AN INTRODUCTION TO LATTICE PATH ENUMERATION

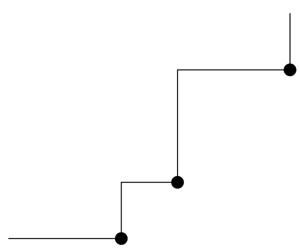
IRA M. GESSEL

ABSTRACT. This tutorial talk will describe some of the basic problems and methods of lattice path enumeration, including the reflection principle, the method of images, the cycle lemma, generating functions, and the kernel method.

1. INTRODUCTION

The simplest lattice path problem is the problem of counting paths in the plane, with unit east and north steps, from the origin to the point (m, n). (When not otherwise specified, our paths will have these steps.) The number of such paths is the binomial coefficient $\binom{m+n}{m}$. We can find more interesting problems by counting these paths according to certain parameters:

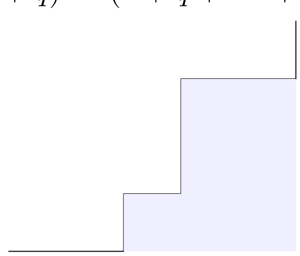
(i) The number of such paths with k left turns (an east step followed by a north step) is $\binom{m}{k}\binom{n}{k}$, as shown by MacMahon:



(ii) The number of such paths in which the area between the path and the x-axis is k is the coefficient of q^k in the *q-binomial coefficient*

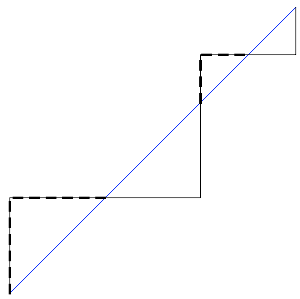
$$\begin{bmatrix} m+n \\ m \end{bmatrix} = \frac{(m+n)!_q}{m!_q\, n!_q},$$

where $m!_q$ is the q-factorial $1 \cdot (1+q) \cdots (1+q+\cdots+q^{n-1})$:

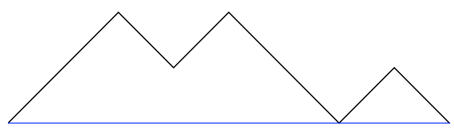


Date: April 20, 2007.

(iii) If $m = n$ then the number of such paths with $2k$ steps above the line $x = y$ is independent of k, for $k = 0, \ldots, n$ and is therefore equal to the *Catalan number* $\frac{1}{n+1}\binom{2n}{n}$. (This is the *Chung-Feller Theorem* [2])



We can also restrict our paths to a subregion of the plane. The simplest, and most fundamental, result in this area is the solution of the *ballot problem*: the number of paths from $(1, 0)$ to (m, n), where $m > n$, that never touch the line $x = y$, is the *ballot number* $\frac{m-n}{m+n}\binom{m+n}{m}$. In the special case $m = n + 1$, this ballot number is the *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. The corresponding paths are often redrawn as paths with northeast and southeast steps that never go below the x-axis; these are called *Dyck paths*:



More generally, we may also consider higher-dimensional paths or paths with other steps.

2. Methods

There are many methods used for counting lattice paths, each applying to a different class of problems. Perhaps the best known is the *reflection principle*, usually attributed to D. André in 1887 [1] in his solution of the ballot problem. The ballot number is obtained in the form

$$\binom{m+n-1}{m-1} - \binom{m+n-1}{m}.$$

The term $\binom{m+n-1}{m-1}$ counts all paths from $(1, 0)$ to (m, n) and the term $\binom{m+n-1}{m}$ is the number of paths from $(0, 1)$ to (m, n). These paths are put in bijection with paths from $(1, 0)$ to (m, n) that touch the line $x = y$ somewhere by reflecting in the line $x = y$ the part of the path from its starting point to its first meeting with this line.

The reflection principle may be used to solve the higher dimensional ballot problem, giving the solution in the form of a determinant. The paths that are counted are equivalent to standard Young tableaux, which are arrangements of the numbers 1, 2, …, n in a Ferrers diagram so that they are increasing in every row and column:

$$\begin{array}{cccc} 1 & 3 & 4 & 7 \\ 2 & 5 & 9 & \\ 6 & 8 & & \end{array}$$

A modification of the reflection principle can be used to count ballot paths according to the number of left turns.

Closely related to the reflection principle is the "method of images" [4] in which we solve recurrence relations with boundary conditions by taking appropriate linear combinations of basic solutions to the recurrence. A typical application shows that the number of paths from $(1,0)$ to (m,n), where $m > rn$, that never touch the line $x = ry$, is equal to

$$\binom{m+n-1}{m-1} - r\binom{m+n-1}{m} = \frac{m-rn}{m+n}\binom{m+n}{m}.$$

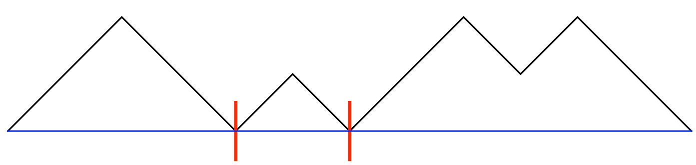
(The case $r = 1$ is the ballot problem.) Both binomial coefficients on the left-hand side satisfy the right recurrence, and the expression on the right shows that the difference takes the correct value of 0 on the line $x = ry$.

Another important method is the "cycle lemma" of Dvoretzky and Motzkin [3]. It may be stated in the following way: For any n-tuple $(a_1, a_2, \ldots, a_n)$ of integers from the set $\{1, 0, -1, -2, \ldots\}$ with sum $k > 0$, there are exactly k values of i for which the cyclic permutation

$$(a_i, \ldots, a_n, a_1, \ldots, a_{i-1})$$

has every partial sum positive. The special case in which each a_i is either 1 or -1 gives the solution to the ballot problem. The Chung-Feller theorem, and some of its generalizations, can be proved by a variation of the cycle lemma.

Generating functions are very useful in lattice path enumeration. They can be applied in many different ways, but the simplest is the derivation of functional equations from combinatorial decompositions. The simplest example is that of Dyck paths. Every Dyck path can be decomposed into "prime" Dyck paths by cutting it at each return to the x-axis:



Moreover, a prime Dyck path consists of an up-step, followed by an arbitrary Dyck path, followed by a down step. It follows that if $c(x)$ is the generating function for Dyck paths (i.e., the coefficient of x^n in $c(x)$ is the number of Dyck paths with $2n$ steps) then $c(x)$ satisfies the equation $c(x) = 1/\big(1 - xc(x)\big)$ which can be solved to give the generating function for the Catalan numbers, $c(x) = \sum_{n=0}^{\infty} C_n x^n = \big(1 - \sqrt{1-4x}\big)/(2x)$.

Many other lattice path results can be proved by similar decompositions. For example, the Chung-Feller theorem can be proved in a similar way by cutting a path at each return to the line $x = y$. This decomposition tells us that the number of paths from $(0, 0)$ to $(2m + 2n, 2m + 2n)$ with $2m$ steps above this line and $2n$ below it is the coefficient of $x^m y^n$ in

$$\frac{1}{1 - xc(x) - yc(y)} = \frac{xc(x) - yc(y)}{x - y} = \sum_{m,n=0}^{\infty} C_{m+n} x^m y^n.$$

More generally, the paths counted by the cycle lemma can also be counted with generating functions. In this case, we obtain a functional equation that generalizes the functional equation $c(x) = 1+xc(x)^2$ for the Catalan numbers. The functional equation can be solved by using the Lagrange inversion formula. Conversely, counting the paths directly by the cycle lemma gives a combinatorial proof of Lagrange inversion, as shown by Raney [6].

Another useful, but very different, application of generating functions is the *kernel method* [5], in which a recurrence with boundary conditions is converted into a functional equation. For example, to solve the ballot problem by this method, we define the ballot numbers $B(m, n)$ by the initial condition $B(1, 0) = 1$, the recurrence $B(m, n) = B(m - 1, n) + B(m, n - 1)$ for $m > n \geq 0$, and the boundary conditions $B(m, n) = 0$ for $m \leq n$ or $n < 0$. Then these defining relations imply that the generating function $\beta(x, y) = \sum_{m,n} B(m, n)x^m y^n$ satisfies

$$(1 - x - y)\beta(x, y) = x - xyc(xy),$$

where the Catalan number generating function $c(x)$ is defined to be $\sum_{n=0}^{\infty} B(n + 1, n)x^n$. To solve this functional equation, we set $y = 1 - x$ which simplifies it to

$$1 - (1 - x)c\big(x(1 - x)\big) = 0,$$

which is easily solved to give the Catalan generating function $c(x) = \big(1 - \sqrt{1 - 4x}\big)/(2x)$, and thus

$$\beta(x, y) = x\frac{1 - yc(xy)}{1 - x - y}.$$

References

[1] D. André, *Solution directed du problème résolu par M. Bertrand*, Comptes Rendus de l'Académie des Sciences, Paris **105** (1887), 436–437.

[2] K. L. Chung and W. Feller, *On fluctuations in coin tossing*, Proc. Nat. Acad. Sci. U.S.A. **35** (1945), 605–608.

[3] A. Dvoretzky and Th. Motzkin, *A problem of arrangements*, Duke Math. J. **14** (1947), 305–313.

[4] J. Peacock, *On "Al Capone and the Death Ray"*, Math. Gazette **26** (1942), 218–219.

[5] H. Prodinger, *The kernel method: a collection of examples*, Sém. Lothar. Combin. **50** (2003/04), Art. B50f, 19 pp.

[6] G. N. Raney, *Functional composition patterns and power series reversion* Trans. Amer. Math. Soc. **94** (1960), 441–451. 05.00 (40.00)

Department of Mathematics, Brandeis University, Waltham, MA 02454-9110
E-mail address: gessel@brandeis.edu