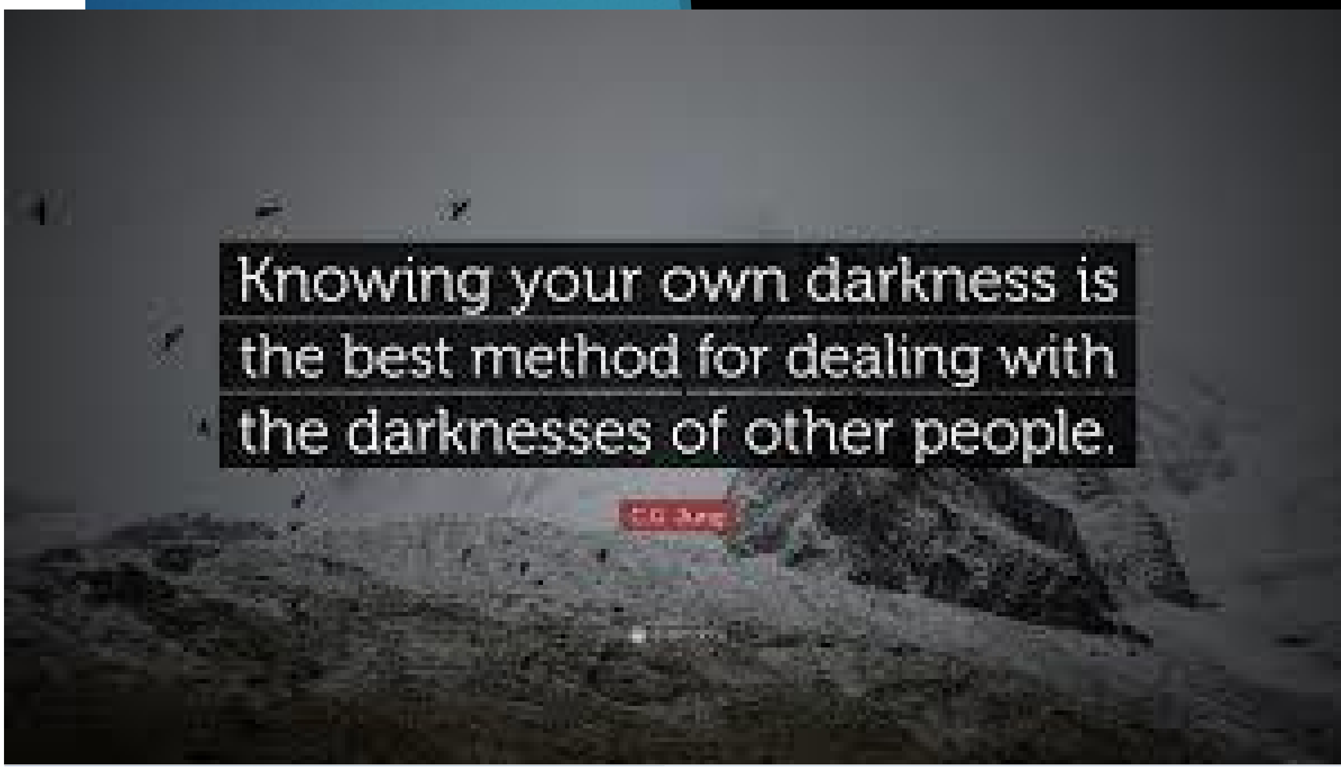
Computational Methods

MATRICES AND VECTORS

QUOTE


Knowing your own darkness is
the best method for dealing with
the darknesses of other people.

Basic Definitions

A **matrix** is a rectangular pattern or array of numbers.

For example,

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ -1 & 2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & -2 \\ 3 & 4 & 0.5 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$$

are all matrices. Note that we usually use a capital letter to denote a matrix, and enclose the array of numbers in brackets. To describe the size of a matrix we quote its number of rows and columns in that order so, for example, an $r \times s$ matrix has r rows and s columns. We say the matrix has **order** $r \times s$.

An $r \times s$ matrix has r rows and s columns.

Basic Definitions

More generally, if the matrix A has m rows and n columns we can write

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

where a_{ij} represents the number or **element** in the ith row and jth column.

A matrix with a single column can also be regarded as a column vector.

Addition, Subtraction & Multiplication

Two matrices can be added (or subtracted) if they have the same shape and size, that is the same order. Their sum (or difference) is found by adding (or subtracting) corresponding elements as the following example shows.

If

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 3 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 1 & 5 & -2 \\ 3 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 7 & -2 \\ 2 & 2 & 5 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1 & 5 & -2 \\ 3 & 1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 3 & -2 \\ 4 & 0 & -3 \end{pmatrix}$$

Addition, Subtraction & Multiplication

On the other hand, the matrices $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ cannot be added or subtracted because they have different orders.

Matrix addition is commutative, that is

$$A + B = B + A$$

Matrix addition is **associative**, that is

$$A + (B + C) = (A + B) + C$$

Addition, Subtraction & Multiplication

Scalar multiplication

Given any matrix A, we can multiply it by a number, that is a scalar, to form a new matrix of the same order as A. This multiplication is performed by multiplying every element of A by the number.

In general we have

$$\text{If } A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \text{ then } kA = \begin{pmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{pmatrix}$$

Addition, Subtraction & Multiplication

If

$$A = \begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} \qquad 2A = 2\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ -4 & 2 \\ 0 & 2 \end{pmatrix}$$

$$-3A = -3\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -3 & -9 \\ 6 & -3 \\ 0 & -3 \end{pmatrix}$$

$$\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{3}{2} \\ -1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Addition, Subtraction & Multiplication

Matrix multiplication

Matrix multiplication is defined in a special way which at first seems strange but is in fact very useful. If A is a $p \times q$ matrix and B is an $r \times s$ matrix we can form the product AB only if $q = r$; that is, only if the number of columns in A is the same as the number of rows in B. The product is then a $p \times s$ matrix C, that is

$$C = AB \qquad \text{where} \qquad \begin{array}{l} A \text{ is } p \times q \\ B \text{ is } q \times s \\ C \text{ is } p \times s \end{array}$$

In general $AB \neq BA$ and so matrix multiplication is not commutative.

Matrix multiplication is associative:

$$(AB)C = A(BC)$$

Addition, Subtraction & Multiplication

Given $A = \begin{pmatrix} 4 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 7 & 6 \\ 5 & 2 & -1 \end{pmatrix}$ can the product AB be formed?

A has size 1×2
B has size 2×3

Because the number of columns in A is the same as the number of rows in B, we can form the product AB. The resulting matrix will have size 1×3 because there is one row in A and three columns in B.

Addition, Subtraction & Multiplication

Find CD when $C = \begin{pmatrix} 1 & 9 & 2 \end{pmatrix}$ and $D = \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix}$.

$$CD = \begin{pmatrix} 1 & 9 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 6 \\ 8 \end{pmatrix} = 1 \times 2 + 9 \times 6 + 2 \times 8 = 2 + 54 + 16 = 72$$

Let us now extend this idea to general matrices A and B. Suppose we wish to find C where $C = AB$.

Addition, Subtraction & Multiplication

If $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$ find, if possible, the matrix C where $C = AB$.

We can form the product

$$C = AB = \underset{\substack{\uparrow \\ 2\times 2}}{\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}} \underset{\substack{\uparrow \\ 2\times 1}}{\begin{pmatrix} 5 \\ -3 \end{pmatrix}}$$

The complete calculation is written as

$$AB = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 1\times 5 + 2\times -3 \\ 4\times 5 + 3\times -3 \end{pmatrix} = \begin{pmatrix} 5-6 \\ 20-9 \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \end{pmatrix}$$

Addition, Subtraction & Multiplication

If $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $C = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ find BC.

B has order 2×3 and C has order 3×1 so clearly the product BC exists and will have order 2×1. BC is formed as follows:

$$BC = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times 2 + 3 \times 4 \\ 4 \times 1 + 5 \times 2 + 6 \times 4 \end{pmatrix} = \begin{pmatrix} 17 \\ 38 \end{pmatrix}$$

Note that the order of the product, 2×1, can be determined at the start by considering the orders of B and C.

Addition, Subtraction & Multiplication

Given

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 3 & 1 \\ 4 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

find, if possible, AB and BA, and comment upon the result.

A and B both have order 3×3 and the products AB and BA can both be formed. Both will have order 3×3.

$$AB = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 \\ 4 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 4 & 2 \\ 4 & 5 & 2 \\ 0 & 14 & 5 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 3 & 1 \\ 4 & -1 & 0 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 2 & 2 \\ 2 & 3 & 4 \\ 9 & 3 & 4 \end{pmatrix}$$

Addition, Subtraction & Multiplication

Given

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ -1 & 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 2 & 9 \\ 1 & 0 & 0 \\ 3 & -2 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 4 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix}$$

find BC, $A(BC)$, AB and $(AB)C$, commenting upon the result.

$$BC = \begin{pmatrix} -1 & 2 & 9 \\ 1 & 0 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 14 & 42 \\ 4 & 1 & 5 \\ 8 & -2 & 18 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 0 & 14 & 42 \\ 4 & 1 & 5 \\ 8 & -2 & 18 \end{pmatrix} = \begin{pmatrix} 12 & 9 & 73 \\ 16 & 38 & 162 \\ 52 & -21 & 63 \end{pmatrix}$$

Addition, Subtraction & Multiplication

$$AB = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & 2 \\ -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} -1 & 2 & 9 \\ 1 & 0 & 0 \\ 3 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 11 \\ 3 & 2 & 29 \\ 19 & -12 & -4 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 4 & -2 & 11 \\ 3 & 2 & 29 \\ 19 & -12 & -4 \end{pmatrix} \begin{pmatrix} 4 & 1 & 5 \\ 2 & 3 & 1 \\ 0 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 9 & 73 \\ 16 & 38 & 162 \\ 52 & -21 & 63 \end{pmatrix}$$

We note that $A(BC) = (AB)C$ so that in this case matrix multiplication is associative.

Some Special Matrices

Square matrices

A matrix which has the same number of rows as columns is called a **square** matrix. Thus

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 3 & 2 & 1 \end{pmatrix} \text{ is a square matrix, while } \begin{pmatrix} -1 & 3 & 0 \\ 2 & 4 & 1 \end{pmatrix} \text{ is not}$$

Diagonal matrices

Some square matrices have elements which are zero everywhere except on the leading diagonal (top-left to bottom-right). Such matrices are said to be **diagonal**. Thus

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ are all diagonal matrices,}$$

$$\text{whereas } \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \text{ is not.}$$

Some Special Matrices

Identity matrices

Diagonal matrices which have only ones on their leading diagonals, for example

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are called **identity** matrices and are denoted by the letter I.

> If A is a square matrix then $IA = AI = A$.

Find IA where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} 2 & 4 & 4 \\ 3 & -1 & 0 \end{pmatrix}$.

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 4 \\ 3 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 4 \\ 3 & -1 & 0 \end{pmatrix}$$

Some Special Matrices

The transpose of a matrix

If A is an arbitrary $m \times n$ matrix, a related matrix is the **transpose** of A, written A^T, found by interchanging the rows and columns of A. Thus the first row of A becomes the first column of A^T and so on. A^T is an $n \times m$ matrix.

If $A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$ find A^T. $\quad A^T = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

If $A = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix}$ find A^T and evaluate AA^T.

$$A^T = \begin{pmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{pmatrix} \qquad AA^T = \begin{pmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{pmatrix} = \begin{pmatrix} 56 & 62 \\ 62 & 114 \end{pmatrix}$$

Some Special Matrices

Symmetric matrices

If a square matrix A and its transpose A^T are identical, then A is said to be a **symmetric** matrix.

If $A = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 6 & 9 \\ 2 & 9 & 13 \end{pmatrix}$ find A^T.

$$A^T = \begin{pmatrix} 5 & -4 & 2 \\ -4 & 6 & 9 \\ 2 & 9 & 13 \end{pmatrix}$$

which is clearly equal to A. Hence A is a symmetric matrix. Note that a symmetric matrix is symmetrical about its leading diagonal.

Some Special Matrices

Skew symmetric matrices

If a square matrix A is such that $A^T = -A$ then A is said to be **skew symmetric**.

If $A = \begin{pmatrix} 0 & 5 \\ -5 & 0 \end{pmatrix}$, find A^T and deduce that A is skew symmetric.

We have $A^T = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}$ which is clearly equal to $-A$.

Hence A is skew symmetric.

Inverse of a Matrix

When we are dealing with ordinary numbers it is often necessary to carry out the operation of division. Thus, for example, if we know that $3x = 4$, then clearly $x = 4/3$. If we are given matrices A and C and know that

$$AB = C$$

how do we find B? It might be tempting to write

$$B = \frac{C}{A}$$

Unfortunately, this would be entirely wrong since division of matrices is not defined.

Inverse of a Matrix

If A is a square matrix and we can find another matrix B with the property that

$$AB = BA = I$$

then B is said to be the **inverse** of A and is written A^{-1}, that is

$$AA^{-1} = A^{-1}A = I$$

Multiplying a matrix by its inverse yields the identity matrix I, that is

$$AA^{-1} = A^{-1}A = I$$

Inverse of a Matrix

If $A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$ show that the matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the inverse of A.

Forming the products

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we see that $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ is the inverse of A.

Inverse of a Matrix

Finding the inverse of a matrix

For 2×2 matrices a simple formula exists to find the inverse of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This formula states

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Inverse of a Matrix

If $A = \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}$ find A^{-1}.

Here we have $ad - bc = 4 - 10 = -6$. Therefore

$$A^{-1} = \frac{1}{-6}\begin{pmatrix} 4 & -5 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{5}{6} \\ \frac{1}{3} & -\frac{1}{6} \end{pmatrix}$$

If $A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$ find A^{-1}.

This time, $ad - bc = 4 - 4 = 0$, so when we come to form $\dfrac{1}{ad - bc}$ we find $1/0$ which is not defined. We cannot form the inverse of A in this case; it does not exist.

Inverse of a Matrix

If A is the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we write its determinant as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. Note that the straight lines $||$ indicate that we are discussing the determinant, which is a scalar, rather than the matrix itself. If the matrix A is such that $|A| = 0$, then it has no inverse and is said to be **singular**. If $|A| \neq 0$ then A^{-1} exists and A is said to be **non-singular**.

A singular matrix A has $|A| = 0$.

A non-singular matrix A has $|A| \neq 0$.

If A and B are square matrices of the same order, $|A||B| = |AB|$.

Inverse of a Matrix

If $A = \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix}$ find $|A|$, $|B|$ and $|AB|$.

$$\begin{aligned}
|A| &= \begin{vmatrix} 1 & 2 \\ 5 & 0 \end{vmatrix} = (1)(0) - (2)(5) = -10 \\
|B| &= \begin{vmatrix} -1 & 2 \\ -3 & 1 \end{vmatrix} = (-1)(1) - (2)(-3) = 5 \\
AB &= \begin{pmatrix} 1 & 2 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -7 & 4 \\ -5 & 10 \end{pmatrix} \\
|AB| &= (-7)(10) - (4)(-5) = -50
\end{aligned}$$

We note that $|A||B| = |AB|$.

Inverse of a Matrix

Orthogonal matrices

A non-singular square matrix A such that $A^T = A^{-1}$ is said to be **orthogonal**.

Consequentuy, if A is orthogonal $AA^T = A^TA = I$.

Find the inverse of $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Deduce that A is an orthogonal matrix.

From the formula for the inverse of a 2×2 matrix we find

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This is clearly equal to the transpose of A. Hence A is an orthogonal matrix.

Determinants

If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, the value of its determinant, $|A|$, is given by

$$|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

If we choose an element of A, a_{ij} say, and cross out its row and column and form the determinant of the four remaining elements, this determinant is known as the **minor** of the element a_{ij}.

A moment's study will therefore reveal that the determinant of A is given by

$$|A| = (a_{11} \times \text{its minor}) - (a_{12} \times \text{its minor}) + (a_{13} \times \text{its minor})$$

This method of evaluating a determinant is known as **expansion along the first row**.

Determinants

Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 5 & 1 & 2 \end{pmatrix}$$

The determinant of A, written as $\begin{vmatrix} 1 & 2 & 1 \\ -1 & 3 & 4 \\ 5 & 1 & 2 \end{vmatrix}$ is found by expanding along its first row:

$$\begin{aligned} |A| &= 1\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} \\ &= 1(2) - 2(-22) + 1(-16) \\ &= 2 + 44 - 16 \\ &= 30 \end{aligned}$$

Determinants

Find the minors of the elements 1 and 4 in the matrix

$$B = \begin{pmatrix} 7 & 2 & 3 \\ 1 & 0 & 3 \\ 0 & 4 & 2 \end{pmatrix}$$

To find the minor of 1 delete its row and column to form the determinant $\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}$.

The required minor is therefore $4 - 12 = -8$.

Similarly, the minor of 4 is $\begin{vmatrix} 7 & 3 \\ 1 & 3 \end{vmatrix} = 21 - 3 = 18$.

Determinants

In addition to finding the minor of each element in a matrix, it is often useful to find a related quantity – the **cofactor** of each element. The cofactor is found by imposing on the minor a positive or negative sign depending upon its position, that is a **place sign**, according to the following rule:

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Determinants

If

$$A = \begin{pmatrix} 3 & 2 & 7 \\ 9 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

find the cofactors of 9 and 7.

The minor of 9 is $\begin{vmatrix} 2 & 7 \\ -1 & 2 \end{vmatrix} = 4 - (-7) = 11$, but since its place sign is negative, the required cofactor is -11.

The minor of 7 is $\begin{vmatrix} 9 & 1 \\ 3 & -1 \end{vmatrix} = -9 - 3 = -12$. Its place sign is positive, so that the required cofactor is simply -12.

The Inverse of a 3×3 Matrix

Given a 3×3 matrix, A, its inverse is found as follows:

(1) Find the transpose of A, by interchanging the rows and columns of A.

(2) Replace each element of A^T by its cofactor; by its minor together with its associated place sign. The resulting matrix is known as the **adjoint** of A, denoted $\text{adj}(A)$.

(3) Finally, the inverse of A is given by

$$\boxed{A^{-1} = \frac{\text{adj}(A)}{|A|}}$$

The Inverse of a 3×3 Matrix

Find the inverse of $A = \begin{pmatrix} 1 & -2 & 0 \\ 3 & 1 & 5 \\ -1 & 2 & 3 \end{pmatrix}$

Solution

$$A^T = \begin{pmatrix} 1 & 3 & -1 \\ -2 & 1 & 2 \\ 0 & 5 & 3 \end{pmatrix}$$

Replacing each element of A^T by its cofactor, we find

$$\text{adj}(A) = \begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

The Inverse of a 3×3 Matrix

The determinant of A is given by

$$\begin{aligned} |A| &= 1\begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} - (-2)\begin{vmatrix} 3 & 5 \\ -1 & 3 \end{vmatrix} + 0\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} \\ &= (1)(-7) + (2)(14) \\ &= 21 \end{aligned}$$

Therefore,

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \frac{1}{21}\begin{pmatrix} -7 & 6 & -10 \\ -14 & 3 & -5 \\ 7 & 0 & 7 \end{pmatrix}$$

The Inverse of a 3×3 Matrix

Express the following equations in the form $AX = B$ and hence solve them:

$$\begin{aligned} 3x + 2y - z &= 4 \\ 2x - y + 2z &= 10 \\ x - 3y - 4z &= 5 \end{aligned}$$

Solution

Using the rules of matrix multiplication, we find

$$\begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$$

which is in the form $AX = B$. The matrix A is called the **coefficient matrix** and is simply the coefficients of x, y and z in the equations.

The Inverse of a 3×3 Matrix

$$\begin{aligned} AX &= B \\ A^{-1}AX &= A^{-1}B \\ IX &= X = A^{-1}B \end{aligned}$$

We must therefore find the inverse of A in order to solve the equations.

To invert A we use the adjoint. If

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & -4 \end{pmatrix}$$

then

$$A^T = \begin{pmatrix} 3 & 2 & 1 \\ 2 & -1 & -3 \\ -1 & 2 & -4 \end{pmatrix}$$

The Inverse of a 3×3 Matrix

$$\operatorname{adj}(A) = \begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}$$

The determinant of A is found by expanding along the first row:

$$\begin{aligned} |A| &= 3\begin{vmatrix} -1 & 2 \\ -3 & -4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 1 & -4 \end{vmatrix} - 1\begin{vmatrix} 2 & -1 \\ 1 & -3 \end{vmatrix} \\ &= (3)(10) - (2)(-10) - (1)(-5) \\ &= 30 + 20 + 5 \\ &= 55 \end{aligned}$$

Therefore,

$$A^{-1} = \frac{\operatorname{adj}(A)}{|A|} = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}$$

The Inverse of a 3×3 Matrix

Finally, the solution X is given by

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}B = \frac{1}{55}\begin{pmatrix} 10 & 11 & 3 \\ 10 & -11 & -8 \\ -5 & 11 & -7 \end{pmatrix}\begin{pmatrix} 4 \\ 10 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

that is, the solution is $x = 3$, $y = -2$ and $z = 1$.

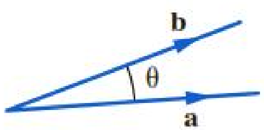
The Scalar/Dot Product

Given any two vectors **a** and **b**, there are two ways in which we can define their product. These are known as the scalar product and the vector product. As the names suggest, the result of finding a scalar product is a scalar whereas the result of finding a vector product is a vector. The **scalar product** of **a** and **b** is written as

$$\mathbf{a} \cdot \mathbf{b}$$

This notation gives rise to the alternative name **dot product**. It is defined by the formula

$$\boxed{\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta}$$



where θ is the angle between the two vectors as shown in Figure.

The Scalar/Dot Product

From the definition of the scalar product, it is possible to show that the following rules hold:

$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$	the scalar product is commutative
$k(\mathbf{a} \cdot \mathbf{b}) = (k\mathbf{a} \cdot \mathbf{b})$	where k is a scalar
$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = (\mathbf{a} \cdot \mathbf{c}) + (\mathbf{b} \cdot \mathbf{c})$	the distributive rule

It is important at this stage to realize that notation is very important in vector work. You should not use a $\times$ to denote the scalar product because this is the symbol we shall use for the vector product.

The Scalar/Dot Product

If $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, show that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$. If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal show that their scalar product is zero.

If $\mathbf{a}$ and $\mathbf{b}$ are parallel then the angle between them is zero. Therefore $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos 0^\circ = |\mathbf{a}||\mathbf{b}|$. If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, then the angle between them is 90° and $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos 90^\circ = 0$.

Similarly we can show that if $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors for which $\mathbf{a} \cdot \mathbf{b} = 0$, then $\mathbf{a}$ and $\mathbf{b}$ must be orthogonal.

The Scalar/Dot Product

If $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors, $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|$.
If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal vectors, $\mathbf{a} \cdot \mathbf{b} = 0$.

An immediate consequence of the previous result is the following useful set of formulae:

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$
$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

The Scalar/Dot Product

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ show that $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

We have

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1\mathbf{i} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) + a_2\mathbf{j} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&\quad\; + a_3\mathbf{k} \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1 b_1 \mathbf{i} \cdot \mathbf{i} + a_1 b_2 \mathbf{i} \cdot \mathbf{j} + a_1 b_3 \mathbf{i} \cdot \mathbf{k} + a_2 b_1 \mathbf{j} \cdot \mathbf{i} + a_2 b_2 \mathbf{j} \cdot \mathbf{j} + a_2 b_3 \mathbf{j} \cdot \mathbf{k} \\
&\quad\; + a_3 b_1 \mathbf{k} \cdot \mathbf{i} + a_3 b_2 \mathbf{k} \cdot \mathbf{j} + a_3 b_3 \mathbf{k} \cdot \mathbf{k} \\
&= a_1 b_1 + a_2 b_2 + a_3 b_3
\end{aligned}$$

as required. Thus, given two vectors in component form their scalar product is the sum of the products of corresponding components.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$.

The Scalar/Dot Product

If $\mathbf{a} = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} + 4\mathbf{j} + \mathbf{k}$ find the scalar product $\mathbf{a} \cdot \mathbf{b}$.

Using the previous result we find

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}) \cdot (-2\mathbf{i} + 4\mathbf{j} + \mathbf{k}) \\ &= (5)(-2) + (-3)(4) + (2)(1) \\ &= -10 - 12 + 2 \\ &= -20 \end{aligned}$$

The Scalar/Dot Product

We note the general result that

$$\boxed{\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2} \qquad \mathbf{a}\cdot\mathbf{a} = a_1^2 + a_2^2 + a_3^2 \qquad |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

If $\mathbf{a} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ find $\mathbf{a}\cdot\mathbf{b}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$.

We have

$$\mathbf{a}\cdot\mathbf{b} = (3)(2) + (1)(1) + (-1)(2) \;= 6 + 1 - 2 \;= 5$$

Furthermore, from the definition of the scalar product $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. Now,

$$|\mathbf{a}| = \sqrt{9+1+1} = \sqrt{11} \qquad \text{and} \qquad |\mathbf{b}| = \sqrt{4+1+4} = 3$$

Therefore, $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{5}{3\sqrt{11}}$ from which we deduce that $\theta = 59.8^\circ$ or 1.04 radians.

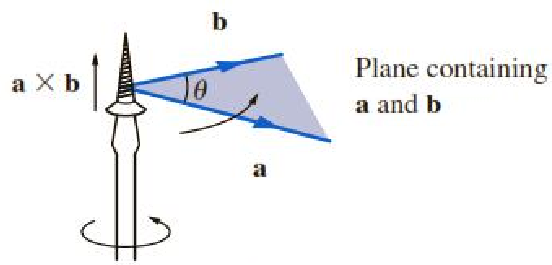
The Vector Product

The result of finding the vector product of **a** and **b** is a vector of length $|\mathbf{a}||\mathbf{b}|\sin\theta$, where θ is the angle between **a** and **b**. The direction of this vector is such that it is perpendicular to **a** and to **b**, and so it is perpendicular to the plane containing **a** and **b**. There are, however, two possible directions for this vector, but it is conventional to choose the one associated with the application of the right-handed screw rule. Imagine turning a right-handed screw in the sense from **a** towards **b** as shown. A right-handed screw is one which, when turned clockwise, enters the material into which it is being screwed. The direction in which the screw advances is the direction of the required vector product. The symbol we shall use to denote the vector product is $\times$.

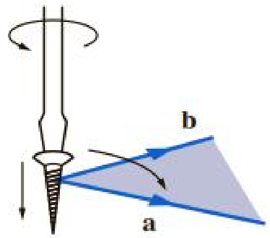
The Vector Product

Formally, we write $$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\mathbf{e}}$$

where $\hat{\mathbf{e}}$ is the unit vector required to define the appropriate direction, that is $\hat{\mathbf{e}}$ is a unit vector perpendicular to **a** and to **b** in a sense defined by the right-handed screw rule. To evaluate $\mathbf{b} \times \mathbf{a}$ we must imagine turning the screw from the direction of **b** towards that of **a**. The screw will advance as shown in Figure.



$\mathbf{a} \times \mathbf{b}$ is perpendicular to the plane containing **a** and **b**. The right-handed screw rule allows the direction of $\mathbf{a} \times \mathbf{b}$ to be found.



Right-handed screw rule allows the direction of $\mathbf{b} \times \mathbf{a}$ to be found.

The Vector Product

We notice immediately that $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$ since their directions are different. From the definition of the vector product, it is possible to show that the following rules hold:

$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$	the vector product is not commutative
$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$	the distributive rule
$k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$	where k is a scalar

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$	$\mathbf{j} \times \mathbf{k} = \mathbf{i}$	$\mathbf{k} \times \mathbf{i} = \mathbf{j}$
$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$	$\mathbf{k} \times \mathbf{j} = -\mathbf{i}$	$\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

The Vector Product

Using determinants to evaluate vector products

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Find the vector product of $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

The two given vectors are represented in the following determinant:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 7 \\ 1 & 2 & 1 \end{vmatrix}$$

Evaluating this determinant we obtain

$$\mathbf{a} \times \mathbf{b} = (3 - 14)\mathbf{i} - (2 - 7)\mathbf{j} + (4 - 3)\mathbf{k} = -11\mathbf{i} + 5\mathbf{j} + \mathbf{k}$$