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Homework 1

Exercice 1

① A rectangle is convex.

Indeed, if $S = \{x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i, i = 1, \dots, n\}$.

For $x, y \in S$ and $0 \leq \theta \leq 1$

$\alpha_i \leq \theta x_i + (1-\theta) y_i \leq \beta_i$, for $i = 1, \dots, n$

So $\theta x + (1-\theta) y \in S$.

② The hyperbolic set $\{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$ is convex.

In fact, consider
$$f : \mathbb{R}_+ \longmapsto \mathbb{R}_+$$
$$x \longmapsto \frac{1}{x}$$

For $x \in \mathbb{R}_+$, $f'(x) = -\frac{1}{x^2}$ and $f''(x) = \frac{2}{x^3} > 0$

f is convex $\Leftrightarrow$ $\text{epi} f = \{(x,t) \in \mathbb{R}_+^2 \mid f(x) \leq t\}$

$= \{(x,t) \in \mathbb{R}_+^2 \mid tx \geq 1\}$

is convex.

③

The set of points closer to a give point than a given set is convex.

Let $S \subseteq \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$.

For $y \in S$, Consider:

$$A_y = \{x \mid \|x - x_0\|_2 \leq \|x - y\|_2\}$$
$$= \{x \mid \|x - x_0\|_2^2 \leq \|x - y\|_2^2\}$$
$$= \{x \mid \underbrace{2(y - x_0)^T}_{u} x \leq \underbrace{\|y\|_2^2 - \|x_0\|_2^2}_{b}\}$$
$$= \{x \mid u^T x \leq b\}$$

Thus, A_y is halfspace, then A_y is convex.

So, $\{x \,/\, \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\} = \bigcap_{y \in S} A_y$

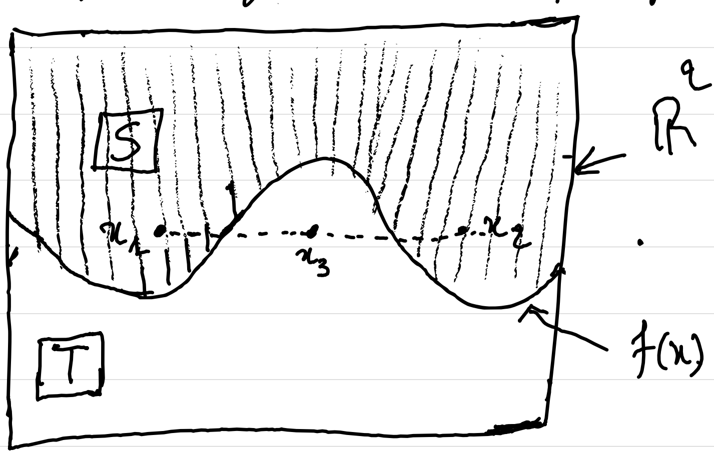
is convex as intersection of convex sets

(4)

Let $S, T \subset \mathbb{R}^n$, the set

$B = \{x \,/\, dist(x, S) \leq dist(x, T)\}$ is <u>not always convex</u>

Consider the folowing counter exemple (for $n = 2$)



by construction, $x_1, x_2 \in S$, so $dist(x_1, S) = dist(x_2, S) = 0$
since $dist(x_1, T) \geq 0$ and $dist(x_2, T) \geq 0$, then
$x_1, x_2 \in B$.
But, by construction, there is $\theta \in (0, 1)$ such that
$x_3 = \theta x_1 + (1 - \theta) x_2$.
since $x_3 \in T$, $dist(x_3, T) = 0$
since x_3 does not belong to the closure of S,
$dist(x_3, S) > 0 = dist(x_3, T)$, then $x_3 \notin S$.

Thus, $x_1, x_2 \in B$, $\theta \in (0, 1)$, but $x_3 = \theta x_1 + (1-\theta) x_2 \notin B$.
$B = \{x \,/\, dist(x, S) \leq dist(x, T)\}$ is not convex.

(5)

The set $C = \{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subset \mathbb{R}^n$ with S_1 convex is convex.

Indeed, $x \in C$ if and only if, for all $y \in S_2$,

$x + y \in S_1 \Leftrightarrow x \in S_1 - y$ for all $y \in S_2$.

$$\Leftrightarrow x \in \bigcap_{y \in S_2} (S_1 - y)$$

For all $y \in S_2$, the function $f_y : x \mapsto x - y$ is an affine function, thus $f_y(S_1)$ is a convex set.

Since $C = \{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} (S_1 - y)$, C is a convex set as an intersection of convex sets.

Exercice 2

(1)

(+) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$ is neither convex nor concave.

Indeed, f is twice differentiable and

$$\nabla^2 f(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues of $\nabla^2 f(x)$ are 1 and -1, then $\nabla^2 f(x)$ is neither positive semidefinite or negative semidefinite.

(+) f is quasiconcave, in fact, as in exercise 1 question 2, we can show that the set $\{x \in \mathbb{R}^2_+ \mid x_1 x_2 \geq \alpha\}$ is convex for all $\alpha \in \mathbb{R}$.

(2)

(+) $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$ is Convex

In fact, f is twice differentiable and for $(x_1, x_2) \in \mathbb{R}^2_{++}$

$$\nabla f(x) = \begin{pmatrix} -\frac{x_2}{(x_1 x_2)^2} \\ -\frac{x_1}{(x_1 x_2)^2} \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} \frac{2x_2^2}{(x_1 x_2)^3} & \frac{1}{(x_1 x_2)^2} \\ \frac{1}{(x_1 x_2)^2} & \frac{2x_1^2}{(x_1 x_2)^3} \end{pmatrix}$$

If λ_1 and λ_2 are the eigenvalues of $\nabla^2 f(x)$,

$$\begin{cases} Tr(\nabla^2 f(x)) = \lambda_1 + \lambda_2 = \frac{2(x_2^2 + x_1^2)}{(x_1 x_2)^3} > 0 \\ det(\nabla^2 f(x)) = \lambda_1 \lambda_2 = \frac{3}{(x_1 x_2)^4} > 0 \end{cases}$$

Thus $\lambda_1 > 0$ and $\lambda_2 > 0$. and $\nabla^2 f(x)$ is positive definite. f is Convex

(+) Since f is convex, f is also quasiconvex

(+) $f(x_1, x_2) = x_1/x_2$ on $\mathbb{R}^2_{++}$ is <u>neither convex or concave</u>.

For $x = (x_1, x_2) \in \mathbb{R}^2_{++}$.

$$\nabla f(x) = \begin{pmatrix} \frac{1}{x_2} \\ -\frac{x_1}{x_2^2} \end{pmatrix}, \quad \nabla^2 f(x) = \begin{pmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{pmatrix}$$

If λ_1 and λ_2 are the eigenvalues of $\nabla^2 f(x)$

$$\begin{cases} Tr(\nabla^2 f(x)) = \lambda_1 + \lambda_2 = \frac{2x_1}{x_2^3} > 0 \\ \det(\nabla^2 f(x)) = \lambda_1 \lambda_2 = -\frac{1}{x_2^4} < 0. \end{cases}$$

Thus $(\lambda_1 < 0$ and $\lambda_2 > 0)$ or $(\lambda_1 > 0$ and $\lambda_2 < 0)$
Then $\nabla^2 f(x)$ is neither positive semidefinite or negative semidefinite.

(+) $f(x_1, x_2)$ is <u>quasilinear</u> (both quasiconvex and quasiconcave)

Indeed, for $\alpha \in \mathbb{R}$,

- $\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid f(x_1, x_2) \leq \alpha\} = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 - \alpha x_2 \leq 0\}$

which is a halfspace, then convex

- $\{(x_1, x_2) \in \mathbb{R}^2_{++} \mid f(x_1, x_2) \geq \alpha\} = \{(x_1, x_2) \in \mathbb{R}^2_{++} \mid x_1 - \alpha x_2 \geq 0\}$

which is also an halfspace, then convex.

Let $0 \leq \alpha \leq 1$. and $f(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$. on $\mathbb{R}^2_{++}$

⊕ <u>If $\alpha = 0$ or $\alpha = 1$, f is an affine function</u>, thus <u>f is convex, concave and quasilinear</u> (both quasiconvex and quasiconcave).

⊕ <u>If $0 < \alpha < 1$, f is concave and quasiconcave.</u>

In fact, for $x = (x_1, x_2) \in \mathbb{R}^2_{++}$.

$$\nabla f(x) = \begin{pmatrix} \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{pmatrix}$$

$$\nabla^2 f(x) = \begin{pmatrix} \alpha(\alpha-1) x_1^{\alpha-2} x_2^{1-\alpha} & \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} \\ \alpha(1-\alpha) x_1^{\alpha-1} x_2^{-\alpha} & -\alpha(1-\alpha) x_1^{\alpha} x_2^{-\alpha-2} \end{pmatrix}$$

If λ_1 and λ_2 are eigenvalue of $\nabla^2 f(x)$.

$$\begin{cases} \operatorname{Tr}(\nabla^2 f(x)) = \lambda_1 + \lambda_2 = \alpha(\alpha-1)\left(x_1^{\alpha-2} x_2^{1-\alpha} + x_1^{\alpha} x_2^{-\alpha-2}\right) < 0 \\ \operatorname{Det}(\nabla^2 f(x)) = \lambda_1 \lambda_2 = (\alpha(\alpha-1))^2 \left[x_1^{2\alpha-2} x_2^{-2\alpha} - x_1^{2\alpha-2} x_2^{-2\alpha}\right] = 0 \end{cases}$$

Thus, $(\lambda_1 + \lambda_2 < 0$ and $\lambda_1 \lambda_2 = 0) \Rightarrow (\lambda_1 = 0$ and $\lambda_2 < 0)$
or $(\lambda_2 = 0$ and $\lambda_1 < 0)$

so $\nabla^2 f(x)$ is negative semidefinite, and f is <u>concave</u>

Exercice 3

Show that following functions are convex.

1) $f(X) = \mathrm{Tr}(X^{-1})$ on $\mathrm{dom} f = S^n_{++}$,

let $Z \in S^n_{++}$, $V \in S^n$, $g(t) = f(Z + tV)$ with $\mathrm{dom}\, g = \{ t \in \mathbb{R} \,/\, Z + tV \in S^n_{++} \}$.

$$g(t) = \mathrm{tr}\left((Z+tV)^{-1}\right)$$
$$= \mathrm{tr}\left(\left[Z^{1/2}(I + t Z^{-1/2} V Z^{-\frac{1}{2}}) Z^{1/2}\right]^{-1}\right)$$
$$= \mathrm{tr}\left(Z^{-\frac{1}{2}}(I + tZ^{-\frac{1}{2}} V Z^{-\frac{1}{2}})^{-1} Z^{-\frac{1}{2}}\right) \quad \left[\text{since } (AB)^{-1} = B^{-1}A^{-1}\right]$$
$$= \mathrm{tr}\left(Z^{-1}(I + t Z^{-1/2} V Z^{-1/2})^{-1}\right) \quad \left[\text{since } \mathrm{Tr}(AB) = \mathrm{Tr}(BA)\right]$$

Since $Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}} \in S^n$, there exist an orthogonal P ($P^{-1} = P^T$) and diagonal matrix D such that

$$Z^{-\frac{1}{2}} V Z^{-\frac{1}{2}} = PDP^T$$

Hence,

$$g(t) = \mathrm{tr}\left(Z^{-1}(I + t PDP^T)^{-1}\right)$$
$$= \mathrm{tr}\left(Z^{-1}\left[P(I + tD)P^T\right]^{-1}\right)$$
$$= \mathrm{tr}\left(Z^{-1} P (I + tD)^{-1} P^T\right)$$
$$= \mathrm{tr}\left(P^T Z^{-1} P (I + tD)^{-1}\right)$$
$$= \sum_{i=1}^{n} \left(P^T Z^{-1} P (I + tD)^{-1}\right)_{ii}$$

let $\lambda_1, \dots, \lambda_n$ the eigenvalue of $Z^{-1/2} V Z^{-1/2}$,

$$D = \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

$$(I + tD)^{-1} = \mathrm{diag}\left(\frac{1}{1+t\lambda_1}, \dots, \frac{1}{1+t\lambda_n}\right)$$

For $i \in \{1, \dots, n\}$.

$$\left[P^T Z^{-1} P (I + tD)^{-1}\right]_{ii} = \sum_{k=1}^{n} (P^T Z^{-1} P)_{ik} (I + tD)^{-1}_{ki}$$
$$= (P^T Z^{-1} P)_{ii} (1 + t\lambda_i)^{-1}$$

$$g(t) = \sum_{i=1}^{n} \frac{(P^T Z^{-1} P)_{ii}}{(1 + t\lambda_i)}$$

the function $g_i : t \longmapsto \frac{1}{1+t\lambda_i}$ is convex since

$$g_i''(t) = \frac{2\lambda_i^2}{(1+t\lambda_i)^3};$$

let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ith. $\in \mathbb{R}^n$

Since $e_i \neq 0_{\mathbb{R}^n}$ and $P^T Z^{-1} P \in S^n_{++}$.

$$(P^T Z^{-1} P)_{ii} = e_i^T (P^T Z^{-1} P) e_i > 0.$$

g is convex as a positive weighted sum of convex function g_i

(2)

$f(X, y) = y^T X^{-1} y$ on $\mathrm{dom} f = S^n_{++} \times \mathbb{R}^n$.
for all $x \in \mathbb{R}^n$, the function

$(X, y) \longmapsto 2x^T y - x^T X x$ is an affine function of (y, X), so it is convex.

Then $g(X, y) = \sup_{x \in \mathbb{R}^n} \{ \underbrace{2x^T y - x^T X x}_{G(x)} \}$ is convex. And,

$\nabla G(x) = 2y - 2Xx$

$\nabla^2 G(x) = -2X \preccurlyeq 0.$

$\nabla G(x^*) = 0 \iff x^* = X^{-1} y.$

$$g(X, y) = G(x^*) = 2 y^T X^{-1} y - y^T X^{-1} y$$
$$= y^T X^{-1} y$$
$$= f(X, y)$$

Hence $f = g$ which is convex.

(3)

$f(X) = \sum_{i=1}^n \sigma_i(X)$ on $\mathrm{dom} f = S^n$, where $\sigma_1(X), \dots, \sigma_n(X)$ are singular values of X.

We know that $\sigma_1^2(X), \dots, \sigma_n^2(X)$ are the eigen values of $X^T X$.
Since $X^T X \in S^n_+$, there exist a matrix $A \in S^n_+$ such that $A^2 = X^T X$ ie $A = (X^T X)^{1/2}$

and $\sigma_1(X), \sigma_2(X), \dots, \sigma_n(X)$ are the eigen values of A.
Now, using the variational characterization, we have

$$f(X) = \sum_{i=1}^n \sigma_i(X) = \sup \{ tr(V^T A V) \,/\, V \in \mathbb{R}^{n \times n}, V^T V = I \}$$
$$= \sup_{V \in \mathcal{A}} \{ tr(V^T A V) \}$$

with $\mathcal{A} = \{ V \in \mathbb{R}^{n \times n} \,/\, V^T V = I \}$

since for all $V \in \mathcal{A}$, $A \mapsto V^T A V$ is affine (convex), we conclude that f est convex.

Exercice 4

$K_{m+} = \{ x \in \mathbb{R}^n \,/\, x_1 \geq x_2 \geq \dots \geq x_n \geq 0 \}$

(1)

⊕ $K_{m+} = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \} \cap \left[\bigcap_{i=1}^{n-1} \{ x \in \mathbb{R}^n \,/\, x_i - x_{i+1} \geq 0 \} \right]$

K_{m+} is an intersection of halfspaces, thus K_{m+} is a polyhedral and then closed

⊕ The interior of K_{m+} is not empty.

$\mathring{K}_{m+} = \{ x \in \mathbb{R}^n \mid x_1 > x_2 > \dots x_n > 0 \} \ni (2n, 2n-1, \dots, n)$

⊕ let $x \in K_{m+}$ such that $-x \in K_{m+}$.

we have

$$\begin{cases} x_1 \geq x_2 \geq \dots \geq x_n \geq 0 \\ -x_1 \geq -x_2 \geq \dots \geq -x_n \geq 0 \end{cases} \implies x_1 = x_2 = \dots = x_n = 0$$

So $x = 0$. and K_{m+} is pointed.

Hence K_{m+} is a proper cone.

(2)

let $y \in \mathbb{R}^n$ such that, $y^T x \geq 0$ for all $x \in K_{m+}$.

$$y^T x = \sum_{i=1}^{n} x_i y_i = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

$$= (x_1 - x_2) y_1 + x_2 y_1 + x_2 y_2 + \dots + x_n y_n$$

$$= (x_1 - x_2) y_1 + x_2 (y_1 + y_2) + x_3 y_3 + \dots + x_n y_n$$

$$= (x_1 - x_2) y_1 + (x_2 - x_3)(y_1 + y_2)$$
$$+ (x_3 - x_4)(y_1 + y_2 + y_3) + (x_4 - x_5)(y_1 + y_2 + y_3 + y_4)$$
$$+ \dots + (x_{n-1} - x_n)(y_1 + \dots + y_{n-1})$$
$$+ x_n (y_1 + \dots + y_n) \geq 0.$$

$\implies y_1 \geq 0, \quad y_1 + y_2 \geq 0, \quad y_1 + y_2 + y_3 \geq 0$

$\dots, \quad y_1 + \dots + y_n \geq 0.$

$$\boxed{K^*_{m+} = \left\{ y \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{k} y_i \geq 0 \quad k = 1, \dots, n \right\}}$$

Exercise 5 Conjugates of functions.

1) $f(x) = \max_{i=1\dots n} x_i$ on $\mathbb{R}^n$.

$y \in \mathbb{R}^n$.

$$f^*(y) = \sup_{x \in \mathbb{R}} \{ x^T y - \max_{i=1\dots n} x_i \}$$

⊕ if $\exists_{j \in \{1,\dots,n\}}$ such that $y_j < 0$, by choosing $t \in \mathbb{R}$ and a vector x with $x_j = -t$, $x_i = 0$ for $i \neq j$ we have

$$x^T y - \max_{i=1\dots n} x_i = -t y_j \xrightarrow[t \to -\infty]{} +\infty$$

⊕ Suppose that for all $i = 1, \dots, n$, $y_i \geq 0$. and $\sum_{i=1}^n y_i > 1$.

For $t \in \mathbb{R}$, we consider the vector $x = (t, t, \dots, t)$

$$x^T y - \max_{i=1\dots n} x_i = \left(\sum_{i=1}^n y_i - 1 \right) t \xrightarrow[t \to +\infty]{} +\infty$$

⊕ If, for all $i = 1, \dots, n$ $y_i \geq 0$ and $\sum_{i=1}^n y_i < 1$

For $t \in \mathbb{R}$, consider $x = (-t, \dots, -t)$

$$x^T y - \max_{i=1\dots n} x_i = t \left(- \sum_{i=1}^n y_i + 1 \right) \xrightarrow[t \to +\infty]{} +\infty$$

⊕ If for all $i = 1, \dots, n$ $y_i \geq 0$ and $\sum_{i=1}^n y_i = 1$

$\forall x \in \mathbb{R}^n$,
$$x^T y - \max_{i=1\dots n} x_i = \sum_{i=1}^n x_i y_i - \max_{i=1\dots n} x_i$$
$$\leq \left(\max_{i=1\dots n} x_i \right) \underbrace{\sum_{i=1}^n y_i}_{1} - \max x_i = 0$$

$\forall x \in \mathbb{R}^n$, $x^T y - \max_{i=1\dots n} x_i \leq 0$

Taking $x_0 = (0, \dots, 0)$, we have $x_0^T y - \max_{i=1\dots n} x_{0i} = 0$

Therefore: $f^*(y) = 0$.

Hence
$$\boxed{f^*(y) = \begin{cases} 0 & \text{if } \sum_{i=1}^n y_i = 0,\ y_i \geq 0\ i = 1 \dots n \\ +\infty & \text{otherwise.} \end{cases}}$$

(2)

2) $f(x) = \sum_{i=1}^{r} x_{[i]}$ on $\mathbb{R}^n$.

Same reasoning as above.

⊕ $y \in \mathbb{R}^n$, if there exist $j \in \{1, \dots, n\}$ such that $y_j < 0$
by choosing the same vector x with $x_j = -t$, $x_i = 0$ $i \neq j$

$$x^T y - f(x) = -t y_j \xrightarrow[t \to -\infty]{} +\infty$$

⊕ if there exist $j \in \{1, \dots, n\}$ such that $y_j > 1$,
by choosing x with $x_j = t$, $x_i = 0$ for $i \neq j$,

$$x^T y - f(x) = t y_j - t = t(y_j - 1) \xrightarrow[t \to +\infty]{} +\infty.$$

⊕ if $\sum_{i=1}^{n} y_i \neq r$, by choosing $x = (t, t, \dots, t)$

$$x^T y - f(x) = t \sum_{i=1}^{n} y_i - rt$$
$$= t\left(\sum_{i=1}^{n} y_i - r\right)$$

- if $\sum_{i=1}^{n} y_i < r$, $x^T y - f(x) \xrightarrow[t \to -\infty]{} +\infty$
- if $\sum_{i=1}^{n} y_i > r$, $x^T y - f(x) \xrightarrow[t \to +\infty]{} +\infty$

⊕ if $\sum_{i=1}^{n} y_i = r$. and $0 \leq y_i \leq 1$, $i = 1 \dots n$

For all $x \in \mathbb{R}^n$.

$$x^T y - f(x) = \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{r} x_{[i]}$$

Since $x_i \leq \frac{1}{r} \sum_{k=1}^{r} x_{[k]}$,

$$x^T y - f(x) \leq \sum_{i=1}^{n} \left(\frac{1}{r} \sum_{k=1}^{r} x_{[k]}\right) y_i - \sum_{i=1}^{r} x_{[i]}$$
$$= \left(\frac{1}{r} \sum_{k=1}^{r} x_{[k]}\right)\underbrace{\left(\sum_{i=1}^{n} y_i\right)}_{r} - \sum_{i=1}^{r} x_{[i]}$$
$$= \sum_{k=1}^{r} x_{[k]} - \sum_{i=1}^{r} x_{[i]} = 0$$

Thus $x^T y - f(x) \leq 0$ $\forall x \in \mathbb{R}^n$.
by taking $x = 0$, we have the equality, then
$f^*(y) = 0$.

Therefore:

$$\boxed{f^*(y) = \begin{cases} 0 & \text{si } \sum_{i=1}^{n} y_i = r,\ 0 \leq y_i \leq 1 \quad i = 1 \dots n. \\ +\infty & \text{, sinon.} \end{cases}}$$

③

$f(x) = \max_{i=1,\dots,m} (a_i x + b_i)$ on $\mathbb{R}$.

we suppose that $a_1 \leq \dots \leq a_m$ and that none of the function $a_i x + b_i$ is redundant.

$$f^*(y) = \sup_x \{ xy - \max_{i=1,\dots,m} (a_i x + b_i) \}.$$

⊕ if $y > a_m$

For $x \geq 0$, $a_i \leq a_m \Rightarrow a_i x + b_i \leq a_m x + b_i$

$$\Rightarrow \max_{i=1,\dots,n} \{a_i x + b_i\} \leq a_m x + \max_{i=1,\dots,n} b_i$$

$$xy - \max(a_i x + b_i) \geq x.y - a_m x - \max_{i=1,\dots,n} b_i$$

$$= x(y - a_m) - \max_{i=1,\dots,n} b_i \xrightarrow[x \to +\infty]{} +\infty$$

⊕ if $y < a_1$

For $x < 0$, $a_i x + b_i \leq a_1 x + b_i$

$$\Rightarrow \max_{i=1,\dots,m} \{a_i x + b_i\} \leq a_1 x + \max_{i=1,\dots,m} b_i$$

$$xy - \max_{i=1,\dots,n} (a_i x + b_i) \geq \underbrace{(y - a_1)}_{<0} x - \max_{i=1,\dots,n} b_i \xrightarrow[x \to -\infty]{} +\infty$$

Therefore, if $y \notin [a_1, a_m]$, $\underline{f^*(y) = +\infty}$

if $a_i \leq y \leq a_{i+1}$, the supremum of the function $x \mapsto xy - \max_{i=1,\dots,n} (a_i x + b_i)$ is obtain at $(b_{i+1} - b_i) / (a_{i+1} - a_i)$,

$$f^*(y) = \frac{y(b_{i+1} - b_i)}{(a_{i+1} - a_i)} - a_i \frac{b_{i+1} - b_i}{a_{i+1} - a_i} - b_i$$

$$= (y - a_i) \frac{b_{i+1} - b_i}{a_{i+1} - a_i} - b_i$$

$$\boxed{f^*(y) = \begin{cases} (y - a_i) \dfrac{b_{i+1} - b_i}{a_{i+1} - a_i} - b_i & \text{if } a_i \leq y \leq a_{i+1} \\ +\infty & \text{si } y \notin [a_1, a_m]. \end{cases}}$$