$$\left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r})\right\| \triangleq \max\left(\|\Pi_{\mathcal{K}}(\boldsymbol{r})\|, \|\Pi_{\mathcal{K}}(-\boldsymbol{r})\|\right)$$

$$\left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r})\right\|^2 + \|\Pi_{\dot{\mathcal{K}}^{\perp}}(\boldsymbol{r})\|^2 = \left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r}) + \Pi_{\dot{\mathcal{K}}^{\perp}}(\boldsymbol{r})\right\|^2$$

$$\left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r})\right\|^2 > \alpha^2\left(\left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r})\right\|^2 + \|\Pi_{\dot{\mathcal{K}}^{\perp}}(\boldsymbol{r})\|^2\right)$$

$$(1-\alpha^2)\left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r})\right\|^2 > \alpha^2\|\Pi_{\dot{\mathcal{K}}^{\perp}}(\boldsymbol{r})\|^2$$

$$\left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r})\right\|^2 > \frac{\alpha^2}{1-\alpha^2}\|\Pi_{\dot{\mathcal{K}}^{\perp}}(\boldsymbol{r})\|^2$$

$$\left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r})\right\| > \sqrt{\frac{1-\epsilon^2}{\epsilon^2}}\|\Pi_{\dot{\mathcal{K}}^{\perp}}(\boldsymbol{r})\|$$

$$\epsilon\left\|\Pi_{\mathcal{K}}^{\pm}(\boldsymbol{r})\right\| > \sqrt{1-\epsilon^2}\|\Pi_{\dot{\mathcal{K}}^{\perp}}(\boldsymbol{r})\|$$

Calcul du gradient cas complexe

$$\begin{aligned}
\|\boldsymbol{x}\|^2 = \boldsymbol{x}^H\boldsymbol{x} &= (\boldsymbol{x}_r - j\boldsymbol{x}_i)^T(\boldsymbol{x}_r + j\boldsymbol{x}_i)\\
&= \boldsymbol{x}_r^T\boldsymbol{x}_r + \boldsymbol{x}_i^T\boldsymbol{x}_i - j\boldsymbol{x}_i^T\boldsymbol{x}_r + j\boldsymbol{x}_r^T\boldsymbol{x}_i\\
&= \boldsymbol{x}_r^T\boldsymbol{x}_r + \boldsymbol{x}_i^T\boldsymbol{x}_i
\end{aligned}$$

$$\begin{aligned}
\|\boldsymbol{y} - \mathcal{A}\boldsymbol{x}\|^2 &= \|\boldsymbol{y}_r - (\mathcal{A}\boldsymbol{x})_r\|^2 + \|\boldsymbol{y}_i - (\mathcal{A}\boldsymbol{x})_i\|^2\\
(\mathcal{A}\boldsymbol{x})_r &= \mathcal{A}_r\boldsymbol{x}_r - \mathcal{A}_i\boldsymbol{x}_i\\
(\mathcal{A}\boldsymbol{x})_i &= \mathcal{A}_r\boldsymbol{x}_i + \mathcal{A}_i\boldsymbol{x}_r
\end{aligned}$$

$$\begin{aligned}
\nabla_{\boldsymbol{x}_r}\|\boldsymbol{y} - \mathcal{A}\boldsymbol{x}\|^2 &= \nabla_{\boldsymbol{x}_r}\underbrace{\|\boldsymbol{y}_r - (\mathcal{A}\boldsymbol{x})_r\|^2}_{=\|\boldsymbol{y}_r + \mathcal{A}_i\boldsymbol{x}_i - \mathcal{A}_r\boldsymbol{x}_r)\|^2} + \nabla_{\boldsymbol{x}_r}\underbrace{\|\boldsymbol{y}_i - (\mathcal{A}\boldsymbol{x})_i\|^2}_{=\|\boldsymbol{y}_i - \mathcal{A}_r\boldsymbol{x}_i - \mathcal{A}_i\boldsymbol{x}_r)\|^2}\\
&= -2\mathcal{A}_r^T(\boldsymbol{y}_r + \mathcal{A}_i\boldsymbol{x}_i - \mathcal{A}_r\boldsymbol{x}_r) - 2\mathcal{A}_i^T(\boldsymbol{y}_i - \mathcal{A}_r\boldsymbol{x}_i - \mathcal{A}_i\boldsymbol{x}_r)\\
&= -2\mathcal{A}_r^T(\boldsymbol{y}_r - (\mathcal{A}\boldsymbol{x})_r) - 2\mathcal{A}_i^T(\boldsymbol{y}_i - (\mathcal{A}\boldsymbol{x})_i)\\
&= -2\,\mathrm{Re}\left\{\mathcal{A}^H(\boldsymbol{y} - \mathcal{A}\boldsymbol{x})\right\}
\end{aligned}$$

$$
\begin{aligned}
\nabla_{\boldsymbol{x}_i} \|\boldsymbol{y} - \mathcal{A}\boldsymbol{x}\|^2 &= \nabla_{\boldsymbol{x}_i} \underbrace{\|\boldsymbol{y}_r - (\mathcal{A}\boldsymbol{x})_r\|^2}_{=\|\boldsymbol{y}_r + \mathcal{A}_i\boldsymbol{x}_i - \mathcal{A}_r\boldsymbol{x}_r)\|^2} + \nabla_{\boldsymbol{x}_i} \underbrace{\|\boldsymbol{y}_i - (\mathcal{A}\boldsymbol{x})_i\|^2}_{=\|\boldsymbol{y}_i - \mathcal{A}_r\boldsymbol{x}_i - \mathcal{A}_i\boldsymbol{x}_r)\|^2} \\
&= 2\mathcal{A}_i^T(\boldsymbol{y}_r + \mathcal{A}_i\boldsymbol{x}_i - \mathcal{A}_r\boldsymbol{x}_r) - 2\mathcal{A}_r^T(\boldsymbol{y}_i - \mathcal{A}_r\boldsymbol{x}_i - \mathcal{A}_i\boldsymbol{x}_r) \\
&= 2\mathcal{A}_i^T(\boldsymbol{y}_r - (\mathcal{A}\boldsymbol{x})_r) - 2\mathcal{A}_r^T(\boldsymbol{y}_i - (\mathcal{A}\boldsymbol{x})_i) \\
&= -2\,\mathrm{Im}\left\{\mathcal{A}^H(\boldsymbol{y} - \mathcal{A}\boldsymbol{x})\right\}
\end{aligned}
$$

Thus:

$$
\nabla_{\boldsymbol{x}_r} \|\boldsymbol{y} - \mathcal{A}\boldsymbol{x}\|^2 = -2\,\mathrm{Re}\left\{\mathcal{A}^H(\boldsymbol{y} - \mathcal{A}\boldsymbol{x})\right\} \tag{1}
$$

$$
\nabla_{\boldsymbol{x}_i} \|\boldsymbol{y} - \mathcal{A}\boldsymbol{x}\|^2 = -2\,\mathrm{Im}\left\{\mathcal{A}^H(\boldsymbol{y} - \mathcal{A}\boldsymbol{x})\right\} \tag{2}
$$

F.-W. blasso

$$J(\boldsymbol{x},t) = \frac{1}{2}\left\|\boldsymbol{y}-\mathcal{A}\boldsymbol{x}\right\|^2 + \lambda t$$

$$\text{s.t.} \qquad \underbrace{\|\boldsymbol{x}\|_1}_{=\sum \text{ modules des } \boldsymbol{x}_i} \leq t \leq \frac{\|\boldsymbol{y}\|^2}{2\lambda} \triangleq M \qquad \equiv \mathcal{C}$$

Differential $J(\boldsymbol{x},t)$**:**

$$J(\boldsymbol{x},t) \;=\; J(\boldsymbol{x}_0,t_0) + \nabla^T_{\boldsymbol{x}_r} J(\boldsymbol{x}_0,t_0)(\boldsymbol{x}_r - \boldsymbol{x}_{0r}) + \nabla^T_{\boldsymbol{x}_i} J(\boldsymbol{x}_0,t_0)(\boldsymbol{x}_i - \boldsymbol{x}_{0i}) + \nabla^T_{t} J(\boldsymbol{x}_0,t_0)(t-t_0)$$

F.-W. step:
Find $(\hat{\boldsymbol{x}}_r, \hat{\boldsymbol{x}}_i, \hat{t}) \in \arg\min_{(\boldsymbol{x}_r,\boldsymbol{x}_i,t)\in\mathcal{C}} \nabla^T_{\boldsymbol{x}_r} J(\boldsymbol{x}_0,t_0)\boldsymbol{x}_r + \nabla^T_{\boldsymbol{x}_i} J(\boldsymbol{x}_0,t_0)\boldsymbol{x}_i + \nabla^T_{t} J(\boldsymbol{x}_0,t_0)t$.
From eq. (1) and eq. (2), we have:

$$\nabla^T_{\boldsymbol{x}_r} J(\boldsymbol{x}_0,t_0)\boldsymbol{x}_r + \nabla^T_{\boldsymbol{x}_i} J(\boldsymbol{x}_0,t_0)\boldsymbol{x}_i = -\operatorname{Re}\left\{\boldsymbol{x}^H\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\}$$

From the Hôlder inequality:

$$\operatorname{Re}\left\{\boldsymbol{x}^H\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\} \leq \|\boldsymbol{x}\|_1 \left\|\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\|_\infty$$

Thus:

$$\begin{aligned}
-\operatorname{Re}\left\{\boldsymbol{x}^H\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\} + \lambda t &\geq -\|\boldsymbol{x}\|_1 \left\|\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\|_\infty + \lambda t \\
&\geq -\|\boldsymbol{x}\|_1 \left(\left\|\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\|_\infty - \lambda\right), \quad \forall (x,t)\in\mathcal{C} \\
&= \|\boldsymbol{x}\|_1 \left(\lambda - \left\|\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\|_\infty\right)
\end{aligned}$$

We have:

$$\min_{0\leq\|\boldsymbol{x}\|_1\leq M} \|\boldsymbol{x}\|_1 \left(\lambda - \left\|\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\|_\infty\right) = \begin{cases} 0 & \text{if } \lambda \geq \left\|\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\|_\infty \\ M\left(\lambda - \left\|\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\|_\infty\right) & \text{otherwise} \end{cases} \tag{3}$$

Note that if we choose

$$\begin{aligned}
\hat{\boldsymbol{x}}(i) &= \begin{cases} M.e^{-j\arg(\boldsymbol{a}_i^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0))} & i = \arg\max_j \left|\boldsymbol{a}_j^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right| \\ 0 & \text{otherwise} \end{cases} \\
\hat{t} &= M
\end{aligned} \tag{4}$$

we have $(\boldsymbol{x},t)\in\mathcal{C}$ and $-\operatorname{Re}\left\{\boldsymbol{x}^H\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\} + \lambda t = M\left(\lambda - \left\|\mathcal{A}^H(\boldsymbol{y}-\mathcal{A}\boldsymbol{x}_0)\right\|_\infty\right)$
$\rightarrow$ attains the lower bounds eq. (3). Equation (4) is thus a minimizer.

1. Current iterate: $(\boldsymbol{x}^{(k)}, t^{(k)})$
2. Atom selection step:

$$\boldsymbol{a}^{(k)} = \arg\max_{\tilde{\boldsymbol{a}}\in\mathcal{A}} \left|\langle \tilde{\boldsymbol{a}}, \boldsymbol{y} - \mathcal{A}\boldsymbol{x}^{(k)}\rangle\right|$$

3. Std F.-W. update:

$$\begin{aligned}
\gamma^{(k)} &= \arg\min_{\gamma\in[0,1]} \frac{1}{2}\left\|\underbrace{\boldsymbol{y} - (1-\gamma)\mathcal{A}\boldsymbol{x}^{(k)} - \gamma\boldsymbol{a}^{(k)}\hat{\boldsymbol{x}}}_{\boldsymbol{r}^{(k)}-\gamma(\boldsymbol{a}(k)-\mathcal{A}\boldsymbol{x}(k))}\right\|^2 + \lambda\left((1-\gamma)t^{(k)} + \gamma M\right)\\
&= \arg\min_{\gamma\in[0,1]} \frac{1}{2}\left(\left\|\boldsymbol{r}^{(k)}\right\|^2 - 2\gamma\,\mathrm{Re}\left\{\langle \boldsymbol{r}^{(k)}, \boldsymbol{a}^{(k)}\hat{\boldsymbol{x}} - \mathcal{A}\boldsymbol{x}(k)\rangle\right\} + \gamma^2\left\|\boldsymbol{a}^{(k)}\hat{\boldsymbol{x}} - \mathcal{A}\boldsymbol{x}^{(k)}\right\|^2\right)\\
&\quad + \lambda\left(t^{(k)} - \gamma(t^{(k)} - M)\right)\\
&\triangleq \arg\min_{\gamma\in[0,1]} f(\gamma)
\end{aligned}$$

$$f'(\gamma) = -\,\mathrm{Re}\left\{\langle \boldsymbol{r}^{(k)}, \boldsymbol{a}^{(k)}\hat{\boldsymbol{x}} - \mathcal{A}\boldsymbol{x}^{(k)}\rangle - \lambda\left(t^{(k)} - M\right)\right\} + \gamma\left\|\boldsymbol{a}^{(k)}\hat{\boldsymbol{x}} - \mathcal{A}\boldsymbol{x}^{(k)}\right\|^2$$

$$\begin{aligned}
&\tilde{\gamma} : f'(\gamma) = 0\\
&= \frac{\mathrm{Re}\left\{\langle \boldsymbol{r}^{(k)}, \boldsymbol{a}^{(k)}\hat{\boldsymbol{x}} - \mathcal{A}\boldsymbol{x}^{(k)}\rangle + \lambda\left(t^{(k)} - M\right)\right\}}{\left\|\boldsymbol{a}^{(k)}\hat{\boldsymbol{x}} - \mathcal{A}\boldsymbol{x}^{(k)}\right\|^2}
\end{aligned}$$

$$\gamma^{(k)} = \begin{cases} 0 & \text{si } \tilde{\gamma} < 0\\ \tilde{\gamma} & \text{si } 0 \le \tilde{\gamma} \le 1\\ 1 & \text{si } \tilde{\gamma} > 1 \end{cases} \tag{5}$$

$$x^{(k+\frac{1}{2})} = (1-\gamma^{(k)})\boldsymbol{x}^{(k)} + \gamma^{(k)}\hat{\boldsymbol{x}} \tag{6}$$

$$t^{(k+\frac{1}{2})} = (1-\gamma^{(k)})t^{(k)} + \gamma^{(k)}M \tag{7}$$

4. Let $\mathcal{S}^{(k)}$ be the current support of $\boldsymbol{x}^{(k+\frac{1}{2})}$

$$\boldsymbol{x}^{(k+1)} = \arg\min_{\boldsymbol{x}_{\mathcal{S}^{(k)}}} \frac{1}{2}\left\|\boldsymbol{y} - \mathcal{A}_{\mathcal{S}^{(k)}}\boldsymbol{x}_{\mathcal{S}^{(k)}}\right\|^2 + \lambda\left\|\boldsymbol{x}_{\mathcal{S}^{(k)}}\right\|_1 \tag{8}$$

5. Joint optimization:

$$\min_{\boldsymbol{x},\theta,\beta} \frac{1}{2}\left\|\boldsymbol{y} - \sum_{i=1}^{\mathrm{card}(\mathcal{S}^{(k)})} \boldsymbol{a}(\theta_i)x_i\right\|^2 + \lambda\sum_{i=1}^{\mathrm{card}(\mathcal{S}^{(k)})}\beta_i \tag{9}$$

$$\text{s.t.} \qquad |x_i| \le \beta_i \equiv \sqrt{\mathrm{Re}^2(x_i) + \mathrm{Im}^2(x_i)} \le \beta_i \equiv \quad \text{contraintes convexe} \tag{10}$$

Borne inf. pour le screening

On cherche τ tel que:

$$\max_{\boldsymbol{a}\in\mathcal{A}} |\langle \boldsymbol{a}, \boldsymbol{r}\rangle| \geq \tau$$

Soit

$$\bar{\boldsymbol{a}} = \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{a}_i, \qquad \boldsymbol{a}_i \in \bar{\mathcal{A}} \subseteq \mathcal{A},$$

alors,

$$\begin{aligned}
|\langle \bar{\boldsymbol{a}}, \boldsymbol{r}\rangle| =& \frac{1}{N}\left|\langle \sum_{i=1}^{N} \boldsymbol{a}_i, \boldsymbol{y}\rangle\right| \\
=& \frac{1}{N}\left|\sum_{i=1}^{N} \langle \boldsymbol{a}_i, \boldsymbol{y}\rangle\right| \\
\leq& \frac{1}{N}\sum_{i=1}^{N} |\langle \boldsymbol{a}_i, \boldsymbol{y}\rangle| \\
\leq& \max_{\boldsymbol{a}\in\mathcal{A}} |\langle \boldsymbol{a}, \boldsymbol{r}\rangle| .
\end{aligned}$$

Donc, $\tau = |\langle \bar{\boldsymbol{a}}, \boldsymbol{r}\rangle|$ est une borne inf. pour $\max\limits_{\boldsymbol{a}\in\mathcal{A}} |\langle \boldsymbol{a}, \boldsymbol{r}\rangle|$.

DoA avec polynôme trigonométrique

$$f(\theta) = \langle \boldsymbol{y}, \boldsymbol{a}(\theta)\rangle = \sum_{k=0}^{M} y_k e^{-itk} \tag{11}$$

On cherche $\theta = \arg\max\limits_{\theta} |f(\theta)|$.
On a

$$f'(\theta) = \sum_{k=0}^{M} \underbrace{-i\theta y_k}_{\alpha_k} \underbrace{(e^{-i\theta})^k}_{x^k} \tag{12}$$

On défini le polynome:

$$p(x) = \sum_{k=0}^{M} \alpha_k x^k$$

Estimation de la DoA en 3 étapes:

1. Recherche de $\mathcal{X} = \{x \quad \text{t.q.} \quad p(x) = 0 \quad \text{et} \quad \|x\| = 1\}$
2. Selection de l'atom: $x = \arg\max\limits_{x\in\mathcal{X}} \left|\sum\limits_{k=0}^{M} y_k x^k\right|$
3. Estimation de la DoA : $\theta = \arg(ix)$