Class 13: Normal distributions

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Today you will learn more about (normal) distributions

Fundamentals of probability theory

Standardization

$$z = \frac{x - \mu}{\sigma}$$

Normal distributions

The mean $\bar{x}$ is the average value

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n} \sum x_i$$

The standard deviation s measures spread

variance: $$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1}$$

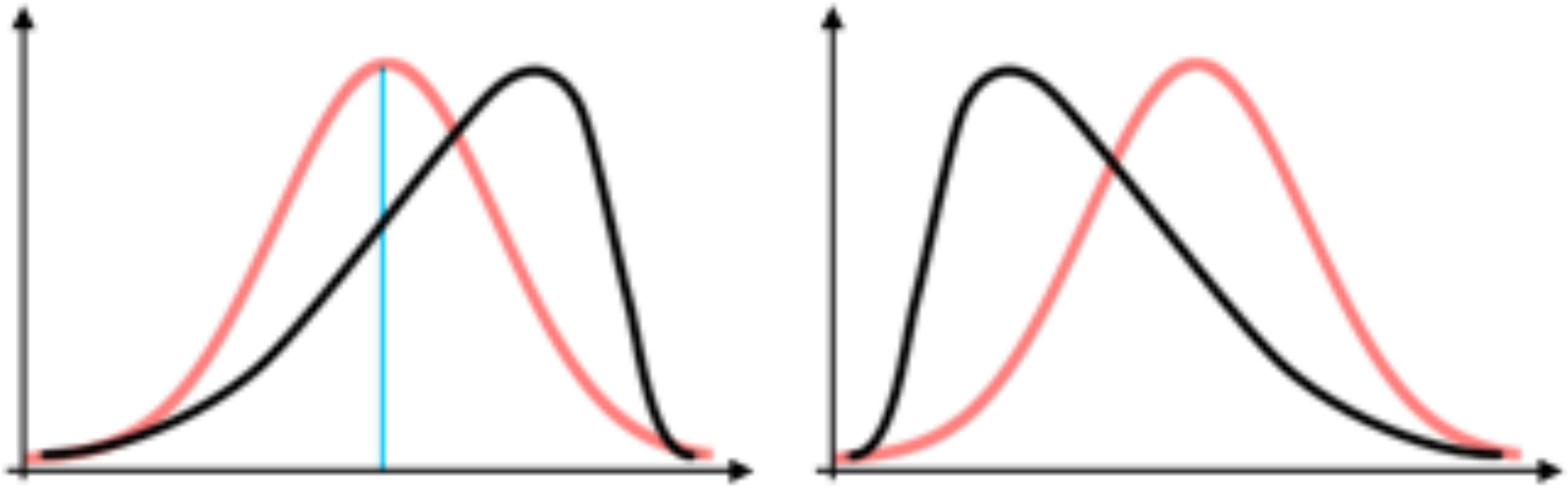
$$= \frac{1}{n-1} \sum (x_i - \bar{x})^2$$

$$s = \sqrt{\frac{1}{n-1} \sum (x_i - \bar{x})^2}$$

Mean and deviation are often not useful measures

1) They are not robust to outliers

2) They are inadequate for skewed distributions:



Negative Skew
(large tail to the left)

Positive Skew
(large tail to the right)

Our recipe for exploring data on a single quantitative variable:

1) Plot a histogram

2) Look for the overall pattern and deviations, outliers

3) Calculate a numerical summary to describe center and spread

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4) Sometimes the distribution is so regular that
we can describe it by a smooth curve

Probability theory

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An experiment produces one outcome (event).

You see only ☼

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You see neither

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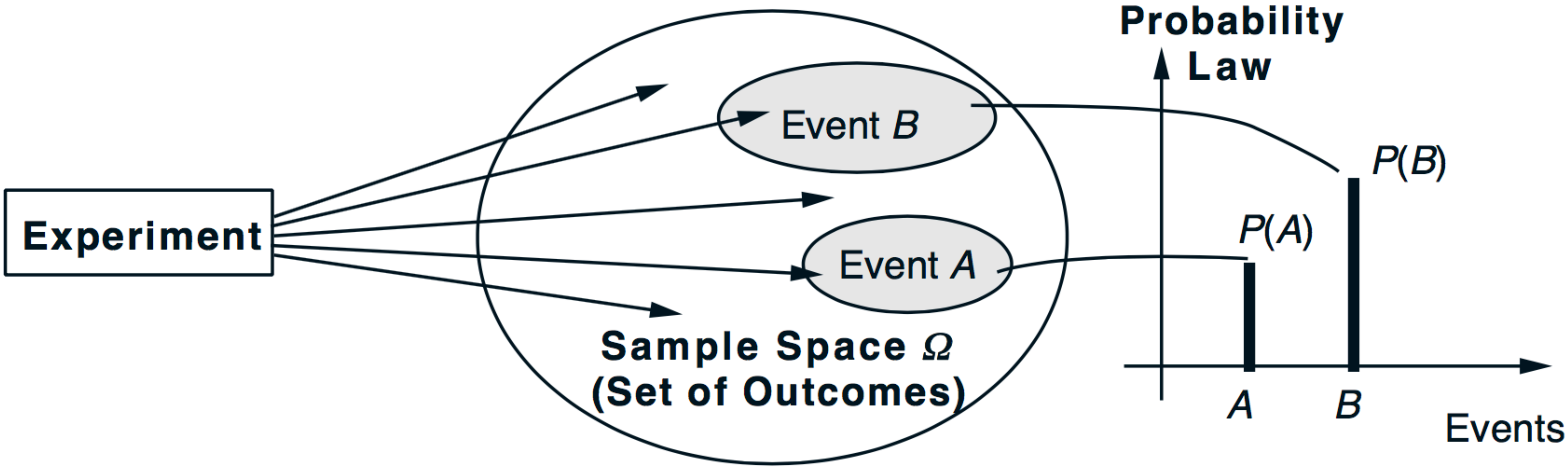
You see both

The set of all possible outcomes is the sample space Ω

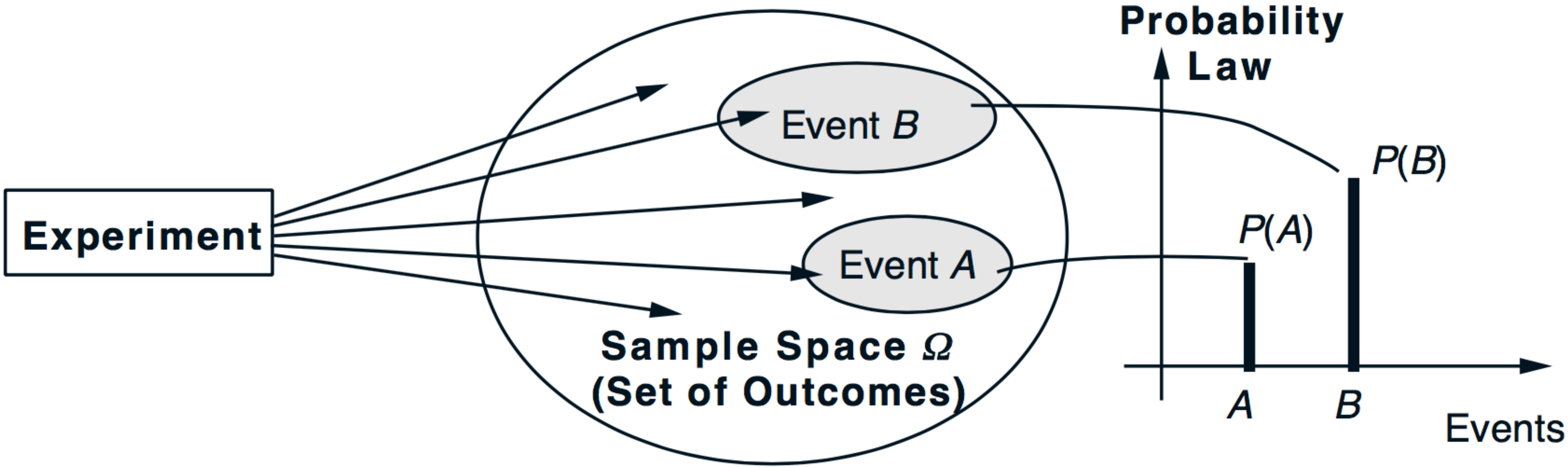
$$\Omega = \{\text{You see only ☼},\\ \text{You see only ☾},\\ \text{You see neither},\\ \text{You see both}\}$$

The possible outcomes are mutually exclusive.

Running an experiment repeatedly allows us to assign a probability $P(A)$ to each event A



This mathematical description of an uncertain situation is called probabilistic model



Probabilities must satisfy 3 axioms

1) Nonnegativity $P(A) \geq 0$

2) Additivity $P(A \cup B) = P(A) + P(B)$
for disjoint sets A and B

3) Normalization $P(\Omega) = 1$

A random variable X assigns to each outcome a numerical value x, formalizing the notion of a measurement

Example: Rolling two 4-sided dice
where X is the maximum roll

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EXERCISE: 5 min in groups of 3:

What is the sample space?
What is x for each possible event?

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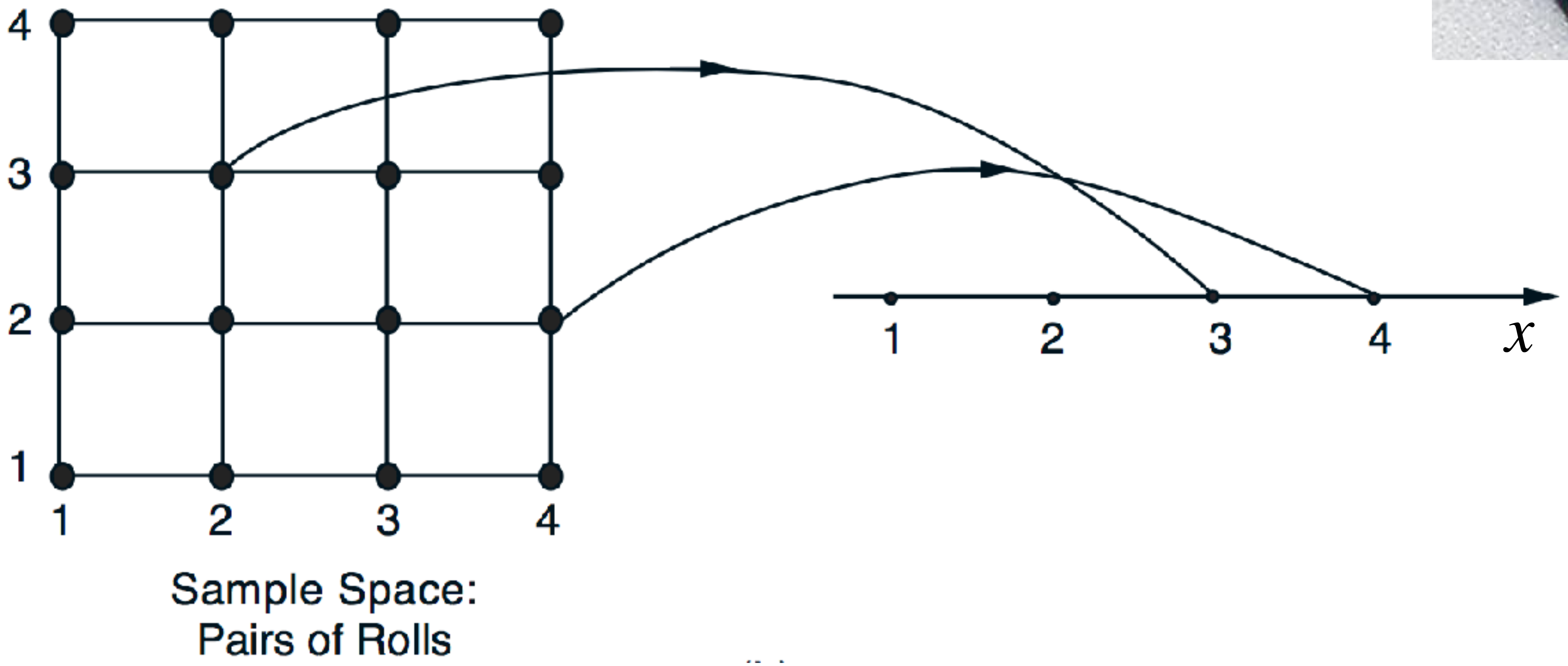
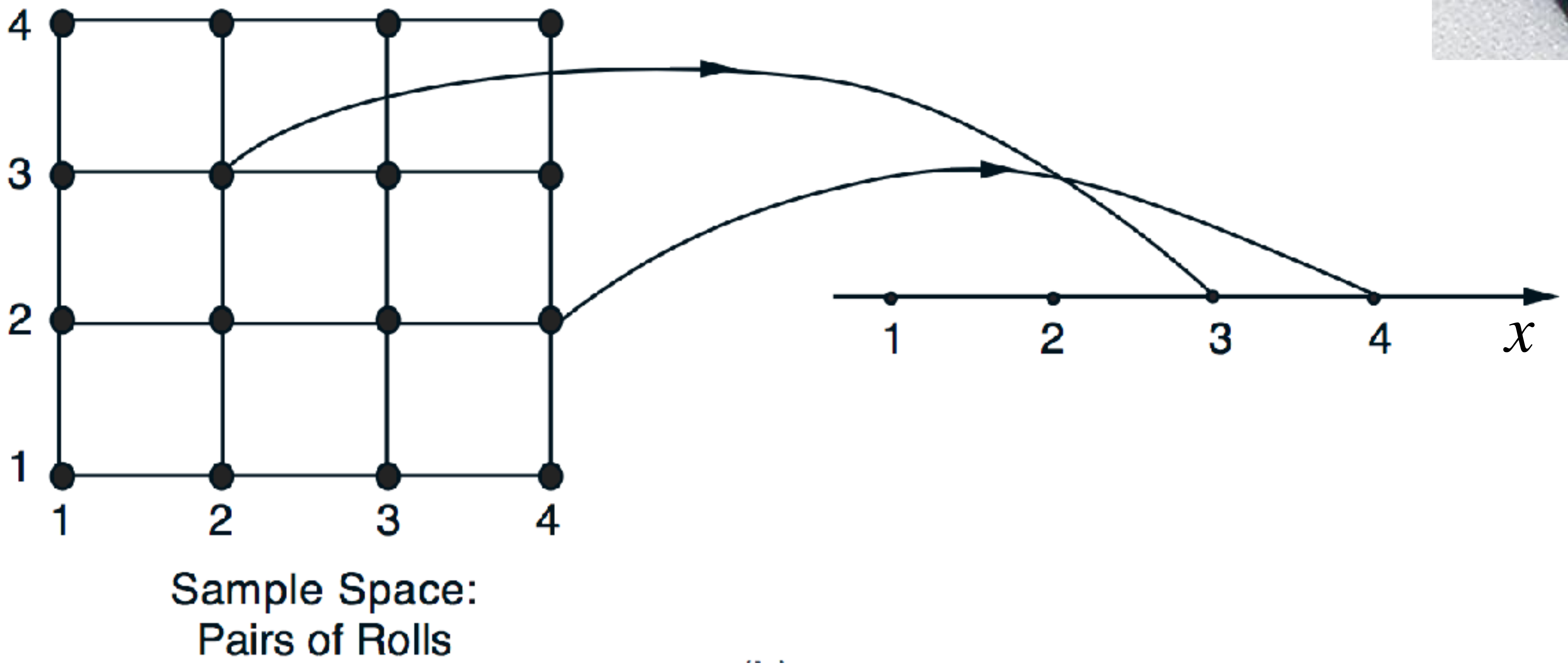
Example: Rolling two 4-sided dice where X is the maximum roll



This example is discrete: x can take only certain values

Example: Rolling two 4-sided dice
where X is the maximum roll



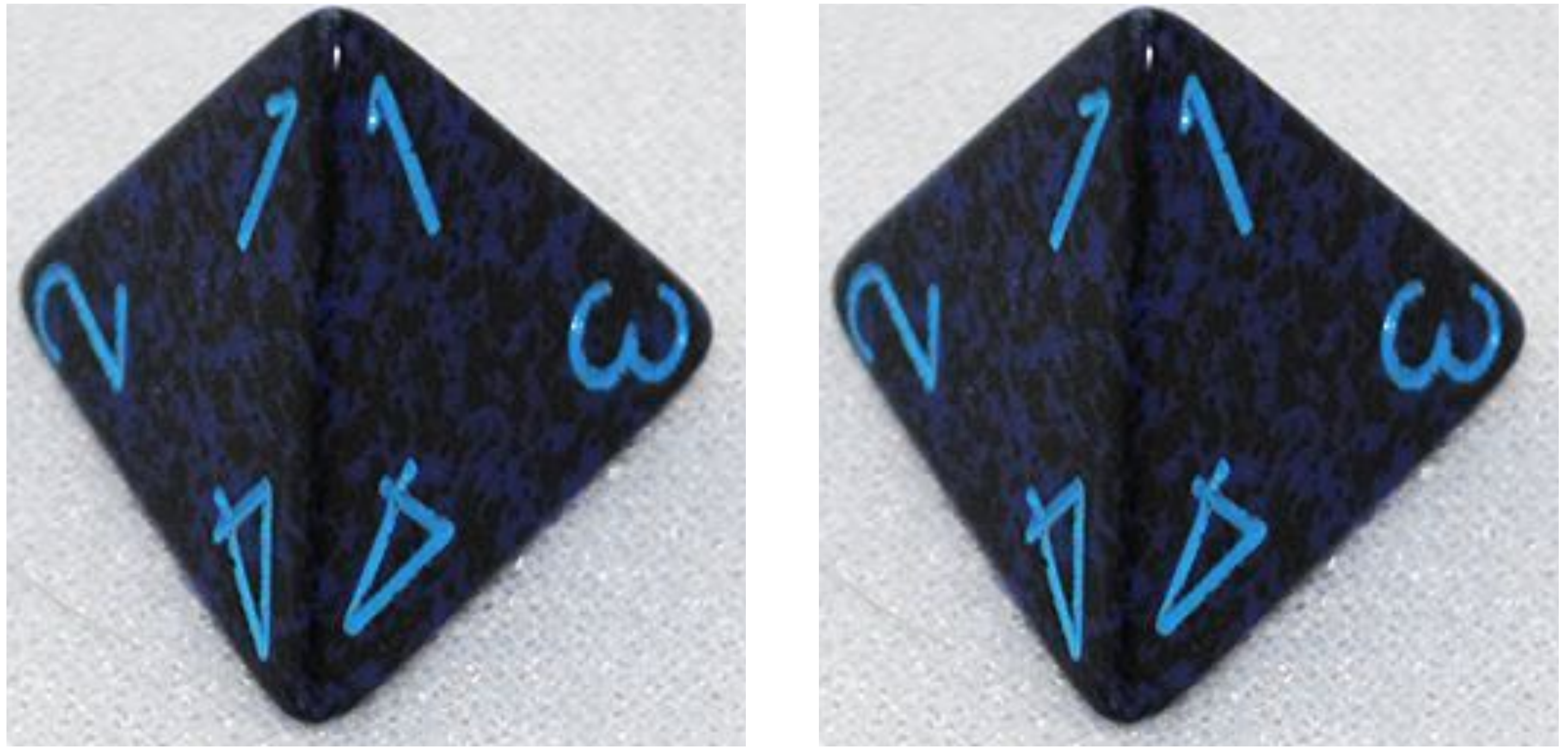
When X is a discrete random variable and the probability $p(x)$ is known for all possible x, then $p(x)$ is called the probability mass function (PMF)

Example: Rolling two 4-sided dice where X is the maximum roll

$p(x)$ is short for: $P(X = x)$

When X is a discrete random variable and the probability $p(x)$ is known for all possible x, then $p(x)$ is called the probability mass function (PMF)

Example: Rolling two 4-sided dice where X is the maximum roll

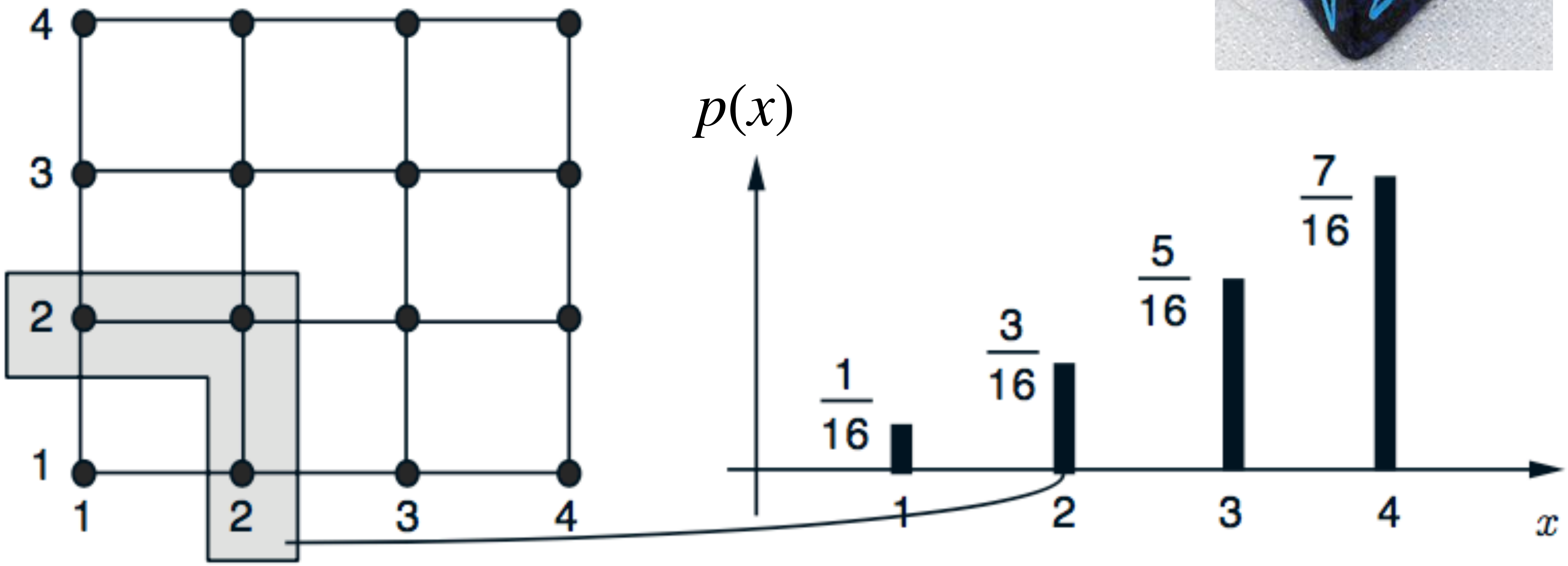


EXERCISE: 3 min in groups of 3:

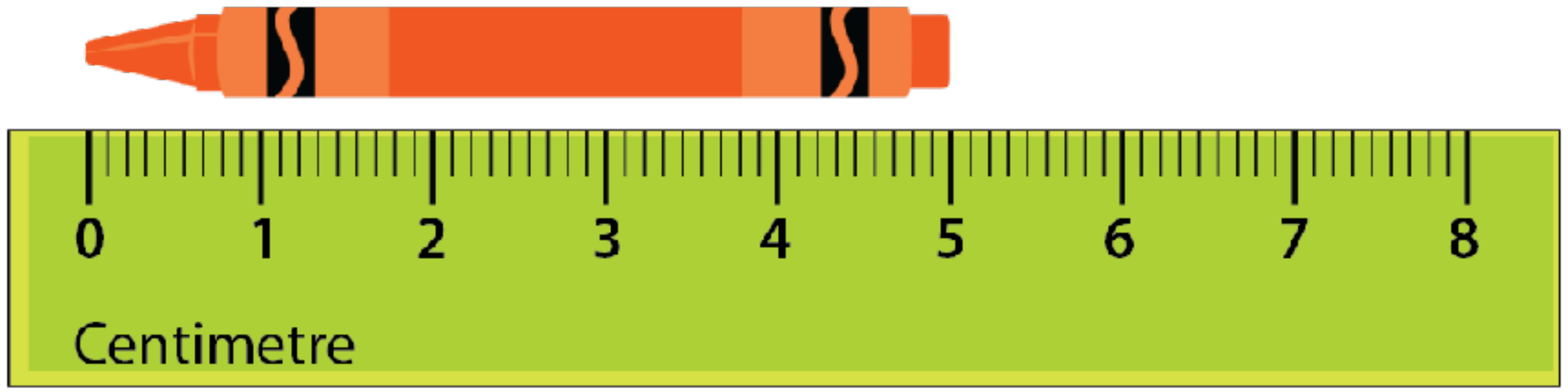
What is $p(x)$ here?

When X is a discrete random variable and the probability $p(x)$ is known for all possible x, then $p(x)$ is called the probability mass function (PMF)

Example: Rolling two 4-sided dice where X is the maximum roll

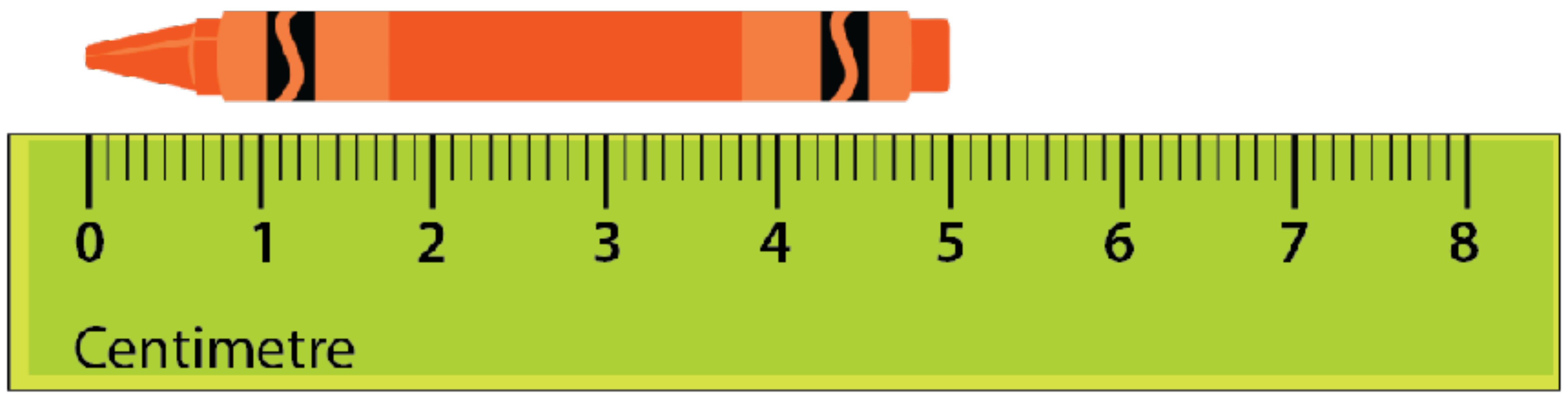


When X can take any value along an interval, it is continuous



Here the probability of measuring a specific value is effectively 0.

When X can take any value along an interval, it is continuous

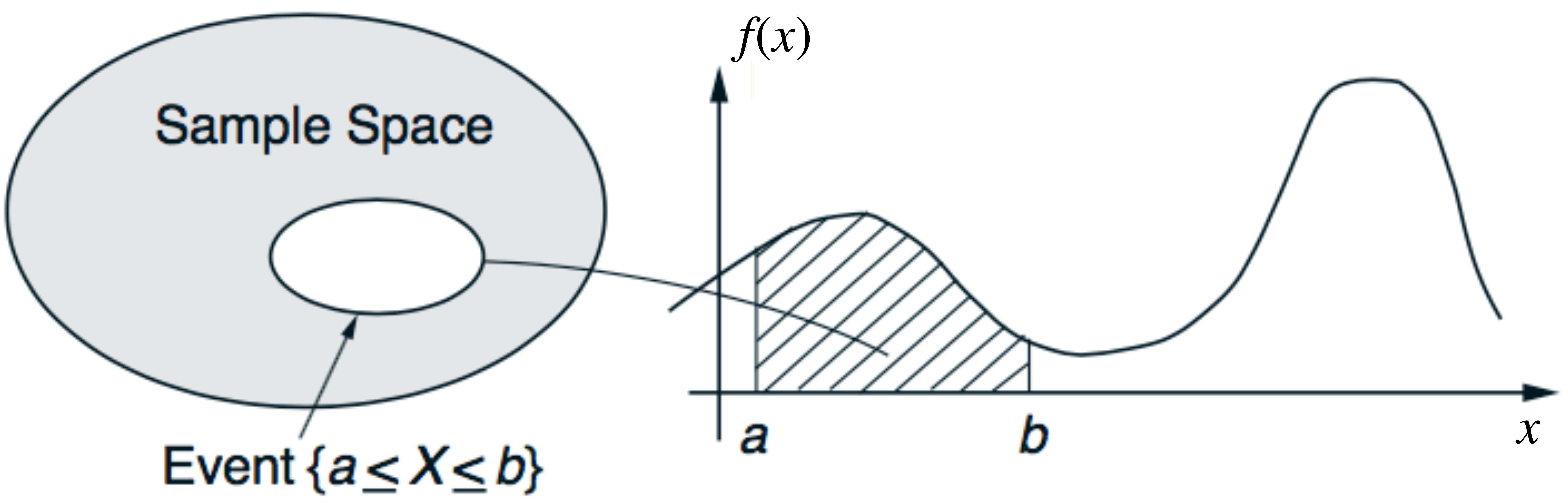


Here the probability of measuring a specific value is effectively 0.

So we use intervals. We ask: What is $P(a \leq X \leq b)$?

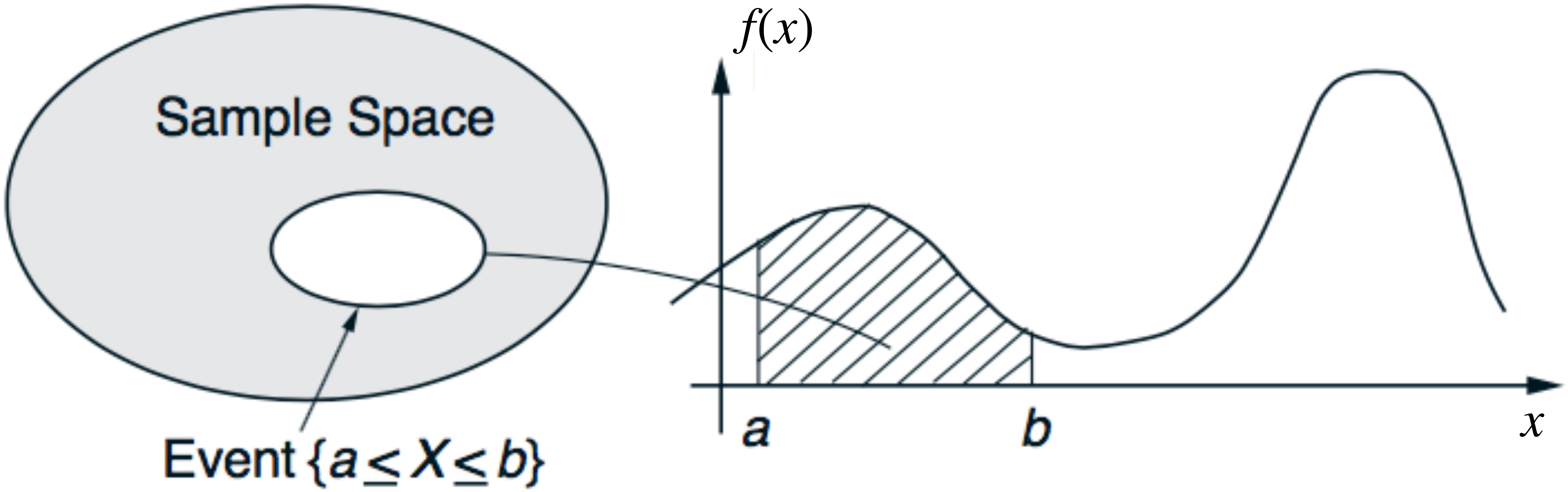
When X is a continuous random variable, we get the probability of X falling into an interval by integrating the probability density function (PDF) $f(x)$

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

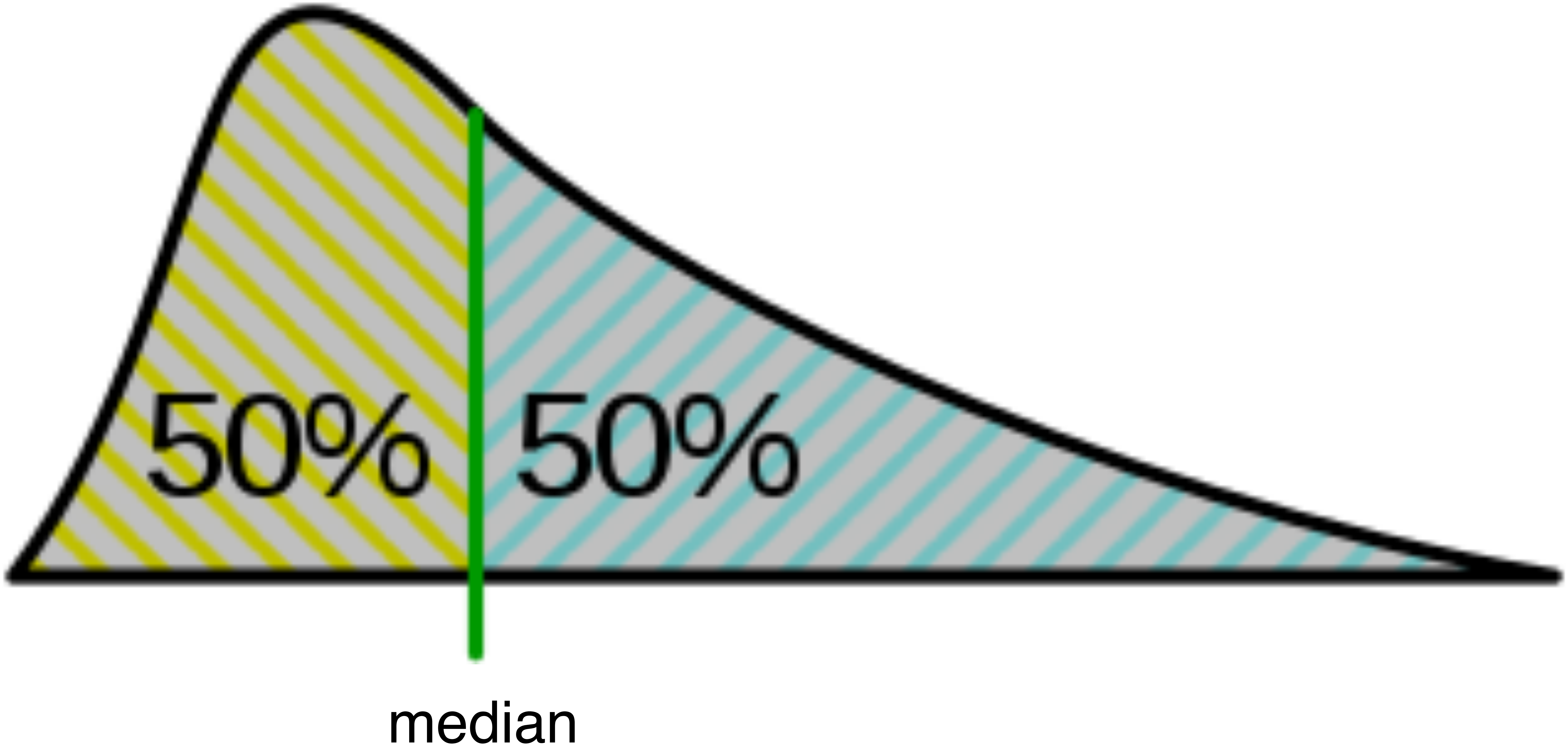


A density curve must be everywhere non-negative and the entire area below it must be 1

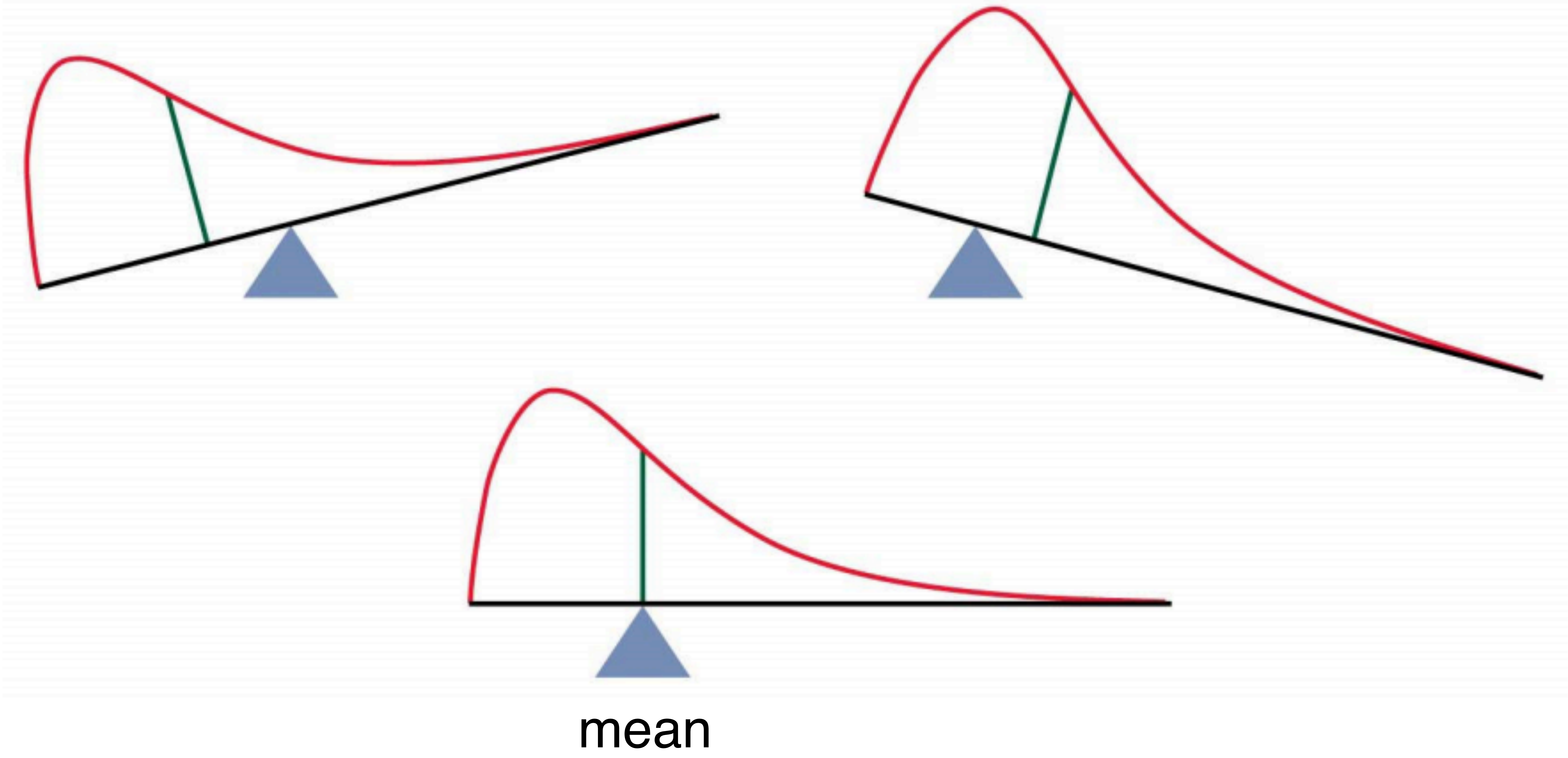
$$\int_{-\infty}^{\infty} f(x)dx = 1$$



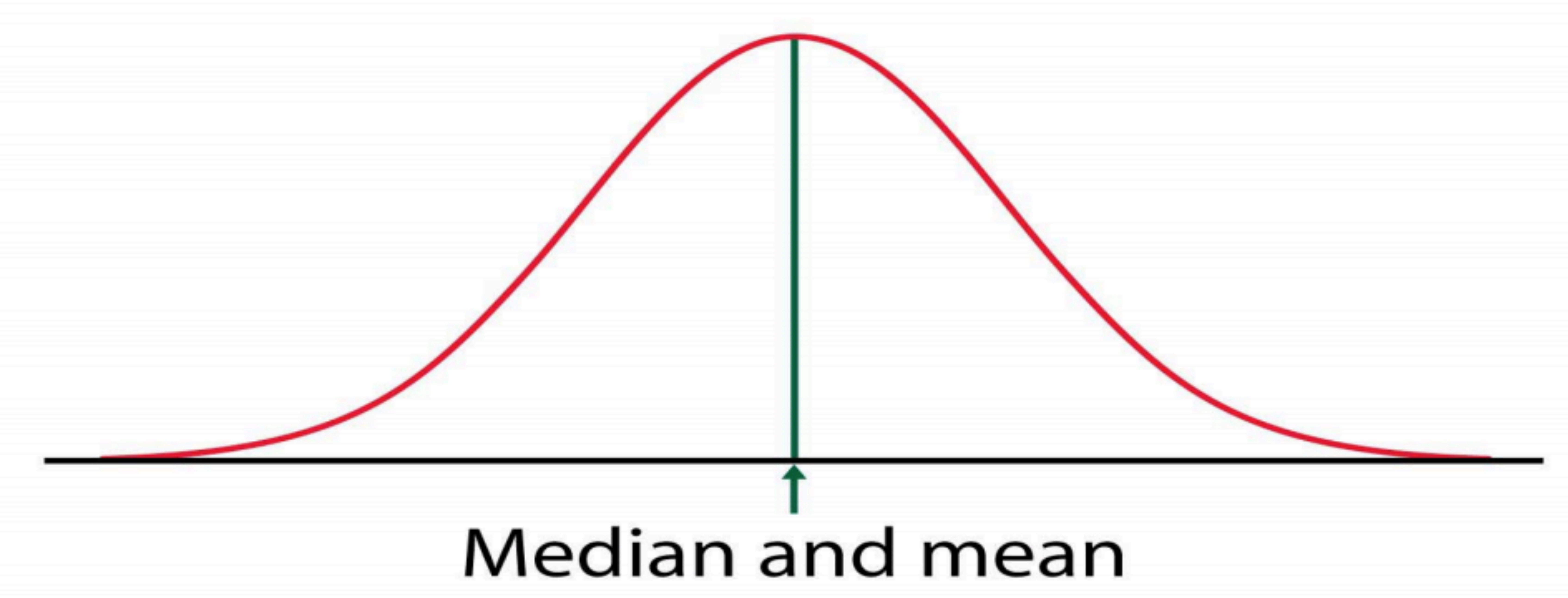
The median of a density curve cuts the area in half

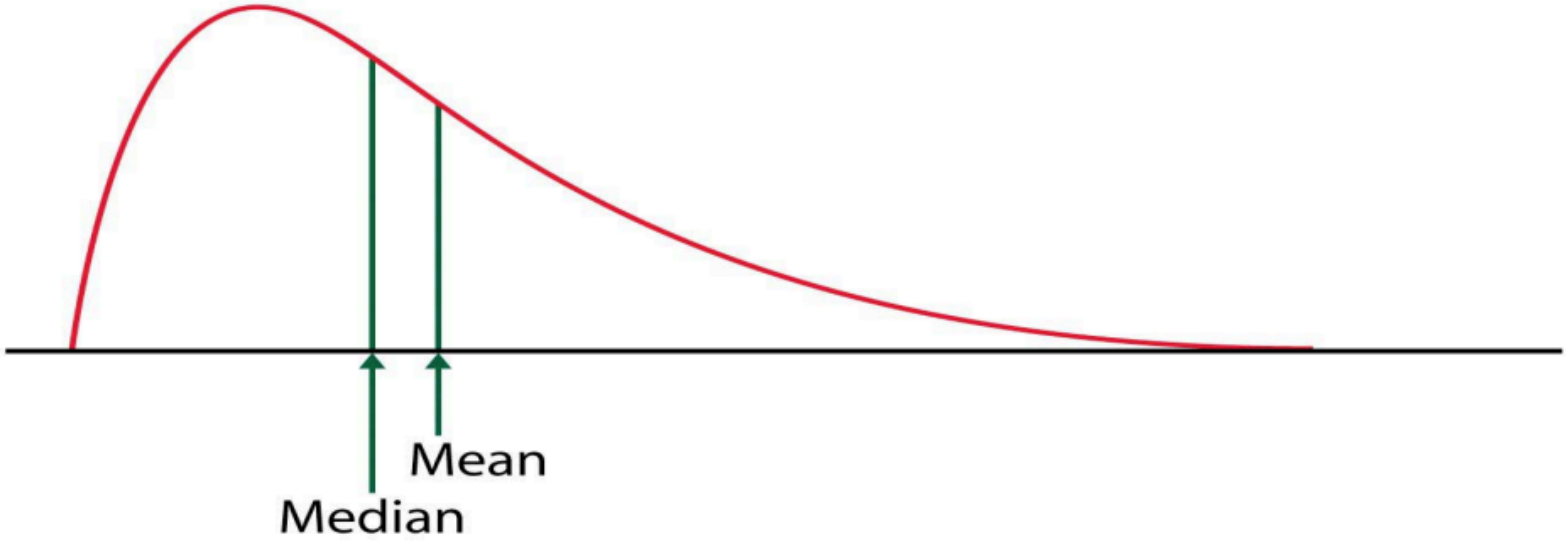


The mean of a density curve is the balance point

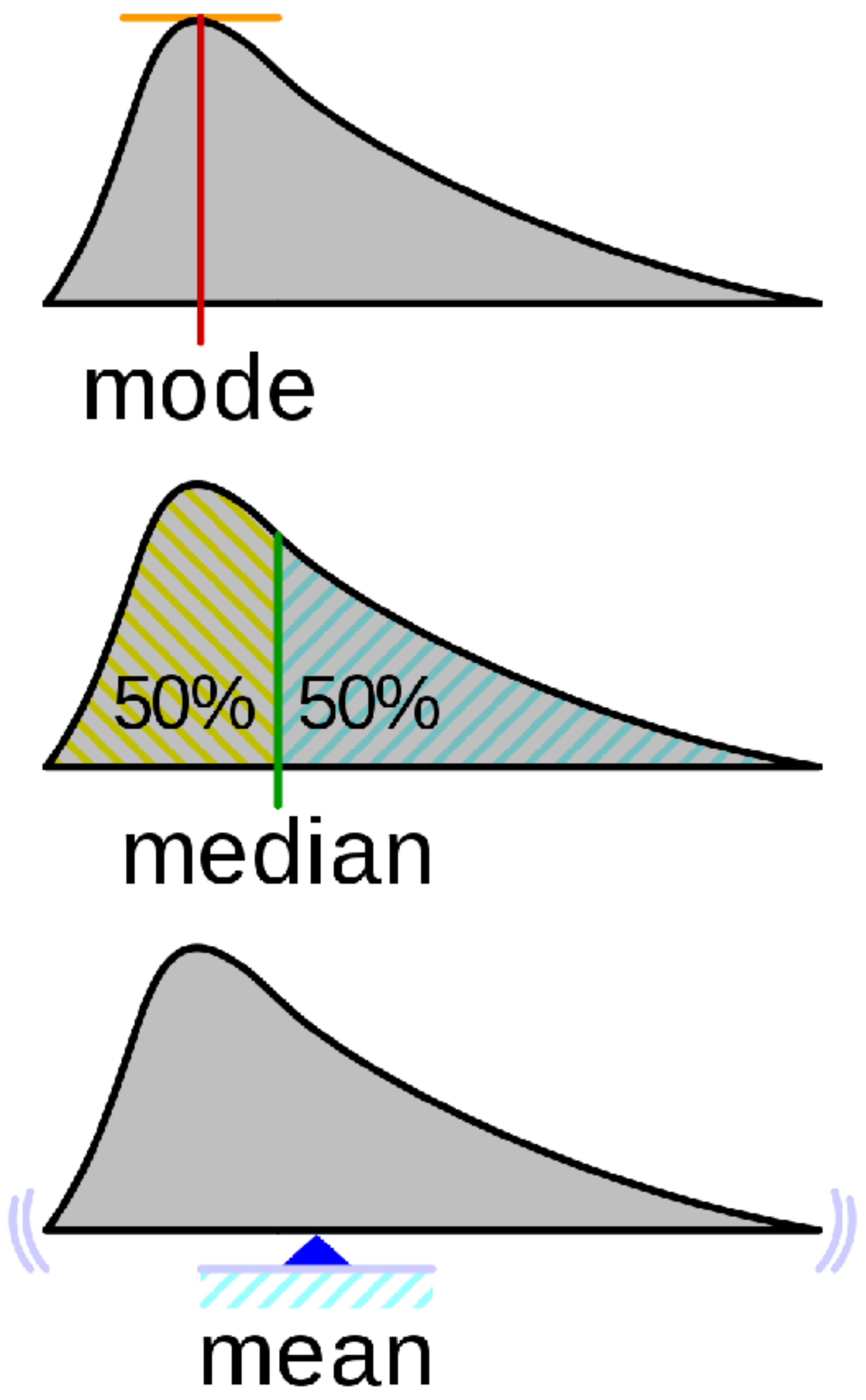


Often mean and median are not the same

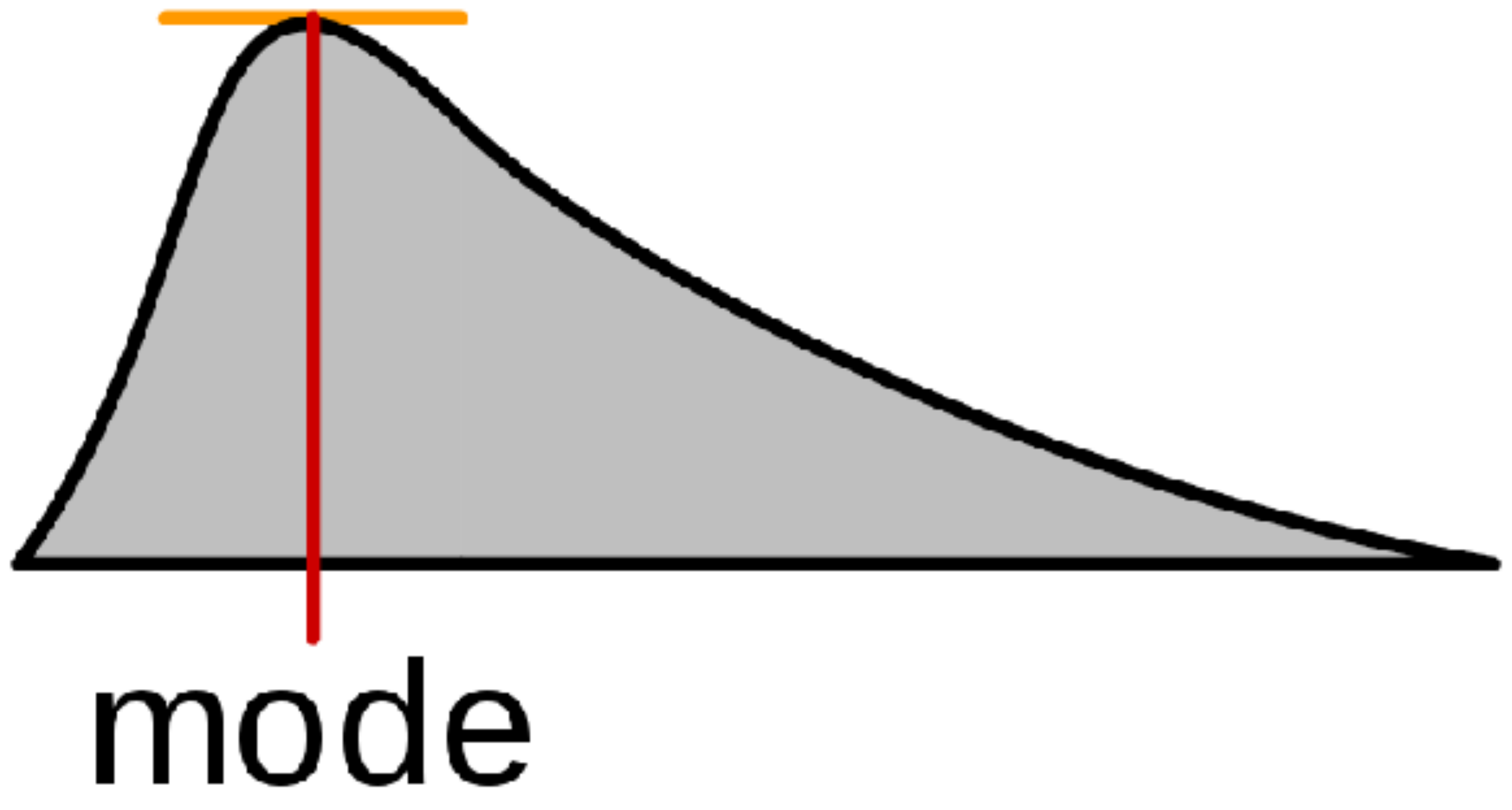




The mode is the value that appears most often



A unimodal distribution has one "hump"
A multimodal distribution has multiple "humps"



unimodal

bimodal

trimodal

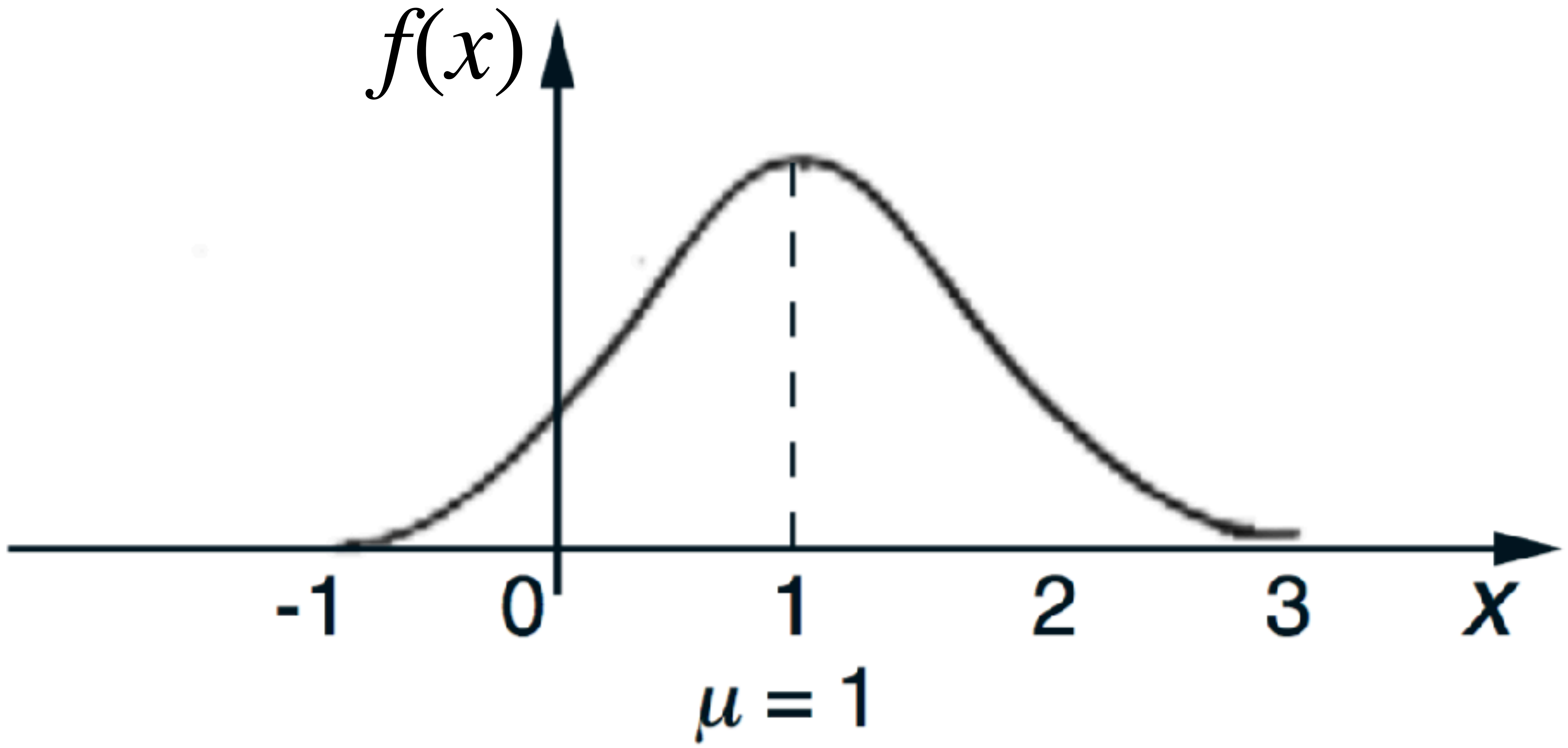
Jupyter

A normal distribution with parameters μ and σ is a continuous random variable X with the PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

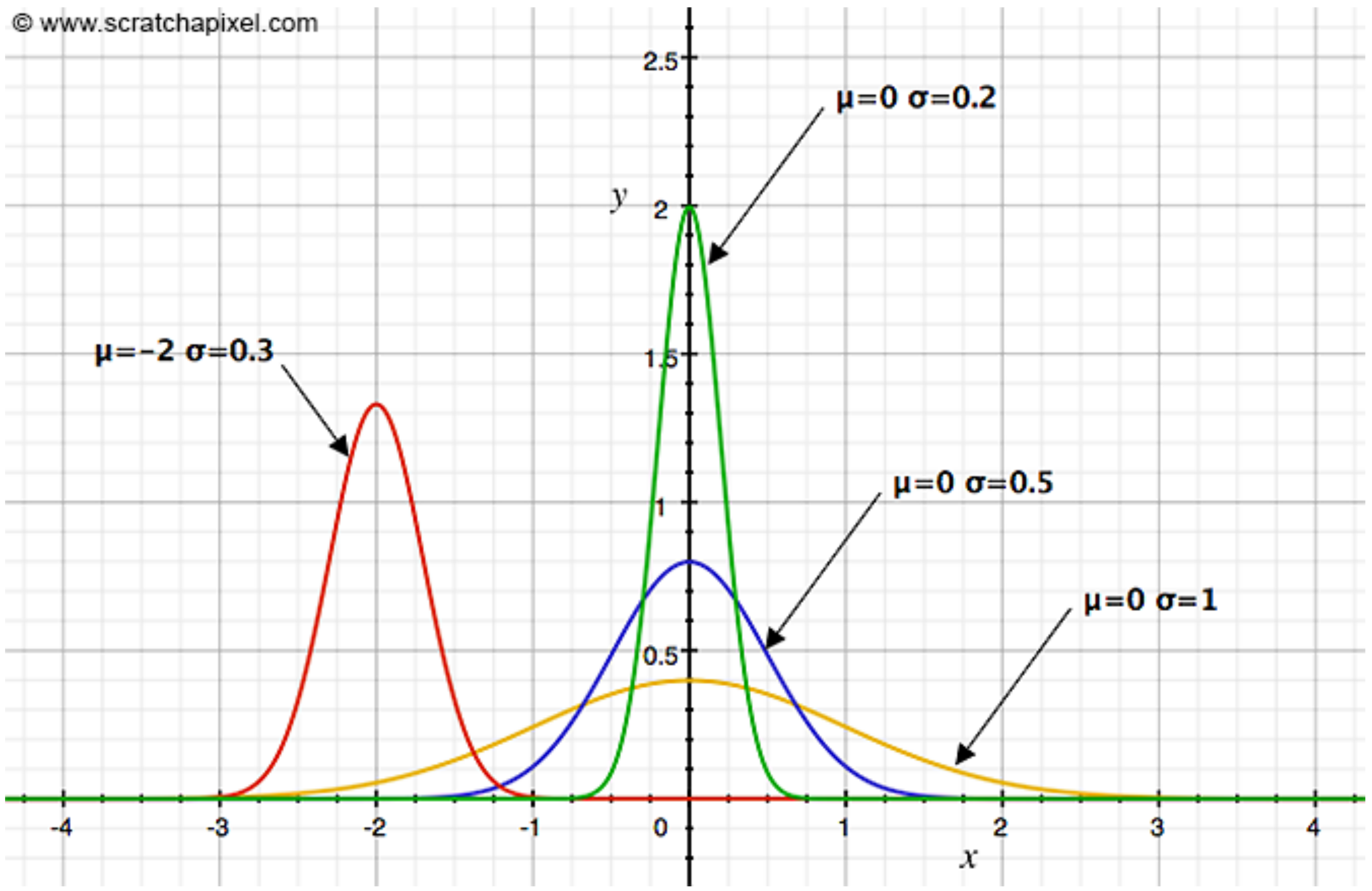
A normal distribution with parameters μ and σ^2 is a continuous random variable X with the PDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$



$$X \sim \mathcal{N}(\mu, \sigma^2)$$

The two parameters completely determine the curve



A parameter is numerical characteristic of a statistical model

Normal distributions are important in statistics

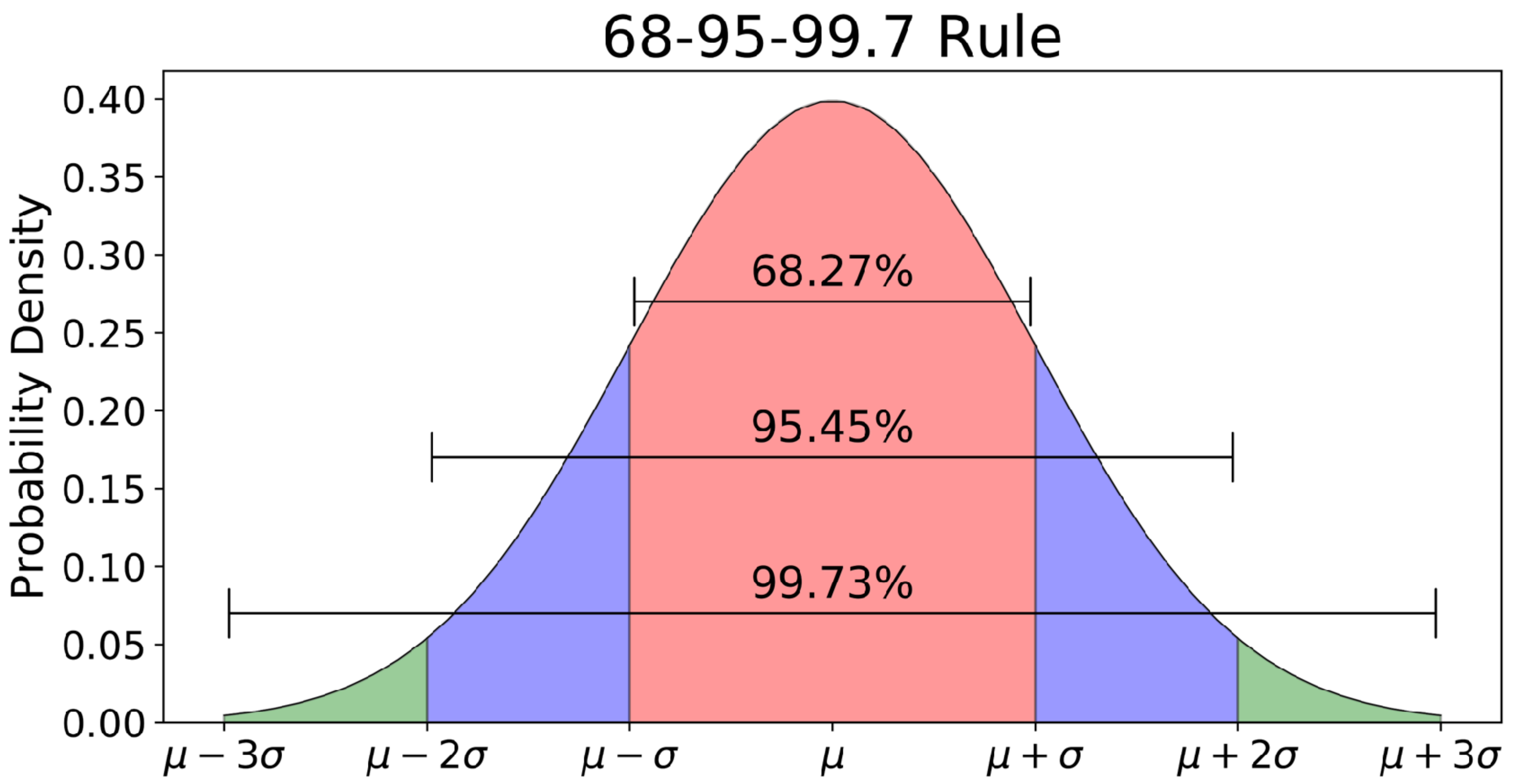
1) They describe well some real data sets

2) They approximate well many chance outcomes

3) They are useful for statistical inference
   of many symmetric distributions

Real data are often VERY MUCH NOT normal

*“Many years ago [in 1893] I called the Laplace-Gaussian curve the **normal** curve, which [...] has the disadvantage of leading people to believe that all other distributions of frequency are in one sense or another abnormal.” -Pearson*



68-95-99.7 Rule
Probability Density
0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
68.27%
95.45%
99.73%
$\mu - 3\sigma$
$\mu - 2\sigma$
$\mu - \sigma$
μ
$\mu + \sigma$
$\mu + 2\sigma$
$\mu + 3\sigma$

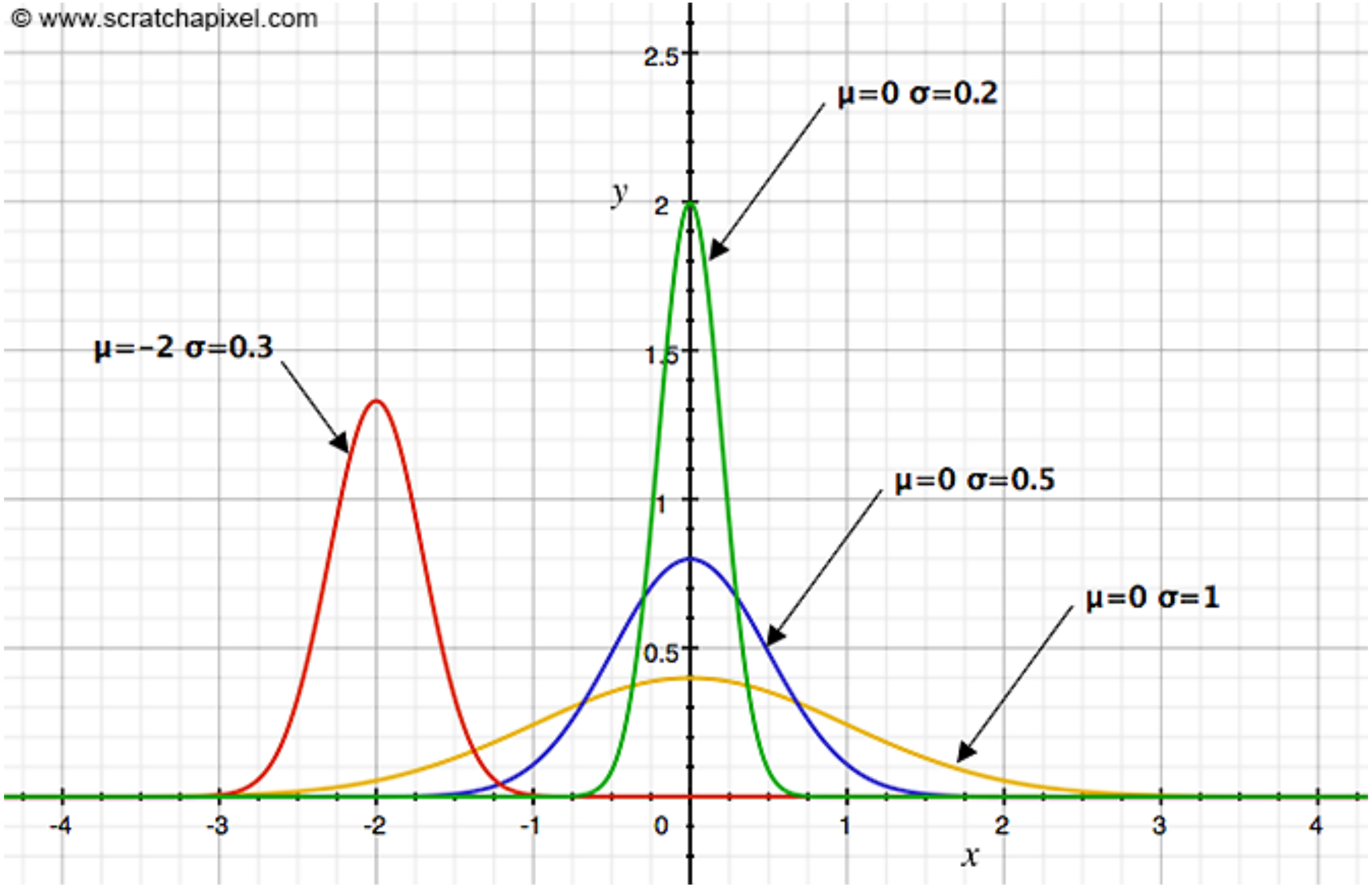
Linear transformations do not change the shape of a distribution

$$x_{\text{new}} = a + bx$$

They only:

- add a to the center
- multiply center and spread by b.

All normal distributions are the same if we measure in units of size σ around the mean μ as center



The standard normal distribution $\mathcal{N}(0,1)$ has $\mu = 0$ and $\sigma = 1$

We can turn any normally distributed variable $X \sim \mathcal{N}(\mu, \sigma)$ into $\mathcal{N}(0,1)$ by standardizing it:

$$Z = \frac{X - \mu}{\sigma}$$

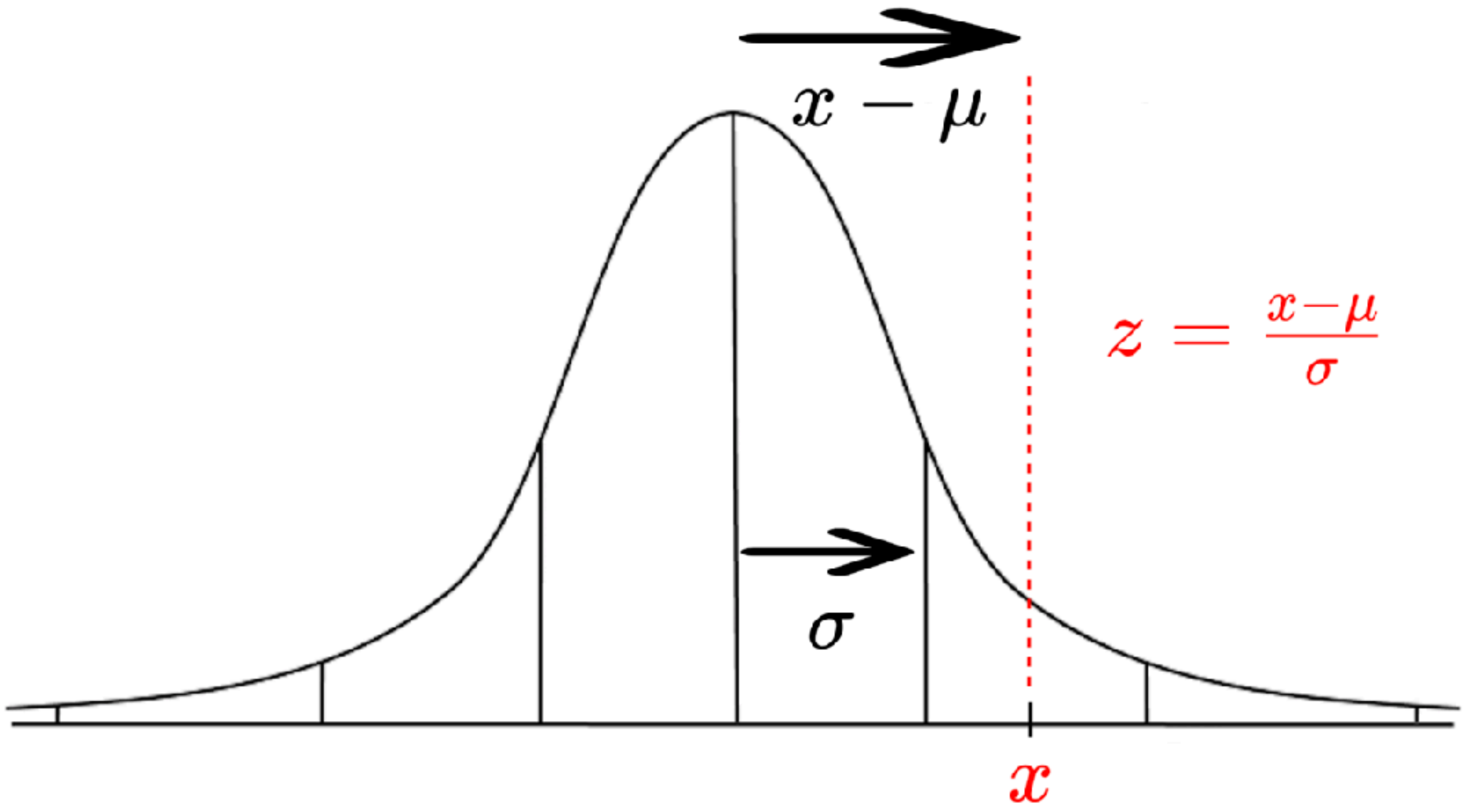
The standard normal distribution $\mathcal{N}(0,1)$ has $\mu = 0$ and $\sigma = 1$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

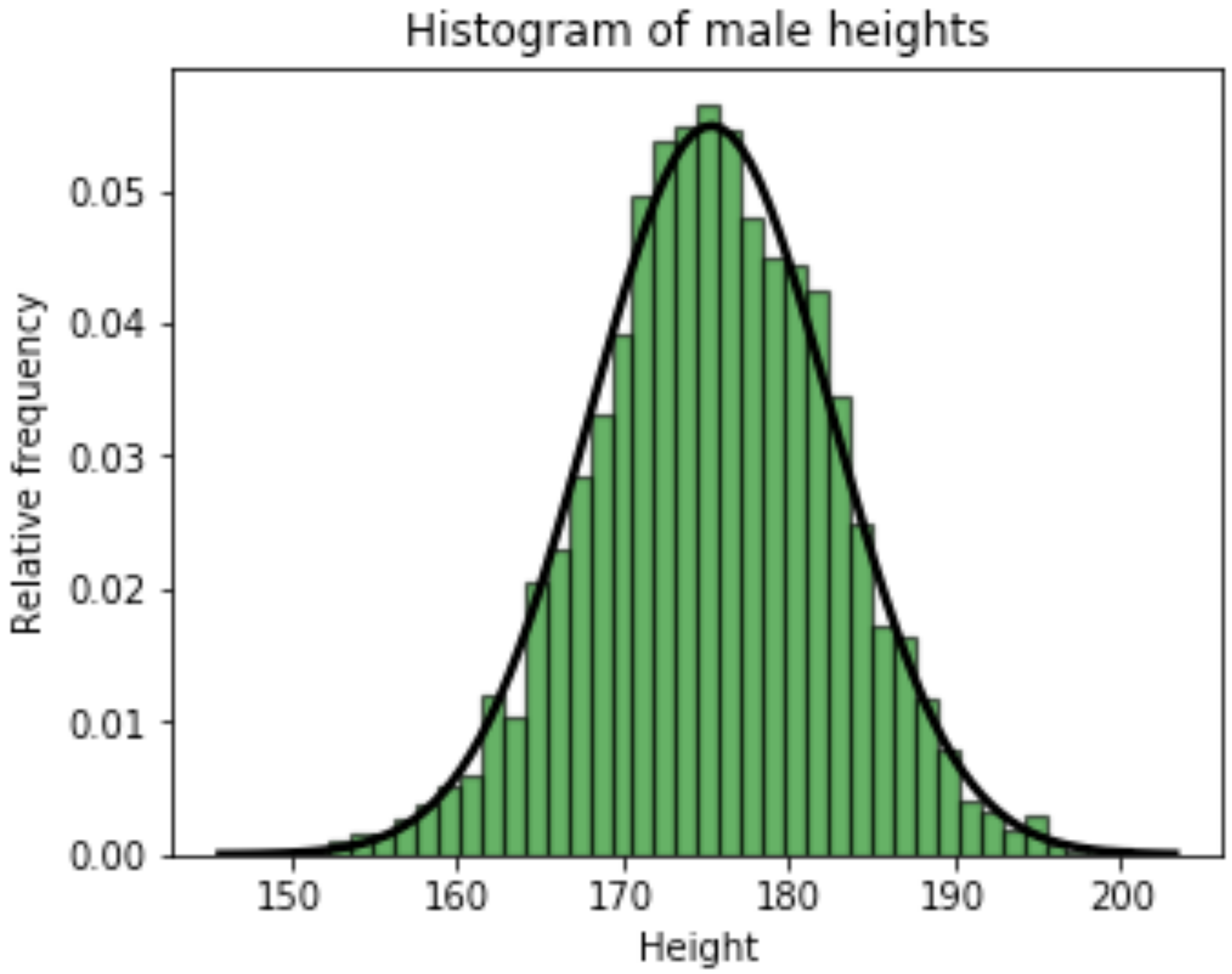
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$$

The z-score (standard score) is the number of standard deviations that an observed value x is away from the mean of a reference distribution

$$z = \frac{x - \mu}{\sigma}$$



Now we know what normal distributions are, but how to reliably spot one?



How can we be sure that the normal distribution is a good model for the data?

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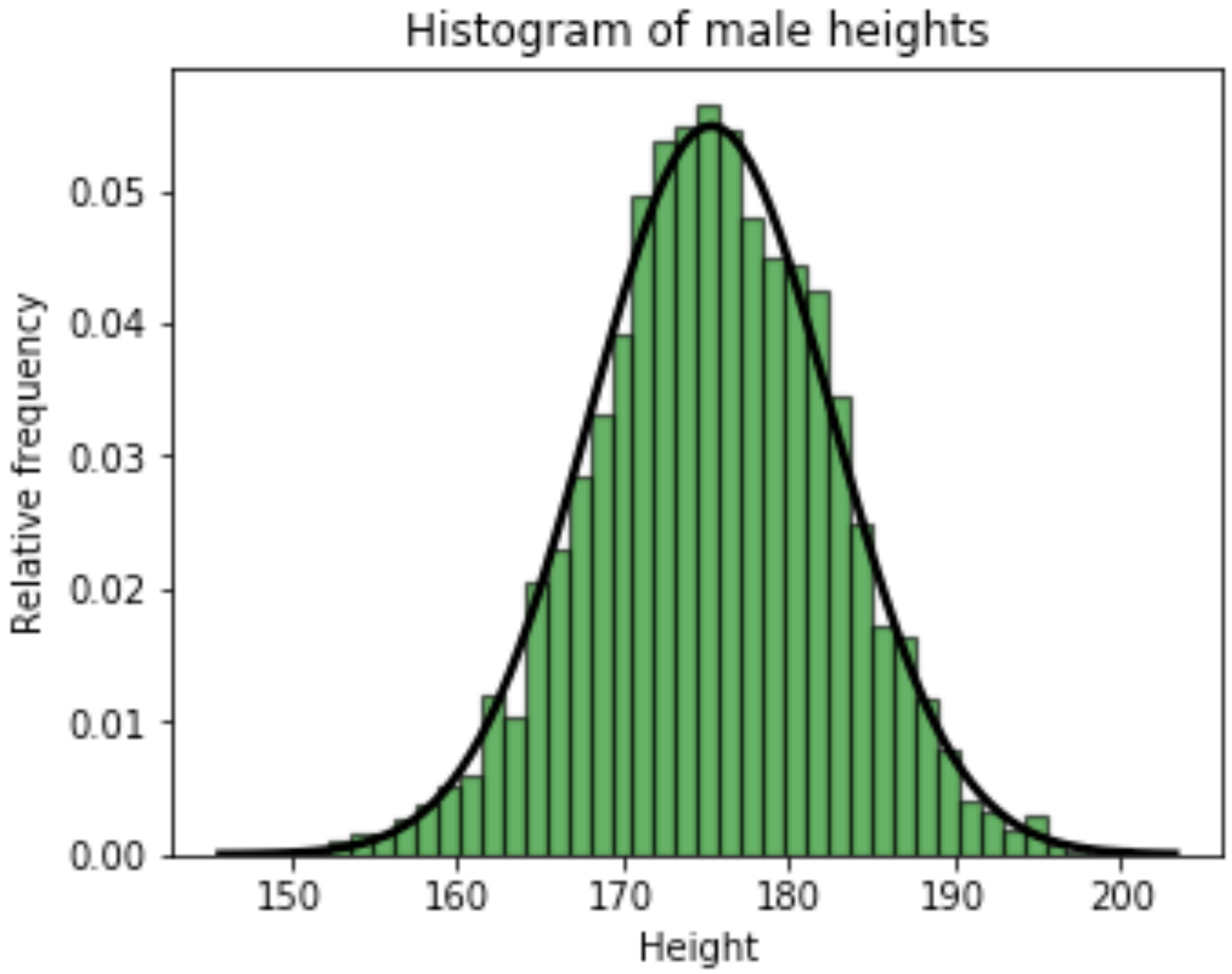
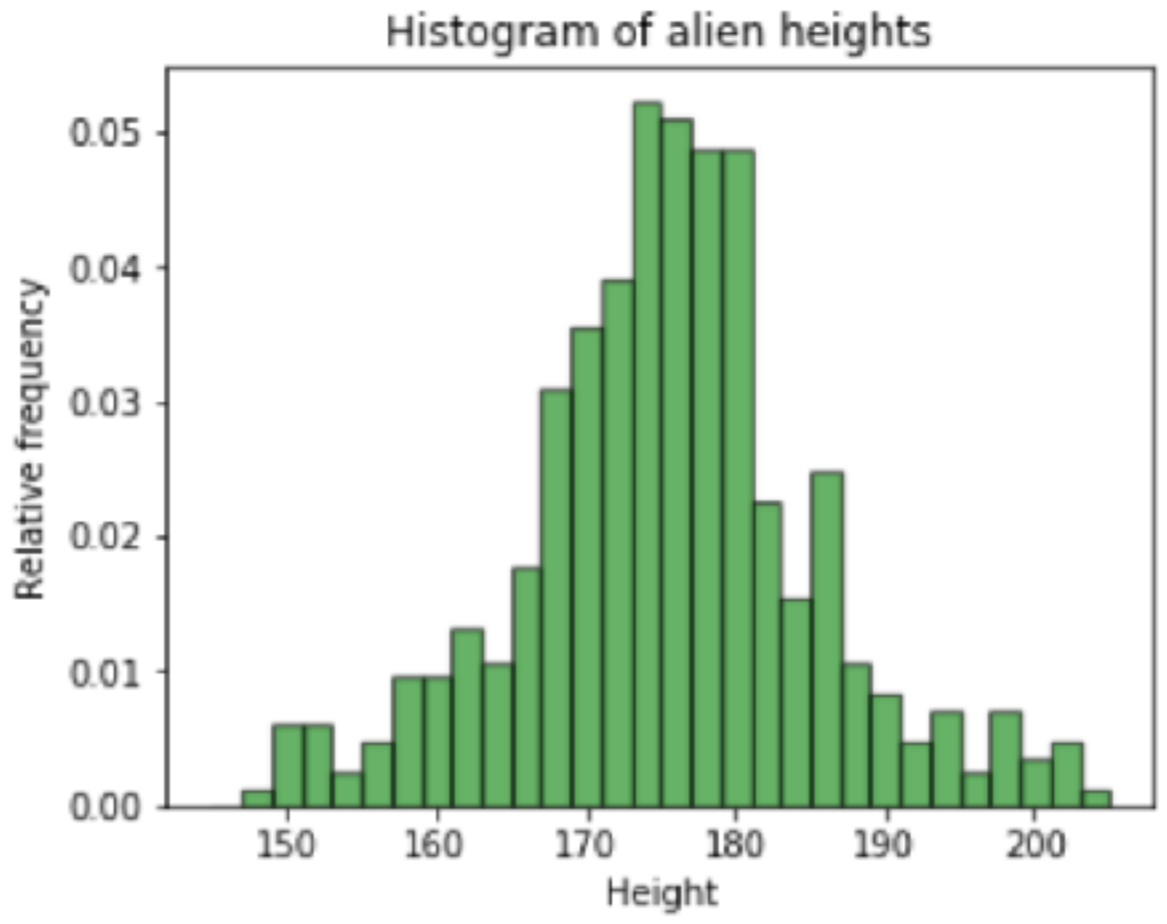
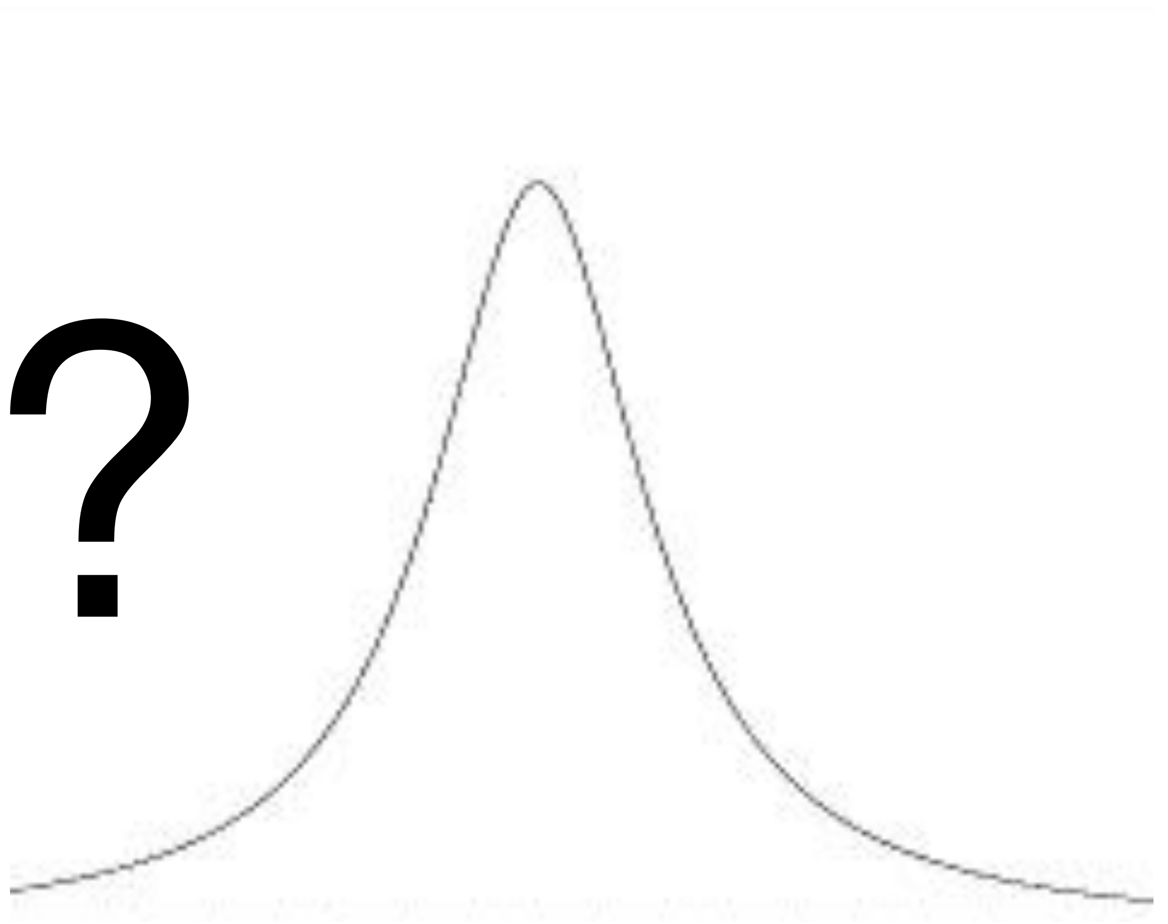


Unimodal
Symmetric

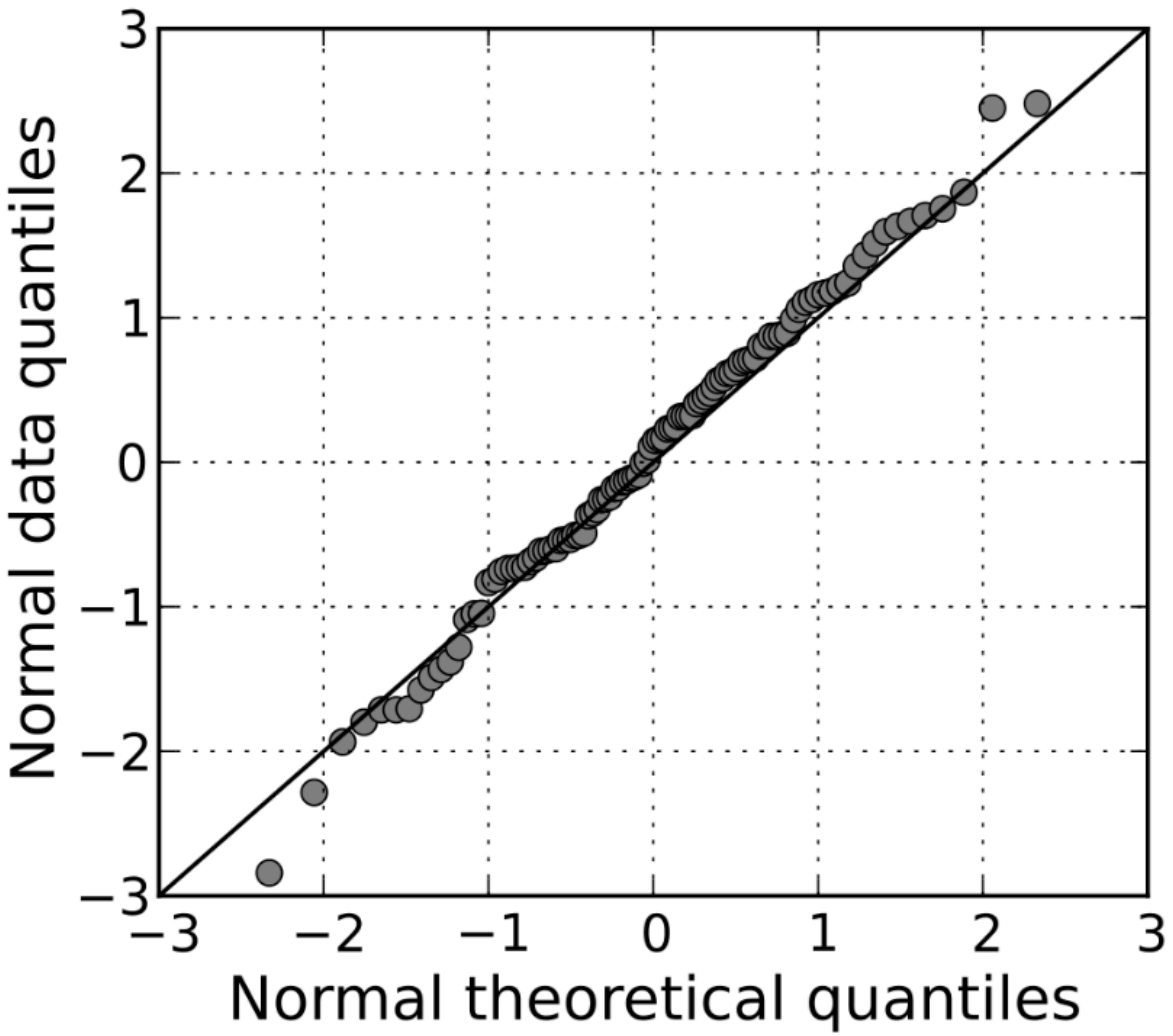


?



How can we be sure that the normal distribution is a good model for the data?

The Q-Q plot (quantile-quantile) assesses normality visually



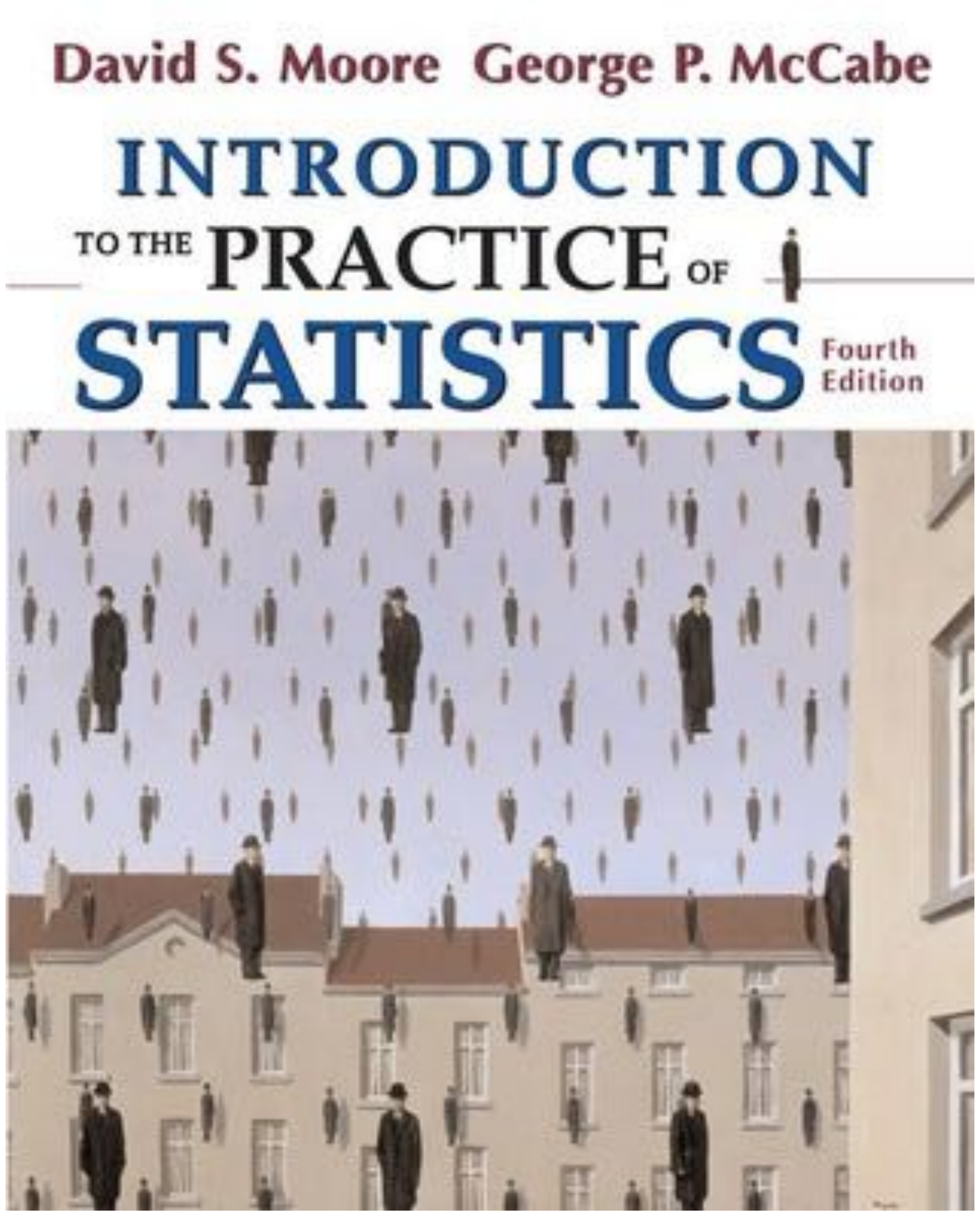
1) Order data points and calculate their quantiles

2) Calculate z-scores of theoretical normal distributions at same quantiles

3) Compare the two. If on the diagonal, we have a normal distribution

https://en.wikipedia.org/wiki/Q%E2%80%93Q_plot

Sources and further materials for today's class



Chapter 1.3

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