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Scaling of Differential Equations

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Preface

Finding proper values of physical parameters in mathematical models is often quite a challenge. While many have gotten away with using just the mathematical symbols when doing science and engineering with pen and paper, the modern world of numerical computing requires each physical parameter to have a numerical value, otherwise one cannot get started with the computations. For example, in the simplest possible transient heat conduction simulation, a case relevant for a real physical material needs values for the heat capacity, the density, and the heat conduction coefficient of the material. In addition, relevant values must be chosen for initial and boundary temperatures as well as the size of the material. With a dimensionless mathematical model, as explained in Chapter 3.2, *no physical quantities* need to be assigned (!). Not only is this a simplification of great convenience, as one simulation is valid for any type of material, but it also actually increases the understanding of the physical problem.

Scaling of differential equations is basically a simple mathematical process, consisting of the chain rule for differentiation and some algebra. The *choice of scales*, however, is a non-trivial topic, which may cause confusion among practitioners without extensive experience with scaling. How to choose scales is unfortunately not well treated in the literature. Most of the times, authors just state scales without proper motivation. The choice of scales is highly problem-dependent and requires knowledge of the characteristic features of the solution or the physics of the problem. The present book aims at explaining "all nuts and bolts" of the scaling technique, including choice of scales, the algebra, the interpretation of dimensionless parameters in scaled models, and how scaling impacts software for solving differential equations.

Traditionally, scaling was mainly used to identify small parameters in mathematical models, such that perturbation methods based on series expansions in terms of the small parameters could be used as an approximate solution method for differential equations. Nowadays, the greatest practical benefit of scaling is related to running numerical simulations, since scaling greatly simplifies the choice of values for the input data and makes the sim-

ulations results more widely applicable. The number of parameters in scaled models may be much less than the number of physical parameters in the original model. The parameters in scaled models are also dimensionless and express *ratios* of physical effects rather than the levels of individual effects. Setting meaningful values of a few dimensionless numbers is much easier than determining physically relevant values for the original physical parameters.

Another great benefit of scaling is the physical insight that follows from dimensionless parameters. Since physical effects enter the problem through a few dimensionless groups, one can from these groups see how different effects compete in their impact on the solution. Ideally, a good physical understanding should provide the same insight, but it is not always easy to "think right" and realize how spatial and temporal scales interact with physical parameters. This interaction becomes clear through the dimensionless numbers, and such numbers are therefore a great help, especially for students, in developing a correct physical understanding.

Since we have a special focus on scaling related to numerical simulations, the book contains a lot of examples on how to program with dimensionless differential equation models. Most numerical models feature quantities with dimension, so we show in particular how to utilize such existing models to solve the equations in the associated scaled model.

Scaling is not a universal mathematical technique as the details depend on the problem at hand. We therefore present scaling in a range of specific applications, starting with simple ODEs, progressing with basic PDEs, before attacking more complicated models, especially from fluid mechanics.

Chapter 1 discusses units and how to make programs that can automatically take care of unit conversion (the most frequent mathematical mistake in industry and science?). Section 2.1 introduces the mathematics of scaling and the thinking about scales in a simple ODE problem modeling exponential decay. The ideas are generalized to nonlinear ODEs and to systems of ODEs. Another ODE example on mechanical vibrations is treated in Section 2.2, where we cover many different physical contexts and different choices of scales. Scaling the standard, linear wave equation is the topic of Chapter 3.1, with discussion of how boundary and initial conditions influence the choice of scales. Another PDE example, the diffusion equation, appears in Chapter 3.2. Here we progress from a simple linear diffusion equation in 1D to the impact on scales of an oscillating boundary condition. Nonlinear diffusion models as well as convection-diffusion PDEs are also discussed. The final Chapter is devoted to many famous PDEs arising from continuum models: elasticity, viscous fluid flow, thermal convection, etc.

Experimental fluid mechanics is a field full of relations involving dimensionless numbers such as the Grashof and Prandtl numbers, but none of the textbooks the authors have seen explain how these numbers actually relate to dimensionless forms of the governing equations. Consequently, this non-trivial topic is particularly highlighted in the fluid mechanics examples.

The mathematics in the first two chapters is very gentle and requires no more background than basic one-variable calculus and preferably some knowledge of differential equation models. The next chapter involves PDEs and assumes familiarity with basic models for wave phenomena, diffusion, and combined convection-diffusion. The final chapter is meant for readers with knowledge of the physics and mathematics of continuum mechanical models. The mathematical level of the text rises quickly after the first two chapters.

The pedagogy is to saturate the reader with lots of detailed examples to provide an understanding for the topic, primarily because the choice of scales depends on the problem at hand. One can also view the book as a reference on how to scale many of the most important differential equation models in physics. For the simpler differential equations in Chapters 2 and 3, we present computer code for many computational examples, but the treatment of the advanced models in Chapter 4 is more superficial to limit the size of that chapter.

The exercises are named either Exercise or Problem. The latter is a stand-alone exercise without reference to the rest of the text, while the former typically extends a topic in the text or refers to sections or formulas in the text.

To conclude, the purpose of this book is to demystify the techniques of scaling and motivate the use of dimensionless differential equations before performing numerical simulations.

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hpl 1: Thank proof readers, Springer, others, ...

Oslo, November 2015 *Hans Petter Langtangen, Geir K. Pedersen*

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Chapter 1
Dimensions and units

A mechanical system undergoing one-dimensional damped vibrations can be modeled by the equation

$$mu'' + bu' + ku = 0, \tag{1.1}$$

where m is the mass of the system, b is some damping coefficient, k is a spring constant, and $u(t)$ is the displacement of the system. This is an equation expressing the balance of three physical effects: mu'' (mass times acceleration), bu' (damping force), and ku (spring force). The different physical quantities, such as m, $u(t)$, b, and k, have all different *dimensions*, measured in different *units*, but mu'', bu', and ku must all have the same dimension, otherwise it would not make sense to add them (you will get the same problem as when trying to add one banana, one apple, and one orange).

1.1 Fundamental concepts

1.1.1 Base units and dimensions

There are seven fundamental (SI) *base units* and corresponding physical quantities: meter (m) for length, kilogram (kg) for mass, second (s) for time, kelvin (K) for temperature, ampere (A) for electric current, candela (cd) for luminous intensity, mole (mol) for the amount of substance.

The dimension of length is written as [L], the dimension of mass is [M], the dimension of time is [T], and the dimension of temperature is $[\Theta]$ (the dimensions of the other base units are omitted in this introduction). The dimension of a *derived unit* like velocity, which is distance (length) divided by time, then becomes $[\mathrm{LT}^{-1}]$. The dimension of force, another derived unit, is the same as the dimension of mass times acceleration, and hence the dimension of force is $[\mathrm{MLT}^{-2}]$.

Let us find the dimensions of the terms in (1.1). A displacement $u(t)$ has dimension [L]. The derivative $u'(t)$ is change of displacement, which has dimension [L], divided by a time interval, which has dimension [T], implying that the dimension of u' is LT^{-1}. This results coincides with the interpretation of u' as velocity and the fact that velocity is defined as distance ([L]) per time ([T]).

Looking at (1.1), and interpreting $u(t)$ as displacement, we realize that the term mu'' (mass times acceleration) has dimension $[\mathrm{MLT}^{-2}]$. The term bu' must have the same dimension, and since u' has dimension $[\mathrm{LT}^{-1}]$, b must have dimension $[\mathrm{MT}^{-1}]$. Finally, ku must also have dimension $[\mathrm{MLT}^{-1}]$, implying that k is a parameter with dimension $[\mathrm{MT}^{-2}]$.

The unit of a physical quantity follows from the dimension expression. For example, since velocity has dimension LT^{-1} and length is measured in m while time is measured in s, the unit for velocity becomes m/s. Similarly, force has dimension $[\mathrm{MLT}^{-2}]$ and unit kg m/s^2. The k parameter in (1.1) is measured in kg s^{-2}.

Dimension of derivatives

The easiest way to realize the dimension of a derivative, is to express the derivative as a finite difference. For a function $u(t)$ we have

$$\frac{du}{dt} \approx \frac{u(t+\Delta t)-u(t)}{\Delta t},$$

where Δt is a small time interval. If u denotes a velocity, its dimension is $[\mathrm{LT}]^{-1}$, and $u(t+\Delta t)-u(t)$ gets the same dimension. The time interval has dimension [T], and consequently, the finite difference gets the dimension $[\mathrm{LT}]^{-2}$. In general, the dimension of the derivative du/dt is the dimension of u divided by the dimension of t.

1.1.2 Dimensions of common physical quantities

Many derived quantities are measured in derived units. Force is one example: Newton (N) is a derived unit for force, equal to kg m/s^2. Another derived unit is Pascal (Pa) for pressure and stress, i.e., force per area. The unit of Pa then equals N/m^2 or kg/ms^2. Below are more derived quantities and their units.

Name	Symbol	Physical quantity	unit
radian	rad	angle	1
hertz	Hz	frequency	s^{-1}
newton	N	force, weight	kg m/s^2
pascal	Pa	pressure, stress	N/m^2
joule	J	energy, work, heat	Nm
watt	W	power	J/s

Some common physical quantities and their dimensions are listed next.

hpl 2: TODO: Check the tables.

Quantity	relation	unit	dimension
stress	force/area	$N/m^2 = Pa$	$[MT^{-2}L^{-1}]$
pressure	force/area	$N/m^2 = Pa$	$MT^{-2}L^{-1}]$
density	mass/volume	kg/m^3	$[ML^{-3}]$
strain	displacement/length	1	[1]
Young's modulus	stress/strain	$N/m^2 = Pa$	$MT^{-2}L^{-1}]$
Poisson's ratio	transverse strain/axial strain	1	[1]
Lame' parameters λ and μ	stress/strain	$N/m^2 = Pa$	$MT^{-2}L^{-1}]$
moment (of a force)	distance $\times$ force	Nm	$[ML^2T^{-2}]$
impulse	force $\times$ time	Ns	$[MLT^{-1}]$
linear momentum	mass $\times$ velocity	kg m/s	$[MLT^{-1}]$
angular momentum	distance $\times$ mass $\times$ velocity	kg m^2/s	$[ML^2T^{-1}]$
work	force $\times$ distance	$Nm = J$	$[ML^2T^{-2}]$
energy	work	$Nm = J$	$[ML^2T^{-2}]$
power	work/time	$Nm/s = W$	$[ML^2T^{-3}]$
heat	work	J	$[ML^2T^{-2}]$
heat flux	heat rate/area	Wm^{-2}	$[MT^{-3}]$
temperature	base unit	K	$[\Theta]$
heat capacity	heat change/temperature change	J/K	$[ML^2T^{-2}\Theta^{-1}]$
specific heat capacity	heat capacity/unit mass	$JK^{-1}kg^{-1}$	$[L^2T^{-2}\Theta^{-1}]$
thermal conductivity	heat flux/temperature gradient	$Wm^{-1}K^{-1}$	$[MLT^{-3}\Theta^{-1}]$
dynamic viscosity	shear stress/velocity gradient	$kgm^{-1}s^{-1}$	$[ML^{-1}T^{-1}]$
kinematic viscosity	dynamic viscosity/density	$[m^2/s$	$L^2T^{-1}]$
surface tension	energy/area	$[J/m^2$	$MT^{-2}]$

Prefixes for units. Units often have prefixes. For example, kilo is a prefix for 1000, so kg is 1000 g. Similarly, GPa means giga pascal or 10^9 Pa.

1.1.3 The Buckingham Pi theorem

Almost all texts on scaling has a treatment of the famous Buckingham Pi theorem, which can be used to derive physical laws based on unit compatibility rather than the underlying physical mechanisms. This book has its focus

on models where the physical mechanisms are already expressed through differential equations. Nevertheless, the Pi theorem has a remarkable position in the literature on scaling, and we will occasionally make references to it so the theorem is briefly discussed below.

The theorem itself is simply stated in two parts. First, if a problem involves n physical parameters in which m independent unit-types (such as length, mass etc.) appear, then the parameters can be combined to exactly $n-m$ independent dimensionless numbers, referred to as Pi's. Secondly, any unit-free relation between the original n parameters can be transformed into a relation between the $n-m$ dimensionless numbers. Such relations may be identities or inequalities stating, for instance, whether or not a given effect is negligible. Moreover, the transformation an equation set into dimensionless form corresponds to expressing the coefficients, as well as the free and dependent variables, in terms of Pi's.

At first glance the Pi theorem may appear as bordering on the trivial. However, it may produce remarkable progress for selected problems, such as turbulent jets, nuclear blasts or similarity solutions, without the detailed knowledge of mathematical or physical models. Hence, to a novice in scaling it may stand out as something very profound, if not magical. Anyhow, as one move on to more complex problems with many parameters the use of the theorem yields comparatively less gain as the number of Pi's become large and they may recombined in countless ways. It then requires either a good physical insight or information conveyed through an equation set to pick the useful dimensionless numbers or the appropriate scaling of the said equation set. Sometimes the scrutiny of the equations also reveals that some of Pi's, obtained by applying the theorem, in fact may be removed from the problem. As a consequence, in mathematical modeling of a complex physical problem the real assessment of scaling and dimensionless numbers will anyhow be an integral part of the analysis of the governing equations instead of a separate issue left with the Pi theorem. In textbooks and articles alike the discussion of scaling in the context of the equations are too often missing or presented in an half-hearted fashion. Hence, herein the focus will be on this process, while we do not provide much in the way of examples on the Pi theorem. We do not allude that the Pi theorem is of little value. In a number of contexts, such as in experiments, it may provide valuable and even crucial guidance, but in this particular textbook we seek to tell the complementary story on scaling. Moreover, as will be shown in this book, the dimensionless numbers in a problem also arise, in a very natural way, from scaling the differential equation(s). Provided one has a model based on differential equation, there is actually no need for classical dimensional analysis.

1.1.4 Absolute errors, relative errors, and units

Mathematically, it does not matter what units we use for a physical quantity. However, when we deal with approximations and errors, units are important. Suppose we work with a geophysical problem where the length scale is typically measured in km and we have an approximation 12.5 km to the exact value 12.52 km. The error is then 0.02 km. Switching units to mm leads to an error of 20,000 mm. A program working in mm would report $2 \cdot 10^5$ as the error, while a program working in km would print 0.02. The absolute error is therefore sensitive to the choice of units. This fact motivates for the use of the *relative error*: (exact - approximate)/exact since then the unit cancels. In the present example, one gets a relative error of $1.6 \cdot 10^{-3}$ regardless of the whether the length is measured in km or mm.

Nevertheless, rather than relying solely on relative errors, it is in general better to scale the problem such that the quantities entering the computations are of unit size (or at least moderate) instead of being very large or very small. The techniques of this appendix show how this can be done.

1.1.5 Units and computers

Traditional numerical computing involves numbers only and therefore require dimensionless mathematical expressions. Usually, an implicit trivial scaling is used. One can, for example, just scale all length quantities by 1 m, all time quantities by 1 s, and all mass quantities by 1 kg, to obtain the dimensionless numbers needed for calculations. This is the most common approach, although it is very seldom explicitly stated.

Symbolic computing packages, such as Mathematica and Maple, allow computations with quantities that have dimension. This is also possible in popular computer languages used for numerical computing (Section 1.1.8 provides a specific example in Python).

1.1.6 Unit systems

Confusion arises quickly when some physical quantities are expressed in SI units while others are in US or British units. Density could, for instance, be given in unit of ounce per teaspoon (see Exercise 2.1 for how to safely convert to a standard unit like $\mathrm{kg\,m^{-3}}$). Although unit conversion tables are frequently met in school, errors in unit conversions between units probably rank highest among all errors committed by scientists and engineers (and when a unit conversion error make an airplane's fuel run out, it is serious!). Having good software tools to assist in unit conversion is therefore paramount,

and this topic is treated in Sections 1.1.8 and 1.2. Readers who are primarily interested in the mathematical scaling technique may safely jump over this material and continue with Section 2.1.

1.1.7 Example on challenges arising from unit systems

A slightly elaborated example on scaling in an actual science/engineering project may hopefully be useful for further motivation. In its full extent, the study of *tsunamis* spans geophysics, geology, history, fluid dynamics, statistics, geodesy, engineering, and civil protection. This complexity reflects in a diversity of practices concerning the use of units, scales, and concepts. If we narrow the scope to the modeling of tsunami propagation, the scaling aspect, at least, may seem simple as we are mainly concerned with length and time. Still, even here the non-uniformity concerning physical units is an encumbrance.

A minor issue is the occasional use of non-SI units such as inches, or in old charts, even fathoms. More important is the non-uniformity in the magnitude of the different variables, and the differences in the inherent horizontal and vertical scales in particular. Typically surface elevations are meters or smaller. For far-field deep water propagation as well as small tsunamis, which are still of scientific interest, surface elevations are often given in cm or even mm. In the deep ocean the characteristic depth is orders of magnitude larger than this, typically 5000 m. Propagation distances, on the other hand, are hundreds or thousands of kilometers. Often locations and computational grids are best described in geographical coordinates (longitude/latitude) which are related to SI units by 1 latitude minute being roughly one nautical mile (1852 m), and 1 longitude minute being this quantity times the cosine of the latitude. Wave periods of tsunamis are mostly from minutes to an hour, hopefully sufficiently short to be well separated from the half-daily period of the tides. Propagation times are typically hours or maybe the better part of a day when the Pacific Ocean is traversed.

The scientists, engineers, and bureaucrats in the tsunami community tend to be particular and non-conform concerning formats and units, as well as the type of data required. To accommodate these demands, a tsunami modeler must produce a diversity of data which are in units and formats which cannot be used internally in her models. On the other hand, she must also be prepared to accept the input data in diversified forms. Some data sets may be large, implying that unnecessary duplication, with different units or scaling, should be avoided. In addition, tsunami models are often bench-marked through comparison with experimental data. The lab scale is generally cm or m, at most, which implies that measured data are provided in different units than real events, or even in volts, with conversion information, as obtained from the measuring gauges.

All the unit particulars in various file formats is clearly a nuisance and give rise to a number of misconceptions and errors that may cause loss of precious time or efforts. To reduce such problems, developers of computational tools should combine a reasonable flexibility concerning units in input and output with a clear and consistent convention for scaling within the tools. In fact, this is also goes for academic tools for in-house use.

The discussion above points to some best practices that this book promotes. First, always compute with scaled differential equation models. This book tells you how to do that. Second, users of software often want to specify input data with dimension and get output data with dimension. The software should then apply tools like `PhysicalQuantity` (Section 1.1.8) or the more sophisticated Parampool package (Section 1.2) to allow input with explicit dimensions and convert the dimensions to the right types if necessary. It is trivial to apply these tools if the computational software is written in Python, but it is even straightforward if the software is written in compiled languages like Fortran, C, or C++. In the latter case one just makes an input reading module in Python that grabs data from a user interface and feeds them into the computational software, either through files or function calls (the relevant functions to be called must be wrapped in Python with tools like f2py, Cython, Weave, SWIG, Instant, or similar, see [7, Appendix C] for basic examples on f2py and Cython wrapping of C and Fortran code).

1.1.8 PhysicalQuantity: a tool for computing with units

This book contains quite some computer code to illustrate how the theory maps in detail to running software. Python is the programming language used, primarily because it is an easy-to-read, powerful, full-fledged language that can be made very close to popular MATLAB-like code as well as class-based code typically used in Java, C#, and C++. The Python ecosystem for scientific computing has in recent years grown fast in popularity and acts as a replacement for more specialized tools like MATLAB, R, and IDL. The coding examples in this book requires only familiarity with basic procedural programming in Python.

Readers without knowledge of Python variables, functions, if tests, and module import should consult, e.g., a brief tutorial on scientific Python, the Python Scientific Lecture Notes, or a full textbook [3] in parallel with reading about Python code in the present book.

This book applies Python 2.7

Python exists in two incompatible versions, numbered 2 and 3. The differences can be made small, and there are tools to write code that runs under both versions.

As Python version 2 is still dominating in scientific computing, we stick to this version, but write code in version 2.7 that is as close as possible to version 3.4 and later. In most of our programs, only the `print` statement differs between version 2 and 3.

hpl 3: Geir, is this sufficient intro about Python? Should we be more specific about the expected competence? **Geir 4**: hpl, no this will do.

Computations with units in Python are well supported by the `PhysicalQuantity` object from the ScientificPython package by Konrad Hinsen. Unfortunately, ScientificPython does not, at the time of this writing, work with NumPy version 1.9 or later, so we have isolated the `PhysicalQuantity` object in a module `PhysicalQuantities` and made it publicly available on GitHub. There is also an alternative package Unum for computing with numbers with units, but we shall stick to the former module here.

Let us demonstrate the usage of the `PhysicalQuantity` object by computing $s = vt$, where v is a velocity given in the unit *yards per minute* and t is time measured in hours. First we need to know what the units are called in `PhysicalQuantities`. To this end, run `pydoc PhysicalQuantities`, or

Terminal

```
Terminal> pydoc Scientific.Physics.PhysicalQuantities
```

if you have entire ScientificPython package installed. The resulting documentation shows the names of the units. In particular, yards are specified by `yd`, minutes by `min`, and hours by `h`. We can now compute $s = vt$ as follows:

```
>>> # With ScientificPython:
>>> from Scientific.Physics.PhysicalQuantities import \
... PhysicalQuantity as PQ
>>> # With PhysicalQuantities as separate/stand-alone module:
>>> from PhysicalQuantities import PhysicalQuantity as PQ
>>>
>>> v = PQ('120 yd/min')   # velocity
>>> t = PQ('1 h')          # time
>>> s = v*t                # distance
>>> print s                # s is string
120.0 h*yd/min
```

The odd unit `h*yd/min` is better converted to a standard SI unit such as meter:

```
>>> s.convertToUnit('m')
>>> print s
6583.68 m
```

Note that `s` is a `PhysicalQuantity` object with a value and a unit. For mathematical computations we need to extract the value as a `float` object. We can also extract the unit as a string:

```
>>> print s.getValue()     # float
6583.68
>>> print s.getUnitName()  # string
m
```

Here is an example on how to convert the odd velocity unit yards per minute to something more standard:

```
>>> v.convertToUnit('km/h')
>>> print v
6.58368 km/h
>>> v.convertToUnit('m/s')
>>> print v
1.8288 m/s
```

As another example on unit conversion, say you look up the specific heat capacity of water to be 1 $\mathrm{cal\,g^{-1}K^{-1}}$. What is the corresponding value in the standard unit $\mathrm{Jg^{-1}K^{-1}}$ where joule replaces calorie?

```
>>> c = PQ('1 cal/(g*K)')
>>> c.convertToUnit('J/(g*K)')
>>> print c
4.184 J/K/g
```

1.2 Parampool: user interfaces with automatic unit conversion

The Parampool package allows creation of user interfaces with support for units and unit conversion. Values of parameters can be set as a number with a unit. The parameters can be registered beforehand with a preferred unit, and whatever the user prescribes, the value and unit are converted so the unit becomes the registered unit. Parampool supports various type of user interfaces: command-line arguments (option-value pairs), text files, and interactive web pages. All of these are described next.

Example application. As case, we want to make software for computing with the simple formula $s = v_0 t + \frac{1}{2}at^2$. We want v_0 to be a velocity with unit m/s, a to acceleration with unit $\mathrm{m/s^2}$, t to be time measured in s, and consequently s is a distance measured in m.

1.2.1 Pool of parameters

First, Parampool requires us to define a *pool* of all input parameters, which is here simply represented by list of dictionaries, where each dictionary holds information about one parameter. It is possible to organize input parameters in a tree structure with subpools having subpools, but for our simple application we just need a flat structure with three input parameters: v_0, a, and t. These parameters are put in a subpool called "Main". The pool is created by the code

```
def define_input():
    pool = [
        'Main', [
            dict(name='initial velocity', default=1.0, unit='m/s'),
            dict(name='acceleration', default=1.0, unit='m/s**2'),
            dict(name='time', default=10.0, unit='s')
            ]
        ]

    from parampool.pool.UI import listtree2Pool
    pool = listtree2Pool(pool)  # convert list to Pool object
    return pool
```

For each parameter we can define a logical name, such as `initial velocity`, a default value, and a unit. Additional properties are also allowed, see the Parampool documentation.

Tip: specify default values of numbers as float objects

Note that we write 1.0 as default value and not just 1. In the latter case, Parampool will interpret that our parameter is an integer and actually convert input like `2.5 m/s` to `2 m/s`. To ensure that a real-valued parameter becomes a `float` object inside the pool, specify the default value as a real number: `1.` or `1.0`. (The type of an input parameter can alternatively be explicitly set by the `str2type` property, e.g., `str2type=float`.)

1.2.2 Fetching pool data for computing

We can make a little function for fetching values from the pool and computing s:

```
def distance(pool):
    v_0 = pool.get_value('initial velocity')
    a = pool.get_value('acceleration')
    t = pool.get_value('time')
```

```
    s = v_0*t + 0.5*a*t**2
    return s
```

The `pool.get_value` function returns the numerical value of the named parameter, after the unit has been converted from what the user has specified to what was registered in the pool. For example, if the user provides the command-line argument `-time '2 h'`, Parampool will convert this quantity to seconds and `pool.get_value('time')` will return 7200.

1.2.3 Reading command-line options

To run the computations, we define the pool, load values from the command line, and call `distance`:

```
pool = define_input()
from parampool.menu.UI import set_values_from_command_line
pool = set_values_from_command_line(pool)

s = distance(pool)
print 's=%g' % s
```

Parameter names with whitespace must use an underscore for whitespace in the command-line option, such as in `--Initial_velocity`. We can now run

Terminal

```
Terminal> python distance.py --initial_velocity '10 km/h' \
          --acceleration 0 --time '1 h
s=10000
```

Notice from the answer (`s`) that 10 km/h gets converted to m/s and 1 h to s.

It is also possible to fetch parameter values as `PhysicalQuantity` objects from the pool through calling

```
v_0 = pool.get_value_unit('Initial velocity')
```

The following variant of the `distance` function computes with values and units:

```
def distance_unit(pool):
    """Compute distance $s = v_0t + \frac{1}{2}at^2$. (DocOnce)"""
    # Compute with units
    from parampool.PhysicalQuantities import PhysicalQuantity as PQ
    v_0 = pool.get_value_unit('initial velocity')
    a = pool.get_value_unit('acceleration')
    t = pool.get_value_unit('time')
    s = v_0*t + 0.5*a*t**2
    return s.getValue(), s.getUnitName()
```

We can then do

```
s, s_unit = distance_unit(pool)
print 's=%g' % s, s_unit
```

and get output with the right unit as well.

1.2.4 Setting default values in a file

In large applications with lots of input parameters one will often like to define a (huge) set of default values specific for a case and then override a few of them on the command-line. Such sets of default values can be set in a file using syntax like

```
subpool Main
initial velocity = 100 ! yd/min
acceleration = 0 ! m/s**2         # drop acceleration
end
```

The unit can be given after the `!` symbol (and before the comment symbol `#`).

To read such files we have to add the lines

```
from parampool.pool.UI import set_defaults_from_file
pool = set_defaults_from_file(pool)
```

before the call to `set_defaults_from_command_line`.

If the above commands are stored in a file `distance.dat`, we give this file information to the program through the option `-poolfile distance.dat`. Running just

Terminal

```
Terminal> python distance.py --poolfile distance.dat
s=15.25 m
```

first loads the velocity 100 yd/min converted to 1.524 m/s and zero acceleration into the pool system and, and then we call `distance_unit`, which loads these values from the pool along with the default value for time, set as 10 s. The calculation is then $s = 1.524 \cdot 10 + 0 = 15.24$ with unit m. We can override the time and/or the other two parameters on the command line:

Terminal

```
Terminal> python distance.py --poolfile distance.dat --time '2 h'
s=10972.8 m
```

The resulting calculations are $s = 1.524 \cdot 7200 + 0 = 10972.8$. You are encouraged to play around with the `distance.py` program.

1.2.5 Specifying multiple values of input parameters

Parampool has an interesting feature: multiple values can be assigned to an input parameter, thereby making it easy for an application to run through all combinations of all parameters. We can demonstrate this feature by making a table of v_0, a, t, and s values. In the compute function, we need to call `pool.get_values` instead of `pool.get_value` to get a list of all the values that were specified for the parameter in question. By nesting loops over all parameters, we visit all combinations of all parameters as specified by the user:

```
def distance_table(pool):
    """Grab multiple values of parameters from the pool."""
    table = []
    for v_0 in pool.get_values('initial velocity'):
        for a in pool.get_values('acceleration'):
            for t in pool.get_values('time'):
                s = v_0*t + 0.5*a*t**2
                table.append((v_0, a, t, s))
    return table
```

In case just a single value was specified for a parameter, `pool.get_values` returns this value only and there will be only one pass in the associated loop.

After loading command-line arguments into our `pool` object, we can call `distance_table` instead of `distance` or `distance_unit` and write out a nicely formatted table of results:

```
table = distance_table(pool)
print '|-----------------------------------------------------|'
print '|      v_0   |      a     |      t     |      s       |'
print '|-----------------------------------------------------|'
for v_0, a, t, s in table:
    print '|%11.3f | %10.3f | %10.3f | %12.3f |' % (v_0, a, t, s)
print '|-----------------------------------------------------|'
```

Here is a sample run,

Terminal

```
Terminal> python distance.py --time '1 h & 2 h & 3 h' \
          --acceleration '0 m/s**2 & 1 m/s**2 & 1 yd/s**2' \
          --initial_velocity '1 & 5'
|-----------------------------------------------------|
|      v_0   |      a     |      t     |      s       |
|-----------------------------------------------------|
|      1.000 |      0.000 |   3600.000 |     3600.000 |
|      1.000 |      0.000 |   7200.000 |     7200.000 |
|      1.000 |      0.000 |  10800.000 |    10800.000 |
|      1.000 |      1.000 |   3600.000 |  6483600.000 |
|      1.000 |      1.000 |   7200.000 | 25927200.000 |
|      1.000 |      1.000 |  10800.000 | 58330800.000 |
|      1.000 |      0.914 |   3600.000 |  5928912.000 |
|      1.000 |      0.914 |   7200.000 | 23708448.000 |
|      1.000 |      0.914 |  10800.000 | 53338608.000 |
|      5.000 |      0.000 |   3600.000 |    18000.000 |
|      5.000 |      0.000 |   7200.000 |    36000.000 |
```

```
|      5.000 |      0.000 |  10800.000 |    54000.000 |
|      5.000 |      1.000 |   3600.000 |  6498000.000 |
|      5.000 |      1.000 |   7200.000 | 25956000.000 |
|      5.000 |      1.000 |  10800.000 | 58374000.000 |
|      5.000 |      0.914 |   3600.000 |  5943312.000 |
|      5.000 |      0.914 |   7200.000 | 23737248.000 |
|      5.000 |      0.914 |  10800.000 | 53381808.000 |
|-----------------------------------------------------|
```

Notice that some of the multiple values have dimensions different from the registered dimension for that parameter, and the table shows that conversion to the right dimension has taken place.

1.2.6 Generating a graphical user interface

For the fun of it, we can easily generate a graphical user interface via Parampool. We wrap the `distance_unit` function in a function that returns the result in some nice-looking HTML code:

```
def distance_unit2(pool):
    # Wrap result from distance_unit in HTML
    s, s_unit = distance_unit(pool)
    return '<b>Distance:</b> %.2f %s' % (s, s_unit)
```

In addition, we must make a file `generate_distance_GUI.py` with the simple content

```
from parampool.generator.flask import generate
from distance import distance_unit2, define_input

generate(distance_unit2, pool_function=define_input, MathJax=True)
```

Running `generate_distance_GUI.py` creates a Flask-based web interface[1] to our `distance_unit` function, see Figure 1.1. The text fields in this GUI allow specification of parameters with numbers and units, e.g., acceleration with unit yards per minute squared, as shown in the figure. Hovering the mouse slightly to the left of the text field causes a little black window to pop up with the registered unit of that parameter.

With examples shown above, you should be able to utilize the `PhysicalQuantity` object and the Parampool package in your programs to work safely with units. For the coming text, where we discuss the craft of scaling in detail, we shall just work in standard SI units and avoid unit conversion so there will be no more use of `PhysicalQuantity` and Parampool.

[1] You need to have Flask and additional packages installed. This is easy to do with a few `pip install` commands, see [5] or [6].

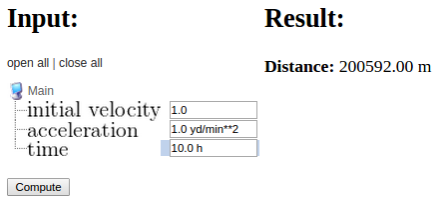


Fig. 1.1 Web GUI where parameters can be specified with units.

Chapter 2
ODE models

This chapter introduces the basic techniques of scaling and the ways to reason about scales. The first class of examples targets exponential decay models, starting with the simple ordinary differential equation (ODE) for exponential decay processes: $u' = -au$, with constant $a > 0$. Then we progress to various generalizations of this ODE, including nonlinear versions and systems of ODEs. The next class of examples concerns second-order ODEs for oscillatory systems, where the simplest ODE reads $mu'' + ku = 0$, with m and k as positive constants. Various extensions with damping and force terms are discussed in detail.

2.1 Exponential decay problems

2.1.1 Fundamental ideas

Scaling is an extremely useful technique in mathematical modeling and numerical simulation. The purpose of the technique is three-fold:

1. Make independent and dependent variables dimensionless.
2. Make the size of independent and dependent variables about unity.
3. Reduce the number of independent physical parameters in the model.

The first and second item mean that for any variable, denote it by q, we introduce a corresponding dimensionless variable

$$\bar{q} = \frac{q - q_0}{q_c},$$

where q_0 is a reference value of q ($q_0 = 0$ is a common choice) and q_c is a characteristic size of $|q|$. Since the numerator and denominator have the same dimension, $\bar{q}$ becomes a dimensionless number.

If q_c is the maximum value of $|q - q_0|$, we see that $0 < |\bar{q}| \leq 1$. How to find q_c is sometimes the big challenge of scaling. Examples will illustrate various approaches to meet this challenge.

The forthcoming text has the following goals.

- Teach the technical steps of making a mathematical model, based on differential equations, dimensionless.
- Describe various techniques for reasoning about the scales, i.e., finding the characteristic sizes of quantities.
- Teach how to identify and interpret dimensionless numbers arising from the scaling process.
- Provide a lot of different examples on making models dimensionless with physically correct scales.
- Demonstrate software tools for computing with numbers with units, including doing unit conversions.
- Introduce software tools for creating user interfaces that can automatically perform unit conversion.
- Use symbolic software (SymPy) to derive exact solutions of differential equations.
- Explain how to run a dimensionless model with software developed for the problem with dimensions.

What this book is and is not

Books covering scaling and non-dimensionalization very often cover topics not treated in the present book. One key topic is dimensional analysis and the famous Buckingham Pi Theorem [1, 9].

Other classical topics closely related to scaling and non-dimensionalization are analytical solution methods like perturbation techniques and similarity solutions. These are not addressed in this book. There are numerous texts on perturbation techniques, and these build on an already scaled differential equations. Similarity solutions do not fit within the present scope since these involve non-dimensional *combinations of the independent variables* to derive new differential equations that are easier to solve.

Our scope is to scale differential equations to simplify the setting of parameters in numerical simulations, and at the same time understand more of the physics through interpretation of the dimensionless numbers that automatically arise from the scaling procedure.

2.1.2 The basic model problem

Processes undergoing exponential reduction can be modeled by the ODE problem

$$u'(t) = -au(t), \quad u(0) = I, \tag{2.1}$$

where $a, I > 0$ are prescribed constants and $u(t)$ is the unknown function. For this particular model, we can easily derive the solution, $u(t) = Ie^{-at}$, which is helpful to have in mind during the scaling process.

Example: Population dynamics. The evolution of a population of humans, animals, cells, etc., under unlimited access to resources, can be modeled by (2.1). Then u is the number of individuals in the population, strictly speaking an integer, but well modeled by a real number in large populations. The parameter a is the increase in the number of individuals per time and per individual.

Example: Decay of pressure with altitude. The simple model (2.1) also governs the pressure in the atmosphere (under many assumptions, such as an ideal gas in equilibrium). In this case u is the pressure, measured in Nm^{-2}; t is the hight in meters; and $a = M/(R^*T)$, where M is the molar mass of the Earth's air (0.029 kg/mol), R^* is the universal gas constant ($8.314\,\frac{\mathrm{Nm}}{\mathrm{mol\ K}}$), and T is the temperature in Kelvin (K). The temperature depends on the hight so we have $a = a(t)$.

2.1.3 The technical steps of the scaling procedure

Step 1: Identify independent and dependent variables. There is one independent variable, time t, and one dependent variable, u.

Step 2: Make independent and dependent variables dimensionless. We introduce a new dimensionless t, called $\bar{t}$, defined by

$$\bar{t} = \frac{t}{t_c}, \tag{2.2}$$

where t_c is a *characteristic value* of t. Similarly, we introduce a dimensionless u, named $\bar{u}$, according to

$$\bar{u} = \frac{u}{u_c}, \tag{2.3}$$

where u_c is a constant *characteristic size* of u. When u has a specific interpretation, say when (2.1) models pressure in an atmospheric layer, u_c would be referred to as characteristic pressure. For a decaying population, u_c may be a characteristic number of members in the population, e.g., the initial population I.

Step 3: Derive the model involving only dimensionless variables. The next task is to insert the new dimensionless variables in the governing mathematical model. That is, we replace t by $t_c\bar{t}$ and u by $u_c\bar{u}$ in (2.1). The derivative with respect to $\bar{t}$ is derived as

$$\frac{du}{dt} = \frac{d(u_c\bar{u})}{d\bar{t}}\frac{d\bar{t}}{dt} = u_c\frac{d\bar{u}}{d\bar{t}}\frac{1}{t_c} = \frac{u_c}{t_c}\frac{d\bar{u}}{d\bar{t}}\,.$$

The model (2.1) now becomes

$$\frac{u_c}{t_c}\frac{d\bar{u}}{d\bar{t}} = -au_c\bar{u}, \quad u_c\bar{u}(0) = I\,. \tag{2.4}$$

Step 4: Make each term dimensionless. Equation (2.4) still has terms with dimensions. To make each term dimensionless, we usually divide by the coefficient in front of the term with the highest time derivative (but dividing by any coefficient in any term will do). The result is

$$\frac{d\bar{u}}{d\bar{t}} = -at_c\bar{u}, \quad \bar{u}(0) = u_c^{-1}I\,. \tag{2.5}$$

Step 5: Estimate the scales. A characteristic quantity like t_c reflects the time scale in the problem. Estimating such a time scale is certainly the most challenging part of the scaling procedure. There are different ways to reason. The first is to aim at a size of $\bar{u}$ and its derivatives that is of order unity. If u_c is chosen such that $|\bar{u}|$ is of size unity, we see from (2.5) that $d\bar{u}/d\bar{t}$ is of the size of $\bar{u}$ (i.e., unity) if we choose $t_c = 1/a$.

Alternatively, we may look at a special case of the model where we have analytical insight. In the present problem we are lucky to know the exact solution for any value of the input data. For exponential decay, $u(t) \sim e^{-at}$, it is common to define a characteristic time scale t_c as the time it takes to reduce u by a factor of $1/e$ (also called the *e-folding time*):

$$e^{-at_c} = \frac{1}{e}e^{-a\cdot 0} \quad \Rightarrow \quad e^{-at_c} = e^{-1},$$

from which it follows that $t_c = 1/a$.

In this example, two different, yet common ways of reasoning, lead to the same value of t_c. However, instead of using the e-folding time we could use the half-time of the exponential decay as characteristic time, which is also a very common measure of the time scale in such processes. The half time is defined as the time it takes to halve u:

$$e^{-at_c} = \frac{1}{2}e^{-a\cdot 0} \quad \Rightarrow \quad t_c = a^{-1}\ln 2\,.$$

There is a factor $\ln 2 = 0.69$ difference from the other t_c value. As long as the factor is not an order of magnitude or more different, we do not pay attention to such small differences. Although $t_c = a^{-1}\ln 2$ is a fine time scale to be used in this problem, it leads to a scaled differential equation $u' = -(\ln 2)u$, which

is fine, but an unusual form. People tend to prefer the simpler ODE $u' = -u$, which arises from $t_c = 1/a$, and we shall therefore use this time scale.

Regarding u_c, we may look at the initial condition and realize that the choice $u_c = I$ makes $\bar{u}(0) = 1$. For $\bar{t} > 0$ we know that $\bar{u}$ is decreasing, so $u_c = I$ gives us $\bar{u} \leq 1$, which is always a goal (alternatively, we may look to analytical insight, $u(t) = Ie^{-at}$, to see that $u \leq I$, such that $u_c = I$ gives $\bar{u} \leq 1$).

With $t_c = 1/a$ and $u_c = I$, we have the final dimensionless model

$$\frac{d\bar{u}}{d\bar{t}} = -\bar{u}, \quad \bar{u}(0) = 1\,. \tag{2.6}$$

This is a remarkable result in the sense that *all physical parameters* (a and I) are removed from the model! Or more precisely, there are no physical input parameters to assign before using the model. In particular, numerical investigations of the original model (2.1) would need experiments with different a and I values, while numerical investigations of (2.6) can be limited to *a single run*! As soon as we have computed the curve $\bar{u}(\bar{t})$, we can find the solution $u(t)$ of (2.1) by

$$u(t) = u_c\bar{u}(t/t_c) = I\bar{u}(at)\,. \tag{2.7}$$

This particular transformation actually means stretching the $\bar{t}$ and $\bar{u}$ axes in a plot of $\bar{u}(\bar{t})$ by the factors a and I, respectively.

It is very common to drop the bars when the scaled problem has been derived and work further with (2.6) simply written as

$$\frac{du}{dt} = -u, \quad u(0) = 1\,.$$

Nevertheless, in this book we have decided to stick to bars for all dimensionless quantities.

2.1.4 Making software for utilizing the scaled model

Software for solving (2.1) could take advantage of the fact that only one simulation of (2.6) is necessary. As soon as we have $\bar{u}(\bar{t})$ accessible, a simple scaling (2.7) computes the real $u(t)$ for any given input data a and I. Although the numerical computation of $u(t)$ from (2.1) is very fast in this simple model problem, using (2.7) is very much faster than computing a full numerical solution in more complicated differential equation problems.

We can compute with the dimensionless model (2.6) in two ways, either make a solver for (2.6) or reuse a solver for (2.1) with the parameters appropriately set ($I = 1$, $a = 1$). The latter approach has the advantage of giving

us software that works both with a dimensionless model and a model with dimensions and all the original physical parameters.

Software for the original problem with dimensions. We base our solver for (2.6) on a solver for (2.1). Assume that we have some module `decay.py` that offers the following functions:

- `solver(I, a, T, dt, theta=0.5)` for returning the solution arrays `u` and `t` for (2.1) solved by the so-called θ rule. This rule includes the Forward Euler scheme ($\theta = 0$), the Backward Euler scheme ($\theta = 1$), or the Crank-Nicolson (centered midpoint) scheme ($\theta = \frac{1}{2}$).
- `read_command_line_argparse()` for reading parameters in the problem from the command line and returning them: `I`, `a`, `T`, `theta` (θ), and a list of Δt values for time steps. (We shall only make use of the first Δt value.)

The basic statements for solving (2.1) are then

```
from decay import solver, read_command_line_argparse
I, a, T, theta, dt_values = read_command_line_argparse()
u, t = solver(I, a, T, dt_values[0], theta)

from matplotlib.pyplot import plot, show
plot(t, u)
show()
```

The module `decay.py` is developed and explained in Section 5.1.7 in [4].

To solve the dimensionless problem, just fix $I = 1$ and $a = 1$:

```
I, a, T, theta, dt_values = read_command_line_argparse()
u, t = solver(I=1, a=1, T=T, dt=dt_values[0], theta=theta)
```

Avoiding unnecessary computations. A key observation, as mentioned, is that we need to solve the problem (2.6) only once. All solutions corresponding to different I and a values in the original physical problem can be recovered by scaling this single solution with formula (2.7). We may therefore want to make software that takes advantage of this fact. When requesting a solution, we see if it has already been computed and stored in a file, and if so, the data can be retrieved from file, otherwise we have to compute a new solution and store it in a file.

A very plain solution to the problem is found in the file `decay_scaled_v1.py`. The `np.savetxt` function saves a two-dimensional array ("table") to a text file, and the `np.loadtxt` function can load the data back into the program. A better solution to this problem is obtained by using the `joblib` package as described next.

Implementation with joblib. The Python package `joblib` has functionality that is very convenient for implementing the `solver_scaled` function. The first time a function is called with a set of arguments, the statements in the function are executed and the return value is saved to file. If the function is called again with the same set of arguments, the statements in the function

are not executed, but the return value is read from file. In computer science, one would say that `joblib` in this way provides *memorization* functionality for Python functions. This functionality is particularly aimed at large-scale computations with arrays that one would like to avoid being recomputed. We illustrate the technique here in a very simple mathematical context.

First we make a `solver_scaled` function for the scaled model that just calls up a `solver_unscaled` for the problem with dimensions:

```
from decay import solver_unscaled
import numpy as np
import matplotlib.pyplot as plt

def solver_scaled(T, dt, theta):
    """
    Solve u'=-u, u(0)=1 for (0,T] with step dt and theta method.
    """
    print 'Computing the numerical solution'
    return solver_unscaled(I=1, a=1, T=T, dt=dt, theta=theta)
```

Then we create some "computer memory on disk", i.e., some disk space to store the result of a call to the `solver_scaled` function. Thereafter, we redefine the name `solver_scaled` to a new function, created by `joblib`, which calls our original `solver_scaled` function if necessary and otherwise loads data from file:

```
import joblib
disk_memory = joblib.Memory(cachedir='temp')
solver_scaled = disk_memory.cache(solver_scaled)
```

The solutions are actually stored in files in the cache directory `temp`.

A typical use case is to read values from the command line, solve the unscaled problem (if necessary), scale the solution, and visualize the solution with dimension:

```
from decay import unscale, read_command_line_argparse

def main():
    # Read parameters, solve and plot
    I, a, T, theta, dt_values = read_command_line_argparse()
    dt = dt_values[0]  # use only the first dt value
    u_scaled, t_scaled = solver_scaled(T, dt, theta)
    u, t = unscale(u_scaled, t_scaled, I, a)

    plt.figure()
    plt.plot(t_scaled, u_scaled)
    plt.xlabel('scaled time'); plt.ylabel('scaled velocity')
    plt.title('Universial solution of scaled problem')
    plt.savefig('tmp1.png');  plt.savefig('tmp1.pdf')

    plt.figure()
    plt.plot(t, u)
    plt.xlabel('t'); plt.ylabel('u')
    plt.title('I=%g, a=%g, theta=%g' % (I, a, theta))
    plt.savefig('tmp2.png'); plt.savefig('tmp2.pdf')
    plt.show()
```

The complete code resides in the file `decay_scaled.py`. A plot of the scaled and unscaled solution appears in Figure 2.1.

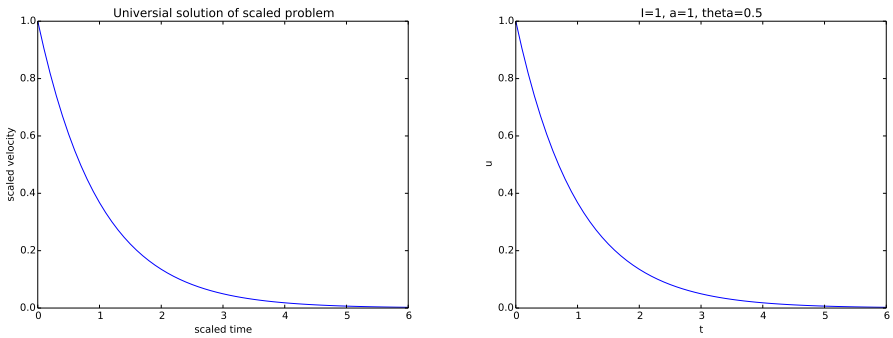


Fig. 2.1 Scaled (left) and unscaled (right) exponential decay.

Note that we write a message `Computing the numerical solution` inside the `solver_scaled` function. We can then easily detect when the solution is actually computed and when it is simply read from file. Here is a demo:

Terminal

```
Terminal> # Very first run
Terminal> python decay_scaled.py --T 7 --a 1 --I 0.5 --dt 0.2
[Memory] Calling __main__--home-hpl...
solver_scaled-alias(7.0, 0.2, 0.5)
Computing the numerical solution

Terminal> # No change of T, dt, theta - can reuse solution in file
Terminal> python decay_scaled.py --T 7 --a 4 --I 2.5 --dt 0.2

Terminal> # Change of dt, must recompute
Terminal> python decay_scaled.py --T 7 --a 4 --I 2.0 --dt 0.5
[Memory] Calling __main__--home-hpl...
solver_scaled-alias(7.0, 0.5, 0.5)
Computing the numerical solution

Terminal> # Change of dt again, but dt=0.2 is already in a file
Terminal> python decay_scaled.py --T 7 --a 0.5 --I 1 --dt 0.2
```

We realize that `joblib` has access to all previous runs and does not recompute unless it is strictly required. Our previous implementation without `joblib` used only one file (for one numerical case) and will therefore perform many more calls to `solver_unscaled`.

On the implementation of a simple memoize function

A memoized function recalls previous results when the same set of arguments is encountered. That is, the function caches its results. A simple

implementation stores the arguments in a function call and the returned results in a dictionary, and if the arguments are seen again, one looks up in the dictionary and return previously computed results:

```
class Memoize:
    def __init__(self, f):
        self.f = f
        self.memo = {}  # map arguments to results

def __call__(self, *args):
        if not args in self.memo:
            self.memo[args] = self.f(*args)
        return self.memo[args]

# Wrap my_compute_function(arg1, arg2, ...)
my_compute_function = Memoize(my_compute_function)
```

The memoize functionality in `joblib.Memory` is more sophisticated and can work very efficiently with large array data structures as arguments. Note that the simple version above can only be used when all arguments to the function `f` are immutable (since the key in a dictionary has to be immutable).

2.1.5 Scaling a generalized problem

Now we consider an extension of the exponential decay ODE to the form

$$u'(t) = -au(t) + b, \quad u(0) = I\,. \tag{2.8}$$

One particular model, with constant a and b, is a spherical micro-organism falling in air,

$$u' = -\frac{3\pi d\mu}{\varrho_b V}u + g\left(\frac{\varrho}{\varrho_b} - 1\right), \tag{2.9}$$

where d, μ, ϱ_b, ϱ, V, and g are physical parameters. The function $u(t)$ represents the vertical velocity, being positive upwards. We shall use this model in the following.

Exact solution. It can be handy to have the exact solution for reference, in case of constant a and b:

$$u_{\rm e}(t) = \frac{e^{-at}}{a}\left(b(e^{at} - 1) + aI\right)\,.$$

Solving differential equations in SymPy

It can be very useful to use a symbolic computation tool such as SymPy to aid us in solving differential equations. Let us therefore demonstrate how SymPy can be used to find this solution. First we define the parameters in the problem as symbols and $u(t)$ as a function:

```
>>> from sympy import *
>>> t, a, b, I = symbols('t a b I', real=True, positive=True)
>>> u = symbols('u', cls=Function)
```

The next task is to define the differential equation, either as a symbolic expression that is to equal zero, or as an equation `Eq(lhs, rhs)` with `lhs` and `rhs` as expressions for the left- and right-hand side):

```
>>> # Define differential equation
>>> eq = diff(u(t), t) + a*u(t) - b
>>> # or
>>> eq = Eq(diff(u(t), t), -a*u(t) + b)
```

The differential equation can be solved by the `dsolve` function, yielding an equation of the form `u(t) == expression`. We want to grab the expression on the right-hand side as our solution:

```
>>> sol = dsolve(eq, u(t))
>>> print sol
u(t) == (b + exp(a*(C1 - t)))/a
>>> u = sol.rhs                    # grab solution
>>> print u
(b + exp(a*(C1 - t)))/a
```

The solution contains the unknown integration constant `C1`, which must be determined by the initial condition. We form the equation arising from the initial condition $u(0) = I$:

```
>>> C1 = symbols('C1')
>>> eq = Eq(u.subs(t, 0), I)   # substitute t by 0 in u
>>> sol = solve(eq, C1)
>>> print sol
[log(I*a - b)/a]
```

The one solution that was found (stored in a list!) must then be substituted back in the expression `u` to yield the final solution:

```
>>> u = u.subs(C1, sol[0])
>>> print u
(b + exp(a*(-t + log(I*a - b)/a)))/a
```

As in mathematics with pen and paper, we strive to simplify expressions also in symbolic computing software. This frequently requires some trial

and error process with SymPy's simplification functions. A very standard first try is to expand everything and run simplification algorithms:

```
>>> u = simplify(expand(u))
>>> print u
(I*a + b*exp(a*t) - b)*exp(-a*t)/a
```

Doing `latex(u)` automatically converts the expression to LaTeX syntax for inclusion in reports.

Theory. The challenges in our scaling is to find the right u_c and t_c scales. From (2.8) we see that if $u' \to 0$ as $t \to \infty$, u approaches the constant value b/a. It can be convenient to let the scaled $\bar{u} \to 1$ as we approach the $d\bar{u}/d\bar{t} = 0$ state. This idea points to choosing

$$u_c = \frac{b}{a} = g\left(\frac{\varrho}{\varrho_b} - 1\right)\left(\frac{3\pi d\mu}{\varrho_b V}\right)^{-1}. \tag{2.10}$$

On the sign of the scaled velocity

A little note on the sign of u_c is necessary here. With $\varrho_b < \varrho$, the buoyancy force upwards wins over the gravity force downwards, and the body will move upwards. In this case, the terminal velocity $u_c > 0$. When $\varrho_b > \varrho$, we get a motion downwards, and $u_c < 0$. The corresponding u is then also negative, but the scaled velocity u/u_c, becomes positive.

Inserting $u = u_c\bar{u} = b\bar{u}/a$ and $t = t_c\bar{t}$ in (2.8) leads to

$$\frac{d\bar{u}}{d\bar{t}} = -t_c a\bar{u} + \frac{t_c}{u_c}b, \quad \bar{u}(0) = I\frac{a}{b}.$$

We want the scales such that $d\bar{u}/d\bar{t}$ and $\bar{u}$ are about unity. To balance the size of $\bar{u}$ and $d\bar{u}/d\bar{t}$ we must therefore choose $t_c = 1/a$, resulting in the scaled ODE problem

$$\frac{d\bar{u}}{d\bar{t}} = -\bar{u} + 1, \quad u(0) = \beta, \tag{2.11}$$

where β is a dimensionless number,

$$\beta = \frac{I}{u_c} = I\frac{a}{b}, \tag{2.12}$$

reflecting the ratio of the initial velocity and the terminal ($t \to \infty$) velocity b/a. Scaling normally ends up with one or more dimensionless parameters,

such as β here, containing ratios of physical effects in the model. Many more examples on dimensionless parameters will appear in later sections.

The analytical solution of the scaled model (2.11) reads

$$\bar{u}_{\rm e}(t) = e^{-t}\left(e^{t} - 1 + \beta\right) = 1 + (\beta - 1)e^{-t}\,. \tag{2.13}$$

The result (2.11) with the solution (2.13) is actually astonishing if a and b are as in (2.9): the six parameters d, μ, ϱ_b, ϱ, V, and g are conjured to one:

$$\beta = I\frac{3\pi d\mu}{\varrho_b V}\frac{1}{g}\left(\frac{\varrho}{\varrho_b} - 1\right)^{-1},$$

which is an enormous simplification of the problem if our aim is to investigate how u varies with the physical input parameters in the model. In particular, if the motion starts from rest, $\beta = 0$, and there are no physical parameters in the scaled model! We can then perform a single simulation and recover all physical cases by the unscaling procedure. More precisely, having computed $\bar{u}(\bar{t})$ from (2.11), we can use

$$u(t) = \frac{b}{a}\bar{u}(at), \tag{2.14}$$

to scale back to the original problem again. We observe that (2.11) can utilize a solver for (2.8) by setting $a = 1$, $b = 1$, and $I = \beta$. Given some implementation of a solver for (2.8), say `solver(I, a, b, T, dt, theta)`, the scaled model is run by `solver(beta, 1, 1, T, dt, theta)`.

Software. We may develop a solver for the scaled problem that uses `joblib` to cache solutions with the same β, Δt, and T. For now we fix $\theta = 0.5$. The module `decay_vc.py` (see ref[Section 3.1.3][in [4][the section Implementation of the generalized model problem [4]] for details) has a function `solver(I, a, b, T, dt, theta)` for solving $u'(t) = -a(t)u(t) + b(t)$ for $t \in (0,T]$, $u(0) = I$, with time step `dt`. We reuse this function and call it with $a = b = 1$ and $I = \beta$ to solve the scaled problem:

```
from decay_vc import solver as solver_unscaled

def solver_scaled(beta, T, dt, theta=0.5):
    """
    Solve u'=-u+1, u(0)=beta for (0,T]
    with step dt and theta method.
    """
    print 'Computing the numerical solution'
    return solver_unscaled(
        I=beta, a=lambda t: 1, b=lambda t: 1,
        T=T, dt=dt, theta=theta)

import joblib
disk_memory = joblib.Memory(cachedir='temp')
solver_scaled = disk_memory.cache(solver_scaled)
```

If we want to plot the physical solution, we need an `unscale` function,

```
def unscale(u_scaled, t_scaled, d, mu, rho, rho_b, V):
    a, b = ab(d, mu, rho, rho_b, V)
    return (b/a)*u_scaled, a*t_scaled

def ab(d, mu, rho, rho_b, V):
    g = 9.81
    a = 3*pi*d*mu/(rho_b*V)
    b = g*(rho/rho_b - 1)
    return a, b
```

Looking at droplets of water in air, we can fix some of the parameters and let the size parameter d be the one for experimentation. The following function sets physical parameters, computes β, runs the solver for the scaled problem (`joblib` detects if it is necessary), and finally plots the scaled curve $\bar{u}(\bar{t})$ and the unscaled curve $u(t)$.

```
def main(dt=0.075, # Time step, scaled problem
         T=7.5,    # Final time, scaled problem
         d=0.001,  # Diameter (unscaled problem)
         I=0,      # Initial velocity (unscaled problem)
         ):
    # Set parameters, solve and plot
    rho = 0.00129E+3  # air
    rho_b = 1E+3      # density of water
    mu = 0.001        # viscosity of water
    # Asumme we have list or similar for d
    if not isinstance(d, (list,tuple,np.ndarray)):
        d = [d]

    legends1 = []
    legends2 = []
    plt.figure(1)
    plt.figure(2)
    betas = []     # beta values already computed (for plot)

    for d_ in d:
        V = 4*pi/3*(d_/2.)**3  # volume
        a, b = ab(d_, mu, rho, rho_b, V)
        beta = I*a/b
        # Restrict to 3 digits in beta
        beta = abs(round(beta, 3))

        print 'beta=%.3f' % beta
        u_scaled, t_scaled = solver_scaled(beta, T, dt)

        # Avoid plotting curves with the same beta value
        if not beta in betas:
            plt.figure(1)
            plt.plot(t_scaled, u_scaled)
            plt.hold('on')
            legends1.append('beta=%g' % beta)
        betas.append(beta)

        plt.figure(2)
        u, t = unscale(u_scaled, t_scaled, d_, mu, rho, rho_b, V)
        plt.plot(t, u)
        plt.hold('on')
        legends2.append('d=%g [mm]' % (d_*1000))
    plt.figure(1)
```

```
    plt.xlabel('scaled time'); plt.ylabel('scaled velocity')
    plt.legend(legends1, loc='lower right')
```

The most complicated part of the code is related to plotting, but this part can be skipped when trying to understand how we work with a scaled model to perform the computations. The complete program is found in the file `falling_body.py`.

Since $I = 0$ implies $\beta = 0$, we can run different d values without any need to recompute $\bar{u}(\bar{t})$ as long as we assume the particle starts from rest.

From the scaling, we see that $u_c = b/a \sim d^{-2}$ and also that $t_c = 1/a \sim d^{-2}$, so plotting of $u(t)$ with dimensions for various d values will involve significant variations in the time and velocity scales. Figure 2.2 has an example with $d = 1, 2, 3$ mm, where we clearly see the different time and velocity scales in the figure with unscaled variables. Note that the scaled velocity is positive because of the sign of u_c (see the box above).

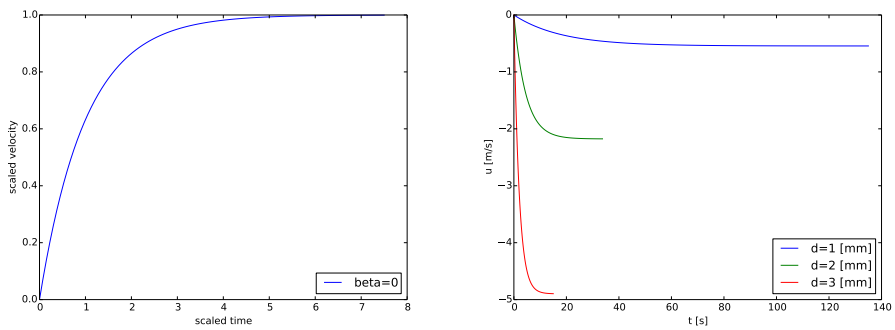


Fig. 2.2 Velocity of falling body: scaled (left) and with dimensions (right).

2.1.6 Variable coefficients

When a prescribed coefficient like $a(t)$ in $u'(t) = -a(t)u(t)$ varies with time one usually also performs a scaling of this a,

$$\bar{a}(\bar{t}) = \frac{a(t) - a_0}{a_c},$$

where the goal is to have the scaled $\bar{a}$ of size unity: $|\bar{a}| \leq 1$. This property is obtained by choosing a_c as the maximum value of $|a(t) - a_0|$ for $t \in [0, T]$, which is usually a quantity that can be estimated since $a(t)$ is known as a function of t. The a_0 parameter can be chosen as 0 here. (It could be tempting to choose $a_0 = \min_t a(t)$ so that $0 \leq \bar{a} \leq 1$, but then there is at least one point where $\bar{a} = 0$ and the differential equation collapses to $u' = 0$.)

As an example, imagine a decaying cell culture where we at time t_1 change the environment (typically the nutrition) such that the death rate increases by a factor 5. Mathematically, $a(t) = d$ for $t < t_1$ and $a(t) = 5d$ for $t \geq t_1$. The model reads $u' = -a(t)u$, $u(0) = I$.

The $a(t)$ function is scaled by letting the characteristic size be $a_c = d$ and $a_0 = 0$:

$$\bar{a}(\bar{t}) = \begin{cases} 1, \bar{t} < t_1/t_c \\ 5, \bar{t} \geq t_1/t_c \end{cases}$$

The scaled equation becomes

$$\frac{u_c}{t_c}\frac{d\bar{u}}{d\bar{t}} = a_c\bar{a}(\bar{t})u_c\bar{u}, \quad u_c\bar{u}(0) = I\,.$$

The natural choice of u_c is I. The characteristic time, previously taken as $t_c = 1/a$, can now be chosen as $t_c = t_1$ or $t_c = 1/d$. With $t_c = 1/d$ we get

$$\bar{u}'(\bar{t}) = -\bar{a}\bar{u}, \quad \bar{u}(0) = 1, \quad \bar{a} = \begin{cases} 1, \bar{t} < \gamma \\ 5, \bar{t} \geq \gamma \end{cases} \tag{2.15}$$

where

$$\gamma = t_1 d$$

is a dimensionless number in the problem. With $t_c = t_1$, we get

$$\bar{u}'(\bar{t}) = -\gamma\bar{a}\bar{u}, \quad \bar{u}(0) = 1, \quad \bar{a} = \begin{cases} 1, \bar{t} < 1 \\ 5, \bar{t} \geq 1 \end{cases}$$

The dimensionless parameter γ is now in the equation rather than in the definition of $\bar{a}$. Both problems involve γ, which is the ratio between the time when the environmental change happens and the typical time for the decay $(1/d)$.

A computation with the scaled model (2.15) and the original model with dimensions appears in Figure 2.3.

2.1.7 Scaling a cooling problem with constant surroundings

The heat exchange between a body at temperature $T(t)$ and the surroundings at constant temperature T_s can be modeled by Newton's law of cooling:

$$T'(t) = -k(T - T_s), \quad T(0) = T_0, \tag{2.16}$$

where k is a prescribed heat transfer coefficient.

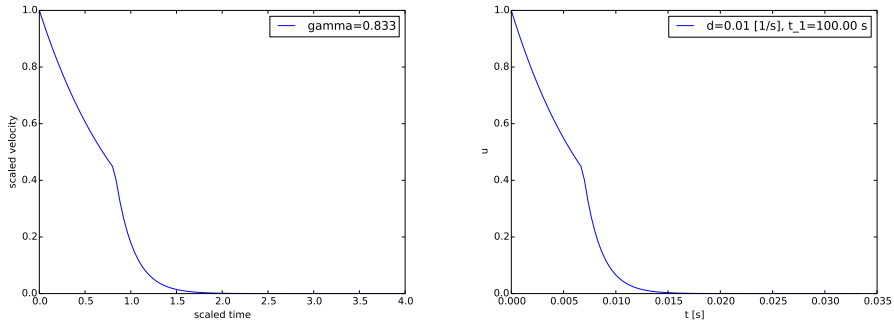


Fig. 2.3 Exponential decay with jump: scaled model (left) and unscaled model (right).

Exact solution. An analytical solution is always handy to have as a control of the choice of scales. The solution of (2.16) is by standard methods for ODEs found to be $T(t) = T_s + (T_0 - T_s)e^{-kt}$.

Scaling. Physically, we expect the temperature to start at T_0 and then to move toward the surroundings (T_s). We therefore expect that T lies between T_0 and T_s. This is mathematically demonstrated by the analytical solution as well. A proper scaling is therefore to scale and translate T according to

$$\bar{T} = \frac{T - T_0}{T_s - T_0}\,. \tag{2.17}$$

Now, $0 \leq \bar{T} \leq 1$.

Scaling time by $\bar{t} = t/t_c$ and inserting $T = T_0 + (T_s - T_0)\bar{T}$ and $t = t_c\bar{t}$ in the problem (2.16) gives

$$\frac{d\bar{T}}{d\bar{t}} = -t_c k(\bar{T} - 1), \quad \bar{T}(0) = 0\,.$$

A natural choice, as argued in other exponential decay problems, is to choose $t_c k = 1$, which leaves us with the scaled problem

$$\frac{d\bar{T}}{d\bar{t}} = -(\bar{T} - 1), \quad \bar{T}(0) = 0\,. \tag{2.18}$$

No physical parameter enters this problem! Our scaling implies that $\bar{T}$ starts at 0 and approaches 1 as $\bar{t} \to \infty$, also in the case $T_s < T_0$. The physical temperature is always recovered as

$$T(t) = T_0 + (T_s - T_0)\bar{T}(k\bar{t})\,. \tag{2.19}$$

Software. An implementation for (2.16) works for (2.18) by setting $k = 1$, $T_s = 1$, and $T_0 = 0$.

Alternative scaling. An alternative temperature scaling is to choose

$$\bar{T} = \frac{T - T_s}{T_0 - T_s}\,. \tag{2.20}$$

Now $\bar{T} = 1$ initially and approaches zero as $t \to \infty$. The resulting scaled ODE problem then becomes

$$\frac{d\bar{T}}{d\bar{t}} = -\bar{T}, \quad \bar{T}(0) = 1\,. \tag{2.21}$$

2.1.8 Scaling a cooling problem with time-dependent surroundings

Let us apply the model (2.16) in case the surrounding temperature varies in time. Say we have an oscillating temperature environment according to

$$T_s(t) = T_m + a\sin(\omega t)\,. \tag{2.22}$$

Exact solution. It is possible to solve the differential equation problem analytically, and such a solution is a good help to see what scales are. In general, using the method of integrating factors for the original differential equation, we have

$$T(t) = T_0 e^{-kt} + e^{-kt} k \int_0^t e^{k\tau} T_s(\tau) d\tau\,.$$

With $T_s(t) = T_m + a\sin(\omega t)$ we can use SymPy to help us with integrations (note that we use `w` for ω in the computer code):

```
>>> from sympy import *
>>> t, k, T_m, a, w = symbols('t k T_m a w', real=True, positive=True)
>>> T_s = T_m + a*sin(w*t)
>>> I = exp(k*t)*T_s
>>> I = integrate(I, (t, 0, t))
>>> Q = k*exp(-k*t)*I
>>> Q = simplify(expand(Q))
>>> print Q
(-T_m*k**2 - T_m*w**2 + a*k*w +
(T_m*k**2 + T_m*w**2 + a*k**2*sin(t*w) -
a*k*w*cos(t*w))*exp(k*t))*exp(-k*t)/((k**2 + w**2))
```

Reordering the result, we get

$$T(t) = T_0 e^{-kt} + T_m(1 - e^{-kt}) + (k^2 + \omega^2)^{-1}(ak\omega e^{-kt} + ak\sin(\omega t) - akw\cos(\omega t))\,.$$

Scaling. The scaling (2.17) brings in a time-dependent characteristic temperature scale $T_s - T_0$. Let us start with a fixed scale, where we take the characteristic temperature variation to be $T_m - T_0$:

$$\bar{T} = \frac{T - T_0}{T_m - T_0}\,.$$

We see from the analytical solution, and realize also by physical reasoning, that T sets out at T_0, but with time, it will oscillate around T_m. The typical average temperature span is therefore $|T_m - T_0|$, unless a is much larger than $|T_m - T_0|$ or T_0 is very close to T_m (see Exercise 2.3 for a discussion of these cases).

We get from the differential equation, with $t_c = 1/k$ as in the former case,

$$k(T_m - T_0)\frac{d\bar{T}}{d\bar{t}} = -k((T_m - T_0)\bar{T} + T_0 - T_m - a\sin(\omega t),$$

resulting in

$$\frac{d\bar{T}}{d\bar{t}} = -\bar{T} + 1 + \alpha\sin(\beta\bar{t}), \quad \bar{T}(0) = 0, \tag{2.23}$$

where we have two dimensionless numbers:

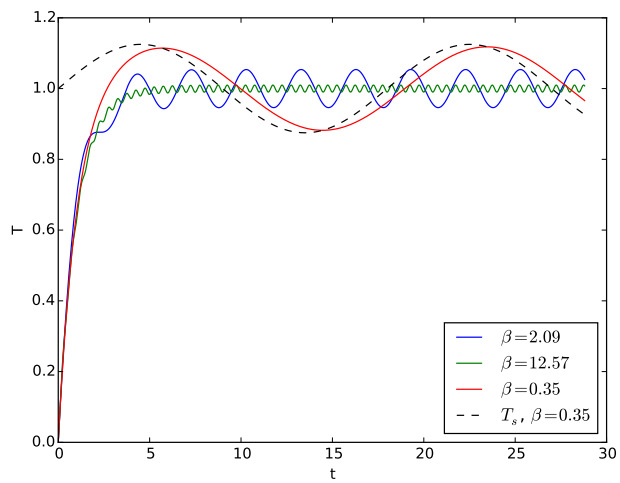
$$\alpha = \frac{a}{T_m - T_0}, \quad \beta = \frac{\omega}{k}\,.$$

The α quantity measures the ratio of temperatures: amplitude of oscillations versus distance from initial temperature to the average temperature for large times. The β number is the ratio of the two time scales: the frequency of the oscillations in T_s and the inverse e-folding time of the heat transfer. For clear interpretation of β we may introduce the period $P = 2\pi/\omega$ of the oscillations in T_s and the e-folding time $e = 1/k$. Then $\beta = 2\pi e/P$ and measures the e-folding time versus the period.

Remark

The original problem features five physical parameters: k, T_0, T_m, a, and ω, but only two dimensionless numbers appear in the scaled model (2.23). In fact, this is an example where application of the Pi theorem (see Section 1.1.3) falls short. Since, only time and temperature are involved as unit types, the theorem predicts that the five parameters yields three dimensionless numbers, not two. Scaling of the differential equations, on the other hand, shows us that the two parameters T_m and T_0 affect the nature of the problem only through their difference.

Software. Implementations of the unscaled problem (2.16) can be reused for the scaled model by setting $k = 1$, $T_0 = 0$, and $T_s(t) = 1 + \alpha\sin(\beta\bar{t})$ ($T_m = 1$, $a = \alpha$, $\omega = \beta$). The file `osc_cooling.py` contains solvers for the problem with dimensions and for the scaled problem. The figure below shows three cases of β values: small, medium, and large.



For the small β value, the oscillations in the surrounding temperature are slow enough compared to k for the heating and cooling process to follow the surrounding temperature, with a small time lag. For larger β, the heating and cooling requires more time, and the oscillations get smaller.

Discussion of the time scale. Looking at the analytical insight we have obtained, we observe that $T(t)$ has two characteristic terms in time: e^{-kt} and $\sin(\omega t)$. The former points to a time scale $t_c = 1/k$, while the latter to $t_c = 1/\omega$. Which one should be chosen? Bringing the temperature from T_0 to the level of the surroundings, T_m, goes like e^{-kt}, so in this process $t_c = 1/k$ is the characteristic time. Thereafter, the body's temperature just responds to the oscillations and the $\sin(\omega t)$ (and $\cos(\omega t)$) term dominates. For these large times, $t_c = 1/\omega$ is the appropriate time scale. Choosing $t_c = 1/\omega$ results in

$$\frac{d\bar{T}}{d\bar{t}} = -\beta^{-1}(\bar{T} - (1 + \alpha\sin(\bar{t}))), \quad \bar{T}(0) = 0\,. \tag{2.24}$$

Let us illustrate another, less effective, scaling. The temperature scale in (2.17) looks natural, so we apply this choice of scale. The characteristic temperature $T_0 - T_s$ now involves a time-dependent term $T_s(t)$. The mathematical steps become a bit more technically involved:

$$T(t) = T_0 + (T_s(t) - T_0)\bar{T},$$

$$\frac{dT}{dt} = \frac{dT_s}{dt}\bar{T} + (T_s - T_0)\frac{d\bar{T}}{d\bar{t}}\frac{d\bar{t}}{dt}\,.$$

With $\bar{t} = t/t_c = kt$ we get from the differential equation

$$\frac{dT_s}{dt}\bar{T} + (T_s - T_0)\frac{d\bar{T}}{d\bar{t}}k = -k(\bar{T}-1)(T_s - T_0),$$

which after dividing by $k(T_s - T_0)$ results in

$$\frac{d\bar{T}}{d\bar{t}} = -(\bar{T}-1) - \frac{dT_s}{dt}\frac{\bar{T}}{k(T_s - T_0},$$

or

$$\frac{d\bar{T}}{d\bar{t}} = -(\bar{T}-1) - \frac{a\omega\cos(\omega\bar{t}/k)}{k(T_m + a\sin(\omega\bar{t}/k) - T_0)}\bar{T}\,.$$

The last term is complicated and becomes more tractable if we factor out dimensionless numbers. To this end, we scale T_s by (e.g.) T_m, which means to factor out T_m in the denominator. We are then left with

$$\frac{d\bar{T}}{d\bar{t}} = -(\bar{T}-1) - \alpha\beta\frac{\cos(\beta\bar{t})}{1+\alpha\sin(\beta\bar{t}) - \gamma}\bar{T}, \tag{2.25}$$

where α, β, and γ are dimensionless numbers characterizing the relative importance of parameters in the problem:

$$\alpha = a/T_m, \quad \beta = \omega/k, \quad \gamma = T_0/T_m\,. \tag{2.26}$$

We notice that (2.25) is not a special case of the original problem (2.16). Furthermore, the original five parameters k, T_m, a, ω, and T_0 are reduced to three dimensionless parameters. We conclude that this scaling is inferior, because using the temperature scale $T_0 - T_m$ enables reuse of the software for the unscaled problem and only two dimensionless parameters appear in the scaled model.

Let us briefly mention another possible temperature scaling: $\bar{T} = T/T_m$, motivated by the fact that as $t \to \infty$, T will oscillate around T_m, so this $\bar{T}$ will oscillate around unity. We get the dimensionless ODE

$$\frac{d\bar{T}}{d\bar{t}} = -(\bar{T} - (1 + \delta\sin(\beta\bar{t}))),$$

with a new dimensionless parameter $\delta = a/T_m$. However, the initial condition becomes $\bar{T}(0) = T_0/T_m$, and the ratio T_0/T_m is a third dimensionless parameter, so this scaling is also inferior to the one above with only two parameters.

2.1.9 Scaling a nonlinear ODE

Exponential growth models, $u' = au$, are not realistic in environments with limited resources. The idea is then to assume that the growth rate a decreases

with u and vanishes when we reach the maximum value M of u the environment can sustain. The initial growth rate is set to r: $a(0) = \varrho$. In general, this reasoning gives rise to models

$$u' = a(u)u, \quad u(0) = I, \tag{2.27}$$

with the logistic model, corresponding to $a(u) = \varrho(1-u/M)$, as the simplest:

$$u' = \varrho u(1-u/M), \quad u(0) = I\,. \tag{2.28}$$

A general choice of a may be $a(u) = \varrho(1-u/M)^p$ for some exponent p.

Scaling. Let us scale (2.27) with $a(u) = \varrho(1-u/M)^p$. The natural scale for u is M ($u_c = M$), since we know that $0 < u \leq M$, and this makes the dimensionless $\bar{u} = u/M \in (0,1]$. The function $a(u)$ is typically varying between 0 and ϱ, so it can be scaled as

$$\bar{a}(\bar{u}) = \frac{a(u)}{\varrho} = (1-\frac{u}{M})^p = (1-\bar{u})^p\,.$$

Time is scaled as $\bar{t} = t/t_c$ for some suitable characteristic time t_c. Inserted in (2.27), we get

$$\frac{u_c}{t_c}\frac{d\bar{u}}{d\bar{t}} = \varrho\bar{a}u_c\bar{u}, \quad u_c\bar{u}(0) = I,$$

resulting in

$$\frac{d\bar{u}}{d\bar{t}} = t_c\varrho(1-\bar{u})^p\bar{u}, \quad \bar{u}(0) = \frac{I}{M}\,.$$

A natural choice is $t_c = 1/\varrho$ as in other exponential growth models since it leads to the term on the right-hand side to be about unity, as the left-hand side, if the scaling is physically correct. Introducing the dimensionless parameter

$$\alpha = \frac{I}{M},$$

measuring the fraction of the initial population compared to the maximum one, we get the dimensionless model

$$\frac{d\bar{u}}{d\bar{t}} = (1-\bar{u})^p\bar{u}, \quad \bar{u}(0) = \alpha\,. \tag{2.29}$$

Here, we have two dimensionless parameters: α and p. A classical logistic model with $p = 1$ has only one dimensionless variable.

Alternative scaling. We could try another scaling of u where we also translate $\bar{u}$:

$$\bar{u} = \frac{u-I}{M}\,.$$

This choice of $\bar{u}$ results in

$$\frac{d\bar{u}}{d\bar{t}} = (1-\alpha-\bar{u})^p\bar{u}, \quad \bar{u}(0) = 0\,. \tag{2.30}$$

The essential difference between (2.29) and (2.30) is that $\bar{u} \in [\alpha, 1]$ in the former and $\bar{u} \in [0, 1-\alpha]$ in the latter. Both models involve the dimensionless numbers α and p. An advantage of (2.29) is that software for the unscaled model can easily be used for the scaled model by choosing $I = \alpha$, $M = 1$, and $\varrho = 1$.

2.1.10 SIR ODE system for spreading of diseases

The field of epidemiology frequently applies ODE systems to describe the spreading of diseases, such as smallpox, measles, plague, ordinary flu, swine flu, and HIV. Different models include different effects, which are reflected in dimensionless numbers. Most of the effects are modeled as exponential decay or growth of the dependent variables.

The simplest model has three categories of people: susceptibles (S) who can get the disease, infectious (I) who are infected and may infect susceptibles, and recovered (R) who have recovered from the disease and gained immunity. We introduce $S(t)$, $I(t)$, and $R(t)$ as the number of people in the categories S, I, and R, respectively. The model, naturally known as the SIR model, takes the form a system of ODEs:

$$\frac{dS}{dt} = -\beta SI, \tag{2.31}$$

$$\frac{dI}{dt} = \beta SI - \nu I, \tag{2.32}$$

$$\frac{dR}{dt} = \nu I, \tag{2.33}$$

where β and ν are empirical constants. The average time for recovering from the disease can be shown to be ν^{-1}, but β is much harder to estimate, so working with a scaled model where β is "scaled away" is advantageous.

hpl 5: Geir asked about the background for this model. A link to Wikipedia is now inserted, but in general all models in this book are just listed without further explanations. Do we need more references? The idea was that "here are the ODEs" and if understanding of the "physics" is necessary, the reader must have the necessary background. Maybe state this in the preface? **Geir 6**: I will look at the example again, but even more on the chemistry example. I guess we need to be pragmatic concerning background and explanations. In short: do not add more references.

Scaling. It is natural to scale S, I, and R by, e.g., $S(0)$:

$$\bar{S} = \frac{S}{S(0)}, \quad \bar{I} = \frac{I}{S(0)}, \quad \bar{R} = \frac{R}{S(0)}.$$

Introducing $\bar{t} = t/t_c$, we arrive at the equations

$$\begin{aligned}
\frac{d\bar{S}}{d\bar{t}} &= -t_c S(0)\beta\bar{S}\bar{I},\\
\frac{d\bar{I}}{d\bar{t}} &= t_c S(0)\beta\bar{S}\bar{I} - t_c\nu\bar{I},\\
\frac{d\bar{R}}{d\bar{t}} &= t_c\nu I,
\end{aligned}$$

with initial conditions $\bar{S}(0) = 1$, $\bar{I}(0) = I_0/S(0) = \alpha$, and $\bar{R}(0) = R(0)/S(0)$. Normally, $R(0) = 0$.

Taking $t_c = 1/\nu$, corresponding to a time unit equal to the time it takes to recover from the disease, we end up with the scaled model

$$\frac{d\bar{S}}{d\bar{t}} = -R_0\bar{S}\bar{I}, \tag{2.34}$$

$$\frac{d\bar{I}}{d\bar{t}} = R_0\bar{S}\bar{I} - \bar{I}, \tag{2.35}$$

$$\frac{d\bar{R}}{d\bar{t}} = I, \tag{2.36}$$

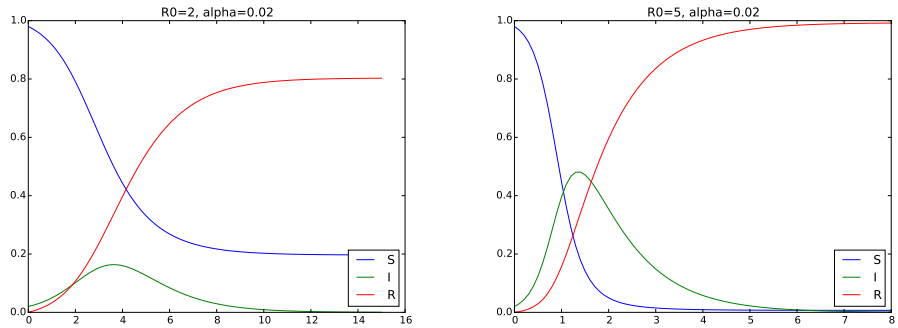
with $\bar{S}(0) = 1$, $\bar{I}(0) = \alpha$, $\bar{R}(0) = 0$, and R_0 as the dimensionless number

$$R_0 = \frac{S(0)\beta}{\nu}. \tag{2.37}$$

We see from (2.35) that to make the disease spreading, $d\bar{I}/d\bar{t} > 0$, and therefore $R_0 S(0) - 1 > 0$ or $R_0 > 1$ since $S(0) = 1$. Therefore, R_0 reflects the disease's ability to spread and is consequently an important dimensionless quantity, known as the basic reproduction number. This number reflects the number of infected people caused by one infectious individual during the time period of the disease.

Looking at (2.32), we see that to increase I initially, we must have $dI/dt > 0$ at $t = 0$, which implies $\beta I(0)S(0) - \nu I(0) > 0$, i.e., $R_0 > 1$.

Software. Any implementation of the SIR model with dimensions can be reused for the scaled model by setting $\beta = R_0$, $\nu = 1$, $S(0) = 1 - \alpha$, and $I(0) = \alpha$. Below is a plot with two cases: $R_0 = 2$ and $R_0 = 5$, both with $\alpha = 0.02$.



Alternative scaling. Adding (2.31)-(2.33) shows that

$$\frac{dS}{dt}+\frac{dI}{dt}+\frac{dR}{dt}=0 \quad \Rightarrow \quad S+I+R=\text{const}=N,$$

where N is the size of the population. We can therefore scale S, I, and R by the total population $N=S(0)+I(0)+R(0)$:

$$\bar{S}=\frac{S}{N}, \quad \bar{I}=\frac{I}{N}, \quad \bar{R}=\frac{R}{N)}\,.$$

With the same time scale, one gets the system (2.34)-(2.36), but with R_0 replaced by the dimensionless number:

$$\tilde{R}_0=\frac{N\beta}{\nu}\,. \tag{2.38}$$

The initial conditions become $\bar{S}(0)=1-\alpha$, $\bar{I}(0)=\alpha$, and $\bar{R}(0)=0$.

For the disease to spread at $t=0$, we must have $\tilde{R}_0\bar{S}(0)>1$, but $\tilde{R}_0\bar{S}(0)=N\beta/\nu\cdot S(0)/N=R_0$, so the criterion is still $R_0>1$. Since R_0 is a more famous number than $\tilde{R}_0$, we can write the ODEs with $R_0/S(0)=R_0/(1-\alpha)$ instead of $\tilde{R}_0$.

Choosing t_c to make the SI terms balance the time derivatives, $t_c=(N\beta)^{-1}$, moves $\tilde{R}_0$ (or R_0 if we scale S, I, and R by $S(0)$) to the I terms:

$$\begin{aligned}
\frac{d\bar{S}}{d\bar{t}} &= -\bar{S}\bar{I},\\
\frac{d\bar{I}}{d\bar{t}} &= \bar{S}\bar{I}-\tilde{R}_0^{-1}\bar{I},\\
\frac{d\bar{R}}{d\bar{t}} &= \tilde{R}_0^{-1}I\,.
\end{aligned}$$

2.1.11 SIRV model with finite immunity

A common extension of the SIR model involves finite immunity: after some period of time, recovered individuals lose their immunity and become susceptibles again. This is modeled as a leakage $-\mu R$ from the R to the S category, where μ^{-1} is the average time it takes to lose immunity. Vaccination is another extension: a fraction pS is removed from the S category by successful vaccination and brought to a new category V (the vaccinated). The ODE model reads

$$\frac{dS}{dt} = -\beta SI - pS + \mu R, \tag{2.39}$$

$$\frac{dI}{dt} = \beta SI - \nu I, \tag{2.40}$$

$$\frac{dR}{dt} = \nu I - \mu R, \tag{2.41}$$

$$\frac{dV}{dt} = pS\,. \tag{2.42}$$

Using $t_c = 1/\nu$ and scaling the unknowns by $S(0)$, we arrive at the dimensionless model

$$\frac{d\bar{S}}{d\bar{t}} = -R_0\bar{S}\bar{I} - \delta S + \gamma\bar{R}, \tag{2.43}$$

$$\frac{d\bar{I}}{d\bar{t}} = R_0\bar{S}\bar{I} - \bar{I}, \tag{2.44}$$

$$\frac{d\bar{R}}{d\bar{t}} = \bar{I} - \gamma\bar{R}, \tag{2.45}$$

$$\frac{d\bar{V}}{d\bar{t}} = \delta\bar{S}, \tag{2.46}$$

with two new dimensionless parameters:

$$\gamma = \frac{\mu}{\nu}, \quad \delta = \frac{p}{\nu}\,.$$

The quantity p^{-1} can be interpreted as the average time it takes to vaccinate a susceptible successfully. Writing $\gamma = \nu^{-1}/\mu^{-1}$ and $\delta = \nu^{-1}/p^{-1}$ gives the interpretation that γ is the ratio of the average time to recover and the average time to lose immunity, while δ is the ratio of the average time to recover and the average time to successfully vaccinate a susceptible.

The plot in Figure 2.4 has $\gamma = 0.05$, i.e., loss of immunity takes 20 weeks if it takes one week to recover from the disease. The left plot corresponds to no vaccination, while the right has $\delta = 0.5$ for a vaccination campaign that

lasts from day 7 to day 15. The value $\delta = 0.5$ reflects that it takes two weeks to successfully vaccinate a susceptible, but the effect of vaccination is still dramatic.

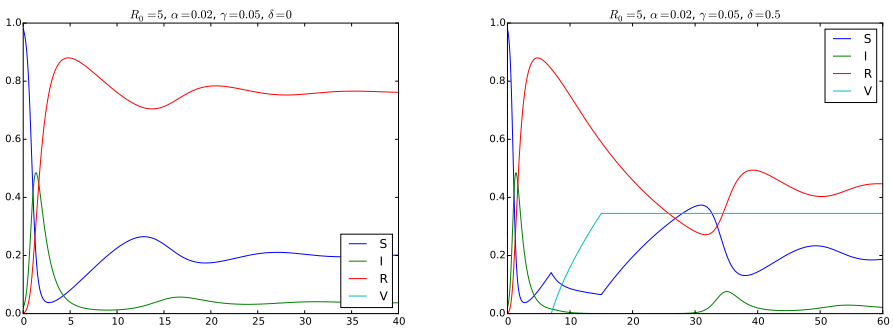


Fig. 2.4 Spreading of a disease with loss of immunity (left) and added vaccination (right).

2.1.12 Michaelis-Menten kinetics for biochemical reactions

A classical reaction model in biochemistry describes how a substrate S is turned into a product P with aid of an enzyme E. S and E react to form a complex ES in the first stage of the reaction. In the second stage, ES is turned into E and P. Introducing the amount of S, E, ES, and P by $[S]$, $[E]$, $[ES]$, and $[P]$, can be written as

$$\frac{d[ES]}{dt} = k_+[E][S] - k_v[ES] - k_-[ES], \tag{2.47}$$

$$\frac{d[P]}{dt} = k_v[ES], \tag{2.48}$$

$$\frac{d[S]}{dt} = -k_+[E][S] + k_-[ES], \tag{2.49}$$

$$\frac{d[E]}{dt} = -k_+[E][S] + k_-[ES] + k_v[ES]\,. \tag{2.50}$$

The initial conditions are $[ES](0) = [P](0) = 0$, and $[S] = S_0$, $[E] = E_0$. Three rate constants are involved: k_+, k_-, and k_v. The above mathematical model is known as Michaelis-Menten kinetics.

The amount of substance is measured in the unit mole (mol). From the equations we can see that k_+ is measured in $\mathrm{s}^{-1}\mathrm{mol}^{-1}$, while k_- and k_v are

measured in s^{-1}. It is convenient to get rid of the mole unit for the amount of a substance. When working with dimensionless quantities, only ratios of the rate constants and not their specific values are needed.

Classical analysis. A common assumption is that the formation of $[ES]$ is very fast and that it quickly reaches an equilibrium state, $[ES]' = 0$. Equation (2.47) then reduces to the algebraic equation

$$k_+[E][S] - k_v[ES] - k_-[ES] = 0,$$

which leads to

$$\frac{[E][S]}{[ES]} = \frac{k_- + k_v}{k_+} = K, \tag{2.51}$$

where K is the famous Michaelis constant - the equilibrium constant between $[E][S]$ and $[ES]$.

Another important observation is that the ODE system implies two conservation equations, arising from simply adding the ODEs:

$$\frac{d[ES]}{dt} + \frac{d[E]}{dt} = 0, \tag{2.52}$$

$$\frac{d[ES]}{dt} + \frac{d[S]}{dt} + \frac{d[P]}{dt} = 0, \tag{2.53}$$

from which it follows that

$$[ES] + [E] = E_0, \tag{2.54}$$

$$[ES] + [S] + [P] = S_0\,. \tag{2.55}$$

We can use (2.54) and (2.51) to express $[E]$ by $[S]$:

$$[E] = E_0 - [ES] = E_0 - \frac{[E][S]}{K} \quad \Rightarrow \quad [E] = \frac{KE_0}{K + [S]}\,.$$

Now (2.49) can be developed to an equation involving $[S]$ only:

$$\begin{aligned}
\frac{d[S]}{dt} &= -k_+[E][S] + k_-[ES] \\
&= (-k_+ + \frac{k_-}{K})[E][S] \\
&= (-k_+ + \frac{k_-}{K})[S]\frac{KE_0}{K + [S]} \\
&= -\frac{k_- E_0}{[S] + K}\,.
\end{aligned} \tag{2.56}$$

We see that the parameter K is central.

Dimensionless ODE system. Let us reason how to make the original ODE system dimensionless. Aiming at $[S]$ and $[E]$ of unit size, two obvious dimensionless unknowns are

$$\bar{S} = \frac{[S]}{S_0}, \quad \bar{E} = \frac{[E]}{E_0}.$$

For the other two unknowns we just introduce scales to be determined later:

$$\bar{P} = \frac{[P]}{P_c}, \quad \bar{Q} = \frac{[ES]}{Q_c}.$$

With $\bar{t} = t/t_c$ the equations become

$$\begin{aligned}
\frac{d\bar{Q}}{d\bar{t}} &= t_c k_+ \frac{E_0 S_0}{Q_c} \bar{E}\bar{S} - t_c(k_v + k_-)\bar{Q}, \\
\frac{d\bar{P}}{d\bar{t}} &= t_c k_v \frac{Q_c}{P_c} \bar{Q}, \\
\frac{d\bar{S}}{d\bar{t}} &= -t_c k_+ E_0 \bar{E}\bar{S} + t_c k_- \frac{Q_c}{S_0} \bar{Q}, \\
\frac{d\bar{E}}{d\bar{t}} &= -t_c k_+ S_0 \bar{E}\bar{S} + t_c(k_- + k_v) \frac{Q_c}{E_0} \bar{Q}.
\end{aligned}$$

Determining scales. Choosing the scales is actually a quite complicated matter that requires extensive analysis of the equations to determine the characteristics of the solutions. Much literature is written about this, but here we shall take a simplistic and pragmatic approach. Besides the Michaelis constant K, there is another important parameter,

$$\epsilon = \frac{E_0}{S_0},$$

because most applications will involve a small ϵ. We shall have K and ϵ in mind while choosing scales such that these symbols appear naturally in the scaled equations.

Looking at the equations, we see that the K parameter will appear if $t_c \sim 1/k_+$. However, $1/k_+$ does not have the dimension $[\mathrm{T}]^{-1}$ as required, so we need to add a factor with dimension mol. A natural choice is $t_c^{-1} = k_+ S_0$ or $t_c^{-1} = k_+ E_0$. Since often $S_0 \gg E_0$, the former t_c is a short time scale and the latter is a long time scale. If the interest is in the long time scale, we set

$$t_c = \frac{1}{k_+ E_0}.$$

The equations then take the form

$$
\begin{aligned}
\frac{d\bar{Q}}{d\bar{t}} &= \frac{S_0}{Q_c}\bar{E}\bar{S} - KE_0^{-1}\bar{Q}, \\
\frac{d\bar{P}}{d\bar{t}} &= \frac{k_v}{k_+E_0}\frac{Q_c}{P_c}\bar{Q}, \\
\frac{d\bar{S}}{d\bar{t}} &= -\bar{E}\bar{S} + \frac{k_-}{k_+E_0}\frac{Q_c}{S_0}\bar{Q}, \\
\frac{d\bar{E}}{d\bar{t}} &= -\epsilon^{-1}\bar{E}\bar{S} + K\frac{Q_c}{E_0^2}\bar{Q}\,.
\end{aligned}
$$

The $[ES]$ variable starts and ends at zero, and its maximum value can be roughly estimated from the equation for $[ES]'$ by setting $[ES]' = 0$, which gives

$$
[ES] = \frac{[E][S]}{K} \approx \frac{E_0S_0}{K},
$$

where we have approximated $[E][S]$ by E_0S_0. This estimate of $[ES]$ at its maximum can be used as the characteristic size Q_c:

$$
Q_c = \frac{E_0S_0}{K}\,.
$$

The equation for $\bar{P}$ simplifies if we choose $P_c = Q_c$. With these assumptions one gets

$$
\begin{aligned}
\frac{d\bar{Q}}{d\bar{t}} &= KE_0^{-1}(\bar{E}\bar{S} - \bar{Q}), \\
\frac{d\bar{P}}{d\bar{t}} &= \frac{k_v}{k_+E_0}\bar{Q}, \\
\frac{d\bar{S}}{d\bar{t}} &= -\bar{E}\bar{S} + \frac{k_-}{k_+E_0}\frac{E_0}{K}\bar{Q}, \\
\frac{d\bar{E}}{d\bar{t}} &= -\epsilon^{-1}\bar{E}\bar{S} + \epsilon^{-1}\bar{Q}\,.
\end{aligned}
$$

We can now identify the dimensionless numbers

$$
\alpha = \frac{K}{E_0}, \quad \beta = \frac{k_v}{k_+E_0}, \quad \gamma = \frac{k_-}{k_+E_0},
$$

where we see that $\alpha = \beta + \gamma$, so γ can be eliminated, leading to the final set of equations:

$$\frac{d\bar{Q}}{d\bar{t}} = \alpha(\bar{E}\bar{S} - \bar{Q}), \tag{2.57}$$

$$\frac{d\bar{P}}{d\bar{t}} = \beta\bar{Q}, \tag{2.58}$$

$$\frac{d\bar{S}}{d\bar{t}} = -\bar{E}\bar{S} + (1-\beta\alpha^{-1})\bar{Q}, \tag{2.59}$$

$$\epsilon\frac{d\bar{E}}{d\bar{t}} = -\bar{E}\bar{S} + \bar{Q}\,. \tag{2.60}$$

The five initial parameters (S_0, E_0, k_+, k_-, and k_v) are reduced to three dimensionless constants:

- α is the dimensionless Michaelis constant, reflecting the ratio of the production of P and E ($k_v + k_-$) versus the production of the complex (k_+), made dimensionless by E_0,
- ϵ is the initial fraction of enzyme relative to the substrate,
- β measures the relative importance of production of P (k_v) versus production of the complex (k_+), made dimensionless by E_0.

Observe that software developed for solving (2.47)-(2.50) cannot be reused for solving (2.57)-(2.60) since the latter system has a slightly different structure.

Analysis of the scaled system. In the scaled system, we may assume ϵ small, which from (2.60) gives rise to the simplification $\epsilon\bar{E}' = 0$, and thereby the relation $\bar{Q} = \bar{E}\bar{S}$. The conservation equation $[ES] + [E] = E_0$ reads $Q_c\bar{Q} + E_0\bar{E} = E_0$ such that $\bar{E} = 1 - Q_c\bar{Q}/E_0 = 1 - \bar{Q}S_0/K = 1 - \epsilon^{-1}\alpha^{-1}\bar{Q}$. The relation $\bar{Q} = \bar{E}\bar{S}$ then becomes

$$\bar{Q} = (1 - \epsilon^{-1}\alpha^{-1}\bar{Q})\bar{S},$$

which can be solved for $\bar{Q}$:

$$\bar{Q} = \frac{\bar{S}}{1 + \epsilon^{-1}\alpha^{-1}\bar{S}}\,.$$

The equation (2.59) for $\bar{S}$ becomes

$$\frac{d\bar{S}}{d\bar{t}} = -\beta\alpha^{-1}\bar{Q} = -\frac{\beta\bar{S}}{\alpha + \epsilon^{-1}\bar{S}}\,. \tag{2.61}$$

This is a more precise analysis than the one leading to (2.56) since we now realize that the mathematical assumption for the simplification is $\epsilon \to 0$.

Is (2.61) consistent with (2.56)? It is easy to make algebraic mistakes when deriving scaled equations, so it is always wise to carry out consistency checks. Introducing dimensions in (2.61) leads to

$$\frac{t_c}{S_0}\frac{dS}{dt} = \frac{d\bar{S}}{d\bar{t}} = -\frac{\beta\bar{S}}{\alpha + \epsilon^{-1}\bar{S}} = -\frac{k_v}{k_+E_0}\frac{S}{KE_0^{-1} + E_0^{-1}S_0\bar{S}} = -\frac{k_v}{k_+}\frac{\bar{S}}{K+S},$$

and hence with $t_c^{-1} = k_+ E_0$,

$$\frac{dS}{dt} = -\frac{k_v E_0 S}{K+S},$$

which is (2.56).

Figure 2.5 shows the impact of ϵ: with a small value (0.1) we see that $\bar{Q} \approx 0$, which justifies the simplifications performed above. We also observe that all the unknowns vary between 0 and about 1, indicating that the scaling is successful for the chosen dimensionless numbers. The simulations made use of a time step $\Delta\bar{t} = 0.1$ with a 4th-order Runge-Kutta method, using $\alpha = 1$, $\beta = 4$ (relevant code is in the `simulate_biochemical_process` function in `session.py`).

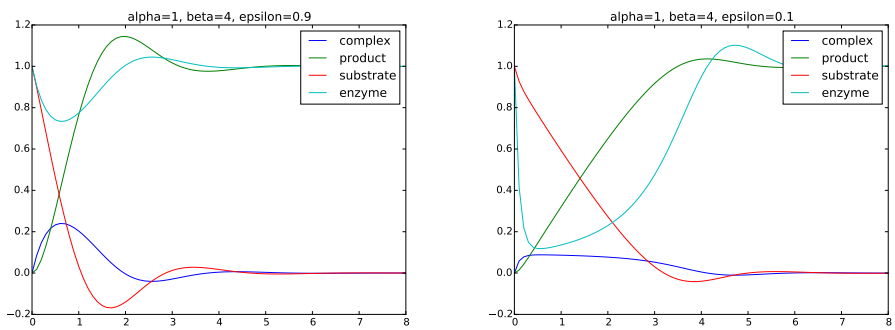


Fig. 2.5 Simulation of a biochemical process.

However, it is of interest to investigate the limit $\epsilon \to 0$. Initially, the equation for $d\bar{E}/d\bar{t}$ reads $d\bar{E}/d\bar{t} = -\epsilon - 1$, which implies a very fast reduction of $\bar{E}$. Using $\epsilon = 0.005$ and $\Delta\bar{t} = 10^{-3}$. The results show that $\bar{E}$ decays to approximately zero at $t = 0.03$ while $\bar{S} \approx 1$ and $\bar{Q} = \bar{P} \approx 0$. This is reasonable since with very little enzyme in comparison with the substrate ($\epsilon \to 0$) very little will happen.

2.2 Vibration problems

We shall in this section address a range of different second-order ODEs for mechanical vibrations and demonstrate how to reason about the scaling in different physical scenarios.

hpl 7: Include more figures.

2.2.1 Undamped vibrations without forcing

The simplest differential equation model for mechanical vibrations reads

$$mu'' + ku = 0, \quad u(0) = I,\ u'(0) = V\,. \tag{2.62}$$

This is a common model for a vibrating body with mass m attached to a linear spring with spring constant k (and force $-ku$). The quantity $u(t)$ measures the displacement of the body. Figure 2.6 shows a typical mechanical sketch of such a system: some mass can move horizontally without friction and is connected to a spring that exerts a force $-ku$ on the body.

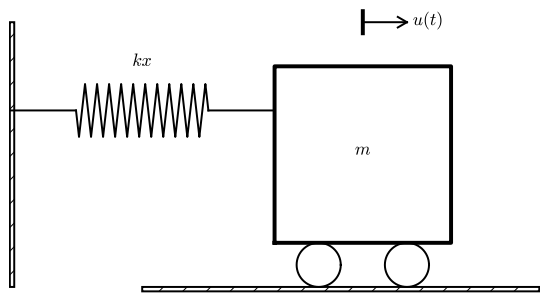


Fig. 2.6 Oscillating body attached to a spring.

The first technical steps of scaling. The problem (2.62) has one independent variable t and one dependent variable u. We introduce dimensionless versions of these variables:

$$\bar{u} = \frac{u}{u_c}, \quad \bar{t} = \frac{t}{t_c},$$

where u_c and t_c are characteristic values of u and t. Inserted in (2.62), we get

$$m\frac{u_c}{t_c^2}\frac{d^2\bar{u}}{d\bar{t}^2} + ku_c\bar{u} = 0, \quad u_c\bar{u}(0) = I, \quad \frac{u_c}{t_c}\frac{d\bar{u}}{d\bar{t}}(0) = V,$$

resulting in

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \frac{t_c^2 k}{m}\bar{u} = 0, \quad \bar{u}(0) = \frac{I}{u_c},\ \bar{u}'(0) = \frac{Vt_c}{u_c}\,. \tag{2.63}$$

What is an appropriate displacement scale u_c? The initial condition $u(0) = I$ is a candidate, i.e., $u_c = I$. But how to choose the time scale? Making the coefficient in front of the $\bar{u}$ unity, such that both terms balance and are of size unity, is a candidate.

The exact solution. To better see what the proper scales of u and t are, we can look into the analytical solution of this problem. Although the exact solution of (2.62) is quite straightforward to calculate by hand, we take the opportunity to make use of SymPy to find $u(t)$. The use of SymPy can later be generalized to vibration ODEs that are harder to solve by hand.

SymPy need all mathematical symbols to be explicitly created:

```
from sympy import *
u = symbols('u', cls=Function)
w = symbols('w', real=True, positive=True)
I, V, C1, C2 = symbols('I V C1 C2', real=True)
```

To specify the ODE to be solved, we can make a Python function returning all the terms in the ODE:

```
# Define differential equation: u'' + w**2*u = 0
def ode(u):
    return diff(u, t, t) + w**2*u

diffeq = ode(u(t))
```

The `diffeq` variable, defining the ODE, can be passed to a SymPy function `dsolve` to find the symbolic solution of the ODE:

```
s = dsolve(diffeq, u(t))
# s is an u(t) == expression (Eq obj.), s.rhs grabs the expression
u_sol = s.rhs
print u_sol
```

The solution that gets printed is `C1*sin(t*w) + C2*cos(t*w)`, indicating that there are two integration constants `C1` and `C2` to be determined by the initial conditions. The result of applying these conditions is a 2×2 linear system of algebraic equations that SymPy can solve by the `solve` function. The code goes as follows:

```
# The solution u_sol contains integration constants C1 and C2
# but these are not symbols, substitute them by symbols
u_sol = u_sol.subs('C1', C1).subs('C2', C2)

# Determine C1 and C2 from the initial conditions
ic = [u_sol.subs(t, 0) - I, u_sol.diff(t).subs(t, 0) - V]
print ic   # 2x2 algebraic system for C1 and C2
s = solve(ic, [C1, C2])
# s is now a dictionary: {C2: I, C1: V/w}
# substitute solution back in u_sol
u_sol = u_sol.subs(C1, s[C1]).subs(C2, s[C2])
print u_sol
```

The `u_sol` variable is now `I*cos(t*w) + V*sin(t*w)/w`. Since symbolic software is far from bug-free and can give wrong results, we should always check the answer. Here, we insert the solution in the ODE to see if the result is zero, and we insert the solution in the initial conditions to see that these are fulfilled:

```
# Check that the solution fulfills the ODE and init.cond.
print simplify(ode(u_sol)),
print u_sol.subs(t, 0) - I, diff(u_sol, t).subs(t, 0) - V
```

There will be many more examples on using SymPy to find exact solutions of differential equation problems.

The solution of the ODE in mathematical notation is

$$u(t) = I\cos(\omega t) + \frac{V}{\omega}\sin(\omega t), \quad \omega = \sqrt{\frac{k}{m}}\,.$$

More insight arises from rewriting such an expression in the form $A\cos(wt - \phi)$:

$$u(t) = \sqrt{I^2 + \frac{V^2}{\omega^2}}\cos(wt - \phi), \quad \phi = \tan^{-1}(V/(\omega I))\,.$$

Now we see that the u corresponds to cosine oscillations with a phase shift ϕ and amplitude $\sqrt{I^2 + (V/\omega)^2}$.

Discussion of the displacement scale. The amplitude of u is $\sqrt{I^2 + V^2/\omega^2}$, and this expression is obviously a candidate for u_c. However, the simpler choice $u_c = \max(I, V/\omega)$ is also relevant and more attractive than the square root expression (but potentially a factor 1.4 wrong compared to the exact amplitude). It is not very important to have $|u| \leq 1$, the point is to avoid $|u|$ very small or large.

Discussion of the time scale. What is an appropriate time scale? Looking at (2.63) and arguing that $\bar{u}''$ and $\bar{u}$ both should be around unity in size, the coefficient $t_c^2 k/m$ must equal unity, implying that $t_c = \sqrt{m/k}$. Also from the analytical solution we see that the solution goes like the sine or cosine of ωt so $1/\omega = \sqrt{m/k}$ can be a characteristic time scale. Likewise, one period of the oscillations, $P = 2\pi/\omega$ can be the characteristic time, leading to $t_c = 2\pi/\omega$.

The dimensionless solution. With $u_c = I$ and $t_c = \sqrt{m/k}$ we get the scaled model

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \bar{u} = 0, \quad \bar{u}(0) = 1,\ \bar{u}'(0) = \alpha, \tag{2.64}$$

where α is a dimensionless parameter:

$$\alpha = \frac{V}{I}\sqrt{\frac{m}{k}}\,.$$

Note that in case $V = 0$, we have "scaled away" all physical parameters. The universal solution without physical parameters is then $\bar{u}(\bar{t}) = \cos\bar{t}$.

The unscaled solution is recovered as

$$u(t) = I\bar{u}(\sqrt{k/m}\bar{t})\,. \tag{2.65}$$

This expressions shows that the scaling is simply a matter of *stretching or shrinking the axes*.

Alternative displacement scale. Using $u_c = V/\omega$, the equation is not changed, but the initial conditions become

$$\bar{u}(0) = \frac{I}{u_c} = \frac{I\omega}{V} = \frac{I}{V}\sqrt{\frac{k}{m}} = \alpha^{-1}, \quad \bar{u}'(0) = 1\,.$$

With $u_c = V/\omega$ and one period as time scale, $t_c = 2\pi\sqrt{m/k}$, we get the alternative model

$$\frac{d^2\bar{u}}{d\bar{t}^2} + 4\pi^2\bar{u} = 0, \quad \bar{u}(0) = \alpha^{-1},\ \bar{u}'(0) = 2\pi\,. \tag{2.66}$$

The unscaled solution is in this case recovered by

$$u(t) = V\sqrt{\frac{m}{k}}\bar{u}(2\pi\sqrt{k/m}\bar{t})\,. \tag{2.67}$$

About frequency and dimensions. The solution goes like $\cos\omega t$, where $\omega = \sqrt{m/k}$ must have dimension 1/s. Actually, ω has dimension *radians per second*: rad/s. A radian is dimensionless since it is arc (length) divided by radius (length), but still regarded as a unit. The period P of vibrations is a more intuitive quantity than the frequency ω. The relation between P and ω is $P = 2\pi/\omega$. The number of oscillation cycles per period, f, is a more intuitive measurement of frequency and also known as *frequency*. Therefore, to be precise, ω should be named *angular frequency*. The relation between f and T is $f = 1/T$, so $f = 2\pi\omega$ and measured in Hz (1/s), which is the unit for counts per unit time.

2.2.2 Undamped vibrations with constant forcing

For vertical vibrations in the gravity field, the model (2.62) must also take the gravity force $-mg$ into account:

$$mu'' + ku = -mg\,.$$

How does the new term $-mg$ influence the scaling? We observe that if there is no movement of the body, $u'' = 0$, and the spring elongation matches the gravity force: $ku = -mg$, leading to a steady displacement $u = -mg/k$. We can then have oscillations around this equilibrium point. A natural scaling for u is therefore

$$\bar{u} = \frac{u - (-mg/k)}{u_c} = \frac{uk + mg}{ku_c}\,.$$

The scaled differential equation with the same time scale as before reads

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \bar{u} - \frac{t_c^2}{u_c}g = -\frac{t_c^2}{u_c}g,$$

leading to

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \bar{u} = 0\,.$$

The initial conditions $u(0) = I$ and $u'(0) = V$ become, with $u_c = I$,

$$\bar{u}(0) = 1 + \frac{mg}{kI}, \quad \frac{d\bar{u}}{d\bar{t}}(0) = \sqrt{\frac{m}{k}}\frac{V}{I}\,.$$

We see that the oscillations around the equilibrium point in the gravity field are identical to the horizontal oscillations without gravity, except for an offset $mg/(kI)$ in the displacement.

2.2.3 Undamped vibrations with time-dependent forcing

Now we add a transient forcing term $F(t)$ to the model (2.62):

$$mu'' + ku = F(t), \quad u(0) = I,\ u'(0) = V\,. \tag{2.68}$$

Take the forcing to be oscillating:

$$F(t) = A\cos(\psi t)\,.$$

The technical steps of the scaling are still the same, with the intermediate result

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \frac{t_c^2 k}{m}\bar{u} = \frac{t_c^2}{mu_c}A\cos(\psi t_c\bar{t}), \quad \bar{u}(0) = \frac{I}{u_c},\ \bar{u}'(0) = \frac{Vt_c}{u_c}\,. \tag{2.69}$$

What are typical displacement and time scales? This is not so obvious without knowing the details of the solution, because there are three parameters (I, V, and A) that influence the magnitude of u. Moreover, there are two time scales, one for the free vibrations of the systems and one for the forced vibrations $F(t)$.

Investigating scales via analytical solutions. We may look into the exact solution to see what the scales are. We continue the SymPy session from the previous section and perform much of the same steps. Note that we use `w` for $\omega = \sqrt{k/m}$ in the computer code (to obtain a more direct visual counterpart to ω). SymPy may get confused when coefficients in differential equations contain several symbols. We therefore rewrite the equation with at

most one symbol in each coefficient. The amplitude A/m in the forcing term is of this reason replaced by the symbol `A1`.

```
A, A1, m, psi = symbols('A A1 m psi', positive=True, real=True)
def ode(u):
    return diff(u, t, t) + w**2*u - A1*cos(psi*t)

diffeq = ode(u(t))
u_sol = dsolve(diffeq, u(t))
u_sol = u_sol.rhs

# Determine the constants C1 and C2 in u_sol
# (first substitute our own declared C1 and C2 symbols,
# then use the initial conditions)
u_sol = u_sol.subs('C1', C1).subs('C2', C2)
eqs = [u_sol.subs(t, 0) - I, u_sol.diff(t).subs(t, 0) - V]
s = solve(eqs, [C1, C2])
u_sol = u_sol.subs(C1, s[C1]).subs(C2, s[C2])

# Check that the solution fulfills the equation and init.cond.
print simplify(ode(u_sol))
print simplify(u_sol.subs(t, 0) - I)
print simplify(diff(u_sol, t).subs(t, 0) - V)

u_sol = simplify(expand(u_sol.subs(A1, A/m)))
print u_sol
```

The output from the last line is

```
A/m*cos(psi*t)/(-psi**2 + w**2) + V*sin(t*w)/w +
(A/m + I*psi**2 - I*w**2)*cos(t*w)/(psi**2 - w**2)
```

With a bit of rewrite this expression becomes

$$u(t) = \frac{A/m}{\omega^2-\psi^2}\cos(\psi t) + \frac{V}{\omega}\sin(\omega t) + \left(\frac{A/m}{\psi^2-\omega^2} + I\right)\cos(\omega t)\,.$$

Obviously, this expression is only meaningful for $\psi \neq \omega$. The case $\psi = \omega$ gives an infinite amplitude in this model, a phenomenon known as resonance. The amplitude becomes finite when damping is included, see Section 2.2.4.

For the case the system starts from rest, $I = V = 0$, and the forcing is the only driving mechanism, we can simplify:

$$\begin{aligned} u(t) &= \frac{A}{m(\omega^2-\psi^2)}\cos(\psi t) + \frac{A}{m(\psi^2-\omega^2)}\cos(\omega t) \\ &= \frac{A}{m(\omega^2-\psi^2)}(\cos(\psi t) - \cos(\omega t))\,. \end{aligned}$$

To gain more insight, $\cos(\psi t) - \cos(\omega t)$ can be rewritten in terms of the mean frequency $(\psi+\omega)/2$ and the difference in frequency $(\psi-\omega)/2$:

$$u(t) = \frac{A}{m(\omega^2-\psi^2)}2\sin\left(\frac{\psi-\omega}{2}t\right)\sin\left(\frac{\psi+\omega}{2}t\right), \tag{2.70}$$

showing that there is a signal with frequency $(\psi+\omega)/2$ whose amplitude has a (much) slower frequency $(\psi-\omega)/2$. Figure 2.7 shows an example on such a signal.

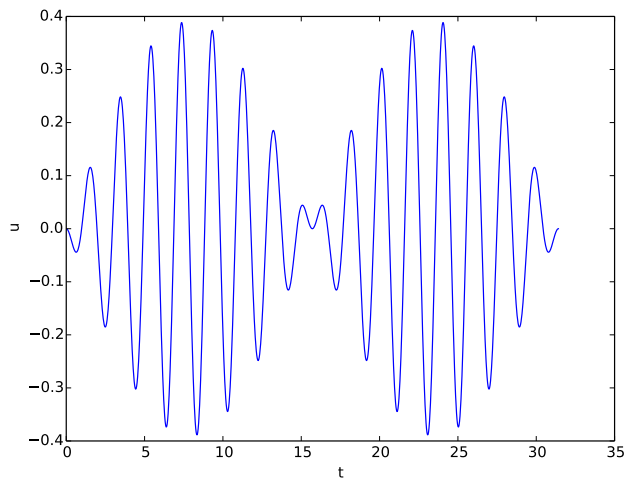


Fig. 2.7 Signal with frequency 3.1 and envelope frequency 0.2.

The displacement and time scales. A characteristic displacement can in the latter special case be taken as $u_c = A/(m(\omega^2-\psi^2))$. This is also a relevant choice in the more general case $I \neq 0, V \neq 0$, unless I or V is so large that it dominates over the amplitude caused by the forcing. With $u_c = A/(m(\omega^2-\psi^2))$ we also have three special cases: $\omega \ll \psi$, $\omega \gg \psi$, and $\psi \sim \omega$. In the latter case we need $u_c = A/(m(\omega^2-\psi^2))$ if we want $|u| \leq 1$. When ω and ψ are significantly different, we may choose one of them and neglect the smaller. Choosing ω means $u_c = A/k$, which is the relevant scale if $\omega \gg \psi$. In the opposite case, $\omega \ll \psi$, $u_c = A/(m\psi^2)$.

The time scale is dominated by the fastest oscillations, which are of frequency ψ or ω when these are close and the largest of them when they are distant. In any case, we set $t_c = 1/\max(\psi,\omega)$.

Finding the displacement scale from the differential equation. Going back to (2.69), we may demand that all the three terms in the differential equation are of size unity. This leads to $t_c = \sqrt{m/k}$ and $u_c = At_c^2/m = A/k$. The formula for u_c is a kind of measure of the ratio of the forcing and the spring force (the dimensionless number $A/(ku_c)$ would be this ratio).

Looking at (2.70), we see that if $\psi \ll \omega$, the amplitude can be approximated by $A/(m\omega^2) = A/k$, showing that the scale $u_c = A/k$ is relevant for small excitation frequency ψ compared to the free vibration frequency ω.

Scaling with free vibrations as time scale. The next step is to work out the dimensionless ODE for the chosen scales. We first select the time scale based on the free oscillations with frequency ω, i.e., $t_c = 1/\omega$. Inserting the expression in (2.69) results in

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \bar{u} = \gamma\cos(\delta\bar{t}), \quad \bar{u}(0) = \alpha,\ \bar{u}'(0) = \beta\,. \tag{2.71}$$

Here we have four dimensionless variables

$$\alpha = \frac{I}{u_c}, \tag{2.72}$$

$$\beta = \frac{Vt_c}{u_c} = \frac{V}{\omega u_c}, \tag{2.73}$$

$$\gamma = \frac{t_c^2 A}{mu_c} = \frac{A}{ku_c}, \tag{2.74}$$

$$\delta = \frac{t_c}{\psi^{-1}} = \frac{\psi}{\omega}\,. \tag{2.75}$$

These dimensionless variables have interpretations as ratios of physical effects:

- α: ratio of the initial displacement and the characteristic response u_c,
- β: ratio of the initial velocity and the typical velocity measure u_c/t_c,
- γ: ratio of the forcing A and the mass times acceleration mu_c/t_c^2 *or* the ratio of the forcing and the spring force ku_c
- δ: ratio of the frequencies or the time scales of the forcing and the free vibrations.

Software. Any solver for (2.69) can be used for (2.71). More details are provided at the end of Section 2.2.4.

Choice of u_c close to resonance. We have chosen not to specify u_c in the formulas above. Now we shall discuss various choices of u_c. Close to resonance, when $\psi \sim \omega$, we may set $u_c = A/(m(\omega^2 - \psi^2))$. The dimensionless numbers become in this case

$$\begin{aligned}
\alpha &= \frac{I}{u_c} = \frac{I}{A/k}(1-\delta^2),\\
\beta &= \frac{V}{\omega u_c} = \frac{V\sqrt{km}}{A}(1-\delta^2),\\
\gamma &= \frac{A}{ku_c} = 1-\delta^2,\\
\delta &= \frac{\psi}{\omega}\,.
\end{aligned}$$

With $\psi = 0.99\omega$, $\delta = 0.99$, $V = 0$, $\alpha = \gamma = 1 - \delta^2 = 0.02$, we have the problem

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \bar{u} = 0.02\cos(0.99\bar{t}), \quad \bar{u}(0) = 0.02,\ \bar{u}'(0) = 0\,.$$

This is a problem with a very small initial condition and a very small forcing, but the state close to resonance brings the amplitude up to about unity, see the result of numerical simulations with $\delta = 0.99$ in Figure 2.8. Neglecting α, the solution is given by (2.70), which here means $A = 1 - \delta^2$, $m = 1$, $\omega = 1$, $\psi = \delta$:

$$\bar{u}(\bar{t}) = 2\sin(-0.005\bar{t})\sin(0.995\bar{t})\,.$$

Note that this is a problem which demands very high accuracy in the numerical calculations. Using 20 time steps per period gives a significant angular frequency error and an amplitude of about 1.4. We used 160 steps per period for the results in Figure 2.8.

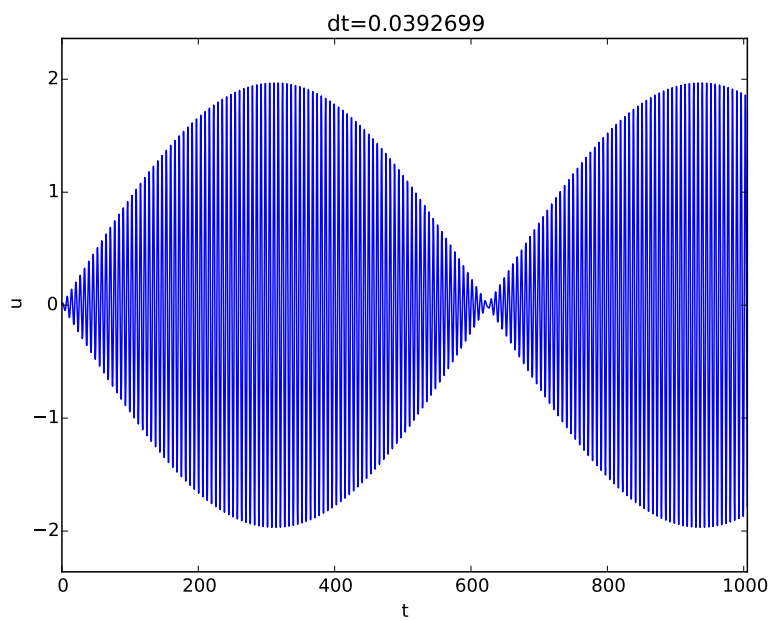


Fig. 2.8 Forced undamped vibrations close to resonance.

Unit size of all terms in the ODE. Using the displacement scale $u_c = A/k$ leads to (2.71) with

$$\alpha = \frac{I}{u_c} = \frac{I}{A/k},$$
$$\beta = \frac{V}{\omega u_c} = \frac{Vk}{A\omega},$$
$$\gamma = \frac{A}{ku_c} = 1,$$
$$\delta = \frac{\psi}{\omega}.$$

Simulating a case with $\delta = 0.5$, $\alpha = 1$, and $\beta = 0$ gives the oscillations in Figure 2.9, which is a case away from resonance, and the amplitude is about unity. However, choosing $\delta = 0.99$ (close to resonance) results in a figure similar to Figure 2.8, except that the amplitude is about 10^2 because of the moderate size of u_c. The present scaling is therefore most suitable away from resonance, and when the terms containing $\cos\omega t$ and $\sin\omega t$ are important (e.g., $\omega \gg \psi$).

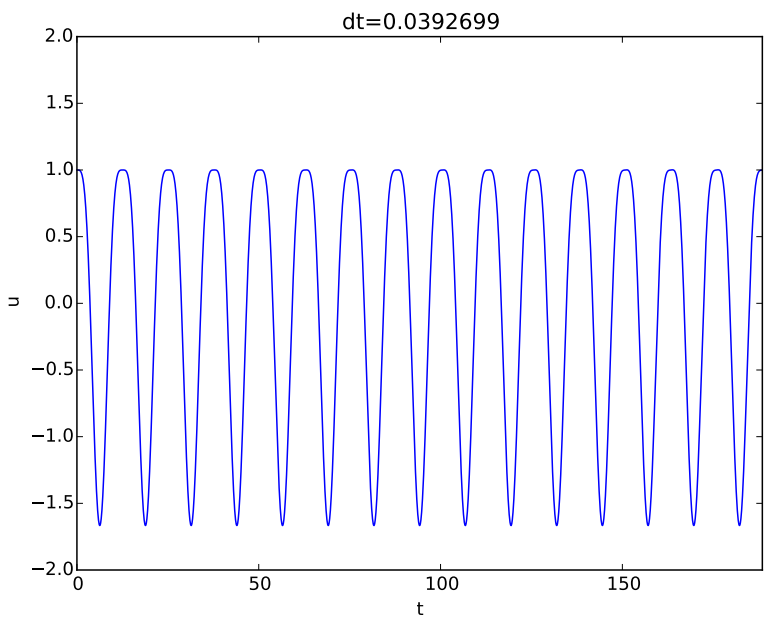


Fig. 2.9 Forced undamped vibrations away from resonance.

Choice of u_c when $\psi \gg \omega$. Finally, we may look at the case where $\psi \gg \omega$ such that $u_c = A/(m\psi^2)$ is a relevant scale (i.e., omitting ω^2 compared to ψ^2 in the denominator), but in this case we should use $t_c = 1/\psi$ since the force

varies much faster than the free vibrations of the system. This choice of t_c changes the scaled ODE to

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \delta^{-2}\bar{u} = \gamma\cos(\bar{t}), \quad \bar{u}(0) = \alpha,\ \bar{u}'(0) = \beta, \tag{2.76}$$

where

$$\begin{aligned}
\alpha &= \frac{I}{u_c} = \frac{I}{A/k}\delta^2,\\
\beta &= \frac{Vt_c}{u_c} = \frac{V\sqrt{km}}{A}\delta,\\
\gamma &= \frac{t_c^2 A}{mu_c} = 1,\\
\delta &= \frac{t_c}{\psi^{-1}} = \frac{\psi}{\omega}.
\end{aligned}$$

In the regime $\psi \gg \omega$, $\delta \gg 1$, thus making α and β large. However, if α and/or β is large, the initial condition dominates over the forcing, and will also dominate the amplitude of u, thereby making the scaling of u inappropriate. In case $I = V = 0$ so that $\alpha = \beta = 0$, (2.70) predicts ($A = m = 1$, $\omega = \delta^{-1}$, $\psi = 1$)

$$\bar{u}(\bar{t}) = (\delta^{-2} - 1)^{-1} 2\sin\left(\frac{1}{2}(1-\delta^{-1})\bar{t}\right)\sin\left(\frac{1}{2}(1+\delta^{-1})\bar{t}\right),$$

which has an amplitude about 2 for $\delta \gg 1$. Figure 2.10 shows a case.

With $\alpha = 0.05\delta^2 = 5$, we get a significant contribution from the free vibrations (the homogeneous solution of the ODE) as shown in Figure 2.11. For larger α values, one must base u_c on I instead. (The graphs in Figure 2.10 and 2.11 were produced by numerical simulations with 160 time steps per period of the forcing.)

Displacement scale based on I. Choosing $u_c = I$ gives

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \bar{u} = \gamma\cos(\delta\bar{t}), \quad \bar{u}(0) = 1,\ \bar{u}'(0) = \beta, \tag{2.77}$$

with

$$\beta = \frac{Vt_c}{u_c} = \frac{V}{I}\sqrt{\frac{m}{k}}, \tag{2.78}$$

$$\gamma = \frac{tc^2 A}{mu_c} = \frac{A}{ku_c} = \frac{A}{kI}. \tag{2.79}$$

This scaling is not relevant close to resonance since then $u_c \gg I$.

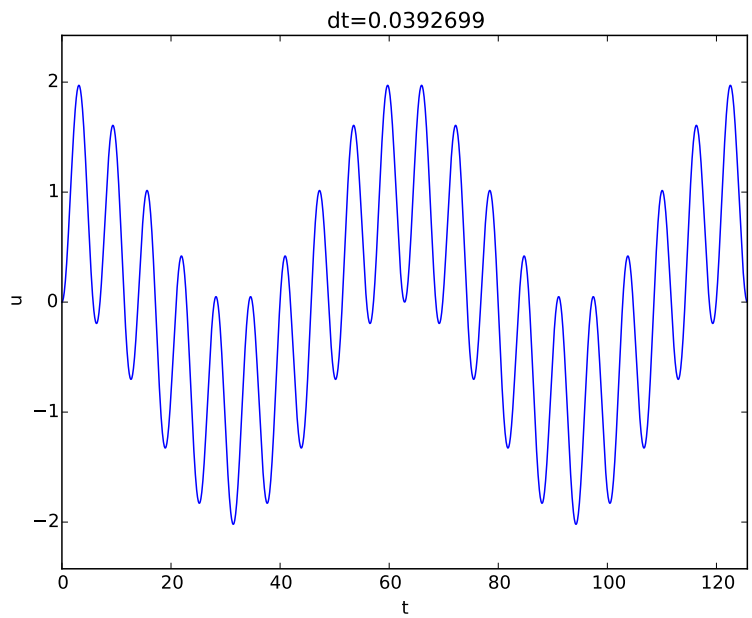


Fig. 2.10 Forced undamped vibrations with rapid forcing.

2.2.4 Damped vibrations with forcing

We now introduce a linear damping force $bu'(t)$ in the equation of motion:

$$mu'' + bu' + ku = A\cos(\psi t), \quad u(0) = I,\ u'(0) = V\,. \tag{2.80}$$

Figure 2.12 shows a typical one-degree-of-freedom mechanical system with a linear dashpot, representing the damper (bu'), a linear spring (ku), and an external force (F).

The standard scaling procedure results in

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \frac{t_c b}{m}\frac{d\bar{u}}{d\bar{t}} + \frac{t_c^2 k}{m}\bar{u} = \frac{t_c^2}{mu_c}A\cos(\psi t_c\bar{t}), \quad \bar{u}(0) = \frac{I}{u_c},\ \bar{u}'(0) = \frac{Vt_c}{u_c}\,. \tag{2.81}$$

The exact solution. To choose scales, it is (as always) a great advantage to look into exact solutions. Using SymPy to solve (2.80) is, in principle, very straightforward:

```
>>> diffeq = diff(u(t), t, t) + b/m*diff(u(t), t) + w**2*u(t)
>>> s = dsolve(diffeq, u(t))
>>> s.rhs
```

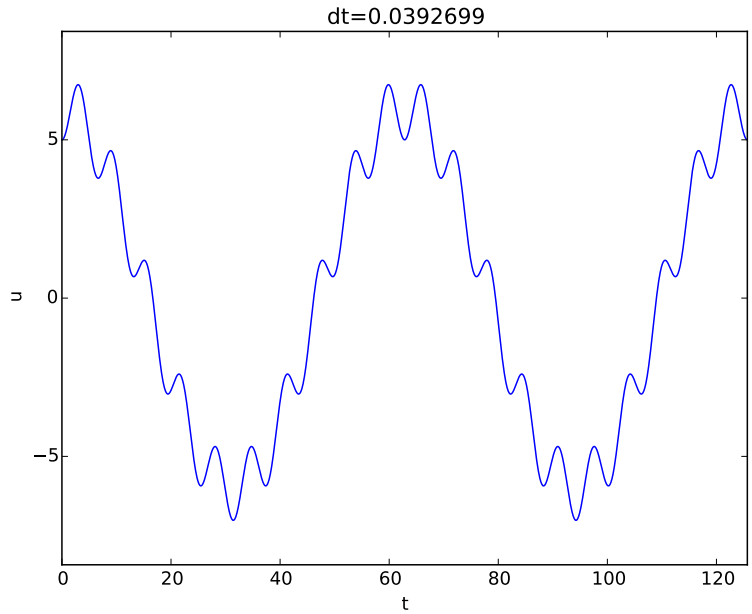


Fig. 2.11 Forced undamped vibrations with rapid forcing and initial displacement of 5.

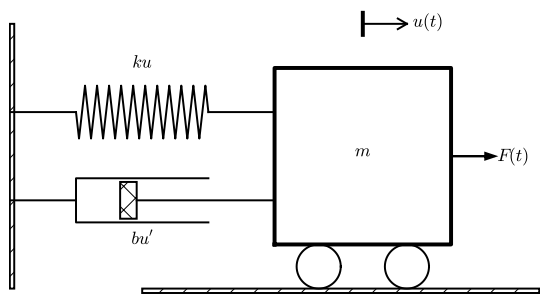


Fig. 2.12 Oscillating body with external force, attached to a spring and damper.

```
C1*exp(t*(-b - sqrt(b - 2*m*w)*sqrt(b + 2*m*w))/(2*m)) +
C2*exp(t*(-b + sqrt(b - 2*m*w)*sqrt(b + 2*m*w))/(2*m))
```

This is indeed the correct solution, but it is on a complex exponential function form, valid for all b, m, and ω. We are interested in the case with *small damping*, $b < 2m\omega$, where the solution is an exponentially damped sinusoidal function. Rewriting the expression in the right form is tricky with SymPy commands. Instead, we demonstrate a common technique when doing symbolic computing: general procedures like `dsolve` are replaced by manual

steps. That is, we solve the ODE "by hand", but use SymPy to assist the calculations.

The solution is composed of a homogeneous solution u_h of $mu'' + bu' + ku = 0$ and one particular solution u_p of the nonhomogeneous equation $mu'' + bu' + ku = A\cos(\psi t)$. The homogeneous solution with damped oscillations (requiring $b < 2\sqrt{mk}$) can be found by the following code. We have divided the differential equation by m and introduced $B = \frac{1}{2}b/m$ and let `A1` represent A/m to simplify expressions and help SymPy with less symbols in the equation (without these simplifications, SymPy stalls in the computations due to too many symbols in the equation).

```
u = symbols('u', cls=Function)
t, w, B, A, A1, m, psi = symbols('t w B A A1 m psi',
                                 positive=True, real=True)

def ode(u, homogeneous=True):
    h = diff(u, t, t) + 2*B*diff(u, t) + w**2*u
    f = A1*cos(psi*t)
    return h if homogeneous else h - f

# Find coefficients in polynomial (in r) for exp(r*t) ansatz
r = symbols('r')
ansatz = exp(r*t)
poly = simplify(ode(ansatz)/ansatz)

# Convert to polynomial to extract coefficients
poly = Poly(poly, r)
# Extract coefficients in poly: a_*t**2 + b_*t + c_
a_, b_, c_ = poly.coeffs()
# Assume b_**2 - 4*a_*c_ < 0
d = -b_/(2*a_)
if a_ == 1:
    omega = sqrt(c_ - (b_/2)**2)  # nicer formula
else:
    omega = sqrt(4*a_*c_ - b_**2)/(2*a_)

# The homogeneous solution is a linear combination of a
# cos term (u1) and a sin term (u2)
u1 = exp(d*t)*cos(omega*t)
u2 = exp(d*t)*sin(omega*t)
C1, C2, V, I = symbols('C1 C2 V I', real=True)
u_h = simplify(C1*u1 + C2*u2)
print 'u_h:', u_h
```

The print out shows

$$u_h = e^{-Bt}\left(C_1\cos(\sqrt{\omega^2 - B^2}t) + C_2\sin(\sqrt{\omega^2 - B^2}t)\right),$$

where C_1 and C_2 must be determined by the initial conditions later. It is wise to check that u_h is indeed a solution of the homogeneous differential equation:

```
assert simplify(ode(u_h)) == 0
```

We have previously just printed the residuals of the ODE and initial conditions after inserting the solution, but it is better in a code to let the program-

ming language test that the residuals are symbolically zero. This is achieved using the `assert` statement in Python. The argument is a boolean expression, and if the expression evaluates to `False`, an `AssertionError` is raised and the program aborts (otherwise `assert` runs silently for a `True` boolean expression). Hereafter, we will use `assert` for consistency checks in computer code.

The ansatz for the particular solution u_p is

$$u_p = C_3 \cos(\psi t) + C_4 \sin(\psi t),$$

which inserted in the ODE gives two equations for C_3 and C_4. The relevant SymPy statements are

```
# Particular solution
C3, C4 = symbols('C3 C4')
u_p = C3*cos(psi*t) + C4*sin(psi*t)
eqs = simplify(ode(u_p, homogeneous=False))

# Collect cos(omega*t) terms
print 'eqs:', eqs
eq_cos = simplify(eqs.subs(sin(psi*t), 0).subs(cos(psi*t), 1))
eq_sin = simplify(eqs.subs(cos(psi*t), 0).subs(sin(psi*t), 1))
s = solve([eq_cos, eq_sin], [C3, C4])
u_p = simplify(u_p.subs(C3, s[C3]).subs(C4, s[C4]))

# Check that the solution is correct
assert simplify(ode(u_p, homogeneous=False)) == 0
```

Using the initial conditions for the complete solution $u = u_h + u_p$ determines C_1 and C_2:

```
u_sol = u_h + u_p  # total solution
# Initial conditions
eqs = [u_sol.subs(t, 0) - I, u_sol.diff(t).subs(t, 0) - V]
# Determine C1 and C2 from the initial conditions
s = solve(eqs, [C1, C2])
u_sol = u_sol.subs(C1, s[C1]).subs(C2, s[C2])
```

Finally, we should check that `u_sol` is indeed the correct solution:

```
checks = dict(
    ODE=simplify(expand(ode(u_sol, homogeneous=False))),
    IC1=simplify(u_sol.subs(t, 0) - I),
    IC2=simplify(diff(u_sol, t).subs(t, 0) - V))
for check in checks:
    msg = '%s residual: %s' % (check, checks[check])
    assert checks[check] == sympify(0), msg
```

Finally, we may take `u_sol = u_sol.subs(A, A/m)` to get the right expression for the solution. Using `latex(u_sol)` results in a huge expression, which should be manually ordered to something like the following:

$$
\begin{aligned}
u =& \frac{Am^{-1}}{4B^2\psi^2+\Omega^2}\left(2B\psi\sin(\psi t)-\Omega\cos(\psi t)\right)+\\
& e^{-Bt}\left(C_1\cos\left(t\sqrt{\omega^2-B^2}\right)+C_2\sin\left(t\sqrt{\omega^2-B^2}\right)\right)\\
C_1 =& \frac{Am^{-1}\Omega+4IB^2\psi^2+I\Omega^2}{4B^2\psi^2+\Omega^2}\\
C_2 =& \frac{-Am^{-1}B\Omega+4IB^3\psi^2+IB\Omega^2+4VB^2\psi^2+V\Omega^2}{\sqrt{\omega^2-B^2}\,(4B^2\psi^2+\Omega^2)},\\
\Omega =& \psi^2-\omega^2\,.
\end{aligned}
$$

The most important feature of this solution is that there are two time scales with frequencies ψ and $\sqrt{\omega^2-B^2}$, but the latter appears in terms that decay as e^{-Bt} in time. The attention is usually on longer periods of time, so in that case the solution simplifies to

$$
\begin{aligned}
u &= \frac{Am^{-1}}{4B^2\psi^2+\Omega^2}\left(2B\psi\sin(\psi t)-\Omega\cos(\psi t)\right)\\
&= \frac{A}{m}\frac{1}{\sqrt{4B^2\psi^2+\Omega^2}}\cos(\psi t+\phi)\frac{(\psi\omega)^{-1}}{(\psi\omega)^{-1}}\\
&= \frac{A}{k}Q\delta^{-1}\left(1+Q^2(\delta-\delta^{-1})\right)^{-\frac{1}{2}}\cos(\psi t+\phi),
\end{aligned}
\tag{2.82}
$$

where we have introduced the dimensionless numbers

$$
Q=\frac{\omega}{2B},\quad \delta=\frac{\psi}{\omega},
$$

and

$$
\phi=\tan^{-1}\left(-\frac{2B}{\omega^2-\psi^2}\right)=\tan^{-1}\left(\frac{Q^{-1}}{\delta^2-1}\right).
$$

Q is commonly called *quality factor* and ϕ is the *phase shift*. Dividing (2.82) by A/k, which is a common scale for u, gives the dimensionless relation

$$
\frac{u}{A/k}=\frac{Q}{\delta}R(Q,\delta)^{\frac{1}{2}}\cos(\psi t+\phi),\quad R(Q,\delta)=\left(1+Q^2(\delta-\delta^{-1})\right)^{-1}. \tag{2.83}
$$

Choosing scales. Much of the discussion about scales in the previous sections are relevant also when damping is included. Although the oscillations with frequency $\sqrt{\omega^2-B^2}$ die out for $t\gg B^{-1}$, we start with using this frequency for the time scale. A highly relevant assumption for engineering applications of (2.80) is that the damping is small. Therefore, $\sqrt{\omega^2-B^2}$ is close

to ω and we simply apply $t_c = 1/\omega$ as before (if not the interest in large t for which the oscillations with frequency ω has died out).

The coefficient in front of the $\bar{u}'$ term then becomes

$$\frac{b}{m\omega} = \frac{2B}{\omega} = Q^{-1}\,.$$

The rest of the ODE is given in the previous section, and the particular formulas depend on the choices of t_c and u_c.

Choice of u_c at resonance. The relevant scale for u_c at or nearby resonance ($\psi = \omega$) becomes different from the previous section, since with damping, the maximum amplitude is a finite value. For $t \gg B^{-1}$ when the $\sin\psi t$ term is dominating, we have for $\psi = \omega$:

$$u = \frac{Am^{-1}2B\psi}{4B^2\psi^2}\sin(\psi t) = \frac{A}{2Bm\psi}\sin(\psi t) = \frac{A}{b\psi}\sin(\psi t)\,.$$

This motivates the choice

$$u_c = \frac{A}{b\psi} = \frac{A}{b\omega}\,.$$

(It is wise during computations like this to stop and check the dimensions: A must be $[\mathrm{MLT}^{-2}]$ from the original equation ($F(t)$ must have the same dimension as mu''), bu' also has dimension $[\mathrm{MLT}^{-2}]$, implying that b has dimension $[\mathrm{MT}^{-1}]$. A/b then has dimension LT^{-1}, and $A/(b\psi)$ gets dimension $[L]$, which matches what we want for u_c.)

The differential equation on dimensionless form becomes

$$\frac{d^2\bar{u}}{d\bar{t}^2} + Q^{-1}\frac{d\bar{u}}{d\bar{t}} + \bar{u} = \gamma\cos(\delta\bar{t}), \quad \bar{u}(0) = \alpha,\ \bar{u}'(0) = \beta, \tag{2.84}$$

with

$$\alpha = \frac{I}{u_c} = \frac{Ib}{A}\sqrt{\frac{k}{m}}, \tag{2.85}$$

$$\beta = \frac{Vt_c}{u_c} = \frac{Vb}{A}, \tag{2.86}$$

$$\gamma = \frac{t_c^2 A}{mu_c} = \frac{b\omega}{k}, \tag{2.87}$$

$$\delta = \frac{t_c}{\psi^{-1}} = \frac{\psi}{\omega} = 1\,. \tag{2.88}$$

Choice of u_c when $\omega \gg \psi$. In the limit $\omega \gg \psi$ and $t \gg B^{-1}$,

$$u \approx \frac{A}{m\omega^2}\cos\psi t = \frac{A}{k}\cos\psi t,$$

showing that $u_c = A/k$ is an appropriate displacement scale. (Alternatively, we get this scale also from demanding $\gamma = 1$ in the ODE.) The dimensionless numbers α, β, and δ are as for the forced vibrations without damping.

Choice of u_c when $\omega \ll \psi$. In the limit $\omega \ll \psi$, we should base t_c on the rapid variations in the excitation: $t_c = 1/\psi$.

Software. It is easy to reuse a solver for a general vibration problem also in the dimensionless case. In particular, we may use the `solver` function in the file `vib.py`:

```
def solver(I, V, m, b, s, F, dt, T, damping='linear'):
```

for solving the ODE problem

$$mu'' + f(u') + s(u) = F(t), \quad u(0) = I,\ u'(0) = V,\ t \in (0,T],$$

with time steps `dt`. With `damping='linear'`, we have $f(u') = bu'$, while the other value is `'quadratic'`, meaning $f(u') = b|u'|u'$. Given the dimensionless numbers α, β, γ, δ, and Q, an appropriate call for solving (2.71) is

```
u, t = solver(I=alpha, V=beta, m=1, b=1.0/Q,
              s=lambda u: u, F=lambda t: gamma*cos(delta*t),
              dt=2*pi/n, T=2*pi*P)
```

where `n` is the number of intervals per period and `P` is the number of periods to be simulated. We way wrap this call in a `solver_scaled` function and wrap it furthermore with `joblib` to avoid repeated calls, as we explained in Section 2.1.4:

```
from vib import solver as solver_unscaled

def solver_scaled(alpha, beta, gamma, delta, Q, T, dt):
    """
    Solve u'' + (1/Q)*u' + u = gamma*cos(delta*t),
    u(0)=alpha, u'(1)=beta, for (0,T] with step dt.
    """
    print 'Computing the numerical solution'
    from math import cos
    return solver_unscaled(I=alpha, V=beta, m=1, b=1./Q,
                           s=lambda u: u,
                           F=lambda t: gamma*cos(delta*t),
                           dt=dt, T=T, damping='linear')

import joblib
disk_memory = joblib.Memory(cachedir='temp')
solver_scaled = disk_memory.cache(solver_scaled)
```

This code is found in `vib_scaled.py` and features an application for running the scaled problem with options on the command-line for α, β, γ, δ, Q, number of time steps per period, and number of periods (see the `main` function). It is an ideal application for exploring scaled vibration models.

2.2.5 Oscillating electric circuits

The differential equation for an oscillating electric circuit is very similar to the equation for forced, damped, mechanical vibrations, and their dimensionless form is identical. This fact will now be demonstrated.

The current $I(t)$ in a circuit with an inductor with inductance L, a capacitor with capacitance C, and overall resistance R, obeys the equation

$$\ddot{I} + \frac{R}{L}\dot{I} + \frac{1}{LC}I = \dot{V}(t), \tag{2.89}$$

where $V(t)$ is the voltage source powering the circuit. We introduce

$$\bar{I} = \frac{I}{I_c}, \quad \bar{t} = \frac{t}{t_c},$$

and get

$$\frac{d^2\bar{I}}{d\bar{t}^2} + \frac{t_c R}{L}\frac{d\bar{I}}{d\bar{t}} + \frac{t_c^2}{LC}\bar{I} = \frac{t_c^2 V_c}{I_c}\bar{V}(t)\,.$$

Here, we have scaled $V(t)$ according to

$$\bar{V}(\bar{t}) = \frac{V(t_c\bar{t})}{\max_t V(t)}\,.$$

The time scale t_c is chosen to make $\ddot{I}$ and $I/(LC)$ balance, $t_c = \sqrt{LC}$. Choosing I_c to make the coefficient in the source term of unit size, means $I_c = LCV_c$. With

$$Q^{-1} = R\sqrt{\frac{C}{L}},$$

we get the scaled equation

$$\frac{d^2\bar{I}}{d\bar{t}^2} + Q^{-1}\frac{d\bar{I}}{d\bar{t}} + \bar{I} = \bar{V}(t), \tag{2.90}$$

which is basically the same as we derived for mechanical vibrations. (Two additional dimensionless variables will arise from the initial conditions for I, just as in the mechanics cases.)

2.3 Exercises

Exercise 2.1: Perform unit conversion

Density (mass per volume: $[\mathrm{ML}^{-3}]$) of water is given as 1.05 ounce per fluid ounce. Use the `PhysicalQuantity` object to convert to $\mathrm{kg\,m^{-3}}$.

Solution. Use `pydoc PhysicalQuantities` to find that `floz` is the name of the volume "fluid ounce" and `oz` is the name of the mass "ounce". Here is an interactive session for the conversion:

```
>>> from PhysicalQuantities import PhysicalQuantity as PQ
>>> d = PQ('1.05 oz/floz')
>>> d.convertToUnit('kg/m**3')
>>> print d
1006.54198946 kg/m**3
```

Filename: `density_conversion`.

Problem 2.2: Scale a simple formula

The height y of a body thrown up in the air is given by

$$y = v_0 t - \frac{1}{2} g t^2,$$

where t is time, v_0 is the initial velocity of the body at $t = 0$, and g is the acceleration of gravity. Scale this formula. Use two choices of the characteristic time: the time it takes to reach the maximum y value and the time it takes to return to $y = 0$.

Solution. We introduce

$$\bar{y} = \frac{y}{y_c}, \quad \bar{t} = \frac{t}{t_c}.$$

Inserted in the formula we get

$$y_c \bar{y} = v_0 t_c \bar{t} - \frac{1}{2} g t_c^2 \bar{t}^2.$$

1. At the maximum point of y, $y' = 0$, so $y' = v_0 - gt = 0$, which means $t = v_0/g$ and $y_{\max} = v_0 v0/g - \frac{1}{2} g v_0^2/g^2 = \frac{1}{2} v_0^2/g$. We choose $t_c = v_0/g$ and $y_c = \frac{1}{2} v_0^2/g$. This gives

$$\frac{1}{2}\frac{v_0^2}{g}\bar{y} = \frac{v_0^2}{g}\bar{t} - \frac{1}{2}\frac{v_0^2}{g}\bar{t}^2 \quad \Rightarrow \quad \bar{y} = 2\bar{t} - \bar{t}^2.$$

2. The body is back at $y=0$ for $v_0t-\frac{1}{2}gt^2=0$, which gives $t_c=2v_0/g$ and $y_c=2y_{\max}=v_0^2/g$. Inserted, we get

$$\frac{v_0^2}{g}\bar{y}=2\frac{v_0^2}{g}\bar{t}-\frac{1}{2}4\frac{v_0^2}{g}\bar{t}^2 \quad\Rightarrow\quad \bar{y}=2\bar{t}(1-\bar{t})\,.$$

Observe that the physical parameters v_0 and g are absent in the scaled formula.
Filename: `vertical_motion`.

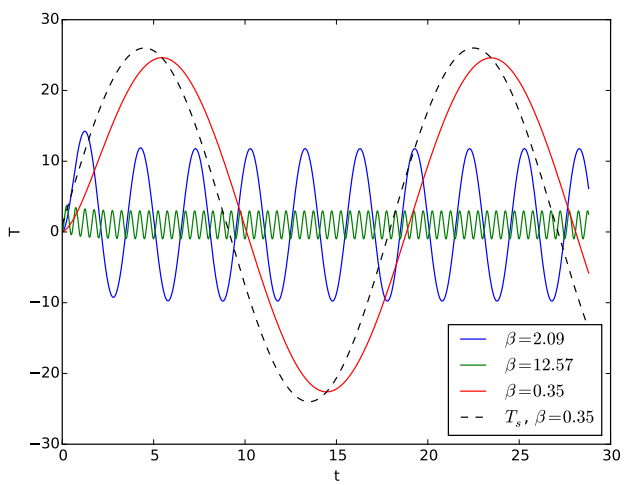
Exercise 2.3: Perform alternative scalings

The problem in Section 2.1.8 applies a temperature scaling

$$\bar{T}=\frac{T-T_0}{T_m-T_0},$$

which is not always suitable.

a) Consider the case $T_0=T_m$ and the fact that $|T_m-T_0|$ does not represent the characteristic temperature scale since it collapses to zero. Formulate a suitable scaling in this case. The figure below corresponds to $T_m=25$ C, $T_0=24.9$ C, and $a=2.5$ C. We clearly see that $\bar{T}$ is not of size unity.



Solution. The typical temperature variations will now be oscillations of amplitude a around $T_m=T_0$, so $2a$ is the typical variation of the surrounding temperature. If the time scale of T_s is sufficiently large (or more precisely, β is small), the temperature will actually reach amplitudes of size a, but for fast oscillations in T_s, there will not be enough time to transfer heat to/from

the body, so the amplitudes of T will be smaller. Taking $2a$ to be the typical temperature range, we can propose the scaling

$$\bar{T} = \frac{T - T_m}{2a}.$$

Inserted in the differential equation, we get with $t_c = 1/k$,

$$k2a\frac{\bar{T}}{d\bar{t}} = -k(2a\bar{T} + T_m - (T_m + a\sin(\omega t))),$$

which simplifies to

$$\frac{\bar{T}}{d\bar{t}} = -(\bar{T} - \frac{1}{2}\sin(\beta t)),$$

where $\beta = \omega/k$. The initial condition becomes

$$\bar{T}(0) = -\frac{1}{2}\alpha,$$

where $\alpha = a/(T_m - T_0)$ is the dimensionless number that appeared in the scaled differential equation in Section 2.1.8.

b) Consider the case where a is much larger than $|T_m - T_0|$. What is an appropriate scaling of the temperature?

Solution. In this case, T will oscillate around T_m and at maximum reach the amplitude a if β is small, see the figure in a). This is the same situation as in a), and we can consequently use the same scaling and obtain the same scaled problem.

Problem 2.4: A nonlinear ODE for vertical motion with air resistance

The velocity $v(t)$ of a body moving vertically through a fluid in the gravity field, with fluid drag and buoyancy, is governed by ODE

$$mv' = -\frac{1}{2}C_D\varrho A|v|v - mg + \varrho V g, \quad v(0) = v_0,$$

where t is time, m is the mass of the body, C_D is a drag coefficient, ϱ is the density of the fluid, A is the cross-sectional area perpendicular to the motion, g is the acceleration of gravity, and V is the volume of the body. Scale this ODE.

Solution. We introduce as usual

$$\bar{v} = \frac{v}{v_c}, \quad \bar{t} = \frac{t}{t_c},$$

but the main challenge is to find values for v_c and t_c. Inserting the scaled quantities gives

$$m\frac{v_c}{t_c}\frac{d\bar{v}}{d\bar{t}} = -\frac{1}{2}C_D\varrho A v_c^2|\bar{v}|\bar{v} - mg + \varrho V g, \quad v_c v(0) = v_0,$$

It is tempting to set $v_c = v_0$, but $v_0 = 0$ is a relevant value so this choice is not good. The motion is of decay type so t_c and v_c should be based on characteristics of the decay. The terminal velocity, defined by $v' = 0$, is

$$v_T = \sqrt{\frac{2(\varrho V - m)g}{C_D\varrho A}},$$

when $\varrho V > m$ such that the buoyancy wins over gravity and the motion is upwards. Otherwise,

$$v_T = -\sqrt{\frac{2(m - \varrho V)g}{C_D\varrho A}}.$$

The two formulas can be combined to

$$v_T = \text{sign}(\varrho V - m)\sqrt{\frac{2|\varrho V - m|g}{C_D\varrho A}}.$$

We take $v_c = |v_T| = \sqrt{\frac{2|\varrho V - m|g}{C_D\varrho A}}$. This results in

$$\frac{d\bar{v}}{d\bar{t}} = -t_c\frac{1}{2m}C_D\varrho A\sqrt{\frac{2|\varrho V - m|g}{C_D\varrho A}}|\bar{v}|\bar{v} - t_c\sqrt{\frac{C_D\varrho g A}{2|\varrho V - m|}}(1 - \frac{\varrho V}{m}),$$

and

$$v(0) = v_0\sqrt{\frac{C_D\varrho A}{2g|\varrho V - m|}}.$$

A natural choice is to assume $d\bar{v}/d\bar{t}$ and $\bar{v}$ to be of the same order, which means that coefficient in front of the nonlinear term $|\bar{v}|\bar{v}$ should be unity. This forces t_c to be

$$t_c = \frac{2m}{\sqrt{2g|\varrho V - m|C_D\varrho A}}.$$

Introducing the dimensionless numbers

$$\alpha = \frac{\varrho V}{m}, \quad \beta = v_0\sqrt{\frac{C_D\varrho A}{2g|\varrho V - m|}} = \frac{v_0}{|v_T|},$$

we get the scaled ODE problem

$$\frac{d\bar{v}}{d\bar{t}} = -|\bar{v}|\bar{v} + \text{sign}(1-\alpha), \quad \bar{v}(0) = \beta\,.$$

Note that, as usual, the dimensionless numbers have simple interpretations: α is the ratio of the mass of the displaced fluid and the mass of the body, while β is the ratio of the initial and terminal velocities.
Filename: `vertical_motion_with_drag`.

Exercise 2.5: Solve a decay ODE with discontinuous coefficient

Make software for the problem in Section 2.1.6 so that you can produce Figure 2.3.

Hint. Follow the ideas for software in Section 2.1.5: use the `decay_vc.py` module as computational engine and modify the `falling_body.py` code.

Solution. We use `joblib` to avoid unnecessary execution of the scaled problem, as explained in Section 2.1.4. A potential code is

```
import sys, os
# Enable loading modules in ../src-scaling
sys.path.insert(0, os.path.join(os.pardir, 'src-scaling'))
from decay_vc import solver as solver_unscaled
from math import pi
import matplotlib.pyplot as plt
import numpy as np

def solver_scaled(gamma, T, dt, theta=0.5):
    """
    Solve u'=-a*u, u(0)=1 for (0,T] with step dt and theta method.
    a=1 for t < gamma and 2 for t > gamma.
    """
    print 'Computing the numerical solution'
    return solver_unscaled(
        I=1, a=lambda t: 1 if t < gamma else 5,
        b=lambda t: 0, T=T, dt=dt, theta=theta)

import joblib
disk_memory = joblib.Memory(cachedir='temp')
solver_scaled = disk_memory.cache(solver_scaled)

def unscale(u_scaled, t_scaled, d, I):
    return I*u_scaled, d*t_scaled

def main(d,
         I,
         t_1,
         dt=0.04, # Time step, scaled problem
         T=4,     # Final time, scaled problem
         ):

    legends1 = []
    legends2 = []
```

```
    plt.figure(1)
    plt.figure(2)

    gamma = t_1*d
    print 'gamma=%.3f' % gamma
    u_scaled, t_scaled = solver_scaled(gamma, T, dt)

    plt.figure(1)
    plt.plot(t_scaled, u_scaled)
    legends1.append('gamma=%.3f' % gamma)

    plt.figure(2)
    u, t = unscale(u_scaled, t_scaled, d, I)
    plt.plot(t, u)
    legends2.append('d=%.2f [1/s], t_1=%.2f s' % (d, t_1))
    plt.figure(1)
    plt.xlabel('scaled time'); plt.ylabel('scaled velocity')
    plt.legend(legends1, loc='upper right')
    plt.savefig('tmp1.png');  plt.savefig('tmp1.pdf')

    plt.figure(2)
    plt.xlabel('t [s]');  plt.ylabel('u')
    plt.legend(legends2, loc='upper right')
    plt.savefig('tmp2.png');  plt.savefig('tmp2.pdf')
    plt.show()

if __name__ == '__main__':
    main(d=1/120., I=1, t_1=100)
```

Filename: `decay_jump`.

Exercise 2.6: Implement a scaled model for cooling

Use software for the unscaled problem (2.16) to compute the solution of the scaled problem (2.23). Let T_s be a function of time.

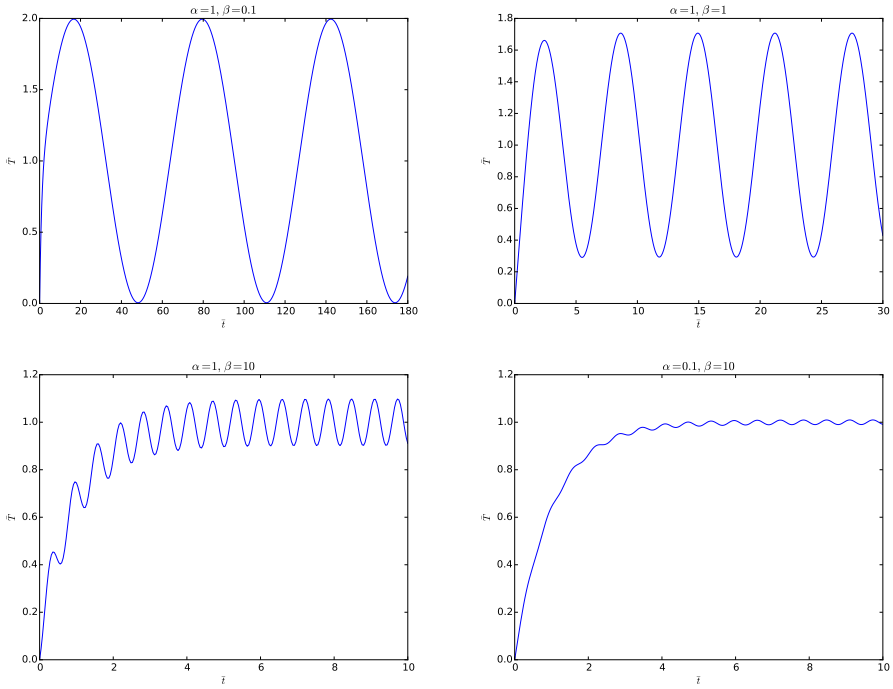
Hint. You may use the general software `decay_vc.py` for computing with the cooling model. See Section 2.1.5 for more ideas.

Solution. The problem (2.16) is just a special case of the general problem $u' = -au + b$ solved by the `decay_vc` module. We can make an implementation of (2.23) in terms of the model $u' = -au + b$:

```
import sys, os
# Enable loading modules in ../src-scaling
sys.path.insert(0, os.path.join(os.pardir, 'src-scaling'))
from decay_vc import solver as solver_unscaled
from math import pi
import matplotlib.pyplot as plt
import numpy as np

def solver_scaled(alpha, beta, t_stop, dt, theta=0.5):
    """
    Solve T' = -T + 1 + alha*sin(beta*t), T(0)=0
    for (0,T] with step dt and theta method.
    """
```

```
    print 'Computing the numerical solution'
    return solver_unscaled(
        I=0, a=lambda t: 1,
        b=lambda t: 1 + alpha*np.sin(beta*t),
        T=t_stop, dt=dt, theta=theta)

import joblib
disk_memory = joblib.Memory(cachedir='temp')
solver_scaled = disk_memory.cache(solver_scaled)

def main(alpha,
         beta,
         t_stop=50,
         dt=0.04
         ):

    T, t = solver_scaled(alpha, beta, t_stop, dt)
    plt.plot(t, T)
    plt.xlabel(r'$\bar t$');  plt.ylabel(r'$\bar T$')
    plt.title(r'$\alpha=%g,\ \beta=%g$' % (alpha, beta))
    filestem = 'tmp_%s_%s' % (alpha, beta)
    plt.savefig(filestem + '.png');  plt.savefig(filestem + '.pdf')
    plt.show()

if __name__ == '__main__':
    import sys
    alpha = float(sys.argv[1])
    beta = float(sys.argv[2])
    t_stop = float(sys.argv[3])
    main(alpha, beta, t_stop)
```

Simulations for $\alpha = 1$ and $\beta = 0.1, 1, 10$ as well as for $\alpha = 0.1$ and $\beta = 1$ are shown below.



Filename: `cooling1`.

Problem 2.7: Decay ODE with discontinuous coefficients

The goal of this exercise is to scale the problem $u'(t) = -a(t)u(t) + b(t)$, $u(0) = I$, when

$$a(t) = \begin{cases} Q, & t < s, \\ Q - A, & t \geq s, \end{cases} \quad b = \begin{cases} \epsilon t, & t < s, \\ 0, & t \geq s, \end{cases}$$

Here, $Q, A, \epsilon > 0$.

Solution. We start by scaling the known functions a and b. Since $Q, A.\epsilon > 0$, $\max|a(t)| = Q$ and $\max|b(t)| = \epsilon s$. Scaled versions of these functions are then

$$\bar{a} = \frac{a}{Q}, \quad \bar{b} = \frac{b}{\epsilon s}.$$

As usual, we scale u and t as

$$\bar{u} = \frac{u}{u_c}, \quad \bar{t} = \frac{t}{t_c}.$$

The scaled ODE reads

$$\frac{d\bar{u}}{d\bar{t}} = -t_c Q\bar{a}(\bar{t}) + u_c^{-1} t_c \epsilon s \bar{b}\,.$$

A natural choice of u_c is $u_c = I$. The a term will reduce $\bar{u}$ from 1, while the b term may have a growth effect.

The time scale is best chosen to reflect the dynamics of the process, i.e., the decay with strength Q, so we set $t_c = 1/Q$. This choice results in

$$\frac{d\bar{u}}{d\bar{t}} = -\bar{a}(\bar{t}) + \alpha\bar{b},$$

with

$$\bar{a}(\bar{t}) = \begin{cases} 1, & \bar{t} < \gamma, \\ 1-\beta, & \bar{t} \geq \gamma, \end{cases}$$

and

$$\bar{b}(\bar{t}) = \begin{cases} \gamma^{-1}\bar{t}, & \bar{t} < \gamma, \\ 0, & \bar{t} \geq \gamma, \end{cases}$$

The initial condition is $\bar{u}(0) = 1$. We have three dimensionless numbers in the problem:

$$\alpha = \frac{\epsilon s}{QI}, \quad \beta = \frac{A}{Q}, \quad \gamma = Qs\,.$$

We realize that α measures the ratio of the b term (ϵs) and the au term (QI), β reflects the relative jump in a, while γ measures the ratio of the transition point $t = s$ and the characteristic time scale.
Filename: `decay_varcoeff`.

Exercise 2.8: Alternative scalings of a cooling model

Implement the scaled model (2.29) and produce a plot with curves corresponding to various values of α and p to summarize how $\bar{u}(\bar{t})$ looks like.

Hint. A centered Crank-Nicolson-style scheme for (2.29) can use an old time value for the nonlinear coefficient:

$$\frac{\bar{u}^{n+1} - \bar{u}^n}{\Delta t} = (1 - \alpha\bar{u}^n)^p \frac{1}{2}(\bar{u}^n + \bar{u}^{n+1})\,.$$

Filename: `growth`.

Exercise 2.9: Projectile motion

We have the following mathematical model for the motion of a projectile in two dimensions:

$$m\ddot{\boldsymbol{x}} + \frac{1}{2}C_D\varrho A|\dot{\boldsymbol{x}}|\dot{\boldsymbol{x}} = -mg\boldsymbol{j}, \quad \boldsymbol{x}(0) = \boldsymbol{0},\ \dot{\boldsymbol{x}}(0) = v_0\cos\theta\boldsymbol{i} + v_0\sin\theta\boldsymbol{j}\,.$$

Here, m is the mass of the projectile, $\boldsymbol{x} = x\boldsymbol{i} + y\boldsymbol{j}$ is the position vector of the projectile, $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors along the x and y axes, respectively, $\ddot{\boldsymbol{x}}$ and $\dot{\boldsymbol{x}}$ is the second- and first-order time derivative of $\boldsymbol{x}(t)$, C_D is a drag coefficient depending on the shape of the projectile (can be taken as 0.4 for a sphere), ϱ is the density of the air, A is the cross section area (can be taken as πR^2 for a sphere of radius R), g is gravity, v_0 is the initial velocity of the projectile in a direction that makes the angle θ with the ground.

a) Neglect the air resistance term proportional to $\dot{\boldsymbol{x}}$ and solve analytically for $\boldsymbol{x}(t)$.

Solution. The vector differential equation reduces to the two component equations

$$m\ddot{x}(t) = 0, \quad m\ddot{y}(t) = -mg\,.$$

Integrating twice yields

$$x(t) = C_1 t + C_2, \quad y(t) = -\frac{1}{2}gt^2 + C_3 t + C_4\,.$$

The condition $\boldsymbol{x}(0) = \boldsymbol{0}$ forces $C_2 = C_4 = 0$. The condition on the derivative gives $C_1 = v_0\cos\theta$ and $C_3 = v_0\sin\theta$. The result is therefore

$$\boldsymbol{x}(t) = v_0\cos(\theta)t\boldsymbol{i} + (v_0\sin(\theta)t - \frac{1}{2}gt^2)\boldsymbol{j}\,.$$

b) Make the model for projectile motion with air resistance non-dimensional. Use the maximum height from the simplification in a) as length scale.

Solution. We introduce dimensionless quantities:

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{t} = \frac{t}{t_c},$$

where the scales L and t_c must be determined. Inserted in the original equation:

$$\frac{mL}{t_c^2}\frac{d^2\bar{\boldsymbol{x}}}{d\bar{t}^2} + \frac{1}{2}C_D\varrho A\frac{L^2}{t_c^2}\left|\frac{d\bar{\boldsymbol{x}}}{d\bar{t}}\right| = -mg\boldsymbol{j}\,.$$

Dividing by mL/t_c^2 gives

$$\frac{d^2\bar{\boldsymbol{x}}}{d\bar{t}^2} + \frac{1}{2}C_D\varrho A\frac{L}{m}\left|\frac{d\bar{\boldsymbol{x}}}{d\bar{t}}\right| = -\frac{gt_c^2}{L}\boldsymbol{j}\,.$$

The maximum height $y_{\max}$ occurs when $\dot{y}=0$:

$$\dot{y} = v_0\sin\theta - gt = 0 \quad \Rightarrow \quad t = g^{-1}v_0\sin\theta\,.$$

The corresponding value is

$$y_{\max} = g^{-1}v_0^2\sin^2\theta - \frac{1}{2}g^{-1}v_0^2\sin^2\theta = \frac{1}{2}g^{-1}v_0^2\sin^2\theta\,.$$

We can take $L = y_{\max}$ and let t_c be the corresponding t value: $t_c = g^{-1}v_0\sin\theta$. Inserted in the scaled problem:

$$\frac{d^2\bar{\boldsymbol{x}}}{d\bar{t}^2} + \frac{1}{2}C_D\varrho A\frac{v_0^2\sin^2\theta}{2mg}\left|\frac{d\bar{\boldsymbol{x}}}{d\bar{t}}\right| = -\boldsymbol{j}\,.$$

We can identify a dimensionless parameter

$$\alpha = \frac{1}{4}m^{-1}g^{-1}C_D\varrho A v_0^2\sin^2\theta,$$

and write the scaled equation as

$$\frac{d^2\bar{\boldsymbol{x}}}{d\bar{t}^2} + \alpha\left|\frac{d\bar{\boldsymbol{x}}}{d\bar{t}}\right| = -\boldsymbol{j},$$

with initial conditions

$$\bar{\boldsymbol{x}}(0) = \boldsymbol{0}, \quad \frac{d\bar{x}}{d\bar{t}}(0) = \frac{t_c}{L}(v_0\cos\theta\boldsymbol{i} + v_0\sin\theta\boldsymbol{j}) = 2\cot\theta\boldsymbol{i} + 2\boldsymbol{j}\,.$$

hpl 8: Check that α is dimensionless. Easy: ϱ goes like m/L^3, A cancels L^2, v_0^2 cancels the remaining dimensions... seems to be correct... **hpl 9**: Need to check the final problem. Solve scaled problem without drag.

c) Make the model dimensionless again, but this time by demanding that the scaled initial velocity is unity in x direction.

Solution. The scaled initial velocity condition is

$$\bar{x}(0) = \frac{t_c}{L}v_0\cos\theta\,.$$

Demanding the scaled velocity to be unity gives

$$L = t_c v_0\cos\theta\,.$$

The scaled initial velocity in y direction becomes

$$\bar{y}(0) = \tan\theta\,.$$

The scaled ODE becomes

$$\frac{d^2\bar{\boldsymbol{x}}}{d\bar{t}^2} + \frac{1}{2}C_D\varrho A\frac{L}{m}\left|\frac{d\bar{\boldsymbol{x}}}{d\bar{t}}\right| = -\frac{gt_c}{v_0\cos\theta}\boldsymbol{j}\,.$$

We can choose t_c such that the source term is unity:

$$t_c = g^{-1}v_0\cos\theta,$$

which makes

$$L = t_c v_0\cos\theta = g^{-1}v_0^2\cos^2\theta\,.$$

The coefficient in the drag term becomes

$$\alpha = \frac{1}{2}m^{-1}g^{-1}C_D\varrho A v_0^2\cos^2\theta\,.$$

d) A soccer ball has diameter $R = 11$ cm and mass 0.43 kg, the density of air is 1.2 kgm^{-3}, a soft kick has velocity 10 km/h, while a hard kick may have 120 km/h. Estimate the dimensionless parameter in the scaled problem for a soft and a hard kick with θ corresponding to 30 degrees. Solve the scaled differential equation for these values and plot the trajectory (y versus x) for the two cases.

Solution. We need to express R, v_0, and θ in standard SI units: $R = 0.11$ m, $\theta = 30\cdot pi/180 \approx 0.52$, $v_0 = 10/3.6 \approx 2.8$ m/s for a soft kick, and $v_0 = 120/3.6 \approx 33.33$ m/s for a hard kick. Using α from b), we ...

hpl 10: Need to fill in more after α is checked - and solve the ODE with Odespy.

Filename: `projectile`.

Problem 2.10: A predator-prey model

The evolution of animal populations with a predator and a prey (e.g., lynx and hares, or foxes and rabbits) can be described by the Lotka-Volterra ODE system

$$H' = H(a - bL), \tag{2.91}$$
$$L' = L(dH - c), \tag{2.92}$$
$$H(0) = H_0, \tag{2.93}$$
$$L(0) = L_0\,. \tag{2.94}$$

Here, H is the number of animals of the prey (say hares) and L is the corresponding measure of the predator population (say lynx). There are six parameters: a, b, c, d, H_0, and L_0.

The terms has the following meanings:

- aH is the exponential population growth of H due to births and deaths and is governed by the access to nutrition,
- $-bHL$ is the loss of preys because they are eaten by predators,
- dHL is the increase of preys because they eat predators (but only a fraction of the eaten preys, bHL, contribute to population growth of the predator and therefore $d < b$),
- $-cL$ is the exponential decay in the predator population because of deaths (the increase is modeled by dHL).

Dimensionless independent and dependent variables are introduced as usual by

$$\bar{t} = \frac{t}{t_c}, \quad \bar{H} = \frac{H}{H_c}, \quad \bar{L} = \frac{L}{L_c},$$

where t_c, H_c, and L_c are scales to be determined. Inserted in the ODE problem we arrive at

$$\frac{H_0}{t_c}\frac{d\bar{H}}{d\bar{t}} = H_0\bar{H}(a - bH_0\bar{L}), \tag{2.95}$$

$$\frac{H_0}{t_c}\frac{d\bar{L}}{d\bar{t}} = H_0\bar{L}(dH_0\bar{H} - c), \tag{2.96}$$

$$H_c\bar{H}(0) = H_0, \tag{2.97}$$

$$L_c\bar{H}(0) = L_0\,. \tag{2.98}$$

a) Consider first a simple, intuitive scaling of H and L based on initial conditions $H_c = H_0$ and $L_c = H_c$. This means that $\bar{H}$ starts out at unity and $\bar{L}$ starts out as the fraction L_0/H_0. Find a time scale and identify dimensionless parameters in the scaled ODE problem.

Solution. With $H_c = L_c = H_0$ in (2.95)-(2.98) we get

$$\begin{aligned}
\frac{d\bar{H}}{d\bar{t}} &= \frac{a}{bH_0}\bar{H} - \bar{L}\bar{H},\\
\frac{d\bar{L}}{d\bar{t}} &= \frac{d}{b}\bar{L}\bar{H} - \frac{c}{bH_0}\bar{L}),\\
\bar{H}(0) &= 1,\\
\bar{L}(0) &= \frac{L_0}{H_0}\,.
\end{aligned}$$

With the dimensionless parameters

$$\alpha = \frac{a}{bH_0}, \quad \beta = \frac{d}{b}, \quad \gamma = \frac{c}{bH_0}, \quad \delta = \frac{H_0}{L_0},$$

we can write the dimensionless problem as

$$\begin{aligned}
\frac{d\bar{H}}{d\bar{t}} &= \alpha\bar{H} - \bar{L}\bar{H}, \\
\frac{d\bar{L}}{d\bar{t}} &= \beta\bar{L}\bar{H} - \gamma\bar{L}), \\
\bar{H}(0) &= 1, \\
\bar{L}(0) &= \delta\,.
\end{aligned}$$

The quantity bH_0 is the number of eaten preys per predator. Then α measures the ratio of natural population growth of the prey, due to nutrition, and the number of eaten preys per predator. The β parameter measures the fraction of the eaten preys and the amount of this that actually leads to population growth of the predator. The number γ reflects the ratio of predator deaths and the eaten preys per predator, and δ is the initial fraction of preys and predators.

b) Try a different scaling where the aim is to adjust the scales such that the ODEs become as simple as possible, i.e, have as few dimensionless parameters as possible. Compare with the scaling in a).

Solution. Dividing by H_c and L_c in (2.95) and (2.98), respectively, and multiply by t_c:

$$\begin{aligned}
\frac{d\bar{H}}{d\bar{t}} &= t_c\bar{H}(a - bL_c\bar{L}), \\
\frac{d\bar{L}}{d\bar{t}} &= t_c\bar{L}(dH_c\bar{H} - c)\,.
\end{aligned}$$

Choosing $t_c = 1/a$ and $t_c a L_c = 1$, i.e., $L_c = a/b$, makes the first equation free of parameters: $\bar{H}' = \bar{H}(1 - \bar{L})$. Factoring out c in the equation for L and choosing $H_c d/c = 1$, i.e., $H_c = c/d$, leaves us with the L equation as $\bar{L}' = (c/a)\bar{L}(\bar{H} - 1)$. The ratio c/a is now called μ and equals γ/α from a).

The initial conditions lead to $\bar{H}(0) = H_0/H_c = H_0 d/c = \beta/\gamma = \nu$, and $\bar{L}(0) = L_0/L_c = L_0 b/a = \delta/\alpha = \omega$.

The dimensionless problem is now

$$\frac{d\bar{H}}{d\bar{t}} = \bar{H}(1-\bar{L}), \tag{2.99}$$

$$\frac{d\bar{L}}{d\bar{t}} = \mu\bar{L}(\bar{H}-1) = \gamma\alpha^{-1}\bar{L}(\bar{H}-1), \tag{2.100}$$

$$\bar{H}(0) = \nu = \beta/\gamma, \tag{2.101}$$

$$\bar{L}(0) = \omega = \delta/\alpha, \tag{2.102}$$

with

$$\mu = \frac{c}{a}, \quad \nu = H_0\frac{d}{c}, \quad \omega = L_0\frac{b}{a}.$$

The unknowns $\bar{H}$ and $\bar{L}$ now has less intuitive scalings,

$$\bar{H} = \frac{Hd}{c}, \quad \bar{L} = \frac{Lb}{a},$$

while time is measured in the units based on the exponential growth due to births and deaths of preys (a). The number of dimensionless parameters is one less since we have one more scale (for L_c) at our disposal. Simplicity in one initial conditions in a) is exchanged with more simplicity in the ODEs, which now have only one dimensionless parameter.

Note that ν and ω must be different from unity to avoid $\bar{H} \neq 0$ and $\bar{L} \neq 0$ because of the factors $1-L$ and $H-1$ in the equations that can make $\bar{H}' = 0$ and $\bar{L}' = 0$.

c) A more mathematical approach to determining suitable scales for H and L consists in finding the stationary points (H, L) of the ODE system, where $H' = L' = 0$, and use such points as characteristic sizes of the dependent variables. Show that $H' = L' = 0$ implies $H = L = 0$ or $L = a/b$ and $H = c/d$. Use $H_c = a/b$, $L_c = c/d$, and find a time scale. Compare with the result in b).

Solution. Setting $H' = L' = 0$ leads to

$$H(a-bL) = 0, \quad L(dH-c) = 0,$$

from which we see that the factors must vanish: $H = L = 0$, $L = a/b$, and $H = c/d$. With Use $H_c = a/b$, $L_c = c/d$, and $t_c = 1/a$ we get the same scaling as in b), but with a different motivation.
Filename: `predator_prey`.

Problem 2.11: A model for competing species

Let $N_1(t)$ and $N_2(t)$ be the number of animals in two competing species. A generalized Lotka-Volterra model is based on a logistic growth of each specie and a predator-prey like interaction (cf. Problem **??**):

$$\frac{dN_1}{dt} = r_1 N_1 \left(1 - \frac{N_1}{M_1} - s_{12}\frac{N_2}{M_1}\right), \tag{2.103}$$

$$\frac{dN_2}{dt} = r_2 N_2 \left(1 - \frac{N_2}{M_2} - s_{21}\frac{N_1}{M_2}\right), \tag{2.104}$$

where r_1, r_2, M_1, M_2, s_{12}, and s_{21} are given constants. The initial conditions specify N_1 and N_2 at $t = 0$. Find suitable scales and derive a dimensionless ODE problem.

Solution. As always, we can introduce dimensionless variables, We use ideas from scaling of ODEs for logistic growth, i.e., we use the carrying capacities M_1 and M_2 as characteristic (maximum) values of N_1 and N_2, respectively. Time can be scaled from the initial exponential growth of N_1 or N_2, i.e., $t_c = 1/r_1$ or $t_c = 1/r_2$. We choose the former here. Introducing

$$\bar{t} = r_1 t, \quad u_1 = \frac{N_1}{M_1}, \quad u_2 = \frac{N_2}{M_2},$$

in the ODE system, leads to

$$\frac{du_1}{d\bar{t}} = u_1\left(1 - u_1 - b_{21}\beta\gamma\right),$$
$$\frac{du_2}{d\bar{t}} = \alpha u_2\left(1 - u_2 - b_{12}\beta^{-1}\gamma^{-1}\right),$$

where the dimensionless numbers are given by

$$\alpha = \frac{r_2}{r_1}, \quad \beta = \frac{M_2}{M_1}, \quad \gamma = \frac{b_{12}}{b_{21}}.$$

We have introduced to separate numbers β and γ since they related to different parameters, but only their product matters. Alternatively, we could introduce the numbers $\mu = b_{12}\gamma$ and $\nu = b_{21}\gamma^{-1}$ in the last term of the first and second ODE, respectively.
Filename: `competing_species`.

Problem 2.12: Find the period of sinusoidal signals

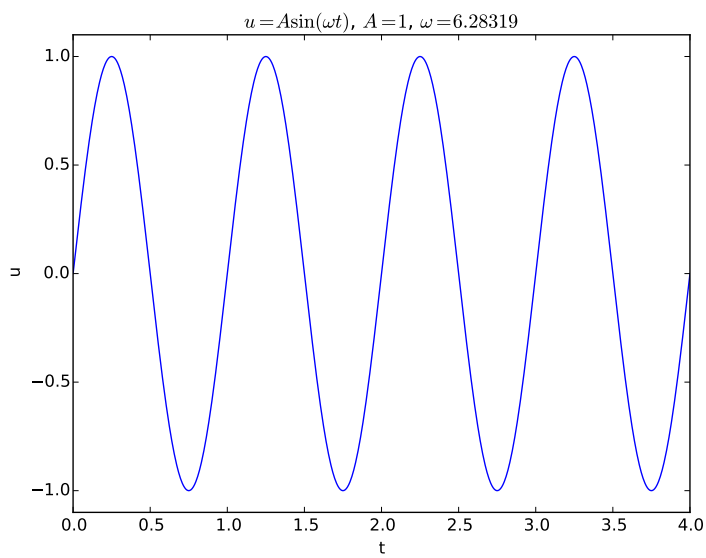
a) Plot the function

$$u(t) = A\sin(\omega t),$$

for $t \in [0, 8\pi/\omega]$. Choose ω and A.

Solution. Appropriate code is

```
import numpy as np
import matplotlib.pyplot as plt

def u(t, A, w, module=np):
    return A*module.sin(w*t)

def a():
    """Plot u."""
    w = 2*np.pi
    A = 1.0
    t = np.linspace(0, 8*np.pi/w, 1001)
    plt.figure()
    plt.plot(t, u(t, A, w))
    plt.xlabel('t');  plt.ylabel('u')
    plt.axis([t[0], t[-1], -1.1, 1.1])
    plt.title(r'$u=A\sin (\omega t)$, $A=%g$, $\omega = %g$'
              % (A, w))
    plt.savefig('tmp1.png');  plt.savefig('tmp1.pdf')
```



b) The *period* P of u is the shortest distance between two peaks (where $u = A$). Show mathematically that

$$P = \frac{2\pi}{\omega}\,.$$

Frequently, P is also referred to as the *wave length* of u.

Solution. Since the sine function has period 2π, we have that

$$\sin(\omega t) = \sin(\omega t + 2\pi)\,.$$

The definition of P is that sine gets its value again after time P:

$$\sin(\omega t) = \sin(\omega(t+P))\,.$$

Combing we get that $\sin(\omega t + 2\pi) = \sin(\omega(t+P))$, so the arguments must be equal:

$$\omega t + 2\pi = \omega(t+P),$$

from which it follows that $P = 2\pi/\omega$.

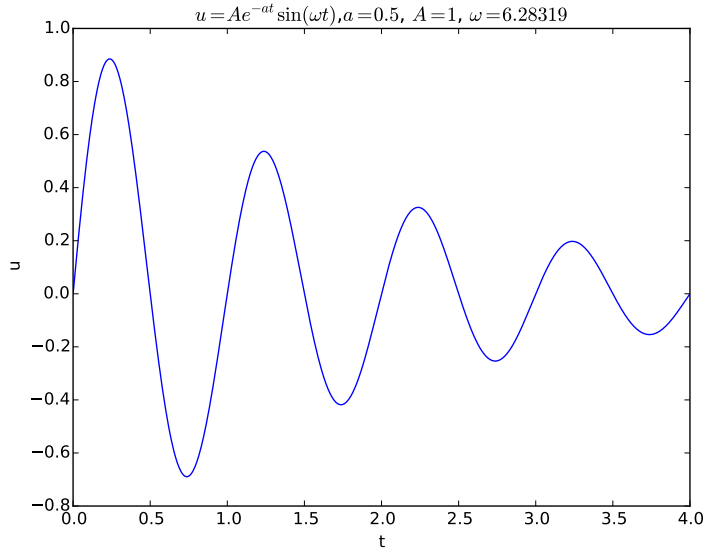
An alternative is to find the peaks as the points where $du/dt = 0$. Since $du/dt = \omega\cos(\omega t)$, this function is zero when $\omega t = n\pi$ for integer n. If $n\pi$ corresponds to a maximum, $(n+1)\pi$ will correspond to a minimum and $(n+2)\pi$ to the next maximum. The period P is the distance in time between two maxima:

$$\omega(t+P) - \omega t = (n + 2\pi - n\pi \quad \Rightarrow \quad P = \frac{2\pi}{\omega}\,.$$

c) Plot the damped signal $u(t) = e^{-at}\sin(\omega t)$ over four periods of $sin(\omega t)$. Choose ω, A, and a.

Solution. Code:

```
def u_damped(t, A, w, a, module=np):
    return A*module.exp(-a*t)*module.sin(w*t)

def c():
    """Plot damped u."""
    w = 2*np.pi
    A = 1.0
    a = 0.5
    t = np.linspace(0, 8*np.pi/w, 100001)
    plt.figure()
    plt.plot(t, u_damped(t, A, w, a))
    plt.xlabel('t');  plt.ylabel('u')
    plt.title(r'$u=Ae^{-at}\sin (\omega t)$,'
              '$a=%g$, $A=%g$, $\omega = %g$' % (a, A, w))
    plt.savefig('tmp2.png');  plt.savefig('tmp2.pdf')

    u_max = []
    u_ = u_damped(t, A, w, a)
    for i in range(1, len(t)-1):
        if u_[i-1] < u_[i] > u_[i+1]:
            u_max.append((t[i], u_[i]))
    print u_max
    for i in range(len(u_max)-1):
        print 'P=', u_max[i+1][0] - u_max[i][0]
```



d) What is the period of $u(t) = e^{-at}\sin(\omega t)$? We define the period P as the shortest distance between two peaks of the signal.

Hint. Use that $v = p\cos(\omega t) + q\sin(\omega t)$ can be rewritten as $v = B\cos(\omega t - \phi)$ with $B = \sqrt{p^2 + q^2}$ and $\phi = \tan^{-1}(p/q)$. Use such a rewrite of u' to find the peaks of u and then the period.

Solution. Finding the extrema from $u' = 0$ leads to

$$u' = -ae^{-at}\sin(\omega t) + e^{-at}\omega\cos(\omega t) = 0\,.$$

Using the hint to rewrite ($p = \omega$, $q = -a$), we get

$$u' = e^{-at}B\cos(\omega t - \phi) = 0, \quad B = \sqrt{\omega^2 + a^2},\ \phi = \tan^{-1}(-\omega/a)\,.$$

Now, e^{-at} is always positive so only the cosine function can cross zero, and that happens when the argument is $n\pi$ for integer n. However, all the maxima only occurs for $2n\pi$ (n integer). Demanding the argument to be $2n\pi$ we get the distance between two nearby peaks as

$$\omega(t + P) - \phi - (\omega t - \phi) = 2(n+1)\pi - 2n\pi,$$

which leads to

$$\omega P = 2\pi \quad \Rightarrow \quad P = \frac{2\pi}{\omega}\,.$$

The period of the damped signal is the same; only ω can alter the period.
Filename: `sine_period`.

Remarks. The *frequency* is the number of up and down cycles in one unit time. Since there is one cycle in a period P, the frequency is $f = 1/P$, measured in Hz. The *angular frequency* ω is then $\omega = 2\pi/P = 2\pi f$.

Problem 2.13: Oscillating mass with sliding friction

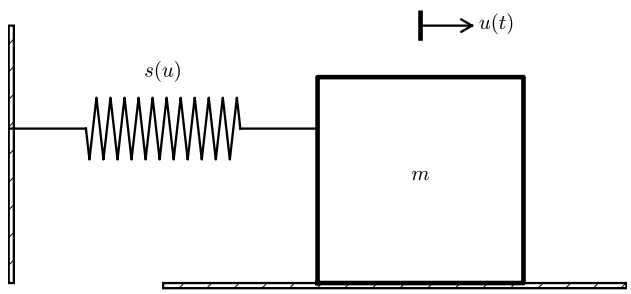


Fig. 2.13 Body sliding on a surface.

A mass attached to a spring is sliding on a surface and subject to a friction force, see Figure 2.13. The spring represents a force $-ku\boldsymbol{i}$, where k is the spring stiffness. The friction force is proportional to the normal force on the surface, $-mg\boldsymbol{j}$, and given by $-f(\dot{u})\boldsymbol{i}$, where

$$f(\dot{u}) = \begin{cases} -\mu mg, & \dot{u} < 0, \\ \mu mg, & \dot{u} > 0, \\ 0, & \dot{u} = 0 \end{cases}$$

Here, μ is a friction coefficient. With the signum function

$$\text{sign(x)} = \begin{cases} -1, & x < 0, \\ 1, & x > 0, \\ 0, & x = 0 \end{cases}$$

we can simply write $f(\dot{u}) = \mu mg\,\text{sign}(\dot{u})$ (the sign function is implemented by `numpy.sign`).

The ODE problem for this one-dimensional oscillatory motion reads

$$m\ddot{u} + \mu mg\,\text{sign}(\dot{u}) + ku = 0, \quad u(0) = I,\ \dot{u}(0) = V\,. \tag{2.105}$$

a) Scale the problem.

Solution. Inserting the dimensionless dependent and independent variables,

$$\bar{u} = \frac{u}{I}, \quad \bar{t} = \frac{t}{t_c},$$

in the problem gives

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \frac{t_c^2\mu g}{I}\text{sign}\left(\frac{d\bar{u}}{d\bar{t}}\right) + \frac{t_c^2 k}{m}\bar{u} = 0, \quad \bar{u}(0) = 1,\ \frac{d\bar{u}}{d\bar{t}}(0) = \frac{Vt_c}{I}.$$

As usual, we base the characteristic time on the friction-free oscillations, which means a balance of the acceleration term and the spring term. That is, $t_c^2 k/m = 1$, and consequently $t_c = \sqrt{m/k}$.

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \frac{\mu mg}{kI}\text{sign}\left(\frac{d\bar{u}}{d\bar{t}}\right) + \bar{u} = 0, \quad \bar{u}(0) = 1,\ \frac{d\bar{u}}{d\bar{t}}(0) = \frac{V\sqrt{mk}}{kI}.$$

Introducing the dimensionless variables

$$\alpha = \frac{\mu mg}{kI}, \quad \beta = \frac{V\sqrt{mk}}{kI},$$

the scaled problem can then be written

$$\frac{d^2\bar{u}}{d\bar{t}^2} + \alpha\text{sign}\left(\frac{d\bar{u}}{d\bar{t}}\right) + \bar{u} = 0, \quad \bar{u}(0) = 1,\ \frac{d\bar{u}}{d\bar{t}}(0) = \beta.$$

The initial set of 6 parameters (μ, m, g, k, I, V) are reduced to 2 dimensionless combinations.

Let us check that the dimensionless parameters really are dimensionless. From the original ODE we know that each term has the dimension of force, i.e., $[\text{MLT}^{-2}]$. Therefore, the friction coefficient μ is dimensionless since mg has dimension $[\text{MLT}^{-2}]$, and k has dimension $[\text{MT}^{-2}]$ since u has dimension $[\text{L}]$. Since I has the same dimension as u, kI has the dimension of $[\text{MLT}^{-2}]$, which is the dimension of mg, and α is dimensionless. The β parameter has dimensions $[\text{LT}^{-1}\text{M}^{1/2}\text{M}^{-1/2}\text{TL}^{-1} = [1]$.

b) Implement the scaled model. Simulate for $\alpha = 0, 0.05, 0.1$ and $\beta = 0$.

Solution. We can use the package Odespy to solve the ODE. This requires rewriting the ODE as a system of two first-order ODEs:

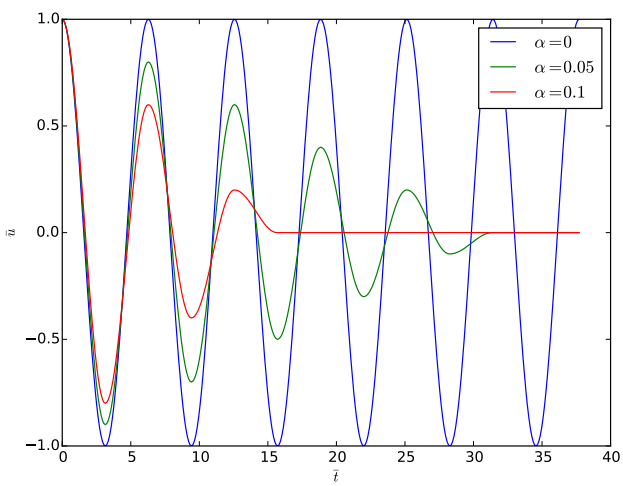
$$v' = -\alpha\text{sign}(v) - \bar{u}, u' \qquad = v,$$

with initial conditions $v(0) = \beta$ and $u(0) = 1$. Here, $u(t)$ corresponds to the previous $\bar{u}(\bar{t})$, while $v(t)$ corresponds to $d\bar{u}/\bar{t}(\bar{t})$. Appropriate code is

```
import matplotlib.pyplot as plt
import numpy as np

def simulate(alpha, beta=0,
             num_periods=8, time_steps_per_period=60):
    # Use oscillations without friction to set dt and T
    P = 2*np.pi
    dt = P/time_steps_per_period
    T = num_periods*P
```

```
    t = np.linspace(0, T, time_steps_per_period*num_periods+1)
    import odespy

    def f(u, t, alpha):
        # Note the sequence of unknowns: v, u (v=du/dt)
        v, u = u
        return [-alpha*np.sign(v) - u, v]

    solver = odespy.RK4(f, f_args=[alpha])
    solver.set_initial_condition([beta, 1])  # sequence must match f
    uv, t = solver.solve(t)
    u = uv[:,1]  # recall sequence in f: v, u
    v = uv[:,0]
    return u, t


if __name__ == '__main__':
    alpha_values = [0, 0.05, 0.1]
    for alpha in alpha_values:
        u, t = simulate(alpha, 0, 6, 60)
        plt.plot(t, u)
        plt.hold('on')
    plt.legend([r'$\alpha=%g$' % alpha for alpha in alpha_values])
    plt.xlabel(r'$\bar t$');  plt.ylabel(r'$\bar u$')
    plt.savefig('tmp.png');   plt.savefig('tmp.pdf')
    plt.show()
```

We find that simulating for 6 periods is relevant for $\alpha = 0, 0.05, 0.1$.



Filename: `sliding_box`.

Problem 2.14: Pendulum equations

The equation for a so-called simple pendulum with a mass m at the end is

$$mL\ddot{\theta} + mg\sin\theta = 0, \tag{2.106}$$

where $\theta(t)$ is the angle with the vertical, L is the length of the pendulum, and g is the acceleration of gravity.

A physical pendulum with moment of inertia I is governed by a similar equation,

$$I\ddot{\theta} + mgL\sin\theta = 0\,. \tag{2.107}$$

Both equations have the initial conditions $\theta(0) = \Theta$ and $\theta'(0) = 0$ (start at rest).

a) Use θ as dimensionless unknown, find a proper time scale, and scale both differential equations.

Solution. Introducing $\bar{t} = t/t_c$ gives

$$mL\frac{1}{t_c^2}\frac{d^2\theta}{d\bar{t}^2} + mg\sin\theta = 0,$$
$$I\frac{1}{t_c^2}\frac{d^2\theta}{d\bar{t}^2} + mgL\sin\theta = 0\,.$$

or on dimensionless form,

$$\frac{d^2\theta}{d\bar{t}^2} + \frac{t_c^2 g}{L}\sin\theta = 0,$$
$$\frac{d^2\theta}{d\bar{t}^2} + \frac{t_c^2 mgL}{I}\sin\theta = 0\,.$$

An obvious choice to make the terms equal are $t_c = \sqrt{L/g}$ in the first equation and $t_c = \sqrt{I/(mgL)}$ in the second. These choices are also compatible with the frequencies if the angle is small: $\ddot{\theta} + g/L\theta = 0$ has solution of the type $\sin(\omega t)$ with $\omega = \sqrt{g/L}$, and then $t_c = 1/\omega$ is a natural scale.

The dimensionless equations become equal in this case:

$$\frac{d^2\theta}{d\bar{t}^2} + \sin\theta = 0\,.$$

b) Some may argue that θ is not dimensionless since it is measured in radians. One may introduce a truly dimensionless angle $\bar{\theta} \in [0,1]$. Set up the scaled ODE problem in this case.

Solution. A $\bar{\theta} \in [0,1]$ is obtained by $\bar{\theta} = \theta/\Theta$. The resulting equation then becomes

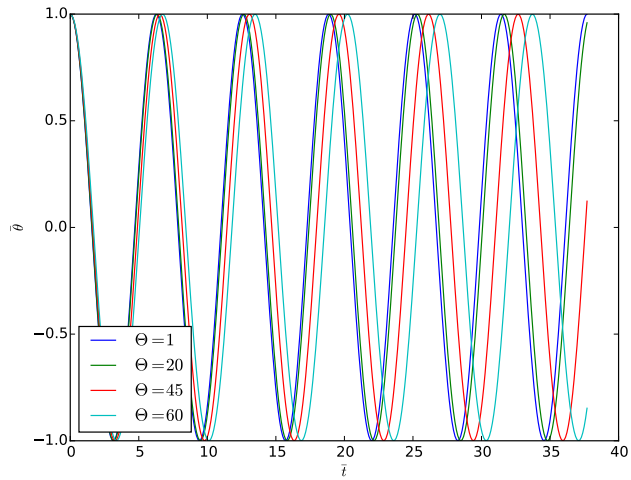
$$\frac{d^2\bar{\theta}}{d\bar{t}^2} + \Theta^{-1}\sin(\Theta\bar{\theta}) = 0,$$

with boundary condition $\bar{\theta}(0) = 1$. That is, the only parameter Θ either remains in the ODE or in the initial condition.

c) Simulate the problem in b) for $\Theta = 1, 20, 45, 60$ measure in degrees.

Solution. We use Odespy to solve the ODE, rewritten as system of first-order ODEs: $\bar{\omega}' = -\Theta^{-1}\sin(\Theta\bar{\theta})$ and $\bar{\theta}' = \bar{\omega}$. Appropriate code is

```
import matplotlib.pyplot as plt
import numpy as np

def simulate(Theta, num_periods=8, time_steps_per_period=60):
    # Use oscillations for small Theta to set dt and T
    P = 2*np.pi
    dt = P/time_steps_per_period
    T = num_periods*P
    t = np.linspace(0, T, time_steps_per_period*num_periods+1)
    import odespy

    def f(u, t, Theta):
        # Note the sequence of unknowns: omega, theta
        # omega = d(theta)/dt, angular velocity
        omega, theta = u
        return [-Theta**(-1)*np.sin(Theta*theta), omega]

    solver = odespy.RK4(f, f_args=[Theta])
    solver.set_initial_condition([0, 1]) # sequence must match f
    u, t = solver.solve(t)
    theta = u[:,1]  # recall sequence in f: omega, theta
    return theta, t


if __name__ == '__main__':
    Theta_values_degrees = [1, 20, 45, 60]
    for Theta_degrees in Theta_values_degrees:
        Theta = Theta_degrees*np.pi/180
        theta, t = simulate(Theta, 6, 60)
        plt.plot(t, theta)
        plt.hold('on')
    plt.legend([r'$\Theta=%g$' % Theta
                for Theta in Theta_values_degrees],
               loc='lower left')
    plt.xlabel(r'$\bar t$');  plt.ylabel(r'$\bar\theta$')
    plt.savefig('tmp.png');   plt.savefig('tmp.pdf')
    plt.show()
```



We clearly see that increasing Θ increases the period of the oscillations.

Remark. The scaling in b) is more suitable for comparing graphs than the scaling in a) since all the curves have the same amplitude, just different frequency/period. With the scaling in a), we would also get a major difference in amplitudes.
Filename: `pendulum`.

Exercise 2.15: ODEs for a binary star

The equations for a binary star, or a planet and a moon, are

$$m_A \ddot{\boldsymbol{x}}_A = \boldsymbol{F}, \tag{2.108}$$
$$m_B \ddot{\boldsymbol{x}}_B = -\boldsymbol{F}, \tag{2.109}$$

where $\boldsymbol{x}_A$ is the position of object (star) A, and $\boldsymbol{x}_B$ is the position object B. The corresponding masses are m_A and m_B. The only force is the gravity force

$$\boldsymbol{F} = \frac{G m_A m_B}{||\boldsymbol{r}||^3}\boldsymbol{r},$$

where

$$\boldsymbol{r}(t) = \boldsymbol{x}_B(t) - \boldsymbol{x}_A(t),$$

and G is the gravitational constant: $G = 6.674 \cdot 10^{-11}$ Nm^2/kg^2. A problem with these equations is that the parameters are very large (m_A, m_B, $||\boldsymbol{r}||$) or very small (G). The rotation time for binary stars can be very small and large as well.

a) Scale the equations.

Solution. A natural length scale could be the initial distance between the objects: $L = \boldsymbol{r}(0)$. We write the dimensionless quantities as

$$\bar{\boldsymbol{x}}_A = \frac{\boldsymbol{x}_A}{L}, \quad \bar{\boldsymbol{x}}_B = \frac{\boldsymbol{x}_B}{L}, \quad \bar{t} = \frac{t}{t_c}.$$

The gravity force is transformed to

$$\boldsymbol{F} = \frac{G m_A m_B}{L^2 ||\bar{\boldsymbol{r}}||^3} \bar{\boldsymbol{r}}, \quad \bar{\boldsymbol{r}} = \bar{\boldsymbol{x}}_B - \bar{\boldsymbol{x}}_A,$$

so the first ODE for $\boldsymbol{x}_A$ becomes

$$\frac{d^2 \bar{\boldsymbol{x}}_A}{d\bar{t}^2} = \frac{G m_B t_c^2}{L^3} \frac{\bar{\boldsymbol{r}}}{||\bar{\boldsymbol{r}}||^3}.$$

Assuming that quantities with a bar and their derivatives are around unity in size, it is natural to choose t_c such that the fraction $G m_B t_c / L^2 = 1$:

$$t_c = \sqrt{\frac{L^3}{G m_B}}.$$

From the other equation for $\boldsymbol{x}_B$ we get another candidate for t_c with m_A instead of m_B. Which mass we choose play a role if $m_A \ll m_B$ or $m_B \ll m_A$. One solution is to use the sum of the masses:

$$t_c = \sqrt{\frac{L^3}{G(m_A + m_B)}}.$$

Taking a look at Kepler's laws of planetary motion, the orbital period for a planet around the star is given by the t_c above, except for a missing factor of 2π, but that means that t_c^{-1} is just the angular frequency of the motion. Our characteristic time t_c is therefore highly relevant. Introducing the dimensionless number

$$\alpha = \frac{m_A}{m_B},$$

we can write the dimensionless ODE as

$$\frac{d^2\bar{\boldsymbol{x}}_A}{d\bar{t}^2} = \frac{1}{1+\alpha}\frac{\bar{\boldsymbol{r}}}{||\bar{\boldsymbol{r}}||^3}, \tag{2.110}$$

$$\frac{d^2\bar{\boldsymbol{x}}_B}{d\bar{t}^2} = \frac{1}{1+\alpha^{-1}}\frac{\bar{\boldsymbol{r}}}{||\bar{\boldsymbol{r}}||^3}. \tag{2.111}$$

In the limit $m_A \ll m_B$, i.e., $\alpha \ll 1$, object B stands still, say $\bar{\boldsymbol{x}}_B = 0$, and object A orbits according to

$$\frac{d^2\bar{\boldsymbol{x}}_A}{d\bar{t}^2} = -\frac{\bar{\boldsymbol{x}}_A}{||\bar{\boldsymbol{x}}_A||^3}.$$

To better see the motion, and that our scaling is reasonable, we introduce polar coordinates r and θ:

$$\bar{\boldsymbol{x}}_A = r\cos\theta\boldsymbol{i} + r\sin\theta\boldsymbol{j},$$

which means $\bar{\boldsymbol{x}}_A$ can be written as $\bar{\boldsymbol{x}}_A = r\boldsymbol{i}_r$. Since

$$\frac{d}{dt}\boldsymbol{i}_r = \dot{\theta}\boldsymbol{i}_\theta, \quad \frac{d}{dt}\boldsymbol{i}_\theta = -\dot{\theta}\boldsymbol{i}_r,$$

we have

$$\frac{d^2\bar{\boldsymbol{x}}_A}{d\bar{t}^2} = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{i}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{i}_\theta.$$

The equation of motion for mass A is then

$$\begin{aligned}\ddot{r} - r\dot{\theta}^2 &= -\frac{1}{r^2},\\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= 0.\end{aligned}$$

The special case of circular motion, $r = 1$, fulfills the equations, since the latter equation then gives $\dot{\theta} = \text{const}$ and the former then gives $\dot{\theta} = 1$, i.e., the motion is $r(t) = 1$, $\theta(t) = t$, with unit angular frequency as expected and period 2π as expected.

b) Solve the scaled equations numerically for two cases:

1. a planet around a star: $\alpha = 10^{-3}$, $\boldsymbol{x}_A(0) = (1,0)$, $\dot{\boldsymbol{x}}_A(0) = (0,1)$, $\boldsymbol{x}_B(0) = 0$, $\dot{\boldsymbol{x}}_B(0) = 0$
2. two stars: $\alpha = \frac{1}{2}$, $\boldsymbol{x}_A(0) = (1,0)$, $\dot{\boldsymbol{x}}_A(0) = (0,\frac{1}{2})$, $\boldsymbol{x}_B(0) = 0$, $\dot{\boldsymbol{x}}_B(0) = (0,-\frac{1}{2})$

Solution. Here is an appropriate program (using SciTools for simpler animation code than required by Matplotlib):

```
import odespy
import numpy as np
#import matplotlib.pyplot as plt
import scitools.std as plt
```

```
import sys

def solver(alpha, ic, T, dt=0.05):
    def f(u, t):
        x_A, vx_A, y_A, vy_A, x_B, vx_B, y_B, vy_B = u
        distance3 = np.sqrt((x_B-x_A)**2 + (y_B-y_A)**2)**3
        system = [
            vx_A,
            1/(1.0 + alpha)*(x_B - x_A)/distance3,
            vy_A,
            1/(1.0 + alpha)*(y_B - y_A)/distance3,
            vx_B,
            -1/(1.0 + alpha**(-1))*(x_B - x_A)/distance3,
            vy_B,
            -1/(1.0 + alpha**(-1))*(y_B - y_A)/distance3,
            ]
        return system

    Nt = int(round(T/dt))
    t_mesh = np.linspace(0, Nt*dt, Nt+1)

    solver = odespy.RK4(f)
    solver.set_initial_condition(ic)
    u, t = solver.solve(t_mesh)
    x_A = u[:,0]
    x_B = u[:,2]
    y_A = u[:,4]
    y_B = u[:,6]
    return x_A, x_B, y_A, y_B, t

def demo_circular():
    # Mass B is at rest at the origin,
    # mass A is at (1, 0) with vel. (0, 1)
    ic = [1, 0, 0, 1, 0, 0, 0, 0]
    x_A, x_B, y_A, y_B, t = solver(alpha=0.001, ic=ic, T=2*np.pi, dt=0.01)
    plt.plot(x_A, x_B, 'r2-', y_A, y_B, 'b2-', legend=['A', 'B'],
             daspectmode='equal')
    plt.savefig('tmp_circular.png');  plt.savefig('tmp_circular.pdf')
    plt.show()

def demo2(animate=True):
    ic = [0.6, 0, 0, 1, 0, 0, 0, -0.5]
    x_A, x_B, y_A, y_B, t = solver(alpha=0.5, ic=ic, T=4*np.pi, dt=0.05)
    if animate:
        for i in range(len(x_A)):
            plt.plot(x_A[:i+1], x_B[:i+1], 'r-', y_A[:i+1], y_B[:i+1], 'b-',
                     [x_A[0], x_A[i]], [x_B[0], x_B[i]], 'r2o',
                     [y_A[0], y_A[i]], [y_B[0], y_B[i]], 'b4o',
                     daspectmode='equal', legend=['A', 'B', 'A', 'B'],
                     axis=[-1, 1, -1, 1], savefig='tmp_%04d.png' % i,
                     title='t=%.2f' % t[i])
    else:
        plt.plot(x_A, x_B, 'r-', y_A, y_B, 'b-',
                 daspectmode='equal', legend=['A', 'B'],
                 axis=[-1, 1, -1, 1], savefig='tmp_two_stars.png')
    #plt.axes().set_aspect('equal')  # mpl
    plt.show()

if __name__ == '__main__':
    if sys.argv[1] == 'circular':
        demo_circular()
    else:
```

```
        demo2(True)
    raw_input()
```

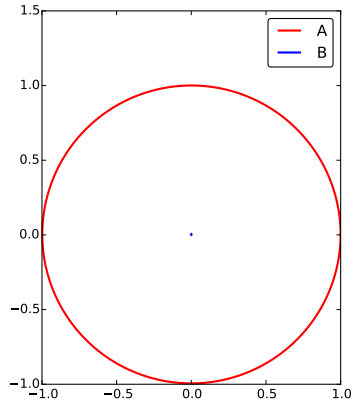


Fig. 2.14 Planet in circular motion around a star.

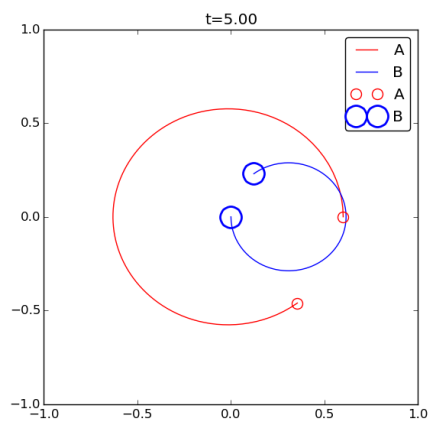


Fig. 2.15 Two rotating stars.

Filename: `binary_star`.

Problem 2.16: Duffing's equation

Duffing's equation is a vibration equation with linear and cubic spring terms:

$$mu'' + k_0 u + k_1 u^3 = 0, \quad u(0) = U_0,\ u'(0) = 0\,.$$

Scale this problem.

Solution. We introduce $\bar{t} = t/t_c$ and $\bar{u} = u/u_c$:

$$mt_c^{-2} u_c \bar{u}'' + k_0 u_c \bar{u} + k_1 u_c^3 \bar{u}^3 =, \quad u_c \bar{u}(0) = U_0,\ u_c \bar{u}'(0) = 0\,.$$

Choosing t_c as in a linear vibration problem, $t_c = \sqrt{m/k_0}$, and $u_c = U_0$, we get

$$\bar{u}'' + \bar{u} + \alpha u^3 = 0, \quad \bar{u}(0) = 1,\ \bar{u}'(0) = 0,$$

where

$$\alpha = U_0^2 \frac{k_1}{k_0},$$

is a dimensionless parameter reflecting the ratio of the cubic spring term $k_1 U_0^3$ and the linear spring term $k_0 U_0$ at maximum displacement.
Filename: `Duffing_eq`.

Problem 2.17: Vertical motion in a varying gravity field

A body (e.g., projectile or rocket) is launched at the earth's surface at velocity V. The body's distance (height) from the earth's surface at time t is represented by the function $h(t)$. Unless h is very much smaller than the earth's radius R, the motion takes place in a varying gravity field. The governing ODE problem for $h(t)$ is then

$$h''(t) = -\frac{R^2 g}{(h+R)^2}, \quad h(0) = 0,\ h'(0) = V, \quad t \in (0,T], \tag{2.112}$$

where g is the acceleration of gravity at the earth's surface.

The goal is to discuss three scalings of this problem. First we introduce

$$\bar{h} = \frac{h}{h_c}, \quad \bar{t} = \frac{t}{t_c},$$

which gives the dimensionless ODE

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{t_c^2}{h_c}\frac{R^2 g}{(h_c\bar{h}+R)^2} = -\frac{t_c^2}{h_c^3}\frac{R^2 g}{\left(\bar{h}+R/h_c\right)^2}$$

and the dimensionless initial condition

$$\frac{d\bar{h}}{d\bar{t}}(0) = \frac{t_c V}{h_c}\,.$$

The key dimensionless variable in this problem turns out to be

$$\epsilon = \frac{V}{\sqrt{Rg}}\,.$$

a) Assume we study the motion over long distances such that h may be of the same size as R. In this case, $h_c = R$ is a reasonable choice. Determine t_c from requiring the initial velocity to be unity. Set up the dimensionless ODE problem.

Solution. The suggested requirement leads to

$$\frac{t_c V}{h_c} = \frac{t_c V}{R} = 1 \quad \Rightarrow \quad t_c = \frac{R}{V}\,.$$

Inserting this t_c in the ODE gives the scaled ODE problem

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{\epsilon^2\left(1+\bar{h}\right)^2}, \quad \bar{h}(0) = 0,\ \frac{d\bar{h}}{d\bar{t}}(0) = 1\,.$$

b) As a), but determine t_c by demanding both terms in the scaled ODE to have unit coefficients.

Solution. We set

$$\frac{t_c^2}{h_c^3}R^2 g = \frac{t_c^2 g}{R} = 1 \quad \Rightarrow \quad t_c = \sqrt{\frac{R}{g}}\,.$$

The initial condition then becomes

$$\frac{d\bar{h}}{d\bar{t}}(0) = \frac{V}{\sqrt{Rg}} = \epsilon\,.$$

The scaled ODE problem is now

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{1}{\left(1+\bar{h}\right)^2}, \quad \bar{h}(0) = 0,\ \frac{d\bar{h}}{d\bar{t}}(0) = \epsilon\,.$$

c) For small initial velocity V, h will be small compared to R. In the limit $h/R \to 0$, the governing equation simplifies to the well-known motion in a constant gravity field: $h'' = -g$. Use this model to suggest a time and length scale, and derive a dimensionless ODE problem.

Solution. The solution of $h'' = -g$ with $h(0)$ and $h'(0) = V$ is easily obtained by integrating twice: $h = -\frac{1}{2}gt^2 + Vt$. The maximum height reached by the body is found by setting $h'(t) = 0$: $V - gt = 0$, which suggests a corresponding characteristic time $t_c = V/g$. The responding maximum height $h(t_c) = \frac{1}{2}V^2/g$ can be used as characteristic height: $h_c = \frac{1}{2}V^2/g$. The factor $\frac{1}{2}$ is not important, and the ODE problem looks nicer without it, so we simply set $h_c = V^2/g$. Inserted in the initial condition, we get

$$\frac{d\bar{h}}{d\bar{t}}(0) = \frac{Vt_c}{h_c} = 1\,.$$

The scaled ODE takes the form

$$\frac{d^2\bar{h}}{d\bar{t}^2} = -\frac{\epsilon^{-4}}{\left(\bar{h} + 2\epsilon^{-2}\right)^2} = -\frac{1}{\left(1 + \epsilon^2\bar{h}\right)^2}\,.$$

d) Give an interpretation of the dimensionless parameter ϵ.

Solution. We know from c) that the characteristic height in the constant gravity limit is V^2/g. We can therefore write

$$\epsilon^2 = \frac{V^2/g}{R},$$

which shows that ϵ^2 is the ratio of the height for small V, i.e., motion in a constant gravity field, and the earth's radius. A small ϵ means that we can neglect varying gravity.

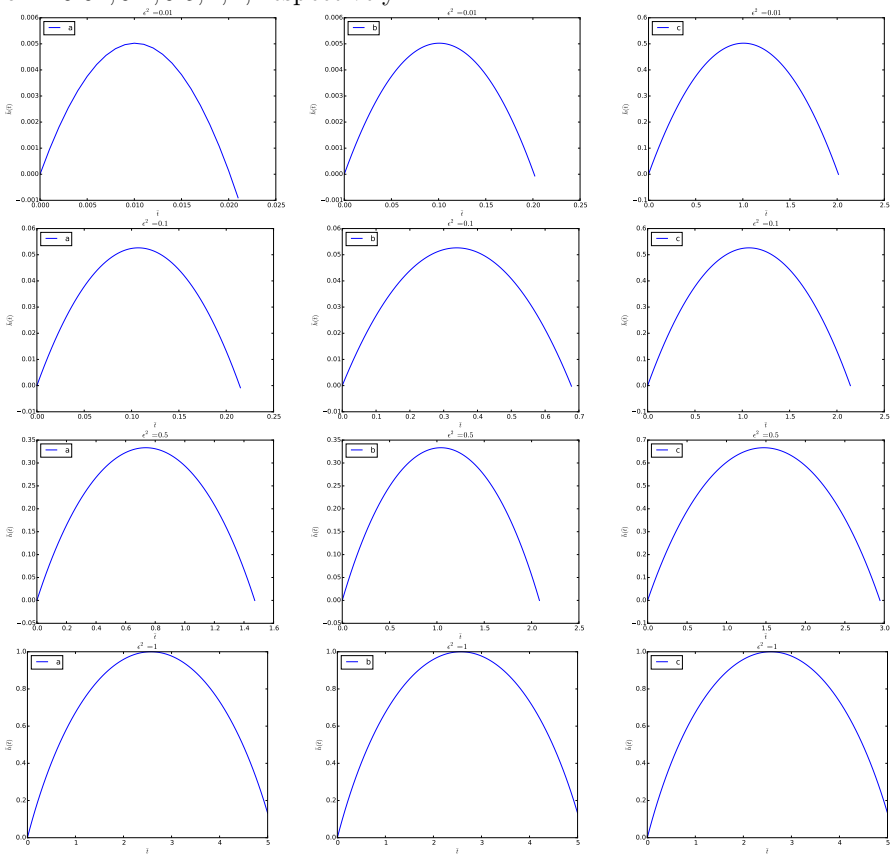
e) Solve numerically for $\bar{h}(\bar{t})$ in each of the three scalings in a), b), and c), with $\epsilon^2 = 0.01, 0.1, 0.5, 1, 2$. When are the various scalings appropriate? (That is, when are $\bar{t}$ and $\bar{h}$ of size unity or at least not very small or big?)

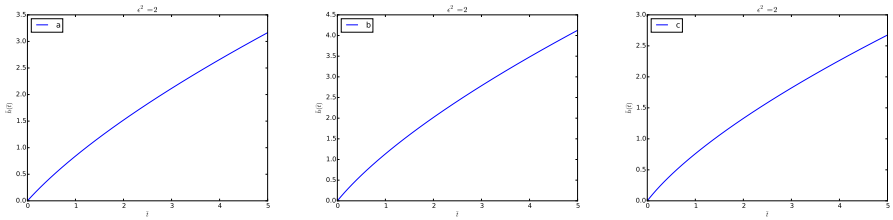
Solution. For numerical solution we rewrite the ODE as a system of two first-order ODEs by introducing a new variable $\bar{v}$ (velocity): $\bar{h}' = \bar{v}$, $\bar{v}' = \ldots$. Here is a code that employs Odespy to solve the system of first-order ODEs:

```
import odespy, numpy as np
import matplotlib.pyplot as plt

def varying_gravity(epsilon):
    def ode_a(u, t):
        h, v = u
        return [v, -epsilon**(-2)/(1 + h)**2]

    def ode_b(u, t):
        h, v = u
        return [v, -1.0/(1 + h)**2]

    def ode_c(u, t):
        h, v = u
        return [v, -1.0/(1 + epsilon**2*h)**2]

    problems = [ode_a, ode_b, ode_c]     # right-hand sides
    ics = [[0, 1], [0, epsilon], [0, 1]] # initial conditions
```

```
    for problem, ic, legend in zip(problems, ics, ['a', 'b', 'c']):
        solver = odespy.RK4(problem)
        solver.set_initial_condition(ic)
        t = np.linspace(0, 5, 5001)
        # Solve ODE until h < 0 (h is in u[:,0])
        u, t = solver.solve(t, terminate=lambda u, t, n: u[n,0] < 0)
        h = u[:,0]

        plt.figure()
        plt.plot(t, h)
        plt.legend(legend, loc='upper left')
        plt.title(r'$\epsilon^2=%g$' % epsilon**2)
        plt.xlabel(r'$\bar t$');  plt.ylabel(r'$\bar h(\bar t)$')
        plt.savefig('tmp_%s.png' % legend)
        plt.savefig('tmp_%s.pdf' % legend)
```

Recall from d) that ϵ^2 is the ratio of the height reached in a constant gravity field and the earth's radius. The figures below shows the results for $\epsilon^2 = 0.01, 0.1, 0.5, 1, 2$, respectively.





For ϵ small ($\epsilon^2 = 0.01, 0.1$), we see that the scaling in c) is most relevant since the scalings in a) and b) give small $\bar{h}$ and $\bar{t}$. For $\epsilon = 1$, all three scalings are equal. For larger ϵ, the body does not return to the earth. The scalings in a) and c) become equal in the limit $\epsilon \to \infty$, but they are already quite similar for $\epsilon = \sqrt{2}$ according to the bottom figure above. The scaling in c) is therefore the most appealing one since it works for small as well as large ϵ and become close to the others for ϵ around unity.

Remark. The present problem is one of the few problems that is discussed at length in the literature, see Logan [9] and Lin and Segel [8]. The standard argument is that the scaling in c) is favorable since it is the only scaling that is valid as $\epsilon \to 0$. However, that it is robust also for the larger relevant values of ϵ is something that is only clear when we solve the three problems numerically.
Filename: `varying_gravity`.

Chapter 3
Basic PDE models

This chapter extends the scaling technique to well-known partial differential equation (PDE) models for waves, diffusion, and transport. We start out with the simplest 1D models of the PDEs and then progress with additional terms, different types of boundary and initial conditions, and generalizations to 2D and 3D.

3.1 The wave equation

A standard, linear, one-dimensional wave equation problem in a homogeneous medium may be written as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad x \in (0, L),\ t \in (0, T], \tag{3.1}$$

where c is the constant wave velocity of the medium. With a briefer notation, where subscripts indicate derivatives, the PDE (3.1) can be written $u_{tt} = c^2 u_{xx}$. This subscript notation will occasionally be used later.

In multi dimensions in heterogeneous media we have the generalization

$$\frac{\partial^2 u}{\partial t^2} = \nabla \cdot \left(c^2 \nabla u\right) + f, \quad x, y, z \in \Omega,\ t \in (0, T]\,. \tag{3.2}$$

How to scale time depends on the PDE, the spatial scale depends on the domain, and the scale of u usually depends on f or the boundary or initial condition.

3.1.1 Simple homogeneous Dirichlet conditions

Let us first start with homogeneous Dirichlet conditions in space and no initial velocity u_t:

$$u(x,0) = I(x), \qquad x \in [0,L], \tag{3.3}$$

$$\frac{\partial}{\partial t}u(x,0) = 0, \qquad x \in [0,L], \tag{3.4}$$

$$u(0,t) = 0, \qquad t \in (0,T], \tag{3.5}$$

$$u(L,t) = 0, \qquad t \in (0,T]. \tag{3.6}$$

The independent variables are x and t, while u is the dependent variable. The rest of the parameters, c, L, T, and $I(x)$, are given data.

We start with introducing dimensionless versions of the independent and dependent variables:

$$\bar{x} = \frac{x}{x_c}, \quad \bar{t} = \frac{t}{t_c}, \quad \bar{u} = \frac{u}{u_c}.$$

Inserting the $x = x_c\bar{x}$, etc., in (3.1) and (3.3)-(3.6) gives

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{t_c^2 c^2}{x_c^2}\frac{\partial^2 \bar{u}}{\partial x^2}, \qquad \bar{x} \in (0, L/x_c),\ \bar{t} \in (0, T/t_c],$$

$$\bar{u}(\bar{x},0) = \frac{I(x_c\bar{x})}{u_c}, \qquad \bar{x} \in [0, L/x_c],$$

$$\frac{\partial}{\partial \bar{t}}\bar{u}(\bar{x},0) = 0, \qquad \bar{x} \in [0, L/x_c],$$

$$\bar{u}(0,\bar{t}) = 0, \qquad \bar{t} \in (0, T/t_c],$$

$$\bar{u}(L/x_c,\bar{t}) = 0, \qquad \bar{t} \in (0, T/t_c].$$

The key question is how to define the scales. A natural choice is $x_c = L$ since this makes $\bar{x} \in [0,1]$. For the spatial scale and the problem governed by (3.1) we have some analytical insight that can help. The solution behaves like

$$u(x,t) = f_R(x-ct) + f_R(x+ct), \tag{3.7}$$

i.e., a right- and left-going wave with velocity c. The initial conditions constrain the choices of f_R and f_L to $f_L + f_R = I$ and $-cf_L' + cf_R' = 0$. The solution is $f_R = f_L = \frac{1}{2}$, and consequently

$$u(x,t) = \frac{1}{2}I(x-ct) + \frac{1}{2}I(x+ct),$$

which tells that the initial condition splits in two, half of it moves to the left and half to the right. This means in particular that we can choose

$u_c = \max_x |I(x)|$ and get $|\bar{u}| \leq 1$, which is a goal. It must be added that boundary conditions may result in reflected waves, and the solution is then more complicated than indicated in the formula above.

Regarding the time scale, we may look at the two terms in the scaled PDE and argue that if $|u|$ and its derivatives are to be of order unity, then the size of the second-order derivatives should be the same, and t_c can be chosen to make the coefficient $t_c^2 c^2/x_c^2$ unity, i.e., $t_c = L/c$. Another reasoning may set t_c as the time it takes the wave to travel through the domain $[0,L]$. Since the wave has constant speed c, $t_c = L/c$.

With the described choices of scales, we end up with the dimensionless initial-boundary value problem

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{\partial^2 \bar{u}}{\partial x^2}, \qquad \bar{x} \in (0,1),\ \bar{t} \in (0,\bar{T}], \tag{3.8}$$

$$\bar{u}(\bar{x},0) = \frac{I(\bar{x}L)}{\max_{x\in(0,L)} |I(x)|}, \qquad \bar{x} \in [0,1], \tag{3.9}$$

$$\frac{\partial}{\partial \bar{t}}\bar{u}(\bar{x},0) = 0, \qquad \bar{x} \in [0,1], \tag{3.10}$$

$$\bar{u}(0,\bar{t}) = 0, \qquad \bar{t} \in (0,\bar{T}], \tag{3.11}$$

$$\bar{u}(1,\bar{t}) = 0, \qquad \bar{t} \in (0,\bar{T}]. \tag{3.12}$$

Here, $\bar{T} = Tc/L$.

The striking feature of (3.8)-(3.12) is that there are *no physical parameters* involved! Everything we need to specify is the shape of the initial condition and then scale it such that it is less than or equal to 1.

The physical solution with dimension is recovered from $\bar{u}(\bar{x},\bar{t})$ through

$$u(x,t) = \max_{x\in(0,L)} I(x)\,\bar{u}(\bar{x}L,\bar{t}L/c) \tag{3.13}$$

3.1.2 Implementation of the scaled wave equation

How do we implement (3.8)-(3.12)? As for the simpler mathematical models, we suggest to implement the model with dimensions and observe how to set parameters to obtain the scaled model. In the present case, one must choose $L = 1$, $c = 1$, and scale I by its maximum value. That's all!

Several implementations of 1D wave equation models with different degree of mathematical and software complexity come along with this book. The simplest version is `wave1D_u0.py` that implements (3.1) and (3.3)-(3.6). This is the code to be used in the following. It is described in Sections **??** in [7].

Waves on a string. As example, we may let the original initial-boundary value problem (3.1)-(3.6) model vibrations of a string on a string instrument.

With u as the displacement of the string, the boundary conditions $u=0$ at the ends are relevant, as well as the zero velocity condition $\partial u/\partial t=0$ at $t=0$. The initial condition $I(x)$ has typically a triangular shape for a picked guitar string. The physical problem needs parameters for the amplitude of $I(x)$, the length L of the string, and the value of c for the string. Only the latter is challenging as it involves relating c to the pitch (i.e., time frequency) of the string. In the scaled problem, we can forget about all this. We simply set $L=1$, $c=1$, and let $I(x)$ have a peak of unity at $x=x_0\in(0,1)$:

$$\frac{I(x)}{\max_x I(x)}=\begin{cases} x/x_0, & x<x_0,\\ (1-x)/(1-x_0), & \text{otherwise}\end{cases}$$

The dimensionless coordinate of the peak, x_0, is the only dimensionless parameter in the problem. For fixed x_0, one single simulation will capture all possible solutions with such a triangular shape.

Detecting an already computed case. The file `wave1D_u0_scaled.py` has functionality for detecting whether a simulation corresponds to a previously run scaled case, and if so, the solution is retrieved from file. The implementation technique makes use of `joblib`, but is more complicated than showed previously in this book since some of the arguments to the function that computes the solution are functions, and one must recognized if the function has been used as argument before or not. There is documentation in the `wave1D_u0_scaled.py` file how this is done.

3.1.3 Time-dependent Dirichlet condition

A generalization of (3.1)-(3.6) is to allow for a time-dependent Dirichlet condition at one end, say $u(0,t)=U_L(t)$. At the other end we may still have $u=0$. This new condition at $x=0$ may model a specified wave that enters the domain. For example, if we feed in a monochromatic wave $A\sin(k(x-ct))$ from the left end, $U_L(t)=A\sin(kct)$. This forcing of the wave motion has its own amplitude and time scale that could affect the choice of u_c and t_c.

The main difference from the previous initial-boundary value problem is the condition at $x=0$, which now reads

$$\bar{u}(0,\bar{t})=\frac{U_L(\bar{t}t_c)}{u_c}$$

in scaled form.

Scaling. Regarding the characteristic time scale, it is natural to base this scale on the wave propagation velocity and not on the time scale of $U_L(t)$, because the time scale of U_L basically determines whether short or long waves are fed in at the boundary. All waves, long or short, propagate with the same velocity c. We therefore continue to use $t_c=L/c$.

The solution u will have one wave contribution from the initial condition I and one from the feeding of waves at $x=0$. This gives us three choices of u_c: $\max_x |I| + \max_t |U_L|$, $\max_x |I|$, or $\max_t |U_L|$. The first seems relevant if the size of I and U_L are about the same, but then we can choose either $\max_x |I|$ or $\max_t |U_L|$ as characteristic size of u since a factor of 2 is not important. If I is much less than U_L, $u_c = \max_t |u_L|$ is relevant, while $u_c = \max_x |I|$ is the choice when I has much bigger impact than U_L on u.

With $u_c = \max_t |U_L(t)|$, we get the scaled problem

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}, \qquad \bar{x} \in (0,1),\ \bar{t} \in (0,\bar{T}], \tag{3.14}$$

$$\bar{u}(\bar{x},0) = \frac{I(x_c\bar{x})}{\max_t |U_L(t)|}, \qquad \bar{x} \in [0,1], \tag{3.15}$$

$$\frac{\partial}{\partial \bar{t}}\bar{u}(\bar{x},0) = 0, \qquad \bar{x} \in [0,1], \tag{3.16}$$

$$\bar{u}(0,\bar{t}) = \frac{U_L(\bar{t}t_c)}{\max_t |U_L(t)|}, \qquad \bar{t} \in (0,\bar{T}], \tag{3.17}$$

$$\bar{u}(1,\bar{t}) = 0, \qquad \bar{t} \in (0,\bar{T}]. \tag{3.18}$$

Also this problem is free of physical parameters like c and L. The input is completely specified by the shape of $I(x)$ and $U_L(t)$.

Software. Software for the original problem with dimensions can be reused for (3.14)-(3.18) by setting $L=1$, $c=1$, and scaling $U_L(t)$ and $I(x)$ by $\max_t |U_L(t)|$.

Specific case. As an example, consider

$$U_L(t) = a\sin(\omega t) \text{ for } 0 \le t \le 2\frac{\omega}{2\pi}, \text{ else } 0,$$
$$I(x) = Ae^{-(x-L/2)^2/\sigma^2}.$$

That is, we start with a Gaussian peak-shaped wave in the center of the domain and feed in a sinusoidal wave at the left end for two periods. The solution will be the sum of three waves: two parts from the initial condition, plus the wave fed in from the left.

Since $\max_t |U_L| = a$ we get

$$\bar{u}(\bar{x},0) = \frac{A}{a}e^{-(L/\sigma)^2(\bar{x}-\frac{1}{2})^2}, \tag{3.19}$$

$$\bar{u}(0,\bar{t}) = \sin(\bar{t}\omega L/c). \tag{3.20}$$

Here, U_L models an incoming wave $a\sin(k(x-ct)$, with k specified. The result is incoming waves of length $\lambda = 2\pi/k$. Since $\omega = kc$, $\bar{u}(0,\bar{t}) = \sin(kL\bar{t}) =$

$\sin(2\pi\bar{t}L/\lambda)$. (This formula demonstrates the previous assertion that the time scale of U_L, i.e., $1/\omega$, determines the wave length $1/\omega = \lambda/(2\pi)$ in space.) We realize from the formulas (3.19) and (3.20) that there are three key dimensionless parameters related to these specific choices of initial and boundary conditions:

$$\alpha = \frac{A}{a}, \quad \beta = \frac{L}{\sigma}, \quad \gamma = kL = 2\pi\frac{L}{\lambda}.$$

With α, β, and γ we can write the dimensionless initial and boundary conditions as

$$\bar{u}(\bar{x},0) = \alpha e^{-\beta^2(\bar{x}-\frac{1}{2})^2},$$
$$\bar{u}(0,\bar{t}) = \sin(\gamma\bar{t}).$$

The dimensionless parameters have the following interpretations:

- α: ratio of initial condition amplitude and amplitude of incoming wave at $x = 0$
- β: ratio of length of domain and width of initial condition
- γ: ratio of length of domain and wave length of incoming wave

Again, these dimensionless parameters tell a lot about the interplay of the physical effects in the problem. And only some ratios count!

We can simulate two special cases:

1. $\alpha = 10$ (large) where the incoming wave is small and the solution is dominated by the two waves arising from $I(x)$,
2. $\alpha = 0.1$ (small) where the incoming waves dominate and the solution has the initial condition just as a small perturbation of the wave shape.

We may choose a peak-shaped initial condition: $\beta = 10$, and also a relatively short incoming wave compared to the domain size: $\gamma = 6\pi$ (i.e., wave length of incoming wave is $L/6$). A function `simulate_Gaussian_and_incoming_wave` in the file `session.py` applies the general unscaled solver in `wave1D_dn.py` for solving the wave equation with constant c, and any time-dependent function or $\partial u/\partial x = 0$ at the end points. This solver is trivially adapted to the present case. Figures 3.1 and 3.2 shows snapshots of how $\bar{u}(\bar{x},\bar{t})$ evolves due to a large/small initial condition and small/large incoming wave at the left boundary.

hpl 11: The HTML version features movies here. Link to them in the LaTeX PDF version as well?

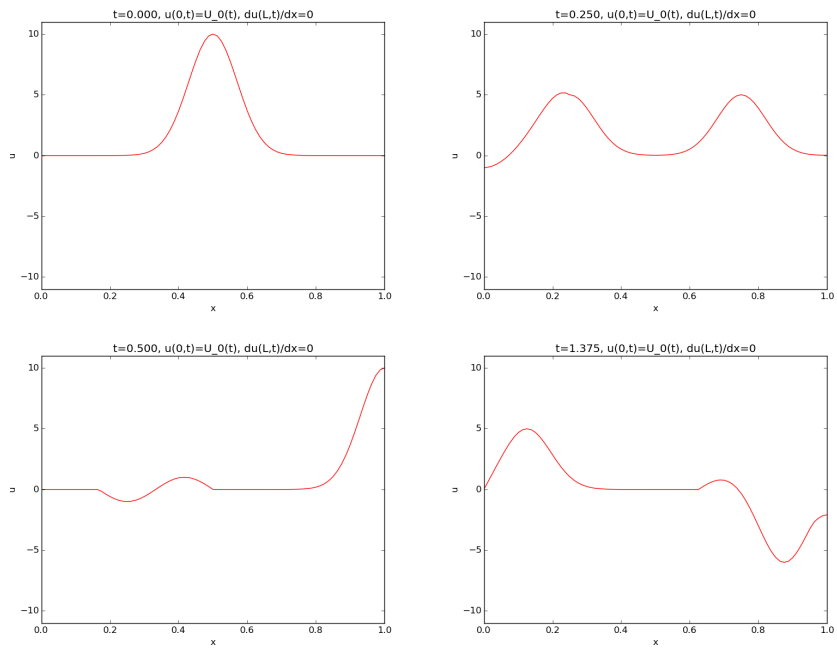


Fig. 3.1 Snapshots of solution with large initial condition and small incoming wave ($\alpha = 10$).

3.1.4 Velocity initial condition

Now we change the initial condition from $u = I$ and $\partial u/\partial t = 0$ to

$$u(x,0) = 0, \tag{3.21}$$

$$\frac{\partial}{\partial t}u(x,0) = V(x)\,. \tag{3.22}$$

Impact problems are often of this kind. The scaled version of $u_t(x,0) = V(x)$ becomes

$$\frac{\partial}{\partial \bar{t}}\bar{u}(\bar{x},0) = \frac{t_c}{u_c}V(\bar{x}x_c)\,.$$

Analytical insight. From (3.7) we now get $f_L + f_R = 0$ and $cf_L' - cf_R' = V$. Introducing $W(x)$ such that $W'(x) = V(x)$, a solution is $-f_L = \frac{1}{2}W$ and $f_R = \frac{1}{2}W$. Hence,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)d\xi\,.$$

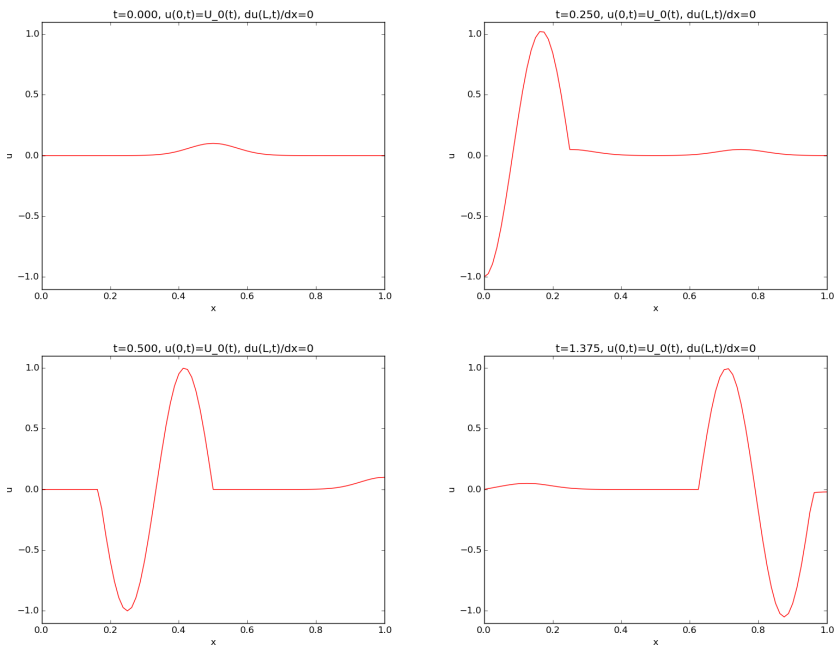


Fig. 3.2 Snapshots of solution with small initial condition and large incoming wave ($\alpha = 0.1$).

Scaling. Since V is the time-derivative of u, the characteristic size of V is typically u_c/t_c, meaning that

$$\max_{x\in(0,L)} |V(x)| = \frac{u_c}{t_c},$$

which gives $u_c = \max_{x\in(0,L)} |V(x)|L/c$. As usual, we base t_c on the wave speed: $t_c = L/c$. We end up with

$$\frac{\partial}{\partial \bar{t}}\bar{u}(\bar{x},0) = \frac{V(\bar{x}x_c)}{\max_x |V(x)|},$$

which by itself looks like a natural scaling of a function V.

Nonzero initial shape. Suppose we change the initial condition $u(x,0) = 0$ to $u(x,0) = I(x)$. The scaled version of this condition with the above u_c based on V becomes

$$\bar{u}(\bar{x},0) = \frac{cI(\bar{x}x_c)}{L\max_x |V(x)|}\,. \tag{3.23}$$

Check that dimensionless numbers are dimensionless!

Is a dimensionless number really dimensionless? It is easy to make errors when scaling equations, so checking that such fractions are dimensionless is wise. The dimension of I is the same as u, here taken to be displacement: [L]. Since V is $\partial u/\partial t$, its dimension is $[\mathrm{LT}^{-1}]$. The dimensions of c and L are $[\mathrm{LT}^{-1}]$ and [L]. The dimension of the right-hand side of (3.23) is then

$$\frac{[\mathrm{LT}^{-1}][L]}{[L][L\mathrm{T}^{-1}]} = 1,$$

demonstrating that the fraction is indeed dimensionless.

One may introduce a dimensionless initial shape, $\bar{I}(\bar{x}) = I(\bar{x}L)/\max_x |I|$. Then

$$\bar{u}(\bar{x},0) = \alpha \bar{I}(\bar{x}),$$

where α the dimensionless number

$$\alpha = \frac{c}{L}\frac{\max_x |I(x)|}{\max_x |V(x)|}.$$

If V is much larger than I, one expects that the influence of I is small. However, it takes time for the initial velocity V to influence the wave motion, so if c is much bigger than L, the initial wave shape I travels quickly through the domain before the effect of V becomes visible. The impact of I may therefore be significant for small t. This is reflected in an α value that is not small since c/L is large and $\max|I|/\max|V|$ is small, resulting in a scaled initial condition $\bar{u}(\bar{x},0)$ that is not small. With c/L about unity, α becomes small, and $\bar{u}(\bar{x},0) \approx 0$ such that not much happens before the effect of V becomes visible. Recall that the dimensionless initial velocity is about unity regardless of other parameters. Again, the scaling and the resulting dimensionless parameter(s) teach us much about the interaction of the various physical effects.

hpl 12: Do experiments. Make exercise or insert here.

hpl 13: Could make paradox: small I, big V, but still significant impact of I in a simulation. Why? Bug? The answer is above. Best as exercise.

3.1.5 Variable wave velocity and forcing

The next problem generalization regards wave propagation in a non-homogeneous medium where the wave velocity c depends on the spatial position: $c = c(x)$.

To simplify the notation we introduce $\lambda(x) = c^2(x)$. We introduce homogeneous Neumann conditions at $x = 0$ and $x = L$. In addition, we add a force term $f(x,t)$ to the PDE, modeling wave generation in the interior of the domain. For example, a moving slide at the bottom of a fjord will generate surface waves and is modeled by such an $f(x,t)$ term (provided the length of the waves is much larger than the depth so that a simple wave equation like (3.24) applies). The initial-boundary value problem can be then expressed as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial u}{\partial x}\right) + f(x,t), \qquad x \in (0,L),\ t \in (0,T], \tag{3.24}$$

$$u(x,0) = I(x), \qquad x \in [0,L], \tag{3.25}$$

$$\frac{\partial}{\partial t}u(x,0) = 0, \qquad x \in [0,L], \tag{3.26}$$

$$\frac{\partial}{\partial x}u(0,t) = 0, \qquad t \in (0,T], \tag{3.27}$$

$$\frac{\partial}{\partial x}u(L,t) = 0, \qquad t \in (0,T]. \tag{3.28}$$

Non-dimensionalization. We make the coefficient λ non-dimensional by

$$\bar{\lambda}(\bar{x}) = \frac{\lambda(\bar{x}x_c)}{\lambda_c}, \tag{3.29}$$

where one normally chooses the characteristic size of λ, λ_c, to be the maximum value such that $|\lambda| \leq 1$:

$$\lambda_c = \max_{x\in(0,L)} \lambda(x)\,.$$

Similarly, f has a scaled version

$$\bar{f}(\bar{x},\bar{t}) = \frac{f(\bar{x}x_c,\bar{t}t_c)}{f_c},$$

where normally we choose

$$f_c = \max_{x,t} |f(x,t)|\,.$$

Inserting dependent and independent variables expressed by their non-dimensional counterparts yields

$$
\begin{aligned}
\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} &= \frac{t_c^2 \lambda_c}{L^2} \frac{\partial}{\partial \bar{x}} \left(\bar{\lambda}(\bar{x}) \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \frac{t_c^2 f_c}{u_c} \bar{f}(\bar{x},\bar{t}), && \bar{x} \in (0,1),\ \bar{t} \in (0,\bar{T}], \\
\bar{u}(\bar{x},0) &= \frac{I(x)}{u_c}, && \bar{x} \in [0,1], \\
\frac{\partial}{\partial \bar{t}} \bar{u}(\bar{x},0) &= 0, && \bar{x} \in [0,1], \\
\frac{\partial}{\partial \bar{x}} \bar{u}(0,\bar{t}) &= 0, && \bar{t} \in (0,\bar{T}], \\
\frac{\partial}{\partial \bar{x}} \bar{u}(1,\bar{t}) &= 0, && \bar{t} \in (0,\bar{T}],
\end{aligned}
$$

with $\bar{T} = Tc/L$.

Choosing the time scale. The time scale is, as before, chosen as $t_c = L/\sqrt{\lambda_c}$. Note that the previous (constant) wave velocity c now corresponds to $\sqrt{\lambda(x)}$. Therefore, $\sqrt{\lambda_c}$ is a characteristic wave velocity.

One could wonder if the time scale of the force term, $f(x,t)$, should influence t_c, but as we reasoned for the boundary condition $u(0,t) = U_L(t)$, we let the characteristic time be governed by the signal speed in the medium, i.e., by $\sqrt{\lambda_c}$ here and not by the time scale of the excitation f which dictates the length of the generated waves and not their propagation speed.

Choosing the spatial scale. We may choose u_c as $\max_x |I(x)|$, as before, or we may fit u_c such that the coefficient in the source term is unity, i.e., all terms balance each other. This latter idea leads to

$$u_c = \frac{L^2 f_c}{\lambda_c}$$

and a PDE without parameters,

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{\partial}{\partial \bar{x}} \left(\bar{\lambda}(\bar{x}) \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \bar{f}(\bar{x},\bar{t}).$$

The initial condition $u(x,0) = I(x)$ becomes in dimensionless form

$$\bar{u}(\bar{x},0) = u_c^{-1} \max_x |I(x)| \bar{I}(\bar{x}) = \beta^{-1} \bar{I}(\bar{x}),$$

where

$$\beta = \frac{L^2}{\lambda_c} \frac{\max_{x,t} |f(x,t)|}{\max_x |I(x)|}.$$

In the case $u_c = \max_x |I(x)|$, $\bar{u}(\bar{x},0) = \bar{I}(\bar{x})$ and the β parameter appears in the PDE instead:

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{\partial}{\partial \bar{x}} \left(\bar{\lambda}(\bar{x}) \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \beta \bar{f}(\bar{x},\bar{t}).$$

With $V=0$, and $u=0$ or $u_x=0$ on the boundaries $x=0,L$, this scaling gives $|\bar{u}|\leq 1$, since initially $|I|\leq 1$, and no boundary condition can increase the amplitude.

Scaling the velocity initial condition. The initial condition $u_t(x,0)=V(x)$ has its dimensionless variant as

$$\bar{V}(\bar{x})=\frac{t_c}{u_c}\frac{V(L\bar{x})}{\max_x|V(x)|},$$

which becomes

$$\frac{\partial\bar{u}}{\partial\bar{t}}(\bar{x},0)=\frac{L}{\sqrt{\lambda_c}}\frac{\max_x|V(x)|}{\max_x|I(x)|}\bar{V}(\bar{x}),\ \text{if } u_c=\max_x|I(x)|,$$

or

$$\frac{\partial\bar{u}}{\partial\bar{t}}(\bar{x},0)=\frac{\sqrt{\lambda_c}}{L}\frac{\max_x|V(x)|}{\max_{x,t}|f(x,t)|}\bar{V}(\bar{x}),\ \text{if } u_c=t_c^2f_c=\frac{L^2}{\lambda_c}\max_{x,t}|f(x,t)|\,.$$

Introducing the dimensionless number α (cf. Section 3.1.4),

$$\alpha^{-1}=\frac{\sqrt{\lambda_c}}{L}\frac{\max_x|V(x)|}{\max_{x,t}|f(x,t)|},$$

we can write

$$\frac{\partial\bar{u}}{\partial\bar{t}}(\bar{x},0)=\begin{cases}\alpha^{-1}\bar{V}(\bar{x}), & u_c=\max_x|I|\\ \alpha^{-1}\beta^{-1}\bar{V}(\bar{x}), & u_c=t_c^2f_c\end{cases}$$

3.1.6 Damped wave equation

A linear damping term $b\partial u/\partial t$ is often added to the wave equation to model energy dissipation and amplitude reduction. Our PDE now reads

$$\frac{\partial^2u}{\partial t^2}+b\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial u}{\partial x}\right)+f(x,t)\,. \tag{3.30}$$

The scaled equation becomes

$$\frac{\partial^2\bar{u}}{\partial\bar{t}^2}+\frac{t_c}{b}\frac{\partial\bar{u}}{\partial\bar{t}}=\frac{t_c^2\lambda_c}{L^2}\frac{\partial}{\partial\bar{x}}\left(\bar{\lambda}(\bar{x})\frac{\partial\bar{u}}{\partial\bar{x}}\right)+\frac{t_c^2f_c}{u_c}\bar{f}(\bar{x},\bar{t})\,.$$

The damping term is usually much smaller than the two other terms involving $\bar{u}$. The time scale is therefore chosen as in the undamped case, $t_c=L/\sqrt{\lambda_c}$. As in Section 3.1.5, we have two choices of u_c: $u_c=\max_x|I|$ or $u_c=t_c^2f_c$. The former choice of u_c gives a PDE with two dimensionless numbers,

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + \gamma \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left(\bar{\lambda}(\bar{x}) \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \beta \bar{f}(\bar{x}, \bar{t}), \tag{3.31}$$

where

$$\gamma = \frac{bL}{\sqrt{\lambda_c}},$$

measures the size of the damping, and β is as given in Section 3.1.5. With $u_c = t_c^2 f_c$ we get a PDE where only γ enters,

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + \gamma \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}} \left(\bar{\lambda}(\bar{x}) \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \bar{f}(\bar{x}, \bar{t}) \,. \tag{3.32}$$

The scaled initial conditions are as in Section 3.1.5, so in this latter case β appears in the initial condition for u.

To summarize, the effects of V, f, and damping are reflected in the dimensionless numbers α, β, and γ, respectively.

3.1.7 A three-dimensional wave equation problem

To demonstrate how the scaling extends to and looks like in three spatial dimensions, we consider

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{\partial}{\partial x} \left(\lambda \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\lambda \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(\lambda \frac{\partial u}{\partial z} \right) . \tag{3.33}$$

We introduce

$$\bar{x} = \frac{x}{x_c}, \quad \bar{y} = \frac{y}{y_c}, \quad \bar{z} = \frac{z}{z_c}, \quad \bar{t} = \frac{t}{t_c}, \quad \bar{u} = \frac{u}{u_c} \,.$$

With $\bar{\lambda} = \lambda(\bar{x}x_c, \bar{y}y_c, \bar{z}z_c)/\lambda_c$, we get

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{t_c^2 \lambda_c}{x_c^2} \frac{\partial}{\partial \bar{x}} \left(\bar{\lambda} \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \frac{t_c^2 \lambda_c}{y_c^2} \frac{\partial}{\partial \bar{y}} \left(\bar{\lambda} \frac{\partial \bar{u}}{\partial \bar{y}} \right) + \frac{t_c^2 \lambda_c}{z_c^2} \frac{\partial}{\partial \bar{z}} \left(\bar{\lambda} \frac{\partial \bar{u}}{\partial \bar{z}} \right) .$$

Often, we will set $x_c = y_c = z_c = L$ where L is some characteristic size of the domain. As before, $t_c = L/\sqrt{\lambda_c}$, and these choices lead to a dimensionless wave equation without physical parameters:

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = \frac{\partial}{\partial \bar{x}} \left(\bar{\lambda} \frac{\partial \bar{u}}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{y}} \left(\bar{\lambda} \frac{\partial \bar{u}}{\partial \bar{y}} \right) + \frac{\partial}{\partial \bar{z}} \left(\bar{\lambda} \frac{\partial \bar{u}}{\partial \bar{z}} \right) . \tag{3.34}$$

The initial conditions remain the same as in the previous one-dimensional examples.

3.2 The diffusion equation

The diffusion equation in a one-dimensional homogeneous medium reads

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,L),\ t \in (0,T], \tag{3.35}$$

where α is the diffusion coefficient. The multi-dimensional generalization to a heterogeneous medium and a source term takes the form

$$\frac{\partial u}{\partial t} = \nabla \cdot (\alpha \nabla u) + f, \quad x,y,z \in \Omega,\ t \in (0,T]\,. \tag{3.36}$$

We first look at scaling the PDE itself, and thereafter we discuss some types of boundary conditions and how to scale the complete initial-boundary value problem.

3.2.1 Homogeneous 1D diffusion equation

To make (3.35) dimensionless, we introduce as usual dimensionless dependent and independent variables:

$$\bar{x} = \frac{x}{x_c}, \quad \bar{t} = \frac{t}{t_c}, \quad \bar{u} = \frac{u}{u_c}\,.$$

Inserting the dimensionless quantities in the one-dimensional PDE (3.35) results in

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{t_c \alpha}{L^2} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}, \quad \bar{x} \in (0,1),\ \bar{t} \in (0, \bar{T} = T/t_c]\,.$$

Arguing as for the wave equation that the scaling should result in

$$\frac{\partial \bar{u}}{\partial \bar{t}} \text{ and } \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}$$

of the same size (about unity), implies $t_c \alpha / L^2 = 1$ and therefore $t_c = L^2/\alpha$.

Insight through an analytical solution can alternatively help with choosing t_c. One can show that $u = Ae^{-pt}\sin(kx)$ is a solution of (3.35) if $p = \alpha k^2$, for any k. This is the typical solution arising from separation of variables and reflects the dynamics of the space and time in the PDE. Exponential decay in time is a characteristic feature of diffusion processes, and the e-folding time can then be taken as a time scale. This means $t_c = 1/p \sim k^{-2}$. Since k is related to the spatial wave length λ through $k = 2\pi/\lambda$, it means that t_c depends strongly on the wave length of the sine term $\sin(kx)$. In particular, short waves (as found in noisy signals) with large k decay very rapidly. For the overall solution we are interested in how the longest meaningful wave

decays and use that time scale for t_c. The longest wave typically has half a wave length over the domain $[0,L]$: $u = Ae^{-pt}\sin(\pi x/L)$ ($k = \pi/L$), provided $u(0,t) = u(L,t) = 0$ (with $u_x(L,t) = 0$, the longest wave is $L/4$, but we look at the case with the wave length $L/2$). Then $t_c = L^2/\alpha\pi^{-2}$, but the factor π^{-2} is not important and we simply choose $t_c = L^2/\alpha$, which equals the time scale we arrived at above. We may say that t_c is the time it takes for the diffusion to significantly change the solution in the entire domain.

Another fundamental solution of the diffusion equation is the diffusion of a Gaussian function: $u(x,t) = (4\pi\alpha t)^{-1/2}\exp(-x^2/(4\alpha t))$. For the diffusion to be significant at a distance $x = L$, we may demand the exponential factor to have a value of $e^{-1} \approx 0.37$, which implies $t = L^2/(4\alpha)$, but the factor 4 is not of importance, so again, a relevant time scale is $t_c = L^2/\alpha$.

The scale u_c is chosen according to the initial condition: $u_c = \max_{x\in(0,L)}|I(x)|$. For a diffusion equation $u_t = \alpha u_{xx}$ with $u = 0$ at the boundaries $x = 0, L$, the solution is bounded by the initial condition $I(x)$. Therefore, the listed choice of u_c implies that $|u| \leq 1$. (The solution $u = Ae^{-pt}\sin(kx)$ is such an example if $k = n\pi/L$ for integer n such that $u = 0$ for $x = 0$ and $x = L$.)

The resulting dimensionless PDE becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}, \quad \bar{x} \in (0,1),\ \bar{t} \in (0,\bar{T}], \tag{3.37}$$

with initial condition

$$\bar{u}(\bar{x},0) = \bar{I}(\bar{x}) = \frac{I(x_c\bar{x})}{\max_x |I(x)|}\,.$$

Notice that (3.37) is without physical parameters, but there may be parameters in $I(x)$.

3.2.2 Generalized diffusion PDE

Turning the attention to (3.36), we introduce the dimensionless diffusion coefficient

$$\bar{\alpha}(\bar{x},\bar{y},\bar{z}) = \alpha_c^{-1}\alpha(x_c\bar{x}, y_c\bar{y}, z_c\bar{z}),$$

typically with

$$\alpha_c = \max_{x,y,z}\alpha(x,y,z)\,.$$

The length scales are

$$\bar{x} = \frac{x}{x_c}, \quad \bar{y} = \frac{y}{y_c}, \quad \bar{z} = \frac{z}{z_c}\,.$$

We scale f in a similar fashion:

$$\bar{f}(\bar{x},\bar{y},\bar{z},\bar{t}) = f_c^{-1} f(\bar{x}x_c,\bar{y}y_c\bar{z}z_c,\bar{t}t_c),$$

with

$$f_c = \max_{x,y,z,t} |f(x,y,z,t)|\,.$$

Also assuming that $x_c = y_c = z_c = L$, and $u_c = \max_{x,y,z}(I(x,y,z)$, we end up with the scaled PDE

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \nabla\cdot\left(\bar{\alpha}\bar{\nabla}\bar{u}\right) + \beta\bar{f}, \quad \bar{x},\bar{y},\bar{z}\in\bar{\Omega},\ \bar{t}\in(0,\bar{T}]\,. \tag{3.38}$$

Here, $\bar{\nabla}$ means differentiation with respect to dimensionless coordinates $\bar{x}$, $\bar{y}$, and $\bar{z}$. The dimensionless parameter β takes the form

$$\beta = \frac{t_c f_c}{u_c} = \frac{L^2}{\alpha}\frac{\max_{x,y,z,t}|f(x,y,z,t)|}{\max_{x,y,z}|I(x,y,z)|}\,.$$

The scaled initial condition is $\bar{u} = \bar{I}$ as in the 1D case.

An alternative choice of u_c is to make the coefficient $t_c f_c/u_c$ in the source term unity. The scaled PDE now becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \nabla\cdot\left(\bar{\alpha}\bar{\nabla}\bar{u}\right) + f, \tag{3.39}$$

but the initial condition features the β parameter:

$$\bar{u}(\bar{x},\bar{y},\bar{z},0) = \frac{I}{t_c f_c} = \beta^{-1}\bar{I}(\bar{x},\bar{y},\bar{z})\,.$$

The β parameter can be interpreted as the ratio of the source term and the terms with u:

$$\beta = \frac{f_c}{u_c/t_c} \sim \frac{|f|}{|u_t|}, \quad \beta = \frac{f_c}{u_c/t_c} = \frac{f_c}{L^2/t_c u_c/L^2} \sim \frac{|f_c|}{|\alpha\nabla^2 u|}\,.$$

We may check that β is really non-dimensional. From the PDE, f must have the same dimensions as $\partial u/\partial t$, i.e., $[\Theta\mathrm{T}^{-1}]$. The dimension of α is more intricate, but from the term αu_{xx} we know that u_{xx} has dimensions $[\Theta\mathrm{L}^{-2}]$, and then α must have dimension $[\mathrm{L}^2\mathrm{T}^{-1}]$ to match the target $[\Theta\mathrm{T}^{-1}]$. In the expression for β we get $[\mathrm{L}^2\Theta\mathrm{T}^{-1}(\mathrm{L}^2\mathrm{T}^{-1}\Theta)^{-1}]$, which equals 1 as it should.

3.2.3 Jump boundary condition

A classical one-dimensional heat conduction problem goes as follows. An insulated rod at some constant temperature U_0 is suddenly heated from one

end ($x=0$), modeled as a constant Dirichlet condition $u(0,t)=U_L\neq U_0$ at that end. That is, the boundary temperature jumps from U_0 to U_1 at $t=0$. All the other surfaces of the rod are insulated such that a one-dimensional model is appropriate, but we must explicitly demand $u_x(L,t)=0$ to incorporate the insulation condition in the one-dimensional model at the end of the domain $x=L$. Heat cannot escape, and since we supply heat at $x=0$, all of the material will eventually be warmed up to the temperature U_1: $u\rightarrow U_1$ as $t\rightarrow\infty$.

The initial-boundary value problem reads

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}, \qquad x\in(0,L),\ t\in(0,T], \tag{3.40}$$

$$u(x,0)=U_0, \qquad x\in[0,L], \tag{3.41}$$

$$u(0,t)=U_1, \qquad t\in(0,T], \tag{3.42}$$

$$\frac{\partial}{\partial x}u(L,t)=0, \qquad t\in(0,T]. \tag{3.43}$$

The diffusion coefficient is related to heat transfer parameters by $\alpha=k/(\varrho c)$, where k is the heat conduction coefficient, ϱ is the density, and c is a specific heat capacity parameter.

The natural dimensionless temperature for this problem is

$$\bar{u}=\frac{u-U_0}{U_1-U_0},$$

since this choice makes $\bar{u}\in[0,1]$. The reason is that u is bounded by the initial and boundary conditions (in the absence of a source term in the PDE), and we have $\bar{u}(\bar{x},0)=0$, $\bar{u}(\bar{x},\infty)=1$, and $\bar{u}(0,\bar{t})=1$.

The choice of t_c is as in the previous cases. We arrive at the dimensionless initial-boundary value problem

$$\frac{\partial \bar{u}}{\partial \bar{t}}=\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}, \qquad \bar{x}\in(0,1),\ \bar{t}\in(0,\bar{T}], \tag{3.44}$$

$$\bar{u}(\bar{x},0)=0, \qquad \bar{x}\in[0,1], \tag{3.45}$$

$$\bar{u}(0,\bar{t})=1, \qquad \bar{t}\in(0,\bar{T}], \tag{3.46}$$

$$\frac{\partial}{\partial \bar{x}}u(1,\bar{t})=0, \qquad \bar{t}\in(0,\bar{T}]. \tag{3.47}$$

The striking feature is that there are no physical parameters left in this problem. One simulation can be carried out for $\bar{u}(\bar{x},\bar{t})$, and the temperature in a rod of any material and any constant initial and boundary temperature can be retrieved by

$$u(x,t)=U_0+(U_1-U_0)\bar{u}(x/L,t\alpha/L^2)\,.$$

hpl 14: Include simulation results.

3.2.4 Oscillating Dirichlet condition

hpl 15: Do we need paragraph headings here?

Now we address a heat equation problem where the temperature is oscillating on the boundary $x = 0$:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \qquad x \in (0, L),\ t \in (0, T], \tag{3.48}$$

$$u(x, 0) = U_0, \qquad x \in [0, L], \tag{3.49}$$

$$u(0, t) = U_0 + A\sin(\omega t), \qquad t \in (0, T], \tag{3.50}$$

$$\frac{\partial}{\partial x} u(L, t) = 0, \qquad t \in (0, T]. \tag{3.51}$$

One important physical application is temperature oscillations in the ground, either day and night variations at a short temporal and spatial scale, or seasonal variations in the Earth's crust. An important modeling assumption is (3.51), which means that the boundary $x = L$ is placed sufficiently far from $x = 0$ such that the solution is much damped and basically constant so $u_x = 0$ is a reasonable condition.

Scaling. Since the boundary temperature is oscillating around the initial condition, we expect $u \in [U_0 - A, U_0 + A]$. The dimensionless temperature is therefore taken as

$$\bar{u} = \frac{u - U_0}{2A},$$

such that $\bar{u} \in [-1, 1]$.

What is an appropriate time scale? There will be two time scales involved, the oscillations $\sin(\omega t)$ with period $P = 2\pi/\omega$ at the boundary and the "speed of diffusion", or more specifically the "speed of heat conduction" in the present context, where $t_c = L^2/\alpha$ is the appropriate scale.

As usual, investigating the exact solution of the model problem can illuminate the involved scales. For this particular initial-boundary value problem the exact solution as $t \to \infty$ (such that the initial condition $u(x, 0) = U_0$ is forgotten) can be shown to be

$$u(x, t) = U_0 - Ae^{-bx}\sin(bx - \omega t), \quad b = \sqrt{\frac{\omega}{2\alpha}}. \tag{3.52}$$

This solution is of the form $e^{-bx}g(x-ct)$, i.e., a damped wave that moves to the right with velocity c and a damped amplitude e^{-bx}. This is perhaps more easily seen if we make a rewrite

$$u(x,t) = U_0 - Ae^{-bx}\sin\left(b(x-ct)\right), \quad c = \omega/b = \sqrt{2\alpha\omega},\ b = \sqrt{\frac{\omega}{2\alpha}}.$$

The boundary oscillations lead to the time scale $t_c = 1/\omega$. The speed of the wave suggests another time scale: the time it takes to propagate through the domain, which is L/c, and hence $t_c = L/c = L/\sqrt{2\alpha\omega}$.

One may argue that L is not the appropriate length scale, because u is damped by e^{-bx} so for $x > 4/b$, u is close to zero. We may instead use $1/b$ as length scale, which is the e-folding distance of the damping factor, and base t_c on the time it takes a signal to propagate one length scale, $t_c^{-1} = bc = \omega$. Similarly, the time scale based on the "speed of diffusion" changes to $t_c^{-1} = b^2\alpha = \frac{1}{2}\omega$ if we employ $1/b$ as length scale.

To summarize, we have three candidates for the time scale: $t_c = L^2/\alpha$, $t_c = 2/\omega$, and $t_c = 1/\omega$.

Let us look at the dimensionless exact solution to see if it can help with the choice of scales. We introduce the dimensionless parameters

$$\beta = bx_c = x_c\sqrt{\frac{\omega}{2\alpha}}, \quad \gamma = \omega t_c\,.$$

The scaled solution becomes

$$\bar{u}(\bar{x},\bar{t};\beta,\gamma) = e^{-\beta\bar{x}}\sin(\beta\bar{x} - \gamma\bar{t})\,.$$

The three choices of γ, implied by the three choices of t_c, are

$$\gamma = \begin{cases} 1, & t_c = 1/\omega, \\ 2, & t_c = 2/\omega, \\ 2\beta^2, & t_c = L^2/\alpha,\ x_c = L \end{cases} \tag{3.53}$$

The former two choices leaves only β as parameter in $\bar{u}$, and with $x_c = 1/b$ as length scale, β becomes unity, and there are no parameters in the dimensionless solution:

$$\bar{u}(\bar{x},\bar{t}) = e^{-\bar{x}}\sin(\bar{x} - \bar{t})\,. \tag{3.54}$$

Therefore, $x_c = 1/b$ and $t_c = 1/\omega$ (or $t_c = 2/\omega$, but the factor 2 is of no importance) are the most appropriate scales.

To further argue why (3.54) demonstrates that these scales are preferred, think of ω as large. Then the wave is damped over a short distance and there will be a thin boundary layer of temperature oscillations near $x = 0$ and little changes in u in the rest of the domain. The scaling (3.54) resolves this problem by using $1/b \sim \omega^{-1/2}$ as length scale, because then the boundary

layer thickness is independent of ω. The length of the domain can be chosen as, e.g., $4/b$ such that $\bar{u} \approx 0$ at the end $x = L$. The length scale $1/b$ helps us to zoom in on the part of u where significant changes take place.

In the other limit, ω small, b becomes small, and the wave is hardly damped in the domain $[0, L]$ unless L is large enough. The imposed boundary condition on $x = L$ in fact requires u to be approximately constant so its derivative vanishes, and this property can only be obtained if L is large enough to ensure that the wave becomes significantly damped. Therefore, the length scale is dictated by b, not L, and L should be adapted to b, typically $L \geq 4/b$.

The resulting scaled problem becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad \bar{x} \in (0, 4/b),\ \bar{t} \in (0, \bar{T}], \tag{3.55}$$

$$\bar{u}(\bar{x}, 0) = 0, \qquad \bar{x} \in [0, 1] \tag{3.56}$$

$$\bar{u}(0, \bar{t}) = \sin(\bar{t}), \qquad \bar{t} \in (0, \bar{T}], \tag{3.57}$$

$$\frac{\partial}{\partial \bar{x}}\bar{u}(1, \bar{t}) = 0, \qquad \bar{t} \in (0, \bar{T}]. \tag{3.58}$$

The coefficient in front of the second-derivative is $\frac{1}{2}$ because

$$\frac{t_c \alpha}{1/b^2} = \frac{b^2 \alpha}{\omega} = \frac{1}{2}.$$

We may, of course, choose $t_c = 2/\omega$ and get rid of the $\frac{1}{2}$ factor, if desired, but then it turns up in (3.57) instead, as $\sin(2\bar{t})$.

3.3 Reaction-diffusion equations

3.3.1 Fisher's equation

Fisher's equation is essentially the logistic equation at each point for population dynamics (see Section 2.1.9) combined with spatial movement through ordinary diffusion:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + \varrho u(1 - u/M). \tag{3.59}$$

This PDE is also known as the KPP equation after Kolmogorov, Petrovsky, and Piskynov (who introduced the equation independently of Fisher).

Setting

$$\bar{x} = \frac{x}{x_c}, \quad \bar{t} = \frac{t}{t_c}, \quad \bar{u} = \frac{u}{u_c},$$

results in

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{t_c \alpha}{x_c^2}\frac{\partial^2 u}{\partial x^2} + t_c \varrho \bar{u}(1 - u_c \bar{u}/M)\,.$$

Balance of all terms. If all terms are equally important, the scales can be determined from demanding the coefficients to be unity. Reasoning as for the logistic ODE in Section 2.1.9, we may choose $t_c = 1/\varrho$. Then the coefficient in the diffusion term dictates the length scale $x_c = \sqrt{t_c \alpha}$. A natural scale for u is M, since M is the upper limit of u in the model (cf. the logistic term). Summarizing,

$$u_c = M, \quad t_c = \frac{1}{\varrho}, \quad x_c = \sqrt{\frac{\alpha}{\varrho}},$$

and the scaled PDE becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial x^2} + \bar{u}(1 - \bar{u})\,. \tag{3.60}$$

With this scaling, the length scale $x_c = \sqrt{\alpha/\varrho}$ is not related to the domain size, so the scale is particularly relevant for infinite domains.

An open question is whether the time scale should be based on the diffusion process rather than the initial exponential growth in the logistic term. The diffusion time scale means $t_c = x_c^2/\alpha$, but demanding the logistic term then to have a unit coefficient forces $x_c^2 \varrho/\alpha = 1$, which implies $x_c = \sqrt{\alpha/\varrho}$ and $t_c = 1/\varrho$. That is, equal balance of the three terms gives a unique choice of the time and length scale.

Fixed length scale. Assume now that we fix the length scale to be L, either the domain size or some other naturally given length. With $x_c = L$, $t_c = \varrho^{-1}$, $u_c = M$, we get

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \beta \frac{\partial^2 \bar{u}}{\partial x^2} + \bar{u}(1 - \bar{u}), \tag{3.61}$$

where β is a dimensionless number

$$\beta = \frac{\alpha}{\varrho L^2} = \frac{\varrho^{-1}}{L^2/\alpha}\,.$$

The last equality demonstrates that β measures the ratio of the time scale for exponential growth in the beginning of the logistic process and the time scale of diffusion L^2/α (i.e., the time it takes to transport a signal by diffusion through the domain). For small β we can neglect the diffusion and spatial movements, and the PDE is essentially a logistic ODE at each point, while for large β, diffusion dominates, and t_c should in that case be based on the diffusion time scale L^2/α. This leads to the scaled PDE

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial x^2} + \beta^{-1}\bar{u}(1-\bar{u}), \tag{3.62}$$

showing that a large β encourages omission of the logistic term, because the point-wise growth takes place over long time intervals while diffusion is rapid. The effect of diffusion is then more prominent and it suffices to solve $\bar{u}_{\bar{t}} = \bar{u}_{\bar{x}\bar{x}}$.

3.3.2 Nonlinear reaction-diffusion PDE

A general, nonlinear reaction-diffusion equation in 1D looks like

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + f(u)\,. \tag{3.63}$$

By scaling the nonlinear reaction term $f(u)$ as $f_c\bar{f}(u_c\bar{u})$, where f_c is a characteristic size of $f(u)$, typically the maximum value, one gets a non-dimensional PDE like

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{t_c\alpha}{x_c^2}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \frac{t_c f_c}{u_c}\bar{f}(u_c\bar{u})\,.$$

The characteristic size of u can often be derived so we first assume that u_c is given. This fact uniquely determines the space and time scales by demanding that all three terms are equally important and of unit size:

$$t_c = \frac{u_c}{f_c}, \quad x_c = \sqrt{\frac{\alpha u_c}{f_c}}\,.$$

The corresponding PDE reads

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \bar{f}(u_c\bar{u})\,. \tag{3.64}$$

If x_c is based on some known length scale L, balance of all three terms can be used to determine u_c and t_c:

$$t_c = \frac{L^2}{\alpha}, \quad u_c = \frac{L^2 f_c}{\alpha}\,.$$

This scaling only works if f is nonlinear, otherwise u_c cancels and there is no freedom to constrain this scale.

With given L and u_c, there are two choices of t_c since it can be based on the diffusion or the reaction time scales. With the reaction scale, $t_c = u_c/f_c$, one arrives a the PDE

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \beta\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \bar{f}(u_c\bar{u}), \tag{3.65}$$

where

$$\beta = \frac{\alpha u_c}{L^2 f_c} = \frac{u_c/f_c}{L^2/\alpha}$$

is a dimensionless number reflecting the ratio of the reaction time scale and the diffusion time scale. On the contrary, with the diffusion time scale, $t_c = L^2/\alpha$, the scaled PDE becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \beta^{-1} \bar{f}(u_c \bar{u}) \,. \tag{3.66}$$

The size of β in an application will determine which of the scalings that is most appropriate.

3.4 The convection-diffusion equation

3.4.1 Convection-diffusion without a force term

We now add a convection term $\boldsymbol{v} \cdot \nabla u$ to the diffusion equation to obtain the well-known convection-diffusion equation:

$$\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u = \alpha \nabla^2 u, \quad x, y, z \in \Omega,\ t \in (0, T] \,. \tag{3.67}$$

The velocity field $\boldsymbol{v}$ is prescribed, and its characteristic size V is normally clear from the problem description.

hpl 16: Drop specification of spatial and temporal domain? Is this basically notational noise when the PDE gets more complicated? Geir says no.

hpl 17: Show a sketch of a couple of cases, can be the same as Navier-Stokes, where different V is obvious.

Inserting

$$\bar{x} = \frac{x}{x_c},\ \bar{y} = \frac{y}{y_c},\ \bar{z} = \frac{z}{z_c},\ \bar{t} = \frac{t}{t_c},\ \bar{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V},\ \bar{u} = \frac{u}{u_c}$$

in (3.67) yields

$$\frac{u_c}{t_c}\frac{\partial \bar{u}}{\partial \bar{t}} + \frac{u_c V}{L} \bar{\boldsymbol{v}} \cdot \bar{\nabla} \bar{u} = \frac{\alpha u_c}{L^2} \bar{\nabla}^2 \bar{u}, \quad \bar{x}, \bar{y}, \bar{z} \in \Omega,\ \bar{t} \in (0, \bar{T}] \,.$$

For u_c we simply introduce the symbol U, which we may estimate from an initial condition. It is not critical here since it vanishes from the scaled equation anyway as long as there is no source term present. With some velocity measure V and length measure L, it is tempting to just let $t_c = L/V$. This is the characteristic time it takes to transport a signal by convection through the domain. The alternative is to use the diffusion length scale $t_c = L^2/\alpha$.

Very often in these kind of problems, the convection term $\boldsymbol{v}\cdot\nabla u$ dominates over the diffusion term $\alpha\nabla^2 u$, so the time scale for convection, which is L/V, is most appropriate of the two. Only when the diffusion term is very much larger than the convection term (corresponding to very small Peclet numbers, see below) we would apply $t_c = L^2/\alpha$.

The non-dimensional form of the PDE with $t_c = L/V$ becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \mathrm{Pe}^{-1}\bar{\nabla}^2\bar{u}, \quad \bar{x},\bar{y},\bar{z}\in\Omega,\ \bar{t}\in(0,\bar{T}], \tag{3.68}$$

where Pe is the *Peclet number*,

$$\mathrm{Pe} = \frac{LV}{\alpha}\,.$$

Estimating the size of the convection term $\boldsymbol{v}\cdot\nabla u$ as VU/L and the diffusion term $\alpha\nabla^2 u$ as $\alpha U/L^2$, we see that the Peclet number measures the ratio of the convection and the diffusion terms:

$$\mathrm{Pe} = \frac{\text{convection}}{\text{diffusion}} = \frac{VU/L}{\alpha U/L^2} = \frac{LV}{\alpha}\,.$$

In case we use the diffusion time scale $t_c = L^2/\alpha$, we get the non-dimensional PDE

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \mathrm{Pe}\,\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \bar{\nabla}^2\bar{u}, \quad \bar{x},\bar{y},\bar{z}\in\Omega,\ \bar{t}\in(0,\bar{T}]\,. \tag{3.69}$$

Discussion of scales and balance of terms in the PDE

We see that (3.68) and (3.69) are not equivalent, and they are based on two different time scales. For moderate Peclet numbers around 1, all terms have the same size in (3.68), i.e., a size around unity. For large Peclet numbers, (3.68) expresses a balance between the time derivative term and the convection term, both of size unity, and then there is a very small term $\mathrm{Pe}^{-1}\bar{\nabla}^2\bar{u}$ term because Pe is large and $\bar{\nabla}^2\bar{u}$ should be of size unity. That the convection term dominates over the diffusion term is consistent with the time scale $t_c = L/V$ based on convection transport. In this case, we can neglect the diffusion term as Pe goes to infinity and work with a pure convection (or advection) equation

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = 0\,.$$

For small Peclet numbers, $\mathrm{Pe}^{-1}\bar{\nabla}^2\bar{u}$ becomes very large and can only be balanced by two terms that are supposed to be unity of size. The time-derivative and/or the convection term must be much larger than

unity, but that means we use suboptimal scales, since right scales imply that $\partial\bar{u}/\partial\bar{t}$ and $\bar{v}\cdot\bar{\nabla}\bar{u}$ are of order unity. Switching to a time scale based on diffusion as the dominating physical effect gives (3.69). For very small Peclet numbers this equation tells that the time-derivative balances the diffusion, while the convection $\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{\boldsymbol{u}}$ is around unity, but multiplied by a very small coefficient Pe, so this term is negligible in the PDE. An approximate PDE for small Peclet numbers is therefore

$$\frac{\partial\bar{u}}{\partial\bar{t}} = \bar{\nabla}^2\bar{u}\,.$$

Scaling can with the above type of reasoning be used to neglect terms from a differential equation under precise mathematical conditions.

3.4.2 Stationary PDE

Suppose the problem is stationary and that there is no need for any time scale. How is this type of convection-diffusion problem scaled? We get

$$\frac{VU}{L}\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \frac{\alpha U}{L^2}\bar{\nabla}^2\bar{u},$$

or

$$\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \mathrm{Pe}^{-1}\bar{\nabla}^2\bar{u}\,. \tag{3.70}$$

This scaling only "works" for moderate Peclet numbers. For very small or very large Pe, either the convection term $\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u}$ or the diffusion term $\bar{\nabla}^2\bar{u}$ must deviate significantly from unity.

Consider the following 1D example to illustrate the point: $\boldsymbol{v} = v\boldsymbol{i}$, $v > 0$ constant, a domain $[0,L]$, with boundary conditions $u(0) = 0$ and $u(L) = U_L$. (The vector $\boldsymbol{i}$ is a unit vector in x direction.) The problem with dimensions is now

$$vu' = \alpha u'', \quad u(0) = 0,\ u(L) = U_L\,.$$

Scaling results in

$$\frac{d\bar{u}}{d\bar{x}} = \mathrm{Pe}^{-1}\frac{d^2\bar{u}}{d\bar{x}^2}, \quad \bar{x}\in(0,1), \quad \bar{u}(0) = 0,\ \bar{u}(1) = 1,$$

if we choose $U = U_L$. The solution of the scaled problem is

$$\bar{u}(\bar{x}) = \frac{1-e^{\bar{x}\mathrm{Pe}}}{1-e^{\mathrm{Pe}}}\,.$$

Figure 3.3 indicates how $\bar{u}$ depends on Pe: small Pe values give approximately a straight line while large Pe values lead to a *boundary layer* close to $x = 1$, where the solution changes very rapidly.

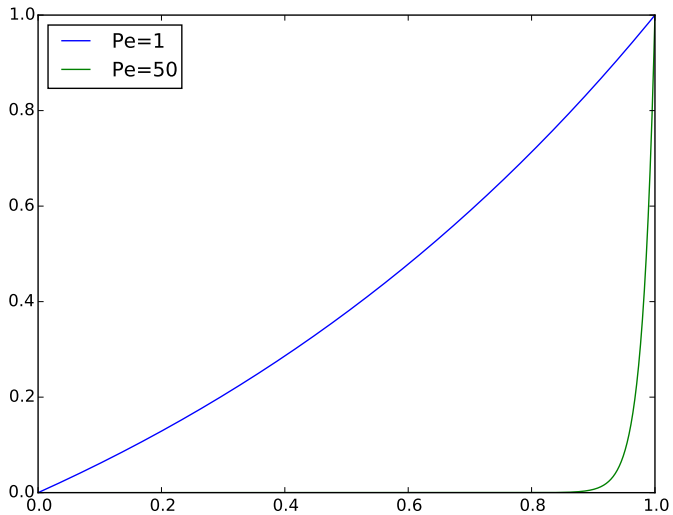


Fig. 3.3 Solution of scaled problem for 1D convection-diffusion.

We realize that for large Pe,

$$\max_{\bar{x}} \frac{d\bar{u}}{d\bar{x}} \approx \mathrm{Pe}, \quad \max_{\bar{x}} \frac{d^2\bar{u}}{d\bar{x}^2} \approx \mathrm{Pe}^2,$$

which are consistent results with the PDE since the double derivative term is multiplied by Pe^{-1}. For small Pe,

$$\max_{\bar{x}} \frac{d\bar{u}}{d\bar{x}} \approx 1, \quad \max_{\bar{x}} \frac{d^2\bar{u}}{d\bar{x}^2} \approx 0,$$

which is also consistent with the PDE since an almost vanishing second-order derivative is multiplied by a very large coefficient Pe^{-1}. However, we have a problem with very large derivatives of $\bar{u}$ when Pe is large.

To arrive at a proper scaling for large Peclet numbers, we need to remove the Pe coefficient from the differential equation. There are only two scales at our disposals: u_c and x_c for u and x, respectively. The natural value for u_c is the boundary value U_L at $x = L$. The scaling of $Vu_x = \alpha u_{xx}$ then results in

$$\frac{d\bar{u}}{d\bar{x}} = \frac{\alpha}{Vx_c}\frac{d^2\bar{u}}{d\bar{x}^2}, \quad \bar{x} \in (0, \bar{L}), \quad \bar{u}(0) = 0,\ \bar{u}(\bar{L}) = 1,$$

where $\bar{L} = L/x_c$. Choosing the coefficient $\alpha/(Vx_c)$ to be unity results in the scale $x_c = \alpha/V$, and $\bar{L}$ becomes Pe. The final, scaled boundary-value problem is now

$$\frac{d\bar{u}}{d\bar{x}} = \frac{d^2\bar{u}}{d\bar{x}^2}, \quad \bar{x} \in (0, \mathrm{Pe}), \quad \bar{u}(0) = 0,\ \bar{u}(\mathrm{Pe}) = 1,$$

with solution

$$\bar{u}(\bar{x}) = \frac{1 - e^{\bar{x}}}{1 - e^{\mathrm{Pe}}}.$$

Figure 3.4 displays $\bar{u}$ for some Peclet numbers, and we see that the shape of the graphs are the same with this scaling. For large Peclet numbers we realize that $\bar{u}$ and its derivatives are around unity ($1 - e^{\mathrm{Pe}} \approx -e^{\mathrm{Pe}}$), but for small Peclet numbers $d\bar{u}/d\bar{x} \sim \mathrm{Pe}^{-1}$.

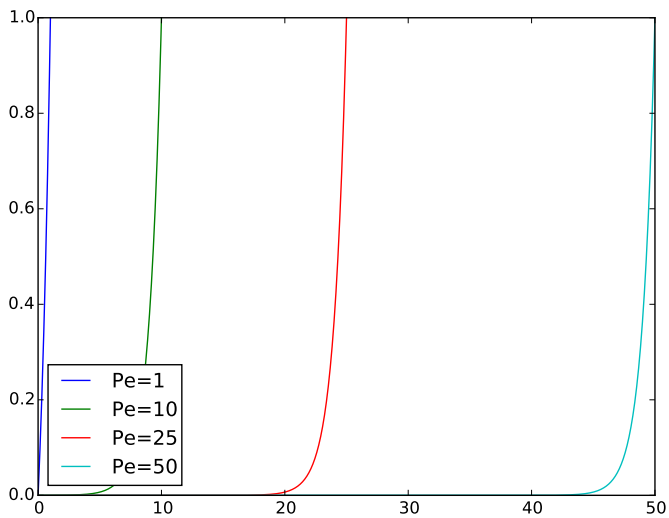


Fig. 3.4 Solution of scaled problem where the length scale depends on the Peclet number.

The conclusion is that for small Peclet numbers, $x_c = L$ is an appropriate length scale. The scaled equation $\mathrm{Pe}\,\bar{u}' = \bar{u}''$ indicates that $\bar{u}'' \approx 0$, and the solution is close to a straight line. For large Pe values, $x_c = \alpha/V$ is an appropriate length scale, and the scaled equation $\bar{u}' = \bar{u}''$ expresses that the terms $\bar{u}'$ and $\bar{u}''$ are equal and of size around unity.

3.4.3 Convection-diffusion with a force term

Let us add a force term $f(\boldsymbol{x},t)$ to the convection-diffusion equation :

$$\frac{\partial u}{\partial t} + \boldsymbol{v}\cdot\nabla u = \alpha\nabla^2 u + f\,. \tag{3.71}$$

The scaled version reads

$$\frac{\partial u}{\partial t} + \frac{t_c V}{L}\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \frac{t_c\alpha}{L^2}\bar{\nabla}^2\bar{u} + \frac{t_c f_c}{u_c}\bar{f}\,.$$

We can base t_c on convective transport: $t_c = L/V$. Now, u_c could be chosen to make the coefficient in the source term unity: $u_c = t_c f_c = L f_c/V$. This leaves us with

$$\frac{\partial u}{\partial t} + \bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \mathrm{Pe}^{-1}\bar{\nabla}^2\bar{u} + \bar{f}\,.$$

In the diffusion limit, we base t_c on the diffusion time scale: $t_c = L^2/\alpha$, and the coefficient of the source term set to unity determines u_c according to

$$\frac{L^2 f_c}{\alpha u_c} = 1 \quad\Rightarrow\quad u_c = \frac{L^2 f_c}{\alpha}\,.$$

The corresponding PDE reads

$$\frac{\partial u}{\partial t} + \mathrm{Pe}\,\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \bar{\nabla}^2\bar{u} + \bar{f},$$

so for small Peclet numbers, which we have, the convective term can be neglected and we get a pure diffusion equation with a source term.

What if the problem is stationary? Then there is no time scale and we get

$$\frac{V u_c}{L}\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \frac{u_c\alpha}{L^2}\bar{\nabla}^2\bar{u} + f_c\bar{f},$$

or

$$\bar{\boldsymbol{v}}\cdot\bar{\nabla}\bar{u} = \mathrm{Pe}^{-1}\bar{\nabla}^2\bar{u} + \frac{f_c L}{V u_c}\bar{f},$$

Again, choosing u_c such that the source term coefficient is unity leads to $u_c = f_c L/V$. Alternatively, u_c can be based on the initial condition, with similar results as found in the sections on the wave and diffusion PDEs.

3.5 Exercises

Problem 3.1: Stationary Couette flow

A fluid flows between two flat plates, with one plate at rest while the other moves with velocity U_0. This classical flow case is known as stationary Couette flow.

a) Directing the x axis in the flow direction and letting y be a coordinate perpendicular to the walls, one can assume that the velocity field simplifies to $\boldsymbol{u} = u(y)\boldsymbol{i}$. Show from the Navier-Stokes equations that the boundary-value problem for $u(y)$ is

$$u''(u) = 0, \quad u(0) = 0,\ u(H) = U_0\,.$$

We have here assumed at $y = 0$ corresponds to the plate at rest and that $y = H$ represents the plate that moves. There are no pressure gradients present in the flow.

b) Scale the problem in a) and show that the result has no physical parameters left in the model:

$$\frac{d^2\bar{u}}{d\bar{y}^2} = 0, \quad \bar{u}(0) = 0,\ \bar{u}(1) = 1\,.$$

c) We can compute $\bar{u}(\bar{y})$ from one numerical simulation (or a straightforward integration of the differential equation). Set up the formula that finds $u(y; H, u_0)$ from $\bar{u}(\bar{y})$ for any values of H and U_0.
Filename: `stationary_Couette`.

Remarks. The problem for u is a classical two-point boundary-value problem in applied mathematics and arises in a number of applications, where Couette flow is just one example. Heat conduction is another example: u is temperature, and the heat conduction equation for an insulated rod reduces to $u'' = 0$ under stationary conditions and no heat source. Controlling the end $x = 0$ at 0 degrees Celsius the other end $x = L$ at U_0 degrees Celsius, gives the same boundary conditions as in the above flow problem. The scaled problem is of course the same whether we have flow of fluid or heat.

Problem 3.2: Starting Couette flow

A fluid is confined in a channel with two planar walls $z = 0$ and $z = H$. The fluid is at rest. At time $t = 0$ the upper wall is suddenly set in motion with a velocity $U\boldsymbol{i}$. We assume that the velocity is directed along the x axis: $\boldsymbol{u} = u(x, z, t)\boldsymbol{i}$. From the equation of continuity, $\nabla \cdot \boldsymbol{u} = 0$, we get that

$\partial u/\partial x = 0$ such that $\boldsymbol{u} = u(z,t)\boldsymbol{i}$. The boundary conditions are $\boldsymbol{u} = 0$ at the lower wall $z = 0$ and $\boldsymbol{u} = U\boldsymbol{i}$ at the upper wall $z = H$. Assume that the pressure is constant everywhere and that there are no body forces.

a) Start with the incompressible Navier-Stokes equations and the assumption $\boldsymbol{u} = u(z,t)\boldsymbol{i}$. Derive an initial-boundary value problem for $u(z,t)$. Scale the problem.

Solution. Inserting the simplified velocity in the original Navier-Stokes equations makes the convection term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ vanish and ∇p vanishes since p is assumed constant (only the upper wall drives the flow). The result becomes

$$\varrho\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial z},$$

or using $\mu/\varrho = \nu$,

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial z},$$

with $u(z,0) = 0,\ u(0,t) = 0$ and $u(H,t) = U$. This is a standard diffusion problem. The natural length scale is H, so $\bar{z} = z/H$. Using the well-established time scale $t_c = H^2/\nu$ and the velocity scale $u_c = U$, we get the dimensionless problem

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{z}},$$

with $\bar{u}(\bar{z},0) = 0,\ u(0,\bar{t}) = 0,\ u(1,t) = 1$. There are no physical parameters. Having computed $\bar{u}(\bar{z},\bar{t})$, the physical solution can be retrieved as

$$u(z,t) = U\bar{u}(\bar{z}H,\bar{t}H^2/\nu)\,.$$

b) Start with the dimensionless Navier-Stokes equations and use the assumption $\bar{\boldsymbol{u}} = \bar{u}(\bar{z},\bar{t})\boldsymbol{i}$ to reduce the problem. The resulting equation now contains a Reynolds number, i.e., one more physical parameter than in a). Why is this an inferior approach to scaling the problem?

Solution. Inserting the simplified velocity in the scaled Navier-Stokes equations leads to

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{1}{\mathrm{Re}}\frac{\partial^2 \bar{u}}{\partial \bar{z}}\,.$$

Here we have the Reynolds number as parameter.

The major difference is the scaling used in time: $t_c = H/U$ (length scale is H here) versus $t_c = H^2/\nu$. The latter is much more suitable as it is based on a diffusion problem and the present problem is indeed a diffusion problem. Normally, ν is very small, so t_c based on diffusion is usually much larger than H/U. With an inappropriate time scale, $\partial\bar{u}/\partial\bar{t}$ is not of unit size, and we need a dimensionless number on the right-hand side to adjust the spatial

derivative term to a non-unity size. The wrong scaling thereby introduces an extra (unnecessary) parameter.

c) Can you construct a heat conduction problem that has the same solution $\bar{u}(\bar{z},\bar{t})$ as in a)?

Solution. Consider a long rod with length H aligned with the z axis. The rod is isolated on the curved circular surface and kept at fixed temperatures U_0 and U_H at the ends $z=0$ and $z=H$, respectively. The initial temperature is U_0. Because of the insulated curved surface, heat can only propagate in the z direction, and a one-dimensional heat conduction equation is appropriate:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial z},$$

with $u(z,0)=U_0$, $u(0,t)=U_0$, and $u(H,t)=U_H$. We introduce a dimensionless temperature

$$\bar{u} = \frac{u-U_0}{U_H-U_0},$$

such that $\bar{u}\in[0,1]$. The standard time scale $t_c = H^2/\alpha$ is used, notifying that the length scale is H. Inserting the dimensionless variables in the governing equation results in the same problem as in a). It means that we from one solution $\bar{u}(\bar{z},\bar{t})$ can get solutions for heat conduction in rods of all lengths and materials, and with all boundary temperatures, as well as flow of any fluid between two walls with any gap and any velocity of the upper wall.

d) Describe how the scaled problem in this exercise can be solved by a program that solves the following diffusion problem with dimensions:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha \frac{\partial^2 u}{\partial z} + f(x,t),\\
u(x,0) &= I(x),\\
u(0,t) &= U_0(t),\\
u(L,t) &= U_L(t).
\end{aligned}$$

Solution. Let z be named x. Set $\alpha=1$, $f=0$, $L=1$, $I(x)=0$, $U_0(t)=0$, $U_L(t)=1$. The resulting problem is our scaled problem from a).
Filename: `starting_Couette`.

Exercise 3.3: Couette flow with pressure gradient

Viscous fluid flow between two infinite flat plates $z=0$ and $z=H$ is governed by

$$\mu u''(z) = -\beta \tag{3.72}$$
$$u(0) = 0, \tag{3.73}$$
$$u(H) = U_0\,. \tag{3.74}$$

Here, $u(z)$ is the fluid velocity in x direction (perpendicular to the z axis), μ is the dynamic viscosity of the fluid, β is a positive constant pressure gradient, and U_0 is the constant velocity of the upper plate $z = H$ in x direction.

a) Find the exact solution $u(z)$. Point out how β and U_0 influence the magnitude of u.

Solution. SymPy can integrate the differential equation twice and fit the integration constants to the boundary conditions:

```
import sympy as sym
mu, beta, z, H = sym.symbols('mu beta z H',
                             real=True, positive=True)
U0, C1, C2 = sym.symbols('U0 C1 C2', real=True)

# Integrate u''(z) = -beta/mu twice and add integration constants
u = sym.integrate(sym.integrate(-beta/mu, z) + C1, z) + C2

# Use the boundary conditions
eq = [sym.Eq(u.subs(z, 0), 0),
      sym.Eq(u.subs(z, H), U0)]
s = sym.solve(eq, [C1, C2])
print s
u = u.subs(C1, s[C1]).subs(C2, s[C2])
u = sym.simplify(sym.expand(u))
```

The result becomes

$$u(z) = \frac{z}{2H\mu}\left(H\beta\left(H - z\right) + 2U_0\mu\right).$$

The maximum value of u is found by

```
# Find max u
dudz = sym.diff(u, z)
s = sym.solve(dudz, z)
print s
umax = u.subs(z, s[0])
umax = sym.simplify(sym.expand(umax))
```

and reads

$$\max_z u = \frac{H^2\beta}{8\mu} + \frac{U_0}{2} + \frac{U_0^2\mu}{2H^2\beta}\,.$$

If the pressure gradient is the domining driving force, we can neglect the U_0 terms: $\max_z u = H^2\beta/(8\mu)$. In case the movement of the upper plate is much more important than the pressure gradient for driving the flow, we can

neglect the β terms. However, we must then resort to the $u(z)$ expression for $\beta = 0$, $u(z) = zU_0/H$, and realize that the maximum then is obtained at the boundary for $z = H$: $\max_z h = U_0$ (as intuitively obvious).

b) Scale the problem.

Solution. Introducing

$$\bar{z} = \frac{z}{z_c}, \quad \bar{u}(\bar{z}) = \frac{u(z_c\bar{z})}{u_c},$$

in the equation gives

$$\frac{d^2\bar{u}}{d\bar{z}^2} = -\frac{z_c^2\beta}{\mu u_c}.$$

The natural scale for z_c is H since that makes $\bar{z} \in [0,1]$. For the two terms in the differential equation to be of order unity (with a correct scaling, the left-hand side should be of order unity), we must have

$$u_c = \frac{H^2\beta}{\mu}.$$

The boundary value problem is

$$\frac{d^2\bar{u}}{d\bar{z}^2} = -1, \qquad \bar{z} \in (0,1), \tag{3.75}$$

$$\bar{u}(0) = 0, \tag{3.76}$$

$$\bar{u}(1) = \alpha, \tag{3.77}$$

where α is a dimensionless number

$$\alpha = \frac{\mu U_0}{H^2\beta}.$$

This is meaningful only for $\beta \neq 0$.

Looking at the exact solution, we see that $\max_z u = H^2\beta/(8\mu)$, and with this $\max_z u$ as u_c we get a differential equation $\bar{u}'' = -8$ instead, and $\bar{u} \in [0,1]$ (if $U_0 = 0$). However, the factor 1 or 8 on the right-hand side is not significant, neither if $\bar{u} \in [0,1]$ or $\bar{u} \in [0,8]$.

The scale u_c used above is relevant if the pressure gradient is the dominating force. If U_0 is more important than β, or $\beta = 0$, we choose $u_c = U_0$ and get instead

$$\frac{d^2\bar{u}}{d\bar{z}^2} = -\alpha^{-1}, \qquad \bar{z} \in (0,1), \tag{3.78}$$

$$\bar{u}(0) = 0, \tag{3.79}$$

$$\bar{u}(1) = 1\,. \tag{3.80}$$

hpl 18: Could extend to time-dependent case, but this will involve three time scales...
Filename: `Couette_wpressure`.

Exercise 3.4: Pulsatile pipeflow

The flow of a viscous fluid in a straight pipe with circular cross section with radius R is governed by

$$\varrho\frac{\partial u}{\partial t} = \frac{\mu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - P(t), \qquad r \in (0,R),\ t \in (0,T], \tag{3.81}$$

$$\frac{\partial u}{\partial r}(0,t) = 0, \qquad t \in (0,T], \tag{3.82}$$

$$u(R,t) = 0, \qquad t \in (0,T], \tag{3.83}$$

$$u(r,0) = 0, \qquad r \in [0,R]. \tag{3.84}$$

The quantity $u(r,t)$ is the fluid velocity, $P(t)$ is a given pressure gradient, ϱ is the fluid density, and μ is the dynamic viscosity.

Assume $P(t) = A\cos\omega t$. Scale the problem and identify appropriate dimensionless numbers. The assume $P(t)$ is a more complicated function, but still period with period p. Discuss how the scaling can be extended to this case.

Solution. We introduce dimensionless quantities:

$$\bar{r} = \frac{r}{R}, \quad \bar{t} = \frac{t}{t_c}, \quad \bar{u} = \frac{u}{u_c}\,.$$

The function $P(t)$ can be scaled as

$$\bar{P}(\bar{t}) = \frac{P(t_c\bar{t})}{A} = \sin(t_c\omega\bar{t})\,.$$

Inserted in the PDE, we get

$$\frac{\partial\bar{u}}{\partial\bar{t}} = \frac{t_c\nu}{R^2}\frac{1}{\bar{r}}\frac{\partial}{\partial\bar{r}}\left(\bar{r}\frac{\partial\bar{u}}{\partial\bar{r}}\right) - \frac{t_cA}{\varrho u_c}\sin(t_c\omega\bar{t}),$$

where $\mu = \mu/\varrho$.

The scale for u can be explored by seeking an analytical solution of the problem. Such solutions do exist, they are typically series expansions of Bessel functions, and it is not so easy to extract a simple expression for the maximum value of $|u(r,t)|$. A simpler approach is to estimate u_c be demanding the coefficient in the pressure term to be of unit size:

$$u_c = \frac{t_c A}{\varrho}\,.$$

There are two choices of time scales: the pressure time scale $t_c = 1/\omega$ and the viscosity (or diffusion) time scale $t_c = R^2/nu$. With the latter we get $u_c = R^2 A/\mu$ and

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{u}}{\partial \bar{r}}\right) - \sin(\alpha \bar{t}),$$

where

$$\alpha = \frac{R^2\omega}{\nu} = \frac{R^2/\nu}{1/\omega},$$

showing that α is the ratio of the viscosity time scale and the pressure oscillation time scale.

With the pressure time scale we have

$$u_c = \frac{\varrho A}{\omega},$$

and the scaled PDE becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \alpha^{-1}\frac{1}{\bar{r}}\frac{\partial}{\partial \bar{r}}\left(\bar{r}\frac{\partial \bar{u}}{\partial \bar{r}}\right) - \sin(\bar{t})\,.$$

In both cases the scaled boundary conditions become

$$\frac{\partial \bar{u}}{\partial \bar{t}} = 0, \quad \bar{u}(1,\bar{t}) = 0,$$

for $t \in (0, T/t_c]$, and $\bar{u}(\bar{r},0) = 0$ for $\bar{r} \in [0,1]$.

If $P(t)$ is not sinusoidal but periodic with period p, we have that P is a function of ωt as above, with $\omega = 2\pi/p$. Everything in the scaling remains the same, just the sin term changes to $P(\alpha\bar{t})$ if the time scale is based on viscosity (diffusion), and $P(\bar{t})$ if the time scale is based on the pressure oscillations. Filename: `pipeflow`.

Exercise 3.5: The linear cable equation

A key PDE in neuroscience is the cable equation, here given in its simplest linear form:

$$\tau\frac{\partial u}{\partial t} = \lambda^2\frac{\partial^2 u}{\partial t^2} - u\,. \tag{3.85}$$

The unknown u is the voltage (measured in volt) associated with an electric current along one-dimensional dendrites ("cables") in neural networks, while τ and λ are given parameters.

Scale (3.85) in three ways: 1) let all terms in the scaled equation have unit coefficients, 2) use the domain size L as spatial scale and base the time scale on diffusion, 3) use the domain size L as spatial scale and base the time scale on reaction, i.e., the $-u$ term.

Solution. Straightforward scaling, with scales u_c, t_c, and x_c, leads in the first step to

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{t_c\lambda^2}{\tau x_c^2}\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} - \frac{t_c}{\tau}\bar{u}\,.$$

Assuming that all terms are equally important and of unit size in the scaled PDE, we get a uniquely determined length and space scale:

$$t_c = \tau, \quad x_c = \lambda\,.$$

The scaled cable equation is then

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} - \bar{u}\,.$$

Let now the spatial scale be fixed as $x_c = L$. Basing t_c on diffusion means $t_c = \tau(L/\lambda)^2$, and the scaled PDE becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{t}^2} - \beta\bar{u},$$

where

$$\beta = \left(\frac{L}{\lambda}\right)^2\,.$$

Basing t_c on the reaction scale, i.e., the balance of the time derivative and the reaction term, gives $t_c = \tau$ and the scaled PDE

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \beta^{-1}\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} - \bar{u}\,.$$

In neuroscience applications of the cable equation to dendrites, it appears that λ is about 1 mm and of the same order of magnitude as the cable length,

so β is around 1 in size. Then there are not big differences in these scalings, and the first one is to be preferred. The two others are more suitable when β is small or large, e.g., such that the term with β can be left out of the PDE. Filename: `cable_eq`.

Exercise 3.6: Heat conduction with discontinuous initial condition

Two pieces of metal at different temperature are brought in contact at $t = 0$. The following initial-boundary value problem governs the temperature evolution in the two pieces:

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u, \qquad \boldsymbol{x} \in \Omega,\ t \in (0,T], \tag{3.86}$$

$$u(\boldsymbol{x},0) = I(x), \boldsymbol{x} \in Omega, \tag{3.87}$$

$$-\alpha \partial u \partial n = h(u - u_S), \qquad x \in \partial\Omega,\ t \in (0,T]. \tag{3.88}$$

Here, $u(\boldsymbol{x},t)$ is the temperature, α the effective heat diffusion coefficient (assuming both pieces are homogeneous and of the same type of metal), and u_S is the surrounding temperature. The domain Ω consists of the two pieces Ω_1 and Ω_2: $\Omega = \Omega_1 \cup \Omega_2$. The initial condition can be specified as

$$I(x) = \begin{cases} U_1,\ \boldsymbol{x} \in \Omega_1, \\ U_2,\ \boldsymbol{x} \in \Omega 2, \end{cases}$$

where U_1 and U_2 are the constant initial temperatures in each piece.

Thinking of two identical pieces Ω_1 and Ω_2 with shapes as bricks, it is tempting to develop a one-dimensional model, especially if the pieces are somewhat slender. We then expect the main temperature variations to take place in the x direction, where the x axis is perpendicular to the contact surface between the pieces. The idea is to integrate the governing PDE in the two other directions, use the Gauss divergence theorem on $\alpha\nabla^2 u$, and apply the cooling boundary condition. Let A be the cross section of the bricks. Integrating over A gives

$$\int_A \frac{\partial u}{\partial t} dydz = \int_A \alpha \left(\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial y^2} \frac{\partial^2 u}{\partial z^2} \right) dydz$$

$$\int_A \alpha \frac{\partial^2 u}{\partial x^2} dydz + \int_A \alpha \left(\frac{\partial^2 u}{\partial y^2} \frac{\partial^2 u}{\partial z^2} \right) dydz$$

$$= \int_A \alpha \frac{\partial^2 u}{\partial x^2} dydz + \alpha \int_{\partial A} \frac{\partial u}{\partial n}$$

$$= \int_A \alpha \frac{\partial^2 u}{\partial x^2} dydz - h(v(x,t) - u_S).$$

The function $v(x,t)$ means $u(\boldsymbol{x},t)$ evaluated at the boundary ∂A. Assuming u to vary little across the cross section A, we can approximate the integrals by u evaluated at ∂A:

$$\int_A \frac{\partial u}{\partial t} dydz \approx A \frac{\partial}{\partial t} v(x,t), \quad \int_A \alpha \frac{\partial^2 u}{\partial x^2} dydz \approx A\alpha \frac{\partial^2 v}{\partial x^2},$$

where A now is the cross-section area. The result is the 1D initial-boundary value problem

$$\frac{\partial v}{\partial t} = \alpha \frac{\partial^2 v}{\partial x^2} - \frac{h}{A}(v(x,t) - u_S), \qquad x \in (0,L),\ t \in (0,T], \tag{3.89}$$

$$v(x,0) = I(x), x \in (0,L), \tag{3.90}$$

$$\alpha \partial v \partial x = h(v(x,t) - u_S), \qquad x = 0,\ t \in (0,T], -\alpha \partial v \partial x = h(v(x,t) - u_S), \quad x = L,\ t \in (0,T], \tag{3.91}$$

with

$$I(x) = \begin{cases} U_1,\ x \in [0, L/2), \\ U_2,\ x \in [L/2, L]. \end{cases}$$

Scale this problem.

Solution. We expect the temperature to start from the discontinuous state with U_1 and U_2 and approach the surrounding temperature u_S in the cooling law as $t \to \infty$. A suitable scaling is then

$$\bar{v} = \frac{v - \min(U_1, U_2)}{u_S - \min(U_1, U_2)},$$

since this implies that u varies from 0 to 1. Without loss of generality we assume $U_1 < U_2$, so

$$\bar{v} = \frac{v - U_1}{u_S - U_1}\,.$$

Furthermore,

$$\bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{t}{t_c}\,.$$

Inserted in the governing PDE, we have

$$\frac{\partial \bar{v}}{\partial \bar{t}} = \frac{t_c \alpha}{L^2}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} - \frac{t_c h}{A(u_S - U_1)}(U_1 + (u_S - U_1)\bar{v}(x,t) - u_S),$$

which simplifies to

$$\frac{\partial \bar{v}}{\partial \bar{t}} = \frac{t_c \alpha}{L^2}\frac{\partial^2 \bar{v}}{\partial \bar{x}^2} - \frac{t_c h}{A}(\bar{v}(x,t) - 1)\,.$$

The natural time scale is that of diffusion: $t_c = L^2/dfc$. This choice results in the scaled PDE

$$\frac{\partial \bar{v}}{\partial \bar{t}} = \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} - \beta(\bar{v}(x,t) - 1),$$

with the dimensionless number

$$\beta = \frac{L^2 h}{A\alpha}\,.$$

The scaled initial condition becomes

$$\bar{v}(\bar{x},0) = 0 \text{ if } 0 \leq x < 1/2 \text{ else } \frac{U_2 - U_1}{U_s - U_1}\,.$$

The scaled boundary conditions take the form

$$\alpha\partial\partial\bar{x}\bar{v}(0,t) = \frac{Lh}{(}\bar{v}(0,t) - 1), \quad -\alpha\partial\partial\bar{x}\bar{v}(L,t) = \frac{Lh}{(}\bar{v}(L,t) - 1)\,.$$

Filename: `metal_pieces`.

Chapter 4
Advanced PDE models

This final chapter addresses more complicated PDE models, including linear elasticity, viscous flow, heat transfer, porous media flow, gas dynamics, and electrophysiology. A range of classical dimensionless numbers are discussed in terms of the scaling.

4.1 The equations of linear elasticity

To the best of the authors' knowledge, it seems that mathematical models in elasticity and structural analysis are almost never non-dimensionalized. This is probably due to tradition, but the following sections will demonstrate the usefulness of scaling also in this scientific field.

We start out with the general, time-dependent elasticity PDE with variable material properties. Analysis based on scaling is used to determine with the acceleration term can be neglected. The stationary elasticity PDE is then discussed, including the simplification to homogeneous media. Scaling of different types of boundary conditions is also treated. At the end, we scale the equations of coupled thermo-elasticity. All the models make the assumption of small displacement gradients.

4.1.1 The general time-dependent elasticity problem

The following vector PDE governs deformation and stress in purely elastic materials, under the assumption of small displacement gradients:

$$\varrho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \nabla((\lambda+\mu)\nabla\cdot\boldsymbol{u}) + \nabla\cdot(\mu\nabla\boldsymbol{u}) + \varrho\boldsymbol{f}\,. \tag{4.1}$$

Here, $\boldsymbol{u}$ is the displacement vector, ϱ is the density of the material, λ and μ are the Lame elasticity parameters, and $\boldsymbol{f}$ is a body force (gravity, centrifugal force, or similar).

We introduce dimensionless variables:

$$\bar{\boldsymbol{u}} = u_c^{-1}\boldsymbol{u}, \quad \bar{x} = \frac{x}{L}, \quad \bar{t} = \frac{f}{t_c},$$

where L is some length scale we used to scale y and z as well. Also the elasticity parameters and the density can be scaled, if they are not constants,

$$\bar{\lambda} = \frac{\lambda}{\lambda_c}, \quad \bar{\mu} = \frac{\mu}{\mu_c}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_c},$$

where the characteristic quantities are typically spatial maximum values of the functions:

$$\lambda_c = \max_{x,y,z} \lambda, \quad \mu_c = \max_{x,y,z} \mu, \quad \varrho_c = \max_{x,y,z} \varrho\,.$$

Finally, we scale $\boldsymbol{f}$ too (if not constant):

$$\bar{\boldsymbol{f}} = f_c^{-1}\boldsymbol{f}, \quad f_c = \max_{x,y,z,t} ||\boldsymbol{f}||\,.$$

Inserting the dimensionless quantities in the governing vector PDE results in

$$\frac{\varrho_c u_c}{t_c^2}\frac{\partial^2 \bar{\boldsymbol{u}}}{\partial \bar{t}^2} = L^{-2}u_c\bar{\nabla}((\lambda_c\bar{\lambda} + \mu_c\bar{\mu})\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + L^{-2}u_c\mu_c\bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \varrho_c f_c\bar{\varrho}\bar{\boldsymbol{f}}\,.$$

Making the terms non-dimensional gives the equation

$$\bar{\varrho}\frac{\partial^2 \bar{\boldsymbol{u}}}{\partial \bar{t}^2} = \frac{t_c^2\lambda_c}{L^2\varrho_c}\bar{\nabla}(\bar{\lambda}\bar{\nabla}\cdot\bar{u}) + \frac{t_c^2\mu_c}{L^2\varrho_c}\bar{\nabla}(\bar{\mu}\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + \frac{t_c^2\mu_c}{L^2\varrho_c}\bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \frac{t_c^2 f_c}{u_c}\bar{\varrho}\bar{\boldsymbol{f}}\,. \quad (4.2)$$

We may choose t_c to make the coefficient in front of any of the spatial derivative terms equal unity. Here we choose the μ term, which implies

$$t_c = L\sqrt{\frac{\varrho_c}{\mu_c}}\,.$$

The scale for $\boldsymbol{u}$ can be chosen from an initial displacement or by making the coefficient in front of the $\bar{\boldsymbol{f}}$ term unity. The latter means

$$u_c = \mu_c^{-1} f_c L^2 \varrho_c\,.$$

The resulting dimensionless PDE becomes

$$\bar{\varrho}\frac{\partial^2\bar{\boldsymbol{u}}}{\partial\bar{t}^2} = \bar{\nabla}((\beta\bar{\lambda}+\bar{\mu})\bar{\nabla}\cdot\bar{u}) + \bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \bar{\varrho}\bar{\boldsymbol{f}}\,. \tag{4.3}$$

The only dimensionless parameter is

$$\beta = \frac{\lambda_c}{\mu_c}\,.$$

If the source term is absent, we must use the initial condition or a known boundary displacement to determine u_c.

Software. Given software for (4.1), we can simulate the dimensionless problem by setting $\varrho = \bar{\varrho}$, $\lambda = \beta\bar{\lambda}$, and $\mu = \bar{\mu}$.

hpl 19: What about setting boundary conditions?

4.1.2 Dimensionless stress tensor

The stress tensor $\boldsymbol{\sigma}$ is a key quantity in elasticity and is given by

$$\boldsymbol{\sigma} = \lambda\nabla\cdot\boldsymbol{u}\boldsymbol{I} + \mu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)\,.$$

This $\boldsymbol{\sigma}$ can be computed as soon as the PDE problem for $\boldsymbol{u}$ has been solved. Inserting dimensionless variables on the right-hand side of the above relation gives

$$\begin{aligned}\boldsymbol{\sigma} &= \lambda_c u_c L^{-2}\bar{\lambda}\bar{\nabla}\cdot\bar{\boldsymbol{u}} + \mu_c u_c L^{-1}\bar{\mu}(\bar{\nabla}\bar{\boldsymbol{u}} + (\bar{\nabla}\bar{\boldsymbol{u}})^T)\\ &= \mu_c u_c L^{-1}\left(\beta\bar{\lambda}\bar{\nabla}\cdot\bar{\boldsymbol{u}} + \bar{\mu}(\bar{\nabla}\bar{\boldsymbol{u}} + (\bar{\nabla}\bar{\boldsymbol{u}})^T)\right)\,.\end{aligned}$$

The coefficient on the right-hand side, $\mu_c u_c L^{-1}$, has dimension of stress, since (according to the second table in Section 1.1.2) $[\mathrm{MT}^{-2}\mathrm{L}^{-1})(\mathrm{L})(\mathrm{L}^{-1})] = [\mathrm{MT}^{-2}\mathrm{L}^{-1}]$, which is the dimension of stress. The quantity $\mu_c u_c L^{-1}$ is therefore the natural scale of the stress tensor:

$$\bar{\boldsymbol{\sigma}} = \frac{\boldsymbol{\sigma}}{\sigma_c}, \quad \sigma_c = \mu_c u_c L^{-1},$$

and we have the dimensionless stress-displacement relation

$$\bar{\boldsymbol{\sigma}} = \beta\bar{\lambda}\bar{\nabla}\cdot\bar{\boldsymbol{u}} + \bar{\mu}(\bar{\nabla}\bar{\boldsymbol{u}} + (\bar{\nabla}\bar{\boldsymbol{u}})^T)\,. \tag{4.4}$$

4.1.3 When can the acceleration term be neglected?

A lot of applications of the elasticity equation involves static or quasi-static deformations where the acceleration term $\varrho \boldsymbol{u}_{tt}$ is neglected. Now we shall see under which conditions the quasi-static approximation holds.

The further discussion will need to look into the time scales of elastic waves, because it turns out that the chosen t_c above is closely linked to the propagation speed of elastic waves in a homogeneous body without body forces. A relevant model for such waves has constant ϱ, λ, and μ, and no force term:

$$\varrho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu) \nabla \nabla \cdot \boldsymbol{u} + \mu \nabla^2 \boldsymbol{u}. \tag{4.5}$$

S waves. Let us take the curl of this PDE and notice that the curl of a gradient vanishes. The result is

$$\frac{\partial^2}{\partial t^2} \nabla \times \boldsymbol{u} = c_S^2 \nabla^2 \nabla \times \boldsymbol{u},$$

i.e., a wave equation for $\nabla \times \boldsymbol{u}$. The wave velocity is

$$c_S = \sqrt{\frac{\mu}{\varrho}}.$$

The corresponding waves are called S waves. The curl of a displacement field is closely related to rotation of continuum elements. S waves are therefore rotation waves, also sometimes referred to as shear waves.

The divergence of a displacement field can be interpreted as the volume change of continuum elements. Suppose this volume change vanishes, $\nabla \cdot \boldsymbol{u} = 0$, which means that the material is incompressible. The elasticity equation then simplifies to

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = c_S^2 \nabla^2 \boldsymbol{u},$$

so each component of the displacement field in this case also propagates as a wave with speed c_S^2. The time it takes for such a wave to travel one characteristic length L is L/c_S, i.e., $L\sqrt{\varrho/\mu}$, which is nothing but our characteristic time t_c.

P waves. We may take the divergence of the PDE instead and notice that $\nabla \cdot \nabla = \nabla^2$ so

$$\frac{\partial^2}{\partial t^2} \nabla \cdot \boldsymbol{u} = c_P^2 \nabla^2 \nabla \cdot \boldsymbol{u},$$

with wave velocity

$$c_P = \sqrt{\frac{\lambda + 2\mu}{\varrho}}\,.$$

That is, the volume change (expansion/compression) propagates as a wave with speed c_P. These type of waves are called P waves. Other names are pressure and expansion/compression waves.

Suppose now that $\nabla \times \boldsymbol{u} = 0$, i.e., there is no rotation ("shear") of continuum elements. Mathematically this condition implies that $\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u})$ (see any book on vector calculus or Wikipedia). Our model equation (4.5) then reduces to

$$\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = c_P^2 \nabla^2 \boldsymbol{u},$$

which is nothing but a wave equation for each component of the displacement field.

Time-varying load. Suppose we have some time-varying boundary condition on $\boldsymbol{u}$ or the stress vector (traction) that has time scale ω (some oscillating movement that goes like $\sin \omega t$ is an example). We choose $t_c = 1/\omega$. The scaling now leads to

$$\gamma \frac{\partial^2 \bar{\boldsymbol{u}}}{\partial \bar{t}^2} = \bar{\nabla}((\beta\bar{\lambda} + \bar{\mu})\bar{\nabla} \cdot \bar{\boldsymbol{u}}) + \bar{\nabla} \cdot (\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) + \bar{\varrho}\bar{\boldsymbol{f}}\,.$$

where we have set

$$u_c = \mu_c^{-1} f_c L^2 \varrho_c,$$

as before, and γ is a new dimensionless number,

$$\gamma = \frac{\varrho_c L^2 \omega^2}{\mu_c} = \left(\frac{L\sqrt{\varrho_c/\mu_c}}{1/\omega}\right)^2\,.$$

The last rewrite shows that $\sqrt{\gamma}$ is the ratio of the time scale for S waves and the time scale for the forced movement on the boundary. The acceleration term can therefore be neglected when $\gamma \ll 1$, i.e., when the time scale for movement on the boundary is much larger than the time it takes for the S waves to travel through the domain. Since the velocity of S waves in solids is very large and the time scale correspondingly small, $\gamma \ll 1$ is very often the case in applications involving structural analysis. Exercise 4.1 explores related models and asks for comparisons of time scales for waves and mechanical vibrations in structures.

4.1.4 The stationary elasticity problem

Scaling of the PDE. We now look at the stationary version of (4.1) where the $\varrho \boldsymbol{u}_{tt}$ term is removed. The first step in the scaling is just inserting the dimensionless variables:

$$0 = L^{-2} u_c \bar{\nabla}((\lambda_c \bar{\lambda} + \mu_c \bar{\mu}) \bar{\nabla} \cdot \bar{\boldsymbol{u}}) + L^{-2} u_c \mu_c \bar{\nabla} \cdot (\bar{\mu} \bar{\nabla} \bar{\boldsymbol{u}}) + \varrho_c f_c \bar{\varrho} \bar{\boldsymbol{f}}\,.$$

Dividing by $L^2 u_c \mu_c$ gives

$$0 = \bar{\nabla}((\beta \bar{\lambda} + \bar{\mu}) \bar{\nabla} \cdot \bar{\boldsymbol{u}}) + \bar{\nabla} \cdot (\bar{\mu} \bar{\nabla} \bar{\boldsymbol{u}}) + \frac{L^2 \varrho_c f_c}{u_c \mu_c} \bar{\varrho} \bar{\boldsymbol{f}}\,.$$

Choosing $u_c = \varrho L^2 f_c / \mu_c$ as before leads to

$$\bar{\nabla}((\beta \bar{\lambda} + \bar{\mu}) \bar{\nabla} \cdot \bar{\boldsymbol{u}}) + \bar{\nabla} \cdot (\bar{\mu} \bar{\nabla} \bar{\boldsymbol{u}}) + \bar{\varrho} \bar{\boldsymbol{f}} = 0\,. \tag{4.6}$$

A homogeneous material with constant λ, μ, and ϱ is an interesting case (this corresponds to $\mu_c = \mu$, $\lambda_c = \lambda$, $\varrho_c = \varrho$, $\bar{\varrho} = \bar{\lambda} = \bar{\mu} = 1$):

$$(1+\beta) \bar{\nabla}(\bar{\nabla} \cdot \bar{\boldsymbol{u}}) + \bar{\nabla}^2 \bar{\boldsymbol{u}}) + \bar{\boldsymbol{f}} = 0\,. \tag{4.7}$$

Now β is defined as

$$\beta = \frac{\lambda}{\mu} = \left(\frac{c_p}{c_s}\right)^2 - 2\,.$$

It shows that in standard, stationary elasticity, λ/μ is the only significant physical parameter.

Scaling of displacement boundary conditions. A typical boundary condition on a part of the boundary is a prescribed displacement. For simplicity we set $\boldsymbol{u} = \boldsymbol{U}_0$ for a constant vector $\boldsymbol{U}_0$ as boundary condition. With $u_c = \varrho L^2 f_c / \mu$, we get the dimensionless condition

$$\bar{\boldsymbol{u}} = \frac{\boldsymbol{U}_0}{u_c} = \frac{\mu \boldsymbol{U}_0}{\varrho L^2 f_c}\,.$$

In the absence of body forces, the expression for u_c has no meaning ($f_c = 0$), so then $u_c = |\boldsymbol{U}_0|$ is a better choice. This gives the dimensionless boundary condition

$$\bar{u} = \frac{\boldsymbol{U}_0}{|\boldsymbol{U}_0|},$$

which is the unit vector in the direction of $\boldsymbol{U}_0$. The new u_c changes the coefficient in front of the body force term, if that term is present, to the dimensionless number

$$\delta = \frac{L^2 \varrho f_c}{\mu |\boldsymbol{U}_0|}.$$

Scaling of traction boundary conditions. The other type of common boundary condition in elasticity is a prescribed traction (stress vector) on a part of the boundary:

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{T}_0,$$

where for simplicity we take $\boldsymbol{T}_0$ as a constant vector. From Section 4.1.2 we have a stress scale $\sigma_c = \mu u_c / L$, but we may alternatively use $|\boldsymbol{T}_0|$ as stress scale. In that case,

$$\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = \frac{\boldsymbol{T}_0}{|\boldsymbol{T}_0|},$$

which is a unit vector in the direction of $\boldsymbol{T}_0$. Many applications involve large traction free areas on the boundary, on which we simply have $\bar{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = 0$.

4.1.5 Quasi-static thermo-elasticity

Heating solids give rise to expansion, i.e., strains, which may cause stress if displacements are constrained. The time scale of temperature changes are usually much larger than the time scales of elastic waves, so the stationary equations of elasticity can be used, but a term depends on the temperature, so the equations must be coupled to a PDE for heat transfer in solids. The resulting system of PDE is known as the equations of *thermo-elasticity* and reads

$$\nabla((\lambda+\mu)\nabla \cdot \boldsymbol{u}) + \nabla \cdot (\mu \nabla \boldsymbol{u}) = \alpha \nabla T - \varrho \boldsymbol{f}, \tag{4.8}$$

$$\varrho c \frac{\partial T}{\partial t} = \nabla \cdot (\kappa \nabla T) + \varrho \boldsymbol{f}_T, \tag{4.9}$$

where T is the temperature, α is a coefficient of thermal expansion, c is a heat capacity, κ is the heat conduction coefficient, and $\boldsymbol{f}_T$ is some heat source. The density ϱ is strictly speaking a function of T and the stress state, but a widely used approximation is to consider ϱ as a constant. Most thermo-elasticity applications have $\boldsymbol{f}_T = 0$ so we drop this term. Most applications also involve some heating from a temperature level T_0 to some level $T_0 + \Delta T$. A suitable scaling for T is therefore

$$\bar{T} = \frac{T - T_0}{\Delta T},$$

so that $\bar{T} \in [0,1]$. The elasticity equation has already been scaled and so has the diffusion equation for T. We base the time scale on the diffusion, i.e., the thermal conduction process:

$$t_c = \varrho c L^2/\kappa_c\,.$$

We imagine that κ is scaled as $\bar{\kappa} = \kappa/\kappa_c$. The dimensionless PDE system then becomes

$$\bar{\nabla}((1+\beta)\bar{\mu}\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + \bar{\nabla}\cdot(\bar{\mu}\bar{\nabla}\bar{\boldsymbol{u}}) = \bar{\nabla}\bar{T} - \epsilon\bar{\varrho}\bar{\boldsymbol{f}}, \tag{4.10}$$

$$\frac{\partial\bar{T}}{\partial\bar{t}} = \bar{\nabla}\cdot(\bar{\kappa}\bar{\nabla}\bar{T})\,. \tag{4.11}$$

Here we have chosen u_c such that the "heating source term" has a unit coefficient, acknowledging that this thermal expansion balances the stress terms with $\bar{\boldsymbol{u}}$. The corresponding displacement scale is

$$u_c = \frac{\alpha L \Delta T}{\mu_c}\,.$$

The dimensionless number in the body force term is therefore

$$\epsilon = \frac{L\varrho_c f_c}{\alpha\Delta T}\,,$$

which measures the ratio of the body force term and the "heating source term".

A homogeneous body with constant ϱ, λ, μ, c, and κ is common. The PDE system reduces in this case to

$$\bar{\nabla}((1+\beta)\bar{\nabla}\cdot\bar{\boldsymbol{u}}) + \bar{\nabla}^2\bar{\boldsymbol{u}}) = \bar{\nabla}\bar{T} - \epsilon\bar{\boldsymbol{f}}, \tag{4.12}$$

$$\frac{\partial\bar{T}}{\partial\bar{t}} = \bar{\nabla}^2\bar{T}\,. \tag{4.13}$$

In the absence of body forces, β is again the key parameter.

The boundary conditions for thermo-elasticity consist of the conditions for elasticity and the conditions for diffusion. Scaling of such conditions are discussed in Section 3.2 and 4.1.4.

4.2 The Navier-Stokes equations

This section presents scalings of various versions of the equations governing incompressible viscous fluid flow. We start with the plain Navier-Stokes equations without body forces and progress with adding the gravity force and a

free surface. We also look at scaling low Reynolds number flow and oscillating flows.

4.2.1 The momentum equation without body forces

The Navier-Stokes equations for incompressible viscous fluid flow without body forces take the form

$$\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}, \tag{4.14}$$

$$\nabla\cdot\boldsymbol{u}=0\,. \tag{4.15}$$

The primary unknowns are the velocity $\boldsymbol{u}$ and the pressure p. Moreover, ϱ is the fluid density, and μ is the dynamic viscosity.

Scaling. We introduce as usual dimensionless independent and dependent variables:

$$\bar{x}=\frac{x}{L},\quad \bar{y}=\frac{y}{L},\quad \bar{z}=\frac{z}{L},\quad \bar{t}=\frac{t}{t_c},\quad \bar{\boldsymbol{u}}=\frac{\boldsymbol{u}}{u_c},\quad \bar{p}=\frac{p}{p_c},$$

where L is some characteristic distance, t_c is some characteristic time, u_c is a characteristic velocity, and p_c is a characteristic pressure. Inserted in the equations,

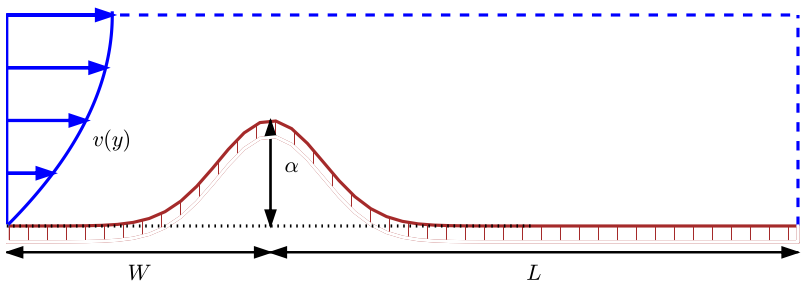
$$\varrho\left(\frac{u_c}{t_c}\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}}+\frac{u_c^2}{L}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}}\right)=-\frac{p_c}{L}\bar{\nabla}\bar{p}+\frac{u_c}{L^2}\mu\bar{\nabla}^2\bar{\boldsymbol{u}}, \tag{4.16}$$

$$\frac{u_c}{L}\bar{\nabla}\cdot\bar{\boldsymbol{u}}=0\,. \tag{4.17}$$

For the velocity it is common to just introduce some U for u_c. This U is normally implied by the problem description. For example, in the flow configuration below, with flow over a bump, we have some incoming flow with a profile $v(y)$ and U can typically be chosen as $U=\max_y v(y)$. The height of the bump influences the wake behind the bump, so a natural length scale is α. For numerical simulations in a domain of finite extent, W must be large enough to avoid feedback on the inlet profile, and L_c must be large enough for the type of outflow boundary condition used. Ideally, $W, L_c\to\infty$, so none of these parameters are useful as length scales.

hpl 20: Geir, check this formulation for choice of length scale. **Geir 21**: L cannot be used as both down-wind length of computational domain and general length scale. Could not change the L on the figure, as it is part of the pdf. A little uncertain concerning the check – it looks mostly fine. The profile

may have a length cale (as implicit in the figur). It may also be beneficial to have a spanwise limit of the computational domain in the figure ?



For flow in a channel or tube, we also have some inlet profile, e.g., $v(r)$ in a tube, where r is the radial coordinate, and a natural choice of characteristic velocity is $U = v(0)$ or the average flow

$$U = \frac{1}{R}\int_0^R v(r)dr,$$

if R is the radius of the tube. Other examples may be flow around a body, where there is some distant constant inlet flow $\boldsymbol{u} = U_0\boldsymbol{i}$, for instance, and $U = U_0$ is an obvious choice. We therefore assume that the flow problem comes with a natural candidate for U.

Having a characteristic distance L and velocity U, an obvious time measure is L/U so we set $t_c = L/U$. Dividing by the coefficient in front of the time derivative term, creates a pressure term

$$\frac{p_c}{\varrho U^2}\bar{\nabla}\bar{p}.$$

The coefficient suggest a choice $p_c = \varrho U^2$ if the pressure gradient term is to have the same size as the acceleration terms. This p_c is a very common pressure scale in fluid mechanics, arising from Bernoulli's equation $p + \frac{1}{2}\varrho\boldsymbol{u}\cdot\boldsymbol{u} = \text{const}$ for stationary flow.

Dimensonless PDEs and the Reynolds number. The discussions so far results in the following dimensionless form of (4.14) and (4.15):

$$\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}} + \bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} = -\bar{\nabla}\bar{p} + \mathrm{Re}^{-1}\nabla^2\boldsymbol{u}, \tag{4.18}$$

$$\bar{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \tag{4.19}$$

where Re is the famous *Reynolds number*,

$$\mathrm{Re} = \frac{\varrho UL}{\mu} = \frac{UL}{\nu}.$$

The latter expression makes use of the kinematic viscosity $\nu = \mu/\varrho$. For viscous fluid flows without body forces there is hence only one dimensionless number, Re.

The Reynolds number can be interpreted as the ratio of convection and viscosity:

$$\frac{\text{convection}}{\text{viscosity}} = \frac{|\varrho \boldsymbol{u} \cdot \nabla \boldsymbol{u}|}{|\mu \nabla^2 \boldsymbol{u}|} \sim \frac{\varrho U^2/L}{\mu U/L^2} = \frac{UL}{\nu} = \mathrm{Re}\,.$$

(We have here used that $\nabla \boldsymbol{u}$ goes like U/L and $\nabla^2 \boldsymbol{u}$ goes like U/L^2.)

4.2.2 Scaling of time for low Reynolds numbers

As we discussed in Section 3.4 for the convection-diffusion equation, there is not just one scaling that fits all problems. Above, we used $t_c = L/U$, which is appropriate if convection is a dominating physical effect. In case the convection term $\varrho \boldsymbol{u} \cdot \nabla \boldsymbol{u}$ is much smaller than the viscosity term $\mu \nabla^2 \boldsymbol{u}$, i.e., the Reynolds number is small, the viscosity term is dominating. However, if the scaling is suitable, the other terms are of order unity, and $\mathrm{Re}^{-1} \nabla^2 \bar{\boldsymbol{u}}$ must then also be of unit size, implying that $\nabla^2 \bar{\boldsymbol{u}}$ is small, but then the scaling is not suitable. In the low-Reynolds number regime, the diffusion effect of $\nabla^2 \bar{\boldsymbol{u}}$ is dominating, and we should use a time scale based on diffusion rather than convection. Such a time scale is $t_c = L^2/(\mu/\varrho) = L^2/\nu$. With this time scale, the dimensionless Navier-Stokes equations look like

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \mathrm{Re}\, \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}} = -\bar{\nabla} p + \nabla^2 \boldsymbol{u}, \tag{4.20}$$

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0\,. \tag{4.21}$$

As stated in the box in Section 3.4, (4.20) is the appropriate PDE for very low Reynolds number flow and suggests neglecting the convection term. If the flow is also steady, the time derivative term can be neglected, and we end up with the so-called *Stokes problem* for steady, slow, viscous flow:

$$-\bar{\nabla} p + \nabla^2 \boldsymbol{u} = 0, \tag{4.22}$$

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0\,. \tag{4.23}$$

This flow regime is also known as *Stokes' flow* or *creeping flow*.

4.2.3 Shear stress as pressure scale

Instead of using the kinetic energy ϱU^2 as pressure scale, one can use the shear stress $\mu U/L$ (U/L reflects the spatial derivative of the velocity, which enters the shear stress expression $\mu \partial u/\partial y$). Using U as velocity scale, L/U as time scale, and $\mu U/L$ as pressure scale, results in

$$\mathrm{Re}\left(\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}}\right) = -\bar{\nabla} \bar{p} + \nabla^2 \bar{\boldsymbol{u}}\,. \tag{4.24}$$

Low Reynolds number flow now suggests neglecting both acceleration terms.

4.2.4 Gravity force and the Froude number

We now add a gravity force to the momentum equation (4.14):

$$\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) = -\nabla p + \mu \nabla^2 \boldsymbol{u} - \varrho g \boldsymbol{k}, \tag{4.25}$$

where g is the acceleration of gravity, and $\boldsymbol{k}$ is a unit vector in the opposite direction of gravity. The new term takes the following form after non-dimensionalization:

$$\frac{t_c}{\varrho u_c} \varrho g \boldsymbol{k} = \frac{Lg}{U^2} \boldsymbol{k} = \mathrm{Fr}^{-2} \boldsymbol{k},$$

where Fr is the dimensionless Froude number,

$$\mathrm{Fr} = \frac{U}{\sqrt{Lg}}\,.$$

This quantity reflects the ratio of inertia and gravity forces:

$$\frac{|\boldsymbol{u} \cdot \nabla \boldsymbol{u}|}{|\varrho g|} \sim \frac{\varrho U^2/L}{\varrho g} = \mathrm{Fr}^2\,.$$

4.2.5 Oscillating boundary conditions and the Strouhal number

Many flows have an oscillating nature, often arising from some oscillating boundary condition. Suppose such a condition at some boundary $x = \text{const}$ takes the specific form

$$\boldsymbol{u} = U \sin(\omega t) \boldsymbol{i}\,.$$

The dimensionless counterpart becomes

$$U\bar{\boldsymbol{u}} = U\sin(\omega\frac{L}{U}\bar{t})\boldsymbol{i},$$

if $t_c = L/U$ is the appropriate time scale. This condition can be written

$$\bar{\boldsymbol{u}} = \sin(\mathrm{St}\,\bar{t}), \tag{4.26}$$

where St is the *Strouhal number*,

$$\mathrm{St} = \frac{\omega L}{U}\,. \tag{4.27}$$

The two important dimensionless parameters in oscillating flows are then the Reynolds and Strouhal numbers.

Even if the boundary conditions are of steady type, as for flow around a sphere or cylinder, the flow may at certain Reynolds numbers get unsteady and oscillating. For $10^2 < \mathrm{Re} < 10^7$, steady inflow towards a cylinder will cause vortex shedding: an array of vortices are periodically shedded from the cylinder, producing an oscillating flow pattern and force on the cylinder. The Strouhal number is used to characterize the frequency of oscillations. The phenomenon, known as *von Karman vortex street*, is particularly important if the frequency of the force on the cylinder hits the free vibration frequency of the cylinder such that resonance occurs. The result can be large displacements of the cylinder and structural failure. A famous case in engineering is the failure of the Tacoma Narrows suspension bridge in 1940, when wind-induced vortex shedding caused resonance with the free torsional vibrations of the bridge.

4.2.6 Cavitation and the Euler number

The dimensionless pressure in (4.18) made use of the pressure scale $p_c = \varrho U^2$. This is an appropriate scale if the pressure level is not of importance, which is very often the case since only the pressure *gradient* enters the flow equation and drives the flow. However, there are circumstances where the pressure level is of importance. For example, in some flows the pressure may become so low that the vapor pressure of the liquid is reached and that vapor cavities form (a phenomenon known as *cavitation*). A more appropriate pressure scale is then $p_c = p_\infty - p_v$, where p_∞ is a characteristic pressure level far from vapor cavities and p_v is the vapor pressure. The coefficient in front of the dimensionless pressure gradient is then

$$\frac{p_\infty - p_v}{\varrho U^2}\,.$$

Inspired by Bernoulli's equation $p+\frac{1}{2}\varrho\boldsymbol{u}\cdot\boldsymbol{u}=\text{const}$ in fluid mechanics, a factor $\frac{1}{2}$ is often inserted in the denominator. The corresponding dimensionless number,

$$\mathrm{Eu}=\frac{p_\infty-p_v}{\frac{1}{2}\varrho U^2}, \tag{4.28}$$

is called the *Euler number*. The pressure gradient term now reads $\frac{1}{2}\mathrm{Eu}\,\bar{\nabla}\bar{p}$. The Euler number expresses the ratio of pressure differences and the kinetic energy of the flow.

4.2.7 Free surface conditions and the Weber number

At a free surface, $z=\eta(x,y,t)$, the boundary conditions are

$$w=\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta, \tag{4.29}$$

$$p-p_0\approx-\sigma\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right), \tag{4.30}$$

where w is the velocity component in the z direction, p_0 is the atmospheric air pressure at the surface, and σ represents the surface tension. The approximation in (4.30) is valid under small deformations of the surface.

The dimensionless form of these conditions starts with inserting the dimensionless quantities in the equations:

$$u_c\bar{w}=\frac{L}{t_c}\frac{\partial\bar{\eta}}{\partial\bar{t}}+u_c\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\eta},$$

$$p_c\bar{p}\approx-\frac{1}{L}\sigma\left(\frac{\partial^2\bar{\eta}}{\partial\bar{x}^2}+\frac{\partial^2\bar{\eta}}{\partial\bar{y}^2}\right).$$

The characteristic length L is usually taken as the depth of the fluid when the surface is flat. We have used $\bar{p}=(p-p_0)/p_c$ for making the pressure dimensionless. Using $u_c=U$, $t_c=L/U$, and $p_c=\varrho U^2$, results in

$$\bar{w}=\frac{\partial\bar{\eta}}{\partial\bar{t}}+\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\eta}, \tag{4.31}$$

$$\bar{p}\approx-\mathrm{We}^{-1}\left(\frac{\partial^2\bar{\eta}}{\partial\bar{x}^2}+\frac{\partial^2\bar{\eta}}{\partial\bar{y}^2}\right), \tag{4.32}$$

where We is the *Weber number*,

$$\mathrm{We} = \frac{\varrho U^2 L}{\sigma}\,. \tag{4.33}$$

The Weber number measures the importance of surface tension effects and is the ratio of the pressure scale ϱU^2 and the surface tension force per area, typically σ/R_x in a 2D problem, which has size σ/L.

4.3 Thermal convection

Temperature differences in fluid flow cause density differences, and since cold fluid is heavier than hot fluid, the gravity force will induce flow due to density differences. This effect is called free thermal convection. Forced convection refers to the case where there is no feedback from the temperature field to the motion, i.e., temperature differences do not create motion, and the temperature distribution is only dependent on a given velocity field.

4.3.1 Forced convection

The model governing forced convection consists of the Navier-Stokes equations and the energy equation for the temperature:

$$\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = -\nabla p + \mu\nabla^2\boldsymbol{u} - \varrho g\boldsymbol{k}, \tag{4.34}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{4.35}$$

$$\varrho c\left(\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T\right) = \kappa\nabla^2 T\,. \tag{4.36}$$

$$\tag{4.37}$$

The symbol T is the temperature, c is a heat capacity, and κ is the heat conduction coefficient for the fluid. The PDE system applies primarily for liquids. For gases one may need a term $-p\nabla\cdot\boldsymbol{u}$ for the pressure work in (4.36) as well as a modified equation of continuity (4.35).

Despite the fact that ϱ depends on T, we treat ϱ as a constant ϱ_0. The major effect of the $\varrho(T)$ dependence is through the buoyancy effect caused by the gravity term $-\varrho(T)g\boldsymbol{k}$. It is common to drop this term in forced convection, and assume the momentum and continuity equations to be independent of the temperature. The flow is driven by boundary conditions (rather than density variations as in free convection), from which we can find a characteristic velocity U.

Dimensionless parameters are introduced as follows:

$$\bar{x} = \frac{x}{L},\ t_c = \frac{L}{U},\ \bar{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U},\ \bar{p} = \frac{p}{\varrho_0 U^2},\ \bar{T} = \frac{T - T_0}{T_c}\,.$$

Other coordinates are also scaled by L. The characteristic temperature T_c is chosen as some range ΔT, which depends on the problem and is often given by the thermal initial and/or boundary conditions. The reference temperature T_0 is also implied by prescribed conditions. Inserted in the equations, we get

$$\begin{aligned}
\varrho_0 \frac{U^2}{L}\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \varrho_0 \frac{U^2}{L}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} &= -\frac{\varrho_0 U^2}{L}\bar{\nabla}\bar{p} + \frac{\mu U}{L^2}\bar{\nabla}^2\bar{\boldsymbol{u}},\\
\frac{U}{L}\bar{\nabla}\cdot\bar{\boldsymbol{u}} &= 0,\\
\varrho_0 c\left(\frac{T_c U}{L}\frac{\partial \bar{T}}{\partial \bar{t}} + \frac{U T_c}{L}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{T}\right) &= \frac{\kappa T_c}{L^2}\bar{\nabla}^2\bar{T}\,.
\end{aligned}$$

Making each term in each equation dimensionless reduces the system to

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} = -\bar{\nabla}\bar{p} + \mathrm{Re}^{-1}\bar{\nabla}^2\bar{\boldsymbol{u}}, \tag{4.38}$$

$$\bar{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \tag{4.39}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{T} = \mathrm{Pe}^{-1}\bar{\nabla}^2\bar{T}\,. \tag{4.40}$$

The two dimensionless numbers in this system are given by

$$\mathrm{Pe} = \frac{\varrho_0 c U L}{\kappa}, \quad \mathrm{Re} = \frac{UL}{\nu} \quad (\nu = \frac{\mu}{\varrho_0})\,.$$

The Peclet number is here defined as the ratio of the convection term for heat $\varrho_0 c U \Delta T/L$ and the heat conduction term $\kappa U/L^2$. The fraction $\kappa/(\varrho_0 c)$ is known as the thermal diffusivity, and if this quantity is given a symbol α, we realize the relation to the Peclet number defined in Section 3.4.

4.3.2 Free convection

Governing equations. The mathematical model for free thermal convection consists of the Navier-Stokes equations coupled to an energy equation governing the temperature:

$$\varrho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}-\varrho g\boldsymbol{k}, \tag{4.41}$$

$$\nabla\cdot\boldsymbol{u}=0, \tag{4.42}$$

$$\varrho c\left(\frac{\partial T}{\partial t}+\boldsymbol{u}\cdot\nabla T\right)=\kappa\nabla^2 T+2\mu\varepsilon_{ij}\varepsilon_{ij}\,. \tag{4.43}$$

$$\tag{4.44}$$

The symbol T is the temperature, c is a heat capacity, κ is the heat conduction coefficient for the fluid. In free convection, the gravity term $-\varrho(T)g\boldsymbol{k}$ is essential since the flow is driven by temperature differences and the fact that hot fluid rises while cold fluid falls.

For slightly compressible gas flow a term $-p\nabla\cdot\boldsymbol{u}$ may be needed in (4.43) and also a modified (4.42).

Heating by viscous effects. We have also included heating of the fluid due to the work of viscous forces, represented by the term $2\mu\varepsilon_{ij}\varepsilon_{ij}$, where ε_{ij} is the strain-rate tensor in the flow, defined by

$$\varepsilon_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=\frac{1}{2}(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T),$$

where u_i is the velocity in direction of x_i ($i=1,2,3$ measures the space directions). The term $2\mu\varepsilon_{ij}\varepsilon_{ij}$ is written with Einstein's summation convention in mind such that there is an implicit sum over i and j. This term is actually much more relevant for forced convection, but was left out in Section 4.3.2 for mathematical simplicity. Heating by the work of viscous forces is often a very small effect and can be neglected, although it plays a major role in forging and extrusion of metals where the viscosity is very large, and those processes require large external forces to drive the flow. The reason behind the inclusion of the work by viscous forces under the heading of free convection is more that we want to scale a more complete, general mathematical model for mixed force and free convection, and arrive at dimensionless numbers that can tell if this extra term is important or not.

Relation between density and temperature. The equations (4.41) and (4.42) has already been made dimensionless in the previous section. The major difference is now that ϱ is no longer a constant, but a function of T. The relationship between ϱ and T is often taken as linear,

$$\varrho=\varrho_0-\varrho_0\beta(T-T_0),$$

where

$$\beta=-\frac{1}{\varrho}\left(\frac{\partial\varrho}{\partial t}\right)_p,$$

is known as the thermal expansion coefficient of the fluid, and ϱ_0 is a reference density when the temperature is at T_0.

Comment on the form of the equation of continuity. It might look strange that the equation of continuity (from the mass conservation principle) is $\nabla\cdot\boldsymbol{u}=0$ when there are density variations in the flow. The reason for using this version of the continuity equation is the assumption that the density of each fluid particle remains constant. Consequently, from the general equation of continuity,

$$\frac{\partial\varrho}{\partial t}+\nabla\cdot(\varrho\boldsymbol{u})=\frac{D\varrho}{dt}+\varrho\nabla\cdot\boldsymbol{u}=0,$$

it follows that if ϱ is constant for a particle, the material derivative $D\varrho/dt=\varrho_t+\boldsymbol{v}\cdot\nabla\varrho=0$, and the equation reduces to $\nabla\cdot\boldsymbol{u}=0$.

The Boussinesq approximation. A very common approximation, called the *Boussinesq approximation*, is to neglect the density variations in all terms except the gravity term. This is a good approximation unless the change in ϱ is large. With the linear $\varrho(T)$ formula and the Boussinesq approximation, (4.41)-(4.43) take the form

$$\varrho_0\left(\frac{\partial\boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p+\mu\nabla^2\boldsymbol{u}-(\varrho_0-\varrho_0\beta(T-T_0))g\boldsymbol{k}, \tag{4.45}$$

$$\nabla\cdot\boldsymbol{u}=0, \tag{4.46}$$

$$\varrho_0 c\left(\frac{\partial T}{\partial t}+\boldsymbol{u}\cdot\nabla T\right)=\kappa\nabla^2T+2\mu\varepsilon_{ij}\varepsilon_{ij}\,. \tag{4.47}$$

$$\tag{4.48}$$

A good justification of the Boussinesq approximation is provided by Tritton [11, Ch. 13].

Scaling. Dimensionless variables are introduced as

$$\bar{x}=\frac{x}{L},\quad t_c=\frac{L}{U},\ \bar{\boldsymbol{u}}=\frac{\boldsymbol{u}}{U},\ \bar{p}=\frac{p}{\varrho U^2},\ \bar{T}=\frac{T-T_0}{\Delta T}\,.$$

The dimensionless y and z coordinates also make use of L as scale. As in forced convection, we assume the characteristic temperature level T_0 and the scale ΔT are given by thermal boundary and/or initial conditions. Contrary to Sections 4.2 and 4.3.2, U is now not given by the problem description, but implied by ΔT.

Replacing quantities with dimensions by their dimensionless counterparts results in

$$\varrho_0 \frac{U^2}{L}\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \varrho_0 \frac{U^2}{L}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} = -\frac{p_c}{L}\bar{\nabla}\bar{p} + \frac{\mu U}{L^2}\bar{\nabla}^2\bar{\boldsymbol{u}} - \varrho_0 g\boldsymbol{k} + \varrho_0\beta T_c\bar{T}g\boldsymbol{k},$$

$$\frac{U}{L}\bar{\nabla}\cdot\bar{\boldsymbol{u}} = 0,$$

$$\varrho_0 c\left(\frac{T_c U}{L}\frac{\partial \bar{T}}{\partial \bar{t}} + \frac{UT_c}{L}\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{T}\right) = \frac{\kappa T_c}{L^2}\bar{\nabla}^2\bar{T} + 2\frac{\mu U}{L}\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij}\,.$$

These equations reduce to

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}} = -\bar{\nabla}\bar{p} + \mathrm{Re}^{-1}\bar{\nabla}^2\bar{\boldsymbol{u}} - \mathrm{Fr}^{-2}\boldsymbol{k} + \gamma\bar{T}\boldsymbol{k}, \tag{4.49}$$

$$\bar{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \tag{4.50}$$

$$\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{T} = \mathrm{Pe}^{-1}\bar{\nabla}^2\bar{T} + 2\delta\bar{\varepsilon}_{ij}\bar{\varepsilon}_{ij}\,. \tag{4.51}$$

The dimensionless numbers, in addition to Re and Fr, are

$$\gamma = \frac{g\beta L\Delta T}{U^2}, \quad \mathrm{Pe}^{-1} = \frac{\kappa}{\varrho_0 cUL}, \quad \delta = \frac{\mu U}{L\varrho_0 c\Delta T}\,.$$

The Peclet number is here defined as the ratio of the convection term for heat $\varrho_0 cU\Delta T/L$ and the heat conduction term $\kappa U/L^2$. The γ number measures the ratio of thermal buoyancy and the convection term:

$$\gamma = \frac{\varrho_0 g\beta\Delta T}{\varrho_0 U^2/L} = \frac{g\beta L\Delta T}{U^2}\,.$$

The Pe parameter is the fraction of the convection term and the thermal diffusion term:

$$\frac{|\varrho_0\boldsymbol{u}\cdot\nabla T|}{|\kappa\nabla^2 T|} \sim \frac{\varrho_0 cU\Delta TL^{-1}}{\kappa L^{-2}\Delta T} = \frac{\varrho cUL}{\kappa} = \mathrm{Pe}\,.$$

The δ parameter is the ratio of the viscous dissipation term and the convection term:

$$\frac{|\mu\nabla^2\boldsymbol{u}|}{|\varrho_0 c\boldsymbol{u}\cdot\nabla T|} \sim \frac{\mu U^2/L^2}{\varrho_0 cU\Delta T/L} = \frac{\mu U}{L\varrho_0 c\Delta T} = \delta\,.$$

4.3.3 The Grashof, Prandtl, and Eckert numbers

The problem with the above dimensionless numbers is that they involve U, but U is implied by ΔT. Assuming that the convection term is much bigger than the viscous diffusion term, the momentum equation features a balance between the buoyancy term and the convection term:

$$|\varrho_0 \boldsymbol{u} \cdot \nabla \boldsymbol{u}| \sim \varrho_0 g \beta \Delta T\,.$$

Translating this similarity to scales,

$$\varrho_0 U^2/L \sim \varrho_0 g \beta \Delta T,$$

gives an U in terms of ΔT :

$$U = \sqrt{\beta L \Delta T}\,.$$

The Reynolds number with this U now becomes

$$\mathrm{Re}_T = \frac{UL}{\nu} = \frac{\sqrt{g\beta L^3 \Delta T}}{\nu^2} = \mathrm{Gr}^{1/2},$$

where Gr is the Grashof number in free thermal convection:

$$\mathrm{Ga} = \mathrm{Re}_T^2 = \frac{g\beta L^3 \Delta T}{\nu^2}\,.$$

The Grashof number replaces the Reynolds number in the scaled equations of free thermal convection. We shall soon look at its interpretations, which are not as straightforward as for the Reynolds and Peclet numbers.

The above choice of U in terms of ΔT results in γ equal to unity:

$$\gamma = \frac{g\beta L \Delta T}{U^2} = \frac{g\beta L \Delta T}{g\beta L \Delta T} = 1\,.$$

The Peclet number can also be rewritten as

$$\mathrm{Pe} = \frac{\varrho c U L}{\kappa} = \frac{\mu c}{\kappa}\frac{\varrho U L}{\mu} = \mathrm{Pr}\mathrm{Re}^{-1} = \mathrm{Pr}\mathrm{Re}_T^{-1},$$

where Pr is the Prandtl number, defined as

$$\mathrm{Pr} = \frac{\mu c}{\kappa}\,.$$

The Prandtl number is the ratio of the momentum diffusivity (kinematic viscosity) and the thermal diffusivity. Actually, more detailed analysis shows that Pr reflects the ratio of the thickness of the thermal and velocity boundary layers: when $\mathrm{Pr} = 1$, these layers coincide, while $\mathrm{Pr} \ll 1$ implies that the thermal layer is much thicker than the velocity boundary layer, and vice versa for $\mathrm{Pr} \gg 1$.

The δ parameter is in free convection replaced by a combination of the Eckert number (Ec) and the Reynolds number. We have that

$$\mathrm{Ec} = \frac{U^2}{c\Delta T} = \delta \mathrm{Re}_T,$$

and consequently

$$\delta = \mathrm{EcRe}_T^{-1} = \mathrm{EcGr}^{-1/2}\,.$$

Writing

$$\mathrm{Ec} = \frac{\varrho_0 U^2}{\varrho_0 c \Delta T},$$

shows that the Eckert number can be interpreted as the ratio of the kinetic energy of the flow and the thermal energy.

We use Ga instead of Re_T in the momentum equations and also instead of Pe in the energy equation (recall that $\mathrm{Pe} = \mathrm{PrRe} = \mathrm{PrRe}_T = \mathrm{PrGr}^{-1/2}$). The resulting scaled system becomes

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}} = -\bar{\nabla} \bar{p} + \mathrm{Gr}^{-1/2} \bar{\nabla}^2 \bar{\boldsymbol{u}} - \mathrm{Fr}^{-2} \boldsymbol{k} + \bar{T} \boldsymbol{k}, \tag{4.52}$$

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0, \tag{4.53}$$

$$\mathrm{Gr}^{1/2} \left(\frac{\partial \bar{T}}{\partial \bar{t}} + \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{T} \right) = \mathrm{Pr}^{-1} \bar{\nabla}^2 \bar{T} + 2\mathrm{EcGr}^{-1/2} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{ij}\,. \tag{4.54}$$

The Grashof number plays the same role as the Reynolds number in the momentum equation in free convection. In particular, it turns out that Gr governs the transition between laminar and turbulent flow. For example, the transition to turbulence occurs in the range $10^8 < \mathrm{Gr} < 10^9$ for free convection from vertical flat plates. Gr is normally interpreted as a dimensionless number expressing the ratio of buoyancy forces and viscous forces.

Interpretations of the Grashof number. Recall that the scaling leading to the Grashof number is based on an estimate of U from a balance of the convective and the buoyancy terms. When the viscous term dominates over convection, we need a different estimate of U, since in this case, the viscous force balances the buoyancy force:

$$\mu \nabla^2 \boldsymbol{u} \sim \varrho_0 g \beta \Delta T \quad \Rightarrow \quad \mu U / L^2 \sim \varrho_0 g \beta \Delta T,$$

This similarity suggests the scale

$$U = \frac{g \beta L^2 \Delta T}{\nu}\,.$$

Now,

$$\frac{|\varrho_0 \boldsymbol{u} \cdot \nabla \boldsymbol{u}|}{|\mu \nabla^2 \boldsymbol{u}|} \sim \frac{UL}{\nu} = \frac{g \beta L^3 \Delta T}{\nu} = \mathrm{Gr}\,.$$

The result means that $\mathrm{Gr}^{1/2}$ measures the ratio of convection and viscous forces when convection dominates, but Gr measures this ratio when viscous forces dominate.

The product of Gr and Pr is the Rayleigh number,

$$\mathrm{Ra} = \frac{g\beta L^3 \Delta T \varrho_0 c}{\nu\kappa},$$

since

$$\mathrm{GrPr} = \mathrm{Re}_T^2 \mathrm{Pr} = \frac{g\beta L^3 \Delta T}{\nu^2}\frac{\mu c}{\kappa} = \frac{g\beta L^3 \Delta T \varrho_0 c}{\nu\kappa} = \mathrm{Ra}\,.$$

The Rayleigh number is the preferred dimensionless number when studying free convection in horizontal layers [2,11]. Otherwise, Gr and Pr are dominate.

4.3.4 Heat transfer at boundaries and the Nusselt number

A common boundary condition, modeling heat transfer to/from the surroundings is

$$-\kappa\frac{\partial T}{\partial n} = h_T(T - T_s), \tag{4.55}$$

where $\partial/\partial n$ means the derivative in the normal direction $(\boldsymbol{n}\cdot\nabla)$, h_T is an experimentally determined heat transfer coefficient, and T_s is the temperature of the surroundings. Scaling (4.55) leads to

$$-\frac{\kappa\Delta t}{L}\frac{\partial \bar{T}}{\partial \bar{n}} = h_T(\Delta T\bar{T} + T_0 - T_s),$$

and further to

$$\frac{\partial \bar{T}}{\partial \bar{n}} = \frac{h_T L}{\kappa}(\bar{T} + \frac{T_s - T_0}{\Delta T}) = \mathrm{Nu}(\bar{T} - \bar{T}_s),$$

where the Nusselt number is defined by

$$\mathrm{Nu} = \frac{h_T L}{\kappa},$$

and $\bar{T}_s$ is simply the dimensionless surrounding temperature,

$$\bar{T}_s = \frac{T_s - T_0}{\Delta T}\,.$$

Heat transfer is a huge engineering field with lots of experimental investigations that are summarized by curves relating various dimensionless numbers such as Gr, Pr, and Nu.

4.4 The bidomain model in electrophysiology

The mechanical functioning of the heart is crucially dependent on correct electric signal propagation through the heart tissue. A widely used mathematical model for the electric signal propagation is the bidomain equations:

$$\chi C_m \frac{\partial v}{\partial t} = \nabla \cdot (\sigma_i \nabla v) + \nabla \cdot (\sigma_i \nabla u_e) - \chi I_{\text{ion}} - \chi I_{\text{app}}, \tag{4.56}$$

$$0 = \nabla \cdot (\sigma_i \nabla v) + \nabla \cdot ((\sigma_i + \sigma_e) \nabla u_e). \tag{4.57}$$

These PDEs are posed in a spatial domain H for $t \in (0,T]$. The symbols in these PDEs have the following meaning: u_e is the extracellular electric potential, v is the transmembrane potential (difference between the extracellular and intracellular potential), C_m is the capacitance of the cell membrane, χ is a membrane area to cell volume ratio, σ_i is an electric conductivity tensor for the intracellular space, and σ_e is an electric conductivity tensor for the extracellular space.

The boundary conditions are of Neumann type, and we drop these from the discussion. The initial condition is typically $u_e = 0, v = v_r$, where v_r is a constant resting potential.

The PDE system is driven by $I_{\text{ion}} + I_{\text{app}}$, where I_{ion} is a reaction term describing ionic currents across the cell membrane, and I_{app} is an externally applied stimulus current. The applied current is normally a prescribed function, typically piecewise constant in time and space, while $I_{\text{ion}} = I_{\text{ion}}(v,s)$, where s is a state vector describing the electro-chemical state of the cells. Typical components of s are intracellular ionic concentrations and so-called gate variables that describe the permeability of the cell membrane, and its dynamics is governed governed by a system of ODEs, see for instance [10] for details. The total current I_{ion} is normally written as a sum of individual ionic currents:

$$I_{\text{ion}}(s,v) = \sum_{j=1}^{n} I_j(s,v),$$

where n is typically between 10 and 20 in recent models of cardiac cells. Most of the individual currents will be on the form $I_j(s,v) = g_j(s)(v - v_j)$, where v_j is the equlibrium potential of the specific ion, and $g_j(s)$ describes the membrane conductance to a specific ion. Without much loss of generality we can assume that this formulation is valid for all I_j, and the total ionic current can then be written in the general form

$$I_{\text{ion}}(s,v) = \sum_{j=1}^{n} I_{\text{j}}(s,v) = g(s)(v - v_{eq}(s)),$$

where $g(s) = \sum_{j=1}^{n} g_j(s)$ and $v_{eq}(s) = (\sum_{j=1}^{n} g_j v_j)/\sum_{j=1}^{n} g_j$. As noted above, the dynamics of s is governed by an ODE system on the form

$$\frac{ds}{dt} = f(v,s)\,.$$

The time scales of individual components of s typically vary greatly, making any scaling of this system highly dependent on the component under study. For the present text, the focus is on tissue-level electrophysiology as described by (4.56)-(4.57), and we will proceed to scale these equations. The bidomain model is a system of reaction-diffusion type, and the scaling will be based on the general non-linear reaction-diffusion equation in Section 3.2.1.

Dimensionless independent variables are introduced by

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{t} = \frac{t}{t_c},$$

where L is the characteristic length scale, and t_c is the characteristic time scale. Dimensionless dependent variables are expressed as

$$\bar{v} = \frac{v - v_r}{v_p - v_r}, \quad \bar{u} = \frac{u_e}{u_c}\,.$$

As noted above, v_r is the resting potential, and v_p is the peak potential. The scaling of v ensures $\bar{v} \in [0,1]$. We introduce the symbol $\Delta v = v_p - v_r$ to save space in the formulas: $\bar{v} = (v - v_r)/\Delta v$. The scale for u_e is u_c, to be determined either from simplicity of the equations or from available analysis of its magnitude.

The variable tensor coefficients σ_i and σ_e depend on the spatial coordinates and are also scaled:

$$\bar{\sigma}_i = \frac{\sigma_i}{\sigma_c}, \quad \bar{\sigma}_e = \frac{\sigma_e}{\sigma_c}\,.$$

For simplicity, we have chosen a common scale σ_c, but the two tensors may employ difference scales, and we may also choose different scales for different directions, to reflect the anisotropic conductivity of the tissue. One typically chooses σ_c as a norm of $\sigma_i + \sigma_e$, e.g., the maximum value.

Finally, we introduce a scaling of the parameters entering the ionic current term

$$\bar{v}_{eq} = (v_{eq} - v_r)/\Delta v, \quad \bar{g} = g/g_c\,.$$

For the characteristic membrane conductance a common choice is $g_c = 1/R_m$, where R_m is the membrane resistance at rest, but we will instead set $g_c = g_{\max}$, the maximum conductance of the membrane. These choices will ensure $\bar{v}_{eq}, \bar{g} \in [0,1]$.

Inserting the dimensionless variables in the equations, the system of governing equations becomes

$$\begin{aligned}\frac{\Delta v}{t_c}\chi C_m\frac{\partial\bar{v}}{\partial\bar{t}} &= \frac{\sigma_c\Delta v}{L^2}\nabla\cdot(\bar{\sigma}_i\bar{\nabla}\bar{v})+\frac{\sigma_c u_c}{L^2}\nabla\cdot(\bar{\sigma}_i\bar{\nabla}\bar{u})-\\ &\quad -\chi g_c\Delta v\bar{g}(s)(\bar{v}-\bar{v}_{eq}(s))-\chi I_{\text{app}},\\ 0 &= \frac{\sigma_c\Delta v}{L^2}\bar{\nabla}\cdot(\bar{\sigma}_i\bar{\nabla}\bar{v})+\frac{\sigma_c u_c}{L^2}\nabla\cdot((\bar{\sigma}_i+\bar{\sigma}_e)\bar{\nabla}\bar{u}),\end{aligned}$$

Multiplying the equations by appropriate factors leads to equations with dimensionless terms only:

$$\begin{aligned}\frac{\partial\bar{v}}{\partial\bar{t}} &= \frac{t_c\sigma_c}{\chi C_m L^2}\nabla\cdot(\bar{\sigma}_i\bar{\nabla}\bar{v})+\frac{t_c\sigma_c u_c}{\Delta v\chi C_m L^2}\nabla\cdot(\bar{\sigma}_i\bar{\nabla}\bar{u})-\\ &\quad \frac{g_c t_c}{C_m}\bar{g}(s)(\bar{v}-\bar{v}_{eq}(s))-\frac{t_c}{C_m\Delta v}I_{\text{app}},\\ 0 &= \bar{\nabla}\cdot(\bar{\sigma}_i\bar{\nabla}\bar{v})+\frac{u_c}{\Delta v}\nabla\cdot((\bar{\sigma}_i+\bar{\sigma}_e)\bar{\nabla}\bar{u}),\end{aligned}$$

The time scale is not so obvious to choose. As noted above, the ODE system that governs s and thereby $\bar{g}(s),\bar{v}_{eq}(s)$ may feature a wide range of spatial scales. Furthermore, even if we focus on the tissue equations and focus on the dynamics of v and u_e, the bidomain equations are characterized by relatively small regions having large spatial and temporal gradients, while in most of the domain variations are slower. The PDEs are of reaction-diffusion nature, and the solution takes the form of a wavefront of activation that propagates through the tissue. For now we choose the time scale based on balancing the reaction and diffusion components. From previous examples in Section 3.2.1, we therefore set the terms in front of the reaction term and the diffusion term to unity. This idea means

$$\frac{t_c\sigma_c}{\chi C_m L^2}=1,\quad \frac{t_c g_c}{C_m}=1,$$

which determines the time and length scales:

$$t_c=\frac{C_m}{g_c},\quad L=\sqrt{\frac{\sigma_c}{g_c\chi}}\,.$$

Two natural dimensionless variables then arise from the second diffusion term and the applied current term:

$$\alpha=\frac{u_c}{\Delta v},\quad \beta=\frac{I_{\text{app}}}{g_c\Delta v}\,.$$

In many cases it will be natural to set $u_c=\Delta v$, which of course removes the need for α, but we include the freedom to have u_c as some specified characteristic size of u_e.

The final dimensionless system becomes

$$\frac{\partial \bar{v}}{\partial \bar{t}} = \nabla \cdot (\bar{\sigma}_i \bar{\nabla} \bar{v}) + \alpha \nabla \cdot (\bar{\sigma}_i \bar{\nabla} \bar{u}) \tag{4.58}$$

$$- \bar{g}(s)(\bar{v} - \bar{v}_{eq}(s)) - \beta \tag{4.59}$$

$$0 = \bar{\nabla} \cdot (\bar{\sigma}_i \bar{\nabla} \bar{v}) + \alpha \nabla \cdot ((\bar{\sigma}_i + \bar{\sigma}_e) \bar{\nabla} \bar{u}) \,. \tag{4.60}$$

The two dimensionless variables in these equations have straightforward interpretations: α is the ratio of the span in the two electric potentials, and β is ratio of the source term with I_{app} and the time-derivative term of v, or the source term and the diffusion term in v.

js 22: preliminary notes We can insert typical values for the parameters to get a feel for the chosen scaling. We have

$$\begin{aligned} C_m &= 1.0\,\mu\text{F}\,\text{cm}^{-2}, \quad g_c = g_{max} = 13.0\text{m}\,\text{S}\mu\text{F}^{-1} = 13.0\text{mS}\,\text{cm}^{-2}, \\ \chi &= 2000\text{cm}^{-1}, \quad u_c = \Delta v = 100\text{mV}, \\ \sigma_c &= 3.0\text{mS}\,\text{cm}^{-1}\,. \end{aligned}$$

This gives the following values of t_c and L:

$$\begin{aligned} t_c &= \frac{1.0\mu\text{F}\,\text{cm}^{-2}}{13.0\mu\text{F}\,\text{cm}^{-2}} = \frac{1.0}{13.0}\frac{\mu\text{F}}{\text{mS}} \approx 0.076\text{ms} \\ L &= \sqrt{\frac{\sigma_c}{\chi g_c}} = \sqrt{\frac{3.0\text{mS}\,\text{cm}^{-2}}{2000\text{cm}^{-1}\mu\text{F}\,\text{cm}^{-}2}} \approx 0.087\text{mm}\,. \end{aligned}$$

These values are both very small, which is probably related to our choice of $g_c = g_{\max}$. This implies that we choose the appropriate scaling based on the upstroke phase of the action potential, when both spatial and temporal variations are extremely high. This may therefore be a "correct" scaling exactly at the wavefront of the electrical potential, but seems less relevant elsewhere. Choosing g_c to be for instance the resting conductance, which is the common choice when scaling the cable equation, may increase t_c, L by factors up to 2500 and 50, respectively.

hpl 23: Joakim, can you say something about the relevance of these scales? A factor of 2500 is quite big...

The conduction velocity is often a quantity of interest, and we could obtain an alternative relation between t_c and L by requiring $v_c = L/t_c$. In human cardiac tissue v_c is known to be about 60 cm/s, while the choices above gives

$$\frac{L}{t_c} = \frac{0.087\text{mm}}{0.076\text{ms}} \approx 144\text{cm/s}\,.$$

Enforcing $L/t_c = 60\text{cm}\,\text{s}^{-}1$ gives the constraint $g_c \approx 4.8\text{mS}\,\text{cm}^{-2}$, and yields $L \approx 0.17$ mm and $t_c = 0.21$ ms. These are still small values, but very close to

the standard values of $\Delta x, \Delta t$ required for accurate numerical solution. That is, this choice of scaling will give $\Delta x \approx \Delta t \approx 1$, instead of $\bar{t}, \bar{x} \approx 1$?

hpl 24: That sounds a big strange, but if the focus is on a smaller scale, choosing L and t_c from this scale leads to that conclusion.

4.5 Two-phase porous media flow

We consider the flow of two incompressible, immiscible fluids in a porous medium with porosity $\phi(\boldsymbol{x})$. The two fluids are referred to as the wetting and non-wetting fluid. In an oil-water mixture, water is usually the wetting fluid. The fraction of the pore volume occupied by the wetting fluid is denoted by $S(\boldsymbol{x}, t)$. The non-wetting fluid then occupies $1 - S$ of the pore volume (or $(1 - S)\phi$ of the total volume). The variable $P(\boldsymbol{x}, t)$ represents the pressure in the non-wetting fluid. It is related to the pressure P_n in the non-wetting fluid through the capillary pressure $p_c = P_n - P$, which is an empirically determined function of S.

From mass conservation of the two fluids and from Darcy's law for each fluid, one can derive the following system of PDEs and algebraic relations that govern the two primary unknowns S and P:

$$\nabla \cdot \boldsymbol{v}_t = -(Q_n + Q_w), \tag{4.61}$$

$$\boldsymbol{v}_t = -\lambda_t \nabla P + \lambda_w p_c'(S) \nabla S + (\lambda_w \varrho_w + \lambda_n \varrho_n) g \boldsymbol{k}, \tag{4.62}$$

$$\phi \frac{\partial S}{\partial t} + f_w'(S) \boldsymbol{v}_t \cdot \nabla S = \nabla \cdot (h_w(S) p_c'(S) \nabla S) + g \frac{\partial G_w}{\partial z} + f_w(Q_n + Q_w) - Q_w, \tag{4.63}$$

$$Q_w = \frac{q_w}{\varrho_w}, \tag{4.64}$$

$$Q_n = \frac{q_n}{\varrho_n}, \tag{4.65}$$

$$\lambda_w(S) = \frac{K}{\mu_w} k_{rw}(S), \tag{4.66}$$

$$\lambda_n(S) = \frac{K}{\mu_n} k_{rn}(S), \tag{4.67}$$

$$\lambda_t(S) = \lambda_w(S) + \lambda_n(S), \tag{4.68}$$

$$k_{rw}(S) = K_{wm} \left[\frac{S - S_{wr}}{1 - S_{nr} - S_{wr}} \right]^a, \tag{4.69}$$

$$k_{rn}(S) = K_{nm} \left[\frac{1 - S - S_{nr}}{1 - S_{nr} - S_{wr}} \right]^b, \tag{4.70}$$

$$f_w(S) = \frac{\lambda_w}{\lambda_t}, \tag{4.71}$$

$$G_w(S) = h_w(S)(\varrho_n - \varrho_w), \tag{4.72}$$

$$h_w(S) = -\lambda_n(S) f_w(S), . \tag{4.73}$$

The permeability of the porous medium is K (usually a tensor, but here taken as a scalar for simplicity); μ_w and μ_n are the dynamic viscosities of the wetting and non-wetting fluid, respectively; ϱ_w and ϱ_n are the densities of the wetting and non-wetting fluid, respectively; q_w and q_n are the injection rates of the wetting and non-wetting fluid through wells, respectively; S_{wr} is the irreducible saturation of the wetting fluid (i.e., $S \geq S_{wr}$); S_{nr} is the corresponding irreducible saturation of the non-wetting fluid (i.e., $(1-S) \geq S_{nr}$), K_{wn} and K_{nr} are the maximum values of the relative permeabilities k_{rw} and k_{rn}, respectively, and a and b are given (Corey) exponents in the expressions for the relative permeabilities.

The two PDEs are of elliptic and hyperbolic/parabolic nature: (4.61) is elliptic since it is the divergence of a vector field, while (4.63) is parabolic ($h_w \geq 0$ because $p_c'(S) \geq 0$ and λ_n as well as f_w are positive since $k_{rn} > 0$ and $k_{rw} > 0$). Very often, p_c' is small so (4.63) is of hyperbolic nature, and S features very steep gradients that become shocks in the limit $p_c' \to 0$ and (4.63) is purely hyperbolic. A popular solution technique is based on operator splitting: (4.61) is solved with respect to P, given S, and (4.63) is solved with respect to S, given P.

The saturation S is a non-dimensional quantity, and so are ϕ, k_{rw}, k_{rn}, K_{wm}, K_{nm}, f_w, and f'_w. The quantity $\boldsymbol{v}_t$ is the total filtration velocity, i.e., the sum of the velocities of the wetting and non-wetting fluid. An associated velocity scale v_c is convenient to define. It is also convenient to introduce dimensionless fractions of wetting and non-wetting fluid properties:

$$
\begin{aligned}
\varrho &\equiv \varrho_w, \\
\varrho_n &= \varrho\alpha, \quad \alpha = \frac{\varrho_n}{\varrho_w}, \\
\mu &\equiv \mu_w, \\
\mu_n &= \mu\beta, \quad \beta = \frac{\mu_n}{\mu_w}, \\
Q &\equiv Q_w = \frac{q_w}{\varrho}, \\
Q_n &= Q\frac{\gamma}{\alpha}, \quad \gamma = \frac{q_n}{q_w}\,.
\end{aligned}
$$

We will benefit from making λ_w, λ_n, and λ_t dimensionless:

$$
\begin{aligned}
\lambda_w(S) &= \frac{K}{\mu}k_{rw}(S) = \lambda_c\bar{\lambda}_w, \quad \lambda_c = \frac{K}{\mu}, \quad \bar{\lambda}_w = k_{rw}, \\
\lambda_n(S) &= \frac{K}{\mu}\beta^{-1}k_{rn}(S) = \lambda_c\beta^{-1}\bar{\lambda}_n, \quad \bar{\lambda}_n = k_{rn}, \\
\lambda_t(S) &= \lambda_w(S) + \lambda_n(S) = \lambda_c\bar{\lambda}_t, \quad \bar{\lambda}_t = \bar{\lambda}_w + \beta^{-1}\bar{\lambda}_n\,.
\end{aligned}
$$

As we see, λ_c is the characteristic size of any "lambda" quantity, and a bar indicates as always a dimensionless variable. The above formulas imply

$$
h_w(S) = -\lambda_c\beta^{-1}\bar{\lambda}_n(S)f_w(S), \quad G_w(S) = h_w(S)\varrho(\alpha-1)\,.
$$

Furthermore, we introduce dimensionless quantities by

$$
\bar{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \quad \bar{\boldsymbol{v}}_t = \frac{\boldsymbol{v}_t}{v_c}, \quad \bar{P} = \frac{P}{P_c}, \quad \bar{p}_c = \frac{p_c}{P_c}\,.
$$

Inserting the above scaled quantities in the governing PDEs results in

$$\bar{\nabla}\cdot\bar{\boldsymbol{v}}_t = -\frac{LQ}{v_c}(1+\alpha^{-1}\gamma), \tag{4.74}$$

$$\bar{\boldsymbol{v}}_t = -\frac{P_c\lambda_c}{v_cL}\bar{\lambda}_t\bar{\nabla}\bar{P} + \frac{\lambda_cP_c}{v_cL}\bar{\lambda}_w\bar{p}_c'(S)\bar{\nabla}S + \frac{g\lambda_c\varrho}{v_c}(\bar{\lambda}_w+\alpha\beta^{-1}\bar{\lambda}_n)\boldsymbol{k}, \tag{4.75}$$

$$\phi\frac{\partial S}{\partial\bar{t}} + \frac{t_cv_c}{L}f_w'(S)\bar{\boldsymbol{v}}_t\cdot\bar{\nabla}S = \frac{t_cP_c\lambda_c}{L^2}\bar{\nabla}\cdot(-\beta^{-1}\bar{\lambda}_n(S)f_w(S)\bar{p}_c'(S)\bar{\nabla}S) + \frac{t_cg}{L}\frac{\partial G_w}{\partial\bar{z}} + t_cf_wQ(1+\alpha^{-1}\gamma) - t_cQ\,. \tag{4.76}$$

As usual, L is taken as the characteristic length of the spatial domain. Since v_c is a velocity scale, a natural time scale is the time it takes to transport a signal with velocity v_c through the domain: $t_c = L/v_c$. The diffusion term in the equation (4.79) then gets a dimensionless fraction

$$\frac{LP_c\lambda_c}{v_cL^2}\,.$$

Forcing this fraction to be unity gives

$$v_c = \lambda_c\frac{P_c}{L}\,.$$

We realize that this is indeed a natural velocity scale if the velocity is given by the pressure term in Darcy's law. This term is K/μ times the pressure gradient:

$$\frac{K}{\mu}|\nabla P| \sim \frac{K}{\mu}\frac{P_c}{L} = \lambda_c\frac{P_c}{L} = v_c\,.$$

We have here dropped the impact of the relative permeabilities $\bar{\lambda}_w$ or $\bar{\lambda}_n$ since these are quantities that are less than or equal to unity.

The other term in Darcy's law is the gravity term that goes like $\lambda_c\varrho g$ (again dropping relative permeabilities). The ratio of the gravity term and the pressure gradient term in Darcy's law is an interesting dimensionless number:

$$\delta = \frac{\lambda_c\varrho g}{\lambda_cP_c/L} = \frac{L\varrho g}{P_c}\,.$$

This number naturally arises when we discuss the term

$$\frac{t_cg}{L}\frac{\partial G_w}{\partial\bar{z}} = -(\alpha-1)\beta^{-1}\delta(\bar{\lambda}_n'(S)f_w(S)+\bar{\lambda}_n(S)f_w'(S))\frac{\partial S}{\partial\bar{z}}$$

Introducing another dimensionless variable,

$$\epsilon = t_cQ = \frac{L^2Q}{\lambda_cP_c},$$

we can write (4.74)-(4.76) in the final dimensionless form as

$$\bar{\nabla}\cdot\bar{\boldsymbol{v}}_t = -\epsilon(1+\alpha^{-1}\gamma), \tag{4.77}$$

$$\bar{\boldsymbol{v}}_t = -\bar{\lambda}_t\bar{\nabla}\bar{P} + \bar{\lambda}_w\bar{p}_c'(S)\bar{\nabla}S + \delta(\bar{\lambda}_w + \alpha\beta^{-1}\bar{\lambda}_n)\boldsymbol{k}, \tag{4.78}$$

$$\begin{aligned}\phi\frac{\partial S}{\partial\bar{t}} + f_w'(S)\bar{\boldsymbol{v}}_t\cdot\bar{\nabla}S = &-\bar{\nabla}\cdot(-\beta^{-1}\bar{\lambda}_n(S)f_w(S)\bar{p}_c'(S)\bar{\nabla}S) - \\ &(\alpha-1)\beta^{-1}\delta(\bar{\lambda}_n'(S)f_w(S) + \bar{\lambda}_n(S)f_w'(S))\frac{\partial S}{\partial\bar{z}} + \\ &\epsilon f_w(1+\alpha^{-1}\gamma) - \epsilon.\end{aligned} \tag{4.79}$$

The eight input parameters L, q_w, q_n, μ_w, μ_n, ϱ_w, ϱ_n, and K are reduced to five dimensionless parameters α, β, γ, δ, and ϵ. There are six remaining dimensionless numbers to be set: K_{wm}, K_{nm}, S_{wr}, S_{nr}, a, and b.

hpl 25: Boundary conditions?

hpl 26: Simplifications: skip gravity, sources, capillary pressure, simplify relative permeabilities.

hpl 27: Exercises: single-phase flow, with and without thermal effects.

4.6 Compressible gas dynamics

4.6.1 The Euler equations of gas dynamics

The fundamental equations for a compressible fluid are based on balance of mass, momentum, and energy. The PDE system, known as the Euler equations of gas dynamics, can be written as

$$\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\boldsymbol{u}) = 0, \tag{4.80}$$

$$\frac{\partial(\varrho\boldsymbol{u})}{\partial t} + \nabla\cdot(\varrho\boldsymbol{u}\boldsymbol{u}^T) = -\nabla p + \varrho\boldsymbol{f}, \tag{4.81}$$

$$\frac{\partial E}{\partial t} + \nabla\cdot(\boldsymbol{u}(E+p)) = 0, \tag{4.82}$$

where the total energy is given by

$$E = \varrho e + \frac{1}{2}\varrho\boldsymbol{u}\cdot\boldsymbol{u}. \tag{4.83}$$

In these equations, $\boldsymbol{u}$ is the fluid velocity, ϱ is the density, p is the pressure, E is the total energy per unit volume, composed of the kinetic energy per unit volume, $\frac{1}{2}\varrho\boldsymbol{u}\cdot\boldsymbol{u}$, and the internal energy per unit volume, ϱe.

Assuming the fluid to be an ideal gas implies the following additional relations:

$$e = c_v T, \tag{4.84}$$

$$p = \varrho R T = \frac{R}{c_v}(E - \frac{1}{2}\varrho \boldsymbol{u}\cdot\boldsymbol{u}), \tag{4.85}$$

where c_v is the specific heat capacity at constant volume (for dry air $c_v = 717.5\,\mathrm{J\,kg^{-1}K^{-1}}$), R is the specific ideal gas constant ($R = 287.14\mathrm{Jkg^{-1}K^{-1}}$), and T is the temperature.

The common way to solve these equations is to propagate ϱ, $\varrho\boldsymbol{u}$, and E by an explicit numerical method in time for (4.80)-(4.82), using (4.85) for p.

We introduce dimensionless independent variables,

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{t} = \frac{t}{t_c},$$

and dimensionless dependent variables,

$$\bar{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U}, \quad \bar{\varrho} = \frac{\varrho}{\varrho_c}, \quad \bar{p} = \frac{p}{p_c}, \quad \bar{E} = \frac{E}{E_c}\,.$$

Inserting these expressions in the governing equations gives

$$\begin{aligned}
\frac{\partial\bar{\varrho}}{\partial\bar{t}} + \frac{t_c U}{L}\bar{\nabla}\cdot(\bar{\varrho}\bar{\boldsymbol{u}}) &= 0,\\
\frac{\partial(\bar{\varrho}\bar{\boldsymbol{u}})}{\partial\bar{t}} + \frac{t_c U}{L}\bar{\nabla}\cdot(\bar{\varrho}\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}^T) &= -\frac{t_c p_c}{U L \varrho_c}\nabla\bar{p} + \frac{t_c f_c}{U}\bar{\varrho}\bar{\boldsymbol{f}},\\
\frac{\partial\bar{E}}{\partial\bar{t}} + \frac{t_c U}{L E_c}\bar{\nabla}\cdot(\bar{\boldsymbol{u}}(E_c\bar{E} + p_c\bar{p})) &= 0,\\
\bar{p} &= \frac{R}{c_v p_c}(E_c\bar{E} - \frac{1}{2}\varrho_c u_c \bar{\varrho}\bar{\boldsymbol{u}}\cdot\bar{\boldsymbol{u}})\,.
\end{aligned}$$

A natural choice of time scale is $t_c = L/U$. A common choice of pressure scale is $p_c = \varrho_c U^2$. The energy equation simplifies if we choose $E_c = p_c = \varrho_c U^2$. With these scales we get

$$\frac{\partial \bar{\varrho}}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\varrho}\bar{\boldsymbol{u}}) = 0,$$
$$\frac{\partial (\bar{\varrho}\bar{\boldsymbol{u}})}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\varrho}\bar{\boldsymbol{u}}\bar{\boldsymbol{u}}^T) = -\nabla \bar{p} + \alpha \bar{\varrho}\bar{\boldsymbol{f}},$$
$$\frac{\partial \bar{E}}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\boldsymbol{u}}(\bar{E} + \bar{p})) = 0,$$
$$\bar{p} = \frac{R}{c_v}(\bar{E} - \frac{1}{2}\bar{\varrho}\bar{u} \cdot \bar{u}),$$

where α is a dimensionless number:

$$\alpha = \frac{L f_c}{U^2}.$$

We realize that the scaled Euler equations look like the ones with dimension, apart from the α coefficient.

hpl 28: Comment on software also for advanced models.

4.6.2 General isentropic flow

Heat transfer can be neglected in isentropic flow, and there is hence an equation of state involving only ϱ and p:

$$p = F(\varrho).$$

The energy equation is now not needed and the Euler equations simplify to

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\boldsymbol{u}\varrho) = 0, \tag{4.86}$$
$$\varrho \frac{\partial \boldsymbol{u}}{\partial t} + \varrho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0. \tag{4.87}$$

Elimination of the pressure. A common equation of state is

$$F(\varrho) = p_0 \left(\frac{\varrho}{\varrho_0}\right)^{\gamma},$$

where $\gamma = 5/3$ for air. The first step is to eliminate p in favor of ϱ so we get a system for ϱ and $\boldsymbol{u}$. To this end, we must calculate ∇p:

$$\nabla p = F'(\varrho)\nabla \varrho, \quad F'(\varrho) = c_0^2 \left(\frac{\varrho}{\varrho_0}\right)^{\gamma-1},$$

where

$$c_0 = \sqrt{\frac{\gamma p_0}{\varrho_0}}$$

is the speed of sound in the fluid in the equilibrium state. Equation (4.87) with eliminated pressure p reads

$$\varrho \frac{\partial \boldsymbol{u}}{\partial t} + \varrho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + c_0^2 \left(\frac{\varrho}{\varrho_0} \right)^{\gamma - 1} \nabla \varrho = 0 \,. \tag{4.88}$$

The governing equations are now (4.86) and (4.88). Space and time are scaled as usual as

$$\bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{L}, \quad \bar{z} = \frac{z}{L}, \quad \bar{t} = \frac{t}{t_c} \,.$$

The scaled dependent variables are

$$\bar{\varrho} = \frac{\varrho}{\varrho_c}, \quad \bar{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U} \,.$$

Then $F'(\varrho) = c_0^2 \bar{\varrho}^{\gamma - 1}$.

Inserting the dimensionless variables in the two governing PDEs leads to

$$\frac{\varrho_c}{t_c} \frac{\partial \bar{\varrho}}{\partial \bar{t}} + \frac{\varrho_c U}{L} \bar{\nabla} \cdot (\bar{\varrho} \bar{\boldsymbol{u}}) = 0,$$

$$\frac{\varrho_c U}{t_c} \bar{\varrho} \frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \frac{\varrho_c U^2}{L} \bar{\varrho} \bar{u} \cdot \bar{\nabla} \bar{\boldsymbol{u}} + \frac{\varrho_c}{L} c_0^2 \bar{\varrho}^{\gamma - 1} \bar{\nabla} \bar{\varrho} = 0 \,.$$

The characteristic flow velocity is U so a natural time scale is $t_c = L/U$. This choice leads to the scaled PDEs

$$\frac{\partial \bar{\varrho}}{\partial \bar{t}} + \bar{\nabla} \cdot (\bar{\varrho} \bar{\boldsymbol{u}}) = 0, \tag{4.89}$$

$$\bar{\varrho} \frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \bar{\varrho} \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}} + \mathrm{M}^{-2} \bar{\varrho}^{\gamma - 1} \bar{\nabla} \bar{\varrho} = 0, \tag{4.90}$$

where the dimensionless number

$$\mathrm{M} = \frac{U}{c_0},$$

is known as the *Mach number*. The boundary conditions specify the characteristic velocity U and thereby the Mach number.

4.6.3 The acoustic approximation for sound waves

Wave nature of isentropic flow with small perturbations. A model for sound waves can be based on (4.86) and (4.88), but in this case there are small pressure, velocity, and density *perturbations* from a ground state at rest where $\boldsymbol{u}=0$, $\varrho=\varrho_0$, and $p=p_0=F(\varrho_0)$. Introducing the perturbations $\hat{\varrho}=\varrho-\varrho_0$ and $\hat{\boldsymbol{u}}$, (4.86) and (4.88) take the form

$$\frac{\partial\hat{\varrho}}{\partial t}+\nabla\cdot(\hat{\boldsymbol{u}}(\varrho_0+\hat{\varrho})=0,$$
$$(\varrho_0+\hat{\varrho})\frac{\partial\hat{\boldsymbol{u}}}{\partial t}+(\varrho_0+\hat{\varrho})\hat{\boldsymbol{u}}\cdot\nabla\hat{\boldsymbol{u}}+c_0^2\left(1+\frac{\hat{\varrho}}{\varrho_0}\right)^{\gamma-1}\nabla\hat{\varrho}=0\,.$$

For small perturbations we can linearize this PDE system by neglecting all products of $\hat{\varrho}$ and $\hat{\boldsymbol{u}}$. Also, $1+\hat{\varrho}/\varrho_0\approx 1$. This leaves us with the simplified system

$$\frac{\partial\hat{\varrho}}{\partial t}+\varrho_0\nabla\cdot\hat{\boldsymbol{u}}=0,$$
$$\varrho_0\frac{\partial\hat{\boldsymbol{u}}}{\partial t}+c_0^2\nabla\hat{\varrho}=0\,.$$

Eliminating $\hat{\boldsymbol{u}}$ by differentiating the first PDE with respect to t and taking the divergence of the second PDE gives a standard wave equation for the density perturbations:

$$\frac{\partial^2\hat{\varrho}}{\partial t^2}=c_0^2\nabla^2\hat{\varrho}.$$

Similarly, $\hat{\varrho}$ can be eliminated and one gets a wave equation for $\hat{\boldsymbol{u}}$, also with wave velocity c_0. This means that the sound perturbations travel with velocity c_0.

Basic scaling for small wave perturbations. Let ϱ_c and u_c be characteristic sizes of the perturbations in density and velocity. The density will then vary in $[\varrho_0-\varrho_c,\varrho_0+\varrho_c]$. An appropriate scaling is

$$\bar{\varrho}=\frac{\varrho-\varrho_0}{\varrho_c}$$

such that $\bar{\varrho}\in[-1,1]$. Consequently,

$$\varrho=\varrho_0+\varrho_c\bar{\varrho}=\varrho_0(1+\alpha\bar{\varrho}),\quad \alpha=\frac{\varrho_c}{\varrho_0}\,.$$

Note that the dimensionless α is expected to be a very small number since $\varrho_c\ll\varrho_0$. The velocity, space, and time are scaled as in the previous section. Also note that ϱ_0 and p_0 are known values, but the scales ϱ_c and U are

not known. Usually these can be estimated from perturbations (i.e., sound generation) applied at the boundary.

Inserting the scaled variables in (4.86) and (4.88) results in

$$\frac{\varrho_0}{t_c}\frac{\partial\bar{\varrho}}{\partial\bar{t}}+\frac{\varrho_0 U}{L}\bar{\nabla}\cdot((1+\alpha\bar{\varrho})\bar{\boldsymbol{u}})=0,$$
$$\frac{\varrho_0 U}{t_c}(1+\alpha\bar{\varrho})\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}}+\frac{\varrho_0 U^2}{L}(1+\alpha\bar{\varrho})\bar{u}\cdot\bar{\nabla}\bar{\boldsymbol{u}}+\frac{\varrho_0}{L}c_0^2(1+\alpha\bar{\varrho})^{\gamma-1}\bar{\nabla}\bar{\varrho}=0\,.$$

Since we now model sound waves, the relevant time scale is not L/U but the time it takes a wave to travel through the domain: $t_c=L/c_0$. This is a much smaller time scale than in the previous section because $c_0\gg U$ (think of speaking: the sound travels very fast but one cannot feel the corresponding very small flow perturbation in the air!). Using $t_c=L/u_0$ we get

$$\frac{\partial\bar{\varrho}}{\partial\bar{t}}+\mathrm{M}\bar{\nabla}\cdot((1+\alpha\bar{\varrho})\bar{\boldsymbol{u}})=0,$$
$$(1+\alpha\bar{\varrho})\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}}+\mathrm{M}(1+\alpha\bar{\varrho})\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}}+\mathrm{M}^{-1}(1+\alpha\bar{\varrho})^{\gamma-1}\bar{\nabla}\bar{\varrho}=0\,.$$

With the assumption of small perturbations, M and α are small numbers. We can first approximate $1+\alpha$ by 1. Thereafter, we realize that multiplying by M in the momentum equation gives

$$\mathrm{M}\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}}+\mathrm{M}^2\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}}+(1+\alpha\bar{\varrho})^{\gamma-1}\bar{\nabla}\bar{\varrho}=0$$

The term $\mathrm{M}^2\bar{\boldsymbol{u}}\cdot\bar{\nabla}\bar{\boldsymbol{u}}$ has size M^2 and is very smaller than the first and third term, suggesting that we omit the nonlinear term. We arrive at the following linearized system of PDEs

$$\frac{\partial\bar{\varrho}}{\partial\bar{t}}+\mathrm{M}\bar{\nabla}\cdot\bar{\boldsymbol{u}}=0, \tag{4.91}$$
$$\mathrm{M}\frac{\partial\bar{\boldsymbol{u}}}{\partial\bar{t}}+\bar{\nabla}\bar{\varrho}=0, \tag{4.92}$$

The velocity $\boldsymbol{u}$ can be eliminated taking the time derivative of (4.91) and the divergence of (4.92):

$$\frac{\partial^2\bar{\varrho}}{\partial\bar{t}^2}=\bar{\nabla}^2\bar{\varrho}, \tag{4.93}$$

which is nothing but a standard dimensionless wave equation with unit wave velocity. Similarly, we can eliminate ϱ by taking the divergence of (4.91) and the time derivative of (4.92):

$$\frac{\partial^2 \bar{\boldsymbol{u}}}{\partial \bar{t}^2} = \bar{\nabla}^2 \bar{\boldsymbol{u}}\,. \tag{4.94}$$

We also observe that there are no physical parameters in the scaled wave equations. The M parameter can be eliminated from (4.91)-(4.92) as well by introducing a new unknown $\bar{\boldsymbol{v}} = M\bar{\boldsymbol{u}}$.

4.7 Surface gravity waves

4.7.1 Equations for surface waves

Provided the Weber number, see section 4.2.7, is sufficiently small, capillary effects may omitted and surface waves are governed by gravity. For large Reynolds numbers viscous effects may also be ignored (save in boundary layers close to the bottom or the surface of the fluid). The flow of an incompressible homogeneous fluid under these assumptions is governed by the Euler equations of motion on the form

$$\nabla \cdot \boldsymbol{u} = 0, \tag{4.95}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{1}{\rho}\nabla p + g\mathbf{k} = 0\,. \tag{4.96}$$

When the free surface position is described as $z = \eta(x,y,t)$, with z as the vertical coordinate, the boundary conditions at the surface read

$$p = p_s, \tag{4.97}$$

$$\frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta = w, \tag{4.98}$$

where p_s is the external pressure applied to the surface. At the bottom, $z = -h(x,y)$, there is the no-flux condition

$$\frac{\partial h}{\partial x}u + \frac{\partial h}{\partial y}v = -w\,.$$

In addition to ρ and g we assume that a typical depth h_c, a typical wavelength λ_c, and a typical surface elevation A, which then per definition is a scale for η, are the given parameters. From these we must derive scales for the coordinates, the velocity components, and the pressure.

First, it is instructive to define a typical wave celerity, c_c, which must be linked to the length and time scale according to $c_c = \lambda_c/t_c$. Since there is no other given parameter that matches the mass dimension of ρ we express c_c

in terms of A, λ_c, h_c, and g. Most of the work on waves in any discipline of physics is devoted to linear or weakly nonlinear waves and the wave celerity must be presumed to remain finite as A goes to zero (see, for instance, Section 4.6.3). Hence, we may assume that c_c must depend on g and either h_c or λ_c. Next, the two horizontal directions are equivalent with regard to scaling, implying that we have a common velocity scale, U, for u and v, a common length scale L for x and y. The obvious choice for L is λ_c, while the "vertical quantities" w and z have scales W and Z, respectively, which may differ from the horizontal counterparts. However, we assume that also the length scale Z remains finite as $A \to 0$ and hence is independent of A. This is less obvious for Z than for c_c and t_c, but may eventually be confirmed by the existence of linear solutions when solving the equation set. From the linear part of (4.98) and (4.95) we obtain two relations between velocity and coordinate scales by demanding the non-dimensionalized terms to be of order unity

$$\frac{A}{t_c} = W, \quad \frac{U}{L} = \frac{W}{Z}\,. \tag{4.99}$$

These relations are indeed useful, but they do not suffice to establish the scaling.

The pressure may regarded as the sum of a large equilibrium part, balancing gravity, and a much smaller dynamic part associated with the presence of waves. To make the latter appear in the equations we define the dynamic pressure, p_d, according to

$$p = p_s - \rho g z + p_d,$$

and the pressure scale $p_c = \rho g A$ for p_d then follows directly from the surface condition (4.97).

The equation set will be scaled according to

$$\bar{t} = \frac{t}{t_c},\ \bar{x} = \frac{x}{L},\ \bar{y} = \frac{y}{L},\ \bar{z} = \frac{z}{Z},\ \bar{\eta} = \frac{\eta}{A},\ \bar{u} = \frac{u}{U},\ \bar{v} = \frac{v}{U},\ \bar{w} = \frac{w}{W},\ \bar{p}_d = \frac{p_d}{p_c}\,.$$

In the further development of the scaling we focus on two limiting cases, namely deep and shallow water.

Waves in deep water. Deep water means that $h_c \gg \lambda_c$. Presumably the waves will not feel the bottom, and h as well as h_c are removed from our equations. The bottom boundary condition is replaced by a requirement of vanishing velocity as $z \to -\infty$. Consequently, c_c must depend upon λ_c and g, leaving us with $c_c = \sqrt{g\lambda_c}$ and $Z = \lambda_c = L$ as the only options. Then, $t_c = \sqrt{\lambda_c/g}$ and (4.99) implies $U = W = c_0 \frac{A}{\lambda_c} = \epsilon c_0$, where we have introduced the non-dimensional number

$$\epsilon = \frac{A}{\lambda_c},$$

which is the wave steepness. The equality of the horizontal and the vertical scale is consistent with the common knowledge that the particle orbits in deep water surface waves are circular.

The scaled equation are now expressed with ϵ as sole dimensionless number

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = 0, \tag{4.100}$$

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \epsilon \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\boldsymbol{u}} + \bar{\nabla} \bar{p}_d = 0\,. \tag{4.101}$$

The surface conditions, at $z = \epsilon\eta$, become

$$\bar{p}_d = \bar{\eta}, \tag{4.102}$$

$$\frac{\partial \bar{\eta}}{\partial \bar{t}} + \epsilon \bar{\boldsymbol{u}} \cdot \bar{\nabla} \bar{\eta} = \bar{w}, \tag{4.103}$$

while the bottom condition is replaced by

$$\bar{\boldsymbol{u}} \to 0, \tag{4.104}$$

as $\bar{z} \to -\infty$.

Long waves in shallow water. In this case the basic assumption is that the wavelength is large compared to the depth: $\lambda_c \gg h_c$. In analogy with the reasoning above, we the presume that the speed of the waves remains finite as $\lambda_c \to \infty$. Then, c_c must be based on g and h_c, which leads to $c_c = \sqrt{gh_c}$ and $t_c = \lambda_c/\sqrt{gh_c}$. The natural choice for the vertical length scale is now the depth; $Z = h_c$. Application of (4.99) then leads to $W = c_c A/\lambda_c$ and $U = c_c A/h_c$.

Introducing the dimensionless numbers

$$\alpha = \frac{A}{h_c}, \quad \mu = \frac{h_c}{\lambda_c},$$

we rewrite the velocity scales as

$$W = \mu\alpha c_c, \quad U = \alpha c_c\,.$$

We observe that $W \ll U$ for shallow water and that particle orbits must be elongated in the horizontal direction.

The equation set is now most transparently written by introducing the horizontal velocity $\bar{\boldsymbol{u}}_h = \bar{u}\mathbf{i} + \bar{v}\mathbf{j}$ and the corresponding vertical component of the gradient operator, ∇_h:

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}}_h + \frac{\partial \bar{w}}{\partial \bar{z}} = 0, \tag{4.105}$$

$$\frac{\partial \bar{\boldsymbol{u}}}{\partial \bar{t}} + \alpha \bar{\boldsymbol{u}} \cdot \bar{\nabla}_h \bar{\boldsymbol{u}}_h + \alpha \bar{w} \frac{\partial \bar{\boldsymbol{u}}_h}{\partial \bar{z}} + \bar{\nabla}_h \bar{p}_d = 0, \tag{4.106}$$

$$\mu^2 \left(\frac{\partial \bar{\boldsymbol{w}}}{\partial \bar{t}} + \alpha \bar{\boldsymbol{u}} \cdot \bar{\nabla}_h \bar{\boldsymbol{w}} + \alpha \bar{w} \frac{\partial \bar{w}}{\partial \bar{z}} \right) + \frac{\partial \bar{p}_d}{\partial \bar{z}} = 0.. \tag{4.107}$$

Surface conditions, at $z = \alpha\eta$, now become

$$\bar{p}_d = \bar{\eta}, \tag{4.108}$$

$$\frac{\partial \bar{\eta}}{\partial \bar{t}} + \alpha \bar{\boldsymbol{u}}_h \cdot \bar{\nabla}_h \bar{\eta} = \bar{w}, \tag{4.109}$$

while the bottom condition is invariant with respect to the present scaling

$$\bar{\nabla}_h \cdot \boldsymbol{u}_h = -\bar{w}\,. \tag{4.110}$$

An immediate consequence is that $\bar{p}_d$ remains equal to $\bar{\eta}$ throughout the water column when $\mu^2 \to 0$, which imply that the pressure is hydrostatic. The above set of equations is the common starting point for perturbation expansions in ϵ and μ^2 that lead to shallow water, KdV, and Boussinesq type equations.

4.8 Exercises

Exercise 4.1: Comparison of vibration models for elastic structures

The time scale for displacement in elastic structures is, according to Section 4.1.1, $t_c = L\sqrt{\varrho/\mu}$ if we assume constant density ϱ and constant shear modulus μ for the structure. The purpose of this exercise is to compare this time scale with the time scales of related models.

a) Longitudinal waves in a bar can be modeled approximately by the PDE

$$\varrho \frac{\partial^2 u}{\partial t^2} + E \frac{\partial^2 u}{\partial x^2} = 0,$$

where $u(x,t)$ is the displacement along the bar, and E is Young's modulus, related to the shear modulus μ through

$$E = 2\mu(1+\nu),$$

where $\nu \in (0, 0.5]$ is Poisson's ratio. Find the time scale for the longitudinal waves and compare with the t_c for displacements in a three-dimensional body.

Solution. Introducing dimensionless dependent and independent variables the usual way gives us

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + \frac{t_c^2 E}{\varrho L^2} \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = 0,$$

where L is the length scale, typically the length of the bar. The natural choice of t_c is to make the coefficient unity,

$$t_c = L\sqrt{\frac{\varrho}{E}} = L\sqrt{\frac{\varrho}{\mu}} \frac{1}{\sqrt{2(1+\nu)}} \approx 0.6L\sqrt{\frac{\varrho}{\mu}},$$

if we take $\nu = 0.3$ as a typical value.

b) Vertical vibrations of a beam are governed by the PDE

$$\rho \frac{\partial^2 u}{\partial t^2} + EI \frac{\partial^4 u}{\partial x^4} = 0,$$

where $u(x,t)$ is the vertical displacement along the beam, ρ is the mass per length of the beam, E is Young's modulus, and I is the moment of inertia. For a rectangular cross section of width b and height h, $I = \frac{1}{12}bh^3$. Compare the time scale for these vibrations with the time scale t_c for three-dimensional elasticity.

Solution. The dimensionless equation becomes

$$\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} + \frac{t_c^2 EI}{\rho L^4} \frac{\partial^4 \bar{u}}{\partial \bar{x}^4} = 0\,.$$

The natural choice of t_c is

$$t_c = L^2\sqrt{\frac{\rho}{EI}} = L\sqrt{\frac{\varrho}{\mu}} L\sqrt{\frac{12}{2h^2(1+\nu)}} \approx 2\frac{L}{h} L\sqrt{\frac{\varrho}{\mu}},$$

where we have set $\nu = 0.3$ and used that $\rho = \varrho bh$ for a rectangular cross section. For a beam, $L \gg h$, so the time scale for vertical vibrations of a beam is much larger than the time scale for elastic waves in a three-dimensional body.

Exercise 4.2: A model for quasi-static poro-elasticity

Flow through a porous elastic medium may induce stress and deformation. This process is known as poro-elasticity and is governed by the following equations for a homogeneous medium:

$$(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})+\mu\nabla^2\boldsymbol{u}=-\alpha\nabla p-\varrho\boldsymbol{f}, \tag{4.111}$$

$$S\frac{\partial p}{\partial t}=\frac{K}{\mu_f}\nabla^2 p+\alpha\frac{\partial}{\partial t}\nabla\cdot\boldsymbol{u}, \tag{4.112}$$

where $\boldsymbol{u}(\boldsymbol{x},t)$ is the displacement field, λ and μ are Lame's elasticity parameters, $\alpha\in[0,1]$, $\boldsymbol{f}$ is the body force, here assumed constant (usually gravity, $\boldsymbol{f}=-g\boldsymbol{k}$, S is a so-called storage coefficient, $p(\boldsymbol{x},t)$ is the fluid pressure, K is the medium's permeability, μ_f is the dynamic viscosity of the fluid, and ϱ is the density of the fluid-solid mixture:

$$\varrho=(1-\phi)\varrho_s+\phi\varrho_f,$$

with ϱ_f being the density of the fluid, ϱ_s the density of the solid, and ϕ the porosity of the elastic medium. The equations are known as Biot's equations of poro-elasticity and written here in a quasi-static form where elastic waves are neglected.

Scale this partial differential equation model, assuming that λ, μ, α, $\boldsymbol{f}$, ϱ, ϕ, ϱ_s, ϱ_f, S, μ_f, and K are all constants.

Hint. The model is very similar to the equations of thermo-elasticity in Section 4.1.5.
Filename: `poroelasticity`.

Exercise 4.3: Suggestions...

Projects:

- Poisson problem, membrane, see Nayfeh chapter 1, stationary and time
- diffusion with oscillating $f(x,t)$, oscillating pressure in tube (channel first)
- Helmholtz
- boundary layer fluid flow problem, with/without thermal effects

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