Matematica esercizi

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Testi

$$\lim_{x\to 0} \frac{x^2+3\sin 2x}{x-2\sin 3x}$$
$$\lim_{x\to 0} \frac{1-e^{x^2}}{x^3+\sqrt{x}} \tag{1}$$

Soluzioni

$$\lim_{x\to 0} \frac{x^2+3\sin 2x}{x-2\sin 3x} = \frac{0^2+3\sin 2(0)}{0-2\sin 3(0)}$$
$$\lim_{x\to 0} \frac{1-e^{x^2}}{x^3+\sqrt{x}} = \frac{1-e^0}{0^3+\sqrt{0}} = \frac{0}{0}$$

1 studio di furnzione

$$f(x) = \frac{e^x}{e^x-1}$$

1.0.1 Dominio

$$x \neq 0$$

Quindi da questa osservazione comprendiamo che la funzione non esiste nell'origine.

$$\forall x \in (-\infty, 0) \vee (0, +\infty)$$

1.1 simmetria

la funzione non è né pari né dispari

1.2 intersezione con gli assi

$$assex = \begin{cases} y = \frac{e^x}{e^x-1} \\ y = 0 \end{cases} \quad \begin{cases} \frac{e^x}{e^x-1} = 0 \\ y = 0 \end{cases}$$

non interseca nessuno dei due assi

1.3 Segno

$$\frac{e^x}{e^x-1} > 0$$

$x > 0$ perché al denominatore è presente un esponenziale.

1.4 Comportamento all'estremo del dominio

$$\lim_{x\to-\infty}\frac{e^x}{e^x-1}=\frac{e^{-\infty}}{e^{-\infty}-1}=\frac{0}{-1}=0$$
$$\lim_{x\to0^-}\frac{e^x}{e^x-1}=\frac{e^0}{e^0-1}=-\infty$$
$$\lim_{x\to0^+}\frac{e^x}{e^x-1}=\frac{e^0}{e^0-1}=+\infty$$
$$\lim_{x\to+\infty}\frac{e^x}{e^x-1}=\frac{e^{+\infty}}{e^{+\infty}-1}=\infty$$

1.5 Derivata prima

$$F'=\frac{e^x*(e^x-1)-e^x*(e^x)}{(e^x)^2}=\frac{-e^x}{(e^x-)^2}$$

1.6 es.2

$$f(x)=\frac{\ln(2x)}{x}$$

1. Dominio

$$x>0$$
$$\forall x\in(0;+\infty)$$

2. Parità

$$\neq f(-x)\text{ pari}$$
$$\neq -f(x)\text{ dispari}$$

3. intersezioni con gli assi

$$assey\begin{cases}f(x)=\frac{\ln(2x)}{x}\\x=0\end{cases}$$

Non interseca l'asse delle ordinate $f(0)=\nexists$

4. segno

$$f(x)=\frac{\ln(2x)}{x}\begin{cases}N\leq 0\to x & \to\frac{1}{2}\\D>0 & \to x>0\end{cases}\tag{2}$$

$$f(x)$$

5. Comportamento

1.7 es.4

$$f(x)=\frac{e^x-2}{x}$$

- Dominio

$$x\neq0$$

$$\forall x\in(-\infty,0)\vee(0,\infty)$$

- simmetrie

$$f(x)\begin{cases}\neq -f(x)\\\neq f(-x)\end{cases}$$

- Int. con gli assi

$$assex\begin{cases}y=\frac{e^x-2}{x}\\y=0\end{cases}\quad\left\{\frac{e^2-2}{\cancel{x}}=0\right.\quad\left\{e^x=2\right.\quad\left\{x=\ln 2\right.$$

asse y

$$\begin{cases} y = \frac{e^x-2}{x} \\ x = 0 \end{cases} \quad \begin{cases} y = \frac{e^0-2}{0} \\ x = 0 \end{cases} \quad \begin{cases} y = \ln 2 \end{cases}$$

- segno

$$\frac{e^x-2}{x} > 0$$
$$x > \ln 2$$

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- comportamento agli estremi

$$\lim_{x\to-\infty} \frac{e^x-2}{x} = \frac{0-2}{-\infty} = 0 \tag{3}$$
$$\lim_{x\to 0} \frac{e^x-2}{x} = \frac{e^0-2}{0} = \infty \tag{4}$$
$$\lim_{x\to\infty} \frac{e^x-2}{x} = \frac{\infty-2}{\infty} = \frac{e^x}{1} \tag{5}$$

2 es.3

$$f(x) = \frac{x-1}{x^2-x-6}$$

1. dominio $\forall x \in \mathbb{R} \setminus \{-2, 3\}$
2. simmetrie $f(x) \begin{cases} \neq -f(x) \\ \neq f(-x) \end{cases}$
3. Int. con gli assi

$$assex \begin{cases} y = \frac{x-1}{x^2-x-6} \\ y = 0 \end{cases} \quad \begin{cases} \frac{x-1}{x^2-x-6} = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 1 \\ y = 0 \end{cases}$$

$$assey \begin{cases} y = \frac{x-1}{x^2-x-6} \\ x = 0 \end{cases} \quad \begin{cases} y = \frac{0-1}{0^2-0-6} \\ x = 0 \end{cases} \quad \begin{cases} y = \frac{1}{6} \\ x = 0 \end{cases}$$

4. segni $-2 < x < 1 \vee x > 3$
5. comportamento agli estremi

$$\lim_{x\to-\infty} \frac{x-1}{x^2-x-6} = \frac{-\infty-1}{+\infty} = \frac{\infty}{\infty} \to \lim_{x\to\infty} \frac{1}{2x-1} = \frac{1}{\infty} = 0 \text{ Assintoto orizzontale} \lim_{x\to-2} \frac{x-1}{x^2-x-6} = \frac{-3}{0} = \infty \text{ Assintoto} \lim_{x\to 3} \frac{x-1}{x^2-x-6} = \frac{2}{9-3-6} = \infty \text{ Assintoto verticale} \tag{6}$$

3 teorema di Roll

$$f(x) = x^2 - 4x + 3$$

$$y = [-1, 5]$$

$$(-1)^2 - 4(-1) + 3 = 1 + 4 + 3 = 8$$

$$(5)^2 - 4(5) + 3 = 25 - 20 + 3 = 8$$

$$\begin{gathered} f(c) = 0 \\ f'(x) = 2x - 4 \\ 2c - 4 = 0 \\ \frac{\cancel{2}c}{\cancel{2}} = \frac{\cancel{4}}{\cancel{2}} \to c = 2 \end{gathered}. \tag{7}$$

3.1 es.2

$$f(x) = x^4 + x^2 + 1$$

Intervallo compreso tra $[-2, 2]$

$$f(-2) = (-2)^4 + (-2)^2 + 1 = 16 + 4 + 1 = 21$$
$$f(2) = (2)^4 + (2)^2 + 1 = 16 + 4 + 1 = 21$$

la funzione è continua

$$f'(x) = 4x^3 + 2x$$
$$f'(c) = 0$$
$$x = c$$
$$4c^3 - 2c = 0 \to 2c(2c^2 + 1) = 0$$
$$2c = 0 \to c = 0$$
$$2x^2 + 1 = 0 \to c = \pm\sqrt{-\tfrac{1}{2}} \text{ NO}$$

4 Teorema di Lagrange

$$f(x) = 2x^2 + x + 1,\ [-2; 3]$$

$$2(-2)^2 - 2 + 1 = 8 - 1 = 7$$
$$2(3)^2 + 3 + 1 = 23 \text{ NO}$$

questa funzione non rispetta i punti del teorema di Lagrange.

4.1 es.2

$$f(x) = \sqrt{x} - x,\ [0,4]$$

$$\sqrt{0} - 0 = 0$$
$$\sqrt{4} - 4 = 2$$

la funzione è continua

$$f' = \frac{1}{2\sqrt{x}} - 1$$

la funzione è derivabile

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{-2 - 0}{4 - 0} = -\frac{1}{2}$$

$$\boxed{\frac{1}{2\sqrt{x}} = -\frac{1}{2}}$$

5 Equazione differenziali

$$y' + 2xy = x\sin(x^2)$$

$$x' = -2xy + x\sin(x^2)$$

$$y' = a(x)b(x)$$

$$y' = -2x + x\sin(x^2)$$

$$y(x) = e^{-A(x)}\left(c + \int e^{A(x)} f(x)dx\right)$$

$$a(x) = 2x; \quad A(x) = \int a(x)dx = x^2$$

$$y = e^{-x^2}\left\{c + \int e^{x^2} x \sin x^2\right\} \tag{8}$$

$$\int e^{x^2} x \sin x^2 dx = \left[x^2 = t; dx = \frac{dt}{2}\right] = \frac{1}{2}\int e^x \sin t dt \tag{9}$$

(con integrazione per parti standard)

$$= \frac{1}{2}e^t(\sin t - \cos t) = \frac{1}{4}e^{x^2}(\sin(x^2) - \cos(x^2)) \tag{10}$$

$$y = e^{e^t}\left\{c + \frac{1}{4}e^{x^2}(\sin x^2 - \cos x^2)\right\} = ce^{-x^2} + \frac{1}{4}(\sin x^2 - \cos x^2)$$

6 integrali di secondo tipo

$$y'' - y' + 2y = 3xe^{-x}$$

$$t = y'$$
$$t^2 - 3t + 2 = 3xe^{-x}$$
$$\Delta = b^2 - 4(a)(c) = 9 - 4(1)(2) = 1$$
$$t_{1,2} = \tfrac{-b\pm\sqrt{\Delta}}{2a} = \tfrac{3\pm 1}{2} = \begin{cases} \tfrac{3+1}{2} = 2 \\ \tfrac{3-1}{2} = 1 \end{cases}$$
$$z(x) = c_1e^x + c_2e^{2x}$$
$$f(x) = 3xe^{-x}$$
$$y(x) = (ax + b)e^{-x}$$
$$y' = e^{-x}(-ax - b + a)$$
$$y'' = e^{-x}(ax + b - 2a)$$
$$e^{-x}[6ax + (6b - 5a)] = 3xe^{-x}$$
$$\begin{cases} 6a = 3 \\ 6b - 5a = 0 \end{cases} \qquad a = \tfrac{1}{2}; \quad b = \tfrac{5}{12};$$
$$y(x) = \left(\tfrac{1}{2}x + \tfrac{5}{12}\right)e^{-x}$$
$$c_1e^x + c_2e^{2x} + \left(\tfrac{1}{2}x + \tfrac{5}{12}\right)e^{-x}$$

7 integrali

$$\int \tfrac{3x-4}{x^2-6x+8}dx$$
$$\Delta = b^2 - 4(a)(c) = 36 - 32 = 4$$
$$\int \tfrac{R(x)}{D(x)} = \tfrac{A}{x-x^1}dx + \int \tfrac{B}{x-x_2}dx = A\ln(|x - x_2|) + B(|x + 1|)$$
$$\int \tfrac{3x-4}{x^2-6x+8}dx = \tfrac{3x-4}{(x-4)(x-2)} = \tfrac{A}{x-4} + \tfrac{B}{x-2} = \tfrac{A(x-2)+B(x-4)}{(x-4)(x-2)}$$
$$A(x-2) + B(x-4) - 2A - 4B \Leftrightarrow \begin{cases} (A+B)x = 3 \\ -2A - 4B = -4 \end{cases} \Rightarrow \begin{cases} A = -B + 3 \\ -2(-B+3) - 4B = -4 \end{cases}$$
$$\Rightarrow \begin{cases} A = -B + 3 \\ 2B - 6 - 4B = -4 \end{cases} \Rightarrow \begin{cases} A = -B + 3 \\ -2B = 2 \end{cases} \Rightarrow \begin{cases} A = -B + 3 \\ \tfrac{-2B}{2} = \tfrac{2}{2} \end{cases} \Rightarrow \begin{cases} A = -B + 3 \\ B = -1 \end{cases}$$
$$\Rightarrow \begin{cases} A = 1 + 3 \to 4 \\ B = -1 \end{cases}$$
$$\tfrac{3x-4}{x^2-6x+8}dx = \tfrac{4}{x-4} - \tfrac{1}{x-2} = \int \left[\tfrac{4}{x-4}\right]dx = 4\log|x-4| - \log|x-2| + c$$

7.0.1 es.2

$$\int \frac{dx}{9x^2-25} = \frac{dx}{(3x-5)(3x+5)}$$

$$\int \frac{R(x)}{D(x)} = \int \frac{A}{x-x_1}dx + \int \frac{B}{x-x_2}dx = \frac{A(x-x_2)+B(x-x_1)}{(x-x_1)(x-x_2)} = A\ln|x-x_2| + B\ln|x+x_1|$$

$$A(3x-5)+B(3x+5) \Rightarrow \begin{cases} (3A+3B)x = 0 \\ -A5+5B = 1 \end{cases} \Rightarrow \begin{cases} 3A+3B = 0 \\ -5A+5B = 1 \end{cases} \Rightarrow \begin{cases} \frac{\cancel{3}A}{\cancel{3}} = \frac{-\cancel{3}B}{\cancel{3}} \\ -5A+5B = 1 \end{cases}$$

$$\begin{cases} A = -B \\ +5B+5B = 1 \end{cases} \begin{cases} A = -B \\ 10B = 1 \end{cases} \rightarrow \begin{cases} A = -B \\ B = \frac{1}{10} \end{cases} \begin{cases} A = -\frac{1}{10} \\ B = \frac{1}{10} \end{cases}$$

$$\int \frac{-\frac{1}{10}}{3x+5} + \int \frac{\frac{1}{10}}{3x-5} = -\frac{1}{10}\int \frac{1}{3x+5} + \frac{1}{10}\int \frac{1}{3x-5} = -\frac{1}{10}\ln(3x+5) + \frac{1}{10}\ln(3x-5)$$

7.0.2 es.3

$$\int \frac{dx}{x^2-x-6}$$

$$\int \frac{dx}{(x-3)(x+2)}$$

$$\int \frac{R(x)}{D(x)} = \frac{A}{x+x_1}dx + \int \frac{B}{x+x_2}dx = A\ln(|x-x_2|) + B(|x+1|)$$

$$\frac{A}{x+3} + \frac{B}{x-2} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$A(x+2)+B(x-3) \rightarrow \begin{cases} (A+B)x = 0 \\ 2A-3B = 1 \end{cases} \rightarrow \begin{cases} A = -B \\ 2A-3B = 1 \end{cases} \rightarrow \begin{cases} A = -B \\ -2B-3B = 1 \end{cases}$$

$$\begin{cases} A = -B \\ -5B = 1 \end{cases} \rightarrow \begin{cases} A = -B \\ -5B = 1 \end{cases} \rightarrow \begin{cases} A = -B \\ \frac{-\cancel{5}B}{\cancel{5}} = -\frac{1}{5} \end{cases} \rightarrow \begin{cases} A = \frac{1}{5} \\ B = -\frac{1}{5} \end{cases}$$

7.1 differenziali

$$\begin{cases} y'' - 8y' + 15y = 2e^{3x} \\ y(1) = 1, y' = 0 \end{cases}$$

$$t = y'$$

$$t^2 - 8t + 15 = 2e^{3x}$$

$$\Delta = b^2 - 4(a)(c) = 64 - 60 = 4$$

$$t_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{8 \pm 2}{2} = \begin{cases} \frac{8+2}{2} = \frac{\cancel{10}}{\cancel{2}} = 5 \\ \frac{8-2}{2} = \frac{\cancel{6}}{\cancel{2}} = 3 \end{cases}$$

$$(\lambda - 3)(\lambda - 5)$$

$$y_p(x) = Axe^{3x}$$

$$y'p(x) = A(3x+1)e^{3x}$$

$$y''(x) = A(9x+6)e^{3x}$$

$$A[9x + 6 - 8(3x+1) + 15x]e^{3x} = 2e^{3x}$$

$$\Delta = -1$$

$$y(x) = c_1e^{3x} + c_2e^{5x} - xe^{3x}$$

$$\begin{cases} y'(x) \\ y(0) = 0 = y'(0) \end{cases} \begin{cases} c_1 + c_2 = 0 \\ 3c_1 + 5c_2 - 1 = 0 \end{cases} \begin{cases} c_1 = -c_2 \\ -3c_2 + 6c_2 - 1 = 0 \end{cases} \begin{cases} c_1 = -c_2 \\ \frac{2}{2} = \frac{1}{2} \end{cases} \rightarrow \begin{cases} c_1 = -\frac{1}{2} \\ c_2 = \frac{1}{2} \end{cases}$$

Soluzione di Cauchy

$$y(x) = -\frac{1}{2}e^{3x} + \frac{1}{2}e^{5x} - xe^{3x}$$

7.1.1 es.2

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 3; y'(0) = 2 \end{cases}$$
$$p(\lambda) = \lambda^2 + 4\lambda + 4\lambda$$
$$\Delta = 4^2 - 4(1)(4) = 16 - 16 = 0$$
$$y(y) = c_1x + c_2x - e^{-2x}$$
$$\begin{cases} y'(x) \\ y(0) = c_2 = 3 \end{cases} \quad \begin{cases} y'(x) = (c_1 - 2c_1x - 2c_2)e^{-2x} \\ y'(0) = c_1 - 2c_2 = c_1 - 6 \end{cases}$$
$$\begin{cases} y'(x) = (c_1 - 2c_1x - 2c_2)e^{-2x} \\ y'(0) = 8 \end{cases}$$
$$y(x) = (8x + 3)e^{-2x}$$

7.2 es.3

$$y'' - 5y' + 6y = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$\Delta = (-5)^2 - 4(1)(6) = 1$$
$$\lambda_{1,2} = \frac{-b \pm \sqrt{1}}{2a} = \begin{cases} \frac{5-1}{2} = 2 \\ \frac{5+1}{2} = 3 \end{cases}$$
$$y(0) = c_1 * e^{2x} + c_2 * e^{3x}$$
$$c_1 \text{ e } c_2 \text{ fanno parte dei reali}$$

$$\begin{cases} y'' - 6y' + 9y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$
$$\lambda^2 - 6\lambda + 9 = 0$$
$$\Delta = (-6)^2 - 4(1)(9) = 36 - 36 = 0$$
$$\lambda_{1,2} = \frac{6 \pm \sqrt{0}}{2a} = \frac{6-0}{2} = 3$$
$$y_0(x) = c_1e^{\lambda_1 x} + c_2xe^{\lambda_2 x} \qquad c_1, c_2 \in \mathbb{R}$$
$$y_0'(x) = 3xc_1e^{3x} + c_2(e^{3x} + x3e^{3x})$$
$$= 3xc_1e^{3x} + c_2e^{3x} + 3xc_2e^{3x}$$
$$\text{sostituire con } \begin{cases} y_0(0) = 1 \\ y_0'(0) = 1 \end{cases}$$

7.3 es.3

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 3; y(0) = 2 \end{cases}$$
$$\lambda^2 + 4\lambda + 4 = 0$$
$$\Delta = 4^2 - 4(1)(4) = 0$$
$$\lambda_{1,2} = \frac{-4 \pm \sqrt{0}}{2} = \frac{-4}{2} = -2$$
$$y_0(x) = c_1e^{\lambda_1 x} + c_2xe^{\lambda_2 x} = c_1e^{-2x} + c_2xe^{-2x}$$
$$y_0'(x) = -2c_1e^{-2x} + c_2(e^{-2x} - 2xe^{-2x})c_1 = 3; c_2 = 2$$
$$y(x) = 3e^{-2x} + 2e^{-2x} - xe^{-2x}$$
$$y(x) = 5e^{-2x} - xe^{-2x}$$