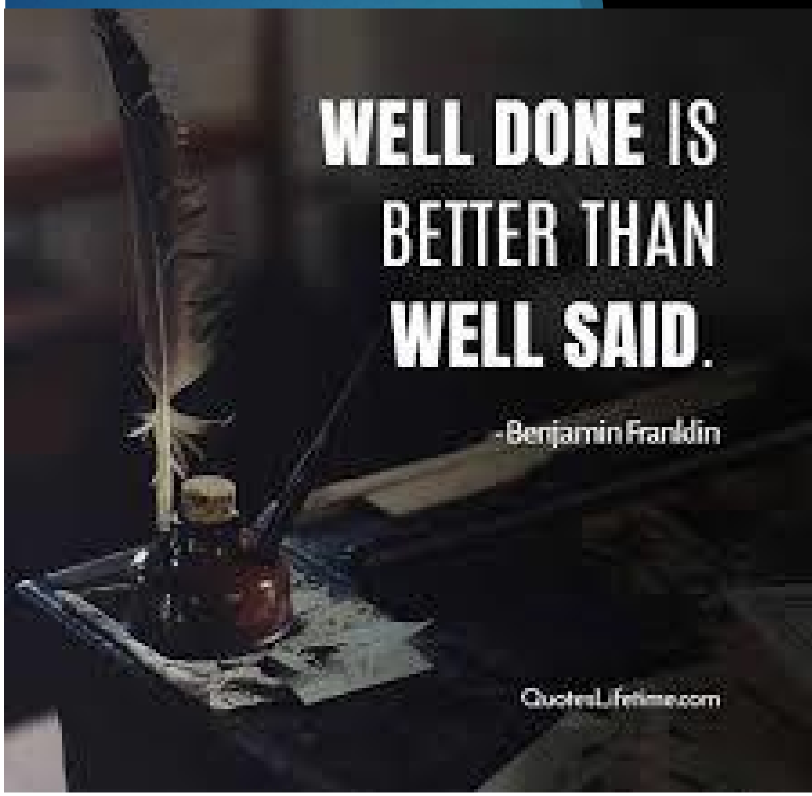
Computational Methods

DIFFERENTIATION AND INTEGRATION

QUOTE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BETTER THAN
WELL SAID.
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Derivatives

Derivatives of commonly used functions.

Function, $y(x)$	*Derivative,* y'
constant	0
x^n	nx^{n-1}
e^x	e^x
e^{-x}	$-e^{-x}$
e^{ax}	ae^{ax}
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$

Function, $y(x)$	*Derivative,* y'
$\cos x$	$-\sin x$
$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$
$\tan(ax+b)$	$a\sec^2(ax+b)$
$\operatorname{cosec}(ax+b)$	$-a\operatorname{cosec}(ax+b)\cot(ax+b)$
$\sec(ax+b)$	$a\sec(ax+b)\tan(ax+b)$
$\cot(ax+b)$	$-a\operatorname{cosec}^2(ax+b)$

Derivatives

Function, y(x)	*Derivative, y′*
$\sin^{-1}(ax+b)$	$\dfrac{a}{\sqrt{1-(ax+b)^2}}$
$\cos^{-1}(ax+b)$	$\dfrac{-a}{\sqrt{1-(ax+b)^2}}$
$\tan^{-1}(ax+b)$	$\dfrac{a}{1+(ax+b)^2}$
$\sinh(ax+b)$	$a\cosh(ax+b)$
$\cosh(ax+b)$	$a\sinh(ax+b)$
$\tanh(ax+b)$	$a\,\mathrm{sech}^2(ax+b)$

Function, y(x)	*Derivative, y′*
$\mathrm{cosech}(ax+b)$	$-a\,\mathrm{cosech}(ax+b)\times\coth(ax+b)$
$\mathrm{sech}(ax+b)$	$-a\,\mathrm{sech}(ax+b)\times\tanh(ax+b)$
$\coth(ax+b)$	$-a\,\mathrm{cosech}^2(ax+b)$
$\sinh^{-1}(ax+b)$	$\dfrac{a}{\sqrt{(ax+b)^2+1}}$
$\cosh^{-1}(ax+b)$	$\dfrac{a}{\sqrt{(ax+b)^2-1}}$
$\tanh^{-1}(ax+b)$	$\dfrac{a}{1-(ax+b)^2}$

Rules of Differentiation

The product rule states: if

$$y(x) = u(x)\,v(x)$$

then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx} = u'v + uv'$$

The chain rule states:

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

The quotient rule states: when

$$y(x) = \frac{u(x)}{v(x)}$$

then

$$y' = \frac{v\left(\frac{du}{dx}\right) - u\left(\frac{dv}{dx}\right)}{v^2} = \frac{vu' - uv'}{v^2}$$

When $y = \ln f(x)$ then $\dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$

Rules of Differentiation

Find y' given

(a) $y = x \sin x$

(b) $y = t^2 e^t$

(a) $y = x \sin x = uv$. Choose $u = x$ and $v = \sin x$. Then $u' = 1$, $v' = \cos x$. Applying the product rule to y yields

$$y' = \sin x + x \cos x$$

(b) $y = t^2 e^t = uv$. Choose $u = t^2$ and $v = e^t$. Then $u' = 2t$ and $v' = e^t$. Applying the product rule to y yields

$$y' = 2te^t + t^2 e^t$$

Rules of Differentiation

Find y' given

(a) $y = \dfrac{\sin x}{x}$ (b) $y = \dfrac{t^2}{2t+1}$

(a) $y = \dfrac{\sin x}{x} = \dfrac{u}{v}$. So $u = \sin x$, $v = x$ and $u' = \cos x$, $v' = 1$. Using the quotient rule the derivative of y is found:

$$y' = \frac{x\cos x - \sin x}{x^2}$$

(b) $y = \dfrac{t^2}{2t+1} = \dfrac{u}{v}$. So $u = t^2$, $v = 2t + 1$ and $u' = 2t$, $v' = 2$. Hence,

$$y' = \frac{(2t+1)2t - (t^2)(2)}{(2t+1)^2} = \frac{2t(t+1)}{(2t+1)^2}$$

Rules of Differentiation

Given $y = z^6$ where $z = x^2 + 1$ find $\frac{dy}{dx}$.

If $y = z^6$ and $z = x^2 + 1$, then $y = (x^2 + 1)^6$. We recognize this as the composition $y(z(x))$. Now $y = z^6$ and so $\frac{dy}{dz} = 6z^5$. Also $z = x^2 + 1$ and so $\frac{dz}{dx} = 2x$.

Using the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dz} \times \frac{dz}{dx} \\ &= 6z^5 2x \\ &= 12x(x^2 + 1)^5\end{aligned}$$

Rules of Differentiation

Find y' when

(a) $y = \ln(x^5 + 8)$ (b) $y = \ln(1 - t)$ (c) $y = 8\ln(2 - 3t)$

(d) $y = \ln(1 + x)$ (e) $y = \ln(1 + \cos x)$

In each case we apply the previous rule.

(a) If $y = \ln(x^5 + 8)$, then $y' = \dfrac{5x^4}{x^5 + 8}$.

(b) If $y = \ln(1 - t)$, then $y' = \dfrac{-1}{1 - t} = -\dfrac{1}{1 - t} = \dfrac{1}{t - 1}$.

(c) If $y = 8\ln(2 - 3t)$, then $y' = 8\dfrac{-3}{2 - 3t} = -\dfrac{24}{2 - 3t} = \dfrac{24}{3t - 2}$.

(d) If $y = \ln(1 + x)$ then $y' = \dfrac{1}{1 + x}$.

(e) If $y = \ln(1 + \cos x)$ then $y' = \dfrac{-\sin x}{1 + \cos x} = -\dfrac{\sin x}{1 + \cos x}$.

Rules of Differentiation

Differentiate

(a) $y = 3e^{\sin x}$ (b) $y = (3t^2 + 2t - 9)^{10}$

In these examples we must formulate the function z ourselves.

(a) Let $z(x) = \sin x$. Then $y(z) = 3e^z$ so $\frac{dy}{dz} = 3e^z$; $z(x) = \sin x$ and so $\frac{dz}{dx} = \cos x$.

The chain rule is used to find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 3e^z \cos x = 3e^{\sin x} \cos x$$

(b) Let $z(t) = 3t^2 + 2t - 9$. Then $y(z) = z^{10}$, $\frac{dy}{dz} = 10z^9$, $\frac{dz}{dt} = 6t + 2$. Using the chain rule $\frac{dy}{dt}$ is found:

$$\frac{dy}{dt} = \frac{dy}{dz} \times \frac{dz}{dt} = 10z^9(6t + 2) = 20(3t + 1)(3t^2 + 2t - 9)^9$$

Parametric Differentiation

In some circumstances both y and x depend upon a third variable, t. This third variable is often called a **parameter**. The derivative $\frac{dy}{dx}$ can still be found using the chain rule.

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

Finding $\frac{dy}{dx}$ by this method is known as **parametric differentiation**.

Parametric Differentiation

Given $y = (1+t)^2, x = 2t$ find $\frac{dy}{dx}$.

Parametric differentiation is an alternative method of finding $\frac{dy}{dx}$ which does not require the elimination of t.

$$\frac{dy}{dt} = 2(1+t) \qquad \frac{dx}{dt} = 2$$

Using the chain rule, we obtain

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{2(1+t)}{2} = 1 + t = 1 + \frac{x}{2}$$

Parametric Differentiation

Given $y = e^t + t$, $x = t^2 + 1$, find $\frac{dy}{dx}$ using parametric differentiation.

$$\frac{dy}{dt} = e^t + 1 \qquad \frac{dx}{dt} = 2t$$

Hence,

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{e^t + 1}{2t}$$

In this example, the derivative is expressed in terms of t. This will always be the case when t has not been eliminated between x and y.

Parametric Differentiation

If $x = \sin t + \cos t$ and $y = t^2 - t + 1$ find $\frac{\mathrm{d}y}{\mathrm{d}x}(t = 0)$.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2t - 1 \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \cos t - \sin t$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t - 1}{\cos t - \sin t}$$

When $t = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-1}{1} = -1.$

Implicit Differentiation

Suppose we are told that

$$y^3 + x^3 = 5\sin x + 10\cos y$$

Although y depends upon x, it is impossible to write the equation in the form $y = f(x)$.

We say y is expressed **implicitly** in terms of x. The form $y = f(x)$ is an **explicit** expression for y in terms of x.

Find

(a) $\dfrac{d}{dx}(y^4)$ (b) $\dfrac{d}{dx}(y^{-3})$

Implicit Differentiation

(a) We make a substitution and let $z = y^4$ so that the problem becomes that of finding $\frac{dz}{dx}$. Now, using the chain rule,

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$$

If $z = y^4$ then $\frac{dz}{dy} = 4y^3$ and so $\frac{dz}{dx} = 4y^3\frac{dy}{dx}$. We conclude that $\frac{d}{dx}(y^4) = 4y^3\frac{dy}{dx}$

(b) We make a substitution and let $z = y^{-3}$ so that the problem becomes that of finding $\frac{dz}{dx}$. If $z = y^{-3}$ then $\frac{dz}{dy} = -3y^{-4}$ and so

$$\frac{d}{dx}(y^{-3}) = \frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx} = -3y^{-4}\frac{dy}{dx}$$

Implicit Differentiation

Find $\frac{d}{dt}(\ln y)$.

We let $z = \ln y$ so that the problem becomes that of finding $\frac{dz}{dt}$. If $z = \ln y$ then $\frac{dz}{dy} = \frac{1}{y}$ and so using the chain rule

$$\frac{dz}{dt} = \frac{dz}{dy} \times \frac{dy}{dt} = \frac{1}{y}\frac{dy}{dt}$$

we conclude that

$$\frac{d}{dt}(\ln y) = \frac{1}{y}\frac{dy}{dt}$$

Implicit Differentiation

Find $\frac{dy}{dx}$ given

(a) $\ln y = y - x^2$

(b) $x^2y^3 - e^y = e^{2x}$

(a) Differentiating the given equation w.r.t. x yields $\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} - 2x$

from which

$$\frac{dy}{dx} = \frac{2xy}{y-1}$$

Implicit Differentiation

(b) Consider $\frac{d}{dx}(x^2y^3)$. Using the product rule we find

$$\frac{d}{dx}(x^2y^3) = \frac{d}{dx}(x^2)y^3 + x^2\frac{d}{dx}(y^3) = 2xy^3 + x^2 3y^2\frac{dy}{dx}$$

Consider $\frac{d}{dx}(e^y)$. Let $z = e^y$ so $\frac{dz}{dy} = e^y$. Hence,

$$\frac{d}{dx}(e^y) = \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = e^y\frac{dy}{dx}$$

So, upon differentiating, the equation becomes $2xy^3 + x^2 3y^2\frac{dy}{dx} - e^y\frac{dy}{dx} = 2e^{2x}$

from which

$$\frac{dy}{dx} = \frac{2e^{2x} - 2xy^3}{3x^2y^2 - e^y}$$

The Laws of Logarithms

Logarithmic expressions can be manipulated using the laws of logarithms. These laws are identical for any base, but it is essential when applying the laws that bases are not mixed.

$$\log_a A + \log_a B = \log_a (AB)$$
$$\log_a A - \log_a B = \log_a \left(\frac{A}{B}\right)$$
$$n \log_a A = \log_a (A^n)$$
$$\log_a a = 1$$

We sometimes need to change from one base to another. This can be achieved using the following rule.

$$\log_a X = \frac{\log_b X}{\log_b a}$$

Logarithmic Differentiation

The technique of **logarithmic differentiation** is useful when we need to differentiate a cumbersome product. The method involves taking the natural logarithm of the function to be differentiated. This is illustrated in the following examples.

Given that $y = t^2(1-t)^8$ find $\dfrac{\mathrm{d}y}{\mathrm{d}t}$.

Solution

The product rule could be used but we will demonstrate an alternative technique. Taking the natural logarithm of both sides of the given equation yields

$$\ln y = \ln(t^2(1-t)^8)$$

Using the laws of logarithms we can write this as

$$\begin{aligned}\ln y &= \ln t^2 + \ln(1-t)^8 \\ &= 2\ln t + 8\ln(1-t)\end{aligned}$$

Logarithmic Differentiation

Both sides of this equation are now differentiated w.r.t. t to give

$$\frac{d}{dt}(\ln y) = \frac{d}{dt}(2\ln t) + \frac{d}{dt}(8\ln(1-t))$$

$$\frac{1}{y}\frac{dy}{dt} = \frac{2}{t} - \frac{8}{1-t}$$

Hence

$$\frac{dy}{dt} = y\left(\frac{2}{t} - \frac{8}{1-t}\right)$$

Finally, replacing y by $t^2(1-t)^8$ we have

$$\frac{dy}{dt} = t^2(1-t)^8\left(\frac{2}{t} - \frac{8}{1-t}\right)$$

Logarithmic Differentiation

Given $y = x^3(1+x)^9 e^{6x}$ find $\frac{dy}{dx}$.

The product rule could be used. However, we will use logarithmic differentiation. Taking the natural logarithm of the equation and applying the laws of logarithms produces

$$\ln y = \ln(x^3(1+x)^9 e^{6x}) = \ln x^3 + \ln(1+x)^9 + \ln e^{6x}$$
$$\ln y = 3\ln x + 9\ln(1+x) + 6x$$

This equation is now differentiated:

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{9}{1+x} + 6$$

and so

$$\frac{dy}{dx} = y\left(\frac{3}{x} + \frac{9}{1+x} + 6\right)$$
$$= 3x^2(1+x)^9 e^{6x} + 9x^3(1+x)^8 e^{6x} + 6x^3(1+x)^9 e^{6x}$$

Logarithmic Differentiation

If $y = \sqrt{1+t^2}\sin^2 t$ find y'.

Taking logarithms we find

$$\begin{aligned}\ln y &= \ln(\sqrt{1+t^2}\sin^2 t)\\ &= \ln\sqrt{1+t^2} + \ln(\sin^2 t)\\ &= \frac{1}{2}\ln(1+t^2) + 2\ln(\sin t)\end{aligned}$$

Differentiation yields

$$\frac{1}{y}y' = \frac{1}{2}\frac{2t}{1+t^2} + 2\frac{\cos t}{\sin t}$$

$$y' = y\left(\frac{t}{1+t^2} + 2\cot t\right) = \sqrt{1+t^2}\sin^2 t\left(\frac{t}{1+t^2} + 2\cot t\right)$$

Integrals of Some Common Functions

$f(x)$	$\int f(x)\,dx$	
k, constant	$kx + c$	
x^n	$\dfrac{x^{n+1}}{n+1} + c$	$n \neq -1$
$x^{-1} = \dfrac{1}{x}$	$\ln \lvert x \rvert + c$	
e^x	$e^x + c$	
e^{-x}	$-e^{-x} + c$	
e^{ax}	$\dfrac{e^{ax}}{a} + c$	

$f(x)$	$\int f(x)\,dx$
$\sin x$	$-\cos x + c$
$\sin ax$	$\dfrac{-\cos ax}{a} + c$
$\sin(ax+b)$	$\dfrac{-\cos(ax+b)}{a} + c$
$\cos x$	$\sin x + c$
$\cos ax$	$\dfrac{\sin ax}{a} + c$
$\cos(ax+b)$	$\dfrac{\sin(ax+b)}{a} + c$

Integrals of Some Common Functions

$f(x)$	$\int f(x)\,dx$
$\tan x$	$\ln\lvert\sec x\rvert + c$
$\tan ax$	$\dfrac{\ln\lvert\sec ax\rvert}{a} + c$
$\tan(ax+b)$	$\dfrac{\ln\lvert\sec(ax+b)\rvert}{a} + c$
$\operatorname{cosec}(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\operatorname{cosec}(ax+b) - \cot(ax+b)\rvert\} + c$

$f(x)$	$\int f(x)\,dx$
$\sec(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\sec(ax+b) + \tan(ax+b)\rvert\} + c$
$\cot(ax+b)$	$\dfrac{1}{a}\{\ln\lvert\sin(ax+b)\rvert\} + c$
$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\dfrac{x}{a} + c$
$\dfrac{1}{a^2+x^2}$	$\dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + c$

Integration by Parts

$$\int u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\mathrm{d}x = uv - \int v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

$$\int_a^b u\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)\mathrm{d}x = [uv]_a^b - \int_a^b v\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)\mathrm{d}x$$

Integration by Parts

Find $\int x \sin x \, dx$.

We recognize the integrand as a product of the functions x and $\sin x$. Let $u = x$, $\frac{dv}{dx} = \sin x$. Then $\frac{du}{dx} = 1$, $v = -\cos x$. Using the integration by parts formula we get

$$\begin{aligned} \int x \sin x \, dx &= x(-\cos x) - \int (-\cos x) 1 \, dx \\ &= -x \cos x + \sin x + c \end{aligned}$$

Integration by Parts

Evaluate

$$\int_0^2 x e^x \, dx$$

We let $u = x$ and $\frac{dv}{dx} = e^x$. Then $\frac{du}{dx} = 1$ and $v = e^x$

Using the integration by parts formula for definite integrals we have

$$\begin{aligned}\int_0^2 x e^x \, dx &= [x e^x]_0^2 - \int_0^2 e^x \cdot 1 \, dx \\ &= 2e^2 - [e^x]_0^2 \\ &= 2e^2 - [e^2 - 1] \\ &= e^2 + 1\end{aligned}$$

Integration by Parts

Evaluate

$$\int_0^2 x^n e^x \, dx$$

for $n = 3, 4, 5$.

The integral may be evaluated by using integration by parts repeatedly. However, this is slow and cumbersome. Instead it is useful to develop a **reduction formula** as is now illustrated. Let $u = x^n$ and $\dfrac{dv}{dx} = e^x$. Then $\dfrac{du}{dx} = nx^{n-1}$ and $v = e^x$.

Using integration by parts we have

$$\int_0^2 x^n e^x \, dx = [x^n e^x]_0^2 - \int_0^2 nx^{n-1} e^x \, dx \;= 2^n e^2 - n \int_0^2 x^{n-1} e^x \, dx$$

Integration by Parts

Writing $I_n = \int_0^2 x^n e^x \, dx$ we see that $I_{n-1} = \int_0^2 x^{n-1} e^x \, dx$

Hence

$$I_n = 2^n e^2 - nI_{n-1}$$

Equation is called a **reduction formula**.

We have already evaluated I_1, that is $\int_0^2 xe^x \, dx$, and found $I_1 = e^2 + 1$

Using the reduction formula with $n = 2$ gives

$$\int_0^2 x^2 e^x \, dx = I_2 = 2^2 e^2 - 2I_1 = 4e^2 - 2(e^2 + 1) = 2e^2 - 2$$

Integration by Parts

With $n = 3$ the reduction formula yields

$$\int_0^2 x^3 e^x \, dx = I_3 = 2^3 e^2 - 3I_2 \ = 8e^2 - 3(2e^2 - 2) \ = 2e^2 + 6$$

With $n = 4$ we have

$$\int_0^2 x^4 e^x \, dx = I_4 = 2^4 e^2 - 4I_3 \ = 16e^2 - 4(2e^2 + 6) \ = 8e^2 - 24$$

With $n = 5$ we have

$$\int_0^2 x^5 e^x \, dx = I_5 = 2^5 e^2 - 5I_4 \ = 32e^2 - 5(8e^2 - 24) \ = 120 - 8e^2$$

Integration by Substitution

Find $\int (3x+1)^{2.7}\,\mathrm{d}x$.

Let $z = 3x + 1$, so that $\frac{\mathrm{d}z}{\mathrm{d}x} = 3$, that is $\mathrm{d}x = \frac{\mathrm{d}z}{3}$. Writing the integral in terms of z, it becomes

$$\int z^{2.7}\frac{1}{3}\,\mathrm{d}z = \frac{1}{3}\int z^{2.7}\mathrm{d}z = \frac{1}{3}\left(\frac{z^{3.7}}{3.7}\right) + c = \frac{1}{3}\frac{(3x+1)^{3.7}}{3.7} + c$$

Integration by Substitution

Evaluate $\int_2^3 t \sin(t^2)\, dt$.

Let $v = t^2$ so $\frac{dv}{dt} = 2t$, that is

$$dt = \frac{1}{2t}\, dv$$

When changing the integral from one in terms of t to one in terms of v, the limits must also be changed. When $t = 2$, $v = 4$; when $t = 3$, $v = 9$. Hence, the integral becomes

$$\int_4^9 \frac{\sin v}{2}\, dv = \frac{1}{2}[-\cos v]_4^9 = \frac{1}{2}[-\cos 9 + \cos 4] = 0.129$$

Integration by Substitution

Evaluate $\int_1^2 \sin t \; \cos^2 t \, \mathrm{d}t$.

Put $z = \cos t$ so that $\dfrac{\mathrm{d}z}{\mathrm{d}t} = -\sin t$, that is $\sin t \, \mathrm{d}t = -\mathrm{d}z$. When $t = 1$, $z = \cos 1$; when $t = 2$, $z = \cos 2$. Hence

$$\int_1^2 \sin t \; \cos^2 t \, \mathrm{d}t = -\int_{\cos 1}^{\cos 2} z^2 \, \mathrm{d}z = -\left[\frac{z^3}{3}\right]_{\cos 1}^{\cos 2}$$

$$= \frac{\cos^3 1 - \cos^3 2}{3} = 0.0766$$

Integration by Substitution

Find $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$.

Put $z = \tan x$. Then $\frac{dz}{dx} = \sec^2 x$, $dz = \frac{dx}{\cos^2 x}$. Hence,

$$\int \frac{e^{\tan x}}{\cos^2 x}\,dx = \int e^z\,dz = e^z + c = e^{\tan x} + c$$

Integration by Substitution

Find

(a) $\displaystyle\int \frac{4}{5x-7}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{t}{t^2+1}\,\mathrm{d}t$

(c) $\displaystyle\int \frac{\mathrm{e}^{t/2}}{\mathrm{e}^{t/2}+1}\,\mathrm{d}t$

$$\int \frac{\mathrm{d}f/\mathrm{d}x}{f}\,\mathrm{d}x = \ln|f| + c$$

The integrands are rewritten so that the numerator is the derivative of the denominator.

(a) $\displaystyle\int \frac{4}{5x-7}\,\mathrm{d}x = \frac{4}{5}\int \frac{5}{5x-7}\,\mathrm{d}x = \frac{4}{5}\ln|5x-7| + c$

(b) $\displaystyle\int \frac{t}{t^2+1}\,\mathrm{d}t = \frac{1}{2}\int \frac{2t}{t^2+1}\,\mathrm{d}t$

$\displaystyle = \frac{1}{2}\ln|t^2+1| + c$

(c) $\displaystyle\int \frac{\mathrm{e}^{t/2}}{\mathrm{e}^{t/2}+1}\,\mathrm{d}t = 2\int \frac{\frac{1}{2}\mathrm{e}^{t/2}}{\mathrm{e}^{t/2}+1}\,\mathrm{d}t = 2\ln|\mathrm{e}^{t/2}+1| + c$

Integration using Partial Fractions

Find

(a) $\displaystyle\int \frac{1}{x^3+x}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{13x-4}{6x^2-x-2}\,\mathrm{d}x$

(a) First express the integrand in partial fractions:

$$\frac{1}{x^3+x} = \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

Then,

$$1 = A(x^2+1) + x(Bx+C)$$

Equating the constant terms: $1 = A$ so that $A = 1$.

Equating the coefficients of x: $0 = C$ so that $C = 0$.

Equating the coefficients of x^2: $0 = A + B$ and hence $B = -1$.

Integration using Partial Fractions

Then,

$$\int \frac{1}{x^3+x}\,\mathrm{d}x = \int \frac{1}{x} - \frac{x}{x^2+1}\,\mathrm{d}x$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + c = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + c$$

(b) $$\int \frac{13x-4}{6x^2-x-2}\,\mathrm{d}x = \int \frac{13x-4}{(2x+1)(3x-2)}\,\mathrm{d}x$$

$$= \int \frac{3}{2x+1} + \frac{2}{3x-2}\,\mathrm{d}x \qquad \text{using partial fractions}$$

$$= \frac{3}{2}\int \frac{2}{2x+1}\,\mathrm{d}x + \frac{2}{3}\int \frac{3}{3x-2}\,\mathrm{d}x$$

$$= \frac{3}{2}\ln|2x+1| + \frac{2}{3}\ln|3x-2| + c$$

Integration using Partial Fractions

Evaluate $\int_0^1 \frac{4t^3 - 2t^2 + 3t - 1}{2t^2 + 1}\, dt$.

Using partial fractions we may write

$$\frac{4t^3 - 2t^2 + 3t - 1}{2t^2 + 1} = 2t - 1 + \frac{t}{2t^2 + 1}$$

Hence,

$$\int_0^1 \frac{4t^3 - 2t^2 + 3t - 1}{2t^2 + 1}\, dt = \int_0^1 2t - 1 + \frac{t}{2t^2 + 1}\, dt = \left[t^2 - t + \frac{1}{4}\ln|2t^2 + 1|\right]_0^1$$

$$= \left[1 - 1 + \frac{1}{4}\ln 3\right] - \left[0 - 0 + \frac{1}{4}\ln 1\right] = 0.275$$

Improper Integrals

There are two cases when evaluation of an integral needs special care:

(1) one, or both, of the limits of an integral are infinite;

(2) the integrand becomes infinite at one, or more, points of the interval of integration.

If either (1) or (2) is true the integral is called an **improper integral**.

Evaluation of improper integrals involves the use of limits.

Improper Integrals

Evaluate $\int_2^{\infty} \frac{1}{t^2}\,\mathrm{d}t$.

$$\int_2^{\infty} \frac{1}{t^2}\,\mathrm{d}t = \left[-\frac{1}{t}\right]_2^{\infty}$$

To evaluate $-\frac{1}{t}$ at the upper limit we consider $\lim_{t\to\infty} -\frac{1}{t}$. Clearly the limit is 0. Hence,

$$\int_2^{\infty} \frac{1}{t^2}\,\mathrm{d}t = 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

Improper Integrals

Evaluate $\int_{-\infty}^{1} e^{2x}\,dx$.

$$\int_{-\infty}^{1} e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_{-\infty}^{1}$$

We need to evaluate $\lim_{x\to-\infty} \frac{e^{2x}}{2}$. This limit is 0. So,

$$\int_{-\infty}^{1} e^{2x}\,dx = \left[\frac{e^{2x}}{2}\right]_{-\infty}^{1} = \frac{e^{2}}{2} - 0 = 3.69$$

Improper Integrals

Evaluate $\int_3^\infty \frac{2}{2t+1} - \frac{1}{t}\,dt.$

$$\int_3^\infty \frac{2}{2t+1} - \frac{1}{t}\,dt = [\ln|2t+1| - \ln|t|]_3^\infty$$

$$= \left[\ln\left|\frac{2t+1}{t}\right|\right]_3^\infty = \left[\ln\left|2+\frac{1}{t}\right|\right]_3^\infty$$

$$= \lim_{t\to\infty}\left[\ln\left(2+\frac{1}{t}\right)\right] - \ln\frac{7}{3}$$

$$= \ln 2 - \ln\frac{7}{3} = \ln\frac{6}{7} = -0.1542$$

Improper Integrals

Evaluate $\int_1^\infty \sin t \, dt$.

$$\int_1^\infty \sin t \, dt = [-\cos t]_1^\infty$$

Now $\lim_{t\to\infty}(-\cos t)$ does not exist, that is the function $\cos t$ does not approach a limit as $t \to \infty$, and so the integral cannot be evaluated.

We say the integral **diverges**.

Improper Integrals

Evaluate $\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$

The integrand, $\frac{1}{\sqrt{x}}$, becomes infinite when $x = 0$, which is in the interval of integration.

The point $x = 0$ is 'removed' from the interval. We consider $\int_b^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ where b is slightly greater than 0, and then let $b \to 0^+$. Now,

$$\int_b^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = [2\sqrt{x}]_b^1 = 2 - 2\sqrt{b}$$

Then,

$$\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{b\to 0^+} \int_b^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{b\to 0^+} (2 - 2\sqrt{b}) = 2$$

The improper integral exists and has value 2.

Improper Integrals

Determine whether the integral $\int_0^2 \frac{1}{x}\,\mathrm{d}x$ exists or not.

As in Example the integrand is not defined at $x = 0$, so we consider $\int_b^2 \frac{1}{x}\,\mathrm{d}x$ for

$b > 0$ and then let $b \to 0^+$.

$$\int_b^2 \frac{1}{x}\,\mathrm{d}x = [\ln|x|]_b^2 = \ln 2 - \ln b$$

So,

$$\lim_{b\to 0}\left(\int_b^2 \frac{1}{x}\,\mathrm{d}x\right) = \lim_{b\to 0}(\ln 2 - \ln b)$$

Since $\lim_{b\to 0} \ln b$ does not exist the integral diverges.

Improper Integrals

Evaluate $\int_{-1}^{2} \frac{1}{x}\,dx$ if possible.

We 'remove' the point $x = 0$ where the integrand becomes infinite and consider two integrals: $\int_{-1}^{b} \frac{1}{x}\,dx$ where b is slightly smaller than 0, and $\int_{c}^{2} \frac{1}{x}\,dx$ where c is slightly larger than 0. If these integrals exist as $b \to 0^-$ and $c \to 0^+$ then $\int_{-1}^{2} \frac{1}{x}\,dx$ converges. If either of the integrals fails to converge then $\int_{-1}^{2} \frac{1}{x}\,dx$ diverges. Now,

$$\int_{-1}^{b} \frac{1}{x}\,dx = \ln|b| - \ln|-1| = \ln|b|$$

$$\lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x}\,dx = \lim_{b\to 0^-} (\ln|b|)$$

This limit fails to exist and so $\int_{-1}^{2} \frac{1}{x}\,dx$ diverges.

Improper Integrals

Find

$$\int_0^{\infty} e^{-st} \sin t \, dt \qquad s > 0$$

Using integration by parts, with $u = e^{-st}$ and $\frac{dv}{dt} = \sin t$, we have

$$\int_0^{\infty} e^{-st} \sin t \, dt = \left[-e^{-st} \cos t\right]_0^{\infty} - s \int_0^{\infty} e^{-st} \cos t \, dt$$

Consider the first term on the r.h.s. We need to evaluate $\left[-e^{-st} \cos t\right]$ as $t \to \infty$ and when $t = 0$. Note that $-e^{-st} \cos t \to 0$ as $t \to \infty$ because we are given that s is positive. When $t = 0$, $-e^{-st} \cos t$ evaluates to -1, and so

$$\int_0^{\infty} e^{-st} \sin t \, dt = 1 - s \int_0^{\infty} e^{-st} \cos t \, dt$$

Improper Integrals

Integrating by parts for a second time yields

$$\int_0^\infty e^{-st}\sin t\,dt = 1 - s\left\{\left[e^{-st}\sin t\right]_0^\infty + s\int_0^\infty e^{-st}\sin t\,dt\right\}$$

$$= 1 - s^2\int_0^\infty e^{-st}\sin t\,dt$$

because $\left[e^{-st}\sin t\right]_0^\infty$ evaluates to zero at both limits. We have

$$\int_0^\infty e^{-st}\sin t\,dt + s^2\int_0^\infty e^{-st}\sin t\,dt = 1$$

$$(1+s^2)\int_0^\infty e^{-st}\sin t\,dt = 1$$

$$\int_0^\infty e^{-st}\sin t\,dt = \frac{1}{1+s^2}$$

Exercises

Use the product rule to differentiate the following functions:

(a) $y = \sin x \cos x$

(b) $y = \ln t \tan t$

(c) $y = (t^3 + 1)e^{2t}$

(d) $y = \sqrt{x}e^x$

(e) $y = e^t \sin t \cos t$

(f) $y = 3 \sinh 2t \cosh 3t$

(g) $y = (1 + \sin t) \tan t$

(h) $y = 4 \sinh(t + 1) \cosh(1 - t)$

Exercises

Use the quotient rule to find the derivatives of the following:

(a) $\dfrac{\cos x}{\sin x}$ (b) $\dfrac{\tan t}{\ln t}$

(c) $\dfrac{e^{2t}}{t^3 + 1}$ (d) $\dfrac{3x^2 + 2x - 9}{x^3 + 1}$

(e) $\dfrac{x^2 + x + 1}{1 + e^x}$ (f) $\dfrac{\sinh 2t}{\cosh 3t}$

(g) $\dfrac{1 + e^t}{1 + e^{2t}}$

Exercises

Use the chain rule to differentiate the following:

(a) $(t^3 + 1)^{100}$

(b) $\sin^3(3t + 2)$

(c) $\ln(x^2 + 1)$

(d) $(2t + 1)^{1/2}$

(e) $3\sqrt{\cos(2x - 1)}$

(f) $\dfrac{1}{t + 1}$

(g) $(at + b)^n$, a and b constants

Exercises

Differentiate each of the following functions:

(a) $y = 5 \sin x$

(b) $y = 5e^x \sin x$

(c) $y = 5e^{\sin x}$

(d) $y = \dfrac{5 \sin x}{e^{-x}}$

(e) $y = (t^3 + 4t)^{15}$

(f) $y = 7e^{-3t^2}$

(g) $y = \dfrac{\sin x}{4 \cos x + 1}$

Exercises

Find $\frac{dy}{dx}$, given

(a) $x = t^2 \qquad y = 1 + t^3$

(b) $x = \sin t \qquad y = e^t$

(c) $x = (1 + t)^3 \qquad y = 1 + t^3$

Exercises

Find $\frac{dy}{dx}$ given

(a) $2y^2 - 3x^3 = x + y$

(b) $\sqrt{y} + \sqrt{x} = x^2 + y^3$

(c) $\sqrt{2x + 3y} = 1 + e^x$

(d) $y = \dfrac{e^x\sqrt{1+x}}{x^2}$

(e) $2xy^4 = x^3 + 3xy^2$

(f) $\sin(x + y) = 1 + y$

(g) $\ln(x^2 + y^2) = 2x - 3y$

(h) $ye^{2y} = x^2e^{x/2}$

Exercises

Use logarithmic differentiation to find the derivatives of the following functions:

(a) $y = x^4 e^x$

(b) $y = \frac{1}{x} e^{-x}$

(c) $z = t^3 (1 + t)^9$

(d) $y = e^x \sin x$

(e) $y = x^7 \sin^4 x$

Exercises

Use logarithmic differentiation to find the derivatives of the following functions:

(a) $z = t^4(1-t)^6(2+t)^4$

(b) $y = \dfrac{(1+x^2)^3 e^{7x}}{(2+x)^6}$

(c) $x = (1+t)^3(2+t)^4(3+t)^5$

(d) $y = \dfrac{(\sin^4 t)(2-t^2)^4}{(1+e^t)^6}$

(e) $y = x^3 e^x \sin x$

Exercises

Evaluate the following definite integrals:

(a) $\int_0^1 x\cos 2x\,\mathrm{d}x$ (b) $\int_0^{\pi/2} x\sin 2x\,\mathrm{d}x$

(c) $\int_{-1}^1 te^{2t}\,\mathrm{d}t$ (d) $\int_1^3 t^2\ln t\,\mathrm{d}t$

(e) $\int_0^2 \frac{2x}{e^{2x}}\,\mathrm{d}x$

Use integration by parts twice to obtain a reduction formula for

$$I_n = \int_0^{\pi/2} t^n \sin t\,\mathrm{d}t$$

Hence find $\int_0^{\pi/2} t^3\sin t\,\mathrm{d}t$, $\int_0^{\pi/2} t^5\sin t\,\mathrm{d}t$ and $\int_0^{\pi/2} t^7\sin t\,\mathrm{d}t$.

Exercises

Use the given substitutions to find the following integrals:

(a) $\int (4x+1)^7 \, dx, \qquad z = 4x+1$

(b) $\int t^2 \sin(t^3+1) \, dt, \qquad z = t^3+1$

(c) $\int 4te^{-t^2} \, dt, \qquad z = t^2$

(d) $\int (1-z)^{1/3} \, dz, \qquad t = 1-z$

(e) $\int \cos t (\sin^5 t) \, dt, \qquad z = \sin t$

Evaluate the following definite integrals:

(a) $\int_1^2 (2t+3)^7 \, dt$

(b) $\int_0^{\pi/2} \sin 2t \cos^4 2t \, dt$

(c) $\int_0^1 3t^2 e^{t^3} \, dt$

(d) $\int_0^2 \sqrt{4+3x} \, dx$

(e) $\int_1^2 \frac{\sin\left(\sqrt{x}\right)}{\sqrt{x}} \, dx$

Exercises

By writing the integrand as its partial fractions find

(a) $\displaystyle\int \frac{x+3}{x^2+x}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{t-3}{t^2-1}\,\mathrm{d}t$

(c) $\displaystyle\int \frac{8x+10}{4x^2+8x+3}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{2t^2+3t+3}{2(t+1)}\,\mathrm{d}t$

(e) $\displaystyle\int \frac{2x^2+x+1}{x^3+x^2}\,\mathrm{d}x$

Evaluate the following integrals:

(a) $\displaystyle\int_1^3 \frac{5x+6}{2x^2+4x}\,\mathrm{d}x$

(b) $\displaystyle\int_0^1 \frac{3x+5}{(x+1)(x+2)}\,\mathrm{d}x$

(c) $\displaystyle\int_1^2 \frac{3-3x}{2x^2+6x}\,\mathrm{d}x$

(d) $\displaystyle\int_{-1}^0 \frac{4x+1}{2x^2+x-6}\,\mathrm{d}x$

(e) $\displaystyle\int_2^3 \frac{x^2+2x-1}{(x^2+1)(x-1)}\,\mathrm{d}x$

Exercises

Evaluate, if possible,

(a) $\displaystyle\int_0^\infty e^{-t}\,dt$

(b) $\displaystyle\int_0^\infty e^{-kt}\,dt$ $\qquad k$ is a constant, $k > 0$

(c) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$

(d) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$

(e) $\displaystyle\int_1^3 \frac{1}{x-2}\,dx$

Exercises

Evaluate the following integrals where possible:

(a) $\displaystyle\int_0^4 \frac{3}{x-2}\,\mathrm{d}x$ (b) $\displaystyle\int_0^3 \frac{1}{x-1} + \frac{1}{x-2}\,\mathrm{d}x$

(c) $\displaystyle\int_0^2 \frac{1}{x^2-1}\,\mathrm{d}x$ (d) $\displaystyle\int_0^\infty \sin 3t\,\mathrm{d}t$

(e) $\displaystyle\int_{-\infty}^3 xe^x\,\mathrm{d}x$

Find

$$\int_0^\infty e^{-st}\cos t\,\mathrm{d}t \qquad s > 0$$