Probability and Random Variables

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Contents

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***Abstract*—This book provides a simple introduction to probability and random variables. The contents are largely based on NCERT textbooks from Class 9-12.**

1 Axioms of Probability

1.1 Boolean Logic

If A and B are two events such that P(A) = $\frac{1}{4}$, P(B) = $\frac{1}{2}$ and P(A ∩ B) = $\frac{1}{8}$. find P (not A and not B).

1.1.1.

$$A'B' = (A+B)' \tag{1.1.1.1}$$
$$\implies \Pr(A'B') = \Pr((A+B)') \tag{1.1.1.2}$$
$$= 1 - \Pr(A+B) \tag{1.1.1.3}$$

1.1.2.

$$\because A+B = A(B+B') + B \tag{1.1.2.1}$$
$$= B(A+1) + AB' \tag{1.1.2.2}$$
$$= B + AB' \tag{1.1.2.3}$$
$$\implies \Pr(A+B) = \Pr(B+AB') \tag{1.1.2.4}$$
$$= \Pr(B) + \Pr(AB') \tag{1.1.2.5}$$
$$\because B(AB') = 0 \tag{1.1.2.6}$$

1.1.3.

$$A = A(B + B') = AB + AB' \quad (1.1.3.1)$$

and

$$(AB)(AB') = 0, \because BB' = 0 \quad (1.1.3.2)$$

Hence, AB and AB' are mutually exclusive and

$$\Pr(A) = \Pr(AB) + \Pr(AB') \quad (1.1.3.3)$$
$$\implies \Pr(AB') = \Pr(A) - \Pr(AB) \quad (1.1.3.4)$$

1.1.4. Substituting (1.1.3.4) in (1.1.2.6),

$$\Pr(A + B) = \Pr(A) + \Pr(B) - \Pr(AB) \quad (1.1.4.1)$$

1.1.5. Substituting (1.1.4.1) in (1.1.1.3)

$$\Pr(A'B') = 1 - \{\Pr(A) + \Pr(B) - \Pr(AB)\} \quad (1.1.5.1)$$
$$= 1 - \left(\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right) \quad (1.1.5.2)$$
$$= \frac{3}{8} \quad (1.1.5.3)$$

1.2 Independence

1.2.1. Prove that if E and F are independent events, then so are the events E and F'.
Solution: If E and F are independent,

$$\Pr(EF) = \Pr(E)\Pr(F) \quad (1.2.1.1)$$

Using Boolean algebra,

$$E = E(F + F') = EF + EF' \quad (1.2.1.2)$$

and

$$(EF)(EF') = 0, \because FF' = 0 \quad (1.2.1.3)$$

Hence, EF and EF' are mutually exclusive and

$$\Pr(E) = \Pr(EF) + \Pr(EF') \quad (1.2.1.4)$$
$$\implies \Pr(EF') = \Pr(E) - \Pr(EF) \quad (1.2.1.5)$$

Substituting from (1.2.1.1) in (1.2.1.5),

$$\Pr(EF') = \Pr(E)(1 - \Pr(F)) \quad = \Pr(E)\Pr(F') \quad (1.2.1.6)$$

$$\because FF' = 0, F + F' = 1 \quad (1.2.1.7)$$
$$\implies \Pr(F) + \Pr(F') = 1 \quad (1.2.1.8)$$

By definition, from (1.2.1.6), we conclude that E and F' are independent.

1.2.2. If A and B are two independent events, then the probability of occurrence of at least one of A and B is given by 1- $P(A')P(B')$
Solution:

$$\because (A + B)(A + B)' = 0 \quad (1.2.2.1)$$
$$\implies 1 = \Pr(A + B) + \Pr((A + B)') \quad (1.2.2.2)$$
$$\implies \Pr(A + B) = 1 - \Pr(A'B') \quad (1.2.2.3)$$
$$= 1 - \Pr(A')\Pr(B') \quad (1.2.2.4)$$

using the definition of independence.

1.3 Conditional Probability

1.3.1. Given that E and F are events such that P(E) = 0.6, P(F) = 0.3 and P(E ∩ F) = 0.2, find P(E/F) and P(F/E)?

1.3.2. Two events A and B will be independent, if
a) A and B are mutually exclusive
b) $P(A'B')$ = [1 − P(A)] [1 − P(B)]
c) P(A) = P(B)
d) P(A) + P(B) = 1
Solution:

1.3.3. If A and B are events such that P(A/B) = P(B/A), then
a) A ⊂ B but A ≠ B
b) A = B
c) A ∩ B = ϕ
d) P(A) = P(B)
Solution:

1.3.4. Which of the following cannot be the probability of an event?
(A)$\frac{2}{3}$(B) −1.5 (C) 15 (D) 0.7

1.3.5. If P(E) = 0.05, what is the probability of 'not E'?

1.3.6. If A and B are two events such that P(A) ≠ 0 and P(B/A) = 1, then (A) A ⊂ B
(B) B ⊂ A
(C) B = ϕ
(D) A = ϕ

1.3.7. If P(A/B) > P(A), then which of the following is correct : (A) P(B/A) < P(B)
(B) P(A ∩ B) < P(A) . P(B)
(C) P(B/A) > P(B)
(D) P(B/A) = P(B)

1.3.8. If A and B are any two events such that P(A) + P(B) – P(A and B) = P(A), then
(A) P(B/A) = 1
(B) P(A/B) = 1
(C) P(B/A) = 0
(D) P(A/B) = 0

1.3.9. Complete the following statements:
(i) Probability of an event E + Probability of the event 'not E' =———— .
(ii) The probability of an event that cannot happen is———- . Such an event is called——— .
(iii) The probability of an event that is certain to happen is ———.
(iv) The sum of the probabilities of all the elementary events of an experiment is———-.
(v) The probability of an event is greater than or equal to and less than or equal to —————.

1.3.10. An electronic assembly consists of two subsystems, say, A and B. From previous testing procedures, the following probabilities are assumed to be known:
P(A fails) = 0.2
P(B fails alone) = 0.15
P(A and B fail) = 0.15

Evaluate the following probabilities
(i) P(A fails—B has failed)
(ii) P(A fails alone)

1.3.11. A and B are two events such that P (A) $\neq$ 0. Find P(B/A), if
(i) A is a subset of B
(ii) A $\cap$ B = ϕ

1.3.12. If A and B are two events such that A $\subset$ B and P(B) $\neq$ 0, then which of the following is correct?

a) P(A/B) = $\frac{P(B)}{P(A)}$
b) $P(A/B) < P(A)$
c) P(A/B) $\geq$ P(A)
d) None of these

1.3.13. Given that the events A and B are such that P(A) = $\frac{1}{2}$, P(A $\cup B$)= $\frac{3}{5}$ and P(B) = p. Find p if they are
(i) mutually exclusive
(ii) independent.

1.3.14. Let A and B be independent events with P(A) = 0.3 and P(B) = 0.4. Find
(i) P(A $\cap$ B)
(ii) P(A $\cup$ B)
(iii) P(A/B)
(iv) P(B/A)

1.3.15. Events A and B are such that P (A) = $\frac{1}{2}$, P(B) = $\frac{7}{12}$ and P(not A or not B) = $\frac{1}{4}$. State whether A and B are independent ?

1.3.16. Given two independent events A and B such that P(A) = 0.3, P(B) = 0.6. Find
(i) P(A and B)
(ii) P(A and not B)
(iii) P(A or B)
(iv) P(neither A nor B)

1.3.17. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event, 'the number is even,' and B be the event, 'the number is red'. Are A and B independent?

1.3.18. A person plays a game of tossing a coin thrice. For each head, he is given Rs 2 by the organiser of the game and for each tail, he has to give Rs 1.50 to the organiser. Let X denote the amount gained or lost by the person. Show that X is a random variable and exhibit it as a function on the sample space of the experiment.

1.3.19. If P(A)=$\frac{7}{13}, P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, Evaluate P(A/B)?

1.3.20. A die is thrown. If E is the event "the number appearing is a multiple of 3" and F be the event "the number appearing is even" then find whether E and F are independent ?

1.3.21. An unbiased die is thrown twice. Let the event A be "odd number on the first throw" and B the event "odd number on the second throw". Check the independence of the events A and B.

1.3.22. Compute P(A/B), if P(B) = 0.5 and P (A $\cap$

B) = 0.32.

1.3.23. If P(A) = 0.8, P(B) = 0.5 and P(B/A) = 0.4, find
(i) P(A ∩ B)
(ii) P(A/B)
(iii) P(A ∪ B)

1.3.24. Evaluate P(A ∪ B), if 2P(A) = P(B) = $\frac{5}{13}$ and P(A/B) = $\frac{2}{5}$.

1.3.25. If P(A) = $\frac{6}{11}$, P(B) = $\frac{5}{11}$ and P(A ∪ B) = $\frac{11}{7}$ find
(i) P(A ∩ B)
(ii) P(A/B)
(iii) P(B/A)

1.3.26. A fair die is rolled. Consider the events E = (1, 3, 5), F = (2, 3) and G = (2, 3, 4, 5) Find
(i) P(E/F) and P(F/E)
(ii) P(E/G) and P(G/E)
(iii) P((E ∪ F)/G) and P((E ∩ F)/G)

1.3.27. Choose the correct answer, if P(A) = $\frac{1}{2}$, P(B) = 0, then P(A/B) is
a) 0
b) $\frac{1}{2}$
c) not defined
d) 1

1.3.28. Let E and F be events with P(E) = $\frac{3}{5}$, P(F) = $\frac{3}{10}$ and P(E ∩ F) = $\frac{1}{5}$. Are E and F independent?

1.3.29. One card is drawn at random from a well shuffled deck of 52 cards. In which of the following cases are the events E and F independent?
(i) E : 'the card drawn is a spade' F : 'the card drawn is an ace'
(ii) E : 'the card drawn is black' F : 'the card drawn is a king'
(iii) E : 'the card drawn is a king or queen' F : 'the card drawn is a queen or jack'.

2 Sum of Independent Random Variables

2.1 The Uniform Distribution

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

2.1.1. *The Uniform Distribution:* Let $X_i \in \{1,2,3,4,5,6\}, i = 1,2,$ be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \le n \le 6 \\ 0 & otherwise \end{cases} \tag{2.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{2.1.1.2}$$

$$\implies X \in \{1,2,\dots,12\} \tag{2.1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{2.1.1.4}$$

2.1.2. *Convolution:* From (2.1.1.2),

$$p_X(n) = \Pr(X_1 + X_2 = n) = \Pr(X_1 = n - X_2) \tag{2.1.2.1}$$

$$= \sum_k \Pr(X_1 = n-k | X_2 = k)\, p_{X_2}(k) \tag{2.1.2.2}$$

after unconditioning. ∵ X_1 and X_2 are independent,

$$\Pr(X_1 = n-k|X_2 = k) = \Pr(X_1 = n-k) = p_{X_1}(n-k) \tag{2.1.2.3}$$

From (2.1.2.2) and (2.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{2.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (2.1.1.1) in (2.1.2.4),

$$p_X(n) = \frac{1}{6}\sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6}\sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{2.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \le 1, k \ge 6. \tag{2.1.2.6}$$

From (2.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{2.1.2.7}$$

Substituting from (2.1.1.1) in (2.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{2.1.2.8}$$

satisfying (2.1.1.4).

2.1.3. *The Z-transform:* The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1.3.1}$$

From (2.1.1.1) and (2.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{2.1.3.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{2.1.3.3}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{2.1.3.4}$$

$$P_X(z) = P_{X_1}(z) P_{X_2}(z) \tag{2.1.3.5}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (2.1.3.3) and (2.1.3.5),

$$P_X(z) = \left\{\frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}\right\}^2 \tag{2.1.3.6}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{2.1.3.7}$$

Using the fact that

$$p_X(n-k) \stackrel{\mathcal{H}}{\longleftrightarrow} Z P_X(z) z^{-k}, \tag{2.1.3.8}$$

$$n u(n) \stackrel{\mathcal{H}}{\longleftrightarrow} Z \frac{z^{-1}}{\left(1 - z^{-1}\right)^2} \tag{2.1.3.9}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \stackrel{\mathcal{H}}{\longleftrightarrow} Z \frac{1}{36}\frac{z^{-2}\left(1 - 2z^{-6} + z^{-12}\right)}{\left(1 - z^{-1}\right)^2} \tag{2.1.3.10}$$

where

$$u(n) = \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases} \tag{2.1.3.11}$$

From (2.1.3.1), (2.1.3.7) and (2.1.3.10)

$$p_X(n) = \frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \tag{2.1.3.12}$$

which is the same as (2.1.2.8). Note that (2.1.2.8) can be obtained from (2.1.3.10) using contour integration as well.

2.1.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 2.1.4.1. The theoretical pmf obtained in (2.1.2.8) is plotted for comparison.

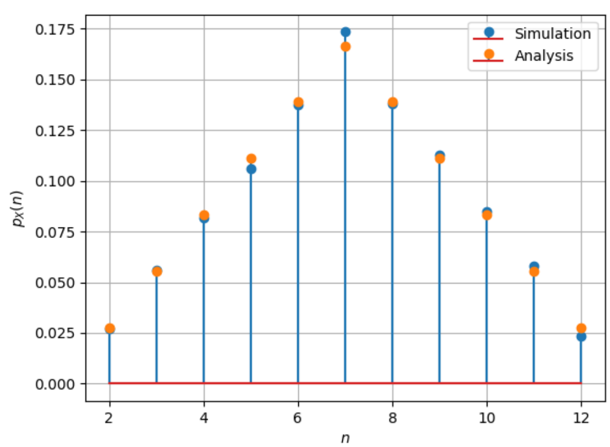

Fig. 2.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

2.1.5. The python code is available in

```
/codes/sum/dice.py
```

TABLE 3.1.3.1

Colour	**X**	**Number**
Blue	0	$n(X = 0)$
Green	1	$n(X = 1)$

3 Cumulative Distribution Function

3.1 The Bernoulli Distribution

3.1.1. Find the probability of getting a head when a coin is tossed once. Also find the probability of getting a tail.
Solution: Let the random variable be $X \in \{0, 1\}$. Then

$$\Pr(X = 0) = \Pr(X = 1) = \frac{1}{2} \tag{3.1.1.1}$$

The following code simulates the event for 100 coin tosses

```
codes/bernoulli/coin.py
```

3.1.2. *Bernoulli Distribution:* In general the binomial distribution is defined using the PMF

$$p_X(n) = \begin{cases} p & n = 1 \\ 1 - p & n = 0 \\ \text{otherwise} \end{cases} \tag{3.1.2.1}$$

3.1.3. A jar contains 24 marbles, some are green and others are blue. If a marble is drawn at random from the jar, the probability that it is green is $\frac{2}{3}$. Find the number of blue balls (marbles) in the jar.
Solution: Let random variable $X \in \{0, 1\}$ denote the outcomes of the experiment of drawing a marble from a jar as shown in Table 3.1.3.1 From the given information,

$$p_X(1) = \frac{2}{3} \tag{3.1.3.1}$$

$$\implies p = 1 - p_X(1) = \frac{1}{3} \tag{3.1.3.2}$$

$$n(X = 0) + n(X = 1) = 24 \tag{3.1.3.3}$$

$\because$

$$p = \frac{n(X = 0)}{n(X = 0) + n(X = 1)}, \tag{3.1.3.4}$$

from (3.1.3.4) and (3.1.3.3),

$$n(X = 0) = p\{n(X = 0) + n(X = 1)\} \tag{3.1.3.5}$$

$$= \frac{1}{3} \times 24 = 8. \tag{3.1.3.6}$$

The following code generates the number of blue marbles

```
codes/bernoulli/bernoulli.py
```

3.2 The Binomial Distribution

In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

3.2.1. Let $X_i \in \{0, 1\}$ represent the *ith* hurdle where 1 denotes a hurdle being knocked down. Then, X_i has a bernoulli distribution with parameter

$$p = 1 - \frac{5}{6} = \frac{1}{6} \tag{3.2.1.1}$$

3.2.2. *The Binomial Distribution:* Let

$$X = \sum_{i=1}^{n} X_i \tag{3.2.2.1}$$

where n is the total number of hurdles. Then X has a binomial distribution. Then, for

$$p_{X_i}(n) \overset{\mathcal{Z}}{\rightleftharpoons} P_{X_i}(z), \tag{3.2.2.2}$$

yielding

$$P_{X_i}(z) = 1 - p + pz^{-1} \tag{3.2.2.3}$$

with Using the fact that X_i are i.i.d.,

$$P_X(z) = \left(1 - p + pz^{-1}\right)^n \tag{3.2.2.4}$$

$$= \sum_{k=0}^{n} {}^nC_k p^k (1 - p)^{n-k} z^{-k} \tag{3.2.2.5}$$

$$\implies p_X(k) = \begin{cases} {}^nC_k p^{n-k} (1 - p)^k & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \tag{3.2.2.6}$$

The cumulative distribution function of X is defined as

$$F_X(r) = \Pr(X \le r) = \sum_{k=0}^{r} {}^nC_k p^k (1-p)^{n-k} \tag{3.2.2.7}$$

upon substituting from (3.2.2.6).

3.2.3. *Evaluationg the Probability:* Substituting from (3.2.1.1) in (3.2.2.7),

$$\Pr(X < 2) = F_X(1) \tag{3.2.3.1}$$
$$= \sum_{k=0}^{1} {}^nC_k \left(\frac{5}{6}\right)^{10-k} \left(\frac{1}{6}\right)^k \tag{3.2.3.2}$$
$$= 3\left(\frac{5}{6}\right)^{10} = 0.4845167486695371 \tag{3.2.3.3}$$

which is the desired probability.

3.2.4. The following code verifies the above result.

```
codes/binomial/binomial.py
```

4 Central Limit Theorem: Gaussian Distribution

4.1 Bernoulli to Gaussian

4.1.1 *Mean :* The mean of the bernoulli distribution is

$$\mu = E(X_i) = \sum_{k=0}^{1} k p_{X_i}(k) = p = \frac{1}{6} \tag{4.1.1}$$

4.1.2 *Moment:* The moment of the distribution is defined as

$$E(X_i^r) = \sum_{k=0}^{1} k^r p_{X_i}(k) = p = \frac{1}{6} \tag{4.2.1}$$

4.1.3 *Variance :* The variance of the bernoulli distribution is defined as

$$\sigma^2 = E(X - E(X))^2 = E\left(X^2\right) - E^2(X) \tag{4.3.1}$$
$$= p - p^2 = p(1-p) = \frac{5}{36} \tag{4.3.2}$$

The standard deviation

$$\sigma = \sqrt{p(1-p)} \tag{4.3.3}$$

4.1.4 *The Gaussian Distribution:* Define

$$G = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{X_i - \mu}{\sigma} \tag{4.4.1}$$

4.1.5 *Approximating Binomial Using Gaussian:* From (4.4.1) and (3.2.2.1),

$$X \approx \sigma\sqrt{n}G + n\mu \tag{4.5.1}$$
$$\implies F_X(k) = \Pr\left(\sigma\sqrt{n}G + n\mu \le k\right) \tag{4.5.2}$$
$$= F_G\left(\frac{k - n\mu}{\sigma\sqrt{n}}\right) \approx \phi\left(\frac{k - n\mu}{\sigma\sqrt{n}}\right) \tag{4.5.3}$$

where

$$\phi_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty \tag{4.5.4}$$

4.1.6 The probability density function (PDF) of G is

$$p_G(x) = \frac{d}{dx} F_X(x) \tag{4.6.1}$$
$$= \frac{1}{\sigma\sqrt{n}} \phi'\left(\frac{k - n\mu}{\sigma\sqrt{n}}\right) \tag{4.6.2}$$

For large n, G is a continuous distribution with probability density function (PDF)

$$p_G(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{4.6.3}$$

4.1.7 *Evaluationg the Probability:* From 4.5.3 and 4.6.2,

$$\Pr(X \le 1) = F_G(1) = p_G(0) + p_G(1) \tag{4.7.1}$$
$$\approx 0.41299463887797094 \tag{4.7.2}$$

which is close to (3.2.3.3).

4.2 Uniform to Gaussian

4.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{4.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
codes/cdf/exrand.c
codes/cdf/coeffs.h
```

4.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What

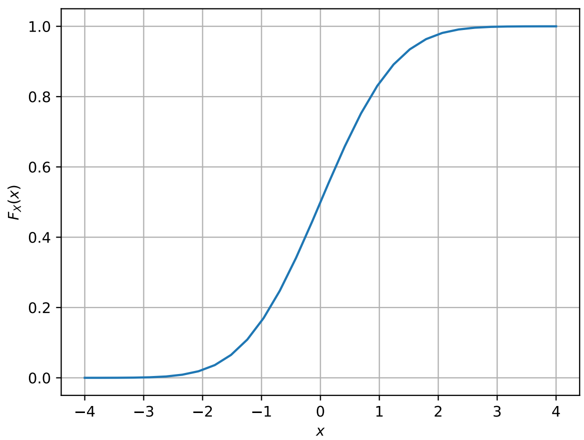

Fig. 4.2: The CDF of X

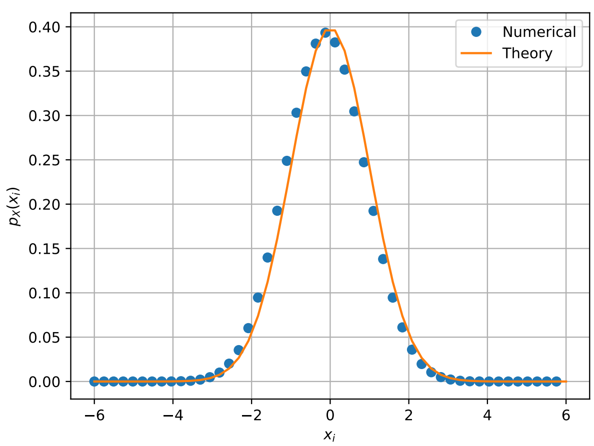

Fig. 4.3: The PDF of X

properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 4.2

4.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{4.3.1}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 4.3 using the code below

```
codes/clt/pdf_plot.py
```

4.2.4 Find the mean and variance of X by writing a C program.

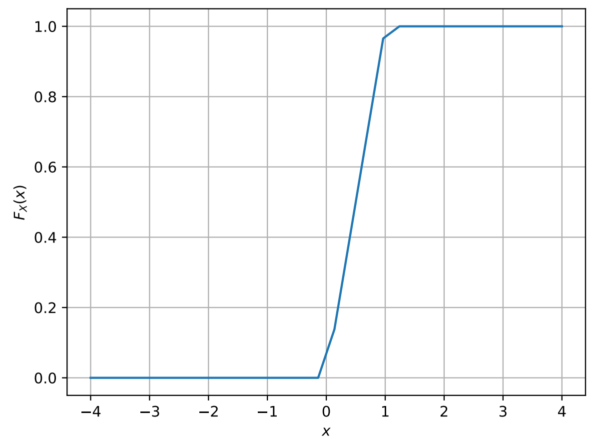

Fig. 4.6: The CDF of U

4.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{4.5.1}$$

repeat the above exercise theoretically. Let U be a uniform random variable between 0 and 1.

4.2.6 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{4.6.1}$$

Solution: The following code plots Fig. 4.6

```
codes/cdf/cdf_plot.py
```

4.2.7 Find a theoretical expression for $F_U(x)$.

4.2.8 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \tag{4.8.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{4.8.2}$$

Write a C program to find the mean and variance of U.

4.2.9 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{4.9.1}$$

5 Stochastic Geometry

Suppose you drop a die at random on the rectangular region shown in Fig. 5.1.1. What is the probability that it will land inside the circle with diameter 1m?

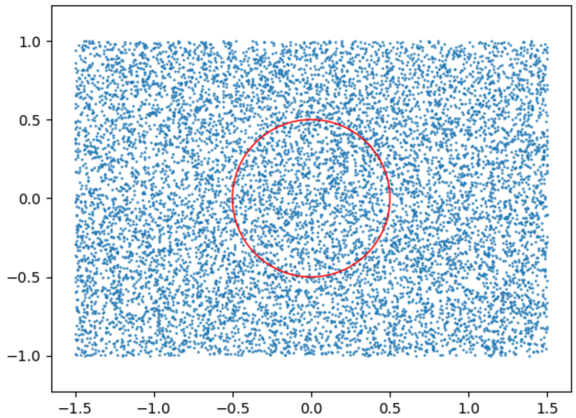

Fig. 5.1.1

5.1. In Fig. 5.1.1, the sample size S is the area of the rectangle given by

$$S = 3 \times 2 = 6m^2 \tag{5.1.1}$$

The event size is the area of the circle given by

$$E = \pi \left(\frac{1}{2}\right)^2 = \frac{\pi}{4} m^2 \tag{5.1.2}$$

The probabilty of the dice landing in the circle is

$$\Pr(E) = \frac{E}{S} = \frac{\pi}{24} \tag{5.1.3}$$

5.2. The python code is available in

```
/codes/stochastic/rect.py
```

The python code generates 10,000 points uniformly within the rectangle of dimensions 3×2 and checks for the number of points within the circle of radius 0.5. The ratio of these is close to $\frac{\pi}{24}$. Note that each time the code is run, the ratio will change, but will still be close to $\frac{\pi}{24}$.

6 Transformation of Variables

6.1 Using Definition

6.1.1. Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1.1.1}$$

6.1.2. If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.1.2.1}$$

find α.

6.1.3. Plot the CDF and PDf of

$$A = \sqrt{V} \tag{6.1.3.1}$$

6.1.4. Find an expression for $F_A(x)$ using the definition. Plot this expression and compare with the result of problem 6.1.3.

6.1.5. Find an expression for $p_A(x)$.

6.2 Using Jacobian

6.2.1. Evaluate the joint PDF of X_1, X_2, given by

$$p_{X_1,X_2}(x_1,x_2) = p_{X_1}(x_1)\, p_{X_2}(x_2) \tag{6.2.1.1}$$

6.2.2. Let

$$X_1 = \sqrt{V}\cos\theta \tag{6.2.2.1}$$

$$X_2 = \sqrt{V}\sin\theta. \tag{6.2.2.2}$$

Evaluate the Jacobian

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial v} & \frac{\partial x_2}{\partial v} \\ \frac{\partial x_1}{\partial \theta} & \frac{\partial x_2}{\partial \theta} \end{vmatrix} \tag{6.2.2.3}$$

6.2.3. Find

$$p_{V,\Theta}(v,\theta) = |J|\, p_{X_1,X_2}(x_1,x_2) \tag{6.2.3.1}$$

6.2.4. Find $p_V(v)$.

6.2.5. Find $p_\Theta(\theta)$.

6.2.6. Are Y and Θ independent?

6.2.7. Find $p_A(x)$ using the Jacobian.

7 Conditional Probability

7.1. Plot

$$P_e = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{7.1.1}$$

for

$$Y = AX + N, \tag{7.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}(0,1), X \in (-1,1)$ for $0 \leq \gamma \leq 10$ dB.

7.2. Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.3. For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x)p_X(x)\,dx \tag{7.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

7.4. Plot P_e in problems 7.1 and 7.3 on the same graph w.r.t γ. Comment.

8 Two Dimensions

8.1. Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{8.1.1}$$

where

$$x \in \left(\mathbf{s}_0, \mathbf{s}_1\right), \mathbf{s}_0 = \begin{pmatrix}1\\0\end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix}0\\1\end{pmatrix} \tag{8.1.2}$$

$$\mathbf{n} = \begin{pmatrix}n_1\\n_2\end{pmatrix}, n_1, n_2 \sim \mathcal{N}\left(0, 1\right). \tag{8.1.3}$$

8.2. Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.2.1}$$

on the same graph using a scatter plot.

8.3. For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.4. Plot

$$P_e = \Pr\left(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0\right) \tag{8.4.1}$$

with respect to the SNR from 0 to 10 dB.

8.5. Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

9 Transform Domain

Let $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

9.1. Find $M_X\left(s\right) = E\left[e^{-sX}\right]$.

9.2. Let

$$N = n_1 - n_2, \quad n_1, n_2 \sim \mathcal{N}\left(0, 1\right). \tag{9.2.1}$$

Find $M_N(s)$, assuming that n_1 and n_2 are independent.

9.3. Show that N is Gaussian. Find its mean and variance. Comment.

10 Uniform to Other

10.1. Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{10.1.1}$$

and plot its CDF. Comment.

10.2. Generate the Rayleigh distribution from Uniform. Verify your result through graphical plots.