Computational Methods

COUNTING TECHNIQUES

QUOTE


"YOU CANNOT HAVE A POSITIVE LIFE AND A NEGATIVE MIND."
- JOYCE MEYER
THE ROAD TRIP EXPERT

Counting

Combinatorics is the branch of mathematics that deals with counting. Counting questions are important whenever we have finite resources (How much storage does a particular database consume? How many users can a given computer configuration support?) or whenever we are interested in efficiency (How many computations does a particular algorithm involve?).

MULTIPLICATION PRINCIPLE

If there are n_1 possible outcomes for a first event and n_2 possible outcomes for a second event, there are $n_1 \times n_2$ possible outcomes for the sequence of the two events.

Counting

The multiplication principle is useful whenever we want to count the total number of possible outcomes for a task that can be broken down into a sequence of successive subtasks.

The last part of your telephone number contains four digits. How many such four-digit numbers are there?

We can construct four-digit numbers by performing a sequence of subtasks: choose the first digit, then the second, the third, and finally the fourth. The first digit can be any one of the 10 digits from 0 to 9, so there are 10 possible outcomes for the first subtask. Likewise, there are 10 different possibilities each for the second digit, the third, and the fourth. Using the multiplication principle, we multiply the number of outcomes for each subtask in the sequence. Therefore there are $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$ different numbers.

Counting

If an element cannot be used again—that is, if repetitions are not allowed—the number of possible outcomes for successive events will be affected.

Referring to Example how many four-digit numbers are there if the same digit cannot be used twice?

Again we have the sequence of subtasks of selecting the four digits, but no repetitions are allowed. There are 10 choices for the first digit, but only 9 choices for the second because we can't use what we used for the first digit, and so on. There are $10 \cdot 9 \cdot 8 \cdot 7 = 5040$ different numbers.

Counting

a. How many ways are there to choose three officers from a club of 25 people?

b. How many ways are there to choose three officers from a club of 25 people if someone can hold more than one office?

In (a), there are three successive subtasks with no repetitions. The first subtask, choosing the first officer, has 25 possible outcomes. The second subtask has 24 outcomes, the third 23 outcomes. The total number of outcomes is $25 \cdot 24 \cdot 23 = 13{,}800$. In (b), the same three subtasks are done in succession, but repetitions are allowed. The total number of outcomes is $25 \cdot 25 \cdot 25 = 15{,}625$.

Counting

Suppose we want to select a dessert from three pies and four cakes. In how many ways can this be done? There are two events, one with three outcomes (choosing a pie) and one with four outcomes (choosing a cake). However, we are not doing a sequence of two events here, since we are getting only one dessert, which must be chosen from the two disjoint sets of possibilities. The number of different outcomes is the total number of choices we have, $3 + 4 = 7$. This illustrates the *addition principle*.

ADDITION PRINCIPLE

If A and B are disjoint events with n_1 and n_2 possible outcomes, respectively, then the total number of possible outcomes for event "A or B" is $n_1 + n_2$.

Counting

A customer wants to purchase a vehicle from a dealer. The dealer has 23 autos and 14 trucks in stock. How many selections does the customer have?

The customer wants to choose a car or truck. These are disjoint events; choosing an auto has 23 outcomes and choosing a truck has 14. By the addition principle, choosing a vehicle has $23 + 14 = 37$ outcomes.

Counting

Using the Principles Together

Frequently the addition principle is used in conjunction with the multiplication principle.

How many four-digit numbers begin with a 4 or a 5?

We can consider the two disjoint cases—numbers that begin with 4 and numbers that begin with 5. Counting the numbers that begin with 4, there is 1 outcome for the subtask of choosing the first digit, then 10 possible outcomes for the subtasks of choosing each of the other three digits. Hence, by the multiplication principle there are $1 \cdot 10 \cdot 10 \cdot 10 = 1000$ ways to get a four-digit number beginning with 4. The same reasoning shows that there are 1000 ways to get a four-digit number beginning with 5. By the addition principle, there are $1000 + 1000 = 2000$ total possible outcomes.

Counting

How many three-digit integers (numbers between 100 and 999 inclusive) are even?

One solution notes that an even number ends in 0, 2, 4, 6, or 8. Taking these as separate cases, the number of three-digit integers ending in 0 can be found by choosing the three digits in turn. There are 9 choices, 1 through 9, for the first digit; 10 choices, 0 through 9, for the second digit; and 1 choice for the third digit, 0. By the multiplication principle, there are 90 numbers ending in 0. Similarly, there are 90 numbers ending in 2, 4, 6, and 8, so by the addition principle, there are $90 + 90 + 90 + 90 + 90 = 450$ numbers.

Another solution takes advantage of the fact that there are only 5 choices for the third digit. By the multiplication principle, there are $9 \cdot 10 \cdot 5 = 450$ numbers.

Counting

Decision Trees

Tony is pitching pennies. Each toss results in heads (H) or tails (T). How many ways can he toss the coin 5 times without having 2 heads in a row?

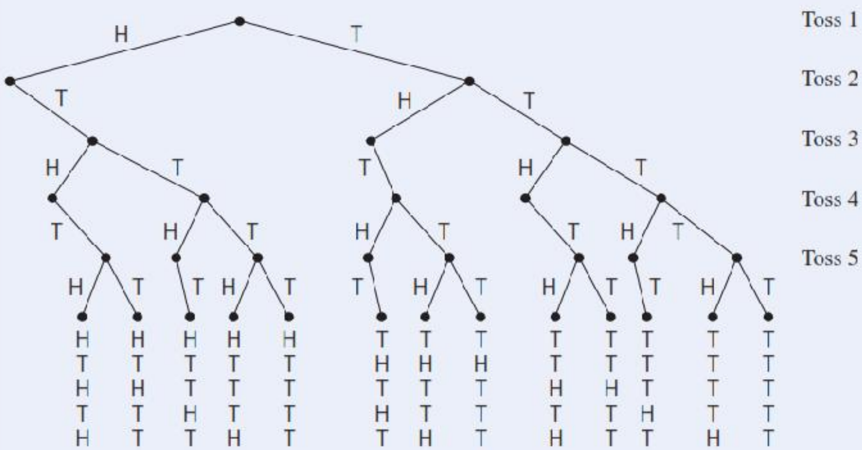


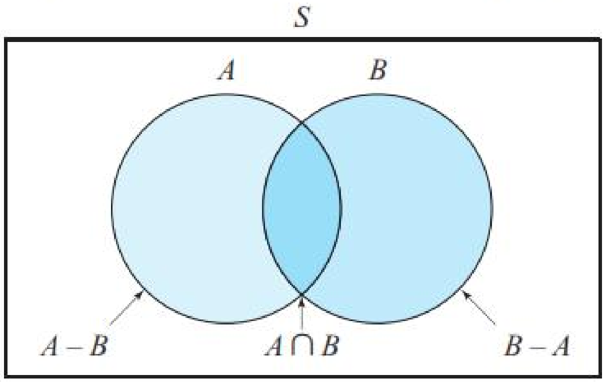
Figure shows the decision tree for this problem. Each coin toss has 2 outcomes; the left branch is labeled H for heads, the right branch is labeled T for tails. Whenever an H appears on a branch, the next level can only contain a right (T) branch. There are 13 possible outcomes, shown at the bottom of the tree.

Counting

Principle of Inclusion and Exclusion

To develop the principle of inclusion and exclusion, we first note that if A and B are any subsets of a universal set S, then $A - B$, $B - A$, and $A \cap B$ are mutually disjoint sets.

For example, if $x \in A - B$, then $x \notin B$, therefore $x \notin B - A$ and $x \notin A \cap B$.



Counting

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

PRINCIPLE OF INCLUSION AND EXCLUSION

Given the finite sets $A_1, \cdots, A_n, n \geq 2$, then

$$|A_1 \cup \cdots \cup A_n| = \sum_{1 \leq i \leq n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|$$

Counting

A pollster queries 35 voters, all of whom support referendum 1, referendum 2, or both, and finds that 14 voters support referendum 1 and 26 support referendum 2. How many voters support both?

If we let A be the set of voters supporting referendum 1 and B be the set of voters supporting referendum 2, then we know that

$$|A \cup B| = 35 \qquad |A| = 14 \qquad |B| = 26$$

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ 35 &= 14 + 26 - |A \cap B| \\ |A \cap B| &= 14 + 26 - 35 = 5 \end{aligned}$$

so 5 voters support both.

Counting

A group of students plans to order pizza. If 13 will eat sausage topping, 10 will eat pepperoni, 12 will eat extra cheese, 4 will eat both sausage and pepperoni, 5 will eat both pepperoni and extra cheese, 7 will eat both sausage and extra cheese, and 3 will eat all three toppings, how many students are in the group?

Let

$$A = \{\text{students who will eat sausage}\}$$

$$B = \{\text{students who will eat pepperoni}\}$$

$$C = \{\text{students who will eat extra cheese}\}$$

Then $|A| = 13$, $|B| = 10$, $|C| = 12$, $|A \cap B| = 4$, $|B \cap C| = 5$, $|A \cap C| = 7$, and $|A \cap B \cap C| = 3$.

$$|A \cup B \cup C| = 13 + 10 + 12 - 4 - 5 - 7 + 3 = 22$$

Counting

A produce stand sells only broccoli, carrots, and okra. One day the stand served 207 people. If 114 people purchased broccoli, 152 purchased carrots, 25 purchased okra, 64 purchased broccoli and carrots, 12 purchased carrots and okra, and 9 purchased all three, how many people purchased broccoli and okra?

Let

$$\begin{aligned} A &= \{\text{people who purchased broccoli}\} \\ B &= \{\text{people who purchased carrots}\} \\ C &= \{\text{people who purchased okra}\} \end{aligned}$$

Then $|A \cup B \cup C| = 207$, $|A| = 114$, $|B| = 152$, $|C| = 25$, $|A \cap B| = 64$, $|B \cap C| = 12$, and $|A \cap B \cap C| = 9$.

$$\begin{aligned} |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \\ 207 &= 114 + 152 + 25 - 64 - |A \cap C| - 12 + 9 \\ |A \cap C| &= 114 + 152 + 25 - 64 - 12 + 9 - 207 = 17 \end{aligned}$$

Counting

The **pigeonhole principle** acquired its quaint name from the following idea: If more than k pigeons fly into k pigeonholes, then at least 1 hole will end up with more than 1 pigeon.

PIGEONHOLE PRINCIPLE
If more than k items are placed into k bins, then at least 1 bin contains more than 1 item.

How many people must be in a room to guarantee that 2 people have last names that begin with the same initial?

There are 26 letters of the alphabet (bins). If there are 27 people, then there are 27 initials (items) to put into the 26 bins, so at least 1 bin will contain more than 1 last initial.

Counting

Permutations

Example discussed the problem of counting all possibilities for the last four digits of a telephone number with no repeated digits. In this problem, the number 1259 is not the same as the number 2951 because the order of the four digits is important. An ordered arrangement of objects is called a **permutation**. Each of these numbers is a permutation of 4 distinct objects chosen from a set of 10 distinct objects (the digits). How many such permutations are there? The answer, found by using the multiplication principle, is $10 \cdot 9 \cdot 8 \cdot 7$—there are 10 choices for the first digit, then 9 for the next digit because repetitions are not allowed, 8 for the next digit, and 7 for the fourth digit. The number of permutations of r distinct objects chosen from n distinct objects is denoted by $P(n, r)$. Therefore the solution to the problem of the four-digit number without repeated digits can be expressed as $P(10, 4)$. In general, $P(n, r)$ is given by the formula

$$P(n, r) = \frac{n!}{(n-r)!} \text{ for } 0 \leq r \leq n$$

Counting

The number of permutations of 3 objects, say a, b, and c, is given by $P(3, 3) = 3! = 3 \cdot 2 \cdot 1 = 6$. The 6 permutations of a, b, and c are

$$abc, acb, bac, bca, cab, cba$$

How many three-letter words (not necessarily meaningful) can be formed from the word "compiler" if no letters can be repeated? Here the arrangement of letters matters, and we want to know the number of permutations of 3 distinct objects taken from 8 objects. The answer is $P(8, 3) = 8!/5! = 336$.

Counting

A library has 4 books on operating systems, 7 on programming, and 3 on data structures. Let's see how many ways these books can be arranged on a shelf, given that all books on the same subject must be together.

We can think of this problem as a sequence of subtasks. First we consider the subtask of arranging the 3 subjects. There are 3! outcomes to this subtask, that is, 3! different orderings of subject matter. The next subtasks are arranging the books on operating systems (4! outcomes), then arranging the books on programming (7! outcomes), and finally arranging the books on data structures (3! outcomes). Thus, by the multiplication principle, the final number of arrangements of all the books is $(3!)(4!)(7!)(3!) = 4{,}354{,}560$.

Counting

Combinations

Sometimes we want to select r objects from a set of n objects, but we don't care how they are arranged. Then we are counting the number of **combinations** of r distinct objects chosen from n distinct objects, denoted by $C(n, r)$. For each such combination, there are $r!$ ways to permute the r chosen objects. By the multiplication principle, the number of permutations of r distinct objects chosen from n objects is the product of the number of ways to choose the objects, $C(n, r)$, multiplied by the number of ways to arrange the objects chosen, $r!$ Thus,

$$C(n, r) \cdot r! = P(n, r)$$

or

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n - r)!} \text{ for } 0 \leq r \leq n$$

Counting

A committee of 8 students is to be formed from a class consisting of 19 freshmen and 34 sophomores.

a. How many committees of 3 freshmen and 5 sophomores are possible?
b. How many committees with exactly 1 freshman are possible?
c. How many committees with at most 1 freshman are possible?
d. How many committees with at least 1 freshman are possible?

Because the ordering of the individuals chosen is not important, these are combinations problems.

For part (a), we have a sequence of two subtasks, selecting freshmen and selecting sophomores. The multiplication principle should be used. (Thinking of a sequence of subtasks may seem to imply ordering, but it just sets up the levels of the decision tree, the basis for the multiplication principle. There is no ordering of the students.) Because there are $C(19, 3)$ ways to choose the freshmen and $C(34, 5)$ ways to choose the sophomores, the answer is

$$C(19, 3) \cdot C(34, 5) = \frac{19!}{3!16!} \cdot \frac{34!}{5!29!} = (969)(278{,}256)$$

Counting

For part (b), we again have a sequence of subtasks: selecting the single freshman and then selecting the rest of the committee from among the sophomores. There are $C(19, 1)$ ways to select the single freshman and $C(34, 7)$ ways to select the remaining 7 members from the sophomores. By the multiplication principle, the answer is

$$C(19, 1) \cdot C(34, 7) = \frac{19!}{1!(19 - 1)!} \cdot \frac{34!}{7!(34 - 7)!} = 19(5{,}379{,}616)$$

For part (c), we get at most 1 freshman by having exactly 1 freshman or by having 0 freshmen. Because these are disjoint events, we use the addition principle. The number of ways to select exactly 1 freshman is the answer to part (b). The number of ways to select 0 freshmen is the same as the number of ways to select the entire 8-member committee from among the 34 sophomores, $C(34, 8)$. Thus the answer is

$$C(19, 1) \cdot C(34, 7) + C(34, 8) = \text{some big number}$$

Counting

We can attack part (d) in several ways. One way is to use the addition principle, thinking of the disjoint possibilities as exactly 1 freshman, exactly 2 freshmen, and so on, up to exactly 8 freshmen. We could compute each of these numbers and then add them. However, it is easier to do the problem by counting all the ways the committee of 8 can be selected from the total pool of 53 people and then eliminating (subtracting) the number of committees with 0 freshmen (all sophomores). Thus the answer is

$$C(53, 8) - C(34, 8)$$

Counting

In general, suppose there are n objects of which a set of n_1 are indistinguishable from each other, another set of n_2 are indistinguishable from each other, and so on, down to n_k objects that are indistinguishable from each other. The number of distinct permutations of the n objects is

$$\frac{n!}{(n_1!)(n_2!)\cdots(n_k!)}$$

a. How many distinct permutations can be made from the characters in the word FLORIDA?
b. How many distinct permutations can be made from the characters in the word MISSISSIPPI?

Counting

Part (a) is a simple problem of the number of ordered arrangements of seven distinct objects, which is 7!. However, the answer to part (b) is not 11! because the 11 characters in MISSISSIPPI are not all distinct. This means that 11! counts some of the same arrangements more than once (the same arrangement meaning that we cannot tell the difference between $MIS_1S_2ISSIPPI$ and $MIS_2S_1ISSIPPI$.)

Consider any one arrangement of the characters. The four *S*'s occupy certain positions in the string. Rearranging the *S*'s within those positions would result in no distinguishable change, so our one arrangement has 4! look-alikes. In order to avoid overcounting, we must divide 11! by 4! to take care of all the ways of moving the *S*'s around. Similarly, we must divide by 4! to take care of the four *I*'s and by 2! to take care of the two *P*'s. The number of distinct permutations is thus

$$\frac{11!}{4!4!2!}$$

Exercises

A frozen yogurt shop allows you to choose one flavor (vanilla, strawberry, lemon, cherry, or peach), one topping (chocolate shavings, crushed toffee, or crushed peanut brittle), and one condiment (whipped cream or shredded coconut). How many different desserts are possible?

A video game is begun by making selections from each of 3 menus. The first menu (number of players) has 4 selections, the second menu (level of play) has 8, and the third menu (speed) has 6. In how many configurations can the game be played?

A user's password to access a computer system consists of 3 letters followed by 2 digits. How many different passwords are possible?

Exercises

How many batting orders are possible for a 9-man baseball team?

How many different ways can 10 flavors of ice cream be arranged in an ice cream store display case?

How many permutations of the characters in COMPUTER are there? How many of the permutations end in a vowel?

In how many ways can first, second, and third prize in a pie-baking contest be given to 15 contestants?

In how many different ways can 19 people be seated in a row?

Quality control wants to test 25 microprocessor chips from the 300 manufactured each day. How many different batches of test chips are possible?

How many juries of 5 men and 7 women can be formed from a panel of 17 men and 23 women?