Introduction to Probability

Random processes

- A *random process* is a situation in which we know what outcomes could happen, but we don't know which particular outcome will happen.
- Examples: coin tosses, die rolls, iTunes shuffle, whether the stock market goes up or down tomorrow, etc.
- It can be helpful to model a process as random even if it is not truly random.

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iTunes: Just how random is random?

By David Braue on 08 March 2007

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- **Say You, Say What?**
- **A role for labels?**
- **The new random**

Think that song has appeared in your playlists just a few too many times? David Braue puts the randomness of Apple's song shuffling to the test -- and finds some surprising results.

Quick -- think of a number between one and 20. Now think of another one, and another, and another.

Starting to repeat yourself? No surprise: in practice, many series of random numbers are far less random than you would think.

Computers have the same problem. Although all systems are able to pick random numbers, the method they use is often tied to specific other numbers -- for example, the time -- that means you could get a very similar series of 'random' numbers in different situations.

This tendency manifests itself in many ways. For anyone who uses their iPod heavily, you've probably noticed that your supposedly random 'shuffling' iPod seems to be particularly fond of the Bee Gees, Melissa Etheridge or Pavarotti. Look at a random playlist that iTunes generates for you, and you're likely to notice several songs from one or two artists, while other artists go completely unrepresented.

http://www.cnet.com.au/itunes-just-how-random-is-random-339274094.htm

Probability

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Frequentist interpretation:

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Bayesian interpretation:

- A Bayesian interprets probability as a subjective degree of belief: For the same event, two separate people could have different viewpoints and so assign different probabilities.
- Largely popularized by revolutionary advance in computational technology and methods during the last twenty years.

Practice

Which of the following events would you be most surprised by?

(a) exactly 3 heads in 10 coin flips

(b) exactly 3 heads in 100 coin flips

(c) exactly 3 heads in 1000 coin flips

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Law of large numbers

Law of large numbers states that as more observations are collected, the proportion of occurrences with a particular outcome, $\hat{p}_n$, converges to the probability of that outcome, p.

Law of large numbers (cont.)

When tossing a fair coin, if heads comes up on each of the first 10 tosses, what do you think the chance is that another head will come up on the next toss? 0.5, less than 0.5, or more than 0.5?

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- The probability is still 0.5, or there is still a 50% chance that another head will come up on the next toss.

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- The coin is not “due” for a tail.
- The common misunderstanding of the LLN is that random processes are supposed to compensate for whatever happened in the past; this is just not true and is also called *gambler’s fallacy* (or *law of averages*).

Probability distributions

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- The probability distribution for the gender of one kid:

Event	Male	Female
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- The probability distribution for the genders of two kids:

Event	*MM*	*FF*	*MF*	*FM*
Probability	*0.25*	*0.25*	*0.25*	*0.25*

Practice

In a survey, 52% of respondents said they are Democrats. What is the probability that a randomly selected respondent from this sample is a Republican?

(a) 0.48

(b) more than 0.48

(c) less than 0.48

(d) cannot calculate using only the information given

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If the only two political parties are Republican and Democrat, then (a) is possible. However it is also possible that some people do not affiliate with a political party or affiliate with a party other than these two. Then (c) is also possible. However (b) is definitely not possible since it would result in the total probability for the sample space being above 1.

Sample space and complements

Sample space is the collection of all possible outcomes of a trial.

- A couple has one kid, what is the sample space for the gender of this kid? S = {M, F}
- A couple has two kids, what is the sample space for the gender of these kids?

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Complementary events are two mutually exclusive events whose probabilities that add up to 1.

- A couple has one kid. If we know that the kid is not a boy, what is gender of this kid? { ~~M~~, *F* } Boy and girl are *complementary* outcomes.
- A couple has two kids, if we know that they are not both girls, what are the possible gender combinations for these kids?

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  S = { MM, ~~FF~~, FM, MF }