Homework #3

Due Time: 2023/05/22 00:00

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Introduction and Rules

1. The judge system is located at https://noj.tw or https://v2.noj.tw, please submit your code by the deadline.
2. Homework Rule
3. Can I refer to resources from the Internet or other sources that are not from textbooks or lecture slides?
   - Yes, but you must include a reference source, such as a website or book title and page number, and attach it as a comment at the top of the code.
   - Although you can refer to external resources, please write your own code after the reference.
   - Remember to specify the references; otherwise we'll view it as cheating.
4. The Non-Programming part is no need to hand in. Just for practicing!

Programming Part

Please go to Normal Online Judge to read the programming problem.

Non-Programming Part

1. Turnip Delivery

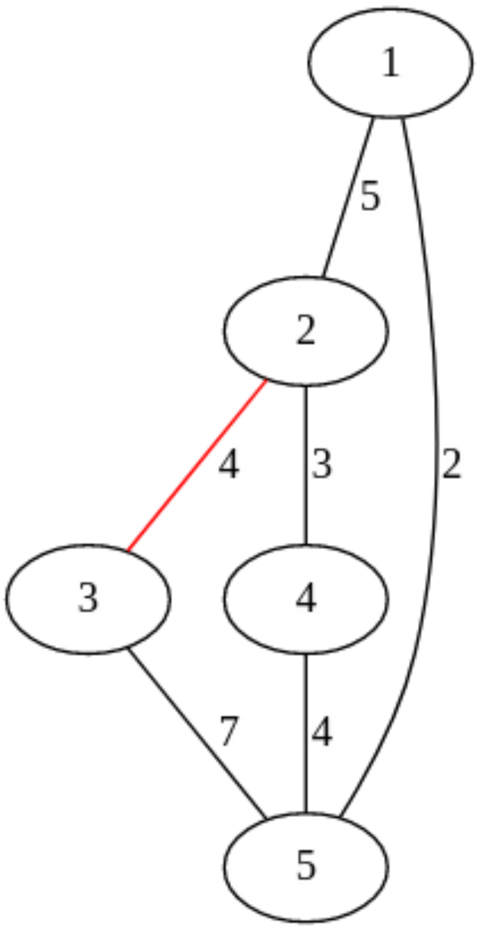
In Animal Crossing, there are many islands, and some of them can reach each other directly by airplane.

An airplane connects two different islands and has a weight limit of w. You want to take some turnips to another island, but the total weight of turnips must not be higher than w, or the airplane will crash.

What is the maximum weight of turnips you can deliver from island A to island B ?

It's guaranteed that all islands are connected, that is, you can reach any island from any other one by some paths.

For example, in $Figure$ 1, the maximum weight of turnips we can deliver from 1 to 5 is 4.



There are several paths from 1 to 5:

1. $1 \to 5$: The maximum weight of turnips we can deliver through this path is 2.

2. $1 \to 2 \to 4 \to 5$: The maximum weight of turnips we can deliver through this path is 3.

3. $1 \to 2 \to 3 \to 5$: The maximum weight of turnips we can deliver through this path is 4.

So, the maximum weight of turnips you can deliver from 1 to 5 is 4.

(1).

Given a connected graph with N nodes, how many edges are in the maximum spanning tree of this graph?

(2).

Please draw down the maximum spanning tree of the graph in $Figure$ 1.

What is the relationship between MST and this problem? After finding out the MST, the path from A to B is unique. Moreover, the answer (the maximum weight of turnips) is the minimum weight limit airplane on the path from A to B.

(3).

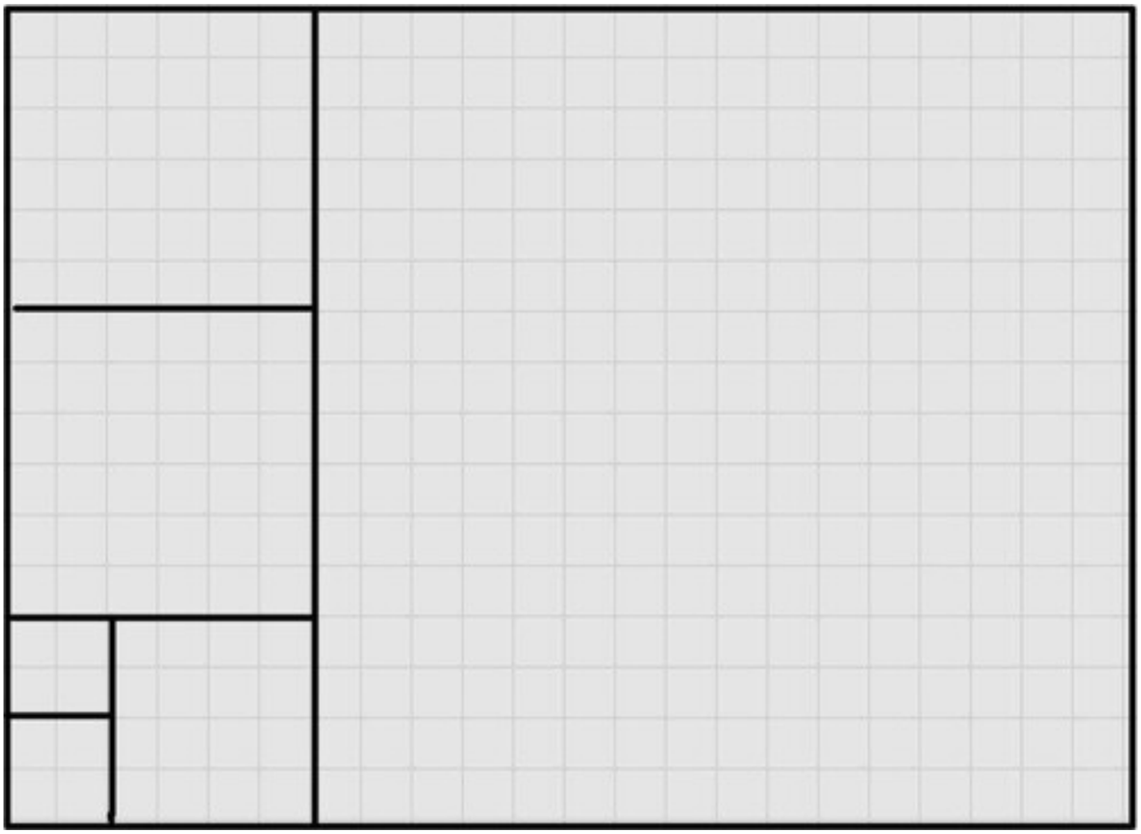
Try to explain the reason why the answer (the essential airplane) is always in MST. What properties of MST can help you to solve this problem?

2. Square

There is a rectangle with width W, and height H, now we wonder how to split the rectangle into the least amounts of squares.

The way we split is that if the size of the rectangle is $i \times j$ assuming W.L.O.G $i < j$, we will split the rectangle into two parts, one will be size $i \times i$, the other will be size $(j - i) \times i$. After that, we will apply the way again until the initial rectangle is split into squares.

For example, a rectangle with $W = 22$, $H = 16$, split into 6 squares.



Another example, a rectangle with $W = 20$, $H = 5$, split into 4 squares.

(1).

Please determine the time complexity (Big-Theta) of the greedy method mentioned above.

(2).

Please prove or disprove the greedy method, does it always give us the least amounts of squares? Proof requires completeness, disproof requires a counter-example otherwise.

3. Lexicographical order

Given two different sequences of the same length, $A = a_1,\ a_2,\ \ldots,\ a_k$ and $B = b_1,\ b_2,\ \ldots,\ b_k$, we say that A is lexicographically smaller than B if $a_i < b_i$ for the first i where a_i and b_i differ. For example, ”a” is smaller than ”b”, ”beddc” is greater than ”beczz”.

To compare sequences of different lengths, the shorter sequence is usually padded at the end with enough blanks (a special symbol that is treated as smaller than any other alphabet). For example, ”aa” is smaller than ”aaa”.

(1).

Fill in the blanks using $>$ or $<$.

card ____________ cast

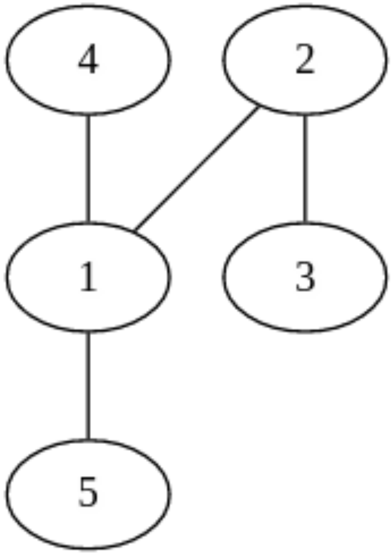
ntu ____________ ntust

ntu ____________ ntnu

When applied to sequences of numbers, lexicographical order is to compare the numerical value of elements. For example, the permutations of $\{1, 22, 18\}$ in lexicographical order are $1, 18, 22 < 1, 22, 18 < 18, 1, 22 < 18, 22, 1 < 22, 1, 18 < 22, 18, 1$.

Now, given an undirected acyclic graph, each node has a number on it. You can choose any node to start traveling the whole graph, but you must travel it by Depth-First Search (Depth-First Traversal). It is guaranteed that the graph is connected, and no multiple edges between any two nodes.

In the following graph, both $2, 1, 4, 5, 3$ and $3, 2, 1, 4, 5$ are valid DFS traversal, and $4, 1, 2, 5, 3$ is not.



Similarly, we can compare two traversals by lexicographical order, for example, $2, 1, 4, 5, 3$ is smaller than $3, 2, 1, 4, 5$.

(2).

In the above graph, to make the minimum lexicographical order DFS traversal, which node should you pick to start with? Explain why this choice is correct.

(3).

A node v has one or more adjacent nodes, denoted by $N(v)$. Assuming the complexity of sorting the adjacent nodes of node v is $O(N(v) \log N(v))$, please prove that $\sum_{i=1}^{n} N(i) \log N(i)$ (the complexity of sorting the adjacent nodes of each node if there are n nodes) $\in O(n \log n)$.

(4).

Please design a $o(n^2)$ greedy algorithm to find the minimum lexicographical order DFS traversal in the undirected acyclic graph that contains n nodes numbered from 1 to n. Prove the correctness of your strategy (why your greedy choice is right?) and describe the time complexity of your algorithm.

Make sure to clearly describe your answer, and you can use the answer in (1). (2). (3). to help you complete the explaining.

4. Yet another lexicographical order

We recommand you to complete the problem ”2. Lexicographical order” first before doing this problem.

Someday, we forget the order of the English alphabet (the 26 letters: $a, b, c, ..., z$). Luckily, some lexicographical orders of some strings had been recorded. Can we analyze the correct order of the letters?

For example:

$$\text{dcd} < \text{dcc} < \text{bddd} < \text{bbbbc} < \text{bbbba} < \text{c}$$

From the relationship dcd < bddd < c, we can know $d < b < c$, because the first letters of them are different.

And from bbbbc < bbbba, we can know $c < a$.

So the correct order of the letters is $d < b < c < a$.

(1).

Another lexicographical order of some strings was found.

$$\text{zyx} < \text{yzx} < \text{yzy} < \text{yxv} < \text{wwww} < \text{wwvx}$$

Please find out the correct order of the letters v, w, x, y, z by observation.

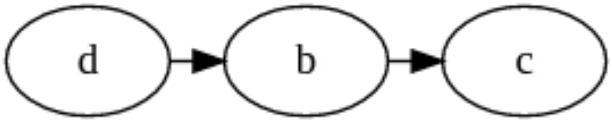
Show your work just like the above demonstration.

Actually, there's an algorithm that can solve this kind of problem.

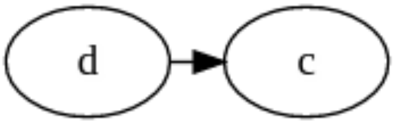
For the same example above, firstly, left align these strings.

1. *dcd*
2. *dcc*
3. *bddd*
4. *bbbbc*
5. *bbbba*
6. *c*

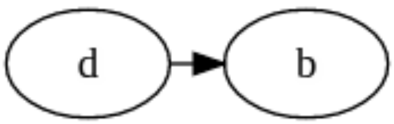
For the first letter of every string, we can create a graph to indicate their relationship. The edges point to the letter that has a higher order.



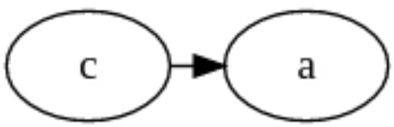
From the second letter, note that you can only create relationships with strings that have the same prefix.



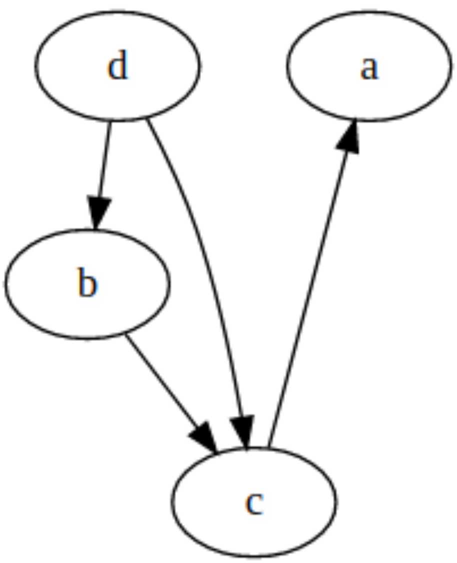
From the third letter between first and second string.



From the second letter between third and fourth string.



From the fifth letter between fourth string and fifth string.

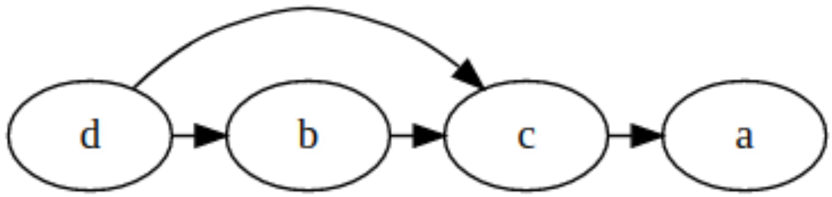


Finally, combine all the relationships.

If there exists at least one solution, the combined graph should be a DAG (Directed

Acyclic Graph).

Find a topological sorting of the DAG.



And we get the answer $d < b < c < a$.

(2).

Same as the previous question, for the following lexicographical order of strings:

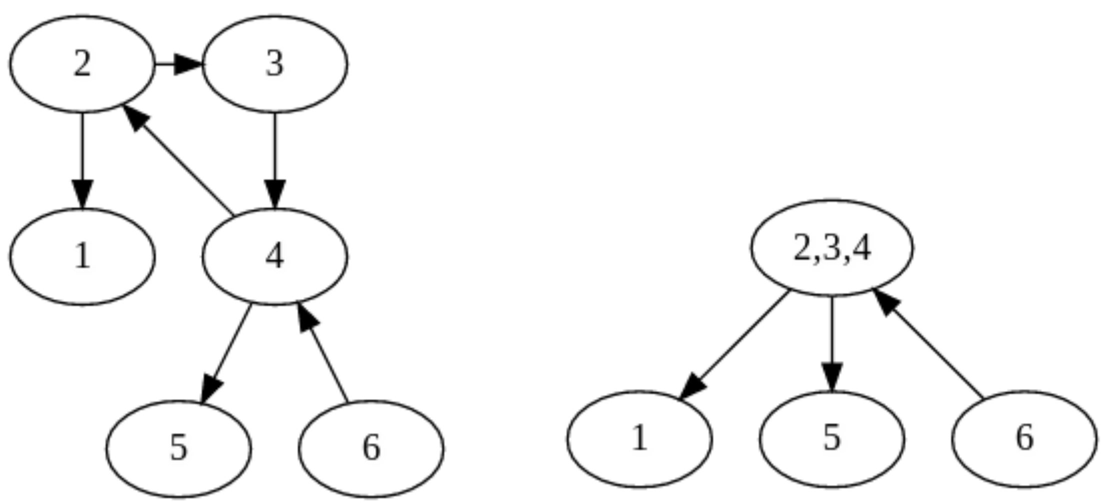
zyx < yzx < yzy < yxv < wwwww < wwvxy

Please find out the correct order of the letters v, w, x, y, z, but this time using the topological sorting and show your work.

5. Strongly Connected Components

A directed graph is strongly connected if there is a path between all pairs of vertices. The strongly connected components of a directed graph form a partition into subgraphs that are themselves strongly connected.

Given a directed graph G, we can shrink the strongly connected component down to a single vertex. Here is an example.

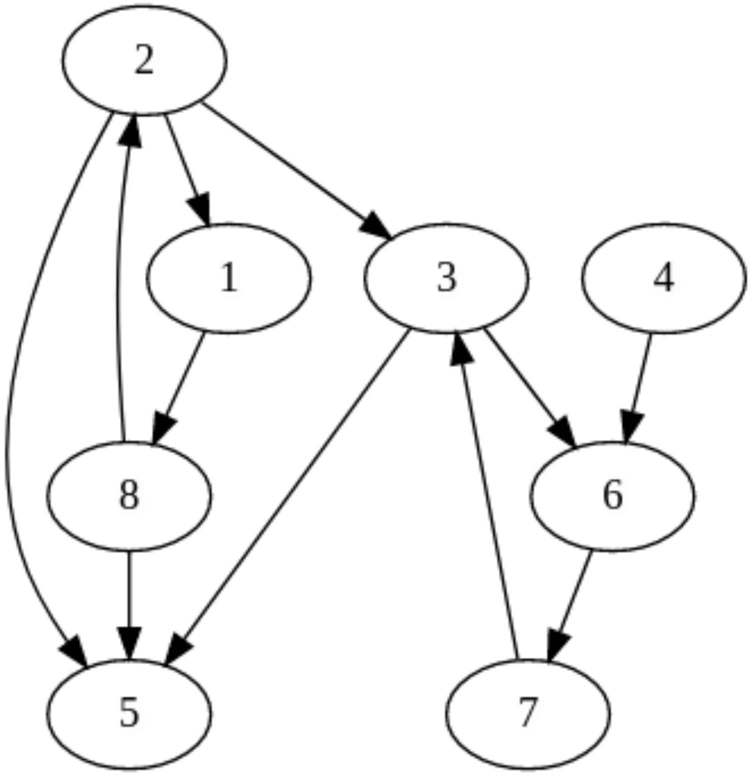


Since the vertices $2, 3, 4$ are strongly connected, they can be shrunk into a vertex (right graph).

After shrinking the graph, you will get a DAG (Directed Acyclic Graph).

(1).

Try to shrink all the SCCs in the graph below, and draw the result. Note that the graph will transform into a DAG.



(2).

Denote the graph whose SCCs are all shrunk as G^{SCC}, and G^T is the transpose of G.

G^T is obtained by reversing the directions of all edges in graph G (in lecture slide: graph.pptx, page 40).

Prove that $(G^{SCC})^T = (G^T)^{SCC}$.