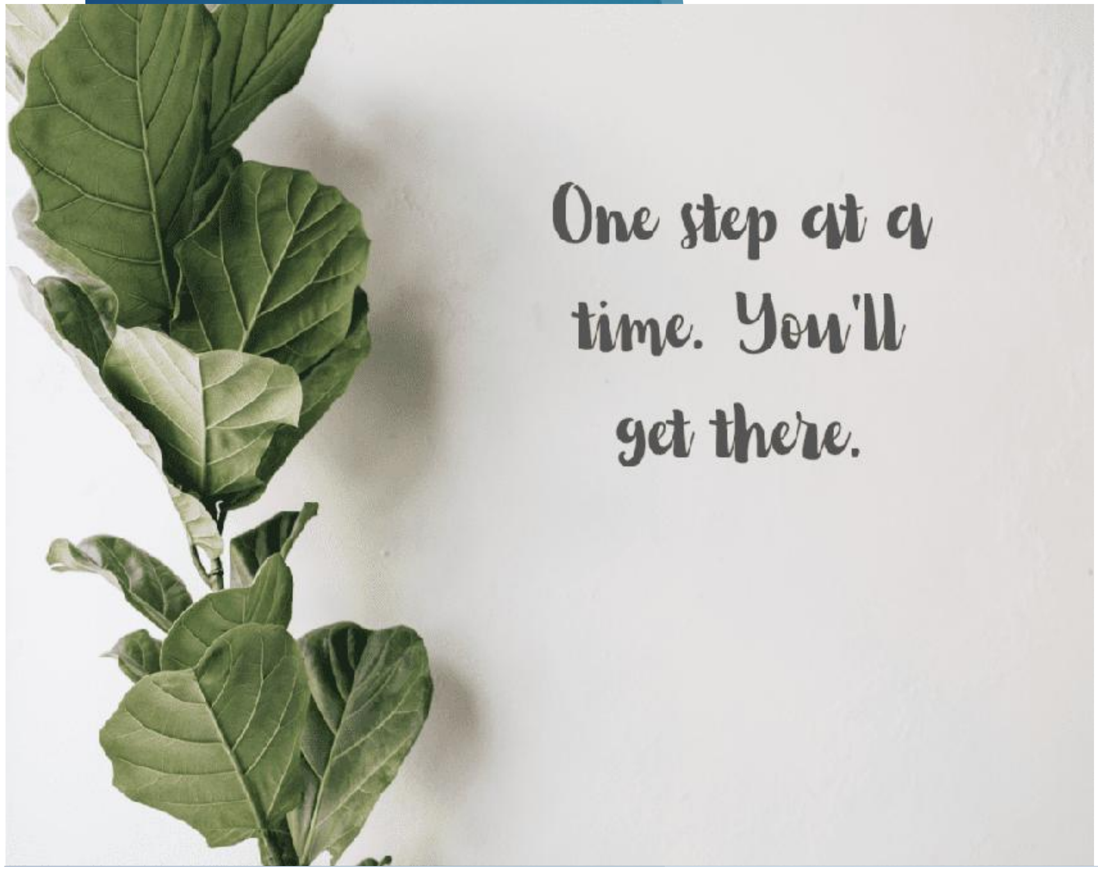
Computational Methods

RECURRENCE RELATIONS

QUOTE



Recurrence Relations

A sequence of numbers $(a_n)_{n\geq 0}$ is said to satisfy a **(homogeneous) linear recurrence relation of order** d if there are scalars $c_1, \ldots, c_d$ such that $c_d \neq 0$, and for all $n \geq d$, we have

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}.$$

Example The Fibonacci numbers f_n are given by the sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$. This isn't really telling you what the general f_n is, so instead let me say that for all $n \geq 2$, we have

$$f_n = f_{n-1} + f_{n-2}.$$

Together with the initial conditions $f_0 = 0, f_1 = 1$, this is enough information to calculate any f_n. So (by definition), the Fibonacci numbers satisfy a linear recurrence relation of order 2.

Recurrence Relations

A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more previous terms of the sequence $a_0, a_1, \ldots, a_{n-1}$. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

The recurrence relation together with the initial conditions uniquely determines a sequence, i.e., a solution.

Recurrence Relations

e.g. Consider the recurrence relation (Fibonacci numbers):

$a_n = a_{n-1} + a_{n-2}$, $n \geq 2$
where $a_0 = 0$ and $a_1 = 1$.

Then $\{0,1,1,2,3,5,8,\ldots\}$ is the solution.

Question: Is it possible to find an explicit formula for a_n?
In some conditions yes!

Recurrence Relations

Definition Let $(a_n)_{n\geq 0}$ be a sequence of numbers. The *generating function* associated to this sequence is the series

$$A(x) = \sum_{n\geq 0} a_n x^n.$$

Example Let p be a positive integer. The generating function associated to the sequence $a_n = \binom{k}{n}$ for $n \leq k$ and $a_n = 0$ for $n > k$ is actually a polynomial:

$$A(x) = \sum_{n\geq 0} \binom{k}{n} x^n = (1+x)^k.$$

Recurrence Relations

Solve $a_n = 8a_{n-1} + 10^{n-1}$ with initial condition $a_0 = 1$.

<u>*Solution:*</u>

Let $G(x)$ be the generating function for the sequence $\{a_n\}$, that is, $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow \quad G(x) = 1 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} (8a_{n-1}x^n + 10^{n-1}x^n) = 1 + 8x \sum_{n=1}^{\infty} a_{n-1}x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1}x^{n-1}$$

$$= 1 + 8x \sum_{n=0}^{\infty} a_n x^n + x \sum_{n=0}^{\infty} 10^n x^n = 1 + 8x \cdot G(x) + x/(1-10x)$$

since by geometric series:

$$\Rightarrow \quad G(x) = \frac{1-9x}{(1-8x)(1-10x)} = \frac{1}{2}\left(\frac{1}{1-8x}\right) + \frac{1}{2}\left(\frac{1}{1-10x}\right)$$

$$1/(1-rx) = \sum_{k=0}^{\infty} r^k x^k$$

$$= \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right) = \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$$

$$\Rightarrow \quad a_n = \frac{1}{2}(8^n + 10^n)$$

Recurrence Relations

Solve $a_k = 3a_{k-1}$ with initial condition $a_0 = 2$.

Solution:

Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$.

$$G(x) = \sum_{k=0}^{\infty} a_k x^k = 2 + \sum_{k=1}^{\infty} 3a_{k-1} x^k \quad \text{(by the recurrence relation } a_k = 3a_{k-1}\text{)}$$

$$= 2 + 3x \sum_{k=1}^{\infty} a_{k-1} x^{k-1} = 2 + 3x \cdot G(x)$$

$$\Rightarrow \quad G(x) - 3x \cdot G(x) = (1 - 3x) \cdot G(x) = 2$$

$$\Rightarrow \quad G(x) = 2/(1 - 3x)$$

Using the identity, $1/(1 - rx) = \sum_{k=0}^{\infty} r^k x^k$,

$$G(x) = 2 \cdot \sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2 \cdot 3^k x^k \quad \therefore \quad a_k = 2 \cdot 3^k$$

Exercises

Solve the below recurrence relations using generating function-:

$$F_n = F_{n-1} + F_{n-2} \quad n \geq 2 \quad F_1 = 1 \quad F_0 = 0$$