additional terms to account for convection. To solve a one-dimensional convection–diffusion problem we write discretised equations of the form (5.14) for all grid nodes. This yields a set of algebraic equations that is solved to obtain the distribution of the transported property ϕ. The process is now illustrated by means of a worked example.

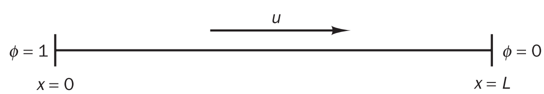
Example 5.1

A property ϕ is transported by means of convection and diffusion through the one-dimensional domain sketched in Figure 5.2. The governing equation is (5.3); the boundary conditions are $\phi_0 = 1$ at $x = 0$ and $\phi_L = 0$ at $x = L$. Using five equally spaced cells and the central differencing scheme for convection and diffusion, calculate the distribution of ϕ as a function of x for (i) Case 1: u = 0.1 m/s, (ii) Case 2: u = 2.5 m/s, and compare the results with the analytical solution

$$\frac{\phi - \phi_0}{\phi_L - \phi_0} = \frac{\exp(\rho u x/\Gamma) - 1}{\exp(\rho u L/\Gamma) - 1} \tag{5.15}$$

(iii) Case 3: recalculate the solution for u = 2.5 m/s with 20 grid nodes and compare the results with the analytical solution. The following data apply: length L = 1.0 m, ρ = 1.0 kg/m^3, Γ = 0.1 kg/m.s.

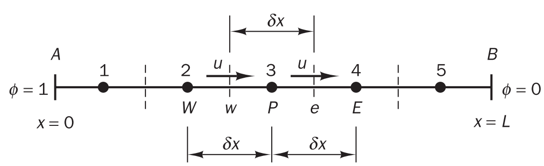
Figure 5.2



Solution

The method of solution is demonstrated using the simple grid shown in Figure 5.3. The domain has been divided into five control volumes giving δx = 0.2 m. Note that $F = \rho u$, $D = \Gamma/\delta x$, $F_e = F_w = F$ and $D_e = D_w = D$ everywhere. The boundaries are denoted by A and B.

Figure 5.3 The grid used for discretisation



The discretisation equation (5.14) and its coefficients apply at internal nodal points 2, 3 and 4, but control volumes 1 and 5 need special treatment since they are adjacent to the domain boundaries. We integrate governing equation (5.3) and use central differencing for both the diffusion terms and the convective flux through the east face of cell 1. The value of ϕ is given at the west face of this cell ($\phi_w = \phi_A = 1$) so we do not need to make any approximations in the convective flux term at this boundary. This yields the following equation for node 1:

$$\frac{F_e}{2}(\phi_P + \phi_E) - F_A\phi_A = D_e(\phi_E - \phi_P) - D_A(\phi_P - \phi_A) \tag{5.16}$$

For control volume 5, the ϕ-value at the east face is known ($\phi_e = \phi_B = 0$). We obtain

$$F_B\phi_B - \frac{F_w}{2}(\phi_P + \phi_W) = D_B(\phi_B - \phi_P) - D_w(\phi_P - \phi_W) \tag{5.17}$$

Rearrangement of equations (5.16) and (5.17), noting that $D_A = D_B = 2\Gamma/\delta x = 2D$ and $F_A = F_B = F$, gives discretised equations at boundary nodes of the following form:

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u} \tag{5.18}$$

with central coefficient

$$\boxed{a_P = a_W + a_E + (F_e - F_w) - S_P}$$

and

Node	a_W	a_E	S_P	S_u
1	0	$D - F/2$	$-(2D + F)$	$(2D + F)\phi_A$
2, 3, 4	$D + F/2$	$D - F/2$	0	0
5	$D + F/2$	0	$-(2D - F)$	$(2D - F)\phi_B$

To introduce the boundary conditions we have suppressed the link to the boundary side and entered the boundary flux through the source terms.

(i) Case 1
$u = 0.1$ m/s: $F = \rho u = 0.1$, $D = \Gamma/\delta x = 0.1/0.2 = 0.5$ gives the coefficients as summarised in Table 5.1.

Table 5.1

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	0.45	$1.1\phi_A$	−1.1	1.55
2	0.55	0.45	0	0	1.0
3	0.55	0.45	0	0	1.0
4	0.55	0.45	0	0	1.0
5	0.55	0	$0.9\phi_B$	−0.9	1.45

The matrix form of the equation set using $\phi_A = 1$ and $\phi_B = 0$ is

$$\begin{bmatrix} 1.55 & -0.45 & 0 & 0 & 0 \\ -0.55 & 1.0 & -0.45 & 0 & 0 \\ 0 & -0.55 & 1.0 & -0.45 & 0 \\ 0 & 0 & -0.55 & 1.0 & -0.45 \\ 0 & 0 & 0 & -0.55 & 1.45 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.19}$$

The solution to the above system is

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 0.9421 \\ 0.8006 \\ 0.6276 \\ 0.4163 \\ 0.1579 \end{bmatrix} \tag{5.20}$$

Comparison with the analytical solution

Substitution of the data into equation (5.15) gives the exact solution of the problem:

$$\phi(x) = \frac{2.7183 - \exp(x)}{1.7183}$$

The numerical and analytical solutions are compared in Table 5.2 and in Figure 5.4. Given the coarseness of the grid the central differencing (CD) scheme gives reasonable agreement with the analytical solution.

Table 5.2

Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentage error*
1	0.1	0.9421	0.9387	−0.003	−0.36
2	0.3	0.8006	0.7963	−0.004	−0.53
3	0.5	0.6276	0.6224	−0.005	−0.83
4	0.7	0.4163	0.4100	−0.006	−1.53
5	0.9	0.1579	0.1505	−0.007	−4.91

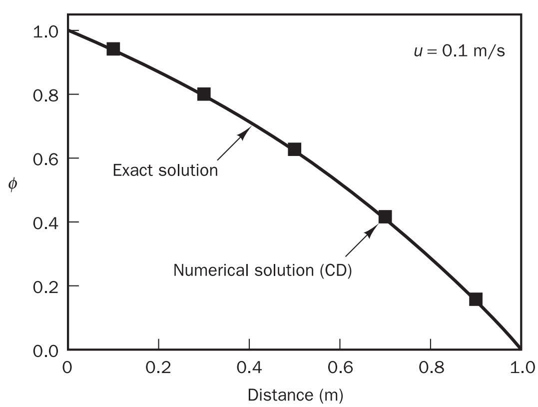


Figure 5.4 Comparison of numerical and analytical solutions for Case 1

(ii) Case 2

$u = 2.5$ m/s: $F = \rho u = 2.5$, $D = \Gamma/\delta x = 0.1/0.2 = 0.5$ gives the coefficients as summarised in Table 5.3.

Table 5.3

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	−0.75	$3.5\phi_A$	−3.5	2.75
2	1.75	−0.75	0	0	1.0
3	1.75	−0.75	0	0	1.0
4	1.75	−0.75	0	0	1.0
5	1.75	0	$-1.5\phi_B$	1.5	0.25

Comparison of numerical and analytical solution

The matrix equations are formed from the coefficients in Table 5.3 by the same method used in Case 1 and subsequently solved. The analytical solution for the data that apply here is

$$\phi(x) = 1 + \frac{1 - \exp(25x)}{7.20 \times 10^{10}}$$

The numerical and analytical solutions are compared in Table 5.4 and shown in Figure 5.5. The central differencing scheme produces a solution that appears to oscillate about the exact solution. These oscillations are often called 'wiggles' in the literature; the agreement with the analytical solution is clearly not very good.

Table 5.4

Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentage error*
1	0.1	1.0356	1.0000	–0.035	–3.56
2	0.3	0.8694	0.9999	0.131	13.05
3	0.5	1.2573	0.9999	–0.257	–25.74
4	0.7	0.3521	0.9994	0.647	64.70
5	0.9	2.4644	0.9179	–1.546	–168.48

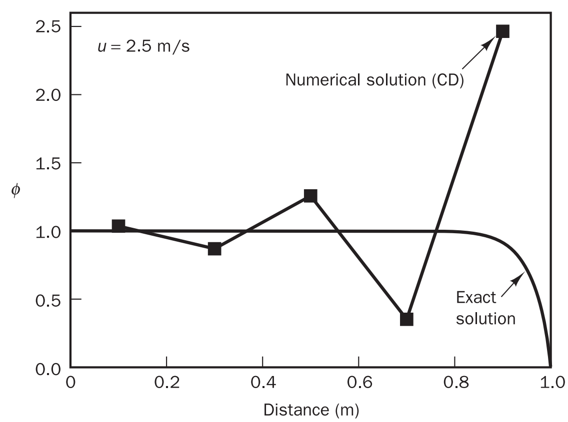


Figure 5.5 Comparison of numerical and analytical solutions for Case 2

(iii) Case 3

u = 2.5 m/s: a grid of 20 nodes gives $\delta x = 0.05$, $F = \rho u = 2.5$, $D = \Gamma/\delta x = 0.1/0.05 = 2.0$. The coefficients are summarised in Table 5.5 and the resulting solution is compared with the analytical solution in Figure 5.6.

Table 5.5

Node	a_W	a_E	S_u	S_P	$a_P = a_W + a_E - S_P$
1	0	0.75	$6.5\phi_A$	–6.5	7.25
2–19	3.25	0.75	0	0	4.00
20	3.25	0	$1.5\phi_B$	–1.5	4.75

or

$$[D_w + (D_e - F_e) + (F_e - F_w)]\phi_P = D_w\phi_W + (D_e - F_e)\phi_E \tag{5.30}$$

Identifying the coefficients of ϕ_W and ϕ_E as a_W and a_E, equations (5.27) and (5.30) can be written in the usual general form

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E} \tag{5.31}$$

with central coefficient

$$\boxed{a_P = a_W + a_E + (F_e - F_w)}$$

and neighbour coefficients

	a_W	a_E
$F_w > 0, F_e > 0$	$D_w + F_w$	D_e
$F_w < 0, F_e < 0$	D_w	$D_e - F_e$

A form of notation for the **neighbour coefficients of the upwind differencing method** that covers both flow directions is given below:

a_W	a_E
$D_w + \max(F_w, 0)$	$D_e + \max(0, -F_e)$

Example 5.2

Solve the problem considered in Example 5.1 using the upwind differencing scheme for (i) $u = 0.1$ m/s, (ii) $u = 2.5$ m/s with the coarse five-point grid.

Solution

The grid shown in Figure 5.3 is again used here for the discretisation. The discretisation equation at internal nodes 2, 3 and 4 and the relevant neighbour coefficients are given by (5.31) and its accompanying tables. Note that in this example $F = F_e = F_w = \rho u$ and $D = D_e = D_w = \Gamma/\delta x$ everywhere.

At the boundary node 1, the use of upwind differencing for the convective terms gives

$$F_e\phi_P - F_A\phi_A = D_e(\phi_E - \phi_P) - D_A(\phi_P - \phi_A) \tag{5.32}$$

And at node 5

$$F_B\phi_P - F_w\phi_W = D_B(\phi_B - \phi_P) - D_w(\phi_P - \phi_W) \tag{5.33}$$

At the boundary nodes we have $D_A = D_B = 2\Gamma/\delta x = 2D$ and $F_A = F_B = F$, and as usual the boundary conditions enter the discretised equations as source contributions:

$$\boxed{a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u} \tag{5.34}$$

with

$$\boxed{a_P = a_W + a_E + (F_e - F_w) - S_P}$$

and

Node	a_W	a_E	S_P	S_u
1	0	D	$-(2D+F)$	$(2D+F)\phi_A$
2, 3, 4	$D+F$	D	0	0
5	$D+F$	0	$-2D$	$2D\phi_B$

The reader will by now be familiar with the process of calculating coefficients and constructing and solving the matrix equation. For the sake of brevity we leave this as an exercise and concentrate on the evaluation of the results. The analytical solution is again given by equation (5.15) and is compared with the numerical, upwind differencing, solution.

Case 1
$u = 0.1$ m/s: $F = \rho u = 0.1$, $D = \Gamma/\delta x = 0.1/0.2 = 0.5$ so $Pe = F/D = 0.2$. The results are summarised in Table 5.6, and Figure 5.12 shows that the upwind differencing (UD) scheme produces good results at this cell Peclet number.

Table 5.6

Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentage error*
1	0.1	0.9337	0.9387	0.005	0.53
2	0.3	0.7879	0.7963	0.008	1.05
3	0.5	0.6130	0.6224	0.009	1.51
4	0.7	0.4031	0.4100	0.007	1.68
5	0.9	0.1512	0.1505	–0.001	–0.02

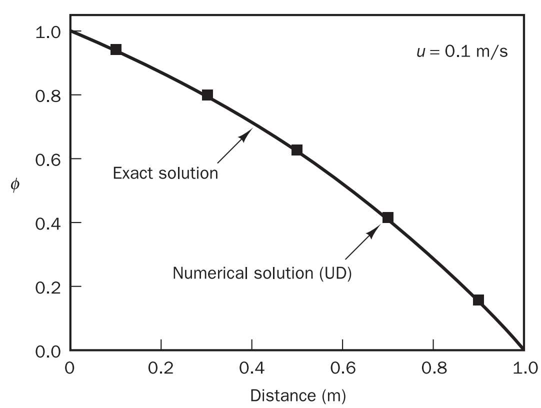


Figure 5.12 Comparison of the upwind difference numerical results and the analytical solution for Case 1

Case 2
$u = 2.5$ m/s: $F = \rho u = 2.5$, $D = \Gamma/\delta x = 0.1/0.2 = 0.5$ now $Pe = 5$. The numerical results are compared with the analytical solution in Table 5.7 and Figure 5.13.

The central differencing scheme failed to produce a reasonable result with the same grid resolution. The upwind scheme produces a much more realistic solution that is, however, not very close to the exact solution near boundary B.

Table 5.7

Node	*Distance*	*Finite volume solution*	*Analytical solution*	*Difference*	*Percentage error*
1	0.1	0.9998	1.0000	0.0002	0.02
2	0.3	0.9987	0.9999	0.001	0.13
3	0.5	0.9921	0.9999	0.008	0.79
4	0.7	0.9524	0.9994	0.047	4.71
5	0.9	0.7143	0.9179	0.204	22.18

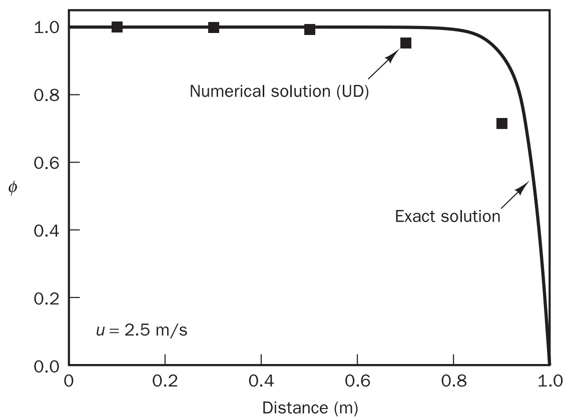


Figure 5.13 Comparison of the upwind difference numerical results and the analytical solution for Case 2

5.6.1 Assessment of the upwind differencing scheme

Conservativeness: The upwind differencing scheme utilises consistent expressions to calculate fluxes through cell faces: therefore it can be easily shown that the formulation is conservative.

Boundedness: The coefficients of the discretised equation are always positive and satisfy the requirements for boundedness. When the flow satisfies continuity the term $(F_e - F_w)$ in a_P (see (5.31)) is zero and gives $a_P = a_W + a_E$, which is desirable for stable iterative solutions. All the coefficients are positive and the coefficient matrix is diagonally dominant, hence no 'wiggles' occur in the solution.

Transportiveness: The scheme accounts for the direction of the flow so transportiveness is built into the formulation.

Accuracy: The scheme is based on the backward differencing formula so the accuracy is only first-order on the basis of the Taylor series truncation error (see Appendix A).

Because of its simplicity the upwind differencing scheme has been widely applied in early CFD calculations. It can be easily extended to