PARSIMONIOUS REPRESENTATION APPLIED ON THE DIRECTIONAL WAVE SPECTRA

This report aims to explain the results of segmentation problem examined on both artificial and real directional wave spectra. These results are obtained by applying a variety of approaches such as segmenting directly on the spectra (Watershed method) or modeling these spectra as a combination of many spectrum units modeled by a specific probability distribution function and utilizing optimization tools to find spare vectors that represents these spectra corresponding to the determined type of distribution function.

In the experiences, the directional wave spectra are modeled as a gaussian distribution function and these spectra can be modeled as the following formula:

$$Ax = y \tag{1}$$

Where the A matrix considered as a dictionary whose columns are generated by applying a gaussian atom, the y vector is the spectrum observation vector and we need to find a x vector so that we can represent the spectrum observation vector with the smallest number of elements of the dictionary A, this turns into resolving the optimization problems. The problem can be represented by several types of optimization problems, but the case more convenient to use expressed as follows:

$$f(x) = \frac{1}{2}||y - Ax||^2 + \gamma ||x||_1 \quad x \in R^n \tag{2}$$

And the objective is to find a x vector to minimize that function. Depending on the size of the A matrix, we address to two different problems. The case, in which the A matrix contains a finite number of columns, is called the Lasso problem which is well resolved by the FISTA method. The other case is called Blasso problem in which the A matrix contain an infinite number of column vectors and this problem are well addressed by the SFW method.

The gaussian atom is described on the 2D coordination whose density function is represented as the following formula:

$$f(x,y) = \frac{1}{\sqrt{2\pi}\sigma_x\sigma_y} e^{\frac{-1}{2\sigma_x^2}(x-u_x)^2 - \frac{-1}{2\sigma_y^2}(y-u_y)^2} \tag{3}$$

Each element of the A matrix is generated by varying at least one of the elements in the set of variables $\sigma_x \ \sigma_y \ u_x \ u_y$.

I. Tests on artificial spectra

1.1 The gaussian atom with the fixed values of variances:

This experience uses the gaussian atom with fixed values of variances to examine the efficiency of the FISTA method and the SFW method recovering the artificial spectrum units.

σ_x^2	0.4
σ_y^2	0.2
u_x	variable
u_y	variable

Table 1.1 Gaussian atom variables.

1.1.1 The artificial spectrum with 3 separated spectra units:

- *Generating an artificial spectrum:*

The artificial spectrum is generated by combining three different spectrum units that are separated from each other. The following table is the information which are used to generate the artificial spectrum:

	1st spec unit	2nd spec unit	3rd spec unit
u_x	-0.46939	1.2463	-2.8278
u_y	-1.1182	-1.9983	-2.3061
coef	1.1544	1.4273	1.4413

Table 1.2 Three separated spectrum units.

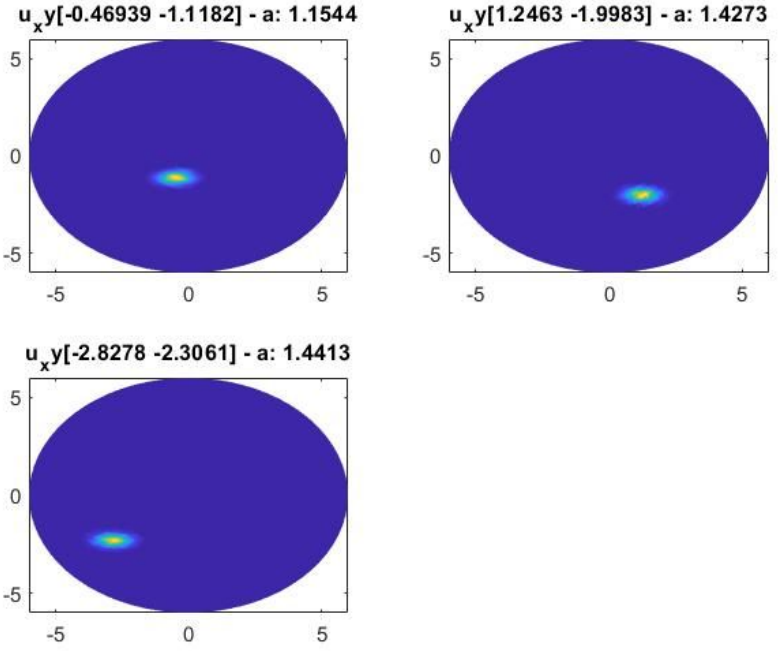

Fig 1.1 Three separated spectrum units.

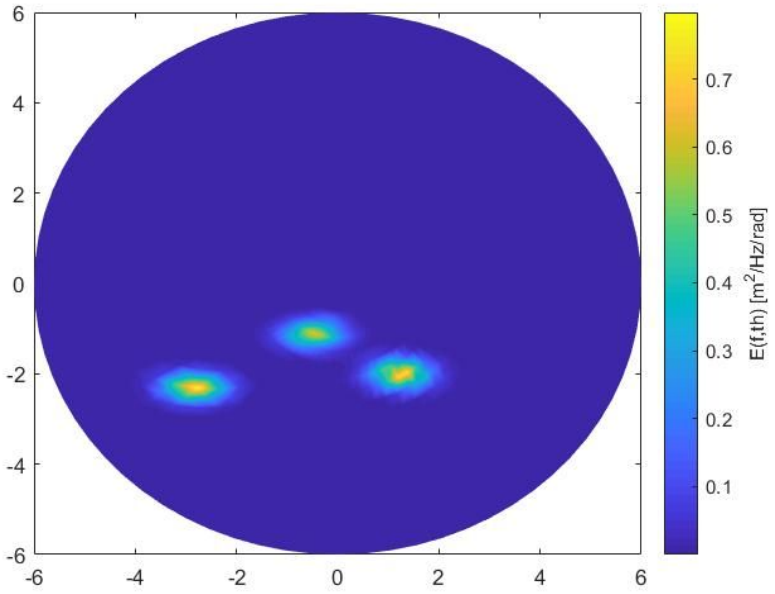

Fig 1.2 Separated spectrum unit combination.

❑ *Recovery the spectrum units by using the discrete dictionary (LASSO problem):*

In this case, the A matrix contains a finite number of column vectors which are generated by applying the different discrete positions of mean values (ux, uy) on the gaussian atom. This dictionary was generated and saved for later usage. The 2D range of mean values must cover the 2D range in which the spectrum units exist. This problem is well addressed by the Fast Iterative Shrinkage-Thresholding Algorithm (FISTA). The following table shows the results obtained via this method:

	u_x	u_y	**coef**
1st spectrum	-3.1128	-2.1796	0.1021
2nd spectrum	-3.1947	-2.2369	0.10963
3rd spectrum	-2.7578	-2.314	1.0769
4th spectrum	-2.8344	-2.3783	0.16829
5th spectrum	-0.50714	-1.10876	0.47211
6th spectrum	-0.5494	-1.1782	0.15523
7the spectrum	-0.41042	-1.1276	0.52732
8th spectrum	1.15	-1.9919	0.65958
9th spectrum	1.2	-2.0785	0.23284
10th spectrum	1.3766	-1.966	0.51895

Table 1.3 spectrum units obtained by the FISTA method.

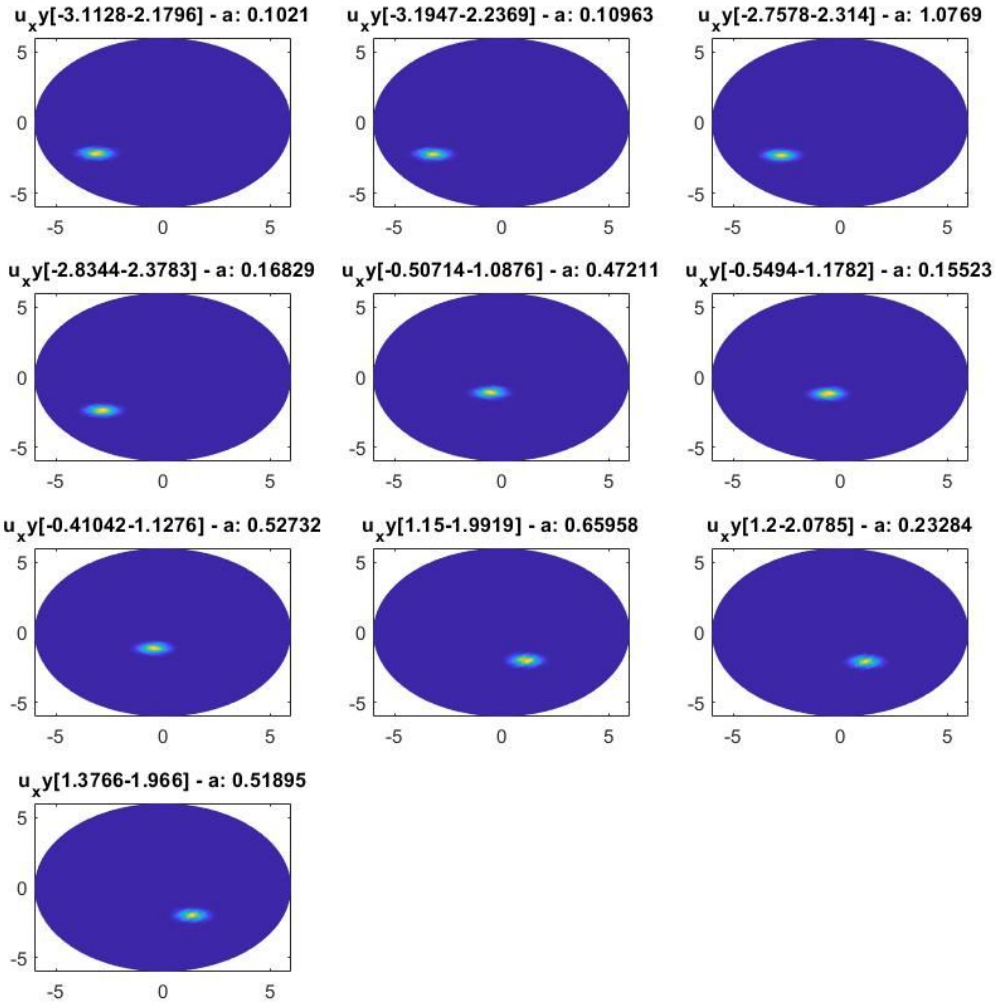

Fig 1.3 Visualization of spectrum units obtained by the FISTA method.

Through observation of spectrum units obtained by FISTA method, we can divide the spectrum units into three differently-colored groups by differences of mean value coordination between them . By comparing to the ground truth of spectrum units in the table 1.2, each ground truth spectrum in this table are represented by a group of spectrum which is near to the ground truth. The total executive time was more than 8 minutes for 20000 iterations and this algorithm doesn't converge to the stopping creatia yet.

The results show the limits of the discrete dictionary that can just approximate the real spectrum units by combining several spectrum units, already existing in the defined dictionary, being near to the ground truth spectrum units. Increasing the number of mean position samples on the certain 2D range is the way to make the results more precise, but the algorithm struggles with the time consuming problem to converge to the stopping criteria as the matrix size is enormous.

❏ *Recovery the spectrum units by using the continuous dictionary (BLASSO problem)*:
Inversely, the mean values (ux, uy) in this case take a continuous value on the certain range. This problem are well addressed by using the Sliding Frank-Wofe algorithm (SWF). Following is the results of spectrum recovery by using this method:

	u_x	u_y	**coef**
1st spectrum	-0.46888	-1.1179	1.1427
2nd spectrum	1.2458	-1.998	1.4046
3rd spectrum	-2.8269	-2.3059	1.4063

Table 1.4 spectrum units obtained by the SFW method.

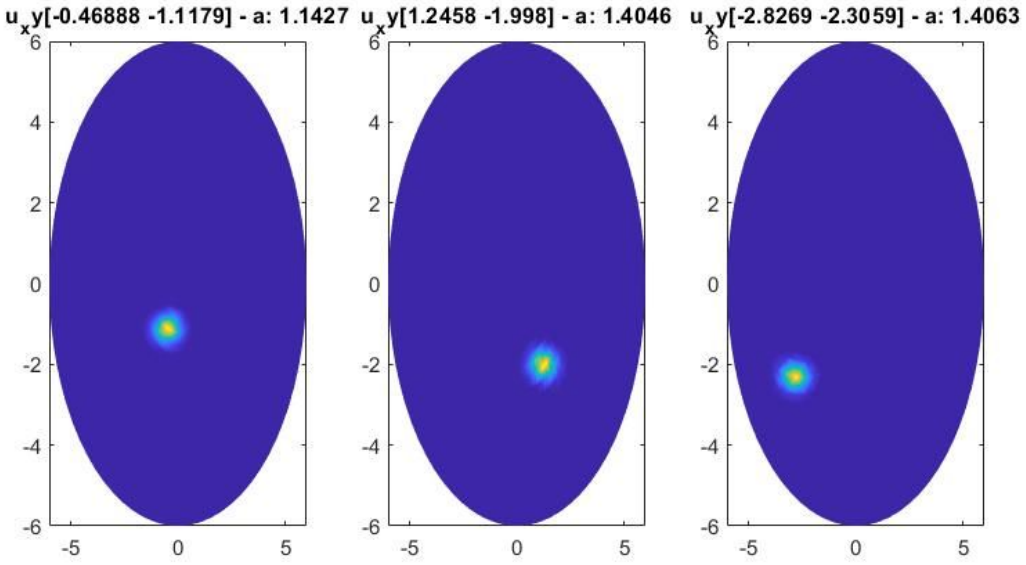

Fig 1.4 Visualization of spectrum units obtained by the SFW method.

By comparing the results with the ground truth in the table 1.2 and Fig 1.1, we observe that the spectrum recovery attains the high accuracy, there are exactly three recovered spectrums corresponding to three ground truth spectrums. They are highly approximate in the mean value coordination and in their coefficients. And the executive time of this method is 0.242432 second that is very fast to converge to the stopping criteria compared with the FISTA method.

1.1.2 The artificial spectrum with 3 closed spectrum units:

❏ *Generating an artificial spectrum:*
In this experience, we try to separate three spectrum units that are close together, i.e their mean values are approximately the same in both axis.

	1st spec unit	2nd spec unit	3rd spec unit
u_x	-2.3566	-2.0135	-2.8283
u_y	-2.4864	-2.6624	-2.724
coef	1.1544	1.4273	1.4413

Table 1.5 Three close spectrum units.

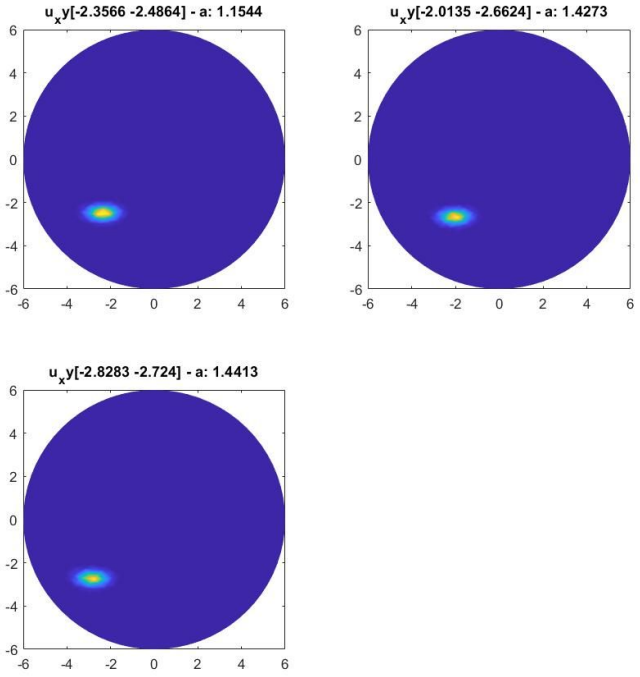

Fig 1.5 Close spectra visualization.

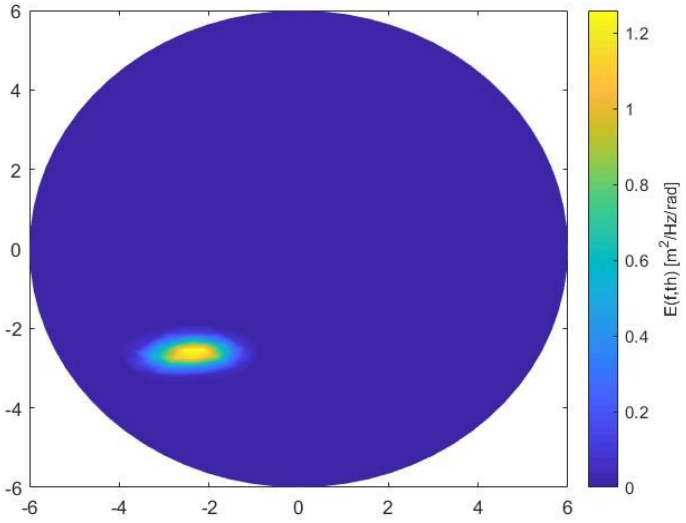

Fig 1.6 Close spectra combination.

❑ *Recovery the spectrum units by using the discrete dictionary (LASSO problem):*

We use the same dictionary as the above Lasso case, the following are the results:

	u_x	u_y	**coef**
1st spectrum	-3.0642	-2.5712	0.22341
2nd spectrum	-2.4749	-2.4749	0.53034
3rd spectrum	-2.687	-2.687	0.23283
4th spectrum	-2.7577	-2.7577	1.0445
5th spectrum	-2.1212	-2.5279	0.57251
6th spectrum	-2.1855	-2.6046	0.78239
7th spectrum	-1.8928	-2.7032	0.59331

Table 1.6 Spectrum units obtained by the FISTA method.

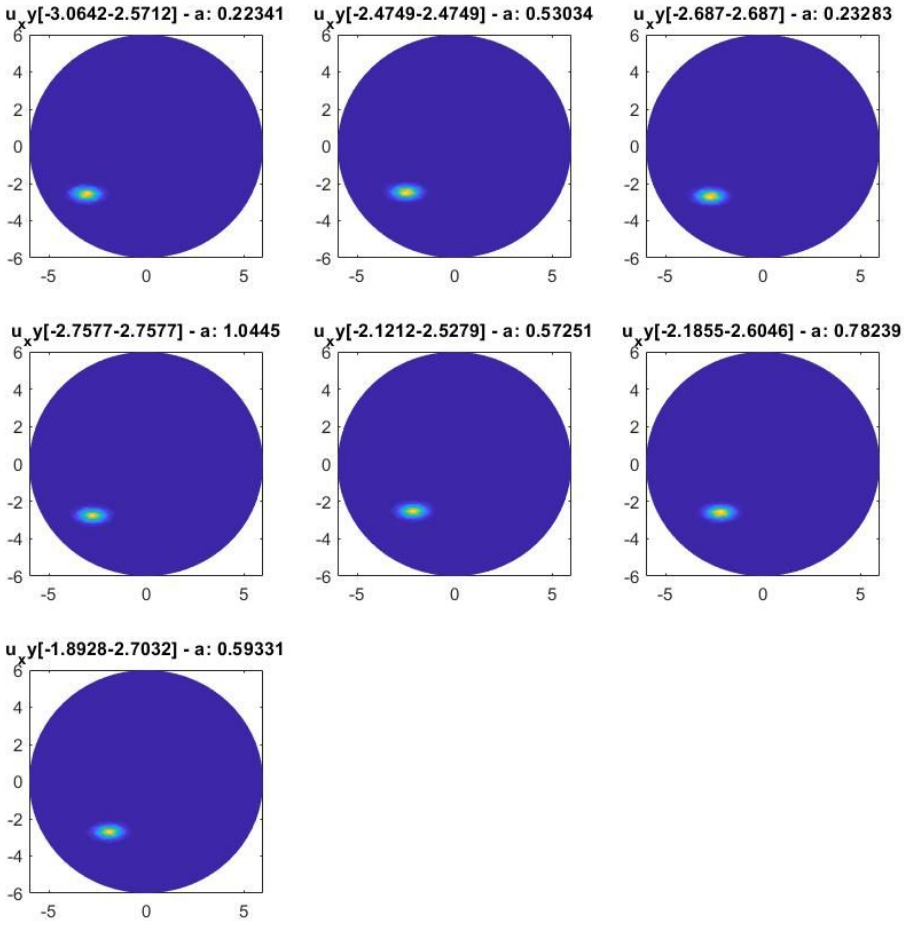

Fig 1.7 Visualization of spectrum units obtained via the FISTA method.

When the spectrum units are very close, the FISTA algorithm has a difficulty in separating individually different ground truth spectrum units. So, the algorithm can't avoid the error as recovering spectra. Comparing to the table 1.5 and the figure 1.5, the first spectrum is out of the ground truth. The rest is divided into three groups corresponding to the ground truth spectrum units in terms of the mean value coordination. As precedently mentioned about the runtime problem, the algorithm takes about 20 minutes to execute 20000 iterations and the algorithm doesn't converge to the criteria yet.

❏ *Recovery the spectrum units by using the continuous dictionary (BLASSO problem)*:

	u_x	u_y	**coef**
1st spectrum	-2.8215	-2.7221	1.4123
2nd spectrum	-2.3906	-2.4804	0.83823
3rd spectrum	-2.0621	-2.6362	1.423
4th spectrum	-1.9012	-2.7261	0.17882

Table 1.7 spectrum units obtained by the SFW method.

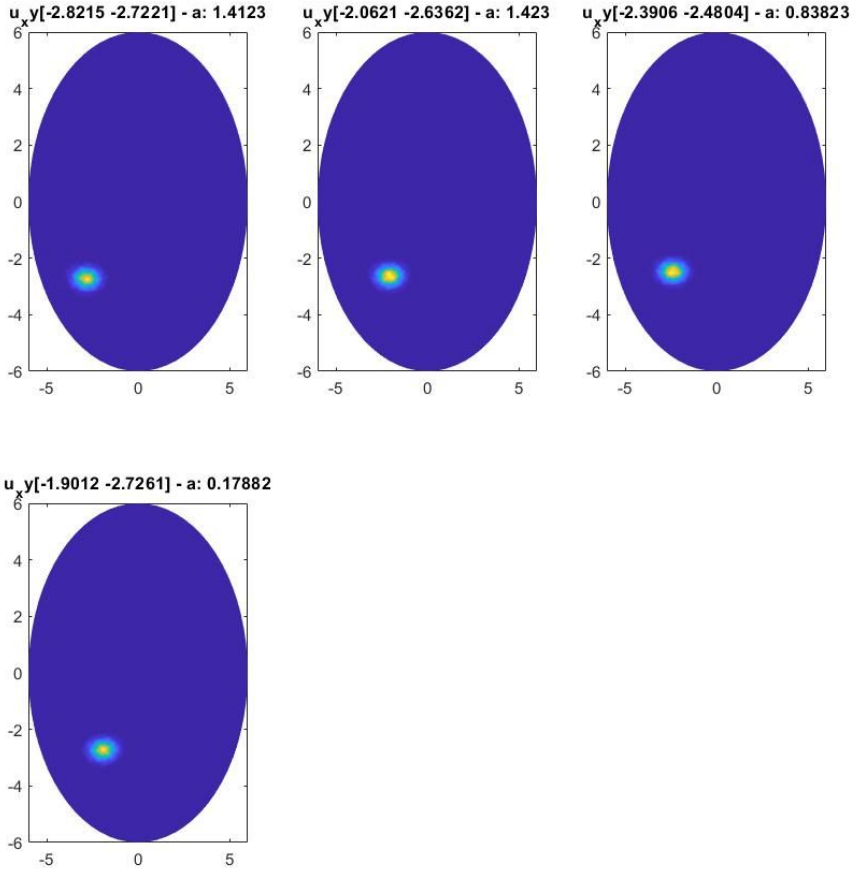

Fig 1.8 Visualization of spectrum units obtained via the SWFmethod.

The results show the efficiency of recovery the highly close spectrum units by using the SFW method. The accuracies in terms of the mean value coordination and the coefficient values are highly achieved. However, there are four spectrum units instead of three spectrum units existing in the ground truth. But to converge to the criteria , the algorithm need to take about 4 seconds. The runtime is significantly reduced compared with the FISTA method (but the FISTA method doesn't converge to the criteria .).

1.2 The gaussian atom with changed variances:

Throughout the above experiences, we have observed that the FISTA method has some limits in accuracy and in time computation and the SWF method achieved better performance in both terms. For generating the discrete dictionary as the variances changing their values, we need to build an enormous matrix which is hard to save and to use on the memory's computer and it will take more time to process than the previous cases. However, the SWF method offers more advantages to overcome the difficulties. The atom is called to generate a dictionary when necessary. The gaussian mean values and the variances could take continuous values on specific ranges. So, in this experience we only use the SFW method to resolve the Blasso problem in the case of continuous dictionary and the atoms used to generate the dictionary's elements has different values of variances. The following table shows ranges of each variable.

σ_x^2	[0.1 1]
σ_y^2	[0.1 1]
u_x	[-4 4]
u_y	[-4 4]

Table 1.8 Gaussian atom variables.

1.2.1 The artificial spectrum with 3 separated spectrum units:

- *Generating an artificial spectrum:*

The following table shows different spectrum units and their gaussian variables used to generate them:

	u_x	u_y	σ_x^2	σ_y^2	coef
1st spectrum	-0.66382	-1.5813	0.26763	0.58494	1.1481
2nd spectrum	1.7626	-2.826	0.411	0.47728	2.8003
3rd spectrum	-3.9991	-3.2613	0.45709	0.7167	0.4806

Table 1.9 Three separated spectrum units.

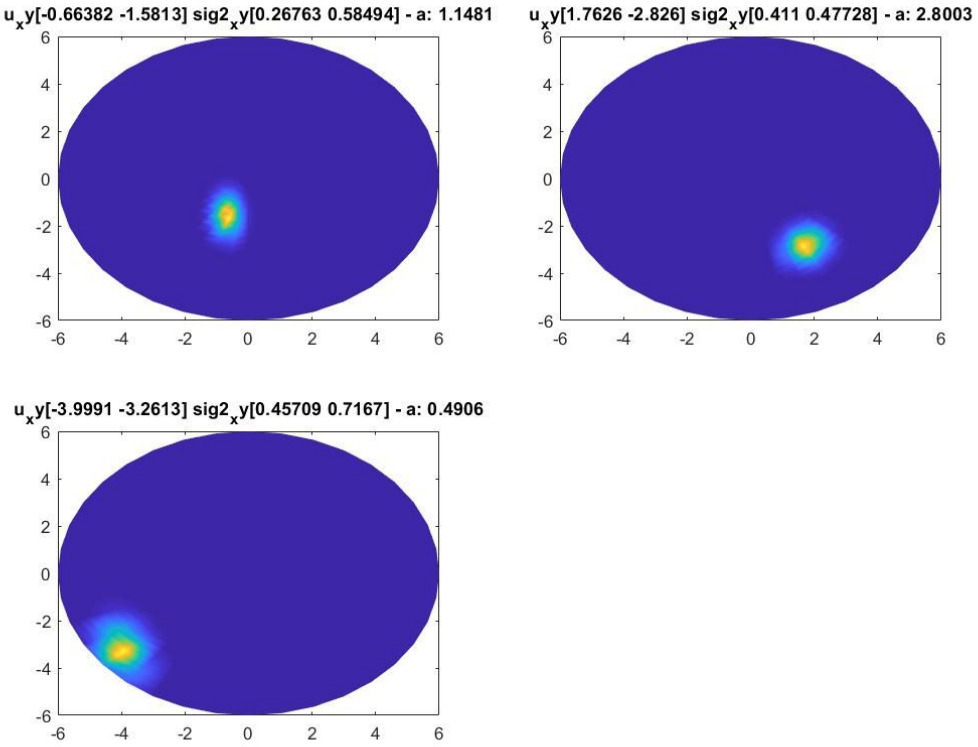

Fig 1.9 Visualization of spectrum units.

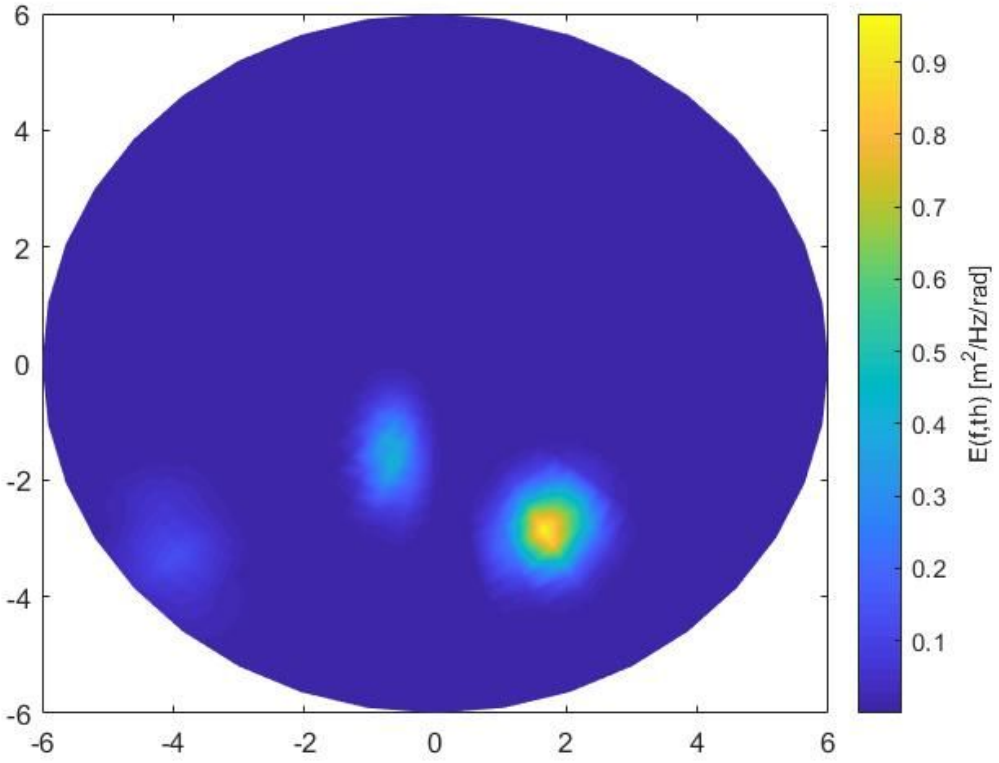

Fig 1.10 Visualization of the spectra.

❏ *Recovery the spectrum units by using the continuous dictionary (BLASSO problem)*:

	u_x	u_y	σ_x^2	σ_y^2	coef
1st spectrum	1.7716	-2.8201	0.36817	0.4396	0.40803
2nd spectrum	1.7377	-2.8683	0.42214	0.53706	0.46802
3rd spectrum	1.1882	-2.7879	0.4375	0.4375	0.33624
4th spectrum	1.7374	-2.8687	0.4375	0.4375	0.27962
5th spectrum	1.7378	-2.7869	0.39731	0.51597	0.56567
6th spectrum	1.7374	-2.7879	0.4375	0.4375	0.2257
7th spectrum	1.8172	-2.8672	0.37884	0.49987	0.32243
8th spectrum	-0.66055	-1.5677	0.23918	0.59131	0.55343
9th spectrum	-0.6866	-1.7373	0.2972	0.64612	0.10974
10th spectrum	-0.60713	-1.5755	0.29787	0.53165	0.14364
11th spectrum	-0.68664	-1.495	0.28672	0.64069	0.12268
12th spectrum	-0.68706	-1.5756	0.29862	0.53232	0.2096
13th spectrum	-3.9999	-3.2473	0.48502	0.73005	0.25136

Table 1.10 spectrum units obtained by the SFW method.

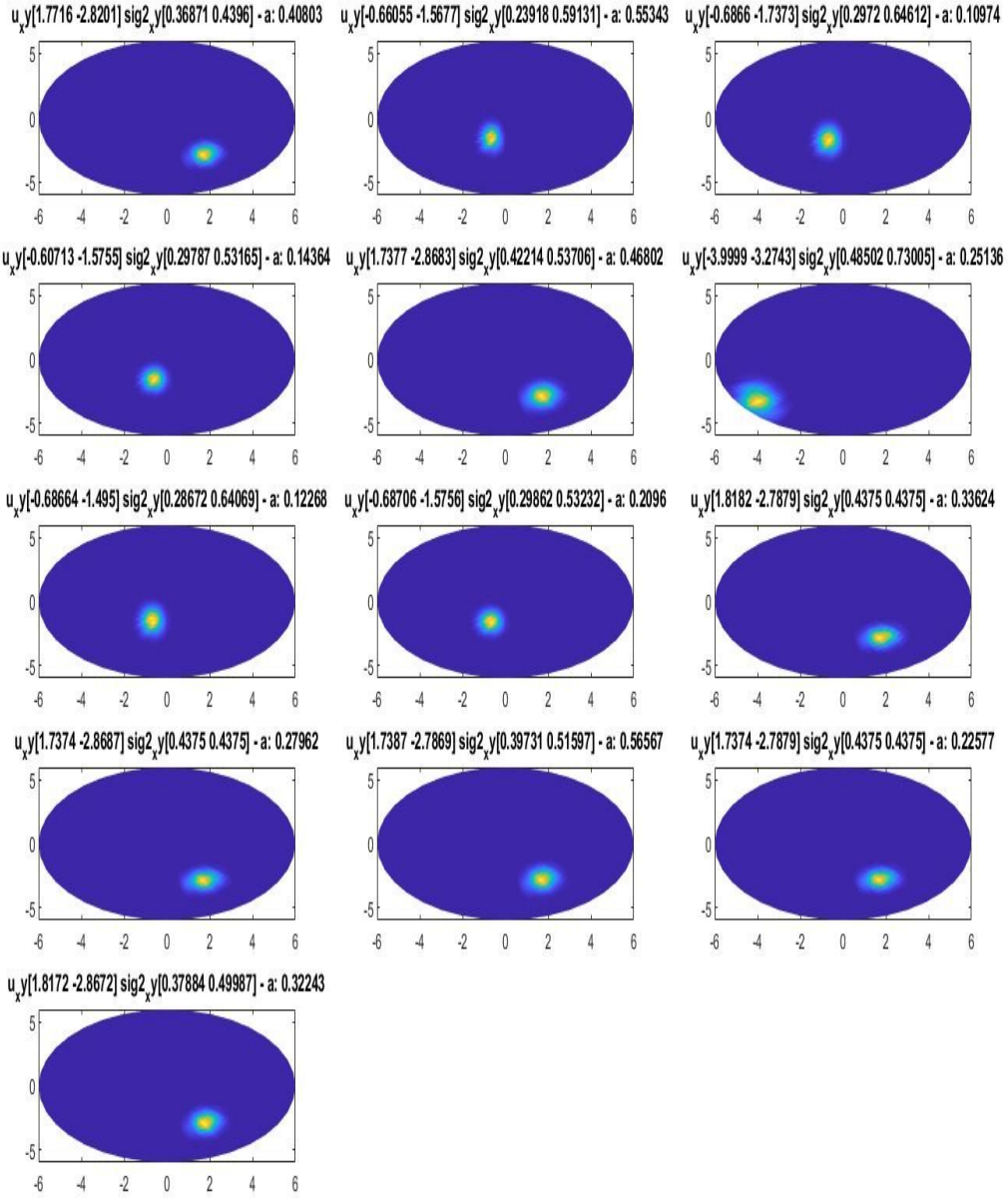

Fig 1.11 Visualization of the spectra obtained by the SFW method.

The spectrum units shown in the table 1.10 and the figure 1.11 have their coefficients being greater than 0.1. There are also many spectrum units that have significantly small coefficients (smaller than 0.1) but we don't show all of the results here. The obtained spectrum units could be divided into three groups corresponding to the ground truths shown in the table 1.9. The ground truths are recovered by summing several spectrum units that have approximate values of gaussian variables. In fact, this is quite a complicated problem when varying all gaussian variables, but the SFW algorithm converge to the criteria after taking 448 seconds to do 99 iterations.

1.2.2 The artificial spectrum with 3 closed spectrum units:

❑ *Generating an artificial spectrum:*

The following table shows different spectrum units and their gaussian variables used to generate them:

	u_x	u_y	σ_x^2	σ_y^2	coef
1st spectrum	-2.6731	-2.9887	0.93125	0.86072	3.9667
2nd spectrum	-3.8123	-3.1313	0.47502	0.37625	2.1471
3rd spectrum	-3.979	-2.4777	0.3591	0.92403	3.0223

Table 1.11 Three close spectrum units.

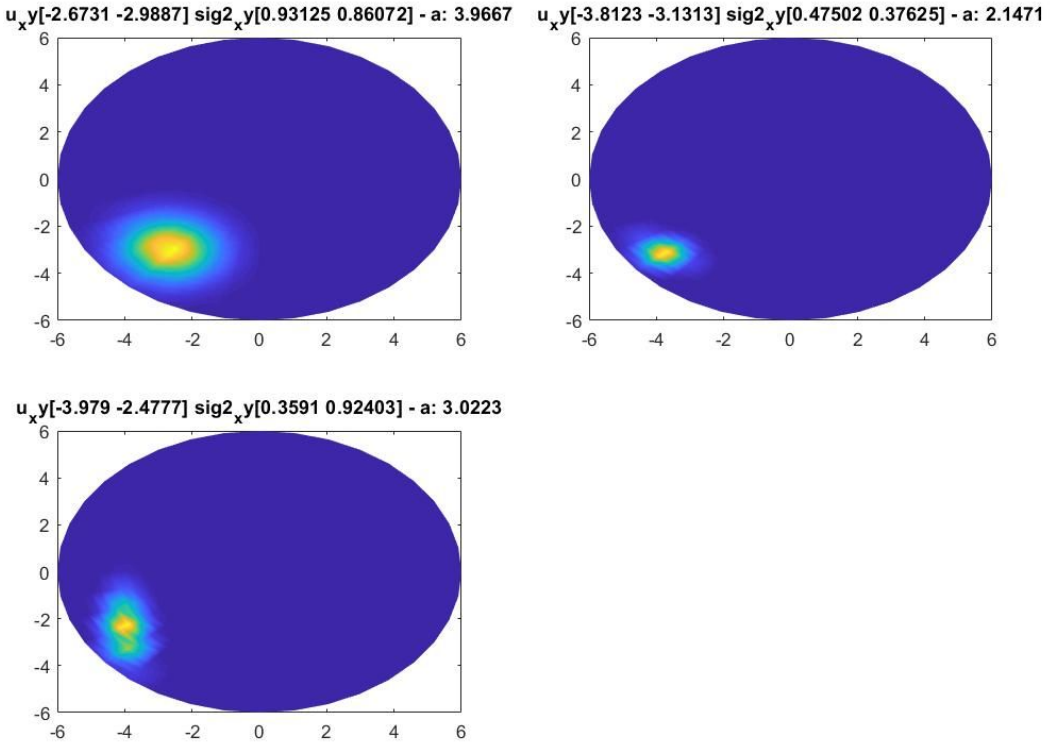

Fig 1.12 Visualization of spectrum units.

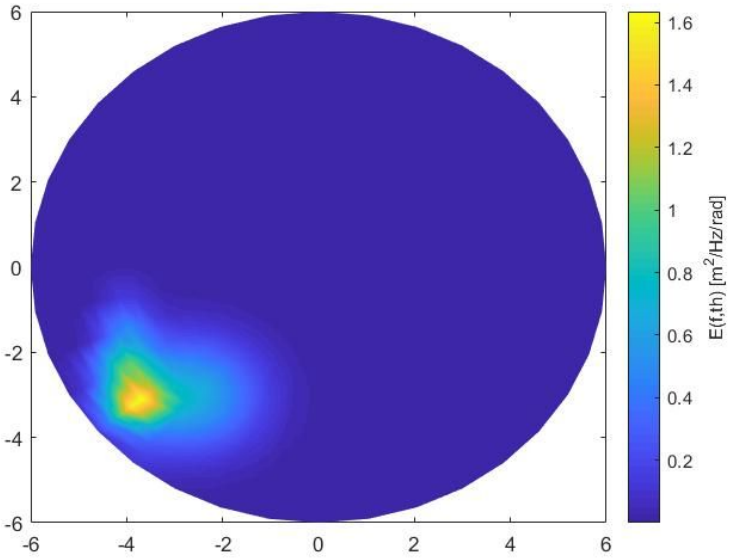

Fig 1.13 Visualization of the spectra.

❑ *Recovery the spectrum units by using the continuous dictionary (BLASSO problem)*:

	u_x	u_y	σ_x^2	σ_y^2	coef
1st spectrum	-3.4505	-2.8953	0.1	0.1	0.21817
2nd spectrum	-3.9101	-3.2809	0.1	0.1	0.23967
3rd spectrum	-4	-3.4375	0.25701	0.15909	0.39115
4th spectrum	-3.677	-3.0854	0.1	0.1	0.37156
5th spectrum	-2.7879	-2.9495	1	0.8875	2.8706
6th spectrum	-3.0303	-3.1111	1	0.6625	0.79317
7th spectrum	-2.7071	-3.0303	1	0.8875	0.14937
8th spectrum	-2.8687	-3.0303	1	0.775	0.2216
9th spectrum	-2.8687	-3.0303	1	0.775	0.12607
10th spectrum	-4	-2.5451	0.40101	0.87706	0.86644
11th spectrum	-4	-2.4646	0.325	1	1.4591
12th spectrum	-4	-2.5455	0.325	0.8875	0.28642
13th spectrum	-3.919	-2.7068	0.40202	0.75469	0.30995

Table 1.12 spectrum units obtained by the SFW method.

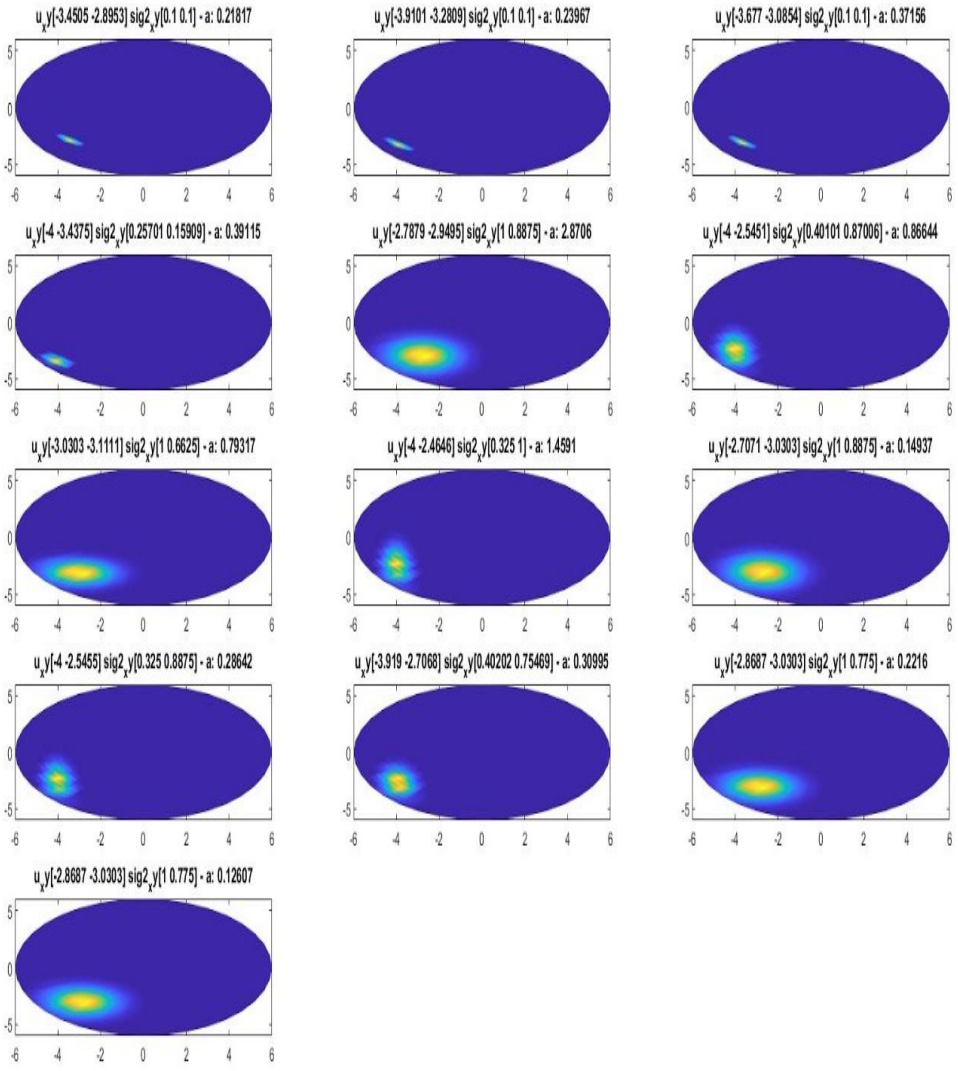

Fig 1.14 Visualization of the spectra obtained by the SFW method.

It's hard to recover the original spectrum units in the case where they are close. The results show an approximation of the original spectrum units by a combination of several spectrum units with different values of gaussian variables. The algorithm converge to the criteria which show that the combination of all recovered spectrum units can represent the original observation vector, but these recovered spectrum units aren't exactly those originally construct the observation vector.

II. Test on real spectra

2.1 Sliding Frank-Wolfe method:

In the real spectrum tests, we suppose that the natural directional wave spectrum is a sum of many directional wave spectra that follow a gaussian possibility distribution. The following table shows a range of each gaussian variable.

σ_x^2	[0.001 2]
σ_y^2	[0.001 2]
u_x	[2*min(min(fx)) 2*max(max(fx))]
u_y	[2*min(min(fy)) 2*max(max(fy))]

Table 1.13 Gaussian atom variables on the real spectrum tests.

Where fx, fy are the spatial frequencies measured on the real spectra. These conditions guarantee that the gaussian variable values can cover all the possible values of gaussian atoms combining the directional wave spectra. Another important attention on choosing the value of the variance is that it should take fairly small values but not get 0, because it might exist several spectra that include many gaussian atoms with small variances. This choice helps the algorithm converging to the stopping criteria and obtaining very approximate result from the ground truth.

- ❑ *Test on the wind sea system:*

This test is implemented on the system where there is only one wind sea appearing in the spectrum. The tested spectrum was measured on the NODE008919(12-Jan-2010 17:00:00).

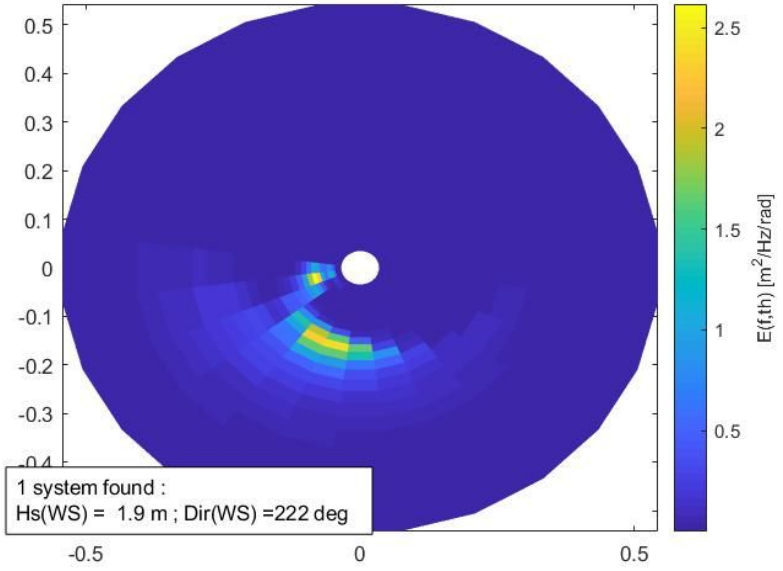

Fig 2.1 Visualization of the real wind sea system.

The SFW method took about twelve minutes to implement 118 iterations to converge to the stopping criteria tolerance. To combine the original spectrum, using this method detects 117 different atoms. The atoms have their coefficients with different values, which decide how much influence (amplitude and direction) that each atom contributes to recover the original spectrum. Approximately, we can examine the atoms which contribute significantly to recover the spectrum. The following table shows the values of these atoms.

	u_x	u_y	σ_x^2	σ_y^2	coef
1st atom	-0.1061	-0.1418	0.0501	0.0501	0.1937
2nd atom	-0.2762	-0.0068	0.2324	0.0001	0.3036
3rd atom	-0.2222	-0.0102	0.2307	0.0001	0.5064
4th atom	0.0165	-0.0102	0.2015	0.001	-0.2073
5th atom	-0.1316	-0.1465	0.0660	0.0001	0.1701
6th atom	-0.1502	-0.1258	0.0660	0.0001	0.1131
7th atom	-0.0674	-0.2335	0.0001	0.1234	0.1095

Table 2.1 Principle atom components.

In this table, we can clearly see that there are not only atoms that have positive impacts (positive direction) on enhancing the energy of the original spectrum but also there exist at the same time the atoms weaken the amplitude of the spectrum's energy (negative direction). We can imagine that, in reality the wind seas can move in any direction and a combination of them creating one final direction that caused by the predominant atoms. The following figure shows a combination of all atoms found by the SFW method.

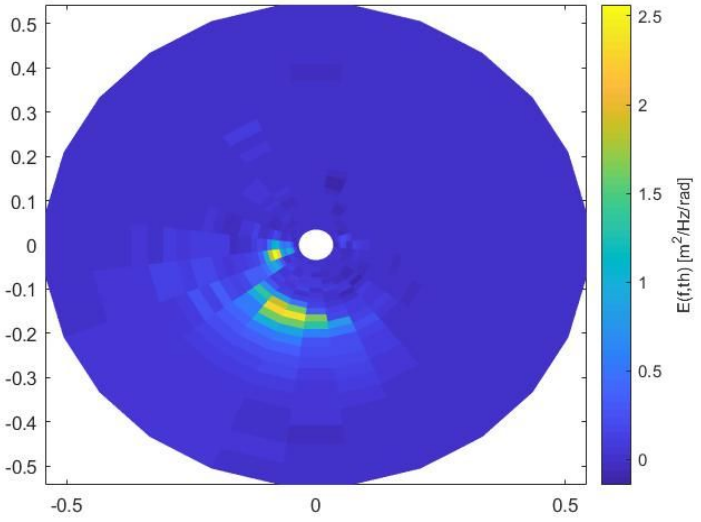

Fig 2.2 Gaussian atom combination found by the SFW method.

Compared with the original spectrum in the figure 2.1, we can see that the recovered spectrum has almost the same altitudes and the same directions as the original one. It proves that the directional wave spectrum can be modeled as a sum of many spectral units that following the gaussian distribution law. Below are several different tests to recovery different spectra which can exist many components in each spectrum.

- ❑ *Test on the swell system:*

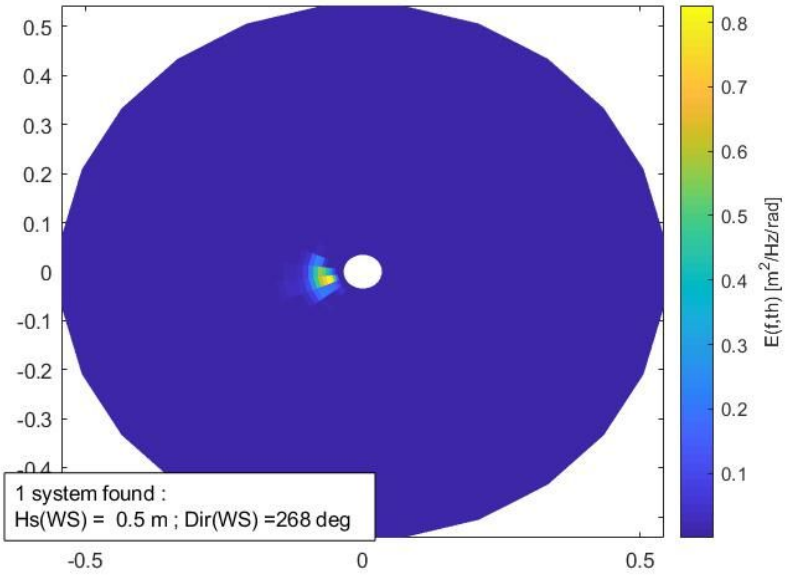

Fig 2.3 The swell system.

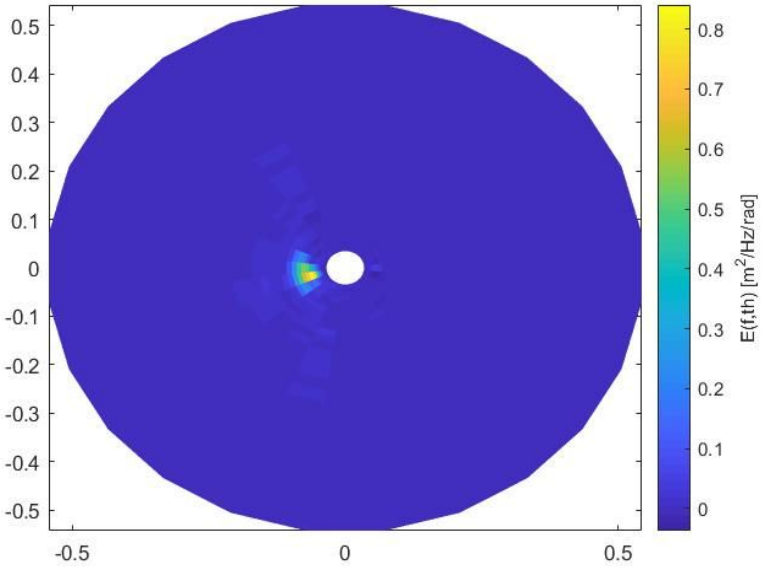

Fig 2.4 The recovered swell system by using the SFW method.

Just like as the above experience, the algorithm recovery successfully the original spectrum. Moreover the algorithm detects more than 60 different atom components. Several of them have very small values of coefficients.

❏ *Test on the system wind sea + swell:*

When the system becomes more complexity, i.e it includes several types of waves, the algorithm struggle to converge to the stopping criteria. For example, in this experience a directional wave spectrum includes a wind sea and swell. To find spectrum units following the gaussian distribution law combining the original spectrum, the system took more than 83 minutes to iterate 200 times, but it didn't reach the stopping criteria. The result shows that there are 172 spectrum components that constructs the original spectrum. Most of the components have almost the same values of coefficients, so it's hard to recognize which ones are dominant in building the original spectrum. Two figures below show the original spectrum and the recovered one by combining several spectrum units found by using the SFW method.

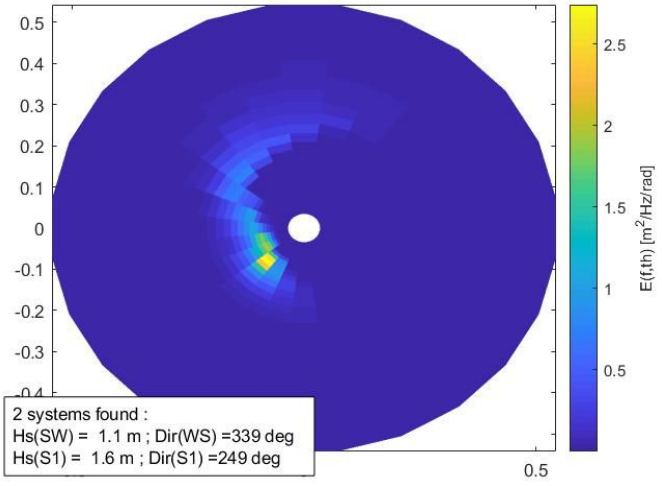

Fig 2.5 The original system (a wind sea + swell).

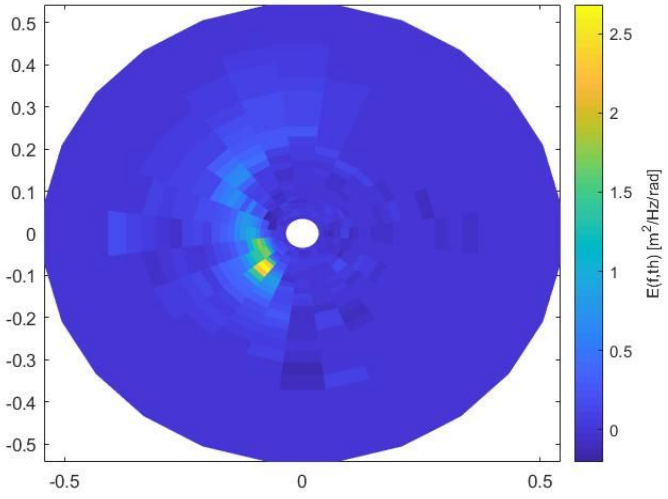

Fig 2.6 The recovered system by using the SFW method.

2.1 Watershed method:

Another approach of segmenting the directional wave spectrum is the watershed method. This is the classical method for segmenting images in general. The figures below show the results obtained by applying the Watershed method on the same directional wave spectra as above.

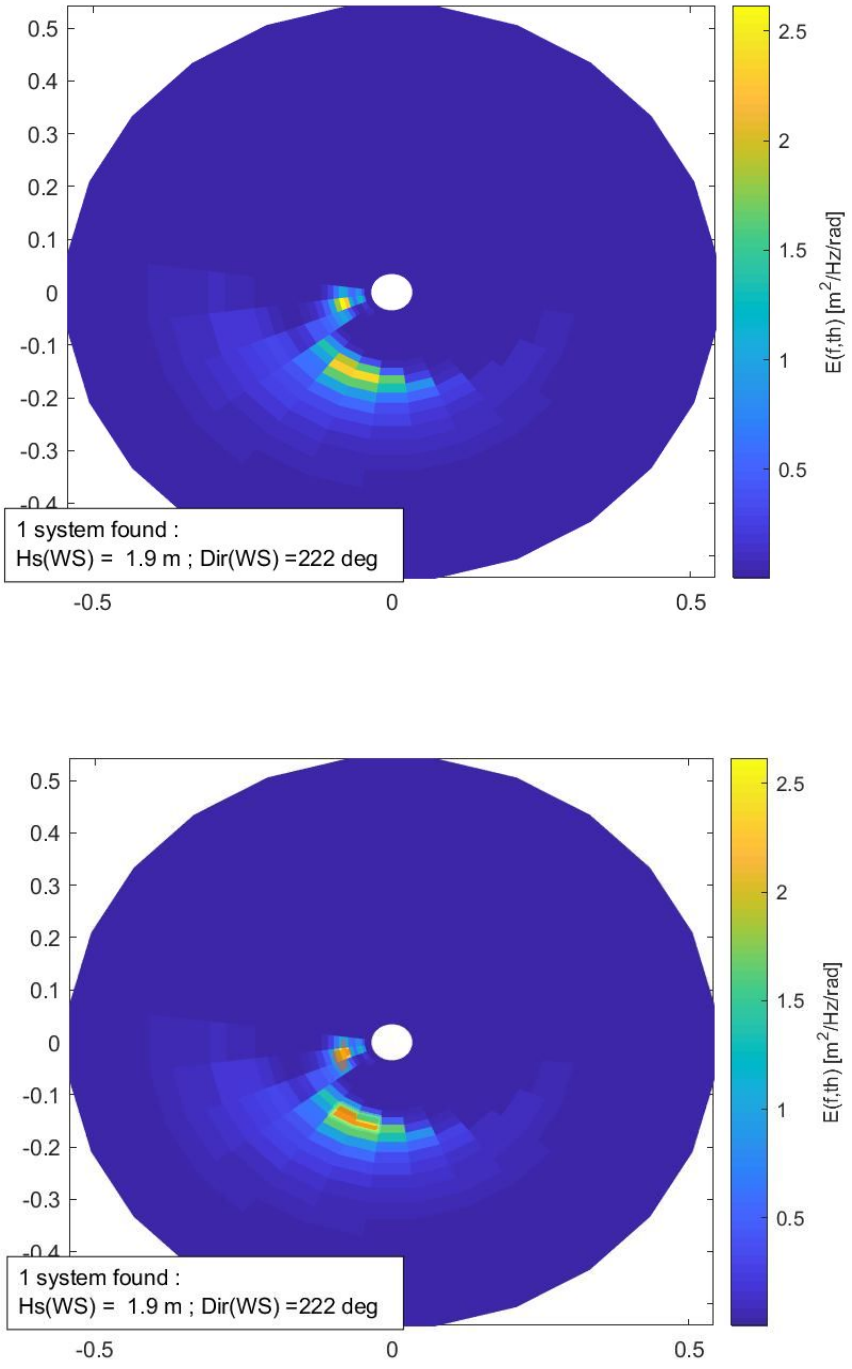

Fig 2.7 The ground truth swell system (above) and its segmented version (below) by using the Watershed method.

The result shows that there are two directional waves in this spectrum instead one win sea shown in the ground truth. This method takes little time to solve the system and to recognize how many directional waves existed in the system. But the drawback of this method, we can not know exactly which distribution law each directional wave follows. It might exist several components, which follow a specific probability distribution law (ex. gaussian), combining each of the segment of the segmented spectrum by using the Watershed method.

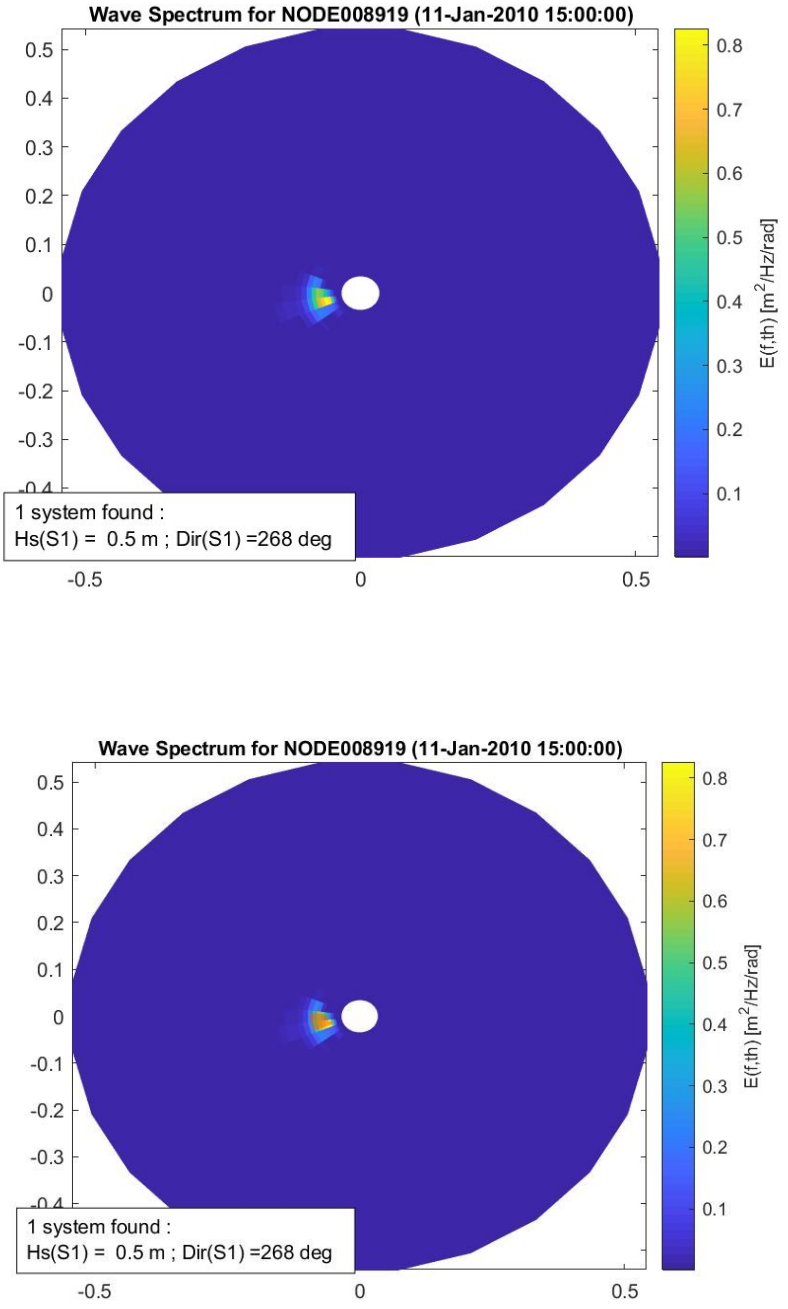

Fig 2.8 The ground truth swell system (above) and a directional wave found in the system (below).

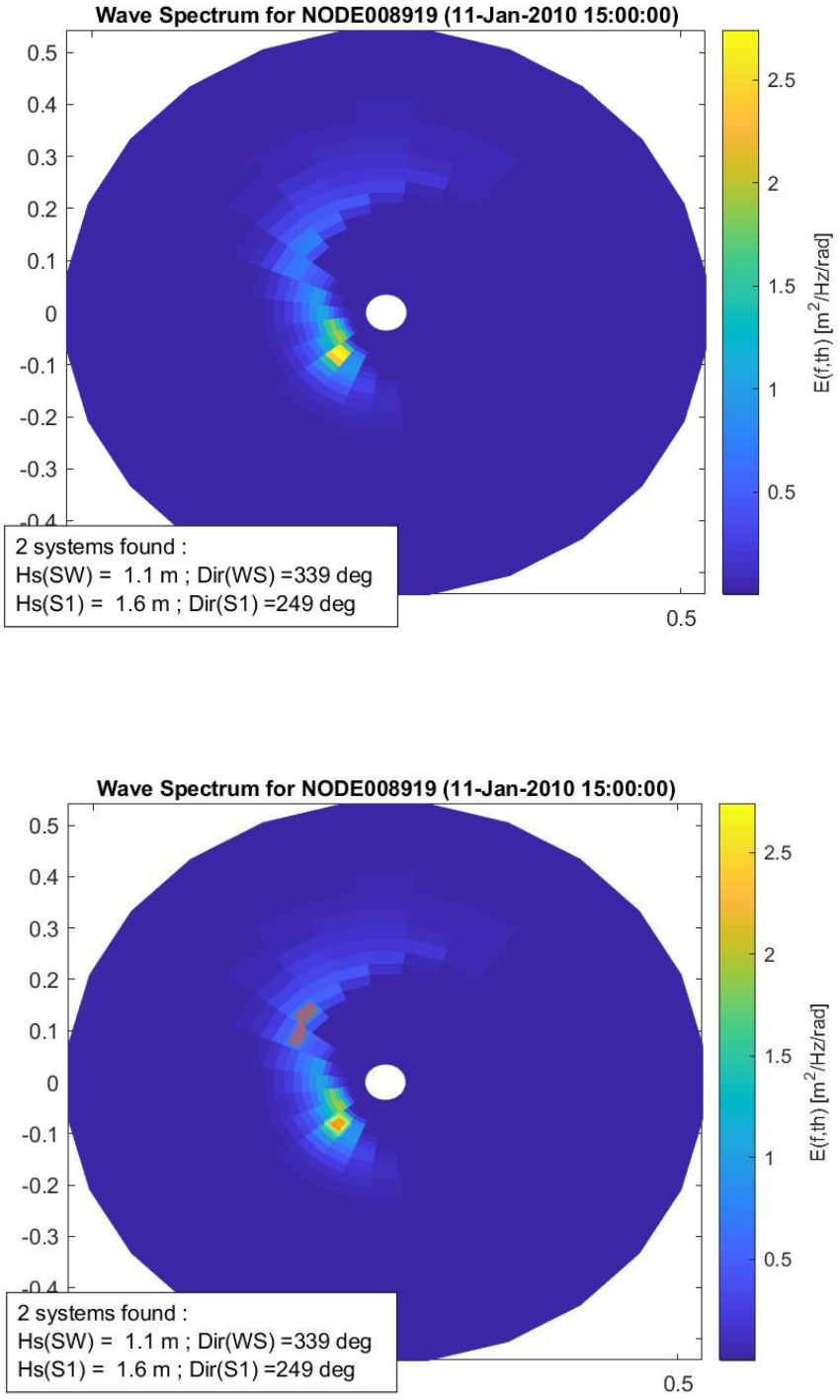

Fig 2.9 The system including a wind sea and swell (above) and its segmented version by using the Watershed method.

The two last results show the effectiveness of this method. The wave components are well localized in both spectra compared with the ground truths. The results also depend forcely on the method's parameters (such as structuring element's radius in a morphological operation) as the variety of directional wave spectrum forms.