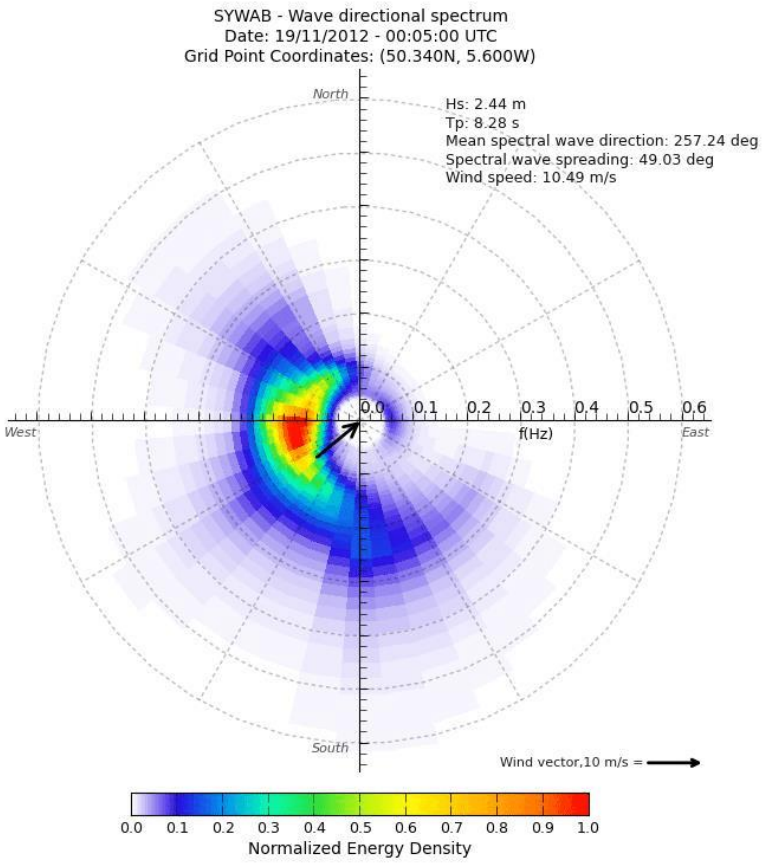
IMT Atlantique
Bretagne-Pays de la Loire
École Mines-Télécom

SUMMER INTERNSHIP REPORT

PARSIMONIOUS REPRESENTATION APPLIED ON THE DIRECTIONAL OCEAN WAVE SPECTRA



VAN-KHOA NGUYEN
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Contents

1. Introduction

This report aims to explain the results of segmentation problem examined on both artificial and real directional wave spectra. These results are obtained by applying a variety of approaches such as segmenting directly on the spectra (Watershed method) or modeling these spectra as a combination of many spectrum units modeled by a specific probability distribution function and utilizing optimization tools to find spare vectors that represents these spectra corresponding to the determined type of distribution function.

In experiences, we primarily use a Gaussian distribution function to model the directional ocean wave spectra. In the later, we will do extension researches on the JONSWAP spectra, which is widely used in modeling the directional ocean wave spectra.

A directional ocean wave spectrum could be though as a combination of several spectrum components, which follow a specific distribution law. Moreover, we can represent these spectra mathematically as the following formula:

$$y = Ax \tag{1}$$

Where the A matrix considered as a dictionary whose columns are generated by applying a type of atom, which follows a probability distribution law. The y vector is the observation vector of the directional ocean wave spectrum. In addition, our tasks are need to find an x vector so that we can represent the observation vector with the smallest number of elements of the dictionary A. We define our objective as the following representation:

$$minimize\ \|x\|_0\ s.t\ Ax = y \tag{2}$$

In the optimization, we call that is the l_0-minimization, the non-convex problem, and the NP-hard problem in general. Extensive researches realized to try to represent the NP-hard optimization problem represented in the [2] formula in different ways, and to find solution for this problem. One of them will be used in our application to the directional ocean wave spectrum shown below:

$$min_x\ f(x) = \frac{1}{2}\|y - Ax\|^2 + \lambda\|x\|_1 \tag{3}$$

Mathematically, the two representations in the formula [2] and [3] will give us almost the same solution. The objective is to find an x vector to minimize the function in the formula [3]. Depending on the size of the A matrix, we address to two different problems. The case, in which the A matrix contains a finite number of columns, is called the LASSO problem which is well resolved by the FISTA method. The other case is called BLASSO problem in which the A matrix contain an infinite number of column vectors and this problem are well addressed by the Sliding Frank-Wolfe method (the development of the Frank-Wolfe method) **[1]**.

2. Gaussian Directional Ocean Wave Spectra

We assume that the atoms used to generate the spectrum components follow the Gaussian probability distribution. These atoms take the means and the variances in the two dimensional Cartesian coordinate system of the distribution as parameters, here is the general two-dimensional Gaussian distribution:

$$f(x,y) = \frac{1}{\sqrt{2\pi}\sigma_x\sigma_y} e^{-\frac{1}{2\sigma_x{}^2}(x-u_x)^2 - \frac{1}{2\sigma_x{}^2}(x-u_x)^2} \quad (4)$$

In our experience, we try to generate artificial directional ocean wave spectra by using the Gaussian atom with changes in its variance and mean values.

2.1. The Gaussian atom with fixed values of variances

This experience uses the Gaussian atom with fixed values of variances to examine the efficiency of the FISTA method and the SFW method recovering the artificial spectrum units.

σ_x^2	0.4
σ_y^2	0.2
u_x	variable
u_y	variable

Table 2.1 Gaussian atom variables.

2.1.1 The artificial Gaussian spectrum with three separating spectrum components

- *Generating an artificial spectrum:*

We generated three different Gaussian spectra, which are far away from each other. The following table show the variable values used to generate the artificial spectrum components:

	1st spec unit	2nd spec unit	3rd spec unit
u_x	-0.46939	1.2463	-2.8278
u_y	-1.1182	-1.9983	-2.3061
coeff	1.1544	1.4273	1.4413

Table 2.2 Three separating spectrum components.

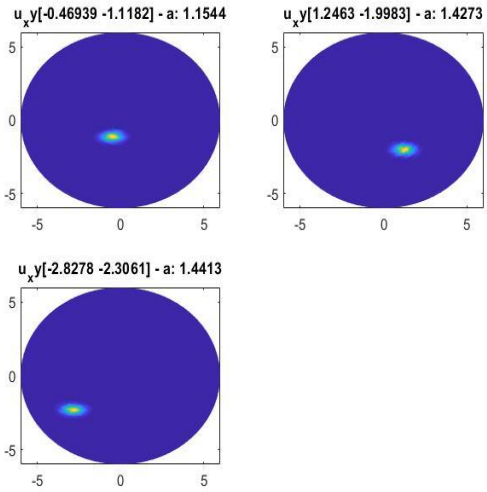


Fig 2.1 Three separating spectrum components.

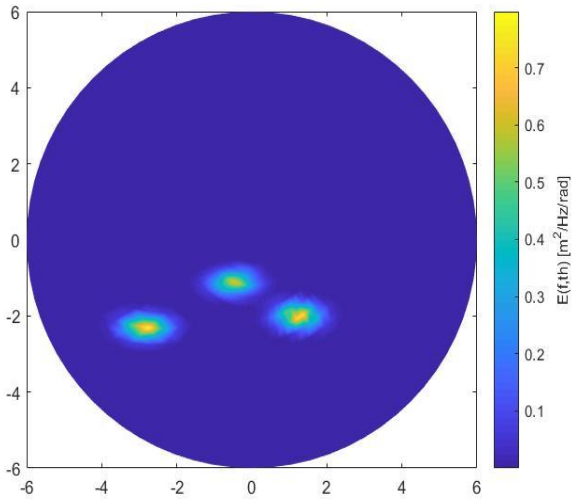


Fig 2.2 Separating spectrum combination.

- *Recovery the spectrum units by using the discrete dictionary (LASSO problem)*

The FISTA method deals with the LASSO problem by using its own grid of variable values. In addition, we use this grid to generate the atoms for the discrete dictionary, so that the dictionary has a limited number of elements in its structure. Here we use the two dimensional grid, whose axis represent several values of means along two axis in the Cartesian coordinate.

Because of the limited resolution of the grid, we do not usually get a good result in this method. The table below show an approximation of several reconstructed components compared with the ground truths. By observing through the reconstructed spectra, there are three different cluster of spectra, which corresponds to the three original spectrum components. Remarkably, the sum of all coefficients in each cluster have the same value as the corresponding spectrum coefficient value. In this study, the FISTA algorithm struggled with the computational cost; it took eight minutes to execute more than two thousands iterations, but did not converge the stopping criteria.

	u_x	u_y	coeff
1st spectrum	-3.1128	-2.1796	0.1021
2nd spectrum	-3.1947	-2.2369	0.10963
3rd spectrum	-2.7578	-2.314	1.0769
4th spectrum	-2.8344	-2.3783	0.16829
5th spectrum	-0.50714	-1.10876	0.47211
6th spectrum	-0.5494	-1.1782	0.15523
7the spectrum	-0.41042	-1.1276	0.52732
8th spectrum	1.15	-1.9919	0.65958
9th spectrum	1.2	-2.0785	0.23284
10th spectrum	1.3766	-1.966	0.51895

Table 2.3 Spectrum components obtained by the FISTA method.

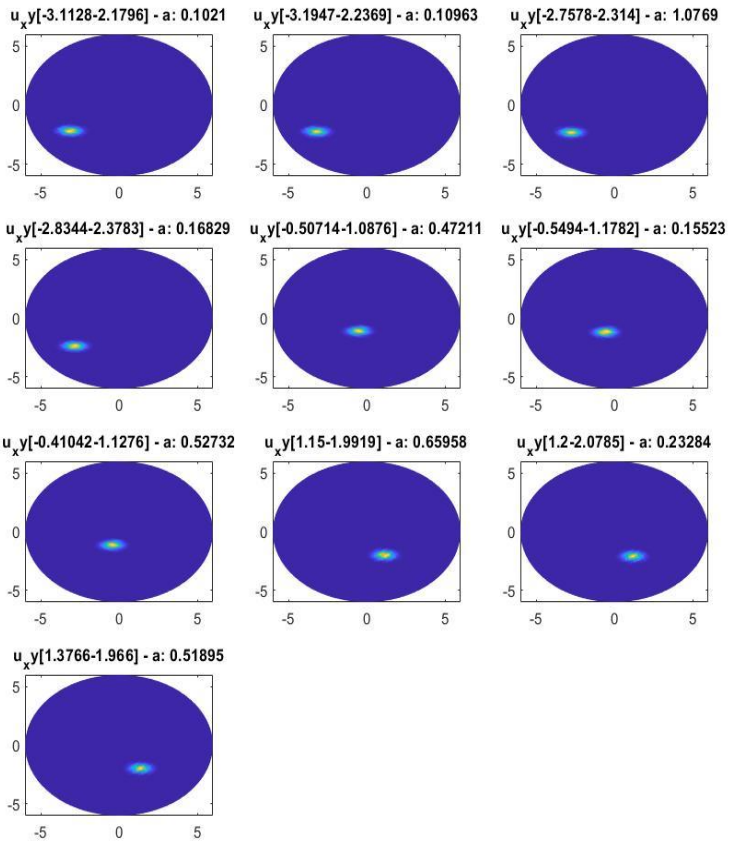


Fig 2.3 Visualization of spectrum units obtained by the FISTA method.

- *Recovery the spectrum units by using the continuous dictionary (BLASSO problem)*

The Sliding Frank-Wolfe method has overcome the limits of the FISTA method. The variables now can take continuous values on certain ranges. This allows us to localize exactly where the atoms are, therefore give us more precis about the coefficient values of each atom. No surprisingly, the method show a very high accuracy in reconstructing the original spectrum components where the atoms vary only their mean values. One more robust of this method is very efficient in the computational time.

	u_x	u_y	coeff
1st spectrum	-0.46888	-1.1179	1.1427
2nd spectrum	1.2458	-1.998	1.4046
3rd spectrum	-2.8269	-2.3059	1.4063

Table 2.4 Spectrum components obtained by the SFW method.

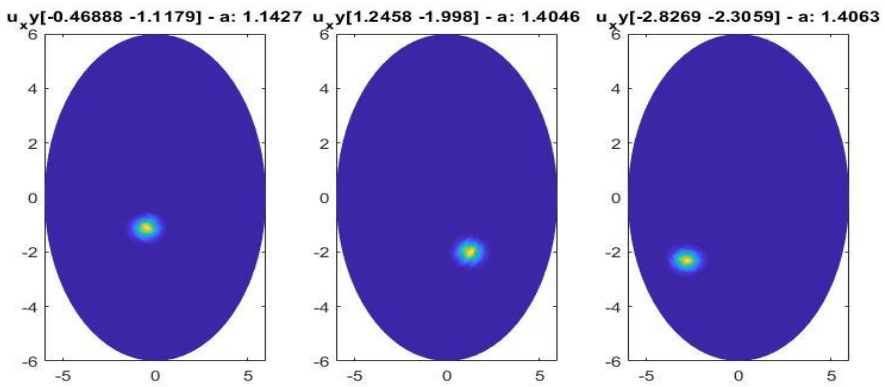


Fig 2.4 Visualization of spectrum units obtained by the SFW method.

2.1.2 The artificial Gaussian spectrum with three closed spectrum components

- *Generating an artificial spectrum:*

To study effectiveness of the two methods, we simulate a more complicated spectrum in which the three spectrum components are closing each other. Here is the simulated values of the variables.

	1st spec unit	2nd spec unit	3rd spec unit
u_x	-2.3566	-2.0135	-2.8283
u_y	-2.4864	-2.6624	-2.724
coeff	1.1544	1.4273	1.4413

Table 2.5 Three closing spectrum components.

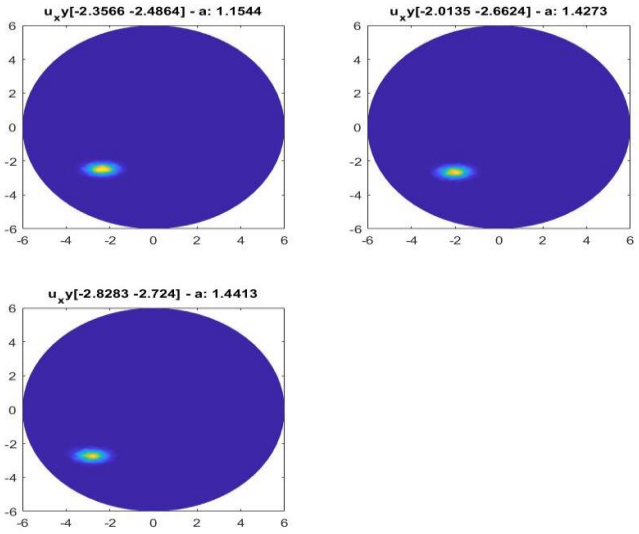


Fig 2.5 Closing spectrum component visualization.

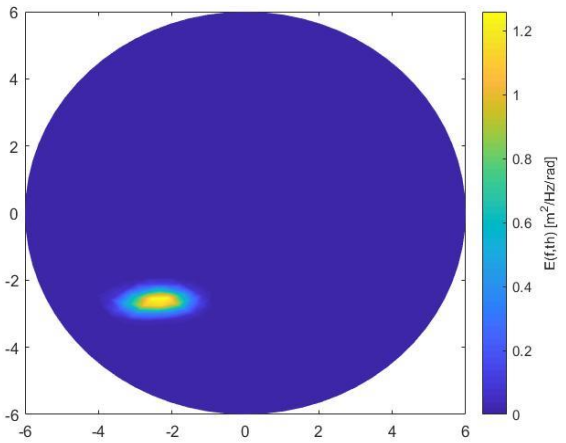


Fig 2.6 Closing spectra combination.

- *Recovery the spectrum units by using the discrete dictionary (LASSO problem)*

When the spectrum components are very closing from each other, we need a grid with high resolution to obtain a more robust result. This signifies that we need more atoms to generate the discrete dictionary, when the dictionary size is enormous that will slow down the algorithm. To evaluate the efficiency of the algorithm, we propose to use the same dictionary as the above case to recovery the original spectrum components. As expected, the algorithm had the difficulty localizing the spectrum component positions and their coefficients and that will give us more spectrum components than the number of the original spectrum components. The result shows four clusters of spectra in total, the first spectrum is out of the three original ones and it deviated the coefficient value determination of the rest.

	u_x	u_y	coeff
1st spectrum	-3.0642	-2.5712	0.22341
2nd spectrum	-2.4749	-2.4749	0.53034
3rd spectrum	-2.687	-2.687	0.23283
4th spectrum	-2.7577	-2.7577	1.0445
5th spectrum	-2.1212	-2.5279	0.57251
6th spectrum	-2.1855	-2.6046	0.78239
7th spectrum	-1.8928	-2.7032	0.59331

Table 2.6 Spectrum components obtained by the FISTA method.

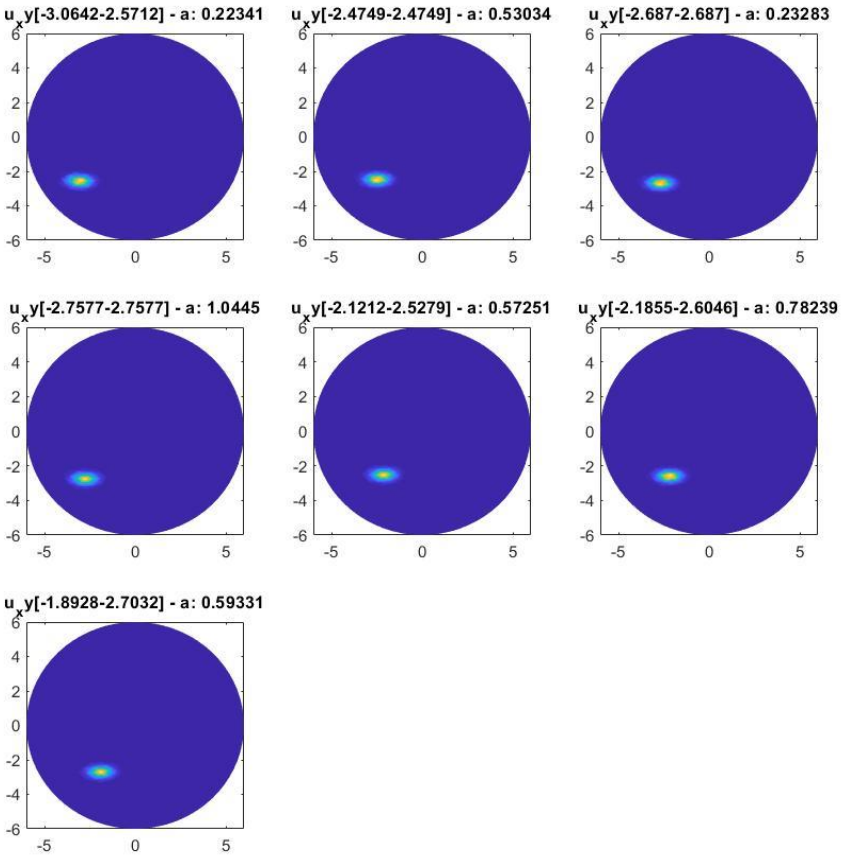


Fig 2.7 Visualization of spectrum units obtained via the FISTA method.

- *Recovery the spectrum units by using the continuous dictionary (BLASSO problem)*

The SFW still proves its efficiency in recovering the original spectrum components even in the case they are closing from each other. The algorithm found four spectrum components, and principally three clusters. We can merge the two last spectrum components into one component

because their similarities in the mean values. Moreover, the algorithm take four second to converge to the stopping criteria, this amount of time is significantly reduced compared with the FISTA method.

	u_x	u_y	coeff
1st spectrum	-2.8215	-2.7221	1.4123
2nd spectrum	-2.3906	-2.4804	0.83823
3rd spectrum	-2.0621	-2.6362	1.423
4th spectrum	-1.9012	-2.7261	0.17882

Table 2.7 Spectrum units obtained by the SFW method.

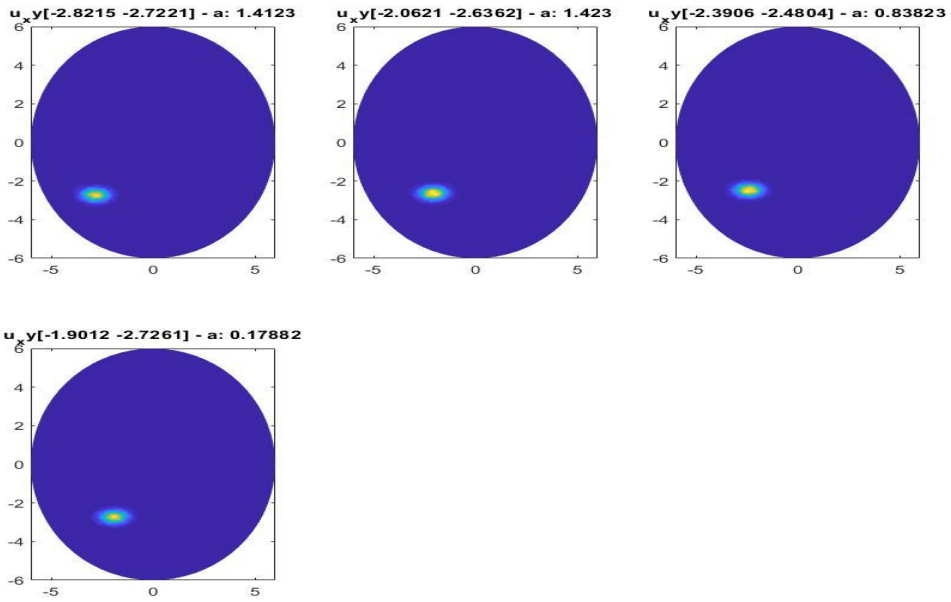


Fig 2.8 Visualization of spectrum units obtained via the SFW method.

2.2. *The Gaussian atom with changes in variance and mean values*

Throughout the above experiences, we have observed that the FISTA method has some limits in accuracies and in computational time, and the SWF method achieved better performance in both terms. For generating the discrete dictionary as we change the variance values, we need to build an enormous matrix which is hard to save and to use on the memory's computer and it will take more time to process than the previous cases. In addition, the SWF method offers more advantages to overcome the difficulties and that is why we will test only the SWF algorithm in the high dimensional case. We will try to know whether the SFW method still perform well in the more than two dimensional space. The table below shows a range of each variable values used to simulate spectra:

σ_x^2	[0.1 1]
σ_y^2	[0.1 1]
u_x	[-4 4]
u_y	[-4 4]

Table 2.8 Gaussian atom variables.

2.2.1. The Gaussian artificial spectrum with three separated spectrum components

- *Generating an artificial spectrum:*

The following table shows different spectrum components and their Gaussian variables:

	u_x	u_y	σ_x^2	σ_y^2	coeff
1st spectrum	-0.66382	-1.5813	0.26763	0.58494	1.1481
2nd spectrum	1.7626	-2.826	0.411	0.47728	2.8003
3rd spectrum	-3.9991	-3.2613	0.45709	0.7167	0.4806

Table 2.9 Three separating spectrum components.

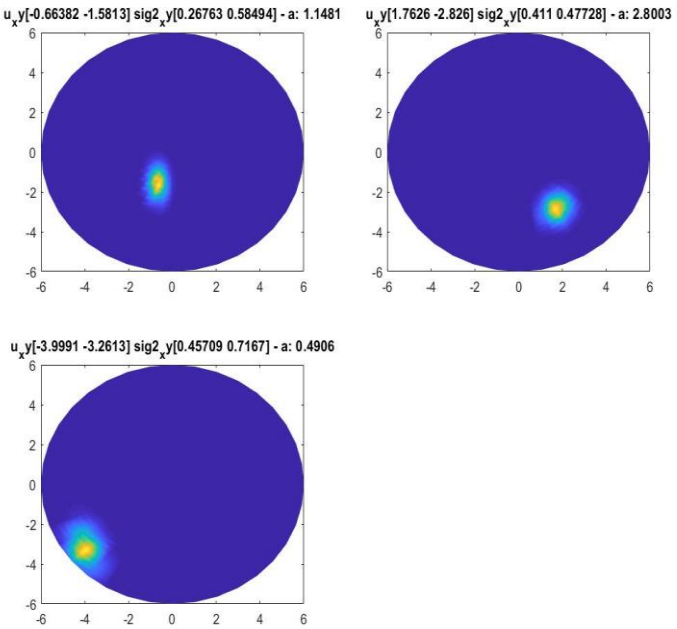


Fig 2.9 Visualization of spectrum components.

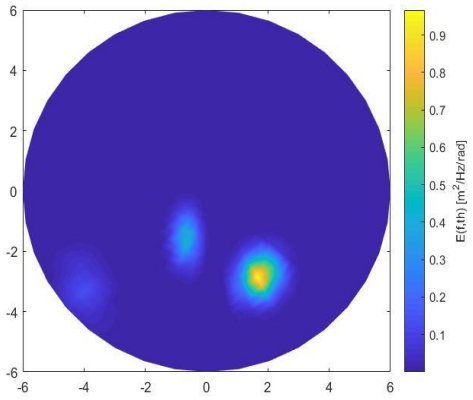


Fig 2.10 Visualization of the spectra.

- *Recovery the spectrum units by using the continuous dictionary (BLASSO problem)*

In the complex Gaussian artificial spectrum, the algorithm need much time to figure out where are the original spectrum components and their variable values. In this case, the algorithm find thirteen main components (those who has the coefficient greater than 0.1) instead of three, and these components compose three main clusters based on their similarity value of the parameters. Some small deviation between the spectrum's mean and variance can create a new spectrum, whose coefficient is a small fraction of the ground truth. And that' why the algorithm show us many spectrum components, we need to find a better condition to merge several closing spectrum into a common spectrum component. The following are the results obtained:

	u_x	u_y	σ_x^2	σ_y^2	coeff
1st spectrum	1.7716	-2.8201	0.36817	0.4396	0.40803
2nd spectrum	1.7377	-2.8683	0.42214	0.53706	0.46802
3rd spectrum	1.1882	-2.7879	0.4375	0.4375	0.33624
4th spectrum	1.7374	-2.8687	0.4375	0.4375	0.27962
5th spectrum	1.7378	-2.7869	0.39731	0.51597	0.56567
6th spectrum	1.7374	-2.7879	0.4375	0.4375	0.2257
7th spectrum	1.8172	-2.8672	0.37884	0.49987	0.32243
8th spectrum	-0.66055	-1.5677	0.23918	0.59131	0.55343
9th spectrum	-0.6866	-1.7373	0.2972	0.64612	0.10974
10th spectrum	-0.60713	-1.5755	0.29787	0.53165	0.14364
11th spectrum	-0.68664	-1.495	0.28672	0.64069	0.12268
12th spectrum	-0.68706	-1.5756	0.29862	0.53232	0.2096
13th spectrum	-3.9999	-3.2473	0.48502	0.73005	0.25136

Table 2.10 Spectrum components obtained by the SFW method.

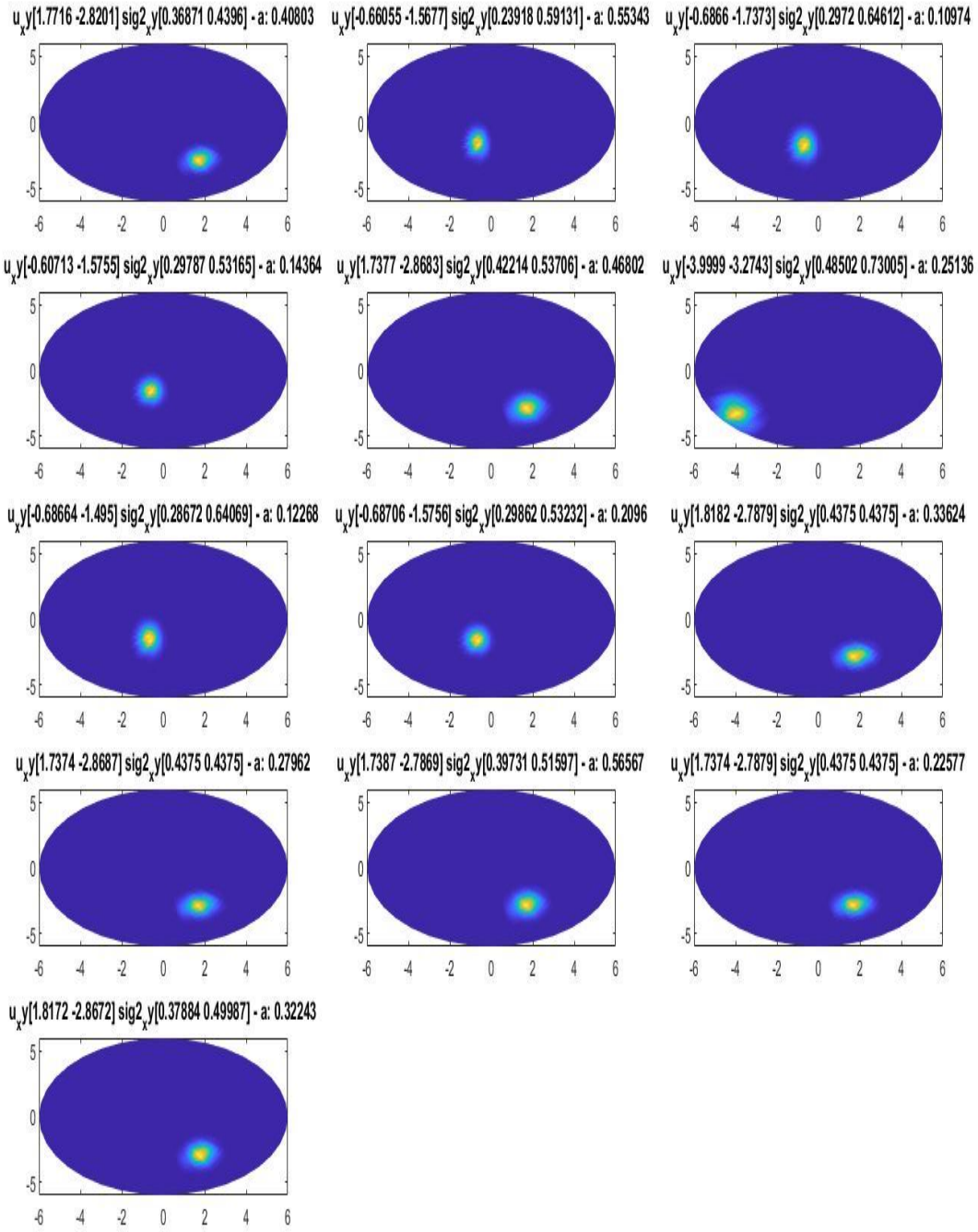


Fig 2.11 Visualization of the spectra obtained by the SFW method.

2.2.2. *The Gaussian artificial spectrum with three closing spectrum components*

3. Real Directional Ocean Wave spectrum Tests

3.1 Sliding Frank-Wolfe algorithm with the Gaussian atom

In the real spectrum tests, we suppose that the natural directional wave spectrum is a sum of many directional wave spectra that follow a Gaussian distribution. The following table shows a range of each Gaussian variable:

σ_x^2	[0.001 2]
σ_y^2	[0.001 2]
u_x	[2*min(min(fx)) 2*max(max(fx))]
u_y	[2*min(min(fy)) 2*max(max(fy))]

Table 3.1 Gaussian atom variables on the real spectrum tests.

Where fx, fy are the spatial frequencies measured on the real spectra. These conditions guarantee that the Gaussian variable values can cover all the possible values of Gaussian atoms combining the directional wave spectra. Another important attention on choosing the value of the variance is that it should take small values but not get zeros, because it might exist several spectra that include many Gaussian atoms with small variances. This choice helps the algorithm converging to the stopping criteria and obtaining approximate result from the ground truth.

- *Test on the wind sea system:*

This test is implemented on the system where there is only one wind sea appearing in the spectrum. The tested spectrum was measured on the NODE008919(12-Jan-2010 17:00:00).

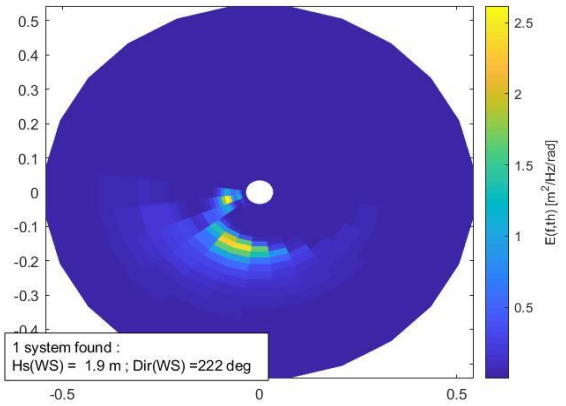


Fig 3.1 Visualization of the real wind sea system.

The SFW method took about twelve minutes to implement 118 iterations to converge to the stopping criteria. The algorithm detects 117 different atoms. The atoms have their coefficients with different values, which decide how much influence (amplitude and direction) that each atom contributes to recover the original spectrum. Approximately, we can examine the atoms, which contribute significantly to recover the spectrum. The following table shows the values of these atoms.

	u_x	u_y	σ_x^2	σ_y^2	coeff
1st atom	-0.1061	-0.1418	0.0501	0.0501	0.1937
2nd atom	-0.2762	-0.0068	0.2324	0.0001	0.3036
3rd atom	-0.2222	-0.0102	0.2307	0.0001	0.5064
4th atom	0.0165	-0.0102	0.2015	0.001	-0.2073
5th atom	-0.1316	-0.1465	0.0660	0.0001	0.1701
6th atom	-0.1502	-0.1258	0.0660	0.0001	0.1131
7th atom	-0.0674	-0.2335	0.0001	0.1234	0.1095

Table 3.2 Principle atom components.

In this table, we can clearly see that there are not only atoms that have positive impacts (positive direction) on enhancing the energy of the original spectrum but also there exist at the same time the atoms weaken the amplitude of the spectrum's energy (negative direction). We can imagine that, in reality the wind seas can move in any direction and a combination of them creating one final direction that caused by the predominant atoms. The following figure shows a combination of all atoms found by the SFW method.

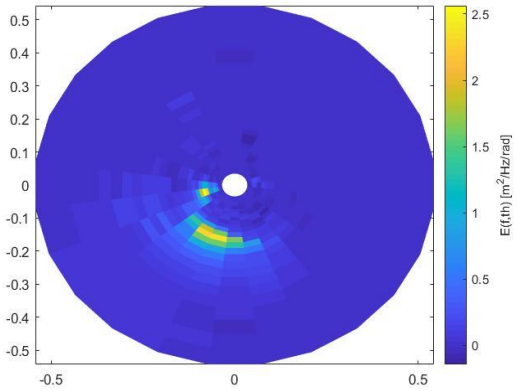


Fig 3.2 Gaussian atom combination found by the SFW method

Compared with the original spectrum in the figure 2.1, we can see that the recovered spectrum has almost the same altitudes and the same directions as the original one. It proves that the directional wave spectrum can be modeled as a sum of many spectral units that following the Gaussian distribution. Below are several different tests recovering different spectra, which can exist many components in each spectrum.

- *Test on the swell system:*

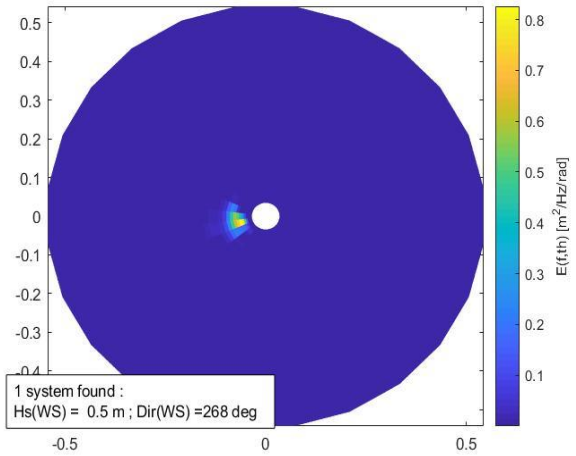


Fig 3.3 The swell system.

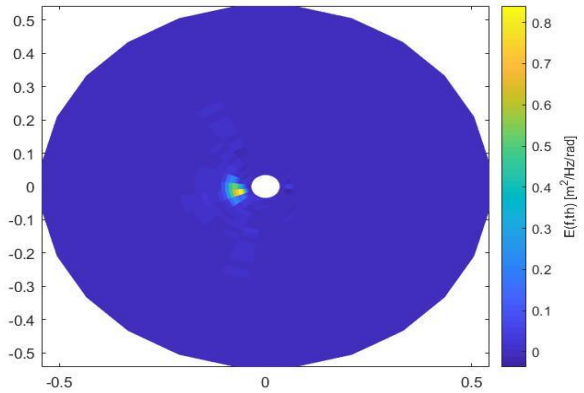


Fig 3.4 The recovered swell system by using the SFW method.

Just like as the above experience, the algorithm recovery successfully the original spectrum. Moreover, the algorithm detects more than 60 different atom components. Several of them have very small values of coefficients.

- *Test on the system wind sea + swell:*

When the system becomes more complexity, i.e. it includes several types of waves, the algorithm struggle to converge to the stopping criteria. For example, in this experience the directional wave spectrum includes a wind sea and swell. To find spectrum components following the Gaussian distribution law combining the original spectrum, the system took more than 83 minutes to iterate 200 times, but it did not reach the stopping criteria. The result shows 172 spectrum components that constructs the original spectrum. Most of the components have almost the same values of coefficients, so it is hard to recognize which ones are dominant in building the original spectrum. Two figures below show the original spectrum and the recovered one by combining several spectrum units found by using the SFW method.

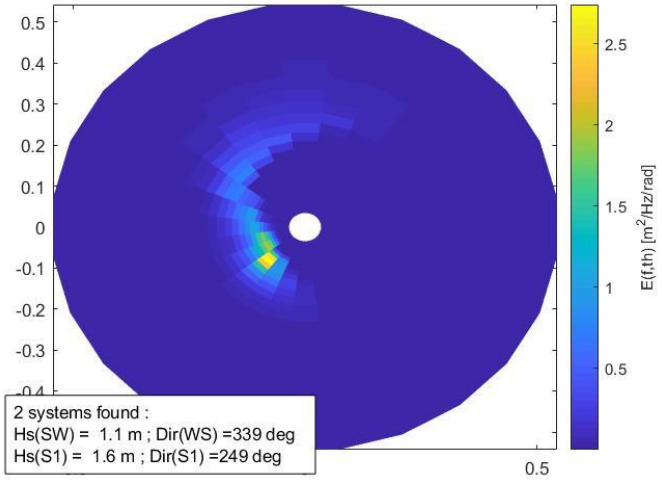


Fig 3.5 The original system (a wind sea + swell).

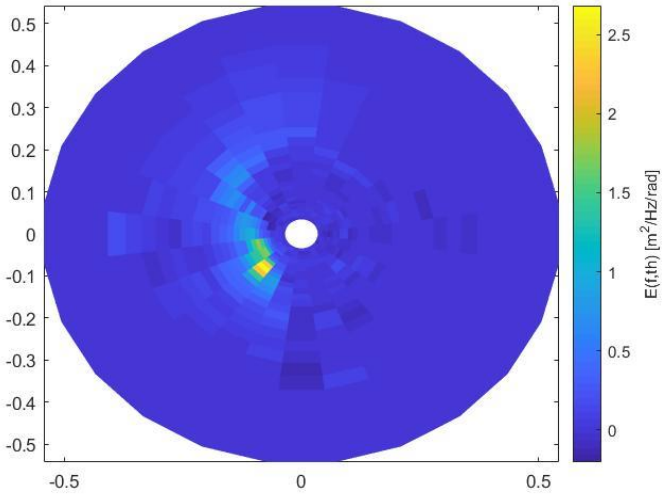


Fig 3.6 The recovered system by using the SFW method.

Until now, it seems that the components of real directional ocean wave spectrum do not follow the Gaussian distribution. All most results show we need more than a decade of spectrum components to reconstruct the original spectrum. We interest the spectrum component structure with which we need only a few of them to reconstruct the original spectrum. A research extension of the report will consider a real directional wave spectrum modeled as the JONSWAP spectrum will discuss in the chapter IV.

3.2 Classical approach of the Watershed method on the directional ocean wave spectrum

Generally, the Watershed algorithm is typically used to segment many kinds of images **[2]**. Now, we apply this approach on the directional ocean wave spectrum; aim to find where the centroid of each spectrum components is.

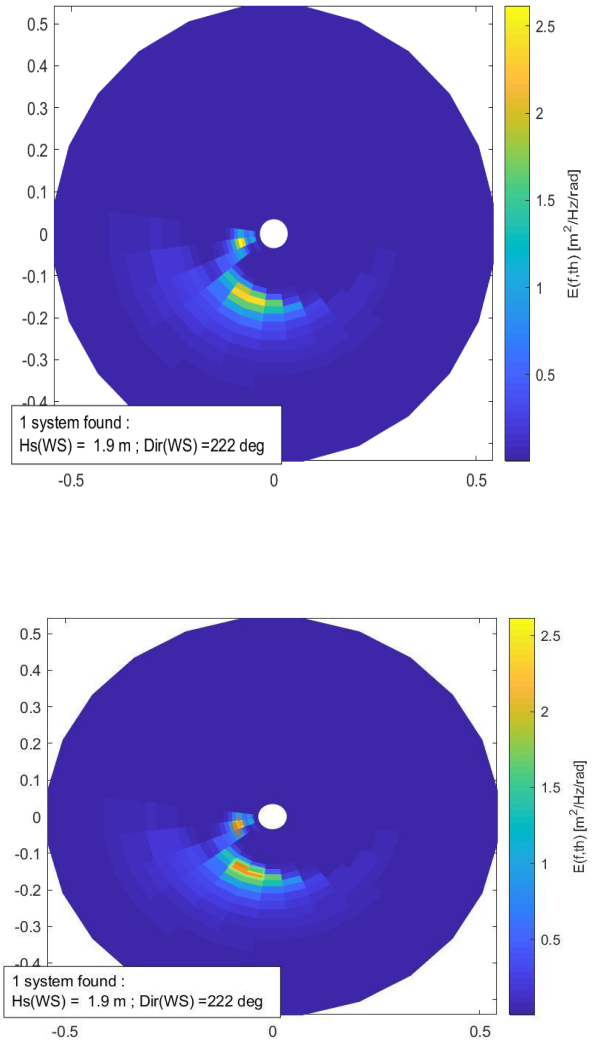


Fig 3.7 The ground truth wind sea system (above) and
its segmented version (below) by using the Watershed method.

The result shows that there are two directional waves in this spectrum instead one wind sea shown in the ground truth. This method takes little time to solve the system and to recognize how many directional waves existed in the system. Nevertheless, the drawback of this method, we cannot know exactly which distribution law each directional wave follows. It might exist several components, which follow a specific probability distribution law (ex. Gaussian), combining each of the segment of the segmented spectrum by using the Watershed method.

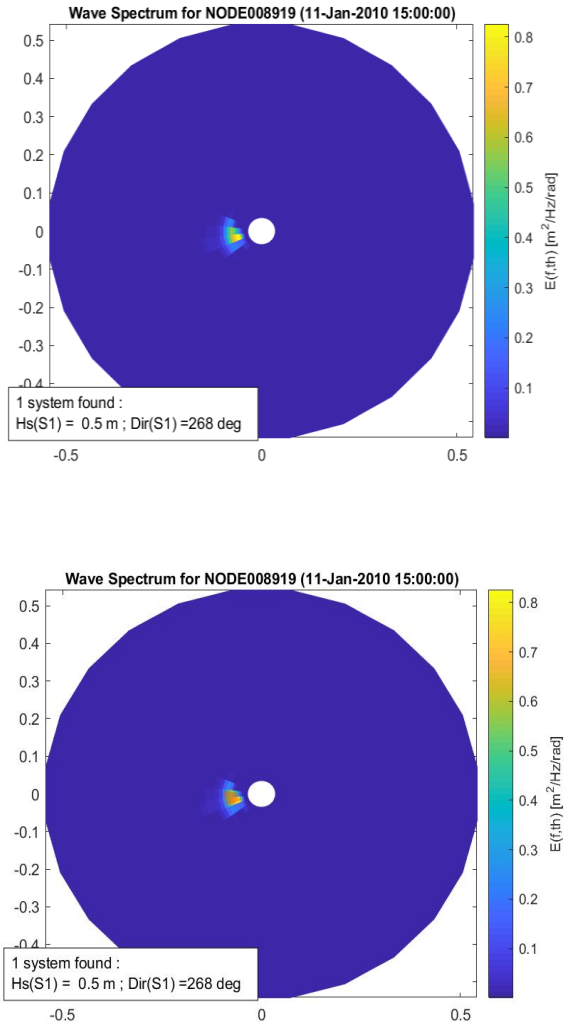
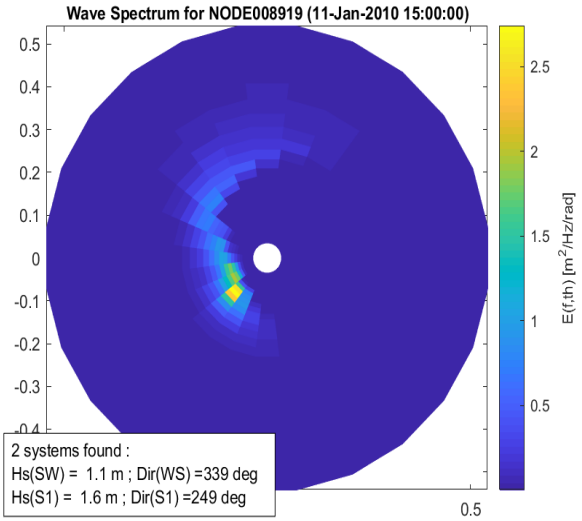


Fig 3.8 The ground truth swell system (above) and its segmented version by the Watershed method(below).



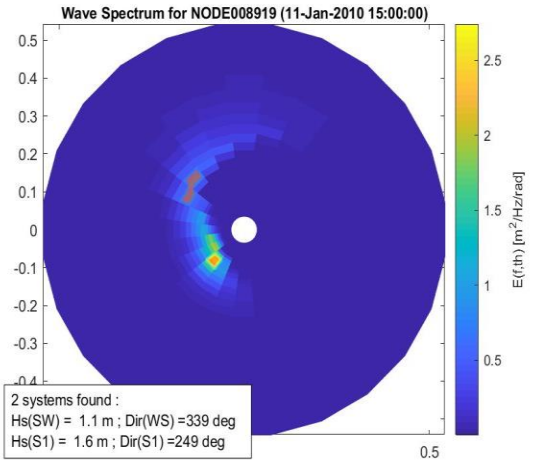


Fig 3.9 The system including a wind sea and swell (above) and its segmented version by using the Watershed method.

Through some experiences with this method, we see that the method successfully segment the directional ocean wave spectrum in which their spectrum components are far away from each other. It is hard to imagine this algorithm can differentiate spectrum components, which are merged together. In addition, the segmentation result depend strongly on the method's parameters (such as structuring element's radius in a morphological operation) as the variety of directional wave spectrum forms.

4. SFW Method with The JONSWAP Frequency Spectrum and Its Implementation on Real Spectra

4.1. A new model of directional ocean wave spectrum

The directional ocean wave spectrum is traditionally factorized as follows:

$$S(\omega,\theta) = S(\omega) * G(\omega,\theta) \tag{5}$$

Where $S(\omega)$ is the frequency spectrum and the so-called spreading function $G(\omega,\theta)$ satisfying $\int_{-\infty}^{\infty} G(\omega,\theta)d\theta = 1$. Throughout many empirical experience researches, we see that the JONSWAP type seems to fit accurately a large range of the recorded deep-water data **[3]**. The parametric form of this frequency spectrum is shown in the following formula:

$$S(\omega) = \frac{\alpha}{\omega^5} e^{-1.25\left(\frac{\omega_p}{\omega}\right)^4} \gamma^{\vartheta(\omega)} \tag{6}$$

with $\vartheta(\omega) = e^{-\frac{(\omega_p-\omega)^2}{2\sigma^2\omega_p{}^2}}$, where σ assumed to be a jump function of ω:

$$\sigma = \begin{cases} 0.07\ if\ \dfrac{\omega}{\omega_p} \leq 1 \\ 0.09\ if\ \dfrac{\omega}{\omega_p} > 1 \end{cases}$$

and the parameter α, the shape parameter γ, and the spectral peak ω_p to be specified as follows:

$$\alpha = \beta_J H_{\frac{1}{3}}^2 \omega_P^2$$

Where:

$$\beta_J = \frac{0.0628(1.094 - 0.01915 \log \gamma)}{0.23 + 0.0336\gamma - 0.185(1.9 + \gamma)^{-1}}$$

$$\omega_p = 2\pi \frac{1 - 0.132(1+\gamma)^{-0.559}}{T_{\frac{1}{3}}}$$

$H_{\frac{1}{3}}$ is a significant height wave (1/3 of highest waves)

$T_{\frac{1}{3}}$ is a average period measured by the zero-down method.

In almost case, the directional wave spectrum depends on the θ. The popular spreading function represents this relation is the Mitsuyasu-type spreading function:

$$G(\omega, \theta) = G(\theta) = G_0 \cos^{2C}\left(\frac{\theta - \theta_m}{2}\right) \quad (7)$$

where:

θ_m is the mean wave direction.

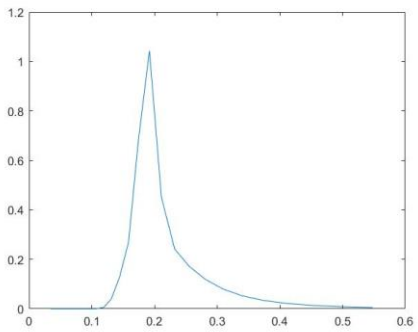
C is the method parameter.

G_0 is the normalization constant.

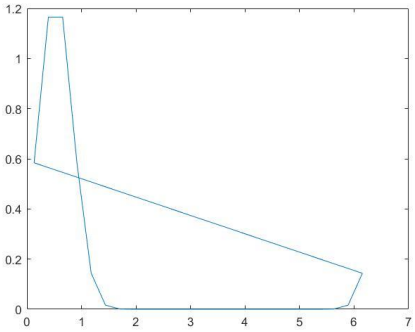
We usually fix the shape parameter value $\gamma = 3.3$ (empirical experiences) to model the directional ocean wave spectrum. Figure below shows a demonstration of the directional wave spectrum, the JONSWAP frequency spectrum and its Mitsuyasu – type spreading function, when:

γ	3.3
$H_{\frac{1}{3}}$	1 m
$T_{\frac{1}{3}}$	5 s
θ_m	π/3
C	20

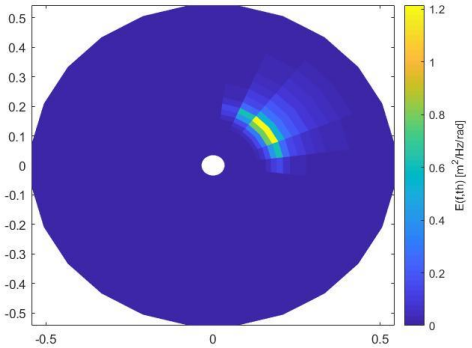
Table 4.1 variable values for illustration



a. JONSWAP frequency spectrum



b. Mitsuyasu-type spreading function



c. Directional wave spectrum

Fig 4.2 The illustration of the new directional wave spectrum.

4.2. SFW implementation on the new type of directional ocean wave spectrum

We experience the new type of directional wave spectrum presented in the formula [5]. We will call atoms of this new type spectrum are the JONSWAP atoms, which we use to generate elements of the continuous dictionary. As mentioned above, we will fix the shape parameter value $\gamma = 3.3$ (empirical experiences) and vary others parameters. The table below shows different parameters for the JONSWAP atoms.

Param	Signification
$H_{\frac{1}{3}}$	Significant height wave (m)
$T_{\frac{1}{3}}$	Significant wave period (s)
θ_m	Mean direction wave (rad)
C	Mitsuyasu-type constant

Table 4.2 Atoms's variable values

In the following experiences, we will also manipulate the parameter λ in the formula [3] to see how the parameter affect the reconstruction of the original ocean wave spectrum (number of elements, their coefficients). We will use a threshold τ to evaluate which spectrum components contributes significally t

4.2.1. Test on the wind sea system

The table below is the variable range values for the test:

Param	Range of value
$H_{\frac{1}{3}}$	[0.1 1.5]
$T_{\frac{1}{3}}$	[1 25]
θ_m	[0 2π]
C	[5 30]

Table 4.3 Atoms's variable range for the wind sea test.

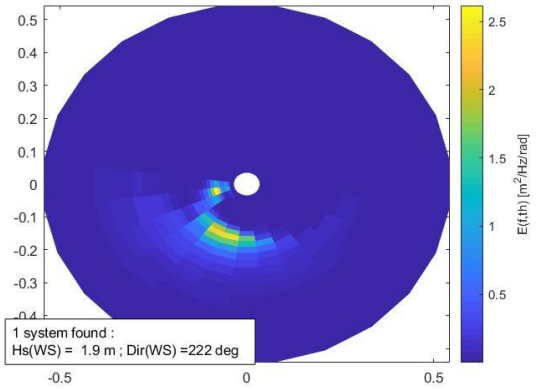
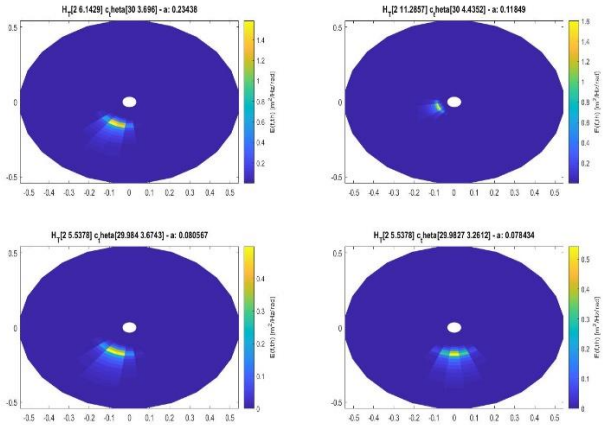


Fig 4.3 Original wind sea spectrum.

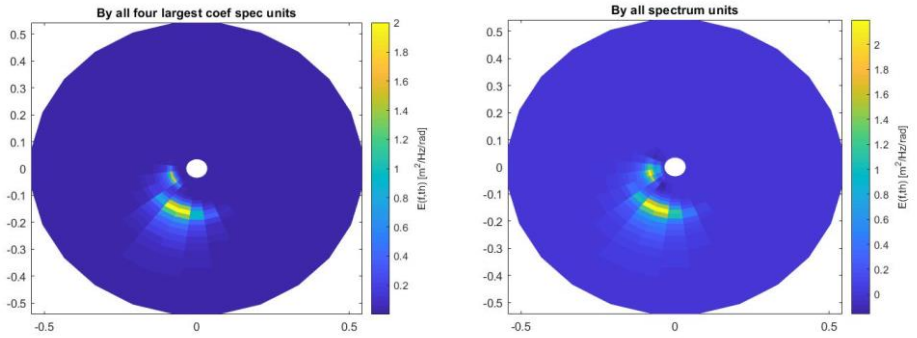
- $\lambda = 0.05$

We use a large value of lambda to test reconstruction of the wind sea spectrum. The following resumes the result of the reconstruction:

Number of atoms found	27
Number of atoms having a coefficient greater than τ = 0.048	4



a) Four largest coefficient spectra.

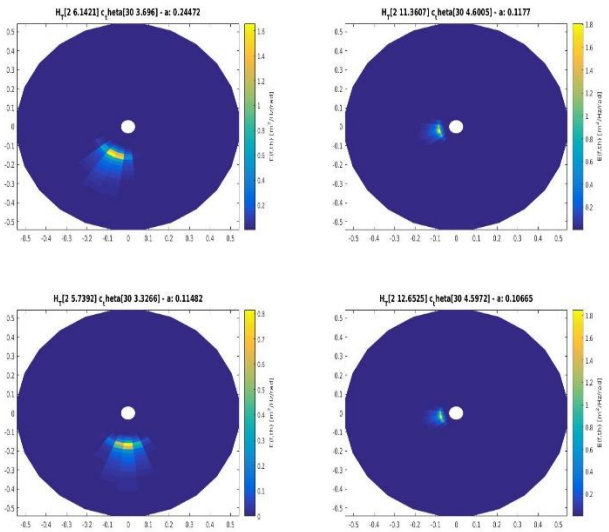


b) The four-spectrum combination. c) All spectrum component combination.

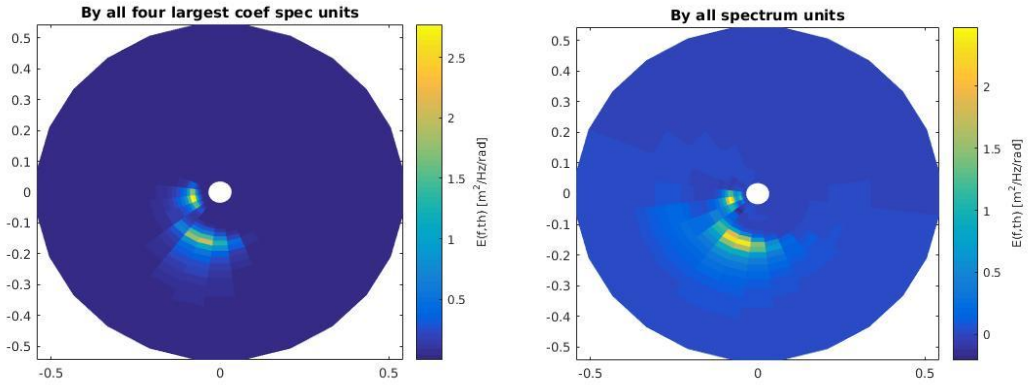
Fig 4.4 Wind sea spectrum reconstruction by the SFW with lambda equal to 0.05.

The algorithm find 27-spectrum components to reconstruct the original wind sea spectrum with the lambda equal to 0.05. By arranging the spectrum's coefficient values in the descending order, we found that from the fifth spectrum to the last spectrum have very small coefficient values descending from 0.048, and some having coefficient's zeros values. Looking at the four largest coefficient values in the fig a), we see the three principal components. The first component is located in 211 degrees; the second component is located in 255 degrees, the last component is located in 191 degrees. The two first principal components have a similar form of spectrum. The fig b and c show two combinations of the four largest coefficient spectra and all spectrum components. By using only four largest spectra, we can have the similar form as the all spectrum component combination. The figs also show that the algorithm's spectrum component combination obtain similar wave directions as well as a similar scale of the sea surface wave directional variance spectral density to the original spectrum.

- $\lambda = 0.005$



a) Four largest coefficient spectra.



b) The four-spectrum combination. c) All spectrum component combination.

Fig 4.5 Wind sea spectrum reconstruction by the SFW with lambda equal to 0.05.

When we fetch a small value of the lambda to the algorithm, we still see principal spectrum components (fig a, b), which are indifferent to the one in the above case Moreover, we obtain more detailed directional wave spectrum in the figure c. it mean that the system output more spectrum components than the previous case.

4.2.2. Test on the swell system

The table below is the variable range values for the test:

Param	Range of value
$H_{\frac{1}{3}}$	[0.1 1.5]
$T_{\frac{1}{3}}$	[1 25]
θ_m	[0 2π]
C	[5 30]

Table 4.4 Atoms's variable range for the wind sea test.

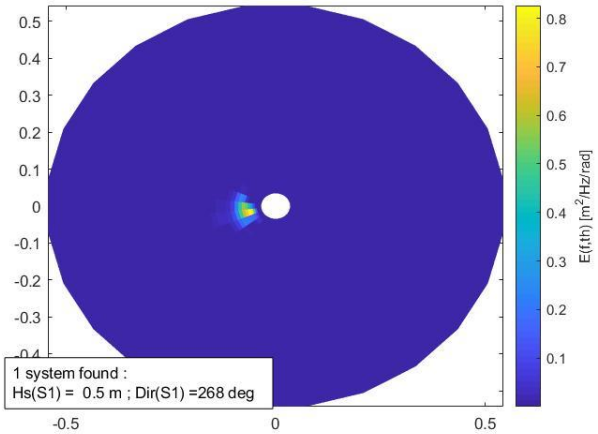
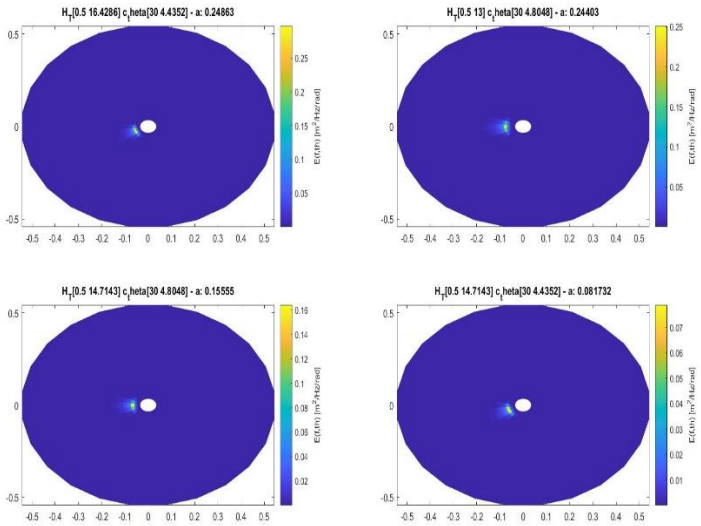


Fig 4.6 Original swell spectrum.

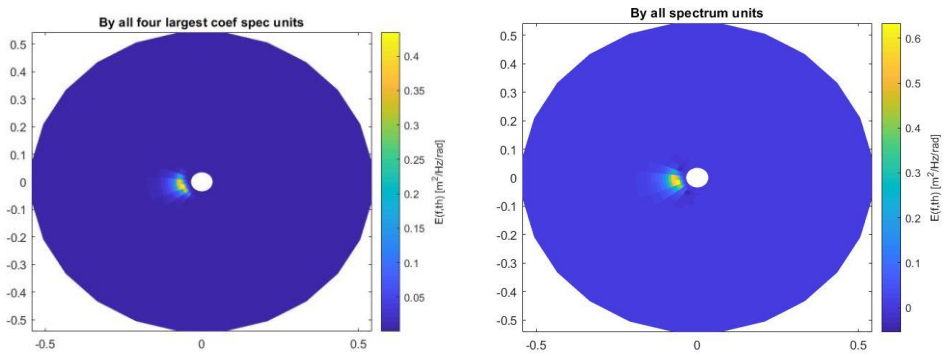
➢ $\lambda = 0.3$

Number of atoms found	5
Number of atoms having the coefficient greater than τ = 0.051	4

For the swell system, which is less complicated than the previous one and by choosing a high value of lambda, the algorithm shows a nearly optimal result. Five components found, the last one whose coefficient is almost vanished. The figure below shows obtained spectrum visualization.



a) Four largest coefficient spectra



b) The four-spectrum combination. c) All spectrum component combination.

Fig 4.7 Swell spectrum reconstruction by the SFW.

The visualization shows that the four principal components are located almost in the same direction, which is not very much different from the original spectrum. The combination of them is similar to the one of all spectrum components, and both of them share similarly a same spectrum's shape and a scale of sea surface wave directional variance spectral density to the original spectrum.

4.2.3. Test on the wind sea and swell system

The table below is the variable range values for the test:

Param	Range of value
$H_{\frac{1}{3}}$	[0 1.6]
$T_{\frac{1}{3}}$	[1 25]
θ_m	[0 2π]
C	[5 30]

Table 4.4 Atoms's variable range for the wind sea test.

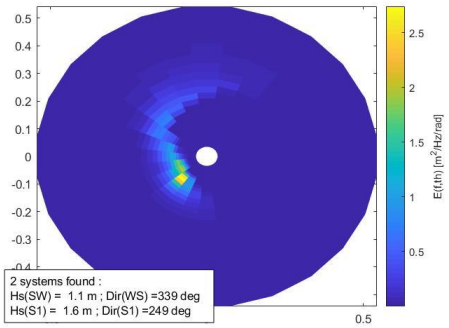
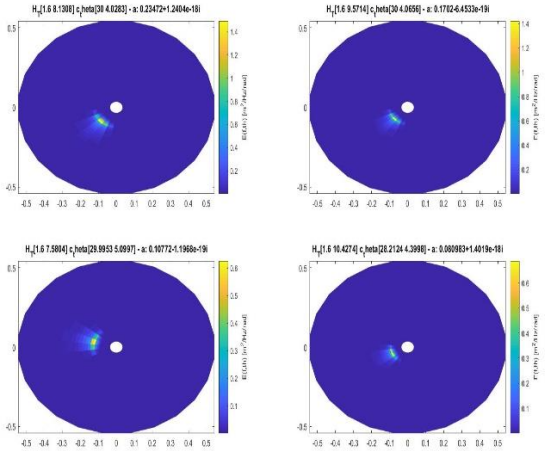


Fig 4.8 Original swell and wind sea spectrum.

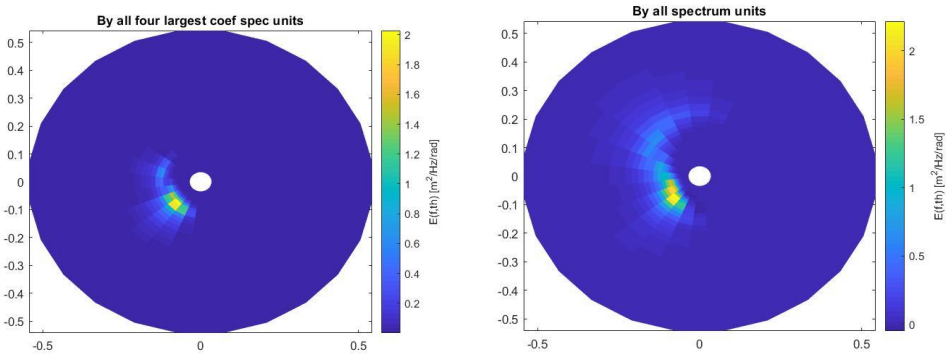
- $\lambda = 0.01$

Number of atoms found	51
Number of atoms having the coefficient greater than τ = 0.066	4

In a more complicated system, it is better to use a small value of lambda to capture all details of spectrum. If someone is only interested in principle components, it is more preferable to use a larger value of lambda. The result obtained for the lambda equal to 0.01 is 51 components in which there are four components having their coefficients greater than 0.066; and the rest whose coefficients are mostly vanished values.



a) Four largest coefficient spectra



b) The four-spectrum combination. c) All spectrum component combination.

Fig 4.9 Swell and Wind sea spectrum reconstruction by the SFW.

The four principal spectra are located in two main directions, which are 241 degrees and 292 degrees. They are quietly different from the ground truth. Nevertheless, the reconstructed spectrum (fig c) is quietly similar to the original spectrum.

5. Conclusion

The results of the SWF method applied on the directional ocean wave spectra depend strongly on the spectrum model. In our experiences, we can clearly see that the JONSWAP spectrum model give us a better result than the Gaussian model in term of the number of parsimonious elements and the wave spectrum form of each component and their combination. Applied this method on different spectra, we recommend trying to change value of the lambda and the merge steps to appropriate values , which will give us more precis positions of the spectrum components and an optimal number of components.

Note: it exists bugs in the original BLASSO codes when setting a high value of the merge steps (such as 1). That is why I could not to merge the closing components found in the JONSWAP spectrum model. Because the merge step is too small, which is 0.01 in the original codes, to apply in the four-dimensional space with Euclidean distance.

6. References

[1] Quentin Denoyelle. Theoretical and Numerical Analysis of Super-Resolution Without Grid. General Mathematics [math.GM]. PSL Research University, 2018.

[2] Matlab toolbox. (17, July 2019). *Marker-Controlled Watershed Segmentation*. Retrieved from https://www.mathworks.com/help/images/marker-controlled-watershed-segmentation.html;jsessionid=a78962e6f53ebc64f35a969c960e#responsive_offcanvas .

[3] Podgorski, Krzysztof & Rychlik, Igor & Machado, Ulla. (2000). Exact distributions for apparent waves in irregular seas. Ocean Engineering. 27. 979-1016. 10.1016/S0029-8018(99)00030-X.