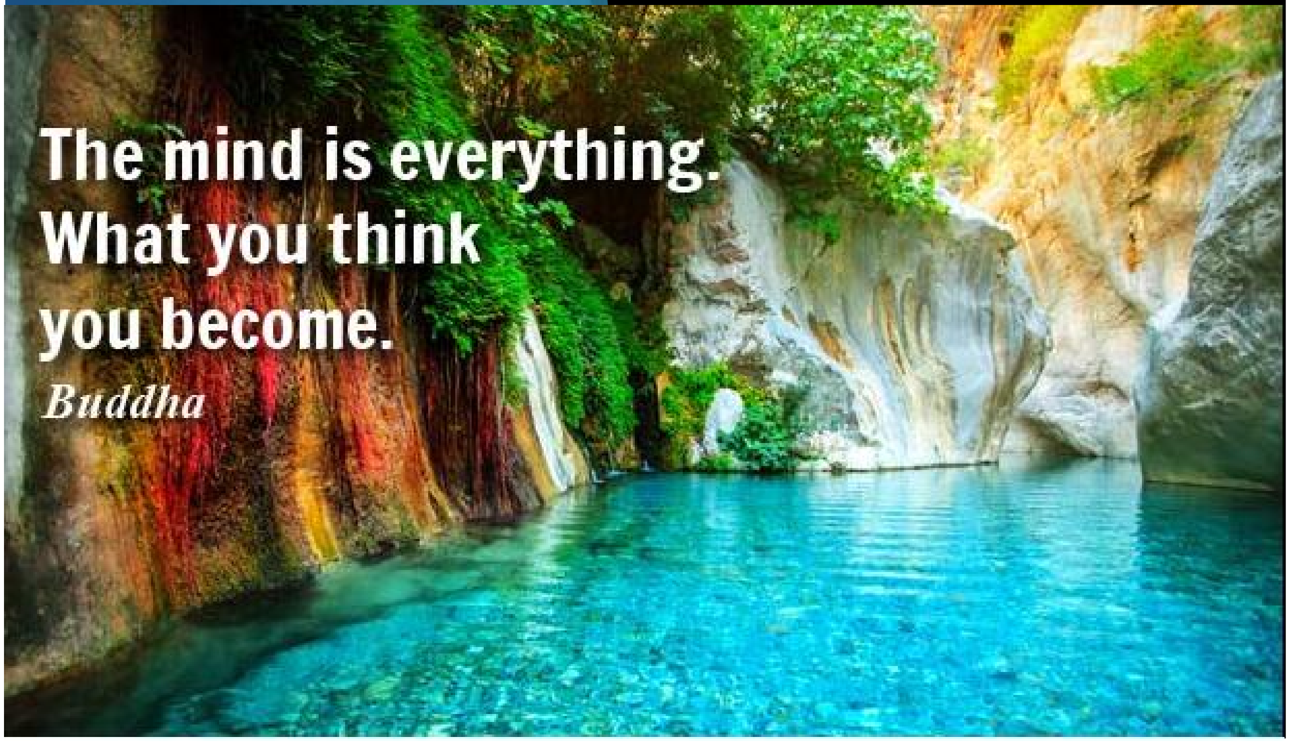
# Computational Methods

RELATIONS AND MAPPINGS

# QUOTE



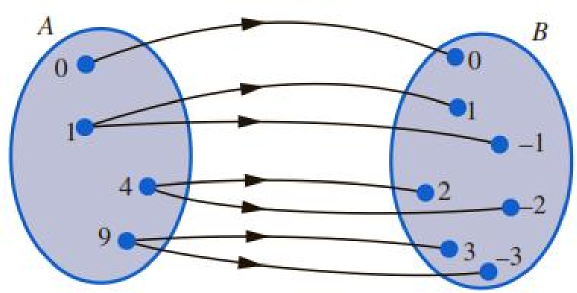
# Relations and Mappings

## Sets and functions

If we are given two sets, $A$ and $B$, a useful exercise is to examine relationships, given by rules, between the elements of $A$ and the elements of $B$. For example, if $A = \{0, 1, 4, 9\}$ and $B = \{-3, -2, -1, 0, 1, 2, 3\}$ then each element of $B$ is plus or minus the square root of some element of $A$.

$r : A \to B$

'take plus or minus the square root of'



# Relations and Mappings



The rule, which, when given an element of $A$, produces an element of $B$, is called a **relation**. If the rule of the relation is given the symbol $r$ we write

$$r : A \to B$$

and say 'the relation $r$ **maps** elements of the set $A$ to elements of the set $B$'. For the example above, we can write $r : 1 \to \pm 1$, $r : 4 \to \pm 2$, and generally $r : x \to \pm\sqrt{x}$.

The set from which we choose our input is called the **domain**; the set to which we map is called the **co-domain**; the subset of the co-domain actually used is called the **range**.

A relation $r$ maps elements of a set $D$, called the domain, to one or more elements of a set $C$, called the co-domain. We write

$$r : D \to C$$

## Relations and Mappings



If $D = \{0, 1, 2, 3, 4, 5\}$ and $E = \{1, 4, 7, 10, 13, 16, 19, 22\}$ and the relation with symbol $s$ is defined by $s : D \rightarrow E$, $s : m \rightarrow 3m + 1$, identify the domain and co-domain of $s$. Draw a mapping diagram to illustrate the relation. What is the range of $s$?

### Solution

The domain of $s$ is the set of values from which we choose our input, that is

$$D = \{0, 1, 2, 3, 4, 5\}.$$

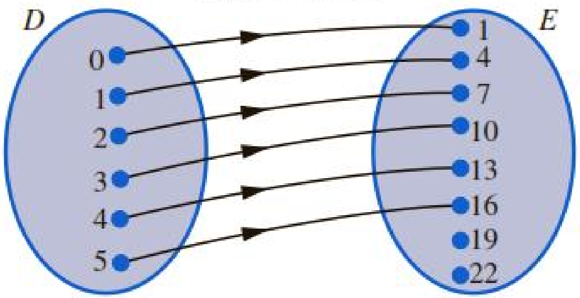
The co-domain of $s$ is the set to which we map, that is

$$E = \{1, 4, 7, 10, 13, 16, 19, 22\}.$$

The rule $s : m \rightarrow 3m+1$ enables us to draw the mapping diagram.

# Relations and Mappings

$$s : D \to E$$
$$s : m \to 3m + 1$$




For example, $s : 3 \to 10$ and so on. The range of $s$ is the subset of $E$ actually used, in this case $\{1, 4, 7, 10, 13, 16\}$.

We note that not all the elements of the co-domain are actually used.

# Relations and Mappings

In fact, a function is a very special form of a relation. Let us recall the definition of a function:

'A function is a rule which when given an input produces a single output'.

If we study the two relations $r$ and $s$, we note that when relation $r$ received input, it could produce two outputs. On the mapping diagram this shows up as two arrows leaving some elements in $A$. When relation $s$ received an input, it produced a single output. This shows up as a single arrow leaving each element in $D$. Hence the relation $r$ is not a function, whereas the relation $s$ is. This leads to the following more rigorous definition of a function.

A function $f$ is a relation which maps each element of a set $D$, called the domain, to a single element of a set $C$, called the co-domain. We write

$$f : D \to C$$

# Relations and Mappings

If $M = \{\text{off, on}\}$, $N = \{0, 1\}$ and we define a relation $r$ by

$$r : M \to N$$

$$r : \text{off} \to 0 \qquad r : \text{on} \to 1$$

then the relation $r$ is a function since each element in $M$ is mapped to a single element in $N$.

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If $P = \{0, 1\}$ and $Q = \{\text{high}\}$ and we define a relation $r$ by

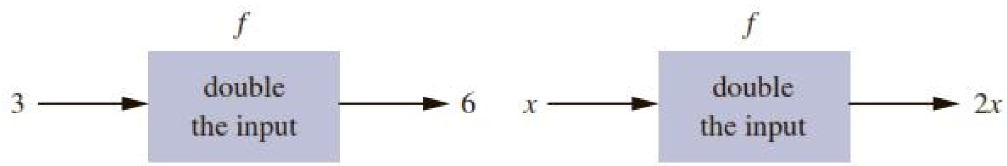
$$r : P \to Q$$

$$r : 1 \to \text{high}$$

then $r$ is not a function since each element in $P$ is not mapped to an element in $Q$.

# Concepts of Functions

If the doubling function has the symbol $f$ we write $f : x \rightarrow 2x$ or more compactly, $f(x) = 2x$.




The last form is often written simply as $f = 2x$. If $f(x)$ is a function of $x$, then the value of the function when $x = 3$, for example, is written as $f(x = 3)$ or simply as $f(3)$.

# Concepts of Functions

Given $f(x) = 2x + 1$ find

(a) $f(3)$

(b) $f(0)$

(c) $f(-1)$

(d) $f(\alpha)$

(e) $f(2\alpha)$

(f) $f(t)$

(g) $f(t+1)$

(a) $f(3) = 2(3) + 1 = 7$

(b) $f(0) = 2(0) + 1 = 1$

(c) $f(-1) = 2(-1) + 1 = -1$

(d) $f(\alpha)$ is the value of $f(x)$ when $x$ has a value of $\alpha$, hence $f(\alpha) = 2\alpha + 1$

(e) $f(2\alpha) = 2(2\alpha) + 1 = 4\alpha + 1$

(f) $f(t) = 2t + 1$

(g) $f(t+1) = 2(t+1) + 1 = 2t + 3$

# Concepts of Functions

## Argument of a function

The input to a function is often called the **argument**.

Given $f(x) = \dfrac{x}{5}$, write down

(a) $f(5x)$

(b) $f(-x)$

(c) $f(x+2)$

(d) $f(x^2)$

(a) $f(5x) = \dfrac{5x}{5} = x$

(b) $f(-x) = -\dfrac{x}{5}$

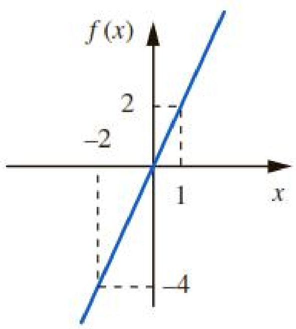
(c) $f(x+2) = \dfrac{x+2}{5}$

(d) $f(x^2) = \dfrac{x^2}{5}$

# Concepts of Functions

### Graph of a function

A function may be represented in graphical form. The function $f(x) = 2x$ is shown




Note that the function values are plotted vertically, the $y$ axis, and the $x$ values horizontally, so that we often write $y = f(x) = 2x$

# Concepts of Functions

Since $x$ and $y$ can have a number of possible values, they are called **variables**: $x$ is the **independent variable** and $y$ is the **dependent variable**. Knowing a value of the independent variable, $x$, allows us to calculate the corresponding value of the dependent variable, $y$. To show this dependence we often write $y(x)$.

# Concepts of Functions

The set of values that $x$ is allowed to take is called the **domain** of the function. A domain is often an interval on the $x$ axis. For example, if

$$f(x) = 3x + 1 \qquad -5 \leqslant x \leqslant 10$$

the domain of the function, $f$, is the closed interval $[-5, 10]$. If the domain of a function is not explicitly given it is taken to be the largest set possible. For example,

$$g(x) = x^2 - 4$$

has a domain of $(-\infty, \infty)$ since $g$ is defined for every value of $x$ and the domain has not been given otherwise. The set of values that the function takes on is called the **range**. The range of $f(x)$ is $[-14, 31]$; the range of $g(x)$ is $[-4, \infty)$.

# Concepts of Functions

## One-to-many

Some rules relating input to output are not functions. Consider the rule: 'take plus or minus the square root of the input', that is

$$x \to \pm\sqrt{x}$$

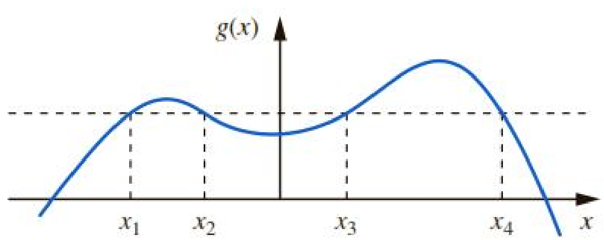
Now, for example, if 4 is the input, the output is $\pm\sqrt{4}$ which can be 2 or $-2$. Thus a single input has produced more than one output. The rule is said to be **one-to-many**, meaning that one input has produced many outputs. Rules with this property are not functions. For a rule to be a function there must be a single output for any given input.

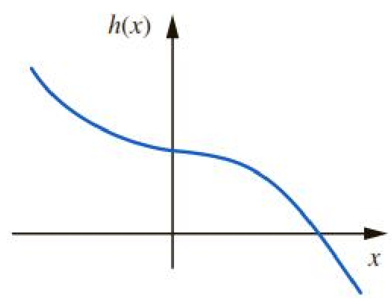
# Concepts of Functions

## Many-to-one and one-to-one functions

The **many-to-one** function means that many inputs produce the same output. A many-to-one function can be recognized from its graph. If a horizontal line intersects the graph in more than one place, the function is many-to-one.

A function is **one-to-one** if different inputs always produce different outputs. A horizontal line will intersect the graph of a one-to-one function in only one place.

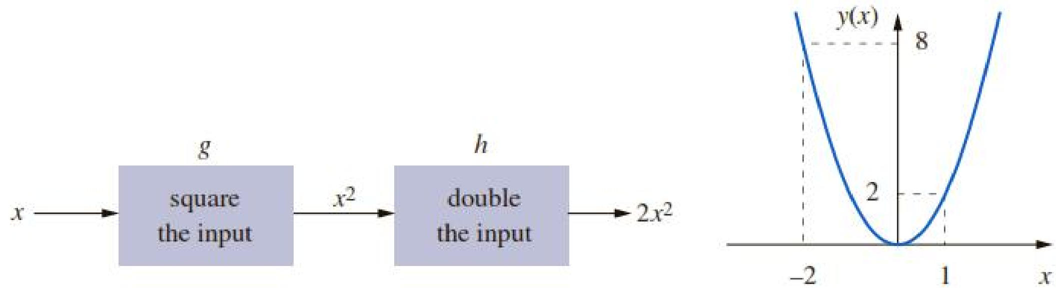


# Concepts of Functions

## Composition of functions

Consider the function $y(x) = 2x^2$. We can think of $y(x)$ as being composed of two functions. One function is described by the rule: 'square the input', while the other function is described by the rule: 'double the input'.




Mathematically, if $h(x) = 2x$ and $g(x) = x^2$ then $y(x) = 2x^2 = 2(g(x)) = h(g(x))$

The form $h(g(x))$ is known as a **composition** of the functions $h$ and $g$. Note that the composition $h(g(x))$ is different from $g(h(x))$.

# Concepts of Functions

If $f(t) = 2t + 3$ and $g(t) = \dfrac{t+1}{2}$ write expressions for the compositions

(a) $f(g(t))$

(b) $g(f(t))$

(a) $f(g(t)) = f\left(\dfrac{t+1}{2}\right)$

The rule describing the function $f$ is: ‘double the input and then add 3’. Hence,

$$f\left(\frac{t+1}{2}\right) = 2\left(\frac{t+1}{2}\right) + 3 = t + 4$$

So

$$f(g(t)) = t + 4$$

# Concepts of Functions



(b) $g(f(t)) = g(2t + 3)$

The rule for $g$ is: 'add 1 to the input and then divide everything by 2'. So,

$$g(2t + 3) = \frac{2t + 3 + 1}{2} = t + 2$$

Hence

$$g(f(t)) = t + 2$$

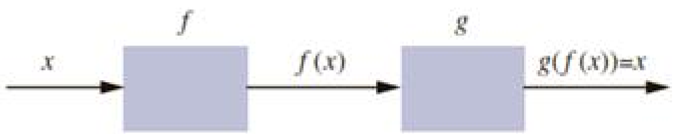
Clearly $f(g(t)) \neq g(f(t))$.

# Concepts of Functions

## Inverse of a function

Consider a function $f(x)$. It can be thought of as accepting an input $x$, and producing an output $f(x)$. Suppose now that this output becomes the input to the function $g(x)$, and the output from $g(x)$ is $x$, that is $g(f(x)) = x$

We can think of $g(x)$ as undoing the work of $f(x)$.




Then $g(x)$ is the **inverse** of $f$, and is written as $f^{-1}(x)$. Since $f^{-1}(x)$ undoes the work of $f(x)$ we have $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

Clearly not all functions have an inverse function. In fact, only one-to-one functions have an inverse function.

# Concepts of Functions

If $f(x) = 5x$ verify that the inverse of $f$ is given by $f^{-1}(x) = \frac{x}{5}$.

The function $f$ receives an input of $x$, and produces an output of $5x$. Hence when the inverse function, $f^{-1}$, receives an input of $5x$, it produces an output of $x$, that is

$$f^{-1}(5x) = x$$

We introduce a new variable, $z$, given by $z = 5x$ so $x = \frac{z}{5}$

Then

$$f^{-1}(z) = x = \frac{z}{5}$$

Writing $f^{-1}$ with $x$ as the argument gives

$$f^{-1}(x) = \frac{x}{5}$$

# Concepts of Functions

If $f(x) = 2x + 1$, find $f^{-1}(x)$.

The function $f$ receives an input of $x$ and produces an output of $2x + 1$. So when the inverse function, $f^{-1}$, receives an input of $2x + 1$ it produces an output of $x$, that is

$$f^{-1}(2x + 1) = x$$

We introduce a new variable, $z$, defined by $z = 2x + 1$. Rearranging gives $x = \dfrac{z-1}{2}$

So $f^{-1}(z) = x = \dfrac{z-1}{2}$

Writing $f^{-1}$ with $x$ as the argument gives

$$f^{-1}(x) = \frac{x-1}{2}$$

# Concepts of Functions

Given $g(x) = \dfrac{x-1}{2}$ find the inverse of $g$.

We know $g(x) = \dfrac{x-1}{2}$, and so $g^{-1}\left(\dfrac{x-1}{2}\right) = x$. Let $y = \dfrac{x-1}{2}$ so that

$$g^{-1}(y) = x$$

But,

$$x = 2y + 1$$

and so

$$g^{-1}(y) = 2y + 1$$

Using the same independent variable as for the function $g$, we obtain

$$g^{-1}(x) = 2x + 1$$

# Exponential Functions

An **exponent** is another name for a power or index. Expressions involving exponents are called **exponential expressions**, for example $3^4$, $a^b$, and $m^n$. In the exponential expression $a^x$, $a$ is called the **base**; $x$ is the exponent. Exponential expressions can be simplified and manipulated using the laws of indices. These laws are summarized here.

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad a^0 = 1 \qquad a^{-m} = \frac{1}{a^m} \qquad (a^m)^n = a^{mn}$$

An **exponential function**, $f(x)$, has the form

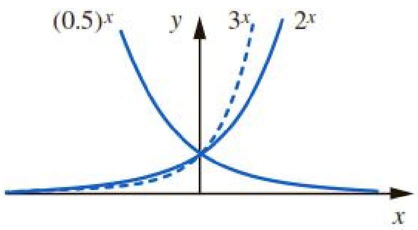
$$f(x) = a^x$$

where $a$ is a positive constant called the base.

# Exponential Functions

Values of $a^x$ for $a = 0.5$, 2 and 3.

| $x$ | $0.5^x$ | $2^x$ | $3^x$ |
|---|---|---|---|
| −3 | 8 | 0.125 | 0.037 |
| −2 | 4 | 0.25 | 0.111 |
| −1 | 2 | 0.5 | 0.333 |
| 0 | 1 | 1 | 1 |
| 1 | 0.5 | 2 | 3 |
| 2 | 0.25 | 4 | 9 |
| 3 | 0.125 | 8 | 27 |




Some typical exponential functions.

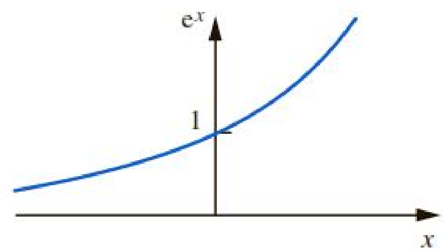
The most widely used exponential function, commonly called **the** exponential function, is

$$f(x) = \mathrm{e}^x$$

where e is an irrational constant ($\mathrm{e} = 2.718\,281\,828\ldots$) commonly called the **exponential constant**.

# Exponential Functions

| $x$ | $e^x$ |
|---|---|
| −3 | 0.050 |
| −2 | 0.135 |
| −1 | 0.368 |
| 0 | 1 |
| 1 | 2.718 |
| 2 | 7.389 |
| 3 | 20.086 |

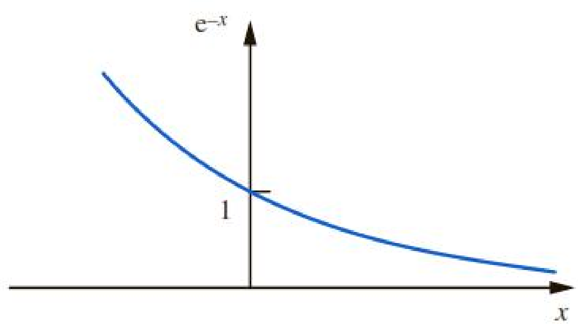



Graph of $y = e^x$ showing exponential growth.

As $x$ increases positively, $e^x$ increases very rapidly; that is, as $x \to \infty$, $e^x \to \infty$. This situation is known as **exponential growth**. As $x$ increases negatively, $e^x$ approaches zero; that is, as $x \to -\infty$, $e^x \to 0$. Thus $y = 0$ is an asymptote. Note that the exponential function is never negative.

# Exponential Functions

| $x$ | $e^{-x}$ |
|---|---|
| −3 | 20.086 |
| −2 | 7.389 |
| −1 | 2.718 |
| 0 | 1 |
| 1 | 0.368 |
| 2 | 0.135 |
| 3 | 0.050 |




Graph of $y = e^{-x}$ showing exponential decay.

As $x$ increases positively, $e^{-x}$ decreases to zero; that is, as $x \to \infty$, $e^{-x} \to 0$.

This is known as **exponential decay**.

# Logarithmic Functions

## Logarithms

The equation $16 = 2^4$ may be expressed in an alternative form using **logarithms**. In logarithmic form we write

$$\log_2 16 = 4$$

and say 'log to the base 2 of 16 equals 4'. Hence logarithms are nothing other than powers. The logarithmic form is illustrated by more examples:

$$125 = 5^3 \quad \text{so} \quad \log_5 125 = 3$$
$$64 = 8^2 \quad \text{so} \quad \log_8 64 = 2$$
$$16 = 4^2 \quad \text{so} \quad \log_4 16 = 2$$
$$1000 = 10^3 \quad \text{so} \quad \log_{10} 1000 = 3$$

In general,

$$\text{if } c = a^b, \text{ then } b = \log_a c$$

# Logarithmic Functions

In practice, most logarithms use base 10 or base e. Logarithms using base e are called **natural logarithms**.

$\text{Log}_{10}x$ and $\log_e x$ are usually abbreviated to $\log x$ and $\ln x$, respectively.

Focusing on base 10 we see that

$$\text{if } y = 10^x \qquad \text{then} \qquad x = \log y$$

Equivalently,

$$\text{if } x = \log y \qquad \text{then} \qquad y = 10^x$$

Using base e we see that

$$\text{if } y = e^x \qquad \text{then} \qquad x = \ln y$$

Equivalently,

$$\text{if } x = \ln y \qquad \text{then} \qquad y = e^x$$

# Logarithmic Functions

Solve the equations

(a) $16 = 10^x$ (b) $30 = e^x$ (c) $\log x = 1.5$ (d) $\ln x = 0.75$

(a)
$$\begin{aligned} 16 &= 10^x \\ \log 16 &= x \\ x &= 1.204 \end{aligned}$$

(b)
$$\begin{aligned} 30 &= e^x \\ \ln 30 &= x \\ x &= 3.401 \end{aligned}$$

(c)
$$\begin{aligned} \log x &= 1.5 \\ x &= 10^{1.5} \\ &= 31.623 \end{aligned}$$

(d)
$$\begin{aligned} \ln x &= 0.75 \\ x &= e^{0.75} \\ &= 2.117 \end{aligned}$$

# Logarithmic Functions

Solve the equations

(a) $50 = 9(10^{2x})$ (b) $3e^{-(2x+1)} = 10$ (c) $\log(x^2 - 1) = 2$ (d) $3\ln(4x + 7) = 12$

(a)
$$\begin{aligned} 50 &= 9(10^{2x}) \\ 10^{2x} &= \frac{50}{9} \\ 2x &= \log\frac{50}{9} \\ x &= \frac{1}{2}\log\frac{50}{9} = 0.372 \end{aligned}$$

(b)
$$\begin{aligned} 3e^{-(2x+1)} &= 10 \\ e^{-(2x+1)} &= \frac{10}{3} \\ -(2x+1) &= \ln\frac{10}{3} \\ 2x &= -\ln\frac{10}{3} - 1 \\ 2x &= -2.204 \\ x &= -1.102 \end{aligned}$$

(c)
$$\begin{aligned} \log(x^2 - 1) &= 2 \\ x^2 - 1 &= 10^2 = 100 \\ x^2 &= 101 \\ x &= \pm 10.050 \end{aligned}$$

(d)
$$\begin{aligned} 3\ln(4x + 7) &= 12 \\ 4x + 7 &= e^4 \\ 4x &= e^4 - 7 \\ x &= \frac{e^4 - 7}{4} = 11.900 \end{aligned}$$

# Logarithmic Functions

Logarithmic expressions can be manipulated using the laws of logarithms. These laws are identical for any base, but it is essential when applying the laws that bases are not mixed.

$$\log_a A + \log_a B = \log_a (AB)$$
$$\log_a A - \log_a B = \log_a \left(\frac{A}{B}\right)$$
$$n \log_a A = \log_a (A^n)$$
$$\log_a a = 1$$

We sometimes need to change from one base to another. This can be achieved using the following rule.

$$\log_a X = \frac{\log_b X}{\log_b a}$$

In particular,

$$\log_2 X = \frac{\log_{10} X}{\log_{10} 2} = \frac{\log_{10} X}{0.3010}$$

# Logarithmic Functions

Simplify

(a) $\log x + \log x^3$

(b) $3\log x + \log x^2$

(c) $5\ln x + \ln\left(\frac{1}{x}\right)$

(d) $\log(xy) + \log x - 2\log y$

(e) $\ln(2x^3) - \ln\left(\frac{4}{x^2}\right) + \frac{1}{3}\ln 27$

(a) Using the laws of logarithms we find

$\log x + \log x^3 = \log x^4$

# Logarithmic Functions

(b) $3\log x + \log x^2 = \log x^3 + \log x^2 = \log x^5$

(c) $5\ln x + \ln\left(\frac{1}{x}\right) = \ln x^5 + \ln\left(\frac{1}{x}\right) = \ln\left(\frac{x^5}{x}\right) = \ln x^4$

(d) $\log xy + \log x - 2\log y = \log xy + \log x - \log y^2$

$$= \log\left(\frac{xyx}{y^2}\right) = \log\left(\frac{x^2}{y}\right)$$

(e) $\ln(2x^3) - \ln\left(\frac{4}{x^2}\right) + \frac{\ln 27}{3} = \ln\left(\frac{2x^3}{4/x^2}\right) + \ln 27^{1/3}$

$$= \ln\left(\frac{2x^3x^2}{4}\right) + \ln 3$$

$$= \ln\left(\frac{x^5}{2}\right) + \ln 3 = \ln\left(\frac{3x^5}{2}\right)$$

# Logarithmic Functions

Find $\log_2 14$.

Using the formula for change of base we have

$$\log_2 14 = \frac{\log_{10} 14}{\log_{10} 2} = \frac{1.146}{0.301} = 3.807$$

# Logarithmic Functions

### Logarithm functions

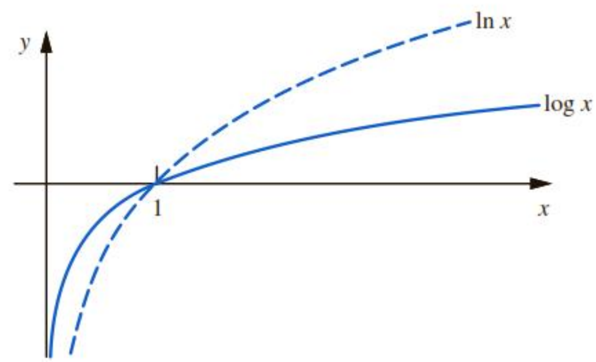
The **logarithm functions** are defined by

$$f(x) = \log_a x \qquad x > 0$$

where $a$ is a positive constant called the base.

Some values for logarithm functions $\log x$ and $\ln x$.

| $x$ | $\log x$ | $\ln x$ |
|---|---|---|
| 0.1 | −1 | −2.303 |
| 0.5 | −0.301 | −0.693 |
| 1 | 0 | 0 |
| 2 | 0.301 | 0.693 |
| 5 | 0.699 | 1.609 |
| 10 | 1 | 2.303 |
| 50 | 1.699 | 3.912 |



Graphs of $\ln x$ and $\log x$.

# Logarithmic Functions

The domain of both of these functions is $(0, \infty)$ and their ranges are $(-\infty, \infty)$.

We observe from the graphs that these functions are one-to-one. It is important to stress that the logarithm functions, $\log_a x$, are only defined for positive values of $x$. The following properties should be noted:

$$\left.\begin{array}{r}\log x \to \infty \\ \ln x \to \infty\end{array}\right\} \text{ as } x \to \infty \qquad \left.\begin{array}{r}\log x \to -\infty \\ \ln x \to -\infty\end{array}\right\} \text{ as } x \to 0$$

$$\log 1 = \ln 1 = 0 \qquad \log 10 = 1 \quad \ln e = 1$$

# Logarithmic Functions

The inverse of the exponential function, $f(x) = a^x$, is the logarithm function, that is $f^{-1}(x) = \log_a x$.

The inverse of the logarithm function, $f(x) = \log_a x$, is the exponential function, that is $f^{-1}(x) = a^x$.

In particular:

If $f(x) = \mathrm{e}^x$, then $f^{-1}(x) = \ln x$.
If $f(x) = \ln x$, then $f^{-1}(x) = \mathrm{e}^x$.
If $f(x) = 10^x$, then $f^{-1}(x) = \log x$.
If $f(x) = \log x$, then $f^{-1}(x) = 10^x$.

# Logarithmic Functions

### Use of log–log and log–linear scales

Suppose we wish to plot

$$y(x) = x^6 \qquad 1 \leqslant x \leqslant 10$$

This may appear a straightforward exercise but consider the variation in the $x$ and $y$ values. As $x$ varies from 1 to 10, then $y$ varies from 1 to 1 000 000.

Several of these points would not be discernible on a graph and so information would be lost. This can be overcome by using a **log scale** which accommodates the large variation in $y$. Thus $\log y$ is plotted against $x$, rather than $y$ against $x$. Note that in this example

$$\log y = \log x^6 = 6 \log x$$

so as $x$ varies from 1 to 10, $\log y$ varies from 0 to 6. A plot in which one scale is logarithmic and the other is linear is known as a **log–linear** graph.

# Logarithmic Functions



Consider $y = 7^x$ for $-3 \leqslant x \leqslant 3$. Plot a log–linear graph of this function.

We have

$$y = 7^x$$

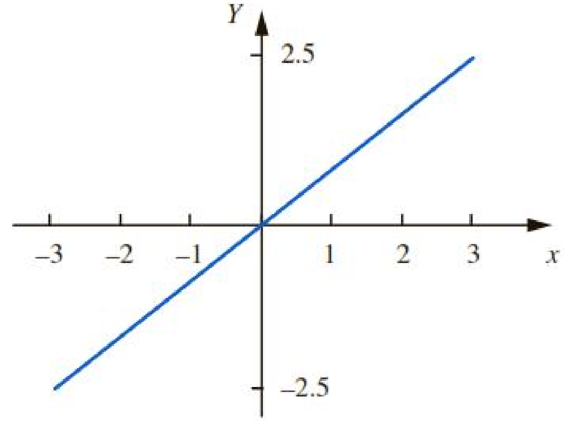
and so

$$\log y = \log(7^x) = x \log 7 = 0.8451x$$

Putting $Y = \log y$ we have $Y = 0.8451x$ which is the equation of a straight line passing through the origin with gradient $\log 7$. Hence when $\log y$ is plotted against $x$ a straight line graph is produced.

# Logarithmic Functions

| $x$ | $y$ | $Y = \log y$ |
|---|---|---|
| $-3$ | 0.003 | $-2.54$ |
| $-2$ | 0.020 | $-1.69$ |
| $-1$ | 0.143 | $-0.85$ |
| 0 | 1 | 0 |
| 1 | 7 | 0.85 |
| 2 | 49 | 1.69 |
| 3 | 343 | 2.54 |




A log–linear plot of $y = 7^x$ produces a straight line graph.

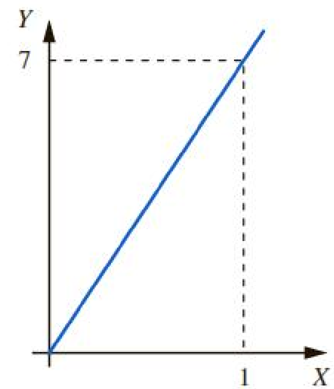
# Logarithmic Functions

Consider $y = x^7$ for $1 \leqslant x \leqslant 10$. Plot a log–log graph of this function.

We have $y = x^7$ and so $\log y = \log(x^7) = 7 \log x$

We plot $\log y$ against $\log x$ in Figure for a log–log plot. Putting $Y = \log y$ and $X = \log x$ we have $Y = 7X$ which is a straight line through the origin with gradient 7.

| $x$ | $y$ | $X = \log x$ | $Y = \log y$ |
|---|---|---|---|
| 1 | 1 | 0 | 0 |
| 2 | 128 | 0.301 | 2.107 |
| 3 | 2 187 | 0.477 | 3.340 |
| 4 | 16 384 | 0.602 | 4.214 |
| 5 | 78 125 | 0.699 | 4.893 |
| 6 | 279 936 | 0.778 | 5.447 |
| 7 | 823 543 | 0.845 | 5.916 |
| 8 | 2 097 152 | 0.903 | 6.322 |
| 9 | 4 782 969 | 0.954 | 6.680 |
| 10 | 10 000 000 | 1 | 7 |




A log–log plot of $y = x^7$ produces a straight line graph.

# Hyperbolic Functions

The **hyperbolic functions** are $y(x) = \cosh x$, $y(x) = \sinh x$, $y(x) = \tanh x$, $y(x) = \operatorname{sech} x$, $y(x) = \operatorname{cosech} x$ and $y(x) = \coth x$. Cosh is a contracted form of 'hyperbolic cosine', sinh of 'hyperbolic sine' and so on. We define $\cosh x$ and $\sinh x$ by

$$y(x) = \cosh x = \frac{e^x + e^{-x}}{2} \qquad y(x) = \sinh x = \frac{e^x - e^{-x}}{2}$$

Note:

$$\cosh(-x) = \frac{e^{-x} + e^x}{2} = \cosh x$$
$$\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\sinh x$$

# Hyperbolic Functions

$$y(x) = \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$y(x) = \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$y(x) = \operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$y(x) = \coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Values of the hyperbolic functions for various $x$ values can be found from a scientific calculator. Usually a Hyp button followed by a Sin, Cos or Tan button is used.

# Hyperbolic Functions

Evaluate

(a) $\cosh 3$ (b) $\sinh(-2)$
(c) $\tanh 1.6$ (d) $\operatorname{sech}(-2.5)$
(e) $\coth 1$ (f) $\operatorname{cosech}(-1)$

(a) $\cosh 3 = 10.07$
(b) $\sinh(-2) = -3.627$
(c) $\tanh(1.6) = 0.9217$
(d) $\operatorname{sech}(-2.5) = 1/\cosh(-2.5) = 0.1631$
(e) $\coth 1 = 1/\tanh 1 = 1.313$
(f) $\operatorname{cosech}(-1) = 1/\sinh(-1) = -0.8509$

# Hyperbolic Functions

### Hyperbolic identities

Several identities involving hyperbolic functions exist.

$$\cosh^2 x - \sinh^2 x = 1$$
$$1 - \tanh^2 x = \text{sech}^2 x$$
$$\coth^2 x - 1 = \text{cosech}^2 x$$
$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

Note also that

$$e^x = \cosh x + \sinh x \qquad e^{-x} = \cosh x - \sinh x$$

# Hyperbolic Functions

Express

(a) $3e^x - 2e^{-x}$ in terms of $\cosh x$ and $\sinh x$,

(b) $2\sinh x + \cosh x$ in terms of $e^x$ and $e^{-x}$.

(a) $3e^x - 2e^{-x} = 3(\cosh x + \sinh x) - 2(\cosh x - \sinh x) = \cosh x + 5\sinh x$.

(b) $2\sinh x + \cosh x = e^x - e^{-x} + \dfrac{e^x + e^{-x}}{2} = \dfrac{3e^x - e^{-x}}{2}$.

# Trigonometric Functions

## DEGREES AND RADIANS

Angles can be measured in units of either degrees or radians. A complete revolution is defined as $360^\circ$ or $2\pi$ radians. It is easy to use this fact to convert between the two measures. We have $360^\circ = 2\pi$ radians

$$1^\circ = \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians} \qquad\qquad 1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57.3^\circ$$

Note that

$$\frac{\pi}{2} \text{ radians} = 90^\circ \qquad \pi \text{ radians} = 180^\circ \qquad \frac{3\pi}{2} \text{ radians} = 270^\circ$$

# Trigonometric Functions

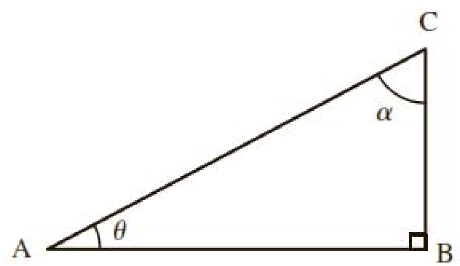
## THE TRIGONOMETRIC RATIOS

Consider the angle $\theta$ in the right-angled triangle ABC, as shown in Figure . We define the trigonometric ratios **sine**, **cosine** and **tangent** as follows:

$$\sin\theta = \frac{\text{side opposite to angle}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\cos\theta = \frac{\text{side adjacent to angle}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\tan\theta = \frac{\text{side opposite to angle}}{\text{side adjacent to angle}} = \frac{BC}{AB}$$



A right-angled triangle, ABC.

# Trigonometric Functions

Note that

$$\sin 0^\circ = 0, \quad \cos 0^\circ = 1, \quad \tan 0^\circ = 0$$

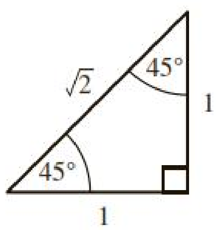
$$\sin 90^\circ = 1, \quad \cos 90^\circ = 0,$$

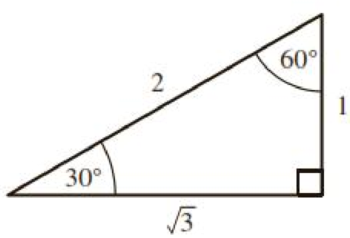
$$\tan\theta \to \infty \quad \text{as } \theta \to 90^\circ$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}}, \quad \cos 45^\circ = \frac{1}{\sqrt{2}}, \quad \tan 45^\circ = 1$$

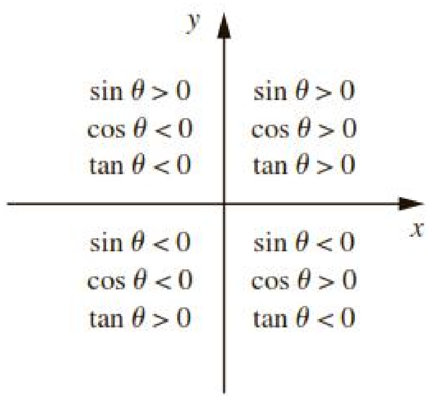
$$\sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}, \quad \tan 60^\circ = \sqrt{3}$$



# Trigonometric Functions




Sign of the trigonometric ratios in each of the four quadrants.

The **cosecant**, **secant** and **cotangent** ratios are defined as the reciprocals of the sine, cosine and tangent ratios.

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

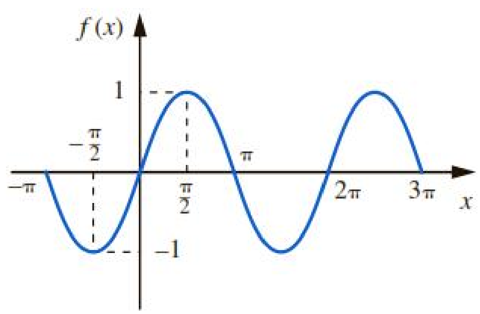
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta}$$

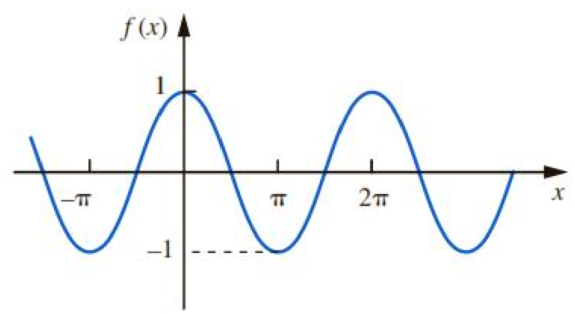
# Trigonometric Functions

## THE SINE, COSINE AND TANGENT FUNCTIONS

The sine, cosine and tangent functions follow directly from the trigonometric ratios. These are defined to be $f(x) = \sin x$, $f(x) = \cos x$ and $f(x) = \tan x$.

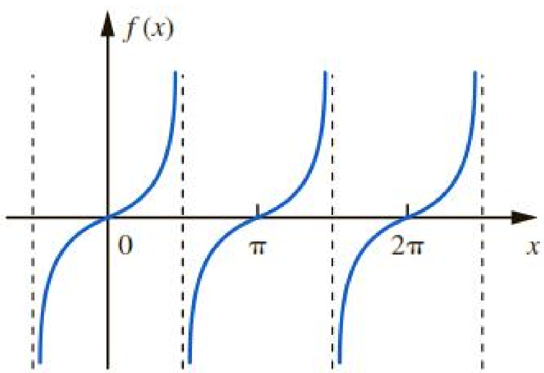



Graph of $f(x) = \sin x$.




Graph of $f(x) = \cos x$.

# Trigonometric Functions




Graph of $f(x) = \tan x$.

# Trigonometric Functions

Common trigonometric identities.

$$\tan A = \frac{\sin A}{\cos A}$$

$$\sin(A \pm B) = \sin A \cos B \pm \sin B \cos A$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$$

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B)$$

$$2 \sin A \sin B = \cos(A - B) - \cos(A + B)$$

# Trigonometric Functions

$$\sin^2 A + \cos^2 A = 1$$

$$1 + \cot^2 A = \operatorname{cosec}^2 A$$

$$\tan^2 A + 1 = \sec^2 A$$

$$\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \cos^2 A - \sin^2 A$$

$$\sin 2A = 2\sin A \cos A$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}$$

$$\cos^2 A = \frac{1 + \cos 2A}{2}$$

*Note:* $\sin^2 A$ is the notation used for $(\sin A)^2$. Similarly $\cos^2 A$ means $(\cos A)^2$.

# Trigonometric Functions

Further trigonometric identities

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$