Short course on dynamical systems and single neuron models

Johnatan (Yonatan) Aljadeff

aljadeff@ucsd.edu

September 22, 2020

Outline

Linear dynamical systems, 1 dimension

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax, \qquad x(T) = ?$$

Linear dynamical systems, 1 dimension

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax, \qquad x(T) = ?$$

$$\frac{\mathrm{d}x}{x} = a\mathrm{d}t$$

Linear dynamical systems, 1 dimension

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= ax, \qquad x(T) = ? \\
\frac{\mathrm{d}x}{x} &= a\mathrm{d}t \\
\int_{x(0)}^{x(T)} \frac{\mathrm{d}x}{x} &= a \int_0^T \mathrm{d}t
\end{aligned}
$$

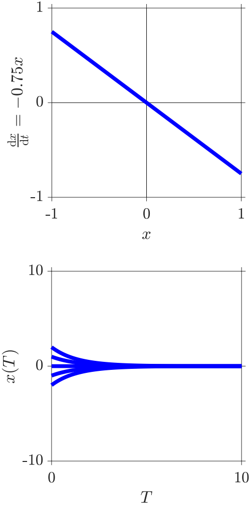
Linear dynamical systems, 1 dimension

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= ax, \qquad x(T) = ? \\
\frac{\mathrm{d}x}{x} &= a\mathrm{d}t \\
\int_{x(0)}^{x(T)} \frac{\mathrm{d}x}{x} &= a \int_0^T \mathrm{d}t \\
\log\left(\frac{x(T)}{x(0)}\right) &= aT
\end{aligned}
$$

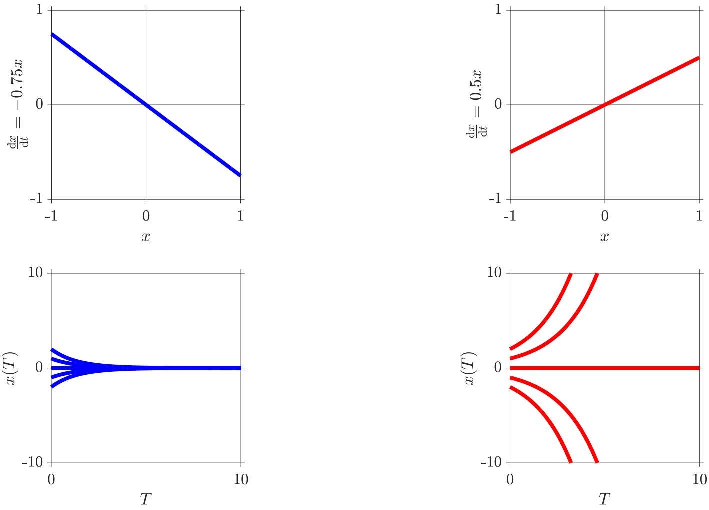
Linear dynamical systems, 1 dimension

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= ax, \qquad x(T) = ? \\
\frac{\mathrm{d}x}{x} &= a\mathrm{d}t \\
\int_{x(0)}^{x(T)} \frac{\mathrm{d}x}{x} &= a\int_0^T \mathrm{d}t \\
\log\left(\frac{x(T)}{x(0)}\right) &= aT \\
x(T) &= \exp(aT)x(0)
\end{aligned}
$$

What do solutions look like?



What do solutions look like?



Even this very simple system goes a long way

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V$$

Even this very simple system goes a long way

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V$$

add input,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V + I$$

Even this very simple system goes a long way

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V$$

add input,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V + I$$

add reset,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= -V + I \\ V &= V_r \text{ if } V > \theta\end{aligned}$$

Even this very simple system goes a long way

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V$$

add input,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V + I$$

add reset,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= -V + I \\ V &= V_r \text{ if } V > \theta\end{aligned}$$

Recall Ashley Juavinett's lecture:
This is a Leaky Integrate and Fire (LIF) neuron.

Even this very simple system goes a long way

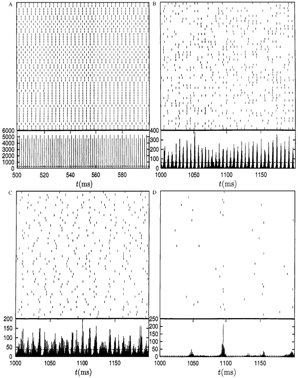
$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V$$

add input,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -V + I$$

add reset,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= -V + I \\ V &= V_r \text{ if } V > \theta\end{aligned}$$



Brunel, 2000

Recall Ashley Juavinett's lecture: This is a Leaky Integrate and Fire (LIF) neuron.

Tomorrow we'll investigate the LIF neuron when the input I is noisy.

Linear dynamical systems, more than 1 dimension

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \qquad \mathbf{x}(T) = \begin{pmatrix} x_1(T) \\ x_2(T) \\ \vdots \\ x_N(T) \end{pmatrix} = ?$$

Linear dynamical systems, more than 1 dimension

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \qquad \mathbf{x}(T) = \begin{pmatrix} x_1(T) \\ x_2(T) \\ \vdots \\ x_N(T) \end{pmatrix} = ?$$

Recall Marcelo Mattar's lecture:

$$A = U\tilde{A}U^{-1}$$

Linear dynamical systems, more than 1 dimension

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \qquad \mathbf{x}(T) = \begin{pmatrix} x_1(T) \\ x_2(T) \\ \vdots \\ x_N(T) \end{pmatrix} = ?$$

Recall Marcelo Mattar's lecture:

$$\begin{aligned} A &= U\tilde{A}U^{-1} \\ &= U \begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & & \\ \vdots & & \ddots & \\ 0 & & & a_N \end{pmatrix} U^{-1} \end{aligned}$$

Linear dynamical systems, more than 1 dimension

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \qquad A = U\tilde{A}U^{-1}, \qquad \tilde{x} = U^{-1}x$$

Linear dynamical systems, more than 1 dimension

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= A\mathbf{x}, \qquad A = U\tilde{A}U^{-1}, \qquad \tilde{x} = U^{-1}x \\
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= U\tilde{A}U^{-1}\mathbf{x}
\end{aligned}$$

Linear dynamical systems, more than 1 dimension

$$
\begin{aligned}
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= A\mathbf{x}, \qquad A = U\tilde{A}U^{-1}, \qquad \tilde{x} = U^{-1}x \\
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= U\tilde{A}U^{-1}\mathbf{x} \\
\frac{\mathrm{d}U^{-1}\mathbf{x}}{\mathrm{d}t} &= U^{-1}U\tilde{A}U^{-1}\mathbf{x}
\end{aligned}
$$

Linear dynamical systems, more than 1 dimension

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x}, \qquad A = U\tilde{A}U^{-1}, \qquad \tilde{x} = U^{-1}x$$

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = U\tilde{A}U^{-1}\mathbf{x}$$

$$\frac{\mathrm{d}U^{-1}\mathbf{x}}{\mathrm{d}t} = U^{-1}U\tilde{A}U^{-1}\mathbf{x}$$

$$\frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \tilde{A}\tilde{\mathbf{x}}$$

Linear dynamical systems, more than 1 dimension

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \tilde{A}\tilde{\mathbf{x}}$$

Was it worth the trouble?

Linear dynamical systems, more than 1 dimension

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \tilde{A}\tilde{\mathbf{x}}$$

Was it worth the trouble? Yes!

$$\begin{pmatrix} \frac{\mathrm{d}\tilde{x}_1}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}\tilde{x}_N}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} a_1\tilde{x}_1 \\ \vdots \\ a_N\tilde{x}_N \end{pmatrix}$$

Linear dynamical systems, more than 1 dimension

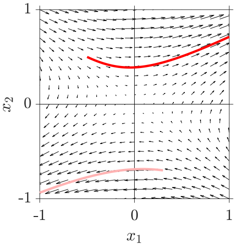
$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = A\mathbf{x} \qquad \Longrightarrow \qquad \frac{\mathrm{d}\tilde{\mathbf{x}}}{\mathrm{d}t} = \tilde{A}\tilde{\mathbf{x}}$$

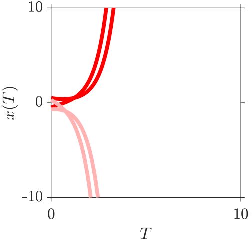
Was it worth the trouble? Yes!

$$\begin{pmatrix} \frac{\mathrm{d}\tilde{x}_1}{\mathrm{d}t} \\ \vdots \\ \frac{\mathrm{d}\tilde{x}_N}{\mathrm{d}t} \end{pmatrix} = \begin{pmatrix} a_1\tilde{x}_1 \\ \vdots \\ a_N\tilde{x}_N \end{pmatrix}, \qquad \mathbf{x}(T) = \exp(AT)\mathbf{x}(0)$$

What do solutions look like?

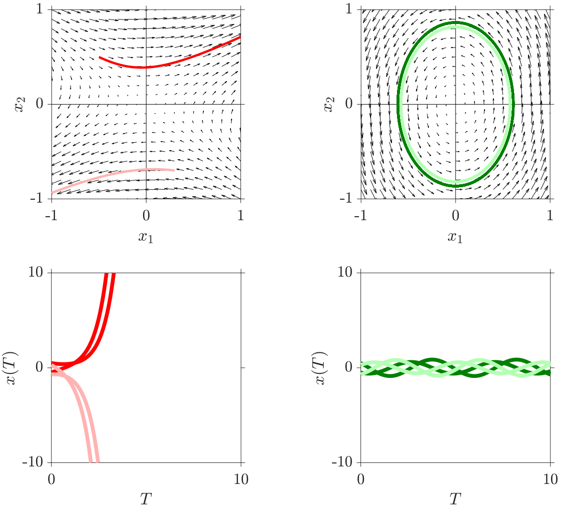
$$a = \begin{pmatrix} 0.5 & 2 \\ 0.8 & 0.1 \end{pmatrix}$$





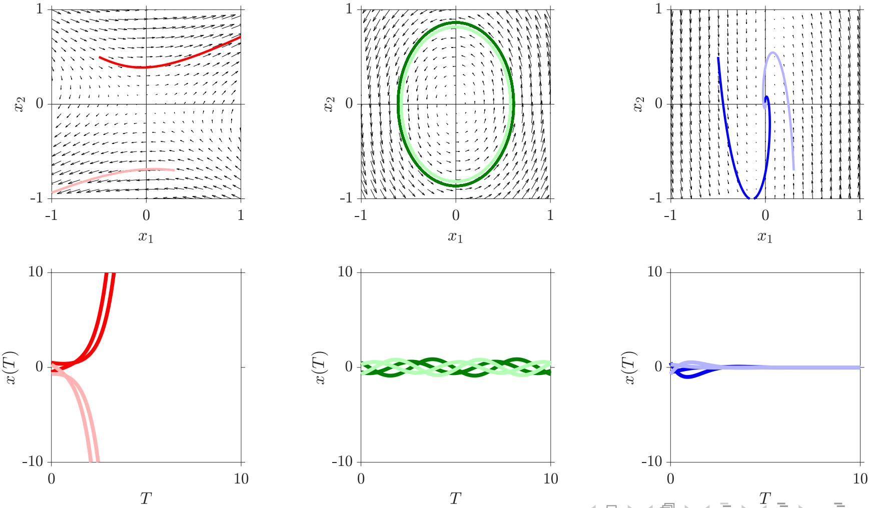
What do solutions look like?

$$a = \begin{pmatrix} 0.5 & 2 \\ 0.8 & 0.1 \end{pmatrix} \qquad a = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$$

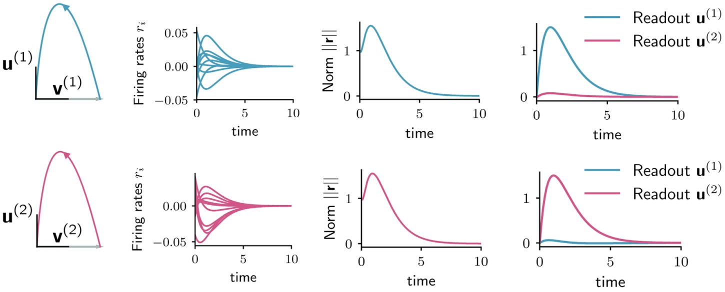


What do solutions look like?

$$a = \begin{pmatrix} 0.5 & 2 \\ 0.8 & 0.1 \end{pmatrix} \qquad a = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \qquad a = \begin{pmatrix} -0.9 & -0.2 \\ 7 & -1 \end{pmatrix}$$



Even this very simple system goes a long way



Bondanelli, Ostojic , 2020

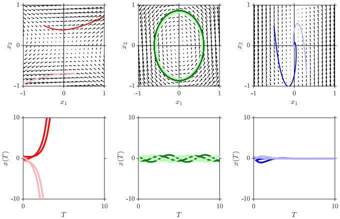
Different initial conditions lead to different transients (before the decay to 0).
It has been suggested that the initial condition is the preparatory neuronal activity before movement.

Linear vs. Nonlinear dynamical systems

In nonlinear dynamical systems, changing the coordinate x can lead to qualitative change in the behavior.

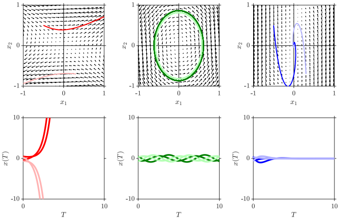
Linear vs. Nonlinear dynamical systems

In nonlinear dynamical systems, changing the coordinate x can lead to qualitative change in the behavior.



Linear vs. Nonlinear dynamical systems

In nonlinear dynamical systems, changing the coordinate x can lead to qualitative change in the behavior.



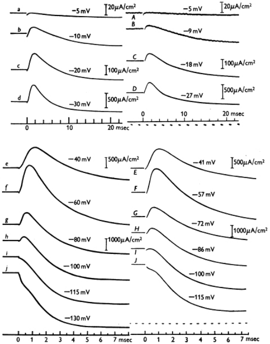


Fig. 11. Left-hand column: time course of membrane current during voltage clamp, calculated for temperature of 4° C from eqn. (26) and subsidiaries and plotted on the same scale as the experimental curves in the right-hand column. Right-hand column: observed time course of membrane currents during voltage clamp. Axon 31 at 4° C; compensated feedback. The time scale changes between *d*, *D* and *e*, *E*. The current scale changes after *b*, *B*; *c*, *C*; *d*, *D* and *f*, *F*.

Hodgkin, Huxley, 1952

Neurons (and networks of neurons) behave nonlinearly !

Nonlinear dynamical systems, 1 dimension

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad x(T) = ?$$

Nonlinear dynamical systems, 1 dimension

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad x(T) = ?$$

We can define a function which is the negative integral of f:

$$U(x) = -\int_0^x f(y)\mathrm{d}y$$

Nonlinear dynamical systems, 1 dimension

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad x(T) = ?$$

We can define a function which is the negative integral of f:

$$U(x) = -\int_0^x f(y)\mathrm{d}y$$

$U(x)$ is sometimes referred to as the “energy landscape”

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}x}U(x)$$

Nonlinear dynamical systems, 1 dimension

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \qquad x(T) = ?$$

We can define a function which is the negative integral of f:

$$U(x) = -\int_0^x f(y)\mathrm{d}y$$

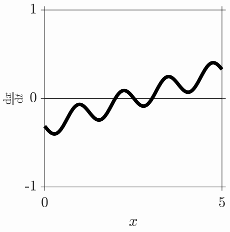
$U(x)$ is sometimes referred to as the "energy landscape"

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}x}U(x)$$

"x is **gradually descending** to the lowest point of U"

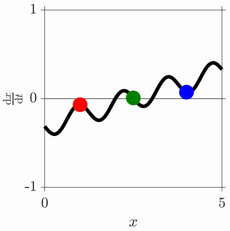
Nonlinear dynamical systems, 1 dimensional example

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{8}\left(x - \sin(5x) - 2.5\right)$$



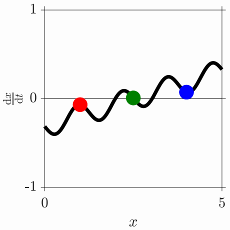
Nonlinear dynamical systems, 1 dimensional example

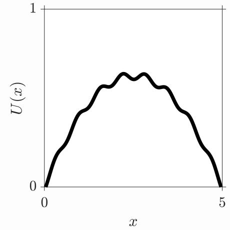
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{8}\left(x - \sin(5x) - 2.5\right)$$



Nonlinear dynamical systems, 1 dimensional example

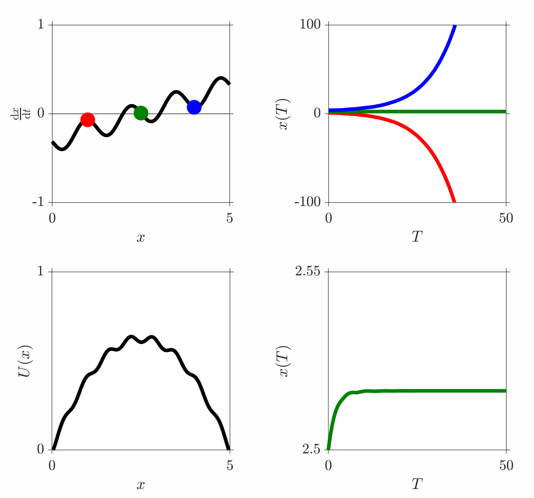
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{8}\left(x - \sin(5x) - 2.5\right)$$





Nonlinear dynamical systems, 1 dimensional example

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{8}\left(x - \sin(5x) - 2.5\right)$$



Nonlinear dynamical systems, more than 1 dimension

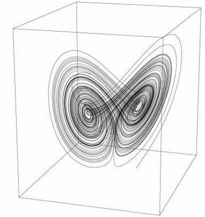
$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

Nonlinear dynamical systems, more than 1 dimension

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f(x)\\ \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(T) = ?\end{aligned}$$

Nonlinear dynamical systems, more than 1 dimension

Now the repertoire of dynamical behaviors is much broader: oscillations, chaos, etc.



Lorenz system

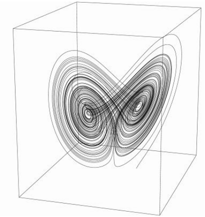
$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= f(x) \\ \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} &= \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(T) = ?\end{aligned}$$

Nonlinear dynamical systems, more than 1 dimension

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x)$$

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x}(T) = ?$$

Now the repertoire of dynamical behaviors is much broader: oscillations, chaos, etc.



Lorenz system

In general it is difficult to analyze the system.
In most cases there is no energy function $U(\mathbf{x})$ that is minimized.

Dynamical model of single neurons, Hodgkin Huxley

$$
\begin{aligned}
C\frac{\mathrm{d}V}{\mathrm{d}t} &= -\bar{g}_{\mathrm{K}}n^4(V - V_{\mathrm{K}}) - \bar{g}_{\mathrm{Na}}m^3h(V - V_{\mathrm{Na}}) - \bar{g}_{\mathrm{l}}(V - V_{\mathrm{l}}) + I \\
&\quad \text{(membrane potential)} \\
\frac{\mathrm{d}n}{\mathrm{d}t} &= \alpha_n(V)(1 - n) + \beta_n(V)n \\
&\quad \text{(potassium activation)} \\
\frac{\mathrm{d}m}{\mathrm{d}t} &= \alpha_m(V)(1 - m) + \beta_m(V)m \\
&\quad \text{(sodium activation)} \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= \alpha_h(V)(1 - h) + \beta_h(V)h \\
&\quad \text{(sodium inactivation)}
\end{aligned}
$$

Dynamical model of single neurons, Hodgkin Huxley

$$
\begin{aligned}
C\frac{\mathrm{d}V}{\mathrm{d}t} &= -\bar{g}_{\mathrm{K}} n^4 (V - V_{\mathrm{K}}) - \bar{g}_{\mathrm{Na}} m^3 h (V - V_{\mathrm{Na}}) - \bar{g}_{\mathrm{l}} (V - V_{\mathrm{l}}) + I \\
&\quad \text{(membrane potential)} \\
\frac{\mathrm{d}n}{\mathrm{d}t} &= \alpha_n(V)(1 - n) + \beta_n(V) n \\
&\quad \text{(potassium activation)} \\
\frac{\mathrm{d}m}{\mathrm{d}t} &= \alpha_m(V)(1 - m) + \beta_m(V) m \\
&\quad \text{(sodium activation)} \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= \alpha_h(V)(1 - h) + \beta_h(V) h \\
&\quad \text{(sodium inactivation)}
\end{aligned}
$$

This is written in the familiar form

$$
\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x})
$$

... but where does all this come from ?

Dynamical model of single neurons, Hodgkin Huxley

Brief walk-through components of HH equations

Total current across the membrane $=$
Capacitance $\times$ Change in membrane potential

$$C\frac{\mathrm{d}V}{\mathrm{d}t} \quad = \quad \text{ionic, leak and input currents}$$

Dynamical model of single neurons, Hodgkin Huxley

Brief walk-through components of HH equations

Total current across the membrane =
Capacitance × Change in membrane potential

$$C\frac{\mathrm{d}V}{\mathrm{d}t} = \text{ionic, leak and input currents}$$

Ionic / Leak current =
conductance × membrane potential relative to reversal potential

$$\begin{aligned}
I_{\mathrm{K}} &= -\bar{g}_{\mathrm{K}} n^4 (V - V_{\mathrm{K}}) \\
I_{\mathrm{Na}} &= -\bar{g}_{\mathrm{Na}} m^3 h (V - V_{\mathrm{Na}}) \\
I_{\mathrm{l}} &= -\bar{g}_{\mathrm{l}} (V - V_{\mathrm{l}})
\end{aligned}$$

Dynamical model of single neurons, Hodgkin Huxley

Brief walk-through components of HH equations

Voltage dependence of ion channel activation/inactivation–

$$
\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}t} &= \alpha_n(V)(1-n) + \beta_n(V)n \\
\alpha_n(V) &= \frac{1 - V/10}{e^{1-V/10} - 1} \\
\beta_n(V) &= 0.125e^{-V/80}
\end{aligned}
$$

... and similarly (with different parameters) for m and h.

The plan for the rest of today:

Investigate the equations,

$$
\begin{aligned}
C\frac{\mathrm{d}V}{\mathrm{d}t} &= -\bar{g}_{\mathrm{K}} n^4 (V - V_{\mathrm{K}}) - \bar{g}_{\mathrm{Na}} m^3 h (V - V_{\mathrm{Na}}) - \bar{g}_{\mathrm{l}} (V - V_{\mathrm{l}}) + I \\
\frac{\mathrm{d}n}{\mathrm{d}t} &= \alpha_n(V)(1-n) + \beta_n(V)n \\
\frac{\mathrm{d}m}{\mathrm{d}t} &= \alpha_m(V)(1-m) + \beta_m(V)m \\
\frac{\mathrm{d}h}{\mathrm{d}t} &= \alpha_h(V)(1-h) + \beta_h(V)h
\end{aligned}
$$

We'll develop some understanding when these details are necessary, and when we can get away with simplifications.