Extending snarks

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mod k-flow

- Let G be a graph.
- Consider the pair (D, φ), where D is an orientation of G and $\varphi : EG \to \{1, \cdots, k-1\}$ a function that assigns to each edge α of D an integer $\varphi(\alpha)$, called the *weight of* α.
- For every vertex $v \in VG$, we say that $\varphi(v)$ is the *net-outflow of* v, such that $\varphi(v)$ is the sum of all edge weights leaving v minus all edge weights entering v.
- We say that vertex v is *balanced* if $\varphi(v) = 0$; A vertex is balanced $\pmod{k}$ if $\varphi(v) \equiv 0 \pmod{k}$.
- A *k-flow* is a pair (D, φ) in which each vertex is balanced. A *mod k-flow* is a pair (D, φ) in which each vertex is balanced $\pmod{k}$.

Conjectures

- Tutte proposed three celebrated conjectures regarding flows of general graphs as a generalization for the face-colouring problems for planar maps. Known as the 3-, 4- and 5-flow conjectures, these are:
  - Every graph free of 1-cuts has a 5-flow.
  - Every graph free of 1-cuts with no Petersen minor has a 4-flow.
  - Every graph free of 1- and 3-cuts has a 3-flow.

Snarks

- Snarks are cubic graphs with no 3-edge-colouring.
- A cubic graph has a 3-edge-colouring if and only if it admits a 4-flow.

About the title

We extend the knowledge of snarks to non-cubic graphs

Definitions and notations

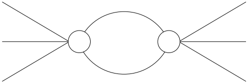
- For the 4-flow problem we have $\varphi : EG \rightarrow \{1, 2, 3\}$. We replace 4-flow by mod 4-flow.
- Every mod 4-flow can be converted to a 4-flow.
- These weights are equivalent to $\varphi : EG \rightarrow \{1, 2, -1\} \pmod 4$
- Therefore, we have unoriented edges of weight 2 (since $2 \equiv -2 \pmod 4$) and oriented edges of weight 1.
- We say that a graph G is *not-4* if it does not admit a 4-flow.

Sparks

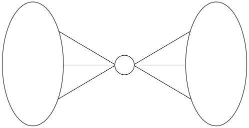
- A *spark* is a not-4 graph which does not have a specified set of simple reductions.
- If there are counterexamples to the 4-flow Conjecture, they must include sparks.

Reducible configurations

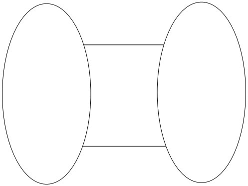
The specified set of simple reductions:



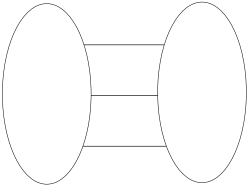
(a) Digon



(b) Cut-vertex



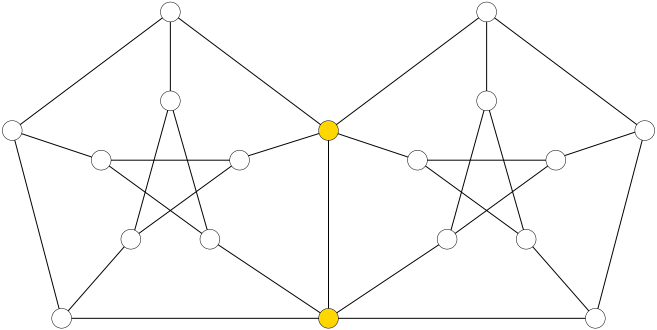
(c) 2-cuts



(d) 3-cuts

2-sum of two Petersen graphs

An example of a spark

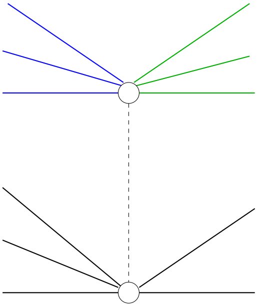


The 2-sum of any two sparks is a spark.

2-sum of sparks is a spark

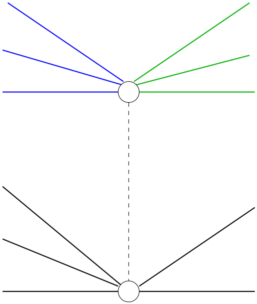
Proof:

- The edge β is the dashed one.
- Let a be the edges in blue and b the edges in green.
- Let the right side be the graph G and the left side be the graph H.
- Suppose $G \cup H$ is not a spark, thus it has a mod 4-flow.



2-sum of sparks is a spark

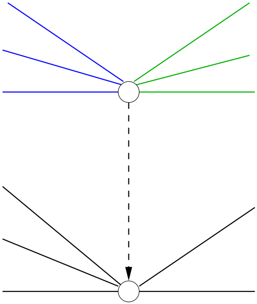
- $\varphi(b) + \varphi(a) + \varphi(\beta) \equiv 0 \pmod 4$.
- $\varphi(\beta) \in \{\pm 1, 2\} \pmod 4$.
- Notice that $\varphi(a) \in \{0, 1, 2, -1\} \pmod 4$.
- Since 1 and -1 are simply the reverse of each other, we may look only for $\{0, 1, 2\} \pmod 4$.



2-sum of sparks is a spark

For $\varphi(\beta) = 1$:

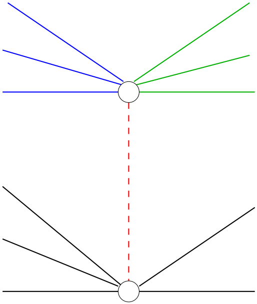
- If $\varphi(a) = 0$, then G has a mod 4-flow.
- If $\varphi(a) = 1$, then H has a mod 4-flow with $\varphi(\beta) = -1$.
- If $\varphi(a) = 2$, then H has a mod 4-flow with $\varphi(\beta) = 2$.



2-sum of sparks is a spark

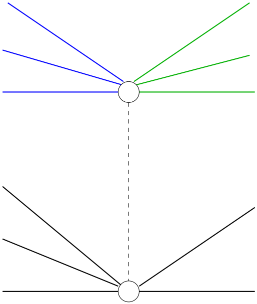
For $\varphi(\beta) = 2$:

- If $\varphi(a) = 0$, then G has a mod 4-flow.
- If $\varphi(a) = 1$, then H has a mod 4-flow with $\varphi(\beta) = -1$.
- If $\varphi(a) = 2$, then H has a mod 4-flow.



2-sum of sparks is a spark

In each case we reach a contradiction. Therefore, the 2-sum of two spark has no mod 4-flow. $\square$



Motivation

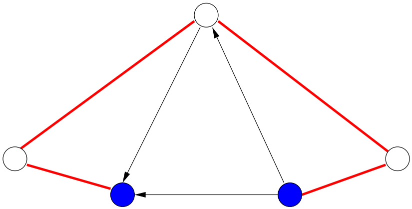


Figure : A mod 4-flow for the graph. Weight two edges are shown in red.

- The complement of a set of weight 2 edges is an interesting object of study.
- After choosing the set of weight 2 edges, if one can orient its complement, then one can find a mod 4-flow for the graph.

Orienting eulerian graphs

A graph is *eulerian* if all of its vertices have even valence. Let G be an eulerian graph with an *even labelling* $\pi : VG \to \{0,1\}$ such that the number of vertices labelled 1 is even. A *mod 4 orientation* of (G, π) is an orientation of the edges of G such that:

$$\begin{cases} \varphi(v) \equiv 2 \pmod 4 \text{ if } \pi(v) = 1 \\ \varphi(v) \equiv 0 \pmod 4 \text{ if } \pi(v) = 0 \end{cases}$$

Lemma 4.1

Let v be any vertex of an eulerian graph G. If all edges incident to v, but α, have a direction, then $\varphi(v) \in \{\pm 1\} \pmod 4$.

Proof Since α is not oriented and G is eulerian, the net-outflow of v is the subtraction of either an even number by an odd number or an odd number by an even number; Therefore, $\varphi(v)$ must be odd and $\varphi(v) \in \{1, 3\} \pmod 4$, and since $3 \equiv -1 \pmod 4$, it follows that $\varphi(v) \in \{\pm 1\} \pmod 4$. $\square$

Corollary 4.2

Let v be any vertex of an eulerian graph G. If all edges incident to v, but α, have a direction, then there is a direction for α such that v balances.

Theorem 4.3

If G is a connected eulerian graph and π an even labelling of VG, G has a mod 4-orientation.

- **Proof** If G contains just one vertex and no edges, then that vertex has label 0 and a mod 4-orientation.
- For a graph with at least one edge, choose two vertices u and v such that $\alpha := (u, v)$. Contract the edge α in G. If $\pi(v) = \pi(u)$, then clearly $\pi(w) = 0$. Otherwise, their flows will not balance when summed and $\pi(w) = 1$. Therefore, $\pi(w) := \pi(u) + \pi(v) \pmod 2$.
- By induction hypothesis, $(G/\alpha, \pi)$ has a mod 4-orientation. Assign the directions of the edges of G/α to G.
- All the vertices of G, except u and v, are balanced. By Lemma 4.1, $\varphi(u), \varphi(v) \in \{\pm 1\}$.

- If $\pi(w) = 0$, then the equation $\varphi(u) \equiv -\varphi(v) \pmod 4$ holds. By Corollary 4.2, there is an orientation for alpha which balances u. Since $\varphi(u) \equiv -\varphi(v) \pmod 4$, v also balances for $\pi(u) = \pi(v) = 0$. For $\pi(u) = \pi(v) = 1$, we reverse the orientation of edge α such that both vertices unbalance in exactly 2.
- If $\pi(w) = 1$, then the equation $\varphi(u) \equiv \varphi(v) \pmod 4$ holds. By Corollary 4.2, there is an orientation for alpha which balances u. Since $\varphi(u) \equiv \varphi(v) \pmod 4$, the direction of alpha will unbalance v in exactly 2, and vice-versa. $\square$

Results

Definition Let G be a 2-edge-connected graph. A set of weight 2 edges is *feasible* in G if the complement subgraph is eulerian.

Definition Let G be a 2-edge-connected graph. Let M be any feasible set of weight 2 edges. A vertex v is labelled 1 if it is incident to an odd number of edges of M, and labelled 0 otherwise.

Results

The Theorem presented yields the following results:

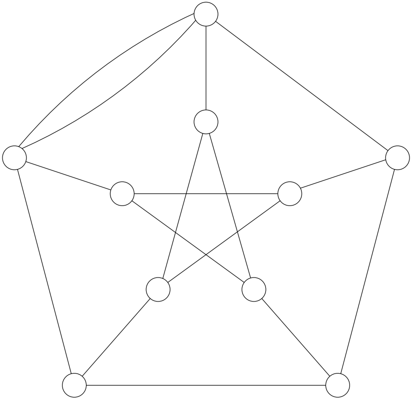
- Let G be a 2-edge-connected graph with no 1-cuts. Let M be any feasible set of weight 2 edges. If every component of $G[EG \setminus M]$ has an even number of 1-vertices, then G has a mod 4-flow.
- Let G be a spark. For every set of feasible weight 2 edges M, the graph $G[EG \setminus M]$ is disconnected and has at least one component with an odd number of vertices labelled 1.

The Algorithm

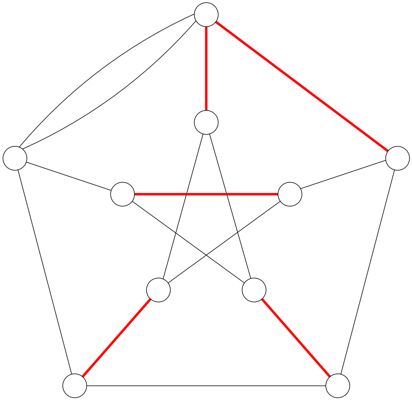
An algorithm for a mod 4-flow

for all set of feasible weight 2 edges M **do**
  $H \leftarrow G[EG \setminus M]$
    $\triangleright$ Label all vertices of H accordingly to its incidence to M
  if $\forall c \in H$, c has an even number of vertices labelled 1 **then**
    $D \leftarrow$ a mod 4-orientation of all components of H
    return $(D, \varphi(D) \cup \varphi(M))$ $\triangleright$ A mod 4-flow of G
  end if
end for
return False

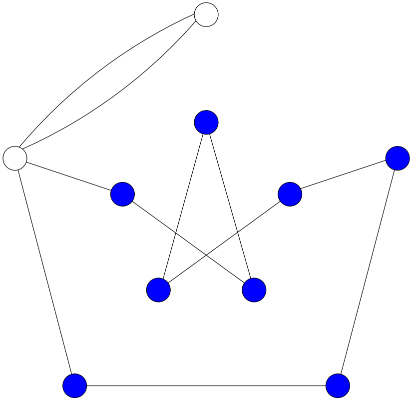
Example of the algorithm



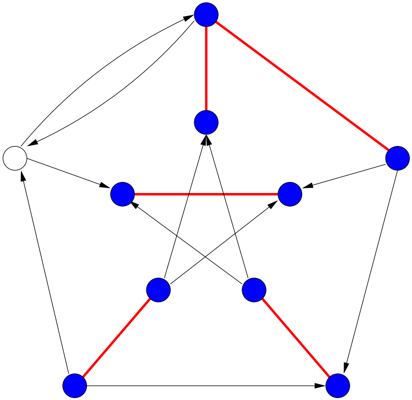
A feasible set of weight-2 edges (shown in red)



The eulerian subgraph (blue vertices are labelled 1)



A mod 4-flow of G



Tests

- Two programs were written and used.
- *Algorithm 1* tests in G all possible weight 1 and weight 2 edges that make up a 4-flow.
- *Algorithm 2* uses the Theorem: tests all possible sets of weight 2 edges and analyzes the eulerian complement of each.

Complexity

- Let $\lambda(G)$ represent the maximum degree of VG

Type	Alg. 1	Alg. 2
Cubic	$O(6^n)$	$O(3^n)$
4-regular	$O(21^n)$	$O(8^n)$
5-regular	$O(60^n)$	$O(15^n)$
General case	$o(\lambda(G)^{3n})$	$o(\lambda(G)^{2n})$

Efficiency

Time in seconds needed to test whether or not a graph is a spark.

Graph	Alg. 1	Alg. 2	Gain
Double-star snark	0.780s	0.137s	82.5%
Flower-snark J_9	12.614s	0.721s	94.3%
(3, 10)-cage	12.537s	0.903s	92.8%
Petersen 2-sum	0.332s	0.074s	77.7%
Vertex-transitive cubic graph on 86	28.046s	1.096s	96%

Thank you!