $z = \bar{z}$ if and only if z is real.

(v) $z = -\bar{z}$ if and only if z is purely imaginary.

(vi) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

(vii) $\overline{z_1 \, z_2} = \bar{z}_1 \, \bar{z}_2$

(viii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$, where $z_2 \neq 0$

Ex. 4 : Express the following complex numbers in the form $a + bi$, $a, b \in R$:

(1) $\dfrac{1+i}{1-i}$ **(GTU : Jan. 2017)**

(2) $\dfrac{5+2i}{2+3i}$ **(GTU : Jan. 2018)**

(3) $\dfrac{2+3i}{1-i}$ **(GTU : March 2021)**

(4) $\dfrac{1+7i}{(2-i)^2}$ **(GTU : Dec. 2019)**

(5) $\dfrac{4+2i}{(3+2i)(5-3i)}$ **(GTU : May 2016)**

(6) $\dfrac{(2-8i)(7+8i)}{1+i}$

(7) $\left(\dfrac{1-i}{1+i}\right)^{100}$

(8) $\dfrac{(2-3i)^2}{(1+i)^3}$

Solution :

(1) $z = \dfrac{1+i}{1-i} = \dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{1+2i+i^2}{1-i^2}$

$= \dfrac{1+2i-1}{1+1} = \dfrac{2i}{2} = i = 0 + i$

Thus, $z = 0 + i$

(2) $z = \dfrac{5+2i}{2+3i} = \dfrac{5+2i}{2+3i} \times \dfrac{2-3i}{2-3i}$

$= \dfrac{10-15i+4i-6i^2}{4-9i^2} = \dfrac{10-11i+6}{4+9}$

$= \dfrac{16-11i}{13} = \dfrac{16}{13} - \dfrac{11}{13}i$

$\therefore \; z = \dfrac{16}{13} - \dfrac{11}{13}i$

(3) $z = \dfrac{2+3i}{1-i} = \dfrac{2+3i}{1-i} \times \dfrac{1+i}{1+i}$

$= \dfrac{2+2i+3i+3i^2}{1-i^2}$

$= \dfrac{2+5i-3}{1+1} = \dfrac{-1+5i}{2} = \dfrac{-1}{2} + \dfrac{5}{2}i$

$\therefore \; z = \dfrac{-1}{2} + \dfrac{5}{2}i$

(4) $z = \dfrac{1+7i}{(2-i)^2} = \dfrac{1+7i}{4-4i+i^2} = \dfrac{1+7i}{4-4i-1} = \dfrac{1+7i}{3-4i}$

$\therefore \; z = \dfrac{1+7i}{3-4i} \times \dfrac{3+4i}{3+4i} = \dfrac{3+4i+12i+28i^2}{9-16i^2}$

$= \dfrac{3+25i-28}{9+16} = \dfrac{-25+25i}{25} = -1+i$

$\therefore \; z = -1 + i$

(5) $z = \dfrac{4+2i}{(3+2i)(5-3i)} = \dfrac{4+2i}{15-9i+10i-6i^2}$

$= \dfrac{4+2i}{15+i+6} = \dfrac{4+2i}{21+i}$

$\therefore \; z = \dfrac{4+2i}{21+i} \times \dfrac{21-i}{21-i} = \dfrac{84-4i+42i-2i^2}{441-i^2}$

$= \dfrac{84+38i+2}{441+1} = \dfrac{86+38i}{442}$

$= \dfrac{86}{442} + \dfrac{38}{442}i$

$= \dfrac{43}{221} + \dfrac{19}{221}i$

(6) $z = \dfrac{(2-8i)(7+8i)}{1+i} = \dfrac{14+16i-56i-64i^2}{1+i}$

$= \dfrac{14-40i+64}{1+i} = \dfrac{78-40i}{1+i}$

$\therefore z = \dfrac{78-40i}{1+i} \times \dfrac{1-i}{1-i} = \dfrac{78-78i-40i+40i^2}{1-i^2}$

$= \dfrac{78-118i-40}{1+1} = \dfrac{38-118i}{2} = 19-59i$

$\therefore z = 19-59\,i$

(7) $z = \left(\dfrac{1-i}{1+i}\right)^{100} = \left(\dfrac{1-i}{1+i} \times \dfrac{1-i}{1-i}\right)^{100}$

$= \left(\dfrac{1-2i+i^2}{1-i^2}\right)^{100} = \left(\dfrac{1-2i-1}{1+1}\right)^{100}$

$= \left(\dfrac{-2i}{2}\right)^{100} = (-i)^{100} = \left[(-i)^4\right]^{25}$

$= 1^{25} = 1 = 1+0i$

$\therefore \quad z = 1+0i$

(8) $z = \dfrac{(2-3i)^2}{(1+i)^3} = \dfrac{4-12i+9i^2}{1+3i+3i^2+i^3}$

$(\because (a+b)^3 = a^3+3a^2b+3ab^2+b^3)$

$= \dfrac{4-12i-9}{1+3i-3-i} = \dfrac{-5-12i}{-2+2i}$

$\therefore \quad z = \dfrac{-5-12i}{-2+2i} \times \dfrac{-2-2i}{-2-2i} = \dfrac{10+10i+24i+24i^2}{4-4i^2}$

$= \dfrac{10+34i-24}{4+4} = \dfrac{-14+34i}{8}$

$= \dfrac{-14}{8} + \dfrac{34}{8}i$

$= -\dfrac{7}{4} + \dfrac{17}{4}i$

Ex. 5 : Do as directed :

(1) $\dfrac{(1-i)^2}{3+i} = x+iy$**, then find the value of** $x-y$**.**

(GTU : June 2014)

(2) If $\dfrac{(1+i)^2}{2-i} = x+iy$**, then find the value of** $x+y$**.**

(GTU : Dec. 2013)

(3) If $(\cos\theta - i\sin\theta)^2 = x - yi$**, then prove that** $2xy = \sin 4\theta$**.**

(4) If $\dfrac{(7-i)(6+i)}{1-i} = x-yi$**, then find the value of** $x+y$**.**

Solution :

(1) $x+yi = \dfrac{(1-i)^2}{3+i} = \dfrac{1-2i+i^2}{3+i} = \dfrac{1-2i-1}{3+i}$

$\therefore\ x+yi = \dfrac{-2i}{3+i} = \dfrac{-2i}{3+i} \times \dfrac{3-i}{3-i}$

$= \dfrac{-6i+2i^2}{9-i^2} = \dfrac{-6i-2}{9+1} = \dfrac{-2-6i}{10}$

$= \dfrac{-2}{10} - \dfrac{6}{10}i$

$= -\dfrac{1}{5} - \dfrac{3}{5}i$

$\therefore\ x = -\dfrac{1}{5}$ and $y = -\dfrac{3}{5}$

$\therefore\ x-y = -\dfrac{1}{5} + \dfrac{3}{5} = \dfrac{2}{5}$

(2) $x+yi = \dfrac{(1+i)^2}{2-i} = \dfrac{1+2i+i^2}{2-i} = \dfrac{1+2i-1}{2-i}$

$x+yi = \dfrac{2i}{2-i} = \dfrac{2i}{2-i} \times \dfrac{2+i}{2+i}$

$= \dfrac{4i+2i^2}{4-i^2} = \dfrac{4i-2}{4+1} = \dfrac{-2+4i}{5}$

$\therefore \quad x + yi = -\dfrac{2}{5} + \dfrac{4}{5}i$

$\therefore \quad x = -\dfrac{2}{5}$ and $y = \dfrac{4}{5}$

$\therefore \quad x + y = -\dfrac{2}{5} + \dfrac{4}{5} = \dfrac{2}{5}$

(3) $x - yi = (\cos\theta - i\sin\theta)^2$

$= \cos^2\theta - 2\sin\theta \cdot \cos\theta\, i + i^2 \sin^2\theta$

$= \cos^2\theta - i\sin 2\theta - \sin^2\theta$

$(\because 2\sin\theta\cos\theta = \sin 2\theta)$

$= \cos^2\theta - \sin^2\theta - i\sin 2\theta$

$= \cos 2\theta - i\sin 2\theta$

$(\because \cos^2\theta - \sin^2\theta = \cos 2\theta)$

$\therefore \quad x = \cos 2\theta$ અને $y = \sin 2\theta$

$\therefore \quad 2xy = 2\cos 2\theta \sin 2\theta = \sin 4\theta$

(4) $x - yi = \dfrac{(7-i)(6+i)}{1-i} = \dfrac{42 + 7i - 6i - i^2}{1-i}$

$= \dfrac{42 + i + 1}{1-i} = \dfrac{43+i}{1-i}$

$\therefore \quad x - yi = \dfrac{43+i}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{43 + 43i + i + i^2}{1 - i^2}$

$= \dfrac{43 + 44i - 1}{1+1} = \dfrac{42 + 44i}{2}$

$= 21 + 22i$

$\therefore \quad x = 21$ and $-y = 22 \qquad \therefore y = -22$

$\therefore \quad x + y = 21 - 22 = -1$

5.10 Modulus of a complex number :

Modulus of a complex number $z = a + bi$ is denoted by $|z|$ and is defined as follows :

$$|z| = \sqrt{a^2 + b^2}$$

Thus, $|z|$ is a real number.

- **Properties of Modulus :**

(i) $|z| \geq 0$

(ii) $|z| = 0 \Leftrightarrow z = 0$

(iii) $|z| \geq \text{Re}(z)$ and $|z| \geq \text{Im}(z)$

(iv) $z\bar{z} = |z|^2 \qquad \therefore z^{-1} = \dfrac{\bar{z}}{|z|^2}$

(v) $|z| = |\bar{z}|$

(vi) $|z_1 z_2| = |z_1|\,|z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, where $z_2 \neq 0$

(viii) $|z_1 + z_2| \leq |z_1| + |z_2|$ (Triangular inequality)

(ix) $|z_1 - z_2| \geq ||z_1| - |z_2||$

(x) $|z^n| = |z|^n$ where $n \in Z$

Ex. 6 : Evaluate :

(1) $|3 - 4i|$ **(GTU : Jan. 2019, March 2021)**

(2) $\left|\dfrac{3}{5} - \dfrac{4}{5}i\right|$ **(GTU : Jan. 2018)**

(3) $|\,(\quad i)^2|$ **(GTU : May 2016)**

(4) $\left|\dfrac{-3+7i}{1+i}\right|$

(5) $|(1+2i)(2-3i)|$

Solution :

(1) $|3 - 4i| = \sqrt{(3)^2 + (-4)^2} = \sqrt{7 + 16} = \sqrt{25} = 5$

(2) $\left|\dfrac{3}{5} - \dfrac{4}{5}i\right| = \sqrt{\left(\dfrac{3}{5}\right)^2 + \left(\dfrac{-4}{5}\right)^2}$

$= \sqrt{\dfrac{9}{25} + \dfrac{16}{25}} = \sqrt{\dfrac{25}{25}} = 1$

(3) $|(3 - 4i)^2| = |9 - 24i + 16i^2| = |9 - 24i - 16|$

$= |-7 - 24i|$

$= \sqrt{(-7)^2 + (-24)^2} = \sqrt{49 + 576} = \sqrt{625} = 25$

OR

$|(3 - 4i)^2| = |3 - 4i|^2 \quad (\because |z^n| = |z|^n)$

$= \left(\sqrt{(3)^2 + (-4)^2}\right)^2 = 9 + 16 = 25$

(4) $\left|\dfrac{-3+7i}{1+i}\right| = \dfrac{|-3+7i|}{|1+i|} \quad \left(\because \left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}\right)$

$= \dfrac{\sqrt{(-3)^2 + (7)^2}}{\sqrt{(1)^2 + (1)^2}} = \dfrac{\sqrt{9+49}}{\sqrt{1+1}} = \sqrt{\dfrac{58}{2}} = \sqrt{29}$

(5) $|(1+2i)(2-3i)| = |1+2i|\,|2-3i|$

$(\because |z_1 z_2| = |z_1|\,|z_2|)$

$= \sqrt{1+4}\,\sqrt{4+9}$

$= \sqrt{5}\,\sqrt{13} = \sqrt{65}$

Ex. 7 : Find conjugate and modulus of the following complex numbers :

(1) $\dfrac{1+i}{1-i}$ **(GTU : Dec. 2013)**

(2) $\dfrac{3+7i}{1-i}$ **(GTU : June 2019)**

(3) $(2+i)^3$

(4) $\dfrac{2+3i}{3+2i}$

Solution :

(1) $z = \dfrac{1+i}{1-i} = \dfrac{1+i}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{1+2i+i^2}{1-i^2} = \dfrac{1+2i-1}{1+1}$

$\therefore \quad z = \dfrac{2i}{2} = i = 0 + i$

$\therefore \quad \bar{z} = 0 - i = -i$

and $|z| = |0+i| = \sqrt{0+1} = 1$

(2) $z = \dfrac{3+7i}{1-i} = \dfrac{3+7i}{1-i} \times \dfrac{1+i}{1-i} = \dfrac{3+3i+7i+7i^2}{1-i^2}$

$\therefore \quad z = \dfrac{3+10i-7}{1+1} = \dfrac{-4+10i}{2} = -2+5i$

$\therefore \quad \bar{z} = -2-5i$ and

$|z| = \sqrt{(-2)^2 + (5)^2} = \sqrt{4+25} = \sqrt{29}$

(3) $z = (2+i)^3$

$= (2)^3 + 3(2)^2 i + 3(2)(i)^2 + i^3$

$= 8 + 12i + 6(-1) - i$

$= 2 + 11i$

$\therefore \quad \bar{z} = 2 - 11i$

and $|z| = \sqrt{(2)^2 + (11)^2} = \sqrt{4+121} = \sqrt{125} = 5\sqrt{5}$

(4) $z = \dfrac{2+3i}{3+2i} = \dfrac{2+3i}{3+2i} \times \dfrac{3-2i}{3-2i} = \dfrac{6-4i+9i-6i^2}{9-4i^2}$

$\therefore \quad z = \dfrac{6+5i+6}{9+4} = \dfrac{12+5i}{13} = \dfrac{12}{13} + \dfrac{5}{13}i$

$\therefore \quad \bar{z} = \dfrac{12}{13} - \dfrac{5}{13}i$

and $|z| = \sqrt{\left(\dfrac{12}{13}\right)^2 + \left(\dfrac{5}{13}\right)^2} = \sqrt{\dfrac{144}{169} + \dfrac{25}{169}}$

$= \sqrt{\dfrac{169}{169}} = 1$

Ex. 8 : Find the multiplication inverse of the following complex numbers :

(1) i **(GTU : Dec. 2019)**

(2) $1+i$ **(GTU : June 2013, Nov. 2020)**

(3) $5-4i$ **(GTU : May 2016)**

(4) $3+4i$ **(GTU : Dec. 2013)**

(5) $\dfrac{2+3i}{4-3i}$ **(GTU : June 2017)**

(**Note :** If $z = a + bi$, then

$z^{-1} = \dfrac{a}{a^2+b^2} - \dfrac{b}{a^2+b^2}i$ OR $\bar{z} = \dfrac{\bar{z}}{|z|^2}$.)

Solution :

(1) Let $z = i$

$\therefore \quad z^{-1} = \dfrac{1}{z} = \dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{i^2}$

$\therefore \quad z^{-1} = \dfrac{i}{-1} = -i$

(2) Let $z = 1 + i$ $\therefore a = 1$ and $b = 1$

$\therefore \quad z^{-1} = \dfrac{a}{a^2+b^2} - \dfrac{b}{a^2+b^2}i =$

$\dfrac{1}{1+1} - \dfrac{1}{1+1}i = \dfrac{1}{2} - \dfrac{1}{2}i$

OR

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{1-i}{\left(\sqrt{1^2+1^2}\right)^2} = \frac{1-i}{2} = \frac{1}{2} - \frac{1}{2}i$$

(3) Let $z = 5 - 4i$

$\therefore \quad \bar{z} = 5 + 4i$ and $|z| = \sqrt{25+16} = \sqrt{41}$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{5+4i}{\left(\sqrt{41}\right)^2} = \frac{5+4i}{41} = \frac{5}{41} + \frac{4}{41}i$$

(4) Let $z = 3 + 4i$

$\therefore \quad \bar{z} = 3 - 4i$ and $|z| = \sqrt{9+16} = \sqrt{25} = 5$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{3-4i}{(5)^2} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i$$

(5) Let $z = \dfrac{2+3i}{4-3i}$

$$\therefore \quad z^{-1} = \frac{4-3i}{2+3i} = \frac{4-3i}{2+3i} \times \frac{2-3i}{2-3i}$$

$$= \frac{8-12i-6i+9i^2}{4-9i^2}$$

$$\therefore \quad z^{-1} = \frac{8-18i-9}{4+9} = \frac{-1-18i}{13} = \frac{-1}{13} - \frac{18}{13}i$$

Ex. 9 : Do as directed :

(1) If $\sqrt{\dfrac{1+i}{1-i}} = a + ib$, then find $a^2 + b^2$.

(GTU : May 2015)

(2) If $\alpha + i\beta = \dfrac{1}{a+ib}$, then prove that

$(\alpha^2 + \beta^2)(a^2 + b^2) = 1$. (GTU : May 2015)

(3) If $z = x + iy$ and $|3z| = |z - 4|$, then prove that $x^2 + y^2 + x = 2$. (GTU : June 2013)

(4) If $z = a + bi$ and $2|z - 1| = |z - 2|$, then prove that $3(a^2 + b^2) = 4a$.

(5) If $z \in C$ and $|2z - 3| = |3z - 2|$, then prove that $|z| = 1$.

(6) If $|z_1| = |z_2| = |z_3| = 1$, then prove that

$$|z_1 + z_2 + z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|.$$

Solution :

(1) $a + ib = \sqrt{\dfrac{1+i}{1-i}}$

$$\therefore \quad (a+ib)^2 = \frac{1+i}{1-i}$$

$$\therefore \quad |(a+ib)^2| = \left|\frac{1+i}{1-i}\right|$$

$$\therefore \quad |a+ib|^2 = \frac{|1+i|}{|1-i|}$$

$$\therefore \quad \left(\sqrt{a^2+b^2}\right)^2 = \frac{\sqrt{1+1}}{\sqrt{1+1}} \qquad \therefore a^2 + b^2 = 1$$

(2) $\alpha + i\beta = \dfrac{1}{a+ib}$

$\therefore \quad (\alpha + i\beta)(a + ib) = 1 = 1 + 0i$

$\therefore \quad |(\alpha + i\beta)(a + ib)| = |1 + 0i|$

$\therefore \quad |\alpha + i\beta|\,|a + ib| = |1 + 0i|$

$\therefore \quad \sqrt{\alpha^2+\beta^2} \cdot \sqrt{a^2+b^2} = \sqrt{1+0}$

$\therefore \quad \sqrt{(\alpha^2+\beta^2)(a^2+b^2)} = 1$

$\therefore \quad (\alpha^2 + \beta^2)(a^2 + b^2) = 1$

($\because$ Squaring both sides)

(3) $z = x + iy$

Now, $|3z| = |z - 4|$

$\therefore \quad |3(x + iy)| = |x + iy - 4|$

$\therefore \quad |3x + 3yi| = |(x - 4) + yi|$

$\therefore \quad \sqrt{9x^2 + 9y^2} = \sqrt{(x-4)^2 + y^2}$

$\therefore \quad 9x^2 + 9y^2 = (x - 4)^2 + y^2$

($\because$ Squaring both sides)

$\therefore \quad 9x^2 + 9y^2 = x^2 - 8x + 16 + y^2$

$\therefore \quad 8x^2 + 8y^2 + 8x = 16$

$\therefore \quad x^2 + y^2 + x = 2$

(4) $z = a + bi$

Now, $2|z - 1| = |z - 2|$

$\therefore \quad 2|a + bi - 1| = |a + bi - 2|$

$\therefore \quad 2|(a - 1) + bi| = |(a - 2) + bi|$

$\therefore \quad 2\sqrt{(a-1)^2 + b^2} = \sqrt{(a-2)^2 + b^2}$

$\therefore \quad 4((a - 1)^2 + b^2) = (a - 2)^2 + b^2$ ($\because$ Squaring both sides)

$\therefore \quad 4(a^2 - 2a + 1 + b^2) = a^2 - 4a + 4 + b^2$

$\therefore \quad 4a^2 + 4b^2 - 8a + 4 = a^2 + b^2 - 4a + 4$

$\therefore \quad 3a^2 + 3b^2 = 4a$

$\therefore \quad 3(a^2 + b^2) = 4a$

(5) Let $z = x + iy \quad \therefore \; |z| = \sqrt{x^2 + y^2}$

Now, $|2z - 3| = |3z - 2|$

$\therefore \quad |2(x + iy) - 3| = |3(x + iy) - 2|$

$\therefore \quad |2x + 2yi - 3| = |3x + 3yi - 2|$

$\therefore \quad |(2x - 3) + 2yi| = |(3x - 2) + 3yi|$

$\therefore \quad \sqrt{(2x-3)^2 + (2y)^2} = \sqrt{(3x-2)^2 + (3y)^2}$

$\therefore \quad (2x - 3)^2 + (2y)^2 = (3x - 2)^2 + (3y)^2$ ($\because$ Squaring both sides)

$\therefore \quad 4x^2 - 12x + 9 + 4y^2 = 9x^2 - 12x + 4 + 9y^2$

$\therefore \quad 4x^2 + 4y^2 - 9x^2 - 9y^2 = 4 - 9$

$\therefore \quad -5x^2 - 5y^2 = -5$

$\therefore \quad x^2 + y^2 = 1$

$\therefore \quad |z|^2 = 1 \qquad \therefore \; |z| = 1$

(6) L.H.S. $= |z_1 + z_2 + z_3|$

$= |\overline{z_1 + z_2 + z_3}| \quad (\because \; |z| = |\bar{z}|)$

$= |\bar{z}_1 + \bar{z}_2 + \bar{z}_3|$

$= \left|\frac{z_1 \bar{z}_1}{z_1} + \frac{z_2 \bar{z}_2}{z_2} + \frac{z_3 \bar{z}_3}{z_3}\right|$

$= \left|\frac{|z_1|^2}{z_1} + \frac{|z_2|^2}{z_2} + \frac{|z_3|^2}{z_3}\right| \quad (\because \; z\bar{z} = |z|^2)$

$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right|$

$(\because \; |z_1| = |z_2| = |z_3| = 1)$

= R.H.S.

Ex. 10 : For $z_1 = 3 + 4i$ and $z_2 = |12 - 5i|$, verify the following results :

(1) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(2) $|z_1 z_2| = |z_1| \, |z_2|$

(3) $|z_1 + z_2| < |z_1| + |z_2|$

Solution :

(1) $z_1 z_2 = (3 + 4i)(12 - 5i) = 36 - 15i + 48i - 20i^2$

$= 36 + 33i + 20$

$= 56 + 33i$

$\therefore \quad \overline{z_1 z_2} = 56 - 33i$ ... (i)

Now, $\bar{z}_1 = 3 - 4i$ and $\bar{z}_2 = 12 + 5i$

$\therefore \quad \bar{z}_1 \cdot \bar{z}_2 = (3 - 4i)(12 + 5i)$

$= 36 + 15i - 48i - 20i^2$

$= 36 - 33i + 20$

$= 56 - 33i$ .... (ii)

From results (i) and (ii), $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

(2) $z_1 z_2 = (3 + 4i)(12 - 5i) = 56 + 33i$

$\therefore \quad |z_1 z_2| = \sqrt{(56)^2 + (33)^2} = \sqrt{3136 + 1089}$

$= \sqrt{4225} = 65$ ... (i)

Now, $|z_1| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

$|z_2| = \sqrt{12^2 + (-5)^2} = \sqrt{144 + 25} = \sqrt{169} = 13$

$\therefore \quad |z_1| \, |z_2| = 5 \times 13 = 65$ .... (ii)

From results (i) and (ii), $|z_1 z_2| = |z_1| \, |z_2|$

(3) $z_1 + z_2 = 3 + 4i + 12 - 5i = 15 - i$

$\therefore \quad |z_1 + z_2| = \sqrt{(15)^2 + (-1)^2} = \sqrt{225 + 1}$

$= \sqrt{226}$ ... (i)

Now, $|z_1| = 5$ and $|z_2| = 13$

$\therefore \quad |z_1| + |z_2| = 5 + 13 = 18 = \sqrt{324}$ ... (ii)

From results (i) and (ii),

$\sqrt{226} < \sqrt{324}$

$\therefore \quad |z_1 + z_2| < |z_1| + |z_2|$

EXERCISE-5.2

1. Express the following complex numbers in the form $a + bi$: $(a, b \in R)$

(1) $(\sqrt{2} - i) - i(1 - \sqrt{2}i)$

(2) $(2 - 3i)(-2 + i)$

(3) $\dfrac{3 + 2i}{5 - 3i}$

(4) $i^3(6 + 3i) - (20 + 5i)(14 + 3i)$

(5) $\left(\dfrac{1 - 5i}{1 + 2i}\right)^2$

2. Find conjugate and modulus of the following complex numbers :

(1) $-3\sqrt{2} + 3\sqrt{2}i$ (2) $(3 - 4i)^2$

(3) $\dfrac{-3 + 7i}{1 + i}$ (4) $(1 + i)^7$

(5) $\dfrac{(8 - 3i)(6 - i)}{2 - 2i}$

3. Find conjugate and modulus of the following complex numbers :

(1) $1 - \sqrt{3}i$ (2) $-3 + 5i$ (3) $(2 - 3i)^2$

(4) $\dfrac{4 + 3i}{5 - 3i}$

(5) $5i$

4. Do as directed :

(1) If $\dfrac{6 + i}{3 - 2i} = x + iy$, then find $3x + 2y$.

(2) If $\dfrac{3 + 2i}{5 - 3i} = x + iy$, then find $2x + 8y$.

5. Do as directed :

(1) If $x + yi = \dfrac{a + bi}{c + di}$, then prove that $(x^2 + y^2)(c^2 + d^2) = (a^2 + b^2)$.

(2) If $(\cos\theta - i\sin\theta)^2 = a + bi$, then prove that $a^2 + b^2 = 1$.

(3) If $z \in C$ and $|z + 3| = |3z + 1|$, then prove that $|z| = 1$.

(4) If $z \in C$ and $|2z - 1| = |z - 2|$, then prove that $|z| = 1$.

(5) If $z_1, z_2 \in C$, then prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$.

(6) If $\dfrac{8}{2 + i\sin\theta} = x + iy$, then prove that $x^2 + y^2 = 4x$.

ANSWERS

1. (1) $0 - 2i$ (2) $-1 + 8i$

(3) $\dfrac{9}{34} + \dfrac{19}{34}i$ (4) $-262 - 136i$

(5) $\dfrac{32}{25} + \dfrac{126}{25}i$

2. (1) $-3\sqrt{2} - 3\sqrt{2}i,\ 6$ (2) $-7 + 24i,\ 25$

(3) $2 - 5i,\ \sqrt{29}$ (4) $8 + 8i,\ 8\sqrt{2}$

(5) $\dfrac{71}{4} - \dfrac{19}{4}i,\ \dfrac{\sqrt{5402}}{4}$

3. (1) $\dfrac{1}{4} + \dfrac{\sqrt{3}}{4}i$ (2) $\dfrac{-3}{34} - \dfrac{5}{34}i$

(3) $\dfrac{-5}{169} + \dfrac{12}{169}i$ (4) $\dfrac{11}{25} - \dfrac{27}{25}i$

(5) $-\dfrac{i}{5}$

4. (1) 6 (2) 5

5.11 Square roots of a complex number :

Suppose square root of a complex number $z = x + iy$ is $a + bi$.

$\therefore \quad (a + bi)^2 = x + yi$

$\therefore \quad a^2 + 2abi + b^2i^2 = x + yi$

$\therefore \quad a^2 + 2abi - b^2 = x + yi$

$\therefore \quad (a^2 - b^2) + 2abi = x + yi$

$\therefore \quad a^2 - b^2 = x$ and $2ab = y$

Now, $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2$

$= x^2 + y^2$

$= |z|^2$

$\therefore \quad a^2 + b^2 = |z|$

Now, from $a^2 + b^2 = |z|$ and $a^2 - b^2 = x$,

$2a^2 = |z| + x$ and $2b^2 = |z| - x$

$$\therefore \quad a^2 = \frac{|z| + x}{2} \text{ and } b^2 = \frac{|z| - x}{2}$$

$$\therefore \quad a = \pm\sqrt{\frac{|z| + x}{2}} \text{ and } b = \pm\sqrt{\frac{|z| - x}{2}}$$

Now if $y = 2ab > 0$, then a and b both are positive or both are negative.

$\therefore$ Square roots of $z = x + yi$ are

$$\pm\left(\sqrt{\frac{|z| + x}{2}} + i\sqrt{\frac{|z| - x}{2}}\right)$$

If $y = 2ab < 0$, then one of a and b is positive and the other is negative.

$\therefore$ Square roots of $z = x + yi$ are

$$\pm\left(\sqrt{\frac{|z| + x}{2}} - i\sqrt{\frac{|z| - x}{2}}\right).$$

Ex. 11 : Find square roots of the following complex numbers :

(1) $5 - 2i$ **(GTU : May 2016)**

(2) $3 + 4i$ **(GTU : Dec. 2015)**

(3) $3 - 4i$ **(GTU : Jan. 2018)**

(4) $7 + 24i$ **(GTU : Jan. 2019)**

(5) $3 - 4\sqrt{10}\, i$ **(GTU : June 2014, May 2018)**

(6) $3 + 4\sqrt{10}\, i$ **(GTU : June 2019)**

(7) $-2 + 2\sqrt{3}\, i$ **(GTU : Nov. 2020)**

(8) $-i$ **(GTU : Dec. 2019)**

Solution :

(1) $z = 5 - 2i$

$\therefore \quad x = 5, y = -2$ and $|z| = \sqrt{25 + 4} = \sqrt{29}$

As $y = -2 < 0$, square roots of $z = 5 - 2i$ are

$$\pm\left(\sqrt{\frac{|z| + x}{2}} - i\sqrt{\frac{|z| - x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{\sqrt{29} + 5}{2}} - i\sqrt{\frac{\sqrt{29} - 5}{2}}\right)$$

(2) $z = 3 + 4i$

$\therefore \quad x = 3, y = 4$ and $|z| = \sqrt{9 + 16} = \sqrt{25} = 5$

As, $y = 4 > 0$, square roots of $z = 3 + 4i$ are

$$\pm\left(\sqrt{\frac{|z| + x}{2}} + i\sqrt{\frac{|z| - x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{5 + 3}{2}} + i\sqrt{\frac{5 - 3}{2}}\right)$$

$$= \pm\left(\sqrt{4} + i\sqrt{1}\right)$$

$$= \pm(2 + i)$$

(3) $z = 3 - 4i$

$\therefore \quad x = 3, \ y = -4$ and $|z| = \sqrt{9 + 16} = \sqrt{25} = 5$

As $y = -4 < 0$, square roots of $z = 3 - 4i$ are

$$\pm\left(\sqrt{\frac{|z| + x}{2}} - i\sqrt{\frac{|z| - x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{5 + 3}{2}} - i\sqrt{\frac{5 - 3}{2}}\right)$$

$$= \pm\left(\sqrt{4} - i\sqrt{1}\right)$$

$$= \pm(2 - i)$$

(4) $z = 7 + 24i$

$\therefore \quad x = 7, \ y = 24$ and

$|z| = \sqrt{49 + 576} = \sqrt{625} = 25$

As, $y = 24 > 0$, square roots of $z = 7 + 24i$ are

$$\pm\left(\sqrt{\frac{|z|+x}{2}} + i\sqrt{\frac{|z|-x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{25+7}{2}} + i\sqrt{\frac{25-7}{2}}\right)$$

$$= \pm\left(\sqrt{16} + i\sqrt{9}\right)$$

$$= \pm(4 + 3i)$$

(5) $z = 3 - 4\sqrt{10}\,i$

$\therefore \quad x = 3,\ y = -4\sqrt{10}$ and

$$|z| = \sqrt{9+160} = \sqrt{169} = 13$$

As, $y = -4\sqrt{10} < 0$, square roots of $z = 3 - 4\sqrt{10}\,i$ are

$$\pm\left(\sqrt{\frac{|z|+x}{2}} - i\sqrt{\frac{|z|-x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{13+3}{2}} - i\sqrt{\frac{13-3}{2}}\right)$$

$$= \pm\left(\sqrt{8} - i\sqrt{5}\right)$$

$$= \pm\left(2\sqrt{2} - i\sqrt{5}\right)$$

(6) $z = 3 + 4\sqrt{10}\,i$

$\therefore \quad x = 3,\ y = 4\sqrt{10}$ and

$$|z| = \sqrt{9+160} = \sqrt{169} = 13$$

As, $y = 4\sqrt{10} > 0$, square roots of $z = 3 + 4\sqrt{10}\,i$ are

$$\pm\left(\sqrt{\frac{|z|+x}{2}} + i\sqrt{\frac{|z|-x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{13+3}{2}} + i\sqrt{\frac{13-3}{2}}\right)$$

$$= \pm\left(\sqrt{8} + i\sqrt{5}\right)$$

$$= \pm\left(2\sqrt{2} + i\sqrt{5}\right)$$

(7) $z = -2 + 2\sqrt{3}\,i$

$\therefore \quad x = -2,\ y = 2\sqrt{3}$ and

$$|z| = \sqrt{4+12} = \sqrt{16} = 4$$

As, $y = 2\sqrt{3} > 0$, square roots of $z = -2 + 2\sqrt{3}\,i$ are

$$\pm\left(\sqrt{\frac{|z|+x}{2}} + i\sqrt{\frac{|z|-x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{4+(-2)}{2}} + i\sqrt{\frac{4-(-2)}{2}}\right)$$

$$= \pm\left(\sqrt{1} + i\sqrt{3}\right)$$

$$= \pm\left(1 + \sqrt{3}\,i\right)$$

(8) $z = -i = 0 - i$

$\therefore \quad x = 0,\ y = -1$ and $|z| = \sqrt{0+1} = \sqrt{1} = 1$

As, $y = -1 < 0$, square roots of $z = -i$ are

$$\pm\left(\sqrt{\frac{|z|+x}{2}} - i\sqrt{\frac{|z|-x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{1+0}{2}} - i\sqrt{\frac{1-0}{2}}\right)$$

$$= \pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

5.12 Geometric Representation of Complex Numbers :

In the begining of this chapter, we have mentioned that a complex number is an order pair of real number, i.e. a complex number is of the form $z = x + iy = (x, y),\ x, y \in R$ and each order pair (x, y) represents a point in the coordinate plane. Thus a complex number $z = x + iy$ can be represented as a point (x, y) in the coordinate plane. Hence there is one-one correspondance between the set of complex numbers and the set of points of coordinate plane. In 1860, a mathematician Argand has started such geometrical representation of complex numbers. So the geometrical representation of complex numbers is known as Argand Diagram and the corresponding coordinate plane is known as complex plane or Argand plane.

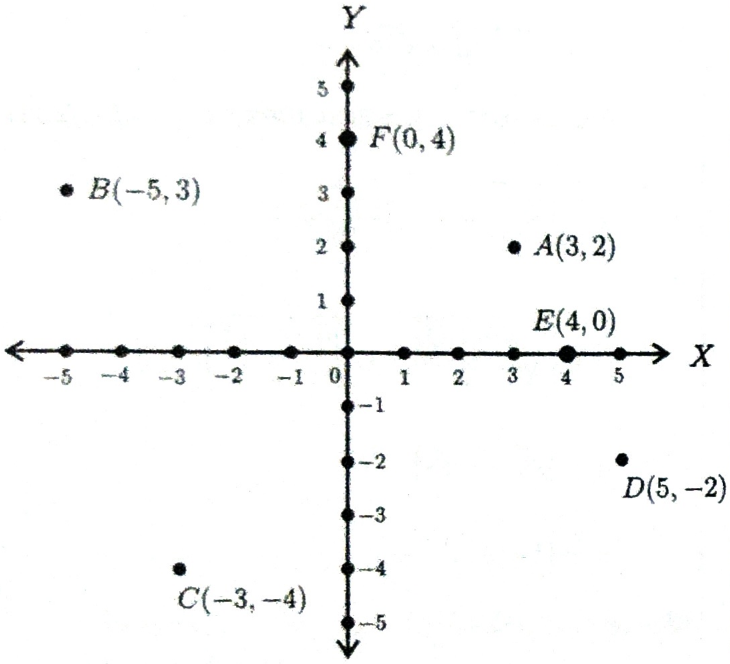

Figure 5.1

Geometrical representation of a complex number $3 + 2i$ is a point whose coordinates are $(3, 2)$, which is shown as A in the figure 5.1. Similarly complex numbers $-5 + 3i$, $-3 -4i$, $5 - 2i$, $4 = 4 + 0i$ and $4i = 0 + 4i$ are represented as $B(-5, 3)$, $C(-3, -4)$, $D(5, -2)$, $E(4, 0)$ and $F(0, 4)$ respectively.

The points on the X-axis correspond to the complex numbers of the form $a + i0$ (real numbers) and the points on the Y-axis correspond to the complex numbers of the form $0 + ib$ (Purely imaginary numbers). The X-axis and Y-axis in the complex plane are called the real axis and the imaginary axis respectively.

Geometrical representation of modulus of a complex number :

As shown in the figure 5.2, the modulus of the complex number $z = x + iy$ is the distance between the point $P(x, y)$ and the origin $O(0, 0)$.

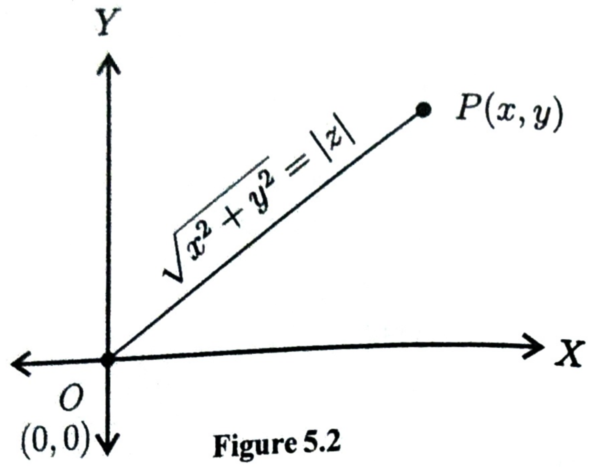

Figure 5.2

Geometrical representation of the conjugate of a complex number :

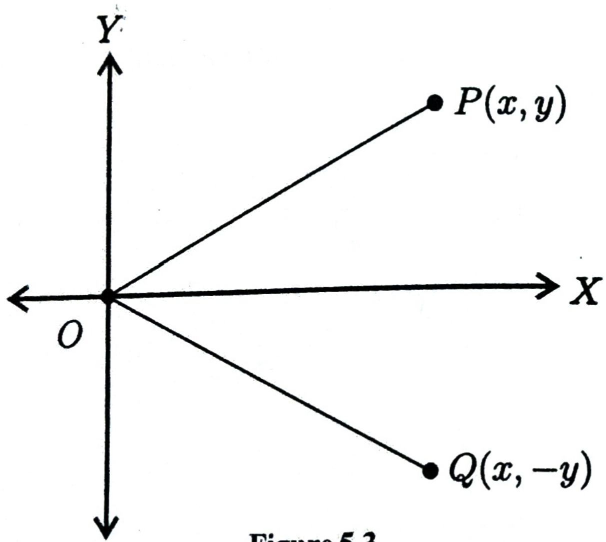

Figure 5.3

The representations of a complex number $z = x + iy$ and its conjugate $\bar{z} = x - iy$ are shown in figure 5.3 as the points $P(x, y)$ and $Q(x, -y)$ respectively. Geometrically, the point $Q(x, -y)$ is called the mirror image of the point $P(x, y)$ with respect to the real axis (X-axis). Thus $\bar{z}$ is a mirror image of z with respect to the real axis in the Argand Plane.

5.13 Polar form of complex numbers :

Now let us try to represent a complex number $z = x + iy$ in an another form.

A point $P(x, y)$ corresponds to a complex number $z = x + iy$ is shown in the figure 5.4.

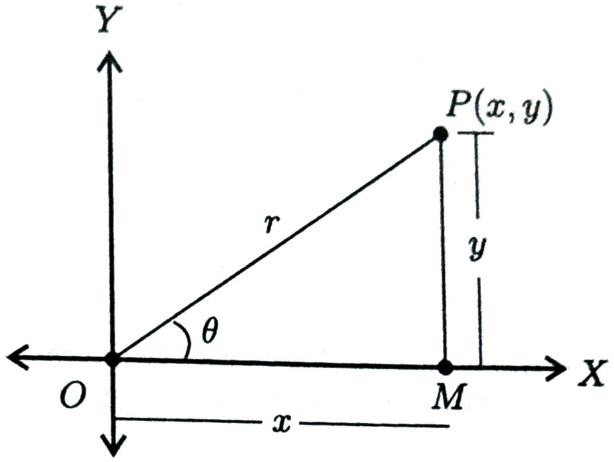

Figure 5.4

Now, draw $\overline{PM} \perp \overrightarrow{OX}$ where $M \in \overrightarrow{OX}$.

$\therefore$ OM = x and PM = y

($\because$ As P(x, y) is in the first quadrant $x > 0$, $y > 0$)

Suppose OP = r and the angle of $\overrightarrow{OP}$ with the positive direction of X-axis = $m\angle$ POM = θ.

$\therefore$ From figure 5.4,

$$\cos\theta = \frac{x}{r} \text{ and } \sin\theta = \frac{y}{r}$$

$\therefore$ $x = r\cos\theta$ and $y = r\sin\theta$

(**Note :** The above result is true even if P(x, y) is in any quadrant).

$$\therefore \quad z = x + iy = r\cos\theta + i\,r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

Also, $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$$= r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2$$

$$\therefore \quad r = \sqrt{x^2 + y^2} = |z|$$

and $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$

The form $z = r(\cos\theta + i\sin\theta)$ is called the polar form of the complex number z. Also θ is known as amplitude or argument of z. It is denoted by arg(z). Since sine and cosine functions are periodic, there are many values of θ satisfying $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$. Each of these θ is an argument of z.

The unique value of θ lies in $(-\pi, \pi]$ for which $x = r\cos\theta$ and $y = r\sin\theta$ is known as principal argument of z. Unless specified, the notation arg(z) means principal argument.

Method of finding principal argument of a complex number :

Case-1 : If a complex number is of the form $z = x + i0$, where $x > 0$, then the point $(x, 0)$ corresponding to z lies on the positive direction of X-axis as shown in the figure 5.5. So its principal argument is 0.

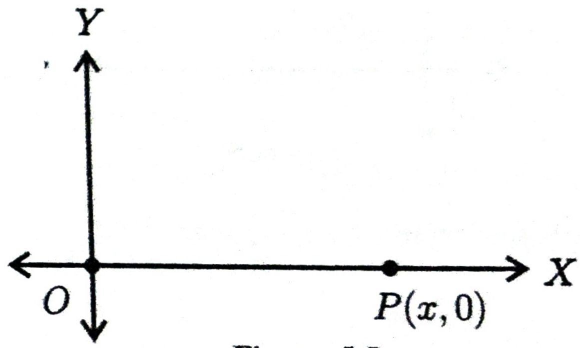

Figure 5.5

For example $arg(5) = 0$, arg $\left(\frac{1}{2}\right) = 0$, $arg(\pi) = 0$

Case-2 : If a complex number is of the form $z = x + i0$, where $x < 0$, then the point $(x, 0)$ corresponding to z lies on the negative direction of X-axis as shown in the figure 5.6. So its principal argument is π.

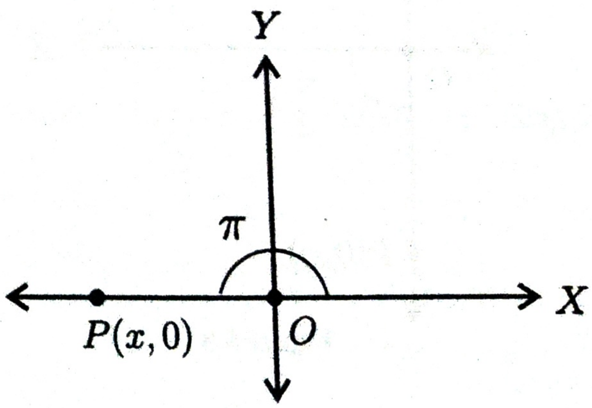

Figure 5.6

For example $\arg(-1) = \pi$, $\arg\left(-\frac{3}{2}\right) = \pi$, $\arg(-7) = \pi$

Case-3 : If a complex number is of the $z = iy = 0 + iy$, where $y > 0$, then the point $(0, y)$ corresponding to z lies on the positive direction of Y-axis as shown in the figure 5.7. So its principal argument is $\frac{\pi}{2}$.

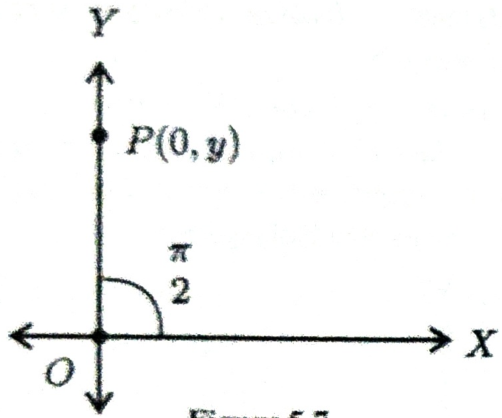

Figure 5.7

For example $\arg(i) = \frac{\pi}{2}$, $\arg(3i) = \frac{\pi}{2}$, $\arg\left(\frac{5}{2}i\right) = \frac{\pi}{2}$.

Case-4 : If a complex number is of the form $z = iy = 0 + iy$, where $y < 0$, then the point $(0, y)$ corresponding to z lies on the negative direction of Y-axis as shown in the figure 5.8. So its principal argument is $-\frac{\pi}{2}$.

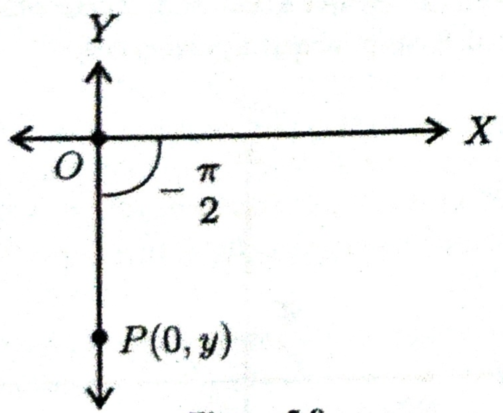

Figure 5.8

For example $\arg(-i) = -\frac{\pi}{2}$, $\arg(-5i) = -\frac{\pi}{2}$, $\arg\left(-\frac{3i}{2}\right) = -\frac{\pi}{2}$.

Case-5 : Argument of the complex number $z = 0 = 0 + 0i$ is not defined.

Case-6 : If a complex number $z = x + iy$, where $x \neq 0$ and $y \neq 0$, then first of all, from $\tan\alpha = \left|\frac{y}{x}\right|$, find $\alpha \in \left(0, \frac{\pi}{2}\right)$. Then,

(i) If $x > 0$ and $y > 0$, then the point $P(x, y)$ corresponding to z lies in the first quadrant, so $\arg(z) = \alpha$.

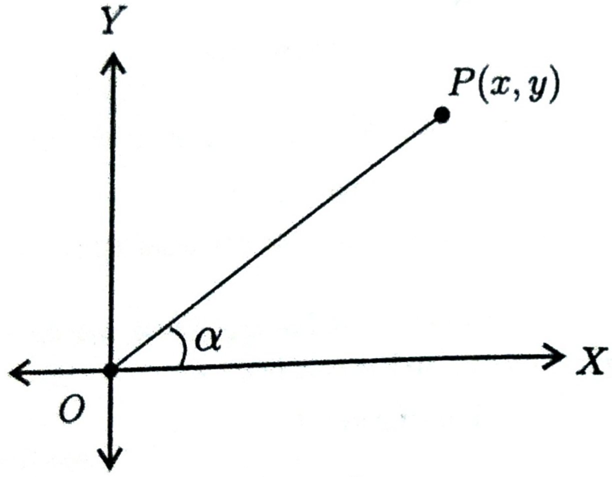

Figure 5.9

(ii) If $x < 0$ and $y > 0$, then the point $P(x, y)$ corresponding to z lies in the second quadrant, so $\arg(z) = \pi - \alpha$.

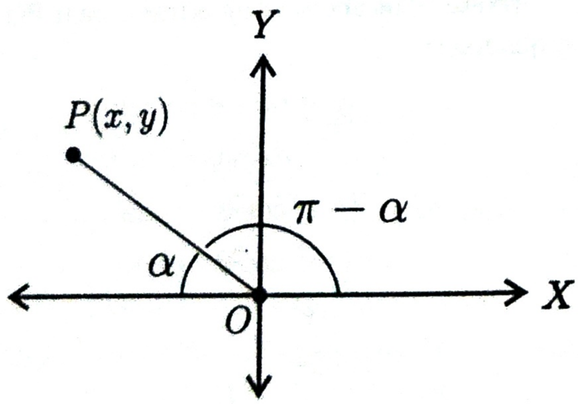

Figure 5.10

(iii) If $x < 0$ and $y < 0$, then the point $P(x, y)$ corresponding to z lies in the third quadrant, so $\arg(z) = \alpha - \pi$.

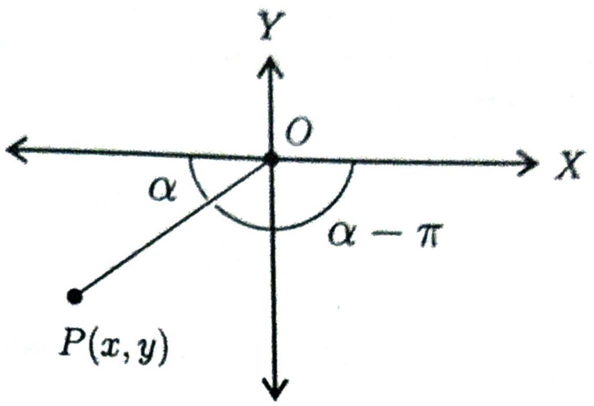

Figure 5.11

(iv) If $x > 0$ and $y < 0$, then the point $P(x, y)$ corresponding to z lies in the fourth quadrant, so $arg(z) = -\alpha$.

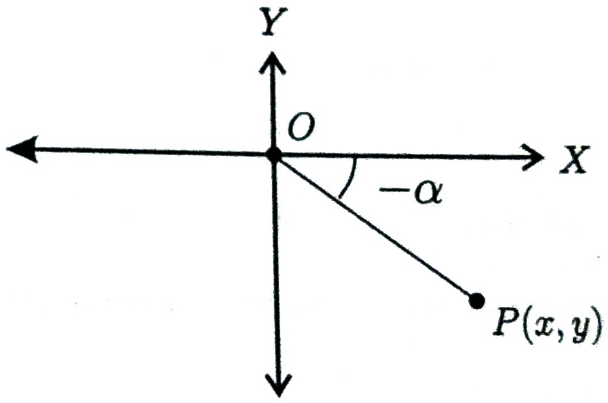

Figure 5.12

Ex. 12 : Determine the modulus and the principal argument of the following complex numbers. Also represent them in polar form :

(1) 35 (GTU : Dec. 2015, Nov. 2020)

(2) –1 (GTU : Jan. 2017)

(3) 2*i* (GTU : Jan. 2019)

(4) –5*i* (GTU : June 2014)

Solution :

(1) $z = 35 = 35 + 0i$, where, $35 > 0$

$\therefore$ $r = |z| = 35$ and $\theta = arg(z) = 0$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$= 35(\cos\theta + i\sin\theta)$

(2) $z = -1 = -1 + 0i$, where, $-1 < 0$

$\therefore$ $r = |z| = 1$ and $\theta = arg(z) = \pi$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$= 1(\cos\pi + i\sin\pi)$

$= \cos\pi + i\sin\pi$

(3) $z = 2i = 0 + 2i$, where, $2 > 0$

$\therefore$ $r = |z| = 2$ and $\theta = arg(z) = \frac{\pi}{2}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

(4) $z = -5i = 0 - 5i$, where, $-5 < 0$

$\therefore$ $r = |z| = 5$ and $\theta = arg(z) = -\frac{\pi}{2}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$= 5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

Ex. 13 : Determine the modulus and the principal argument of the following complex numbers. Also represent them in polar form :

(1) $1+\sqrt{3}i$ (GTU : May 2015, Jan. 2017, Nov. 2020)

(2) $-1+\sqrt{3}i$ (GTU : Dec. 2019)

(3) $-1-\sqrt{3}i$

(4) $1-\sqrt{3}i$ (GTU : Jan. 2019)

Solution :

(1) $z = 1 + \sqrt{3}i$

$\therefore$ $x = 1 > 0$ and $y = \sqrt{3} > 0$

$\therefore$ $r = |z| = \sqrt{1+3} = \sqrt{4} = 2$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{\sqrt{3}}{1}\right| = \sqrt{3}$ $\therefore$ $\alpha = \frac{\pi}{3}$

Since $x > 0$ and $y > 0$, the principal argument of z is

$\theta = arg(z) = \alpha = \frac{\pi}{3}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

(2) $z = -1 + \sqrt{3}i$

$\therefore$ $x = -1 < 0$ and $y = \sqrt{3} > 0$

$\therefore \quad r = |z| = \sqrt{1+3} = \sqrt{4} = 2$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{\sqrt{3}}{-1}\right| = \sqrt{3} \quad \therefore \quad \alpha = \frac{\pi}{3}$

Since $x < 0$ and $y > 0$, the principal argument of z is

$\theta = arg(z) = \pi - \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

(3) $z = -1 - \sqrt{3}i$

$\therefore \quad x = -1 < 0$ and $y = -\sqrt{3} < 0$

$\therefore \quad r = |z| = \sqrt{1+3} = \sqrt{4} = 2$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{-\sqrt{3}}{-1}\right| = \sqrt{3} \quad \therefore \quad \alpha = \frac{\pi}{3}$

Since $x < 0$ and $y < 0$, the principal argument of z is

$\theta = arg(z) = \alpha - \pi = \frac{\pi}{3} - \pi = \frac{-2\pi}{3}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$$= 2\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

(4) $z = 1 - \sqrt{3}i$

$\therefore \quad x = 1 > 0$ and $y = -\sqrt{3} < 0$

$\therefore \quad r = |z| = \sqrt{1+3} = \sqrt{4} = 2$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{-\sqrt{3}}{1}\right| = \sqrt{3} \quad \therefore \quad \alpha = \frac{\pi}{3}$

Since $x > 0$ and $y < 0$, the principal argument of z is

$\theta = arg(z) = -\alpha = -\frac{\pi}{3}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$$= 2\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$$

Ex. 14 : Determine the modulus and the principal argument of the following complex numbers. Also represent them in polar form :

(1) $1 + i$ **(GTU : Dec. 2015)**

(2) $\sqrt{3} + i$ **(GTU : June 2013, May 2018)**

(3) $3 + 2i$ **(GTU : March 2021)**

(4) $\frac{1+i}{1-i}$ **(GTU : Dec. 2014, Jan. 2019)**

(5) $\frac{1+7i}{(2-i)^2}$ **(GTU : Jan. 2018)**

Solution :

(1) $z = 1 + i$

$\therefore \quad x = 1 > 0$ and $y = 1 > 0$

$\therefore \quad r = |z| = \sqrt{1+1} = \sqrt{2}$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{1}{1}\right| = 1 \quad \therefore \quad \alpha = \frac{\pi}{4}$

Since $x > 0$ and $y > 0$, the principal argument of z is

$\theta = arg(z) = \alpha = \frac{\pi}{4}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(2) $z = \sqrt{3} + i$

$\therefore \quad x = \sqrt{3} > 0$ and $y = 1 > 0$

$\therefore \quad r = |z| = \sqrt{3+1} = \sqrt{4} = 2$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{1}{\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \quad \therefore \quad \alpha = \frac{\pi}{6}$

Since $x > 0$ and $y > 0$, the principal argument of z is

$\theta = arg(z) = \alpha = \frac{\pi}{6}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

(3) $z = 3 + 2i$

$\therefore \quad x = 3 > 0$ and $y = 2 > 0$

$\therefore \quad r = |z| = \sqrt{9+4} = \sqrt{13}$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{2}{3}\right| = \frac{2}{3} \quad \therefore \alpha = \tan^{-1}\left(\frac{2}{3}\right)$

Since $x > 0$ and $y > 0$, the principal argument of z is

$\theta = arg(z) = \alpha = \tan^{-1}\left(\frac{2}{3}\right)$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$= \sqrt{13}\left(\cos\left(\tan^{-1}\frac{2}{3}\right) + i\sin\left(\tan^{-1}\frac{2}{3}\right)\right)$

(4) $z = \frac{1+i}{1-i}$

$= \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{1+2i+i^2}{1-i^2} = \frac{1+2i-1}{1+1} = \frac{2i}{2}$

$\therefore \quad z = i = 0 + i$

$\therefore \quad x = 0$ and $y = 1 > 0$

$\therefore \quad r = |z| = 1$ and the principal argument of z is

$\theta = arg\,(z) = \frac{\pi}{2}$

$\therefore$ Polar form of $z = r\,(\cos\theta + i\sin\theta)$

$= 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

(5) $z = \frac{1+7i}{(2-i)^2} = \frac{1+7i}{4-4i+i^2} = \frac{1+7i}{4-4i-1} = \frac{1+7i}{3-4i}$

$\therefore \quad z = \frac{1+7i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{3+4i+21i+28i^2}{9-16i^2}$

$= \frac{3+25i-28}{9+16} = \frac{-25+25i}{25}$

$\therefore \quad z = -1 + i$

$\therefore \quad x = -1 < 0$ and $y = 1 > 0$

$\therefore \quad r = |z| = \sqrt{1+1} = \sqrt{2}$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{1}{-1}\right| = 1 \quad \therefore \alpha = \frac{\pi}{4}$

Since $x < 0$ and $y > 0$, the principal argument of z is

$\theta = arg(z) = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

$\therefore$ Polar form of $z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

Ex. 15 : Find a complex number whose modulus is 2 and the principal argument is $\frac{2\pi}{3}$.

Solution : Let the required complex number is z.

$\therefore \quad r = |z| = 2$ and $\theta = arg(z) = \frac{2\pi}{3}$

$\therefore \quad z = r\,(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

Now, $\cos\frac{2\pi}{3} = \cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$

and $\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore \quad z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -1 + \sqrt{3}\,i$

5.14 De Moivre's Theorem :

If $z_1 = \cos\theta_1 + i\sin\theta_1$ and

$z_2 = \cos\theta_2 + i\sin\theta_2$, then

$$\begin{aligned} z_1 . z_2 &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 \\ &\qquad + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2 \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \end{aligned}$$

Thus

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$
$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) \quad \text{... (i)}$$

Substituting $\theta_1 = \theta_2 = \theta$ in result (i),

$$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$
$$= \cos(\theta+\theta) + i\sin(\theta+\theta)$$
$$\therefore \quad (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin\theta \quad \text{... (ii)}$$

Multiplying both sides of resul (ii) by $\cos\theta + i\sin\theta$,

$$(\cos\theta + i\sin\theta)^3 = (\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta)$$
$$= \cos 3\theta + i\sin 3\theta \text{ ... (iii) (From(i))}$$

Multiplying both sides of result (iii) by $\cos\theta + i\sin\theta$,

$$(\cos\theta + i\sin\theta)^4 = (\cos 3\theta + i\sin 3\theta)(\cos\theta + i\sin\theta)$$
$$= \cos 4\theta + i\sin 4\theta \text{ ... (iv)}$$

(From (i))

By repeating this process n times, after n steps we get the following result.

$$\boxed{(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta} \quad \text{... (v)}$$

We have derived the above result for positive integer n. Actually the result (v) is also true for negative integrs and rational numbers.

This result (v) is known as De Moivre's Theorem.

Now,

$$(\cos\theta - i\sin\theta)^n = (\cos(-\theta) + i\sin(-\theta))^n$$
$$= \cos(-n\theta) + i\sin(-n\theta) \qquad (\because \text{From (v)})$$
$$\therefore \quad \boxed{(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta} \text{... (vi)}$$

Now,

$$(\cos\theta + i\sin\theta)^{-1} = \frac{1}{\cos\theta + i\sin\theta}$$
$$= \frac{1}{\cos\theta + i\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$
$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta - i^2\sin^2\theta}$$
$$= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}$$
$$\therefore \quad (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta \quad \text{... (vii)}$$

Taking power n on both sides of result (vii)

$$[(\cos\theta + i\sin\theta)^{-1}]^n = (\cos\theta - i\sin\theta)^n$$
$$\therefore \quad \boxed{(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta} \text{... (viii)}$$

(From (vi))

Taking power $-n$ on both sides of result (vii),

$$[(\cos\theta + i\sin\theta)^{-1}]^{-n} = (\cos\theta - i\sin\theta)^{-n}$$
$$\therefore \quad (\cos\theta - i\sin\theta)^{-n} = (\cos\theta + i\sin\theta)^n$$
$$\therefore \quad \boxed{(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta} \text{... (ix)}$$

(From (v))

Ex. 16 : Simplify :

(1) $\dfrac{\cos 6\theta + i\sin 6\theta}{\cos 2\theta + i\sin 2\theta}$ **(GTU : Jan. 2017)**

(2) $\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta - i\sin 2\theta}$ **(GTU : March 2021)**

(3) $\left(\dfrac{\cos 3\theta + i\sin 3\theta}{\cos\theta - i\sin\theta}\right)^2$ **(GTU : Dec. 2015)**

(4) $\dfrac{(\cos 2\theta + i\sin 2\theta)^{-3} \cdot (\cos 3\theta - i\sin 3\theta)^2}{(\cos 2\theta - i\sin 2\theta)^{-7} \cdot (\cos 5\theta - i\sin 5\theta)^3}$

(GTU : June 2013)

(5) $\dfrac{(\cos 3\theta + i\sin 3\theta)^{-4} \cdot (\cos\theta - i\sin\theta)^5}{(\cos 2\theta - i\sin 2\theta)^6 \cdot (\cos 13\theta + i\sin 13\theta)}$

(GTU : June 2019)

Solution :

(1)
$$\frac{\cos 6\theta + i\sin 6\theta}{\cos 2\theta + i\sin 2\theta} = \frac{(\cos\theta + i\sin\theta)^6}{(\cos\theta + i\sin\theta)^2}$$
$$= (\cos\theta + i\sin\theta)^4$$
$$= \cos 4\theta + i\sin 4\theta$$

(2)
$$\frac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta - i\sin 2\theta} = \frac{(\cos\theta + i\sin\theta)^5}{(\cos\theta + i\sin\theta)^{-2}}$$
$$= (\cos\theta + i\sin\theta)^7$$
$$= \cos 7\theta + i\sin 7\theta$$

(3)
$$\left(\frac{\cos 3\theta + i\sin 3\theta}{\cos\theta - i\sin\theta}\right)^2 = \left(\frac{(\cos\theta + i\sin\theta)^3}{(\cos\theta + i\sin\theta)^{-1}}\right)^2$$

$$= [(\cos\theta + i\sin\theta)^4]^2$$
$$= (\cos\theta + i\sin\theta)^8$$
$$= \cos 8\theta + i\sin 8\theta$$

(4) $$\frac{(\cos 2\theta + i\sin 2\theta)^{-3} \cdot (\cos 3\theta - i\sin 3\theta)^2}{(\cos 2\theta - i\sin 2\theta)^{-7} \cdot (\cos 5\theta - i\sin 5\theta)^3}$$

$$= \frac{[(\cos\theta + i\sin\theta)^2]^{-3}\,[(\cos\theta + i\sin\theta)^{-3}]^2}{[(\cos\theta + i\sin\theta)^{-2}]^{-7}\,[(\cos\theta + i\sin\theta)^{-5}]^3}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-6} \cdot (\cos\theta + i\sin\theta)^{-6}}{(\cos\theta + i\sin\theta)^{14} \cdot (\cos\theta + i\sin\theta)^{-15}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-12}}{(\cos\theta + i\sin\theta)^{-1}}$$

$$= (\cos\theta + i\sin\theta)^{-11}$$
$$= \cos 11\theta - i\sin 11\theta$$

(5) $$\frac{(\cos 3\theta + i\sin 3\theta)^{-4} \cdot (\cos\theta - i\sin\theta)^5}{(\cos 2\theta - i\sin 2\theta)^6 \cdot (\cos 13\theta + i\sin 13\theta)}$$

$$= \frac{[(\cos\theta + i\sin\theta)^3]^{-4} \cdot [(\cos\theta + i\sin\theta)^{-1}]^5}{[(\cos\theta + i\sin\theta)^{-2}]^6 \cdot (\cos\theta + i\sin\theta)^{13}}$$

$$= \frac{(\cos\theta + i\sin\theta)^{-12} \cdot (\cos\theta + i\sin\theta)^{-5}}{(\cos\theta + i\sin\theta)^{-12} \cdot (\cos\theta + i\sin\theta)^{13}}$$

$$= (\cos\theta + i\sin\theta)^{-18}$$
$$= \cos 18\theta - i\sin 18\theta$$

Ex. 17 : Do as directed :

(1) If $z = \cos\theta + i\sin\theta$, then prove that

$$z^n + \frac{1}{z^n} = 2\cos n\theta \text{ and } z^n - \frac{1}{z^n} = 2i\sin n\theta$$

(GTU : Jan. 2019)

(2) Prove that

$$(1 + \cos\theta + i\sin\theta)^n = 2^n \cdot \cos^n\left(\frac{\theta}{2}\right)\left[\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right]$$

(GTU : March 2018, Dec. 2014)

(3) Prove that

$$\frac{1 - \cos\theta + i\sin\theta}{1 + i\sin\theta + \cos\theta} = i\tan\frac{\theta}{2}(\cos\theta - i\sin\theta)$$

(GTU : June 2014)

(4) Prove that $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = 2^{n+1} \cdot \cos\frac{n\pi}{6}$

(GTU : June 2017)

(5) Prove that $(1 + i)^{2n} + (1 - i)^{2n} = 2^{n+1} \cdot \cos\left(\frac{n\pi}{2}\right)$

(6) If $z = \cos\theta + i\sin\theta$, then prove that $\frac{z^{2n} - 1}{z^{2n} + 1}$

$= i\tan n\theta$**, where** $\theta \neq (2k + 1)\frac{\pi}{2}$, $k \in z$.

Solution :

(1) $$\text{L.H.S.} = z^n + \frac{1}{z^n}$$
$$= z^n + z^{-n}$$
$$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$
$$= 2\cos n\theta$$
$$= \text{R.H.S.}$$

$$\text{L.H.S.} = z^n - \frac{1}{z^n}$$
$$= z^n - z^{-n}$$
$$= (\cos\theta + i\sin\theta)^n - (\cos\theta + i\sin\theta)^{-n}$$
$$= (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$$
$$= \cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta$$
$$= 2i\sin n\theta$$
$$= \text{R.H.S.}$$

(2) $$\text{L.H.S.} = (1 + \cos\theta + i\sin\theta)^n$$

$$= \left(2\cos^2\frac{\theta}{2} + i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n$$

$$= \left[2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)\right]^n$$

$$= 2^n \cos^n \frac{\theta}{2} \cdot \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}\right)^n$$

$$= 2^n \cos^n \frac{\theta}{2} \left(\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2}\right)$$

= R.H.S.

(3) L.H.S. $= \dfrac{1 - \cos\theta + i \sin\theta}{1 + i\sin\theta + \cos\theta}$

$$= \frac{1 - \cos\theta + i\sin\theta}{1 + \cos\theta + i\sin\theta}$$

$$= \frac{2\sin^2 \frac{\theta}{2} + i\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2} + i\, 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$$

$$= \frac{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)}{2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}$$

$$= \tan\frac{\theta}{2} \cdot \frac{\left(-i^2 \sin\frac{\theta}{2} + i\cos\frac{\theta}{2}\right)}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}} \quad (\because i^2 = -1)$$

$$= i\tan\frac{\theta}{2}\left(\frac{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}\right)$$

$$= i\tan\frac{\theta}{2}\, \frac{(\cos\theta + i\sin\theta)^{-\frac{1}{2}}}{(\cos\theta + i\sin\theta)^{\frac{1}{2}}}$$

$$= i\tan\frac{\theta}{2}\,(\cos\theta + i\sin\theta)^{-1}$$

$$= i\tan\frac{\theta}{2}\,(\cos\theta - i\sin\theta)$$

= R.H.S.

(4) L.H.S. $= (\sqrt{3} + i)^n + (\sqrt{3} - i)^n$

$$= \left(2 \cdot \frac{\sqrt{3}}{2} + 2 \cdot \frac{i}{2}\right)^n + \left(2 \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{i}{2}\right)^n$$

(Here, $|\sqrt{3} + i| = 2$)

$$= 2^n\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^n + 2^n\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^n$$

$$= 2^n\left[\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^n + \left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)^n\right]$$

$$= 2^n\left[\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n + \left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^n\right]$$

$$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6} + \cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$$

$$= 2^n\left(2\cos\frac{n\pi}{6}\right)$$

$$= 2^{n+1} \cdot \cos\frac{n\pi}{6}$$

= R.H.S.

(5) L.H.S. = $i)^{2n}[+ (1 - i)^{2n}$

$$= \left(\sqrt{2} \cdot \frac{1}{\sqrt{2}} + \sqrt{2}\,\frac{i}{\sqrt{2}}\right)^{2n}$$

$$+ \left(\sqrt{2} \cdot \frac{1}{\sqrt{2}} - \sqrt{2} \cdot \frac{i}{\sqrt{2}}\right)^{2n}$$

(Here, $|1 + i| = \sqrt{2}$)

$$= \left(\sqrt{2}\right)^{2n}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{2n} + \left(\sqrt{2}\right)^{2n}$$

$$\cdot \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)^{2n}$$

$$= 2^n\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{2n} + 2^n\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{2n}$$

$$\left(\because \left(\sqrt{2}\right)^{2n} = \left[\left(\sqrt{2}\right)^2\right]^n = 2^n\right)$$

$$= 2^n\left[\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{2n} + \left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)^{2n}\right]$$

$$= 2^n\left[\cos\frac{2n\pi}{4} + i\sin\frac{2n\pi}{4} + \cos\frac{2n\pi}{4} - i\sin\frac{2n\pi}{4}\right]$$

$$= 2^n\left(2\cos\frac{n\pi}{2}\right)$$

$$= 2^{n+1}\cdot\cos\frac{n\pi}{2}$$

= R.H.S.

(6) L.H.S. $= \dfrac{z^{2n}-1}{z^{2n}+1}$

$$= \frac{(\cos\theta + i\sin\theta)^{2n} - 1}{(\cos\theta + i\sin\theta)^{2n} + 1}$$

$$= \frac{\cos 2n\theta + i\sin 2n\theta - 1}{\cos 2n\theta + i\sin 2n\theta + 1}$$

$$= \frac{-(1-\cos 2n\theta) + i\sin 2n\theta}{(1+\cos 2n\theta) + i\sin 2n\theta}$$

$$= \frac{-2\sin^2 n\theta + i\cdot 2\sin n\theta\cdot\cos n\theta}{2\cos^2 n\theta + i\cdot 2\sin n\theta\cdot\cos n\theta}$$

$$= \frac{2i^2\sin^2 n\theta + 2i\sin n\theta\cdot\cos n\theta}{2\cos^2 n\theta + 2i\sin n\theta\cdot\cos n\theta}$$

$$= \frac{2i\sin n\theta\,(i\sin n\theta + \cos n\theta)}{2\cos n\theta\,(\cos n\theta + i\sin n\theta)}$$

$= i\tan n\theta$

= R.H.S.

OR

L.H.S. $= \dfrac{z^{2n}-1}{z^{2n}+1}$

$$= \frac{z^n - \frac{1}{z^n}}{z^n + \frac{1}{z^n}}$$

($\because$ Dividing numerator and denominator by z^n)

$$= \frac{2i\sin n\theta}{2\cos n\theta} \qquad \text{(From Ex. 17(i))}$$

$= i\tan n\theta$

= R.H.S.

EXERCISE-5.3

1. **Find square roots of the following complex numbers :**

(1) $5-12i$ (2) $6+2\sqrt{7}\,i$

(3) $\sqrt{3}+i$ (4) $-1+2\sqrt{6}\,i$

(5) $40-42i$ (6) $3-4\sqrt{10}\,i$

(7) $13+20\sqrt{3}\,i$ (8) $-4i$

2. **Determine the modulus and the principal argument of the following complex numbers. Also represent them in polar form :**

(1) $\sqrt{5}$ (2) -16

(3) $7i$ (4) $-\sqrt{3}\,i$

(5) $\sqrt{3}-i$ (6) $-\sqrt{3}+i$

(7) $-1-i$ (8) $\dfrac{1-i}{1+i}$

(9) $\left(\dfrac{2+i}{3-i}\right)^2$ (10) $\dfrac{14-2i}{(3+i)^2}$

3. Find a complex number whose modulus is 4 and the principal argument is $\dfrac{5\pi}{6}$.

4. **Simplify :**

(1) $\dfrac{\cos 7\theta - i\sin 7\theta}{\cos 2\theta + i\sin 2\theta}$

(2) $\left(\dfrac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta - i\sin 2\theta}\right)^2$

(3) $\dfrac{(\cos 2\theta - i\sin 2\theta)^5\cdot(\cos 3\theta + i\sin 6\theta)^6}{(\cos 4\theta + i\sin 4\theta)^7\cdot(\cos\theta - i\sin\theta)^8}$

(4) $\dfrac{(\cos 3\theta + i \sin 3\theta)^{-2} \cdot (\cos 2\theta - i \sin 2\theta)^{3/2}}{(\cos 5\theta - i \sin 5\theta)^{3} \cdot (\cos 2\theta + i \sin 2\theta)^{7}}$

(5) $\dfrac{(\cos 2\theta + i \sin 2\theta)^{-3} \cdot (\cos 3\theta - i \sin 5\theta)^{-5}}{(\cos 4\theta - i \sin 4\theta)^{9} \cdot (\cos 5\theta - i \sin 5\theta)^{-9}}$

5. Do as directed :

(1) Prove that

$$(1 + \cos\theta + i \sin\theta)^n + (1 + \cos\theta - i \sin\theta)^n = 2^{n+1} \cdot \cos^n \frac{\theta}{2} \cdot \cos \frac{n\theta}{2}$$

(2) Prove that $\left(\dfrac{1 + \cos\theta + i \sin\theta}{1 + \cos\theta - i \sin\theta}\right)^n = \cos n\theta + i \sin n\theta$

(3) Prove that $\left(\dfrac{1 + \cos \frac{\pi}{n} + i \sin \frac{\pi}{n}}{1 + \cos \frac{\pi}{n} - i \sin \frac{\pi}{n}}\right)^n = -1$

(4) Prove that $\left(1 + \sqrt{3}i\right)^{6n} + \left(1 - \sqrt{3}i\right)^{6n} = 2^{6n+1}$

(5) Prove that $(1 + i)^{2n} - (1 - i)^{2n} = 2^n i \sin \dfrac{n\pi}{2}$

ANSWERS

1. (1) $\pm (3 - 2i)$

(2) $\pm (\sqrt{7} + i)$

(3) $\pm \left(\dfrac{\sqrt{3} + 1}{2} + i \dfrac{\sqrt{3} - 1}{2}\right)$

(4) $\pm \left(\sqrt{2} + \sqrt{3}i\right)$

(5) $\pm (7 - 3i)$

(6) $\pm \left(2\sqrt{2} - \sqrt{5}i\right)$

(7) $\pm \left(5 + 2\sqrt{3}i\right)$

(8) $\pm \left(\sqrt{2} - \sqrt{2}i\right)$

2. (1) $\sqrt{5}, 0, \sqrt{5} (\cos 0 + i \sin 0)$

(2) $16, \pi, 16(\cos \pi + i \sin \pi)$

(3) $7, \frac{\pi}{2}, 7\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

(4) $\sqrt{3}, -\frac{\pi}{2}, \sqrt{3}\left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right)$

(5) $2, -\frac{\pi}{6}, 2\left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right)$

(6) $2, \frac{5\pi}{6}, 2\left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right)$

(7) $\sqrt{2}, -\frac{3\pi}{4}, \sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i \sin\left(-\frac{3\pi}{4}\right)\right)$

(8) $1, -\frac{\pi}{2}, \cos\left(-\frac{\pi}{2} + i \sin\left(-\frac{\pi}{2}\right)\right)$

(9) $\frac{1}{2}, \frac{\pi}{2}, \frac{1}{2}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right)$

(10) $\sqrt{2}, -\frac{\pi}{2}, \sqrt{2}\left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right)$

3. $-2\sqrt{3} + 2i$

4. (1) $\cos 9\theta - i \sin 9\theta$

(2) $\cos 14\theta + i \sin 14\theta$

(3) $\cos 12\theta - i \sin 12\theta$

(4) $\cos 8\theta - i \sin 8\theta$

(5) $\cos 5\theta + i \sin 5\theta$

Multiple Choice Questions (Solution with Explanation)

(1) $\sqrt{-9}$ = ...

(GTU : June 2014, Dec. 2015, Jan. 2018)

(A) 3 (B) –3 (C) $3i$ (D) $-3i$

Ans. : (C) $3i$

Explanation : $\sqrt{-9} = \sqrt{9} \cdot \sqrt{-1} = 3i$

(2) i^9 = ... (GTU : May 2015, 2018, Nov. 2020)

(A) i (B) $-i$ (C) 1 (D) –1

Ans. : (A) i

Explanation : $i^9 = i^{4(2)+1} = i^{4(2)} \cdot i = (i^4)^2 \cdot i = (1)^2 \cdot i = i$

(3) $i + i^2 + i^3 + i^4$ = ...

(GTU : June 2013, 2017, 2019)

(A) 1 (B) i (C) $-i$ (D) 0

Ans. : (D) 0

Explanation : $i + i^2 + i^3 + i^4 = i + (-1) + i^2 \cdot i + (i^2)^2$

$= i - 1 + (-1)\, i + (-1)^2$

$= i - 1 - i + 1 = 0$

(4) If $3x + 2yi = 6 + 4i$, then x = ... and y = ...

(GTU : June 2019)

(A) 2, 2 (B) 2, 3 (C) 3, 2 (D) 1, 1

Ans. : (A) 2, 2

Explanation : $3x + 2yi = 6 + 4i$

Comparing real and imaginary parts of both sides,

$3x = 6$ and $2y = 4$

$\therefore$ $x = 2$ and $y = 2$

(5) If $x + 4iy = xi + y + 3$, then (x, y) = ...

(GTU : June 2017)

(A) (1, 4) (B) (4, 1) (C) (2, 1) (D) (4, 2)

Ans. : (B) 4, 1

Explanation :

$x + 4iy = xi + y + 3$

$\therefore$ $x + 4yi = (y + 3) + xi$

Comparing real and imaginary parts of both sides,

$x = y + 3$ and $4y = x$

$\therefore$ $4y = y + 3$

$\therefore$ $3y = 3$

$\therefore$ $y = 1$

$\therefore$ $x = y + 3 = 1 + 3 = 4$

Thus, $x = 4$ and $y = 1$

(6) If $z = \frac{3}{5} - \frac{4}{5}i$, then $|z|$ = ...

(GTU : Jan. 2018)

(A) 1 (B) $-\frac{1}{5}$ (C) $\frac{7}{10}$ (D) $-\frac{7}{25}$

Ans. : (A) 1

Explanation : $\left|\frac{3}{5} - \frac{4}{5}i\right| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{-4}{5}\right)^2}$

$= \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}} = 1$

(7) $|(3 - 4i)^2|$ = ... (GTU : May 2016)

(A) 25 (B) 24 (C) 5 (D) 12

Ans. : (A) 25

Explanation :

$|(3 - 4i)^2| = |9 - 24i + 16i^2| = |9 - 24i - 16|$

$= |-7 - 24i|$

$= \sqrt{(-7)^2 + (-24)^2} = \sqrt{49 + 576} = \sqrt{625} = 25$

OR

$|(3 - 4i)^2| = |3 - 4i|^2$ $(\because |z^n| = |z|^n)$

$= \left(\sqrt{(3)^2 + (-4)^2}\right)^2 = 9 + 16 = 25$

(8) Multiplicative inverse of a complex number i is ....

(GTU : Dec. 2019)

(A) i (B) $-i$ (C) 1 (D) –1

Ans. : (B) $-i$

Explanation : Let $z = i$

$\therefore$ $z^{-1} = \frac{1}{z} = \frac{1}{i} = \frac{1}{i} \times \frac{i}{i} = \frac{i}{i^2}$

$\therefore$ $z^{-1} = \frac{i}{-1} = -i$

(9) $(1+i)^{-1} = $ ....

(GTU : June 2013, Nov. 2020)

(A) $\frac{1}{2}+\frac{i}{2}$ (B) $\frac{1}{2}-\frac{i}{2}$

(C) $-\frac{1}{2}+\frac{i}{2}$ (D) $-\frac{1}{2}-\frac{i}{2}$

Ans. : (B) $\frac{1}{2}-\frac{i}{2}$

Explanation : Let $z = 1+i$ $\therefore$ $a = 1$ and $b = 1$

$$\therefore \quad z^{-1} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i = \frac{1}{1+1} - \frac{1}{1+1}i = \frac{1}{2}-\frac{1}{2}i$$

OR

$$z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{1-i}{\left(\sqrt{1^2+1^2}\right)^2} = \frac{1-i}{2} = \frac{1}{2}-\frac{1}{2}i$$

(10) Multiplicative inverse of a complex number $5-4i$ is ....

(GTU : May 2016)

(A) $\frac{-5-4i}{9}$ (B) $\frac{5+4i}{41}$

(C) $\frac{5-4i}{41}$ (D) $\frac{-5+4i}{41}$

Ans. : (B) $\frac{5+4i}{41}$

Explanation : Let, $z = 5-4i$

$\therefore$ $\bar{z} = 5+4i$ and $|z| = \sqrt{25+16} = \sqrt{41}$

$$\therefore \quad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{5+4i}{\left(\sqrt{41}\right)^2} = \frac{5+4i}{41} = \frac{5}{41}+\frac{4}{41}i$$

(11) If $z^2 = -i$, then $z = $ .... **(GTU : Dec. 2019)**

(A) $\frac{1}{\sqrt{2}}(1+i)$ (B) $\pm\frac{1}{\sqrt{2}}(1+i)$

(C) $\pm\frac{1}{\sqrt{2}}(1-i)$ (D) $\frac{1}{\sqrt{2}}(1-i)$

Ans. : (C) $\pm\frac{1}{\sqrt{2}}(1-i)$

Explanation : $z = -i = 0-i$

$\therefore$ $x = 0,\ y = -1$ and $|z| = \sqrt{0+1} = \sqrt{1} = 1$

Since $y = -1 < 0$,

Square roots of $z = -i$ are

$$\pm\left(\sqrt{\frac{|z|+x}{2}} - i\sqrt{\frac{|z|-x}{2}}\right)$$

$$= \pm\left(\sqrt{\frac{1+0}{2}} - i\sqrt{\frac{1-0}{2}}\right)$$

$$= \pm\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)$$

(12) If $z = -5i$, then arg $(z) = $ .... **(GTU : June 2014)**

(A) $\frac{\pi}{2}$ (B) $-\frac{\pi}{2}$ (C) 0 (D) π

Ans. : (B) $-\frac{\pi}{2}$

Explanation : $z = -5i = 0-5i$, where $-5 < 0$

$\therefore$ $\theta = arg(z) = -\frac{\pi}{2}$

(13) arg (35) = ... **(GTU : Dec. 2015, Nov. 2020)**

(A) 0 (B) π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{3}$

Ans. : (A) 0

Explanation : $z = 35 = 35+0i$, where $35 > 0$

$\therefore$ $\theta = arg\,(z) = 0$

(14) arg(–1) = ... **(GTU : Jan. 2017)**

(A) π (B) $-\pi$ (C) 0 (D) $\frac{\pi}{2}$

Ans. : (A) π

Explanation : $z = -1 = -1+0i$, where $-1 < 0$

$\therefore$ $\theta = arg\,(z) = \pi$

(15) If $z = 2i$, then $\arg(z) =$ .... **(GTU : Jan. 2019)**

(A) $-\frac{\pi}{2}$ (B) $\frac{\pi}{2}$ (C) π (D) 0

Ans. : (B) $\frac{\pi}{2}$

Explanation : $z = 2i = 0 + 2i$, where $2 > 0$

$\therefore \quad \theta = arg(z) = \frac{\pi}{2}$

(16) If $z = 1 - i\sqrt{3}$, then arg $(z) =$ ....

(GTU : June 2017, Dec. 2014)

(A) $\frac{\pi}{6}$ (B) $-\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $-\frac{\pi}{3}$

Ans. : (D) $-\frac{\pi}{3}$

Explanation : $z = 1 - \sqrt{3}\,i$

$\therefore \quad x = 1 > 0$ and $y = -\sqrt{3} < 0$

$\therefore \quad r = |z| = \sqrt{1+3} = \sqrt{4} = 2$

and $\tan\alpha = \left|\frac{y}{x}\right| = \left|\frac{-\sqrt{3}}{1}\right| = \sqrt{3} \quad \therefore \alpha = \frac{\pi}{3}$

Since $x > 0$ and $y < 0$, the principal argument of z is

$\theta = arg\,(z) = -\alpha = -\frac{\pi}{3}$

(17) $(\cos\theta + i\sin\theta)^4 + (\cos\theta + i\sin\theta)^{-4} =$ ....

(GTU : Dec. 2014)

(A) $2\sin 4\theta$ (B) $2i\sin 4\theta$

(C) $2i\cos 4\theta$ (D) $2\cos 4\theta$

Ans. : (D) $2\cos 4\theta$

Explanation :

$(\cos\theta + i\sin\theta)^4 + (\cos\theta + i\sin\theta)^{-4}$

$= (\cos 4\theta + i\sin 4\theta) + (\cos 4\theta - i\sin 4\theta)$

$= 2\cos 4\theta$

(18) If $z = \cos\theta + i\sin\theta$, then $z^n - (\bar{z})^n =$ ....

(A) $2\cos n\theta$ (B) $2i\cos n\theta$

(C) $2i\sin n\theta$ (D) $2\sin n\theta$

Ans. : (C) $2i\sin n\theta$

Explanation :

$z = \cos\theta + i\sin\theta \qquad \therefore \bar{z} = \cos\theta - i\sin\theta$

$\therefore \quad z^n - (\bar{z})^n = (\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n$

$= (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$

$= \cos n\theta + i\sin\theta - \cos n\theta + i\sin n\theta$

$= 2\,i\sin n\theta$

Multiple Choice Questions (For Self-Study)

(1) $\sqrt{-4} =$ .... **(GTU : May 2016)**

(A) 2 (B) -2 (C) $2i$ (D) $-2i$

(2) $i^8 =$ .... **(GTU : Jan. 2019)**

(A) -1 (B) 1 (C) i (D) $-i$

(3) $i^{11} =$ .... **(GTU : March 2021)**

(A) -1 (B) 1 (C) i (D) $-i$

(4) $(2 + 3i)(3 - 2i) =$ .... **(GTU : June 2014)**

(A) $-12 + 5i$ (B) $12 - 5i$

(C) $12 + 5i$ (D) $-12 - 5i$

(5) $(1 + i)^2 =$ .... **(GTU : Jan. 2018)**

(A) $2 + 2i$ (B) $2i$

(C) 2 (D) $2 - 2i$

(6) $\frac{1-i}{1+i} =$ .... **(GTU : May 2015)**

(A) 1 (B) -1 (C) i (D) $-i$

(7) If $z = 3i - 2$, then $\bar{z} =$ .... **(GTU : May 2016)**

(A) $3i + 2$ (B) $-2 - 3i$

(C) $2 - 3i$ (D) $-2 + 3i$

(8) If $z = 5 - 2i$, then $\bar{z} =$ ....

(GTU : June 2017, May 2018)

(A) $-5 + 2i$ (B) $-5 - 2i$

(C) $5 + 2i$ (D) none of these

(9) If $z = 5i - 3$, then $\bar{z}$ = ... **(GTU : Dec. 2013)**

(A) $3i - 5$ (B) $5i + 3$

(C) $-5i - 3$ (D) none of these

(10) If $z = \sqrt{3} - i$, then $\bar{z}$ = ...

(GTU : Jan. 2019, Dec. 2019)

(A) $-\sqrt{3} - i$ (B) $\sqrt{3} + i$

(C) $-\sqrt{3} + i$ (D) $\dfrac{1}{\sqrt{3} - i}$

(11) If $z = 3 + 2i$, then $\bar{z}$ = ...

(GTU : March 2021)

(A) $2 + 3i$ (B) $3 - 2i$

(C) $-2 - 3i$ (D) $2 - 3i$

(12) The conjugate complex number of $\dfrac{2-i}{2+i}$ is ....

(GTU : Jan. 2017)

(A) $\dfrac{4+3i}{5}$ (B) $\dfrac{4-3i}{5}$

(C) $\dfrac{3+4i}{5}$ (D) $\dfrac{3-4i}{5}$

(13) If $z = \cos\theta + i\sin\theta$, then $z + \bar{z}$ = ...

(GTU : June 2014, May 2015)

(A) -1 (B) $\cos\theta - i\sin\theta$

(C) $2\cos\theta$ (D) $2i\sin\theta$

(14) $z + \bar{z}$ = ... **(GTU : June 2013, Jan. 2019, Nov. 2020, March 2021)**

(A) $-2\text{Re}(z)$ (B) $\text{Re}(z)$

(C) $2\text{Re}(z)$ (D) $2i\,\text{Im}(z)$

(15) $z - \bar{z}$ = ... **(GTU : Jan. 2018)**

(A) $\text{Im}(z)$ (B) $i\,\text{Im}(z)$

(C) $2\text{Im}(z)$ (D) $2i\,\text{Im}(z)$

(16) If $z = 3 - 4i$, then $|z|$ = ...

(GTU : June 2019, March 2021)

(A) -5 (B) 25

(C) 5 (D) 0

(17) If $|\bar{z}| = 16$, then $|z|$ = ...

(GTU : May 2018, Dec. 2015)

(A) 16 (B) 4

(C) 256 (D) 1

(18) If $z_1 = 3 - 2i$ and $z_2 = -3 - 2i$, then $|z_1 + z_2|$ = ...

(GTU : June 2017)

(A) 1 (B) 5

(C) 0 (D) $\sqrt{5}$

(19) If $z_1 = 3 - 2i$ and $z_2 = -2 + 2i$, then $|z_1 + z_2|$ = ...

(GTU : Dec. 2015)

(A) 0 (B) 1

(C) $\sqrt{5}$ (D) $\sqrt{5}$

(20) Multiplicative inverse of a complex number $3 + 4i$ is ....

(GTU : Dec. 2013)

(A) $\dfrac{3+4i}{5}$ (B) $\dfrac{3-4i}{25}$

(C) $\dfrac{3+4i}{25}$ (D) $\dfrac{4+3i}{25}$

(21) An Argument of $1 + i$ = ... **(GTU : May 2015)**

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{3\pi}{4}$

(C) $\dfrac{5\pi}{4}$ (D) $\dfrac{7\pi}{4}$

(22) If $z = \cos\theta + i\sin\theta$, then $z^3 + \dfrac{1}{z^3}$ = ...

(GTU : June 2017, Dec. 2019)

(A) $2\cos 3\theta$ (B) $2i\cos 3\theta$

(C) $2i\sin 3\theta$ (D) $2\sin 3\theta$

ANSWERS

(1) C	(2) B	(3) D	(4) C	(5) B
(6) D	(7) B	(8) C	(9) C	(10) B
(11) B	(12) C	(13) C	(14) C	(15) D
(16) C	(17) A	(18) A	(19) B	(20) B
(21) A	(22) A			

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ગુજરાત ટેક્નોલોજીકલ યુનિવર્સિટી
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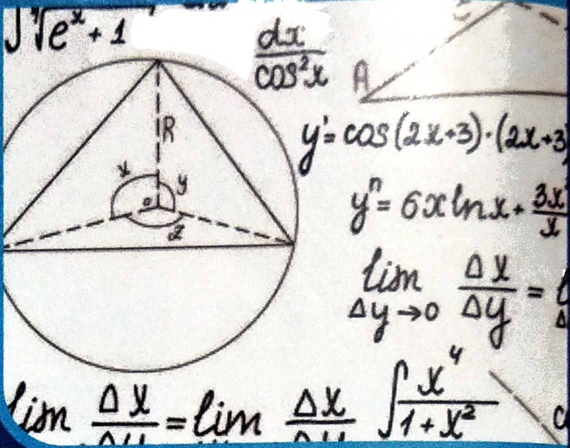
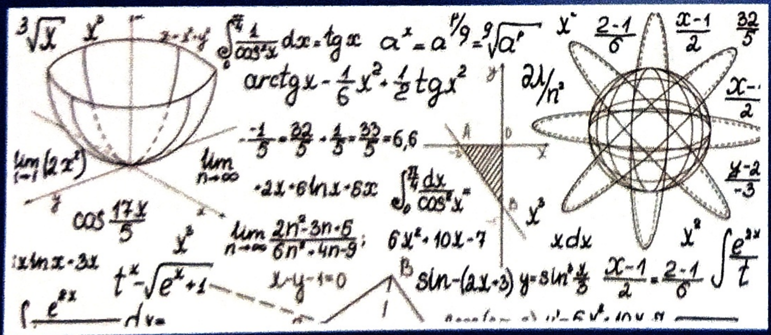


Semester-II

Sr. No.	Subject
1	Physics
2	Indian Constitution
3	Static Web Page Design
4	Engineering Mathematics
5	Electronic and Computer Workshop Practice
6	Basic Object Oriented Programming
7	Basics of Digital Electronics



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