Computational Methods

NUMERICAL SOLUTION OF NONLINEAR EQUATIONS

QUOTE

We cannot
Become
What we want
By remaining
What we are

-Max Depree

Nonlinear Equations in One Variable

The growth of a population can often be modeled over short periods of time by assuming that the population grows continuously with time at a rate proportional to the number present at that time. Suppose that $N(t)$ denotes the number in the population at time t and λ denotes the constant birth rate of the population. Then the population satisfies the differential equation

$$\frac{dN(t)}{dt} = \lambda N(t),$$

whose solution is $N(t) = N_0 e^{\lambda t}$, where N_0 denotes the initial population.

This exponential model is valid only when the population is isolated, with no immigration. If immigration is permitted at a constant rate v, then the differential equation becomes

$$\frac{dN(t)}{dt} = \lambda N(t) + v, \quad \text{whose solution is} \quad N(t) = N_0 e^{\lambda t} + \frac{v}{\lambda}(e^{\lambda t} - 1).$$

Nonlinear Equations in One Variable

Suppose a certain population contains $N(0) = 1{,}000{,}000$ individuals initially, that 435,000 individuals immigrate into the community in the first year, and that $N(1) = 1{,}564{,}000$ individuals are present at the end of one year. To determine the birth rate of this population, we need to find λ in the equation

$$1{,}564{,}000 = 1{,}000{,}000e^{\lambda} + \frac{435{,}000}{\lambda}(e^{\lambda} - 1).$$

It is not possible to solve explicitly for λ in this equation, but numerical methods can be used to approximate solutions of equations of this type to an arbitrarily high accuracy.

Nonlinear Equations in One Variable

Newton's (or the *Newton-Raphson*) **method** is one of the most powerful and well-known numerical methods for solving a root-finding problem.

Suppose that $f \in C^2[a,b]$. Let $p_0 \in [a,b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is "small." Consider the first Taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$.

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)),$$

where $\xi(p)$ lies between p and p_0. Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

Nonlinear Equations in One Variable

Newton's method is derived by assuming that since $|p-p_0|$ is small, the term involving $(p-p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0) f'(p_0).$$

Solving for p gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)} \equiv p_1.$$

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.$$

Nonlinear Equations in One Variable

The stopping-technique inequalities are, select a tolerance $\varepsilon > 0$,

and construct $p_1, \ldots p_N$ until

$$|p_N - p_{N-1}| < \varepsilon,$$

$$\frac{|p_N - p_{N-1}|}{|p_N|} < \varepsilon, \quad p_N \neq 0,$$

or

$$|f(p_N)| < \varepsilon.$$

Nonlinear Equations in One Variable

Example

Consider the function $f(x) = \cos x - x = 0$. Approximate a root of f using Newton's Method

Solution

To apply Newton's method to this problem we need $f'(x) = -\sin x - 1$. Starting with $p_0 = \pi/4$, we generate the sequence defined, for $n \geq 1$, by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})} = p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}.$$

This gives the approximations in Table .

Newton's Method

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

An excellent approximation is obtained with $n = 3$. Because of the agreement of p_3 and p_4 we could reasonably expect this result to be accurate to the places listed.

Exercises

1. Let $f(x) = x^2 - 6$ and $p_0 = 1$. Use Newton's method to find p_2.
2. Let $f(x) = -x^3 - \cos x$ and $p_0 = -1$. Use Newton's method to find p_2. Could $p_0 = 0$ be used?
3. Use Newton's method to find solutions accurate to within 10^{-4} for the following problems.
   a. $x^3 - 2x^2 - 5 = 0, \quad [1, 4]$
   b. $x^3 + 3x^2 - 1 = 0, \quad [-3, -2]$
   c. $x - \cos x = 0, \quad [0, \pi/2]$
   d. $x - 0.8 - 0.2 \sin x = 0, \quad [0, \pi/2]$
4. Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.
   a. $e^x + 2^{-x} + 2\cos x - 6 = 0 \quad$ for $1 \le x \le 2$
   b. $\ln(x - 1) + \cos(x - 1) = 0 \quad$ for $1.3 \le x \le 2$