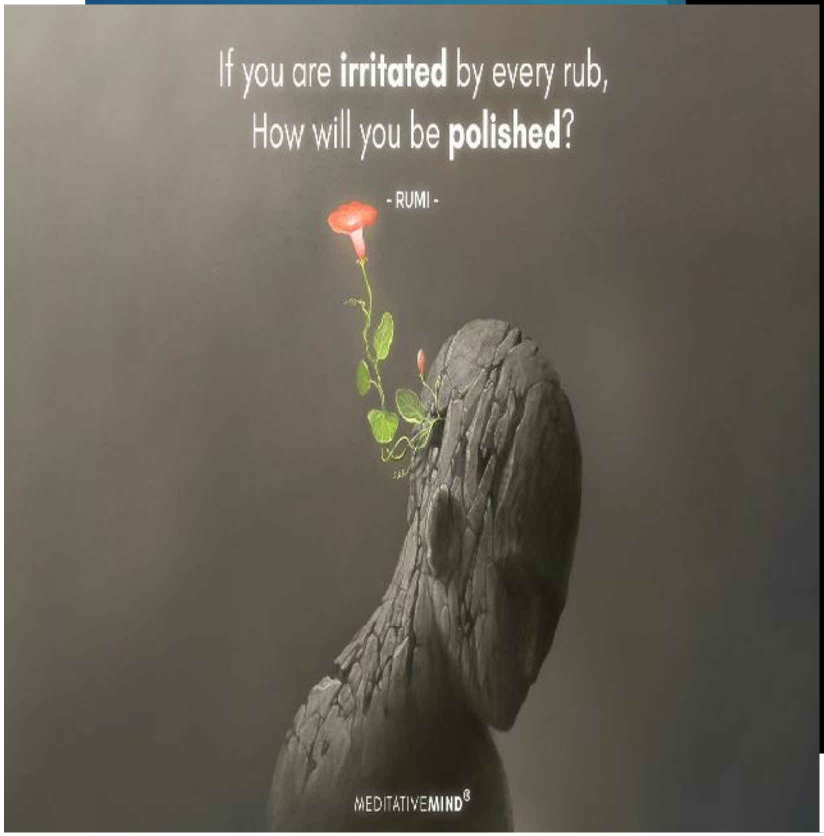
Computational Methods

TAYLOR SERIES EXPANSION

QUOTE


If you are **irritated** by every rub,
How will you be **polished**?
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First-Order Taylor Polynomials

Suppose we know that y is a function of x and we know the values of y and y' when $x = a$, that is $y(a)$ and $y'(a)$ are known. We can use $y(a)$ and $y'(a)$ to determine a linear polynomial which approximates to $y(x)$. Let this polynomial be

$$p_1(x) = c_0 + c_1 x$$

We choose the constants c_0 and c_1 so that

$$p_1(a) = y(a)$$

$$p_1'(a) = y'(a)$$

that is, the values of p_1 and its first derivative evaluated at $x = a$ match the values of y and its first derivative evaluated at $x = a$.

First-Order Taylor Polynomials

Then,

$$p_1(a) = y(a) = c_0 + c_1 a$$

$$p_1'(a) = y'(a) = c_1$$

Solving for c_0 and c_1 yields

$$c_0 = y(a) - ay'(a) \qquad c_1 = y'(a)$$

Thus,

$$p_1(x) = y(a) - ay'(a) + y'(a)x$$

$$p_1(x) = y(a) + y'(a)(x - a)$$

$p_1(x)$ is the **first-order Taylor polynomial** generated by y at $x = a$.

First-Order Taylor Polynomials

A function, y, and its first derivative are evaluated at $x = 2$.

$$y(2) = 1 \qquad y'(2) = 3$$

(a) State the first-order Taylor polynomial generated by y at $x = 2$.

(b) Estimate $y(2.5)$.

(a) $p_1(x) = y(2) + y'(2)(x - 2) = 1 + 3(x - 2) = -5 + 3x$

(b) We use the first-order Taylor polynomial to estimate $y(2.5)$:

$$p_1(2.5) = -5 + 3(2.5) = 2.5$$

Hence, $y(2.5) \approx 2.5$.

First-Order Taylor Polynomials

Find a linear approximation to $y(t) = t^2$ near $t = 3$.

We require the equation of the tangent to $y = t^2$ at $t = 3$, that is the first-order Taylor polynomial about $t = 3$. Note that $y(3) = 9$ and $y'(3) = 6$.

$$\begin{aligned} p_1(t) &= y(a) + y'(a)(t-a) = y(3) + y'(3)(t-3) \\ &= 9 + 6(t-3) \\ &= 6t - 9 \end{aligned}$$

At $t = 3$, $p_1(t)$ and $y(t)$ have an identical value. Near to $t = 3$, $p_1(t)$ and $y(t)$ have similar values, for example $p_1(2.8) = 7.8$, $y(2.8) = 7.84$.

Second-Order Taylor Polynomials

Suppose that in addition to $y(a)$ and $y'(a)$, we also have a value of $y''(a)$. With this information a second-order Taylor polynomial can be found, which provides a quadratic approximation to $y(x)$. Let

$$p_2(x) = c_0 + c_1 x + c_2 x^2$$

We require

$$p_2(a) = y(a)$$

$$p_2'(a) = y'(a)$$

$$p_2''(a) = y''(a)$$

that is, the polynomial and its first two derivatives evaluated at $x = a$ match the function and its first two derivatives evaluated at $x = a$.

Second-Order Taylor Polynomials

Hence

$$p_2(a) = c_0 + c_1 a + c_2 a^2 = y(a)$$

$$p_2'(a) = c_1 + 2c_2 a = y'(a)$$

$$p_2''(a) = 2c_2 = y''(a)$$

Solving for c_0, c_1 and c_2 yields

$$c_2 = \frac{y''(a)}{2}$$

$$c_1 = y'(a) - ay''(a)$$

$$c_0 = y(a) - ay'(a) + \frac{a^2}{2}y''(a)$$

Second-Order Taylor Polynomials

Hence,

$$p_2(x) = y(a) - ay'(a) + \frac{a^2}{2}y''(a) + \{y'(a) - ay''(a)\}x + \frac{y''(a)}{2}x^2$$

Finally,

$$p_2(x) = y(a) + y'(a)(x-a) + y''(a)\frac{(x-a)^2}{2}$$

$p_2(x)$ is the **second-order Taylor polynomial** generated by y about $x = a$.

Second-Order Taylor Polynomials

Given $y(1) = 0$, $y'(1) = 1$, $y''(1) = -2$, estimate

(a) $y(1.5)$ (b) $y(2)$ (c) $y(0.5)$

using the second-order Taylor polynomial.

The second-order Taylor polynomial is $p_2(x)$:

$$p_2(x) = y(1) + y'(1)(x-1) + y''(1)\frac{(x-1)^2}{2}$$
$$= x - 1 - 2\frac{(x-1)^2}{2} = x - 1 - (x-1)^2 = -x^2 + 3x - 2$$

We use $p_2(x)$ as an approximation to $y(x)$.

(a) The value of $y(1.5)$ is approximated by $p_2(1.5)$: $y(1.5) \approx p_2(1.5) = 0.25$

(b) The value of $y(2)$ is approximated by $p_2(2)$: $y(2) \approx p_2(2) = 0$

(c) The value of $y(0.5)$ is approximated by $p_2(0.5)$: $y(0.5) \approx p_2(0.5) = -0.75$

Second-Order Taylor Polynomials

(a) Calculate the second-order Taylor polynomial, $p_2(x)$, generated by $y(x) = x^3 + x^2 - 6$ about $x = 2$.

(b) Verify that $y(2) = p_2(2)$, $y'(2) = p_2'(2)$ and $y''(2) = p_2''(2)$.

(c) Compare $y(2.1)$ and $p_2(2.1)$.

(a) We need to calculate $y(2)$, $y'(2)$ and $y''(2)$. Now

$$y(x) = x^3 + x^2 - 6, \; y'(x) = 3x^2 + 2x, \; y''(x) = 6x + 2$$

and so $y(2) = 6, \; y'(2) = 16, \; y''(2) = 14$

The required second-order Taylor polynomial, $p_2(x)$, is thus given by

$$p_2(x) = y(2) + y'(2)(x-2) + y''(2)\frac{(x-2)^2}{2} \quad = 6 + 16(x-2) + 14\frac{(x-2)^2}{2}$$

$$= 6 + 16x - 32 + 7(x^2 - 4x + 4) \quad = 7x^2 - 12x + 2$$

Second-Order Taylor Polynomials

(b) Using (a) we can see that

$$p_2(x) = 7x^2 - 12x + 2,\ p_2'(x) = 14x - 12,\ p_2''(x) = 14$$

and so

$$p_2(2) = 6,\ p_2'(2) = 16,\ p_2''(2) = 14$$

Hence

$$y(2) = p_2(2),\ y'(2) = p_2'(2),\ y''(2) = p_2''(2)$$

(c) We have

$$y(2.1) = (2.1)^3 + (2.1)^2 - 6 = 7.671$$

$$p_2(2.1) = 7(2.1)^2 - 12(2.1) + 2 = 7.67$$

Clearly there is a very close agreement between values of $y(x)$ and $p_2(x)$ near to $x = 2$.

Taylor Polynomials of the nth Order

If we know y and its first n derivatives evaluated at $x = a$, that is $y(a)$, $y'(a)$, $y''(a)$, $\ldots$, $y^{(n)}(a)$, then the ***n*th-order Taylor polynomial**, $p_n(x)$, may be written as

$$p_n(x) = y(a) + y'(a)(x-a) + y''(a)\frac{(x-a)^2}{2!} + y^{(3)}(a)\frac{(x-a)^3}{3!} + \cdots + y^{(n)}(a)\frac{(x-a)^n}{n!}$$

This provides an approximation to $y(x)$. The polynomial and its first n derivatives evaluated at $x = a$ match the values of $y(x)$ and its first n derivatives evaluated at $x = a$, that is

$$p_n(a) = y(a) \quad p_n'(a) = y'(a) \quad p_n''(a) = y''(a) \quad \cdots \quad p_n^{(n)}(a) = y^{(n)}(a)$$

Taylor Polynomials of the nth Order

Given $y(0) = 1$, $y'(0) = 1$, $y''(0) = 1$, $y^{(3)}(0) = -1$, $y^{(4)}(0) = 2$, obtain a fourth-order Taylor polynomial generated by y about $x = 0$. Estimate $y(0.2)$.

In this example $a = 0$ and hence

$$\begin{aligned} p_4(x) &= y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y^{(3)}(0)\frac{x^3}{3!} + y^{(4)}(0)\frac{x^4}{4!} \\ &= 1 + x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} \end{aligned}$$

The Taylor polynomial can be used to estimate $y(0.2)$:

$$p_4(0.2) = 1 + 0.2 + \frac{(0.2)^2}{2} - \frac{(0.2)^3}{6} + \frac{(0.2)^4}{12} = 1.2188$$

$$y(0.2) \approx 1.2188$$

Taylor's Formula and the Remainder Term

Suppose $p_n(x)$ is an nth-order Taylor polynomial generated by $y(x)$ about $x = a$.

Then **Taylor's formula** states:

$$y(x) = p_n(x) + R_n(x)$$

where $R_n(x)$ is called the **remainder of order *n*** and is given by

$$\text{remainder of order } n = R_n(x) = \frac{y^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$$

for some number c between a and x.

The remainder of order n is also called the **error term**. The error term in effect gives the difference between the function, $y(x)$, and the Taylor polynomial generated by $y(x)$.

Taylor's Formula and the Remainder Term

Calculate the error term of order 5 due to $y(x) = e^x$ generating a Taylor polynomial about $x = 0$.

Here $n = 5$ and so $n + 1 = 6$. In this example $a = 0$. We see that $y = e^x$ and so

$$y'(x) = y''(x) = \cdots = y^{(5)}(x) = y^{(6)}(x) = e^x$$

and so $y^{(6)}(c) = e^c$. The remainder term of order 5, $R_5(x)$, is given by

$$R_5(x) = \frac{e^c x^6}{6!} \qquad \text{for some number } c \text{ between 0 and } x$$

Taylor's Formula and the Remainder Term

(a) Calculate the fourth-order Taylor polynomial, $p_4(x)$, generated by $y(x) = \sin 2x$ about $x = 0$.

(b) State the fourth-order error term, $R_4(x)$.

(c) Calculate an upper bound for this error term given $|x| < 1$.

(d) Compare $y(0.5)$ and $p_4(0.5)$.

(a)

$$\begin{aligned}
y(x) &= \sin 2x, & y(0) &= 0 \\
y'(x) &= 2\cos 2x, & y'(0) &= 2 \\
y''(x) &= -4\sin 2x, & y''(0) &= 0 \\
y^{(3)}(x) &= -8\cos 2x, & y^{(3)}(0) &= -8 \\
y^{(4)}(x) &= 16\sin 2x, & y^{(4)}(0) &= 0
\end{aligned}$$

Taylor's Formula and the Remainder Term

The fourth-order Taylor polynomial, $p_4(x)$, is

$$\begin{aligned} p_4(x) &= y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y^{(3)}(0)\frac{x^3}{3!} + y^{(4)}(0)\frac{x^4}{4!} \\ &= 0 + 2x + 0 - \frac{8x^3}{6} + 0 \\ &= 2x - \frac{4x^3}{3} \end{aligned}$$

(b) We note that $y^{(5)}(x) = 32\cos 2x$ and so $y^{(5)}(c) = 32\cos(2c)$. The error term, $R_4(x)$, is given by

$$\begin{aligned} R_4(x) &= y^{(5)}(c)\frac{x^5}{5!} \\ &= \frac{32\cos(2c)x^5}{120} \\ &= \frac{4}{15}\cos(2c)x^5 \qquad \text{where } c \text{ is a number between 0 and } x \end{aligned}$$

Taylor's Formula and the Remainder Term

(c) In order to calculate an upper bound for this error term we note that $|\cos(2c)| \leqslant 1$ for any value of c. Hence an upper bound for $R_4(x)$ is given by

$$|R_4(x)| \leqslant \left|\frac{4x^5}{15}\right|$$

We know $|x| < 1$, and so $\frac{4x^5}{15}$ is never greater than $\frac{4}{15}$. Hence an upper bound for the error term is $\frac{4}{15}$. If we use $p_4(x)$ to approximate $y = \sin 2x$ the error will be no greater than $\frac{4}{15}$ provided $|x| < 1$.

Taylor's Formula and the Remainder Term

(d) We let $x = 0.5$.

$$y(0.5) = \sin 1 = 0.8415$$

$$p_4(0.5) = 2(0.5) - \frac{4}{3}(0.5)^3 = 0.8333$$

The difference between $y(0.5)$ and $p_4(0.5)$ can never be greater than an upper bound of the error term evaluated at $x = 0.5$. This is verified numerically.

$$y(0.5) - p_4(0.5) = 0.8415 - 0.8333 = 0.0082$$

and

$$|R_4(0.5)| \leqslant \frac{4}{15}(0.5)^5 = 0.0083$$

Taylor and Maclaurin Series

As more and more terms are included in the Taylor polynomial, we obtain an infinite series, called a **Taylor series**. We denote this infinite series by $p(x)$.

Taylor series:

$$p(x) = y(a) + y'(a)(x-a) + y''(a)\frac{(x-a)^2}{2!} + y^{(3)}(a)\frac{(x-a)^3}{3!} + \cdots + y^{(n)}(a)\frac{(x-a)^n}{n!} + \cdots$$

A special, commonly used, case of a Taylor series occurs when $a = 0$. This is known as the **Maclaurin series**.

Maclaurin series:

$$p(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y^{(3)}(0)\frac{x^3}{3!} + \cdots + y^{(n)}(0)\frac{x^n}{n!} + \cdots$$

Taylor and Maclaurin Series

Determine the Maclaurin series for $y = e^x$.

In this example $y(x) = e^x$ and clearly $y'(x) = e^x$ too. Similarly,

$$y''(x) = y^{(3)}(x) = \cdots = y^{(n)}(x) = e^x$$

for all values of n. Evaluating at $x = 0$ yields $y(0) = y'(0) = y''(0) = \cdots = y^{(n)}(0) = 1$

and so

$$p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

As mentioned earlier, the series and the generating function are equal for all values of x. Hence,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \text{for all values of } x \text{, that is} \quad e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Taylor and Maclaurin Series

Obtain the Maclaurin series for $y(x) = \sin x$.

$$\begin{aligned}
y(x) &= \sin x, & y(0) &= 0 \\
y'(x) &= \cos x, & y'(0) &= 1 \\
y''(x) &= -\sin x, & y''(0) &= 0 \\
y^{(3)}(x) &= -\cos x, & y^{(3)}(0) &= -1 \\
y^{(4)}(x) &= \sin x, & y^{(4)}(0) &= 0
\end{aligned}$$

$$p(x) = y(0) + y'(0)x + \frac{y''(0)x^2}{2!} + \frac{y^{(3)}(0)x^3}{3!} + \cdots \quad = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Since the generating function and series are equal for all values of x we have

$$\sin x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Taylor and Maclaurin Series

Obtain the Maclaurin series for $y(x) = \cos x$.

$$\begin{aligned} y(x) &= \cos x, & y(0) &= 1 \\ y'(x) &= -\sin x, & y'(0) &= 0 \\ y''(x) &= -\cos x, & y''(0) &= -1 \\ y^{(3)}(x) &= \sin x, & y^{(3)}(0) &= 0 \\ y^{(4)}(x) &= \cos x, & y^{(4)}(0) &= 1 \end{aligned}$$

and so on. Therefore

$$p(x) = y(0) + y'(0)x + y''(0)\frac{x^2}{2!} + y^{(3)}(0)\frac{x^3}{3!} + y^{(4)}(0)\frac{x^4}{4!} + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

Since the series and the generating function are equal for all values of x then

$$\cos x = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Taylor and Maclaurin Series

Find the Maclaurin series for the following functions:

(a) $y = e^{2x}$ (b) $y = \sin 3x$ (c) $y = \cos\left(\frac{x}{2}\right)$

We use the previously stated series.

(a) We note that

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

Substituting $z = 2x$ we obtain

$$\begin{aligned} e^{2x} &= 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots \\ &= 1 + 2x + 2x^2 + \frac{4x^3}{3} + \frac{2x^4}{3} + \cdots \end{aligned}$$

Taylor and Maclaurin Series

(b) We note that

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

By putting $z = 3x$ we obtain

$$\sin 3x = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \cdots = 3x - \frac{9}{2}x^3 + \frac{81}{40}x^5 - \frac{243}{560}x^7 + \cdots$$

(c) We note that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

Putting $z = \frac{x}{2}$ we obtain

$$\cos\left(\frac{x}{2}\right) = 1 - \frac{(x/2)^2}{2!} + \frac{(x/2)^4}{4!} - \frac{(x/2)^6}{6!} + \cdots = 1 - \frac{x^2}{8} + \frac{x^4}{384} - \frac{x^6}{46\,080} + \cdots$$

Taylor and Maclaurin Series

Find the Maclaurin series for $y(x) = x \cos x$.

The Maclaurin series, $p(x)$, for $\cos x$ is

$$\cos x = p(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

So the Maclaurin series for $x \cos x$ is $xp(x)$, that is

$$x \cos x = xp(x) = x - \frac{x^3}{2!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \cdots$$

Exercises

Calculate the first-order Taylor polynomial generated by $y(x) = e^x$ about

(a) $x = 0$ (b) $x = 2$ (c) $x = -3$

Calculate the first-order Taylor polynomial generated by $y(x) = \sin x$ about

(a) $x = 0$ (b) $x = 1$ (c) $x = -0.5$

Calculate the first-order Taylor polynomial generated by $y(x) = \cos x$ about

(a) $x = 0$

(b) $x = 1$

(c) $x = -0.5$

(a) Find a linear approximation, $p_1(t)$, to $h(t) = t^3$ about $t = 2$.

(b) Evaluate $h(2.3)$ and $p_1(2.3)$.

(a) Find a linear approximation, $p_1(t)$, to $R(t) = \dfrac{1}{t}$ about $t = 0.5$.

(b) Evaluate $R(0.7)$ and $p_1(0.7)$.

Exercises

(a) Obtain the second-order Taylor polynomial, $p_2(x)$, generated by $y(x) = 3x^4 + 1$ about $x = 2$.

(b) Verify that $y(2) = p_2(2)$, $y'(2) = p_2'(2)$ and $y''(2) = p_2''(2)$.

(c) Evaluate $p_2(1.8)$ and $y(1.8)$.

(a) Calculate the second-order Taylor polynomial, $p_2(x)$, generated by $y(x) = \sin x$ about $x = 0$.

(b) Calculate the second-order Taylor polynomial, $p_2(x)$, generated by $y(x) = \cos x$ about $x = 0$.

(c) Compare your results from (a) and (b) with the small-angle approximations given in Section 6.5.

A function, $x(t)$, satisfies the equation

$$\dot{x} = x + \sqrt{t+1} \qquad x(0) = 2$$

(a) Estimate $x(0.2)$ using a first-order Taylor polynomial.

(b) Differentiate the equation w.r.t. t and hence obtain an expression for $\ddot{x}$.

(c) Estimate $x(0.2)$ using a second-order Taylor polynomial.

Exercises

A function, $y(x)$, has $y(0) = 3$, $y'(0) = 1$, $y''(0) = -1$ and $y^{(3)}(0) = 2$.

(a) Obtain a third-order Taylor polynomial, $p_3(x)$, generated by $y(x)$ about $x = 0$.

(b) Estimate $y(0.2)$.

A function, $h(t)$, has $h(2) = 1$, $h'(2) = 4$, $h''(2) = -2$, $h^{(3)}(2) = 1$ and $h^{(4)}(2) = 3$.

(a) Obtain a fourth-order Taylor polynomial, $p_4(t)$, generated by $h(t)$ about $t = 2$.

(b) Estimate $h(1.8)$.

Exercises

The function, $y(x)$, is given by $y(x) = \sin x$.

(a) Calculate the fifth-order Taylor polynomial generated by y about $x = 0$.

(b) Find an expression for the remainder term of order 5.

(c) State an upper bound for your expression in (b).

(a) Find the third-order Taylor polynomial generated by $h(t) = \frac{1}{t}$ about $t = 2$.

(b) State the error term.

(c) Find an upper bound for the error term given $1 \leqslant t \leqslant 4$.

Exercises

Use the Maclaurin series for $\sin x$ to write down the Maclaurin series for $\sin 5x$.

Use the Maclaurin series for $\cos x$ to write down the Maclaurin series for $\cos 3x$.

Use the Maclaurin series for e^x to write down the Maclaurin series for $\frac{1}{e^x}$.

Find the Taylor series for $y(x) = \sqrt{x}$ about $x = 1$.

(a) Find the Maclaurin series for $y(x) = x^2 + \sin x$.

(b) Deduce the Maclaurin series for $y(x) = x^n + \sin x$ for any positive integer n.

(a) Obtain the Maclaurin series for $y(x) = xe^x$.

(b) State the range of values of x for which $y(x)$ equals its Maclaurin series.

Find the Taylor series for $y(x) = x + e^x$ about $x = 1$.

Find the Maclaurin series for $y(x) = \ln(1 + x)$.