Computational Fluid Dynamics Exercises

Fachgebiet Strömungsmechanik

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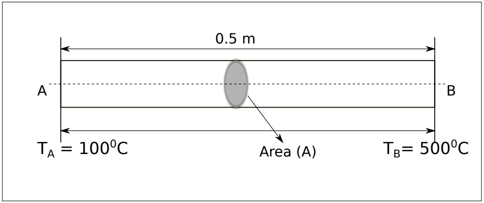
BERGISCHE UNIVERSITÄT WUPPERTAL

LSM

Problem: 1-D Diffusion

- Ends of an insulated rod are maintained at constant temperatures of $100^0\ C$ and $500^0\ C$. Data for the rod is as follows:
  - $k = 1000\ \mathrm{W/mK}$
  - Area, $A = 10 \times 10^{-3}\ \mathrm{m}^2$

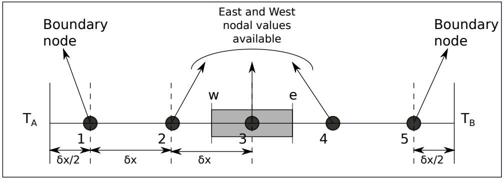


⇒ Calculate the temperature field using CFD and compare your results with the analytical solution

1. Identify the governing equation for the problem

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\underbrace{\cancel{S}}_{\text{no sources}}=0 \tag{1}$$

2. Discretize the domain



3. Discretized equation for central nodes, node 3
   - Integral equation

$$\int_{dV} \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) dV = \int_{dx} \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) Adx = 0 \qquad (2)$$

   - Discretized equation,

$$\left(kA\frac{\partial T}{\partial x}\right)_e - \left(kA\frac{\partial T}{\partial x}\right)_w \qquad (3)$$

   - Interpolation of individual terms

$$\left(kA\frac{\partial T}{\partial x}\right)_e = kA\left(\frac{T_4 - T_3}{\delta x}\right) \qquad (4)$$

$$\left(kA\frac{\partial T}{\partial x}\right)_w = kA\left(\frac{T_3 - T_2}{\delta x}\right) \qquad (5)$$

4. By substitution, (equation 4 and equation 5 in equation 3), the discretized equation for node 3,

$$T_4 - 2T_3 + T_2 = 0 \tag{6}$$

Similarly, discretized equation for node 2,

$$T_3 - 2T_2 + T_1 = 0 \tag{7}$$

Discretized equation for node 4,

$$T_5 - 2T_4 + T_3 = 0 \tag{8}$$

5. Treatment of boundary nodes slightly differ due to the presence of only one central node
   - Discretized equation for boundary node 1,

$$kA\left(\frac{T_2 - T_1}{\delta x} - \frac{T_1 - T_A}{\frac{\delta x}{2}}\right) \tag{9}$$

Node 1,

$$T_2 - 3T_1 + 2T_A = 0 \tag{10}$$

Node 5,

$$2T_B - 3T_5 + T_4 = 0 \tag{11}$$

6. Build the discretized matrix,

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} -2T_A \\ 0 \\ 0 \\ 0 \\ -2T_B \end{bmatrix}$$

7. Solving the system of equations by Gaussian Elimination,

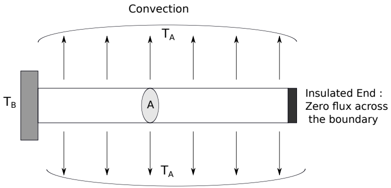
$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 140 \\ 220 \\ 300 \\ 380 \\ 460 \end{bmatrix}$$

Task 1 Find the analytical solution and compare the results with CFD solution

Task 2 How does the CFD results change when the domain is discretized into 4 nodes and 8 nodes (Use Scilab to solve the system of equations)

Problem: 1-D Diffusion with Sink

- A circular fin is cooled by convective heat transfer along its length. The fin has a temperature of 100^0 C at one end and is insulated at the other as shown in the figure.
- Data:
  - $\frac{hP}{kA} = 25\ \mathrm{m}^{-2}$; h ⇒ Convective transfer Coeff.; k ⇒ Thermal conductivity; P ⇒ Perimeter, A ⇒ Cross-sectional area
  - Length of the fin, L$= 1\ \mathrm{m}$



⇒ Calculate the temperature field using CFD and compare your results with the analytical solution

⇒ Analytical solution is given by:

$$\frac{T - T_A}{T_B - T_A} = \frac{cosh[n(L - x)]}{cosh(nL)} \qquad (12)$$

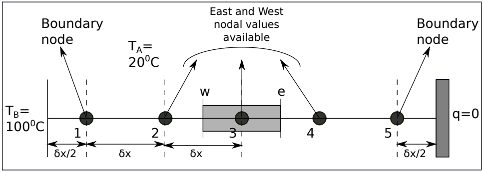
Solution: 1-D Diffusion with Sink

1. Identify the governing equation for the problem

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) - hP(T - T_A) = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) - n^2(T - T_A) = 0 \quad (13)$$

where, $n^2 = \frac{hP}{kA}$

2. Discretize the domain



3. Discretized equation for central nodes, node 3
   - Integral equation

$$\int_{dV} \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) dV - \int_{dV} n^2 (T - T_A) dV = 0 \tag{14}$$

$$\int_{dx} \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) A dx - \int_{dx} n^2 (T - T_A) A dx = 0 \tag{15}$$

   - Discretized equation,

$$\left[\left(\frac{\partial T}{\partial x}\right)_e - \left(\frac{\partial T}{\partial x}\right)_w\right] - n^2 (T_P - T_A)\delta x \tag{16}$$

4. Final discretized equation for node 2,

$$T_3 - 3T_2 + T_1 = -T_A \tag{17}$$

Similarly, discretized equation for node 3,

$$T_4 - 3T_3 + T_2 = -T_A \tag{18}$$

Discretized equation for node 4,

$$T_5 - 3T_4 + T_3 = -T_A \tag{19}$$

5. Boundary nodes
   - Discretized equation for boundary node 1,

$$\left(\frac{T_2 - T_1}{\delta x} - \frac{T_1 - T_B}{\frac{\delta x}{2}} \right) - n^2 (T_1 - T_A)\delta x = 0 \tag{20}$$

Node 1,

$$-4T_1 + T_2 = -2T_B - T_A \tag{21}$$

For node 5, flux in the east boundary is zero. Hence

$$-2T_5 + T_4 = -T_A \tag{22}$$

Solution: 1-D Diffusion with Sink

6. Build the discretized matrix,

$$\begin{bmatrix} -4 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} -220 \\ -20 \\ -20 \\ -20 \\ -20 \end{bmatrix}$$

7. Solving the system of equations by Gaussian Elimination,

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} 64.23 \\ 36.91 \\ 26.50 \\ 22.60 \\ 21.30 \end{bmatrix}$$

Task 3 How would the temperature field change if the thermal conductivity, k of the material was 4 times greater than the material in the previous example and a coarser discretization with only 3 nodes were used? Compare the results with the analytical solution.