Three dimensional magnetostatic problem

B. Bandelier†‡, C. Daveau†‡, J. Laminie §,
S. M. Mefire§ and F. Rioux-Damidau†
† U2R2M, URA CNRS 2212, Bâtiment 220, 91405 Orsay Cedex, France,
‡ Université Cergy-Pontoise, Rue d'Eragny, 95031 Cergy Pontoise Cedex, France,
§ Laboratoire d'Analyse Numérique, Université Paris-Sud, Bâtiment 425, 91405 Orsay, France.

Abstract

The paper is devoted to an approximation of the solution of Maxwell's equations in three-dimensional space. We present two methods which couple a finite element method inside the magnetic materials with a boundary integral method which uses Poincaré-Steklov's operator to describe the exterior domain. A computer code has been implemented fo each method and a number of numerical experiments have been performed to validate each proposed methodology. Namely, we present numerical results concerning a non-linear magnetostatic problem in $\mathbb{R}^3$.

Key words. magnetostatic, finite element, boundary integral method

1 Introduction

The purpose of this paper is to analyze a procedure obtained by coupling the boundary integral method and the usual finite element method. Such coupled procedures have been proposed for instance by Silvester-Hsieh [14] and by Zienkiewicz et al. [17] for the numerical solutions of problems in unbounded domains.

We are specially interested in Maxwell's equations when applied to magnetostatics in the whole three-dimensional space. In 1989, F. Kikuchi proposed a formulation working directly with the magnetic field and using known boundary conditions [10]. A similar mixed method using the magnetic induction and scalar potential as unknown quantities was established by R. Stenberg and P. Trouvé; it also uses boundary conditions [15].

In this research, we consider a magnetostatic problem in $\mathbb{R}^3$. The aim is to obtain a mathematical model and a numerical scheme which give the magnetic field in $\mathbb{R}^3$ without having to impose boundary conditions; it makes use of conforming finite elements in $H(\text{curl})$ and boundary elements.

An outline of the paper is as follows: in Section 2, we describe the problem and introduce some notations. In Section 3, we present two finite element methods (FEM) for the magnetostatic problem in $\mathbb{R}^3$, based on a mixed formulation and a penalty formulation. In Section 4, by using a boundary integral method, we re-formulate the mixed formulation on Ω. Section 5 is devoted to the approximation of the problem. Numerical tests are presented in Section 6 to validate the proposed method. In section 7, we describe a non linear magnetostatic problem in $I\!R^3$. We explain the iterative method which takes into account the nonlinearity of the magnetic materials and we finally compare numerical results obtained by the mixed method and the penalty method with the experimental measures which are available.

2 The mathematical model

2.1 Description of the problem

We consider a domain Ω made of materials having a permeability μ. At first, we consider that μ only depends on point $x = (x_1, x_2, x_3) \in I\!R^3$ and later we will solve a nonlinear problem (TEAM problem 13) where μ depends on $\mathbf{h}$. Then, in Ω the permeability verifies:

$$0 < \mu_0 \leq \mu(x) \leq m$$

where μ_0 is the permeability of the vacuum and m is a real. Besides, we denote by Γ the boundary of Ω and we assume each connected component of Ω is simply connected.

We also consider an inductor Ω^s in which flows a current of density j independent upon time; it creates a source field h^s when Ω is missing. Domains Ω and Ω^s are bounded and disjoint. Ω^s is thus included in Ω^c, the complementary of Ω and its permeability is μ_0.

First, the magnetostatic problem can be written as follows: Find $\mathbf{h}$ such as

$$\begin{aligned} \operatorname{curl} \mathbf{h} &= \mathrm{j}, \\ \operatorname{div} \mathbf{b} &= 0, \\ \mathbf{b} &= \mu \mathbf{h}. \end{aligned}$$

The current density j is given and it verifies $\operatorname{div} \mathrm{j} = 0$. The vector fields $\mathbf{h}(x)$ and $\mathbf{b}(x)$ respectively represent the magnetic field and the magnetic induction at any point in $I\!R^3$. In addition, we assume that at infinity all fields are equal to zero.

We now want to formulate the problem by bringing h^s to the fore . For this, we define the new variable:

$$\mathbf{h}^r = \mathbf{h} - \mathrm{h}^s$$

called the magnetic reaction field.
We note h^s the source field which verifies:

$$\begin{aligned} \operatorname{curl} \mathrm{h}^s &= \mathrm{j}, \\ \operatorname{div} \mathrm{h}^s &= 0. \end{aligned}$$

We define $(L^2(I\!R^3))^3$ as the space of square integrable fields over $I\!R^3$ and

$$(H^1(I\!R^3))^3 = \{u \in (L^2(I\!R^3))^3, \text{ such as } \operatorname{grad} u \in (L^2(I\!R^3))^3\}.$$

With a Fourier transform, we can show that if $\mathrm{j} \in (L^2(I\!R^3))^3$ then $\mathrm{h}^s \in (H^1(I\!R^3))^3$.

The problem then becomes:
Problem (A):
For a given h^s in $(H^1(I\!R^3))^3$, find $\mathbf{h}^r \in H(\mathrm{curl}, I\!R^3)$ such that:

$$\operatorname{div} \mu(\mathbf{h}^r + \mathrm{h}^s) = 0 \text{ in } I\!R^3, \tag{1}$$

$$\operatorname{curl} \mathbf{h}^r = 0 \text{ in } I\!R^3. \tag{2}$$

In the framework of L^2-theory, we will establish a weak mixed formulation of problem (A).

2.2 Notations and Spaces

We call n the inward normal to Γ. Let $(L^2(I\!R^3))^3$ and $(H^1(I\!R^3))^3$ be the usual Hilbert spaces equipped with the natural scalar products $(\,.\,,\,.\,)_{(L^2(I\!R))^3}$ and $(\,.\,,\,.\,)_{(H^1(I\!R^3))^3}$ respectively. We also consider the usual Sobolev spaces $(H^1(\Omega))^3$ and $(H^{\frac{1}{2}}(\Gamma))^3$.

Following Duvaut and Lions [8], we define:

$$H(\mathrm{curl}, I\!R^3) = \{\mathrm{u} \in (L^2(I\!R^3))^3; \operatorname{curl} \mathrm{u} \in (L^2(I\!R^3))^3\};$$

$$H(\mathrm{curl}, \Omega) = \{\mathrm{u} \in (L^2(\Omega))^3; \operatorname{curl} \mathrm{u} \in (L^2(\Omega))^3\}$$

and the associated norms $||\ \ ||_{H(curl,\, I\!R^3)}$ and $||\ \ ||_{H(curl,\, \Omega)}$.
We also consider the following spaces:

- H_0 defined by $H_0 = \{\mathrm{u} \in H(\mathrm{curl}, I\!R^3);\ \operatorname{curl} \mathrm{u} = 0 \text{ in } \Omega^c\}$.
  It is an Hilbert space equipped with the norm:

  $$\|u\|_{H_0} = (\|\mathrm{u}\|^2_{(L^2(I\!R^3))^3} + \|\operatorname{curl} \mathrm{u}\|^2_{(L^2(\Omega))^3})^{\frac{1}{2}}\,.$$

- $W_0^1(\Omega^c)$ defined in Ω^c[13]:

  $$W_0^1(\Omega^c) = \{\psi \in D'(\Omega^c)\,;\ (1+r^2)^{\frac{-1}{2}}\psi \in L^2(\Omega^c),\ \frac{\partial \psi}{\partial x_i} \in L^2(\Omega^c),\ \ i = 1, 2, 3\},$$

  where x_i are the components of vectors $\{x_1, x_2, x_3\}$. Besides, this space is equipped with the usual norm $||\ \ ||_{W_0^1(\Omega^c)}$.

- and also the classical space $H^{1/2}(\Gamma)$ of scalar functions defined on the boundary of Ω with the associated norm:

  $$\|\varphi\|_{H^{\frac{1}{2}}(\Gamma)} = \inf_{\psi \in W_0^1(\Omega^c);\, \psi|_\Gamma = \varphi} \|\operatorname{grad} \psi\|_{(L^2(\Omega^c))^3}\,;$$

  which is the Hilbert space of the traces of functions in $W_0^1(\Omega^c)$.

We denote by $H^{\frac{-1}{2}}(\Gamma)$ the dual space of $H^{\frac{1}{2}}(\Gamma)$ and the duality product between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$ is denoted by $<\,,\,>$.

3 Types of FEM Formulations

3.1 Mixed formulation

Using Poincaré's lemma, [5], we first introduce from (1) $\mathrm{v} \in (W_0^1(I\!R^3))^3$ such as:

$$\mu(\mathbf{h}^r + \mathrm{h}^s) = \mathrm{curl}\,\mathrm{v}, \tag{3}$$

v is unique if one imposes: $\mathrm{div}\,\mathrm{v} = 0$.
We have, using Stokes's formula from (3):

$$(\mu\mathbf{h}^r, \mathrm{h}')_{(L^2(I\!R^3))^3} - (\mathrm{curl}\,\mathrm{h}', \mathrm{v})_{(L^2(\Omega))^3} = -(\mu\mathrm{h}^s, \mathrm{h}')_{(L^2(I\!R^3))^3} \quad \forall\,\mathrm{h}' \in H_0.$$

On the other hand, if we take $\mathbf{h}^{\,r} \in H_0$ it comes from (2):

$$(\mathrm{curl}\,\mathbf{h}^r, \mathrm{v}')_{(L^2(I\!R^3))^3} = (\mathrm{curl}\,\mathbf{h}^r, \mathrm{v}')_{(L^2(\Omega))^3} = 0 \quad \forall \mathrm{v}' \in H(\mathrm{curl}, I\!R^3).$$

We thus obtain a mixed formulation for the magnetostatic problem in $I\!R^3$ written as follows.

Problem (B):

$$\left\{ \begin{array}{l} \text{For a given } \mathrm{h}^s \text{ in } (H^1(I\!R^3))^3\ , \\ \text{find } (\mathbf{h}^r, \mathrm{v}) \in H_0 \times H(\mathrm{curl}, \Omega) \text{ such that:} \\ (\mu\mathbf{h}^r, \mathrm{h}')_{L^2(I\!R^3))^3} \ - (\mathrm{curl}\,\mathrm{h}', \mathrm{v})_{(L^2(\Omega))^3} \ = -(\mu\mathrm{h}^s, \mathrm{h}')_{(L^2(I\!R^3))^3} \ \forall \mathrm{h}' \in H_0\,, \\ -(\mathrm{curl}\,\mathbf{h}^r, \mathrm{v}')_{(L^2(\Omega))^3} \ = 0 \ \ \forall \mathrm{v}' \in H(\mathrm{curl}, \Omega)\,. \end{array} \right. \tag{4}$$

Conversely, let $(\mathbf{h}^r, \mathrm{v}) \in H_0 \times H(\mathrm{curl}, \Omega)$ be a solution of Problem (B). In the first equation, taking $\mathrm{h}' = \mathrm{grad}\,\phi$ with $\phi \in W_0^1(I\!R^3)$, with Green's formula we have:

$$\mathrm{div}(\mu(\mathbf{h}^r + \mathrm{h}^s)) = 0 \text{ in } I\!R^3.$$

On the other hand, the second equation implies that $\mathrm{curl}\,\mathbf{h}^r = 0$ in $I\!R^3$. Thus, $\mathbf{h}^r$ is the solution of Problem (A).

3.2 Penalty Formulation

We can perturb problem (B) [6]. Let us call $\tau > 0$ a small parameter, we obtain:

$$\left\{ \begin{array}{l} \text{For } \mathrm{h}^s \in \{H^1(I\!R^3)\}^3 \text{ given,} \\ \text{find } \mathbf{h}^r \in H_0 \text{ such that :} \\ (\mu\mathbf{h}^r, \mathrm{h}')_{(L^2(I\!R^3))^3} + \frac{1}{\tau}(\mathrm{curl}\ \ \mathbf{h}^r, \mathrm{curl}\,\mathrm{h}')_{(L^2(\Omega))^3} = -(\mu\mathrm{h}^s, \mathrm{h}')_{(L^2(I\!R^3))^3} \ \ \forall\,\mathrm{h}' \in H_0. \end{array} \right. \tag{5}$$

The penalty method has already been studied in [1]. Consequently we shall from now on only study the mixed method. We shall use a boundary integral method to establish a new mixed formulation in Ω. We shall then compare numerical results for both methods.

4 Boundary method and new mixed hybrid formulation

We now use Poincaré-Steklov's operator also called the Dirichlet-to-Neumann map which is a symetric integral operator to take the exterior domain into account. In Ω^c, the unknown quantity is now a scalar potential ψ instead of the magnetic field such that [3]:

$$\psi \in W_0^1(\Omega^c), \tag{6}$$

$$\Delta\psi = 0 \text{ in } D'(\Omega^c), \tag{7}$$

$$\psi = \varphi \text{ on } \Gamma, \tag{8}$$

$$\frac{\partial\psi}{\partial n} = \mathcal{R}(\varphi) \tag{9}$$

where $\mathcal{R}$ is Poincaré-Steklov's operator; it is an integral symmetric operator combining the trace of ψ on the boundary to its normal derivative. Operator $\mathcal{R}$ is an isometry from $H^{\frac{1}{2}}(\Gamma)$ onto $H^{-\frac{1}{2}}(\Gamma)$.

We know that if we call $[.]_{/\Gamma}$ the jump of the quantity . through Γ then:

$$[\psi]_{/\Gamma} = 0 \text{ and } [\frac{\partial\psi}{\partial n}]_{/\Gamma} \neq 0\,.$$

Of course the jump of the normal derivative represents the discontinuity of the normal component of the magnetic field through Γ.

According to relations (6)-(8) and using Green's formula, we obtain a new mixed formulation from problem (B).

Problem (C):

$$\left\{\begin{array}{l} \text{For a given } \mathrm{h}^s \text{ in } (H^1(I\!R^3))^3, \\ \text{find } (\mathbf{h}, \mathrm{v}, \varphi) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega) \times H^{\frac{1}{2}}(\Gamma) \text{ such that:} \\ \forall \mathrm{h}' \in H(\mathrm{curl}, \Omega),\ \forall \varphi' \in H^{\frac{1}{2}}(\Gamma) \text{ and } \forall \mathrm{v}' \in H(\mathrm{curl}, \Omega) \\ (\mu\mathbf{h}, \mathrm{h}')_{(L^2(\Omega))^3} + < \mu_0\ \mathcal{R}\ (\varphi), \varphi' > -(\mathrm{curl}\, \mathrm{h}', \mathrm{v})_{(L^2(\Omega))^3} = - < \mu_0 \mathrm{h}^s.\mathrm{n}, \varphi' >, \\ -(\mathrm{curl}\, \mathbf{h}, \mathrm{v}')_{(L^2(\Omega))^3} = 0 \end{array}\right. \tag{10}$$

where $\mathbf{h}$ is the total magnetic field and φ the reaction potential.

We can show that problems (B) and (C) are equivalent in the sense: if ($\mathbf{h}_{/\Omega}$, v, φ) is a solution of problem (C), then ($\mathbf{h}$, v) is a solution of problem (B). Conversely if ($\mathbf{h}$, v) is a solution of problem (B), there exists ψ such as ($\mathbf{h}_{/\Omega}$, v, φ) is a solution of problem (C) and ψ satisfies relations (6)-(8).

On the other hand, we are going to show the existence and uniqueness of couple $(\mathbf{h}, \varphi)$ of solution $(\mathbf{h}, \varphi, \mathrm{v})$ of problem (C).

We take $Z = H(\mathrm{curl}, \Omega) \times H^{\frac{1}{2}}(\Gamma)$ Hilbert space equipped with the natural norm $|| \; ||_Z$ and we introduce:

- on $Z \times Z$ the bilinear form

$$\tilde{a}((\mathrm{h}, \varphi), (\mathrm{h}', \varphi')) = (\mu \mathrm{h}, \mathrm{h}')_{(L^2(\Omega))^3} + < \mu_0 \mathcal{R}(\varphi), \varphi' > ;$$

- on $Z \times H(\mathrm{curl}, \Omega)$, the bilinear form

$$\tilde{b}((\mathrm{h}, \varphi), \mathrm{v}) = -(\mathrm{curl}\, \mathrm{h}, \mathrm{v})_{(L^2(\Omega))^3}$$

- and on Z the linear form

$$\tilde{f}(\mathrm{h}, \varphi) = < -\mu_0 \mathrm{h}^s.\mathrm{n}, \varphi > .$$

Let V be the null space of the bilinear continuous form $\tilde{b}$:

$$V = \{(\mathrm{h}, \varphi) \in Z ; \tilde{b}((\mathrm{h}, \varphi), \mathrm{v}) = 0 \;\; \forall \mathrm{v} \in H(\mathrm{curl}, \Omega)\},$$
$$= \{(\mathrm{h}, \varphi) \in Z ; \; \mathrm{curl}\, \mathrm{h} = 0 \text{ in } \Omega\}.$$

Let us consider the problem as follows:
Problem (D):
For a given $\mathrm{h}^s \in (H^1(I\!R^3))^3$, find $(\mathbf{h}, \varphi) \in V$, such as:

$$\tilde{a}((\mathbf{h}, \varphi), (\mathrm{h}', \varphi')) = \tilde{f}(\mathrm{h}', \varphi') \; \forall (\mathrm{h}', \varphi') \in V.$$

We can clearly show that, if $(\mathbf{h}, \mathrm{v}, \varphi)$ is solution of problem (C), $(\mathbf{h}, \varphi)$ is then the solution of problem (D). On the other hand, we have:
- the bilinear form $\tilde{a}$ is coercive on V:

$$\forall (\mathrm{h}, \varphi) \in Z, \;\; \tilde{a}((\mathrm{h}, \varphi), (\mathrm{h}, \varphi)) = (\mu \mathrm{h}, \mathrm{h})_{(L^2(\Omega))^3} + < \mu_0 \mathcal{R}(\varphi), \varphi >;$$

then:

$$\tilde{a}((\mathrm{h}, \varphi), (\mathrm{h}, \varphi)) = ||\mu \mathrm{h}||^2_{(L^2(\Omega))^3} + \mu_0 ||\mathrm{grad}\, \varphi||^2_{(L^2(\Omega^c))^3},$$

and therefore:

$$\tilde{a}((\mathrm{h}, \varphi), (\mathrm{h}, \varphi)) \geq \alpha ||(\mathrm{h}, \varphi)||^2_Z \text{ with } \alpha > 0.$$

- there exists $M > 0$ such as:

$$|\tilde{a}((\mathrm{h}, \varphi), (\mathrm{h}', \varphi'))| \leq M ||(\mathrm{h}, \varphi)||_Z ||(\mathrm{h}', \varphi')||_Z, \quad \forall ((\mathrm{h}, \varphi), (\mathrm{h}', \varphi')) \in Z \times Z;$$

- $\tilde{f}$ is a continuous linear form on Z.
Lax-Milgram's theorem then gives a result of uniqueness and existence for problem (D).

Potential vector v, which is a Lagrange's multiplier is however not unique [2]. Indeed if v is solution of Problem (C), v+grad ϕ is also a solution. But we solve this problem thanks to Uzawa's method which is quite efficient in the solution of a problem of the type of problem (C) when operator $\tilde{A}$ associated, by Riesz theorem, to bilinear form $\tilde{a}$ is invertible.

We finally note that the fundamental unknown quantity is the reaction magnetic field in problem (B) and the total magnetic field in problem (C). So contrary to problem (B), problem (C) permits to determine the total magnetic field in whole $I\!R^3$ with the help of the reaction potential.

Remark:
With the same boundary integral method, we have from equation (5) the penalty formulation [1]:

$$\left\{\begin{array}{l} \text{For a given } \mathrm{h}^s \text{ in } (H^1(I\!R^3))^3, \\ \text{find } (\mathbf{h},\varphi) \in H(\mathrm{curl},\Omega) \times H^{\frac{1}{2}}(\Gamma) \text{ such that:} \\ \forall \mathrm{h}' \in H(\mathrm{curl},\Omega),\ \forall \varphi' \in H^{\frac{1}{2}}(\Gamma) \\ (\mu\mathbf{h},\mathrm{h}')_{(L^2(\Omega))^3} + < \mu_0\ \mathcal{R}\ (\varphi),\varphi' > + \frac{1}{\tau}(\mathrm{curl}\,\mathbf{h},\mathrm{curl}\,h')_{(L^2(\Omega))^3} = - < \mu_0 \mathrm{h}^s.\mathrm{n},\varphi' > . \end{array}\right. \tag{11}$$

5 Approximation of problem (C) and Uzawa's method

5.1 Conforming Finite Element Approximation

Let be $h > 0$ the step of discretization. Following Nédélec's ideas
Next, with all tetrahedra which have at least three vertices on Γ and thus at least one face on Γ, we build a triangulation $\mathcal{T}_h^2$ of boundary Γ: collection of triangles T on Γ.

To discretize field $\mathbf{h}$, we choose a finite element conforming in $H(\mathrm{curl},\ \Omega)$:

$$W_h = \{\mathrm{h}_h \in H(\mathrm{curl},\Omega)\,;\, \mathrm{h}_h|_K = \alpha + \beta \times x,\ \ \forall\, K \in \mathcal{T}_h^1,\ \alpha \in (P_0)^3,\ \beta \in (P_0)^3\}$$

where P_0 is the space of polynomials of degree 0. Then we take for variable φ:

$$Y_h = \{\varphi_h\ \in C^0(\Gamma);\ \varphi_h|_T\ \in P_1, \forall\, T \in \mathcal{T}_h^2\ \}.$$

We know that $Y_h \subset H^1(\Gamma)$, so $Y_h \subset H^{\frac{1}{2}}(\Gamma)$. For space Z, we associate the finite dimension space $Z_h = W_h \times Y_h$.

In a tetrahedron without any vertex on Γ, the magnetic field is written as [3] :

$$\mathbf{h} = \sum_{l=1}^{6} h_l(\lambda_i\,\mathrm{grad}\,\lambda_j - \lambda_j\,\mathrm{grad}\,\lambda_i)$$

where λ_n is the barycentric coordinate associated to vertex n, $l = \{i, j\}$ is the edge of vertices i, j, h_l the circulation of h along edge l:

$$h_l = \int_i^j \mathbf{h} \cdot \mathrm{dl}.$$

The discretization of v is the same. For φ, one has:

$$\varphi = \sum_{i=1}^{3} \varphi_i \, \lambda_i$$

where φ_i is the nodal variable associated to vertex i of a triangle of the boundary.

The degrees of freedom are then :

- the circulation of the magnetic field inside Ω,
- the scalar reaction potential at the vertices of Γ,
- the circulation of potential vector v on all edges of $\overline{\Omega}$.

We can remark that the finite element used for the discretization of the magnetic field in Ω is conforming in $H(\mathrm{curl})$, [6]:

$$\forall (\mathrm{h},\, \varphi) \in Z_h, \ \text{ on every face } F \text{ of } \Gamma, \ \text{ we have } \mathrm{h} \times \mathrm{n} = \mathrm{grad}\, \varphi \times \mathrm{n},$$

where n is the normal vector of F.

On the other hand, to evaluate the boundary integral in Problem (C), we compute an exterior stiffness matrix which relies $\frac{\partial \varphi}{\partial n}$ to φ which is the discretization of Poincaré-Steklov's operator. In order to obtain a symmetric problem, which will facilitate the incorporation of the coupled procedure into finite elements codes, we compute a symmetrical exterior matrix [16].

In the next, we present a numerical formalism using iterative Uzawa's algorithm which is very efficient in the solution of problems of this type.

5.2 Uzawa's method

We shall first consider problem (C) and develop a matrix form suited to numerical computation. We shall set the finite dimension spaces Z_h and W_h with $N = \dim Z_h$, $M = \dim W_h$ and we use a basis of these spaces, namely, $\{z_{ih} | \, 1 \le i \le N\}$ for Z_h and $\{v_{ih} | \, 1 \le i \le M\}$ for V_h. We can now define:

$$\tilde{a}_{ij} = \tilde{a}(z_{jh},\, z_{ih}),$$

$$\tilde{b}_{ij} = \tilde{b}(z_{jh},\, v_{ih}),$$

$$\tilde{f}_i = < \tilde{f},\, z_{ih} > .$$

We put $\tilde{A}_{N \times N} = \tilde{a}_{ij}$, $\tilde{B}_{M \times N} = \tilde{b}_{ij}$, $\tilde{f}_N = (\tilde{f}_i)$ and $z = \{\alpha_i\}$, $v = \{\beta_i\}$ the vectors of $I\!R^N$ and $I\!R^M$ formed by the A coefficients of z_h and v_h:

$$z_h = \sum_{i=1}^{N} \alpha_i z_{ih},$$

$$v_h = \sum_{i=1}^{M} \beta_i v_{ih}.$$

Problem (C) can now be written:

$$\begin{cases} \tilde{A}z & + & \tilde{B}^t v & = & \tilde{f}, \\ & & \tilde{B}z & = & \tilde{g}. \end{cases} \tag{12}$$

Let us note that l. h. s. $\tilde{g}$ comes from the fact that, on Γ, the circulation of the magnetic field h along the edge ij is related to that of the source field h^s and to the reaction potential φ

$$\int_i^j \mathrm{h} \cdot \mathrm{dl} = \int_i^j (\mathrm{h}^s + \mathrm{grad}\,\varphi) \cdot \mathrm{dl} = \int_i^j \mathrm{h}^s \cdot \mathrm{dl} + (\varphi_j - \varphi_i).$$

We solve the system with Schur's complement method: we can eliminate variable z from this system because matrix $\tilde{A}$ is invertible. One gets:

$$z = \tilde{A}^{-1}\tilde{f} - \tilde{A}^{-1}\tilde{B}^t v.$$

We have then to solve:

$$\tilde{B}\tilde{A}^{-1}\tilde{B}^t v = \tilde{B}\tilde{A}^{-1}\tilde{f} + \tilde{B}\tilde{g}.$$

In this case, matrix $\tilde{B}\tilde{A}^{-1}\tilde{B}^t$ is positive semidefinite for matrix $\tilde{A}$ is positive definite:

$$< \tilde{B}\tilde{A}^{-1}\tilde{B}^t v,\, v >_{I\!R^M} = < \tilde{A}^{-1}\tilde{B}^t v,\, \tilde{B}^t v >_{I\!R^M} \geq \alpha ||\tilde{B}^t v||_{I\!R^M}, \quad \alpha > 0.$$

Then the system is solved with conjugate gradient method. The iterations are stopped when ratio $\frac{||\tilde{B}\tilde{A}^{-1}\tilde{B}^t v - \tilde{B}\tilde{A}^{-1}\tilde{f}||}{||\tilde{B}\tilde{A}^{-1}\tilde{f}||}$ falls below a given ϵ_{uz}. On the other hand, matrix A^{-1} is calculated by solving a system with conjugate gradient method and we call ϵ_{gr} the parameter which stops the iterations.
In the next, we shall study the effect of two parameters ϵ_{uz} and ϵ_{gr} on the numerical formalism.

6 Numerical tests

A 3-D finite element program has been developed to test the viability of the method using the mixed method. We shall then compare numerical results for both methods: mixed method and penalty method.

6.0.1 Mixed formulation

The magnetic field is computed for a sphere of constant permeability $\mu = \mu_0\,\mu_r$ and of radius $R = 1m$, put in a constant external field, h^s. The analytical solution of this problem is known [7]:

$$\mathrm{h_a} = \frac{3\mathrm{h}^s}{\mu + 2}.$$

We can also study the error on the computation of the magnetic field and its rotational which is physically nil. We thus define the errors:

$$\mathrm{ErrH} = [\frac{1}{V}\int_\Omega \frac{|\mathrm{h_n} - \mathrm{h}_a|^2}{|\mathrm{h_a}|^2}\, dx]^{\frac{1}{2}},$$

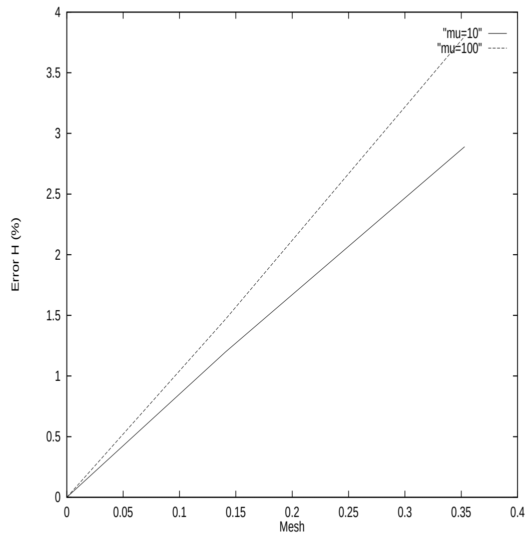

Figure 1: The error on the magnetic field as a function of the mesh.

$$\mathrm{Curl} = [\frac{1}{V}\int_{\Omega} \frac{|\mathrm{curl}\,\mathrm{h_n}|^2}{|\mathrm{h_a}/R|^2}\,dx]^{\frac{1}{2}}$$

where $\mathrm{h_n}$ is the numerical solution provided by the code.

For this study, we use two different meshings of the sphere, the mean mesh of which is m:

- S1 has 360 tetraedra in Ω and 120 triangles on Γ and $m = .353$m,
- S2 has 6192 tetraedra in Ω, 720 triangles on Γ and $m = .141$m.

Besides, the other characteristics of the problem are:

- $\epsilon_{gr} = 10^{-12}$,
- $\epsilon_{uz} = 2.10^{-4}$,
- inside the sphere, $\mu_r = 100$.

The Figure 1 shows that error ErrH is proportional to mean mesh m.
We afterwards look at the effect of the two parameters of the resolution: ϵ_{gr} and ϵ_{uz}. The tests are made on S1 with μ_r=100. We obtain:

- for a given ϵ_{uz}, the results do not depend on ϵ_{gr} (see Table (1)).
- for a given ϵ_{gr}, the results depend on ϵ_{uz} (see Table (2)), but the error on **h** is nearly constant as soon as ϵ_{uz} becomes equal or smaller than 2.10^{-3}. Error Curl decreases when parameter ϵ_{uz} becomes smaller.

Table 1: Errors as a function of ϵ_{gr} ($\epsilon_{uz} = 2.10^{-3}$).

ϵ_{gr}	ErrH	Curl	CPU
10^{-12}	3.59	.11	2.5
10^{-16}	3.59	.11	2.9

Table 2: Errors as a function of ϵ_{uz} ($\epsilon_{gr} = 10^{-12}$).

ϵ_{uz}	ErrH	Curl	CPU
$2.10^{-}2$	5.16	.81	1.26
$2.10^{-}3$	3.59	.11	2.56
$2.10^{-}4$	3.56	.018	3.84
$2.10^{-}5$	3.56	.0015	5.51

We also want to test the proposed method with various values of μ_r. The characteristics of the problem are now:

- $\epsilon_{gr} = 10^{-12}$,
- $\epsilon_{uz} = 2.10^{-3}$ and 2.10^{-4},
- inside the sphere, $\mu_r = 10, 30, 100, 1000$.

Table (3) presents the results. When μ_r increases, the errors increase a little and CPU time becomes a little greater. We see that Curl remains always very small if ϵ_{uz} is small enough.
Besides, we study CPU times when we vary the permeability and the meshing of the sphere. Table (4) gives the results. We remark that, as it may be expected, CPU times increases quite fastly when the mean mesh becomes smaller.
We shall now present the results obtained with the penalty method to compare the methods described in the section " Types of FEM formulations."

Table 3: Errors as a function of μ_r ($\epsilon_{gr} = 10^{-12}$).

ϵ_{uz}	2.10^{-3}			2.10^{-4}		
μ_r	ErrH	Curl	CPU	ErrH	Curl	CPU
10	2.82	.084	2.19	2.79	.0084	3.26
30	3.37	.106	2.36	3.34	.011	3.62
100	3.59	.109	2.56	3.56	.018	3.71
1000	3.69	.127	2.68	3.65	.018	4.09

Table 4: CPU times for the 2 meshings.

μ_r	$m = .35$	$m = .14$
10	3.3	1076
100	3.7	848

Table 5: Errors as a function of penalty parameter.

p	ϵ_{gr}	ErrH	Curl	CPU
Uzawa	10^{-12}	3.56	.018	2.6
.1	10^{-12}	35.9	3.4	.83
"	10^{-14}	"	"	.87
B "	10^{-16}	"	"	.90
1.	10^{-12}	6.07	.28	1.2
"	10^{-14}	"	"	1.3
"	10^{-16}	"	"	1.4
10.	10^{-12}	3.62	.28 10−1	1.2
"	10^{-14}	3.61	"	1.7
"	10^{-16}	"	"	1.8
100.	10^{-12}	4.55	.29 10−2	1.2
"	10^{-14}	3.57	"	1.4
"	10^{-16}	3.56	"	2.1
1000.	10^{-12}	44.2	.29 10−3	1.0
"	10^{-14}	4.69	"	1.4
"	10^{-16}	3.57	"	1.6

6.0.2 Comparison with penalty formulation

The quantity which appears for the choice of the small parameter τ is, [6]:

$$p = \frac{1}{\mu_0 \mu_r \tau l^2}$$

where l is a characteristic length of the system. We choice in the case of the sphere $l = R$.
Table (5) gives for meshing S1 when $\mu_r = 100$, the error on $\mathbf{h}$, ErrH, as a function of parameter p for different values of ϵ_{gr}. We see, as it was established in [1], [6], that the results depend on the value of p and that a good value is $p \approx 10\mu_r$.

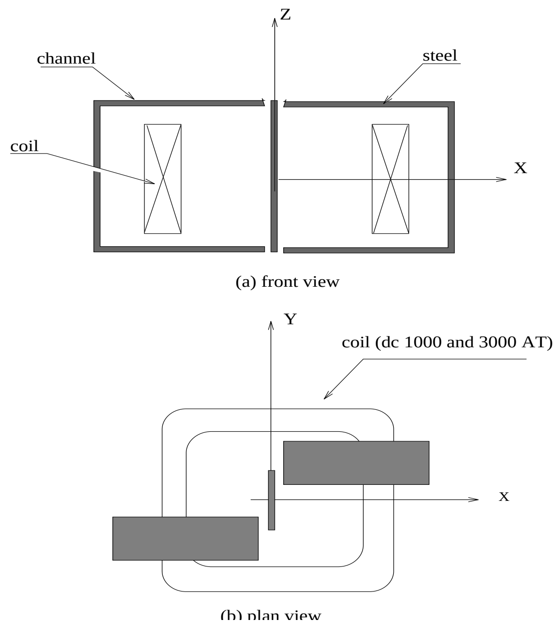

Figure 2: 3-D nonlinear magnetostatic model, (Team workshop Problem 13).

7 Test with an International Problem: Problem 13 of TEAM workshop

We tested the proposed methodology, the mixed formulation, with *Problem 13 of TEAM workshop* [11] which is a 3-D non-linear magnetostatic model (Figure 2). The coil is excited by dc currents. The ampere turns are chosen (1000 AT) such that the field in the steel leads to use the curve $\mu(\mathbf{h})$ where μ greatly depends on $\mathbf{h}$. The meshing of one quarter of the system had 7392 tetraedra.

With these data, the complete matrix which is symmetrical had about 900 000 nonzero elements and there were 8652 degrees of freedom. We have to solve a nonlinear system by a method of substitution on the permeability. We then introduce a relaxation parameter ω. There is no method for finding the optimum relaxation factor ω but we know that this number is between 0 and 2. We choose $\omega = 0.6$. Besides, we define the permeability $\mu_r(x)$ at iteration k, noted μ^k by:

$$\mu^k = \mu^{k-1} + \omega[\mu(X^{k-1}) - \mu^{k-1}]$$

where X^{k-1} is the estimation of the solution at iteration $k-1$.

We compute the solution at iteration k by solving a system of the form:

$$A(\mu^{k-1})X^k = B(\mu^k).$$

Figure 3 and 4 compare the measured and computed values of the average flux

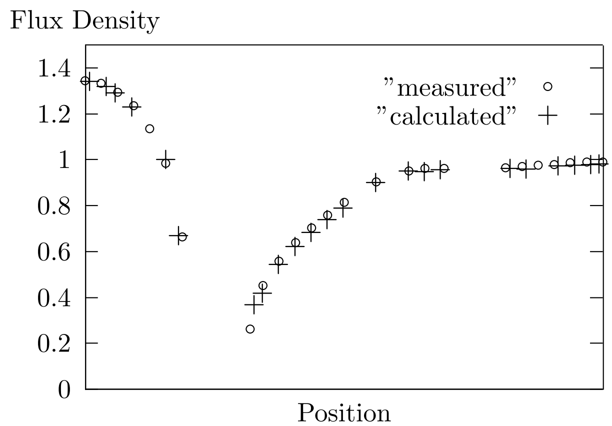

Figure 3: Average flux density in the steel: mixed formulation.

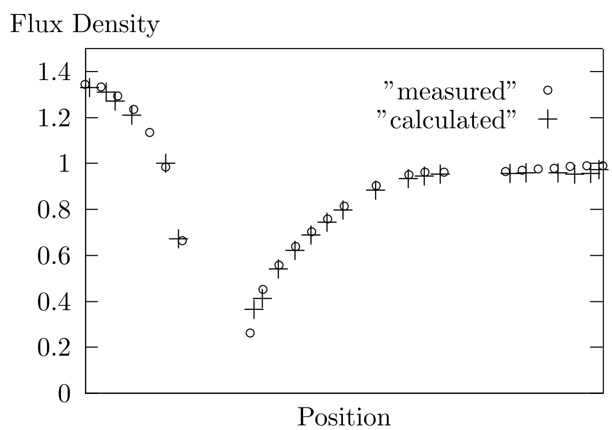

Figure 4: Average flux density in the steel: penalty formulation, ($l = 10^{-3}$).

densities $b = \mu\mathbf{h}$ for the two methods described in section 3. CPU time was 149 s for the mixed method and 125 s for the penalty method.

8 Conclusion

We present two numerical methods, a mixed method and a penalty method, in which magnetic field $\mathbf{h}$ is the variable. These methods allow us to computate the total field $\mathbf{h}$ in the whole $I\!R^3$ by using a reaction potential on the boundary of domain Ω. To conclude, we can say that both methods determinate strength $\mathbf{h}$ with a good accuracy. But a penalty parameter is needed for the second method whereas the first method does not need any. Both require similar CPU times.

References

[1] B. Bandelier, C. Daveau, F. Rioux-Damidau "An h-formulation for the computation of magnetostatic fields. Implementation by combining a finite element method and a boundary element method. " *J. Phys. III France 3, 1993,* pp 995-1004.

[2] A. Bossavit, "A Rationale For "Edge-Elements" in 3-D fields computations," *IEEE Trans. Magn., vol 24, No. 1, 1988* pp 74-79.

[3] A. Bossavit, "Whitney forms: a class of finite elements for three-dimensional computations in electromagnetism," *IEE Proc., 135*, Pt. A, pp 493-500 (1988).

[4] F. Brezzi, M. Fortin, "Mixed and Hybrid Finite Element Methods." *Springer, New-York, 1991.*

[5] M. Cessenat, "Exemples en électromagnétisme et en physique quantique." *Analyse mathématique et calcul numérique pour les sciences et les techniques, Ed by R. Dautray and J.L. Lions, Masson 1988* vol 5, chap IX, pp 235-273.

[6] C. Daveau, "Etablissement d'une formulation variationnelle en champ pour la magnétostatique. Mise en oeuvre numérique dans le cas tridimentionnel par association d'éléments finis et d'intégrale de frontière." *PhD Thesis à l'Université de Tours, 1994.*

[7] E. Durand, "Magnétostatique", *Masson, Paris, Dunod 1968.*

[8] G. Duvaut and J. L. Lions, "Les inégalités en mécanique et en physique," *Dunod, Paris, 1972*

[9] M. Fortin, R. Glowinski, "Augmented Lagrangian Methods." *North Holland, Amsterdam, 1983.*

[10] F. Kikuchi,"Mixed formulations for finite element analysis of magnetostatic and electrostatic problems, " *Japan J. Appl. Math. 6, 1989*, pp 209-221.

[11] T. Nakata, N. Takahashi, K. Fujiwara, P. Olssewski, K. Muramatsu, "Analysis of Magnetic Fields of 3-D Non-Linear Magnetstatic Model (Problem 13)," *Proceedings of the European TEAM Workshop and International Seminar on Electromagnetic Field Analysis, 107, 1990.*

[12] J. C. Nédélec : "Mixed finite elements in $\mathbb{R}^3$.", *Numer. Math., 35, 1980*, pp 315-341.

[13] J. C. Nédélec : "A New Family of Mixed finite elements in $\mathbb{R}^3$, ", *Numer. Math., 35, 1986*, pp 57-80.

[14] P. Silvester, M. S. Hsieh, " Finite element solution of 2-dimensional exterior field problems, " *Proc. Inst. Electr. Engrs., vol 118, 1971,* pp 1743-1746.

[15] R. Stenberg and P. Trouvé, "A Mixed Finite Element Method for the Magnetic Field, " *Proc. of the Fifth. Int. Symp. Numer. Meth. Engng., vol 2, R. Gruber, J. Périaux, and R. P. Shaw Ed, Computational Mechanics and Springer-Verlag 1989*, pp 137-141.

[16] A. Bossavit, J. C. Vérité, "A mixed FEM-BIEM method to solve 3-D eddy current problem by using h as state variable," *IEEE Trans. Magn., MAG-18, 2, 1982*, pp 431-435.

[17] O. C. Zienkiewicz, D. W. Kelly, P. Bettess, " The coupling of the finite element method and boundary solutions procedures," *Internat. J. Numer. Methods Engrg, vol 11, 1977,* pp 355-375.