Matematica esercizi

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Testi

$$\lim_{x\to 0} \frac{x^2+3\sin 2x}{x-2\sin 3x}$$
$$\lim_{x\to 0} \frac{1-e^{x^2}}{x^3+\sqrt{x}} \tag{1}$$

Soluzioni

$$\lim_{x\to 0} \frac{x^2+3\sin 2x}{x-2\sin 3x} = \frac{0^2+3\sin 2(0)}{0-2\sin 3(0)}$$
$$\lim_{x\to 0} \frac{1-e^{x^2}}{x^3+\sqrt{x}} = \frac{1-e^0}{0^3+\sqrt{0}} = \frac{0}{0}$$

1 studio di furnzione

$$f(x) = \frac{e^x}{e^x-1}$$

1.0.1 Dominio

$$x \neq 0$$

Quindi da questa osservazione comprendiamo che la funzione non esiste nell'origine.

$$\forall x \in (-\infty, 0) \vee (0, +\infty)$$

1.1 simmetria

la funzione non è né pari né dispari

1.2 intersezione con gli assi

$$assex = \begin{cases} y = \frac{e^x}{e^x-1} \\ y = 0 \end{cases} \quad \begin{cases} \frac{e^x}{e^x-1} = 0 \\ y = 0 \end{cases}$$

non interseca nessuno dei due assi

1.3 Segno

$$\frac{e^x}{e^x-1} > 0$$

$x > 0$ perché al denominatore è presente un esponenziale.

1.4 Comportamento all'estremo del dominio

$$\lim_{x\to-\infty} \frac{e^x}{e^x-1} = \frac{e^{-\infty}}{e^{-\infty}-1} = \frac{0}{-1} = 0$$
$$\lim_{x\to0^-} \frac{e^x}{e^x-1} = \frac{e^0}{e^0-1} = -\infty$$
$$\lim_{x\to0^+} \frac{e^x}{e^x-1} = \frac{e^0}{e^0-1} = +\infty$$
$$\lim_{x\to+\infty} \frac{e^x}{e^x-1} = \frac{e^{+\infty}}{e^{+\infty}-1} = \infty$$

1.5 Derivata prima

$$F' = \frac{e^x * (e^x - 1) - e^x * (e^x)}{(e^x)^2} = \frac{-e^x}{(e^x -)^2}$$

1.6 es.2

$$f(x) = \frac{\ln(2x)}{x}$$

1. Dominio

$$x > 0$$
$$\forall x \in (0; +\infty)$$

2. Parità

$$\neq f(-x) \text{ pari}$$
$$\neq -f(x) \text{ dispari}$$

3. intersezioni con gli assi

$$assey \begin{cases} f(x) = \frac{\ln(2x)}{x} \\ x = 0 \end{cases}$$

Non interseca l'asse delle ordinate $f(0) = \nexists$

4. segno

$$f(x) = \frac{\ln(2x)}{x} \begin{cases} N \leq 0 \to x & \to \frac{1}{2} \\ D > 0 & \to x > 0 \end{cases} \quad (2)$$

$$f(x)$$

5. Comportamento

1.7 es.4

$$f(x) = \frac{e^x - 2}{x}$$

- Dominio

$$x \neq 0$$

$$\forall x \in (-\infty, 0) \vee (0, \infty)$$

- simmetrie

$$f(x) \begin{cases} \neq -f(x) \\ \neq f(-x) \end{cases}$$

- Int. con gli assi

$$assex \begin{cases} y = \frac{e^x-2}{x} \\ y = 0 \end{cases} \quad \left\{ \frac{e^2-2}{\cancel{x}} = 0 \right. \quad \left\{ e^x = 2 \right. \quad \left\{ x = \ln 2 \right.$$

asse y

$$\begin{cases} y = \frac{e^x - 2}{x} \\ x = 0 \end{cases} \quad \begin{cases} y = \frac{e^0 - 2}{0} \\ x = 0 \end{cases} \quad \begin{cases} y = \ln 2 \end{cases}$$

- segno

$$\frac{e^x-2}{x} > 0$$
$$x > \ln 2$$

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- comportamento agli estremi

$$\lim_{x\to-\infty} \frac{e^x - 2}{x} = \frac{0-2}{-\infty} = 0 \tag{3}$$
$$\lim_{x\to 0} \frac{e^x - 2}{x} = \frac{e^0-2}{0} = \infty \tag{4}$$
$$\lim_{x\to\infty} \frac{e^x - 2}{x} = \frac{\infty-2}{\infty} = \frac{e^x}{1} \tag{5}$$

2 es.3

$$f(x) = \frac{x-1}{x^2-x-6}$$

1. dominio $\forall x \in \mathbb{R} \setminus \{-2, 3\}$
2. simmetrie $f(x) \begin{cases} \neq -f(x) \\ \neq f(-x) \end{cases}$
3. Int. con gli assi

$$assex \begin{cases} y = \frac{x-1}{x^2-x-6} \\ y = 0 \end{cases} \quad \begin{cases} \frac{x-1}{x^2-x-6} = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 1 \\ y = 0 \end{cases}$$

$$assey \begin{cases} y = \frac{x-1}{x^2-x-6} \\ x = 0 \end{cases} \quad \begin{cases} y = \frac{0-1}{0^2-0-6} \\ x = 0 \end{cases} \quad \begin{cases} y = \frac{1}{6} \\ x = 0 \end{cases}$$

4. segni $-2 < x < 1 \vee x > 3$
5. comportamento agli estremi

$$\lim_{x\to-\infty} \frac{x-1}{x^2-x-6} = \frac{-\infty-1}{+\infty} = \frac{\infty}{\infty} \to \lim_{x\to\infty} \frac{1}{2x-1} = \frac{1}{\infty} = 0 \text{ Assintoto orizzontale} \lim_{x\to-2} \frac{x-1}{x^2-x-6} = \frac{-3}{0} = \infty \text{ Assintoto}$$
$$\lim_{x\to 3} \frac{x-1}{x^2-x-6} = \frac{2}{9-3-6} = \infty \text{ Assintoto verticale} \tag{6}$$

3 teorema di Roll

$$f(x) = x^2 - 4x + 3$$

$$y = [-1, 5]$$

$$(-1)^2 - 4(-1) + 3 = 1 + 4 + 3 = 8$$

$$(5)^2 - 4(5) + 3 = 25 - 20 + 3 = 8$$

$$\begin{aligned} f(c) &= 0 \\ f'(x) &= 2x - 4 \\ 2c - 4 &= 0 \\ \frac{\cancel{2}c}{\cancel{2}} &= \frac{\cancel{4}}{\cancel{2}} \to c = 2 \end{aligned} \tag{7}$$

3.1 es.2

$$f(x) = x^4 + x^2 + 1$$

Intervallo compreso tra $[-2, 2]$

$$f(-2) = (-2)^4 + (-2)^2 + 1 = 16 + 4 + 1 = 21$$
$$f(2) = (2)^4 + (2)^2 + 1 = 16 + 4 + 1 = 21$$

la funzione è continua

$$f'(x) = 4x^3 + 2x$$
$$f'(c) = 0$$
$$x = c$$
$$4c^3 - 2c = 0 \rightarrow 2c(2c^2 + 1) = 0$$
$$2c = 0 \rightarrow c = 0$$
$$2x^2 + 1 = 0 \rightarrow c = \pm\sqrt{-\tfrac{1}{2}} \text{ NO}$$

4 Teorema di Lagrange

$$f(x) = 2x^2 + x + 1,\ [-2; 3]$$

$$2(-2)^2 - 2 + 1 = 8 - 1 = 7$$
$$2(3)^2 + 3 + 1 = 23 \text{ NO}$$

questa funzione non rispetta i punti del teorema di Lagrange.

4.1 es.2

$$f(x) = \sqrt{x} - x,\ [0,4]$$

$$\sqrt{0} - 0 = 0$$
$$\sqrt{4} - 4 = 2$$

la funzione è continua

$$f' = \frac{1}{2\sqrt{x}} - 1$$

la funzione è derivabile

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{-2 - 0}{4 - 0} = -\frac{1}{2}$$

$$\boxed{\frac{1}{2\sqrt{x}} = -\frac{1}{2}}$$

5 Equazione differenziali

$$y' + 2xy = x\sin(x^2)$$

$$x' = -2xy + x\sin(x^2)$$

$$y' = a(x)b(x)$$

$$y' = -2x + x\sin(x^2)$$

$$y(x) = e^{-A(x)}\left(c + \int e^{A(x)} f(x)dx\right)$$

$$a(x) = 2x; \quad A(x) = \int a(x)dx = x^2$$

$$y = e^{-x^2}\left\{c + \int e^{x^2} x \sin x^2\right\} \tag{8}$$

$$\int e^{x^2} x \sin x^2 dx = \left[x^2 = t; dx = \frac{dt}{2}\right] = \frac{1}{2}\int e^x \sin t dt \tag{9}$$

(con integrazione per parti standard)

$$= \frac{1}{2}e^t(\sin t - \cos t) = \frac{1}{4}e^{x^2}(\sin(x^2) - \cos(x^2)) \tag{10}$$

$$y = e^{e^t}\left\{c + \frac{1}{4}e^{x^2}(\sin x^2 - \cos x^2)\right\} = ce^{-x^2} + \frac{1}{4}(\sin x^2 - \cos x^2)$$

6 integrali di secondo tipo

$$y'' - y' + 2y = 3xe^{-x}$$

$$t = y'$$
$$t^2 - 3t + 2 = 3xe^{-x}$$
$$\Delta = b^2 - 4(a)(c) = 9 - 4(1)(2) = 1$$
$$t_{1,2} = \tfrac{-b\pm\sqrt{\Delta}}{2a} = \tfrac{3\pm 1}{2} = \begin{cases} \tfrac{3+1}{2} = 2 \\ \tfrac{3-1}{2} = 1 \end{cases}$$
$$z(x) = c_1e^x + c_2e^{2x}$$
$$f(x) = 3xe^{-x}$$
$$y(x) = (ax+b)e^{-x}$$
$$y' = e^{-x}(-ax - b + a)$$
$$y'' = e^{-x}(ax + b - 2a)$$
$$e^{-x}[6ax + (6b - 5a)] = 3xe^{-x}$$
$$\begin{cases} 6a = 3 \\ 6b - 5a = 0 \end{cases} \qquad a = \tfrac{1}{2}; \quad b = \tfrac{5}{12};$$
$$y(x) = \left(\tfrac{1}{2}x + \tfrac{5}{12}\right)e^{-x}$$
$$c_1e^x + c_2e^{2x} + \left(\tfrac{1}{2}x + \tfrac{5}{12}\right)e^{-x}$$

7 integrali

$$\int \tfrac{3x-4}{x^2-6x+8}dx$$
$$\Delta = b^2 - 4(a)(c) = 36 - 32 = 4$$
$$\int \tfrac{R(x)}{D(x)} = \tfrac{A}{x-x^1}dx + \int \tfrac{B}{x-x_2}dx = A\ln(|x-x_2|) + B(|x+1|)$$
$$\int \tfrac{3x-4}{x^2-6x+8}dx = \tfrac{3x-4}{(x-4)(x-2)} = \tfrac{A}{x-4} + \tfrac{B}{x-2} = \tfrac{A(x-2)+B(x-4)}{(x-4)(x-2)}$$
$$A(x-2) + B(x-4) - 2A - 4B \Leftrightarrow \begin{cases} (A+B)x = 3 \\ -2A - 4B = -4 \end{cases} \Rightarrow \begin{cases} A = -B + 3 \\ -2(-B+3) - 4B = -4 \end{cases}$$
$$\Rightarrow \begin{cases} A = -B + 3 \\ 2B - 6 - 4B = -4 \end{cases} \Rightarrow \begin{cases} A = -B + 3 \\ -2B = 2 \end{cases} \Rightarrow \begin{cases} A = -B + 3 \\ \tfrac{-2B}{2} = \tfrac{2}{2} \end{cases} \Rightarrow \begin{cases} A = -B + 3 \\ B = -1 \end{cases}$$
$$\Rightarrow \begin{cases} A = 1 + 3 \to 4 \\ B = -1 \end{cases}$$
$$\tfrac{3x-4}{x^2-6x+8}dx = \tfrac{4}{x-4} - \tfrac{1}{x-2} = \int \left[\tfrac{4}{x-4}\right]dx = 4\log|x-4| - \log|x-2| + c$$

7.0.1 es.2

$$\int \frac{dx}{9x^2-25} = \frac{dx}{(3x-5)(3x+5)}$$
$$\int \frac{R(x)}{D(x)} = \int \frac{A}{x-x_1}dx + \int \frac{B}{x-x_2}dx = \frac{A(x-x_2)+B(x-x_1)}{(x-x_1)(x-x_2)} = A\ln|x-x_2| + B\ln|x+x_1|$$
$$A(3x-5)+B(3x+5) \Rightarrow \begin{cases}(3A+3B)x=0\\ -A5+5B=1\end{cases} \Rightarrow \begin{cases}3A+3B=0\\ -5A+5B=1\end{cases} \Rightarrow \begin{cases}\frac{\cancel{3}A}{\cancel{3}} = \frac{-\cancel{3}B}{\cancel{3}}\\ -5A+5B=1\end{cases}$$
$$\begin{cases}A=-B\\ +5B+5B=1\end{cases} \quad \begin{cases}A=-B\\ 10B=1\end{cases} \rightarrow \begin{cases}A=-B\\ B=\frac{1}{10}\end{cases} \quad \begin{cases}A=-\frac{1}{10}\\ B=\frac{1}{10}\end{cases}$$
$$\int \frac{-\frac{1}{10}}{3x+5} + \int \frac{\frac{1}{10}}{3x-5} = -\frac{1}{10}\int \frac{1}{3x+5} + \frac{1}{10}\int \frac{1}{3x-5} = -\frac{1}{10}\ln(3x+5) + \frac{1}{10}\ln(3x-5)$$

7.0.2 es.3

$$\int \frac{dx}{x^2-x-6}$$
$$\int \frac{dx}{(x-3)(x+2)}$$
$$\int \frac{R(x)}{D(x)} = \frac{A}{x+x_1}dx + \int \frac{B}{x+x_2}dx = A\ln(|x-x_2|) + B(|x+1|)$$
$$\frac{A}{x+3} + \frac{B}{x-2} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$
$$A(x+2)+B(x-3) \rightarrow \begin{cases}(A+B)x=0\\ 2A-3B=1\end{cases} \rightarrow \begin{cases}A=-B\\ 2A-3B=1\end{cases} \rightarrow \begin{cases}A=-B\\ -2B-3B=1\end{cases}$$
$$\begin{cases}A=-B\\ -5B=1\end{cases} \rightarrow \begin{cases}A=-B\\ -5B=1\end{cases} \rightarrow \begin{cases}A=-B\\ \frac{-\cancel{5}B}{\cancel{5}} = -\frac{1}{5}\end{cases} \rightarrow \begin{cases}A=\frac{1}{5}\\ B=-\frac{1}{5}\end{cases}$$

7.0.3 es. 4

$$\int \frac{2x-1}{x^2-5x+6}dx$$
$$\frac{2x-1}{(x-2)(x-3)}$$
$$\int \frac{R(x)}{D(x)} = \int \frac{A}{x+x_1} + \int \frac{B}{x+x_2} = \frac{A(x+x_2)+B(x+x_1)}{(x+x_1)(x+x_2)} = A\ln|x+x_2| + B\ln|x+x_1|$$
$$A(x-3)+B(x-2) \rightarrow \begin{cases}(A+B)x=2\\ -3A-2B=1\end{cases} \rightarrow \begin{cases}A=-B+2\\ -3(-B+2)-2B=1\end{cases} \rightarrow \begin{cases}A=-B+2\\ 3B-6-2B=1\end{cases}$$
$$\begin{cases}A=-B+2\\ B=7\end{cases} \rightarrow \begin{cases}A=-7+2\\ B=7\end{cases} \rightarrow \begin{cases}A=-5\\ B=7\end{cases}$$
$$\int \frac{-5}{x-2} + \int \frac{7}{x-3} = -5\ln|x-2| + 7\ln|x-3|$$

7.1 differenziali

$$\begin{cases} y'' - 8y' + 15y = 2e^{3x} \\ y(1) = 1, y' = 0 \end{cases}$$

$$t = y'$$

$$t^2 - 8t + 15 = 2e^{3x}$$

$$\Delta = b^2 - 4(a)(c) = 64 - 60 = 4$$

$$t_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{8 \pm 2}{2} = \begin{cases} \frac{8+2}{2} = \frac{\cancel{10}}{\cancel{2}} = 5 \\ \frac{8-2}{2} = \frac{\cancel{6}}{\cancel{2}} = 3 \end{cases}$$

$$(\lambda - 3)(\lambda - 5)$$

$$y_p(x) = Axe^{3x}$$

$$y'p(x) = A(3x+1)e^{3x}$$

$$y''(x) = A(9x+6)e^{3x}$$

$$A[9x + 6 - 8(3x+1) + 15x]e^{3x} = 2e^{3x}$$

$$\Delta = -1$$

$$y(x) = c_1 e^{3x} + c_2 e^{5x} - xe^{3x}$$

$$\begin{cases} y'(x) \\ y(0) = 0 = y'(0) \end{cases} \quad \begin{cases} c_1 + c_2 = 0 \\ 3c_1 + 5c_2 - 1 = 0 \end{cases} \quad \begin{cases} c_1 = -c_2 \\ -3c_2 + 6c_2 - 1 = 0 \end{cases} \quad \begin{cases} c_1 = -c_2 \\ \frac{2}{2} = \frac{1}{2} \end{cases} \rightarrow \begin{cases} c_1 = -\frac{1}{2} \\ c_2 = \frac{1}{2} \end{cases}$$

Soluzione di Cauchy

$$y(x) = -\frac{1}{2}e^{3x} + \frac{1}{2}e^{5x} - xe^{3x}$$

7.1.1 es.2

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 3; y'(0) = 2 \end{cases}$$

$$p(\lambda) = \lambda^2 + 4\lambda + 4\lambda$$

$$\Delta = 4^2 - 4(1)(4) = 16 - 16 = 0$$

$$y(y) = c_1 x + c_2 x - e^{-2x}$$

$$\begin{cases} y'(x) \\ y(0) = c_2 = 3 \end{cases} \quad \begin{cases} y'(x) = (c_1 - 2c_1 x - 2c_2)e^{-2x} \\ y'(0) = c_1 - 2c_2 = c_1 - 6 \end{cases}$$

$$\begin{cases} y'(x) = (c_1 - 2c_1 x - 2c_2)e^{-2x} \\ y'(0) = 8 \end{cases}$$

$$y(x) = (8x+3)e^{-2x}$$

7.2 es.3

$$y'' - 5y' + 6y = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\Delta = (-5)^2 - 4(1)(6) = 1$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{1}}{2a} = \begin{cases} \frac{5-1}{2} = 2 \\ \frac{5+1}{2} = 3 \end{cases}$$

$$y(0) = c_1 * e^{2x} + c_2 * e^{3x}$$

c_1 e c_2 fanno parte dei reali

$$\begin{cases} y'' - 6y' + 9y = 0 \\ y(0) = 1 \\ y'(0) = 1 \end{cases}$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$\Delta = (-6)^2 - 4(1)(9) = 36 - 36 = 0$$

$$\lambda_{1,2} = \tfrac{6 \pm \sqrt{0}}{2a} = \tfrac{6-0}{2} = 3$$

$$y_0(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_2 x} \qquad\qquad c_1, c_2 \in \mathbb{R}$$

$$y_0'(x) = 3xc_1 e^{3x} + c_2(e^{3x} + x3e^{3x})$$

$$= 3xc_1 e^{3x} + c_2 e^{3x} + 3xc_2 e^{3x}$$

$$\text{sostituire con } \begin{cases} y_0(0) = 1 \\ y_0'(0) = 1 \end{cases}$$

7.3 es.4

$$\begin{cases} y'' + 4y' + 4y = 0 \\ y(0) = 3; y(0) = 2 \end{cases}$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\Delta = 4^2 - 4(1)(4) = 0$$

$$\lambda_{1,2} = \tfrac{-4 \pm \sqrt{0}}{2} = \tfrac{-4}{2} = -2$$

$$y_0(x) = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_2 x} = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y_0'(x) = -2c_1 e^{-2x} + c_2(e^{-2x} - 2xe^{-2x})c_1 = 3; c_2 = 2$$

$$y(x) = 3e^{-2x} + 2e^{-2x} - xe^{-2x}$$

$$y(x) = 5e^{-2x} - xe^{-2x}$$

7.4 es.5

$$\begin{cases} y'' - 5y' + 6y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$\Delta = -5^2 - 4(1)(6) = 25 - 24 = 1$$
$$\lambda_{1,2} = \tfrac{5 \pm \sqrt{1}}{2a} = \begin{cases} \frac{6}{2} = 3 \\ \frac{4}{2} = 2 \end{cases}$$
$$y_0(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 e^{3x} + c_2 e^{2x}$$
$$y_0'(x) = 3xc_1 e^{3x} + c_2(e^{2x} + 2e^{2x})$$
$$z_1(x) = e^{3x};\ z_2(x) = e^{2x}$$
$$\begin{cases} y(0) = c_1(1) + c_2(1) \\ y'(0) = 3(0)c_1 e^{3(0)} + c_2(e^{2(0)} + 2e^{2(0)}) \end{cases} \rightarrow \begin{cases} y(0) = c_1(1) + c_2(1) \\ y'(0) = c_2(3) = 1 \rightarrow c_2 = \frac{1}{3} \end{cases}$$
$$\rightarrow \begin{cases} y(0) = c_1(1) + c_2(1) \rightarrow c_1 = -c_2 \rightarrow c_1 = -\frac{1}{3} \\ c_2 = \frac{1}{3} \end{cases}$$
$$y(x) = -\tfrac{1}{3} e^{3x} + \tfrac{1}{3} e^{2x}$$

7.4.1 es.6

$$\begin{cases} y'' - 2y - 8y = 0 \\ y(1) = 1,\ y'(1) = 0 \end{cases}$$
$$\lambda^2 - 2\lambda - 8$$
$$\Delta = b^2 - 4(a)(c) = 4 + 32 = 36$$
$$x_{1,2} = \tfrac{2 \pm \sqrt{36}}{2} = \begin{cases} \frac{2+6}{2} = 4 \\ \frac{2-6}{2} = -2 \end{cases}$$
$$y_0(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} = c_1 e^{4x} + c_2 e^{-2x}$$
$$y_0' = 4xc_1 e^{4x} + c_2\left(e^{-2x} - 2e^{-2x}\right) z_1(x) = e^{4x}; z_2(x) = e^{-2x}$$
$$\begin{cases} y(1) = c_1 e^4 + c_2 e^{-2} = 1 \\ y'(1) = 4c_1 e^4 + c_2(e^{-2} - 2e^{-2}) = 0 \end{cases}$$
$$\text{se } x = 0\ e^x = 1$$
$$\begin{cases} y(1) = c_1 = -c_2 \\ y'(1) = 4(-c_2) + c_2(e^{-2} - 2e^{-2}) = 0 \end{cases} \rightarrow \begin{cases} y(1) = c_1 = -c_2 \\ y'(1) = 4(-c_2) + c_2 = 0 \end{cases} \rightarrow \begin{cases} y(1) = c_1 = -c_2 \\ y'(1) = -4c_2 + c_2 = 0 \end{cases}$$