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RANDOM WALK WITH PERSISTENCE AND EXTERNAL BIAS

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The partial differential equation of the random walk problem with persistence of direction and external bias is derived. By persistence of direction or internal bias we mean that the probability a particle will travel in a given direction need not be the same for all directions, but depends solely upon the particle's previous direction of motion. The external bias arises from an anisotropy of the medium or an external force on the particle. The problem is treated by considering that the net displacement of a particle arises from two factors, namely, that neither the probability of the particle traveling in any direction after turning nor the distance the particle travels in a given direction need be the same for all directions. A modified Fokker-Planck equation is first obtained using the assumptions that the particles have a distribution of travel times and speeds and that the average time of travel between turns need not be zero. The final equation incorporating the assumption of a persistence of direction and an external bias is then derived. Applications to the study of diffusion and to long-chain polymers are then made.

Introduction. The classical random walk problem deals with a particle moving in a series of steps, where the length of the step, time between steps, and direction of the step are independent of each other and of preceding steps. This problem was first formulated by K. Pearson (1900), although it had been treated and solved in a different form as early as 1880 by Lord Rayleigh (1945). For some excellent reviews of the history and work done on this problem the articles by S. Chandrasekhar (1943), G. E. Uhlenbeck and L. S. Ornstein (1930), L. Infeld (1940), M. C. Wang and G. E. Uhlenbeck (1945), and W. Feller (1949, 1950) may be consulted. Classical random walk, as defined above, may be considered to be a Markoff process, and as such may be described by the Fokker-Planck equation (Kolmogoroff, 1931), the solutions of which yield the probability of a particle being at any point at time t.

Random walk has been applied to numerous different physical situations, many of which are of interest to biology. In the diffusion of gases,

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a method analogous to the classical random walk treatment was used and was thought to yield the correct results (Meyer, 1899), but it has been shown that this method is not really applicable (Chapman and Cowling, 1939; Jeans, 1940; Furry, 1948), since the postulated independence between the steps is not valid for gas particles. However for diffusion in dilute solutions and for Brownian motion, the random walk method does seem to be applicable and yields useful results (Einstein, 1926; Jost, 1952). In the treatment of genetic variability in populations (Wright, 1945; Feller, 1951), the random walk method has been found to be applicable. However, as yet the equations have not been capable of solution for any but a few restricted cases.

Another interesting application of the random walk method is found in the study of long-chain molecules. One may perhaps assume as a first approximation that there is no fixed bond angle between atoms. This assumption is of course false, but one may consider a group of atoms as a unit, and this assumption will hold fairly well for the unit. Then by considering the chain as having been formed by a particle executing a random walk, where the atoms of the molecule are to be placed at each point where the particle made a turn, the probability distribution of the possible lengths of the molecule may be calculated (Guth and Mark, 1934). Thus upon stretching the molecule (as in rubber or muscle), the entropy changes may be calculated (Treloar, 1942).

As a final example, it may be pointed out that the motion of organisms under the influence of various stimuli (Fraenkel and Gunn, 1940) may, to a good approximation, be treated by this method. This has been done by K. Pearson (1906) for the case of random migration of animals without any stimulus, and the results were applied to the migration of mosquitoes into a new region. A more general treatment has been given by J. G. Skellam (1951), who took the population growth into account. The random walk method has also been used by C. A. Coulson (1947) in an application to the movement of some Mollusca in the presence of the stimulus of non-isotropic lighting, and by D. H. Wilkinson (1952) in a discussion of a theory of pigeon-homing by means of random search.

A more generalized problem may be considered, however. That is, the individual particle's steps need *not* be independent of each other, i.e., the random walk is not a Markoff process. The most familiar example of this, although applied to a different situation, is J. Jeans' treatment (1940) of the "persistence of velocity."

The problem of non-independence of steps has been treated by various authors with different limiting conditions, all of which include the fact