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Grupa 4140.

Temă seminar 1 - BE2

① $\vec{r}_1 = x_1 \cdot \vec{u} + y_1 \cdot \vec{j} + z_1 \cdot \vec{k}$

$\bar{r}_2 = x_2 \cdot \bar{u} + y_2 \cdot \bar{j} + z_2 \cdot \bar{k}$

$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1$; $\bar{r}_{13} = \bar{r}_3 - \bar{r}_1$

$\bar{r}_{12} \times \bar{r}_{13} = ?$

$\bar{r}_{12} = (x_2 - x_1) \cdot \bar{u} + (y_2 - y_1) \cdot \bar{j} + (z_2 - z_1) \cdot \bar{k}$

$\bar{r}_{13} = (x_3 - x_1) \cdot \bar{u} + (y_3 - y_1) \bar{j} + (z_3 - z_1) \cdot \bar{k}$

*Bănuiesc că $r_3 = x_3 \cdot \bar{u} + y_3 \cdot \bar{j} + z_3 \cdot \bar{k}$ (nu apare în enunț)

$$\bar{r}_{12} \times \bar{r}_{13} = \begin{vmatrix} \bar{u} & \bar{j} & \bar{k} \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} =$$

$$= \bar{u}\left[(y_2 - y_1)(z_3 - z_1) - (z_2 - z_1)(y_3 - y_1)\right] +$$

$$+ \bar{j}\left[-(x_2 - x_1)(z_3 - z_1) + (x_3 - x_1)(z_2 - z_1\right] + \bar{k}\left[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y\right.$$

$A = \frac{1}{2} | \bar{r}_{12} \times \bar{r}_{13} |$

Să se deseneze vectorii $r_1, r_2, r_3, r_{12}, r_{13}$

și să se calc. aria triunghiului pentru cazurile:

a) $r_1 = 0\cdot\bar{i} + 0\cdot\bar{j} + 0\cdot\bar{k}$, $r_2 = 0\cdot\bar{i} + 4\cdot\bar{j} + 0\cdot\bar{k}$

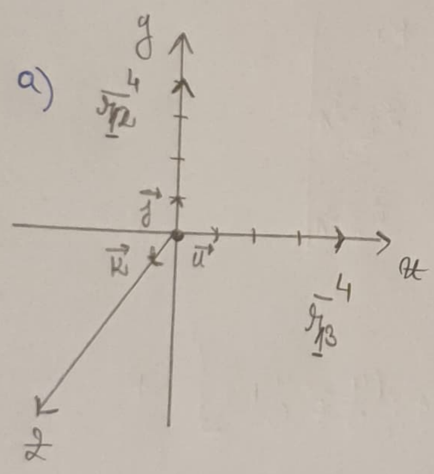
$r_3 = 4\bar{i} + 0\bar{j} + 0\cdot\bar{k}$

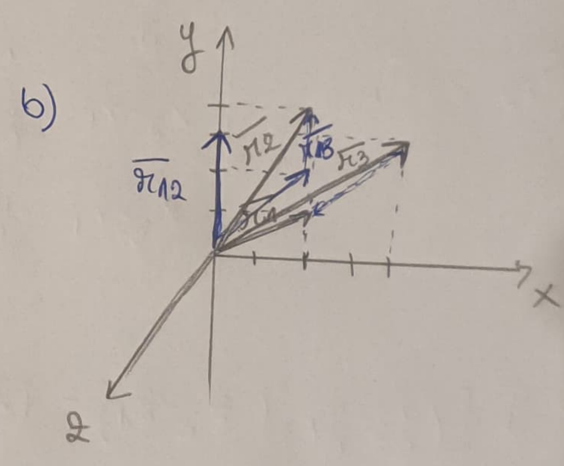
$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1 = 0\cdot\bar{i} + 4\cdot\bar{j} + 0\cdot\bar{k}$

$\bar{r}_{13} = \bar{r}_3 - \bar{r}_1 = 4\cdot\bar{i} + 0\cdot\bar{j} + 0\cdot\bar{k}$.

$$\bar{r}_{12} \times \bar{r}_{13} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{vmatrix} = -16\bar{k} + 0\cdot\bar{j} + 0\cdot\bar{i}$$

$|\bar{r}_{12} \times \bar{r}_{13}| = \sqrt{16^2} = 16 \Rightarrow A = \frac{1}{2}\cdot 16 = 8$





b) $\bar{r}_1 = 2\bar{i} + 1\bar{j} + 0\bar{k}$, $\bar{r}_2 = 2\bar{i} + 4\bar{j} + 0\bar{k}$, $\bar{r}_3 = 4\bar{i} + 3\bar{j} + 0\bar{k}$

$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1 = 0\cdot\bar{i} + 3\cdot\bar{j} + 0\cdot\bar{k}$, $\bar{r}_{13} = 2\cdot\bar{i} + 2\cdot\bar{j} + 0\cdot\bar{k}$

$$\bar{r}_{12} \times \bar{r}_{13} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 3 & 0 \\ 2 & 2 & 0 \end{vmatrix} = -6\bar{k} + 0\cdot\bar{j} + 0\cdot\bar{i}$$

$|\bar{r}_{12} \times \bar{r}_{13}| = \sqrt{(-6)^2} = 6$

$A = \frac{1}{2}\cdot 6 = 3$

c) $\bar{r}_1 = 2\bar{i} + 2\bar{j} + 2\bar{k}$, $\bar{r}_2 = 4\bar{i} + 6\bar{j} + 4\bar{k}$, $\bar{r}_3 = 4\bar{i} + 4\bar{j} + 6\bar{k}$

$\bar{r}_{12} = \bar{r}_2 - \bar{r}_1 = 2\bar{i} + 4\bar{j} + 2\bar{k}$

$\bar{r}_{13} = \bar{r}_3 - \bar{r}_1 = 2\bar{i} + 2\bar{j} + 4\bar{k}$

$$\bar{r}_{12} \times \bar{r}_{13} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix} = 16\bar{i} + 4\bar{k} + 4\bar{j} - 8\bar{k} - 4\bar{i} - 8\bar{j} = 12\bar{i} - 4\bar{j} - 4\bar{k}$$

$$|\bar{r}_{12} \times \bar{r}_{13}| = \sqrt{144 + 32} = \sqrt{176} = 2\sqrt{43}$$

$$\begin{array}{r|l} 176 & 2 \\ 86 & 2 \\ 43 & \end{array}$$

$$A = \frac{1}{2} \cdot 2\sqrt{43} = \sqrt{43}$$

(2) $A = A_x \cdot \bar{i} + A_y \cdot \bar{j} + A_z \cdot \bar{k}$; $B = B_x \cdot \bar{i} + B_y \cdot \bar{j} + B_z \cdot \bar{k}$

$C = C_x \cdot \bar{i} + C_y \cdot \bar{j} + C_z \cdot \bar{k}$

Volumul paralelipipedului generat = modul prod. mixt = ?

$$\boxed{V = \bar{A} \cdot (\bar{B} \times \bar{C})}$$

$A = 4\bar{i} + \bar{j} + 0\bar{k}$, $B = \bar{i} + 4\bar{j} + 0\bar{k}$, $C = \bar{i} + \bar{j} + 4\bar{k}$

$V = \bar{A} \cdot (\bar{B} \times \bar{C})$

$$\bar{B} \times \bar{C} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 4 & 0 \\ 1 & 1 & 4 \end{vmatrix} = 16\bar{i} + 1\bar{k} - 4\bar{k} - 4\bar{j} = 16\bar{i} - 4\bar{j} - 3\bar{k}$$

$$V = (4\bar{i} + \bar{j} + 0\bar{k}) \cdot (16\bar{i} - 4\bar{j} - 3\bar{k}) = 64 - 4 + 0 = 60 \text{ m}^3.$$

③ $\bar{W} = (\bar{A} \times \bar{B}) \cdot (\bar{C} \times \bar{D}) = ?$

(produs scalar de produse vectoriale)

$$W = (\bar{A} \times \bar{B}) \cdot (\bar{C} \times \bar{D}) = \bar{C} \cdot (\bar{D} \times (\bar{A} \times \bar{B})) =$$

$$= \bar{C} \cdot [\bar{A}(\bar{D} \cdot \bar{B}) - \bar{B}(\bar{D} \cdot \bar{A})]$$

④ $\bar{r} = x \cdot \bar{u} + y \cdot \bar{j} + z \cdot \bar{k}$, $r = \sqrt{x^2 + y^2 + z^2}$

Demonstrați sau calculați:

a) $\text{grad}(r^2) = ?$

$$\text{grad}(r^2) = \text{grad}(\underbrace{x^2 + y^2 + z^2}_{Q}) = \frac{\partial Q}{\partial x} \cdot \bar{u} + \frac{\partial Q}{\partial y} \cdot \bar{j} + \frac{\partial Q}{\partial z} \cdot \bar{k}$$

$$= 2x \cdot \bar{u} + 2y \cdot \bar{j} + 2z \cdot \bar{k}$$

b) $\text{div}(\bar{r}) = 3$

$$\text{div}(\bar{r}) = \nabla \cdot \bar{r} = r_x \cdot \bar{u} + \bar{r}_y \cdot \bar{j} + r_z \cdot \bar{k} =$$

$$= \frac{\partial}{\partial x} \cdot r_x + \frac{\partial}{\partial y} \cdot r_y + \frac{\partial}{\partial z} \cdot r_z = \frac{\partial}{\partial x} \cdot x + \frac{\partial}{\partial y} \cdot y + \frac{\partial}{\partial z} \cdot z =$$

$$= 1 + 1 + 1 = 3$$

c) $rot(\bar{r}) = 0$

$$rot(\bar{r}) = \nabla \cdot \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r_x & r_y & r_z \end{vmatrix} =$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0 \cdot \bar{i} + 0 \bar{k} + 0 \cdot \bar{j} - 0 \cdot \bar{k} - 0 \cdot \bar{i} - 0 \cdot \bar{j} =$$

$$= 0.$$

d) $div\left(\frac{\bar{r}}{r^3}\right) = ?$

$$div\left(\frac{1}{r^3} \cdot \bar{r}\right) = \nabla \cdot \left(\frac{1}{r^3} \cdot \bar{r}\right) + \nabla \cdot \left(\frac{1}{r^3} \cdot \bar{r}\right) =$$

$$= \bar{r}\, grad \frac{1}{r^3} + \frac{1}{r^3} \cdot \underbrace{div\, \bar{r}}_{\substack{=3 \\ \text{(calc. outerior)}}}$$

$$grad\left(\frac{1}{r^3}\right) = grad\Big[\underbrace{(x^2+y^2+z^2)^{-\frac{3}{2}}}_{Q}\Big] = \frac{\partial Q}{\partial x} \cdot \bar{i} + \frac{\partial Q}{\partial y} \cdot \bar{j} + \frac{\partial Q}{\partial z} \cdot \bar{k} =$$

$$= -\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot \cancel{2}x \cdot \bar{i} - \frac{3}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{5}{2}} \cdot \cancel{2}y \cdot \bar{j}$$

$$- \frac{3}{\cancel{2}} \cdot (x^2+y^2+z^2)^{-5/2} \cdot \cancel{2}z \cdot \bar{k} = -3 \left[\frac{1}{\sqrt{(x^2+y^2+z^2)^5}} \cdot x \cdot \bar{i} + \right.$$

$$\left. + \frac{1}{\sqrt{(x^2+y^2+z^2)^5}} \cdot y \cdot \bar{j} + \frac{1}{\sqrt{(x^2+y^2+z^2)^5}} \cdot z \cdot \bar{k} \right]$$

$$\operatorname{div}\left(\frac{1}{r^3}\cdot\bar{r}\right) = \bar{r}\ \operatorname{grad}\frac{1}{r^3} + \frac{1}{r^3}\operatorname{div}\bar{r} =$$

$$= 3\left[\left(\frac{-x}{\sqrt{(x^2+y^2+z^2)}}\,\bar{i} + \frac{-y}{\sqrt{(x^2+y^2+z^2)}}\,\bar{j} + \frac{-z}{\sqrt{(x^2+y^2+z^2)}}\cdot\bar{k}\right)\bar{r} + \frac{1}{r^3}\right].$$

Sau

$$\frac{\bar{r}}{r^3} = \underbrace{\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}}_{P}\cdot\bar{i} + \underbrace{\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}}_{Q}\,\bar{j} + \underbrace{\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}}_{R}\cdot\bar{k}$$

$$\operatorname{div}\Big(\underbrace{\frac{\bar{r}}{r^3}}_{\text{vector}}\Big) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\frac{\partial P}{\partial x} = \frac{(x^2+y^2+z^2)^{\frac{3}{2}} - x\cdot\frac{3}{2}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot 2x}{(x^2+y^2+z^2)^3} =$$

$$= \frac{(x^2+y^2+z^2)^{\frac{1}{2}}\left[(x^2+y^2+z^2) - 3x^2\right]}{(x^2+y^2+z^2)^3} =$$

$$= \frac{(x^2+y^2+z^2) - 3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$ calc. la fel și pt $\frac{\partial Q}{\partial y}$, $\frac{\partial R}{\partial z}$.

e) $rot\left(\frac{\bar{r}}{r^3}\right) = ?$

$$\frac{\bar{r}}{r^3} = \frac{1}{r^3}\cdot(x\cdot\bar{i} + y\cdot\bar{j} + z\cdot\bar{k}) = \underbrace{\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}}_{P}\cdot\bar{i} +$$

$$+ \underbrace{\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}}_{Q}\cdot\bar{j} + \underbrace{\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}}_{R}\cdot\bar{k}$$

$$rot\underbrace{\left(\frac{\bar{r}}{r^3}\right)}_{\text{vector}} = \nabla\cdot\frac{\bar{r}}{r^3} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{(x^2+y^2+z^2)^3}} & \frac{y}{\sqrt{\ }} & \frac{z}{\sqrt{\ }} \end{vmatrix} =$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\cdot\bar{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\bar{j} +$$

$$+ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\cdot\bar{k}$$

$$\frac{\partial R}{\partial y} = z\cdot\left(+\cancel{2}y\cdot\frac{-3}{\cancel{2}}(x^2+y^2+z^2)^{-\frac{5}{2}}\right) = -\frac{3yz}{\sqrt{(x^2+y^2+z^2)^5}}$$

$$\frac{\partial Q}{\partial z} = \frac{-3yz}{\sqrt{(x^2+y^2+z^2)^5}}$$

$$\frac{\partial P}{\partial z} = \frac{-3xz}{\sqrt{(x^2+y^2+z^2)^5}}$$

$$\begin{cases} \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = \frac{-3xz}{\sqrt{(x^2+y^2+z^2)^5}} \\ \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = \frac{-3xy}{(x^2+y^2+z^2)^5} \\ \frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z} \end{cases}$$

$$\Rightarrow rot\left(\frac{\bar{r}}{r^3}\right) = 0\cdot\bar{i} + 0\bar{j} + 0\cdot\bar{k}$$