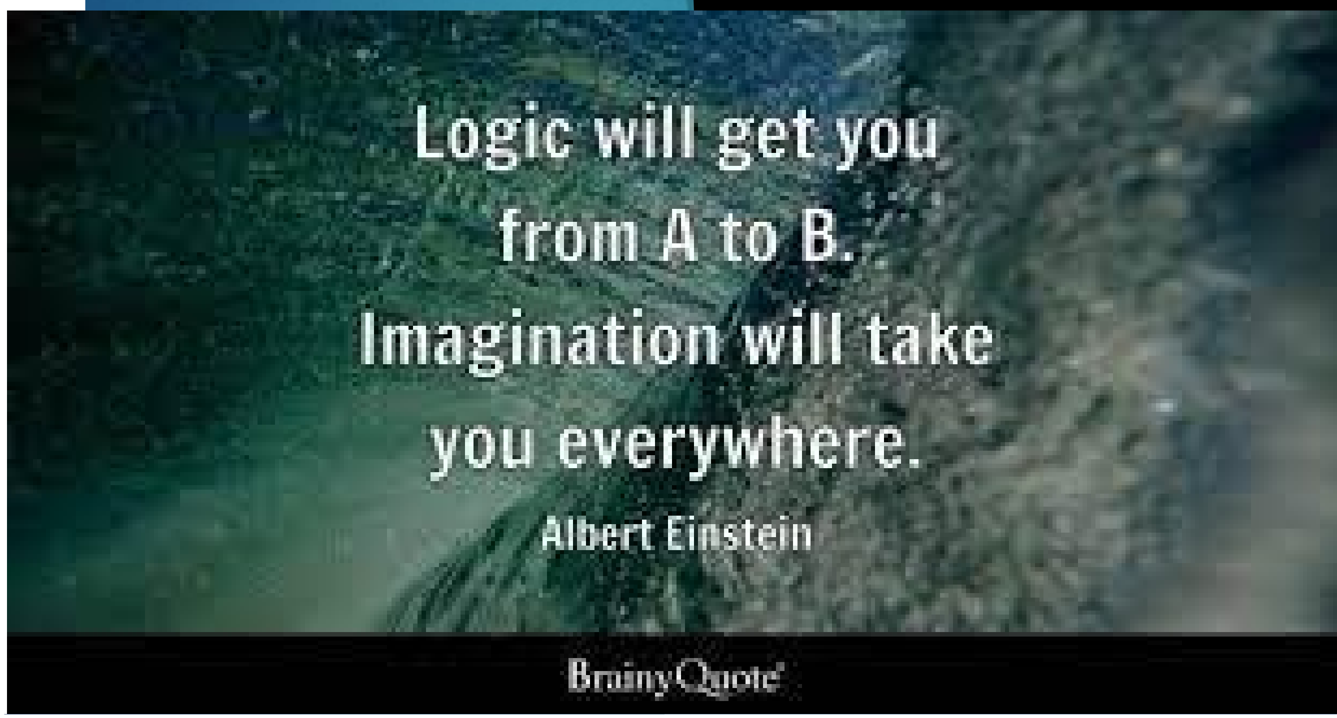
# Computational Methods

LOGICAL OPERATIONS

# QUOTE




Logic will get you
from A to B.
Imagination will take
you everywhere.
Albert Einstein
BrainyQuote

# Statements

Formal logic can represent the statements we use in English to communicate facts or information. A **statement** (or **proposition**) is a sentence that is either true or false.

Consider the following:

a. Ten is less than seven.
b. Cheyenne is the capital of Wyoming.
c. She is very talented.
d. There are life forms on other planets in the universe.

Sentence (a) is a statement because it is false. Sentence (b) is a statement because it is true. Sentence (c) is neither true nor false because "she" is not specified; therefore (c) is not a statement. Sentence (d) is a statement because it is either true or false; we do not have to be able to decide which.

# Statements

In English, simple statements are combined with connecting words like *and* to make more interesting compound statements. The truth value of a compound statement depends on the truth values of its components and which connecting words are used. If we combine the two true statement, "Elephants are big," and, "Baseballs are round," we would consider the resulting statement, "Elephants are big and baseballs are round," to be true.

Capital letters near the beginning of the alphabet, such as $A$, $B$, and $C$, are used to represent statements and are called **statement letters**; the symbol $\wedge$ is a **logical connective** representing *and*. We agree, then, that if $A$ is true and $B$ is true, $A \wedge B$ (read "$A$ and $B$") should be considered true.

The expression $A \wedge B$ is called the **conjunction** of $A$ and $B$, and $A$ and $B$ are called the **conjuncts** of this expression.

# Statements

Another connective is the word *or*, denoted by the symbol $\vee$. The expression $A \vee B$ (read "$A$ or $B$") is called the **disjunction** of $A$ and $B$, and $A$ and $B$ are called the **disjuncts** of this expression. If $A$ and $B$ are both true, then $A \vee B$ would be considered true.

| $A$ | $B$ | $A \wedge B$ |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | F |

| $A$ | $B$ | $A \vee B$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | F |

# Statements

Statements may be combined in the form "if statement 1, then statement 2." If $A$ denotes statement 1 and $B$ denotes statement 2, the compound statement would be denoted by $A \rightarrow B$ (read "$A$ implies $B$"). The logical connective here is **implication**, and it conveys the meaning that the truth of $A$ implies or leads to the truth of $B$. In the implication $A \rightarrow B$, $A$ stands for the **antecedent** statement and $B$ stands for the **consequent** statement.

The **equivalence** connective is symbolized by $\leftrightarrow$. Unlike conjunction, disjunction, and implication, the equivalence connective is not really a fundamental connective but a convenient shortcut. The expression $A \leftrightarrow B$ stands for $(A \rightarrow B) \wedge (B \rightarrow A)$.

# Statements

The connectives we've seen so far are called **binary connectives** because they join two expressions together to produce a third expression. Now let's consider a **unary connective**, a connective acting on one expression to produce a second expression. **Negation** is a unary connective. The negation of $A$—symbolized by $A'$—is read "not $A$."

| $A$ | $B$ | $A \wedge B$ | $A \vee B$ | $A \rightarrow B$ | $A \leftrightarrow B$ | $A'$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | F |
| T | F | F | T | F | F | |
| F | T | F | T | T | F | T |
| F | F | F | F | T | T | |

# Statements

| English Word | Logical Connective | Logical Expression |
|---|---|---|
| and; but; also; in addition; moreover | Conjunction | $A \wedge B$ |
| or | Disjunction | $A \vee B$ |
| If $A$, then $B$.<br>$A$ implies $B$.<br>$A$, therefore $B$.<br>$A$ only if $B$.<br>$B$ follows from $A$.<br>$A$ is a sufficient condition for $B$.<br>$B$ is a necessary condition for $A$. | Implication | $A \rightarrow B$ |
| $A$ if and only if $B$.<br>$A$ is necessary and sufficient for $B$. | Equivalence | $A \leftrightarrow B$ |
| not $A$<br>It is false that $A$ ...<br>It is not true that $A$ ... | Negation | $A'$ |

# Statements

| Statement | Correct Negation | Incorrect Negation |
|---|---|---|
| It will rain tomorrow. | It is false that it will rain tomorrow.<br><br>It will not rain tomorrow. | |
| Peter is tall and thin. | It is false that Peter is tall and thin.<br><br>Peter is not tall or he is not thin.<br><br>Peter is short or fat. | Peter is short and fat.<br><br>Too strong a statement. Peter fails to have both properties (tallness and thinness) but may still have one property. |
| The river is shallow or polluted. | It is false that the river is shallow or polluted.<br><br>The river is neither shallow nor polluted.<br><br>The river is deep and unpolluted. | The river is not shallow or not polluted.<br><br>Too weak a statement. The river fails to have either property, not just fails to have one property. |

# Statements

We can string statement letters, connectives, and parentheses (or brackets) together to form new expressions, as in

$$(A \to B) \wedge (B \to A)$$

An expression that is a legitimate string is called a **well-formed formula**, or **wff**. To reduce the number of parentheses required in a wff, we stipulate an order in which connectives are applied. This *order of precedence* is

1. connectives within parentheses, innermost parentheses first
2. $'$
3. $\wedge, \vee$
4. $\to$
5. $\leftrightarrow$

# Statements

This means that the expression $A \vee B'$ stands for $A \vee (B')$, not $(A \vee B)'$. Similarly, $A \vee B \to C$ means $(A \vee B) \to C$, not $A \vee (B \to C)$. However, we often use parentheses anyway, just to be sure that there is no confusion.

In a wff with a number of connectives, the connective to be applied last is the **main connective**. In

$$A \wedge (B \to C)'$$

the main connective is $\wedge$. In

$$((A \vee B) \wedge C) \to (B \vee C')$$

the main connective is $\to$. Capital letters near the end of the alphabet, such as $P$, $Q$, $R$, and $S$, are used to represent wffs. Thus $P$ could represent a single statement letter, which is the simplest kind of wff, or a more complex wff. We might represent

$$((A \vee B) \wedge C) \to (B \vee C') \quad \text{as} \quad P \to Q$$

# Tautologies



A wff-like item whose truth values are always true, is called a **tautology**. A tautology is "intrinsically true" by its very structure; it is true no matter what truth values are assigned to its statement letters. A simpler example of a tautology is $A \vee A'$; consider, for example, the statement "Today the sun will shine or today the sun will not shine," which must always be true because one or the other of these must happen.

A wff like item whose truth values are always false, is called a **contradiction**. A contradiction is "intrinsically false" by its very structure. A simpler example of a contradiction is $A \wedge A'$; consider "Today is Tuesday and today is not Tuesday," which is false no matter what day of the week it is.

# Tautologies

Some Tautological Equivalences

| | | |
|---|---|---|
| 1a. $A \vee B \Leftrightarrow B \vee A$ | 1b. $A \wedge B \Leftrightarrow B \wedge A$ | (commutative properties) |
| 2a. $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$ | 2b. $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$ | (associative properties) |
| 3a. $A \vee (B \wedge C) \Leftrightarrow$ $(A \vee B) \wedge (A \vee C)$ | 3b. $A \wedge (B \vee C) \Leftrightarrow$ $(A \wedge B) \vee (A \wedge C)$ | (distributive properties) |
| 4a. $A \vee 0 \Leftrightarrow A$ | 4b. $A \wedge 1 \Leftrightarrow A$ | (identity properties) |
| 5a. $A \vee A' \Leftrightarrow 1$ | 5b. $A \wedge A' \Leftrightarrow 0$ | (complement properties) |

**DE MORGAN'S LAWS**

$$(A \vee B)' \Leftrightarrow A' \wedge B' \quad \text{and} \quad (A \wedge B)' \Leftrightarrow A' \vee B'$$

# Exercises

Which of the following sentences are statements?

a. The moon is made of green cheese.
b. He is certainly a tall man.
c. Two is a prime number.
d. The game will be over by 4:00.
e. Next year interest rates will rise.
f. Next year interest rates will fall.
g. $x^2 - 4 = 0$

# Exercises

Given the truth values $A$ true, $B$ false, and $C$ true, what is the truth value of each of the following wffs?

a. $A \wedge (B \vee C)$

b. $(A \wedge B) \vee C$

c. $(A \wedge B)' \vee C$

d. $A' \vee (B' \wedge C)'$

# Exercises

Rewrite each of the following statements in the form "If $A$, then $B$."

a. Healthy plant growth follows from sufficient water.

b. Increased availability of information is a necessary condition for further technological advances.

c. Errors were introduced only if there was a modification of the program.

d. Fuel savings implies good insulation or storm windows throughout.

# Exercises

Write the negation of each statement.

a. If the food is good, then the service is excellent.
b. Either the food is good or the service is excellent.
c. Either the food is good and the service is excellent, or else the price is high.
d. Neither the food is good nor the service excellent.
e. If the price is high, then the food is good and the service is excellent.

# Exercises

Using the letters indicated for the component statements, translate the following compound statements into symbolic notation.

a. *A*: prices go up; *B*: housing will be plentiful; *C*: housing will be expensive
If prices go up, then housing will be plentiful and expensive; but if housing is not expensive, then it will still be plentiful.

b. *A*: going to bed; *B*: going swimming; *C*: changing clothes
Either going to bed or going swimming is a sufficient condition for changing clothes; however, changing clothes does not mean going swimming.

c. *A*: it will rain; *B*: it will snow
Either it will rain or it will snow but not both.

d. *A*: Janet wins; *B*: Janet loses; *C*: Janet will be tired
If Janet wins or if she loses, she will be tired.

e. *A*: Janet wins; *B*: Janet loses; *C*: Janet will be tired
Either Janet will win or, if she loses, she will be tired.

# Exercises

Construct truth tables for the following wffs. Note any tautologies or contradictions.

a. $(A \rightarrow B) \leftrightarrow A' \vee B$

b. $(A \wedge B) \vee C \rightarrow A \wedge (B \vee C)$

c. $A \wedge (A' \vee B')'$

d. $A \wedge B \rightarrow A'$

e. $(A \rightarrow B) \rightarrow [(A \vee C) \rightarrow (B \vee C)]$