Supplementary Material

Before the analysis, the following assumptions are made: 1) solutions have been normalized within the interval [0, 1]; 2) solutions are mapped into the hyper-spherical coordinates system; 3) Partitions are located in the the first quadrant of the coordinate system. Here, the two-dimensional case is considered firstly and then it is extended to *N*-dimensional space.

In the two-dimensional case, as shown in Fig.1, the sum S_{sum} of $S_{\Delta OPC}$ and $S_{\Delta OPD}$ is calculated as

$$\begin{aligned} S_{sum} &= \mathrm{S}_{\Delta OPC} + S_{\Delta OPD} = \frac{1}{2} * r_p * \sin(\phi_p - \phi_1) \\ &* \frac{r_p * \sin\phi_p}{\sin\phi_1} + \frac{1}{2} * r_p * \sin(\phi_2 - \phi_p) * \frac{r_p * \cos\phi_p}{\cos\phi_2} \end{aligned} \tag{7}$$

Then, the partial derivative of the function S_{sum} with respect to the variable r_p is calculated as

$$\begin{aligned} \frac{\partial S_{sum}}{\partial r_p} &= r_p [\frac{\sin(\phi_2 - \phi_p)\cos\phi_p}{\cos\phi_2} \\ &+ \frac{\sin(\phi_p - \phi_1)\sin\phi_p}{\sin\phi_1}] \end{aligned} \tag{8}$$

From above equation, it is apparent that $r_p \geq 0$. Therefore, r_p does not affect the monotonicity of S_{sum}. And the polynomial in the right of Eq.(8) is denoted as

$$G(\phi_p) = \frac{\sin(\phi_2 - \phi_p)\cos\phi_p}{\cos\phi_2} + \frac{\sin(\phi_p - \phi_1)\sin\phi_p}{\sin\phi_1} \tag{9}$$

$$\because 0 \leq \phi_1 \leq \phi_p < \phi_2 \leq \frac{\pi}{2}$$

$$\therefore G(\phi_p) \geq 0$$

$$\therefore \frac{\partial S_{sum}}{\partial r_p} \geq 0$$

Thus, S_{sum} is monotonically increasing relative to r_p.

Next, the sum of the heights of $\Delta\mathrm{OPC}$ and $\Delta\mathrm{OPD}$, termed as H_{sum} (the sum of *PF* and *PE* in Fig.1), is calculated as

$$H_{sum} = r_p \sin(\phi_2 - \phi_p) + r_p \sin(\phi_p - \phi_1), \tag{10}$$

and its inverse function is

$$r_p = \frac{H_{sum}}{\sin(\phi_2 - \phi_p) + \sin(\phi_p - \phi_1)} \tag{11}$$

Accordingly, the partial derivative of function r_p with respect to H_{sum} is

$$\frac{\partial r_p}{\partial \mathrm{H}_{sum}} = \frac{1}{\sin(\phi_2 - \phi_p) + \sin(\phi_p - \phi_1)} \tag{12}$$

$$\because 0 \leq \phi_1 \leq \phi_p < \phi_2 \leq \frac{\pi}{2}$$

$$\therefore \frac{\partial r_p}{\partial \mathrm{H}_{sum}} > 0$$

Thus, r_p is monotonically increasing relative to H_{sum} as

$$\frac{\partial S_{sum}}{\partial H_{sum}} = \frac{\partial S_{sum}}{\partial r_p} * \frac{\partial r_p}{\partial H_{sum}} = \frac{H_{sum}}{\sin(\phi_2 - \phi_p) + \sin(\phi_p - \phi_1)}$$
$$* [\frac{\sin(\phi_2 - \phi_p)\cos\phi_p}{\cos\phi_2} + \frac{\sin(\phi_p - \phi_1)\sin\phi_p}{\sin\phi_1}] * \frac{1}{\sin(\phi_2 - \phi_p) + \sin(\phi_p - \phi_1)} \quad (13)$$

Finally, S_{sum} is monotonically increasing relative to H_{sum} according to $\frac{\partial S_{sum}}{\partial H_{sum}} > 0$. It is apparent that *PPR*(*p*) is maximum if r_p =0. Based on above, the H_{sum} can be seen as a evaluation indicator of *DArea*, and then the equation for *PPR* is proved.

In order to extend the two-dimensional case to the *N*-dimensional one, the deductions are given as follows. In two dimensional cartesian coordinate system, a straight line *L* across a point (0,0) is defined as

$$a * x + b * y = 0 \quad (14)$$

Then, the distance from a point (x_p,y_p) to line *L* is

$$dist = \frac{|a * x_p + b * y_p|}{\sqrt{a^2 + b^2}} \quad (15)$$

where (a, b) is the normal vector of *L* and point (x_p,y_p) is outside the line *L*.

Next, we define *V* and *W* to represent distance as

$$V = (x_p - x_0, y_p - y_0)$$
$$W = (a, b) \quad (16)$$

where (x_0,y_0) is a point in the line *L*. Then, the following equations exist

$$\forall (x_0, y_0) \in L,$$
$$s.t (a,b) * (x_0, y_0) = 0 \quad (17)$$

Accordingly, the distance from (x_0,y_0) to *L* can be defined as

$$\frac{|W^T * V|}{|W|} = \frac{|(a,b)^T * (x_p - x_0, y_p - y_0)|}{\sqrt{a^2 + b^2}}$$
$$= \frac{|a * x_p + b * y_p - a * x_0 - b * y_0|}{\sqrt{a^2 + b^2}}$$
$$\because (x_0, y_0) \in L$$
$$\therefore a * x_0 + b * y_0 = 0$$
$$\therefore \frac{|W^T * V|}{|W|} = \frac{|a * x_p + b * y_p|}{\sqrt{a^2 + b^2}} = dist \quad (18)$$

and considering the following relationship

$$x = r * \cos\theta$$
$$y = r * \sin\theta \quad , \quad (19)$$

we can obtain the equation of a straight line across (0,0)

$$\cos\theta * r + \sin\theta * r = 0$$
$$\therefore dist = \frac{|W^T * V|}{|W|} = \frac{|\sin\theta * r_p * \cos\theta_p - \cos\theta * r_p * \sin\theta_p|}{|W|} \quad (20)$$
$$W = (\tan\theta, -1)$$

Geometrically, the coordinate of the point multiply the vertical vector of the line, i.e., the distance between a point and a line. Accordingly, in the N-dimensional space the coordinate of the point multiply the vertical vector of the hyperplane, i.e., the distance between a point and a hyperplane. Therefore, it is extended to N dimensional space as

$$dist = \frac{\left|W^T * V\right|}{\left|W\right|} \qquad (21)$$

$$W = (\tan\theta_1, \tan\theta_2, \cdots, -1)$$