HERMITE INTERPOLATION

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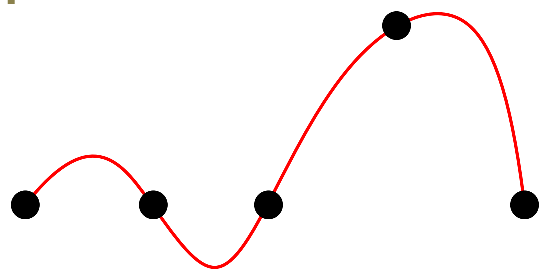
Recall: Issues with polynomial interpolation



- The high degree of the polynomial produces a curve with higher roughness (i.e., it can wiggle a lot) than probably desired

  $\rightarrow$ The variation diminishing property is not satisfied!

- Intuitively: adding data points should improve the resulting polynomial curve. But that is not always the case! This is known as *Runge's phenomenon*

- Lagrange's formula requires $\Theta(n^2)$ additions and products, which is quite a lot (although more efficient versions exist)

- If one has computed $\gamma(t)$ for n points and needs to add one extra point, everything needs to be recomputed

- Lagrange's formula is not numerically stable: small variations in the input points can produce large variations in the final curve

- The method is not easy to make interactive: if the curve is not what one wants, (and you cannot modify the data points) all you can do is to add more points

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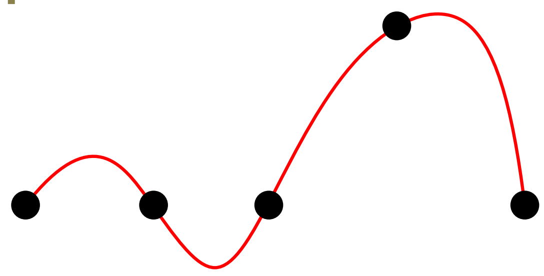

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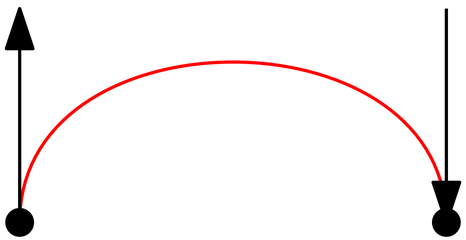

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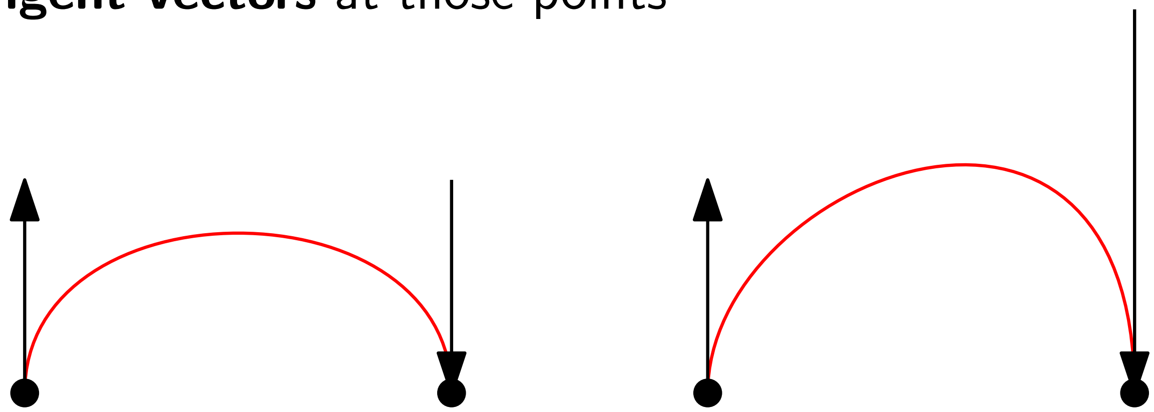

Note the effect of modifying one of the tangent vectors

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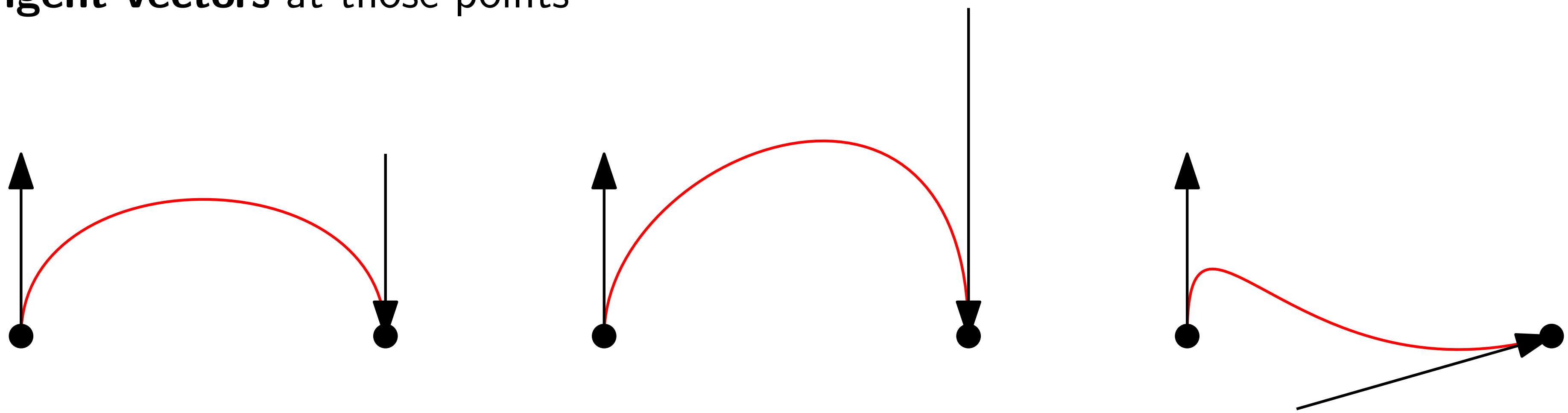

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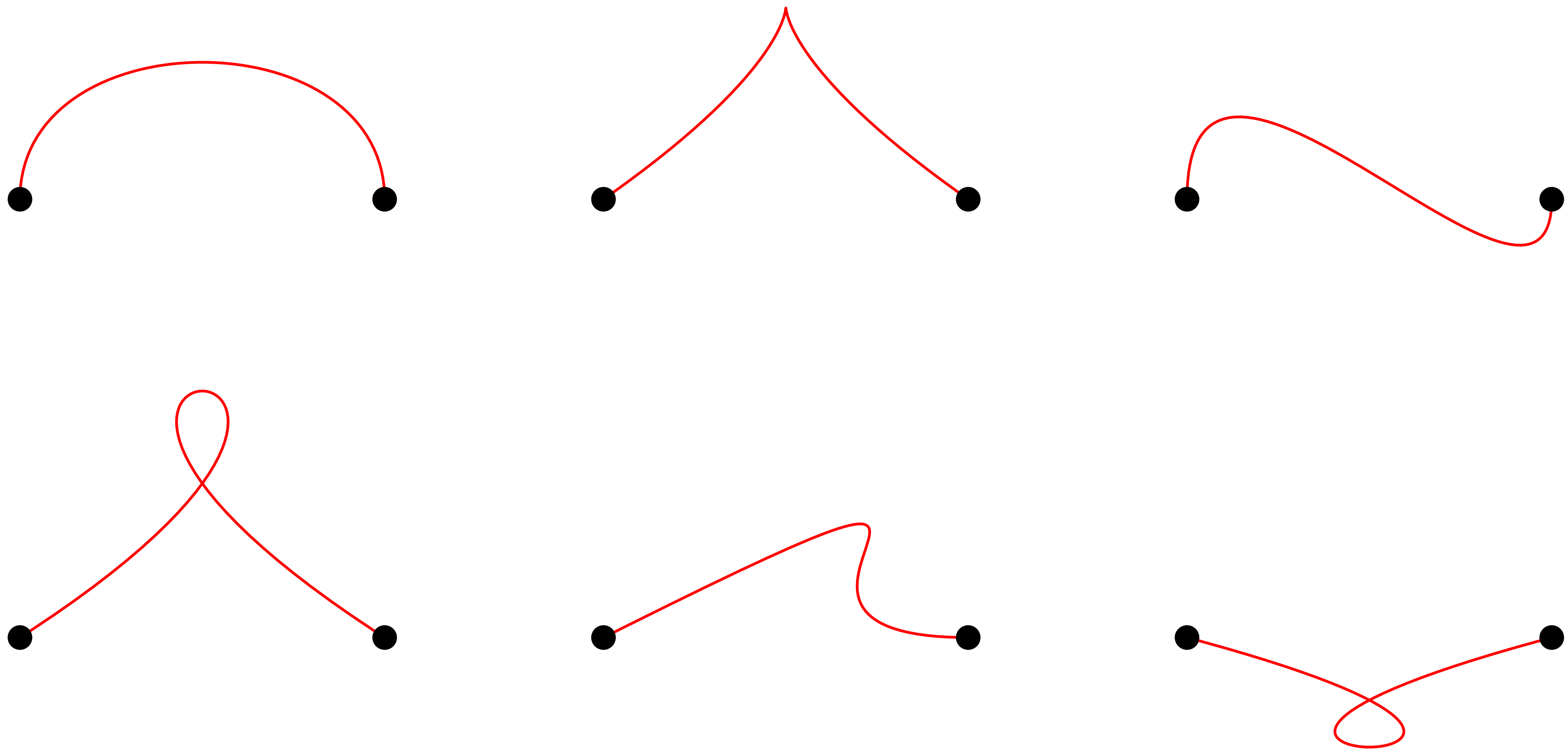
HERMITE INTERPOLATION

A single Hermite curve can take many different shapes



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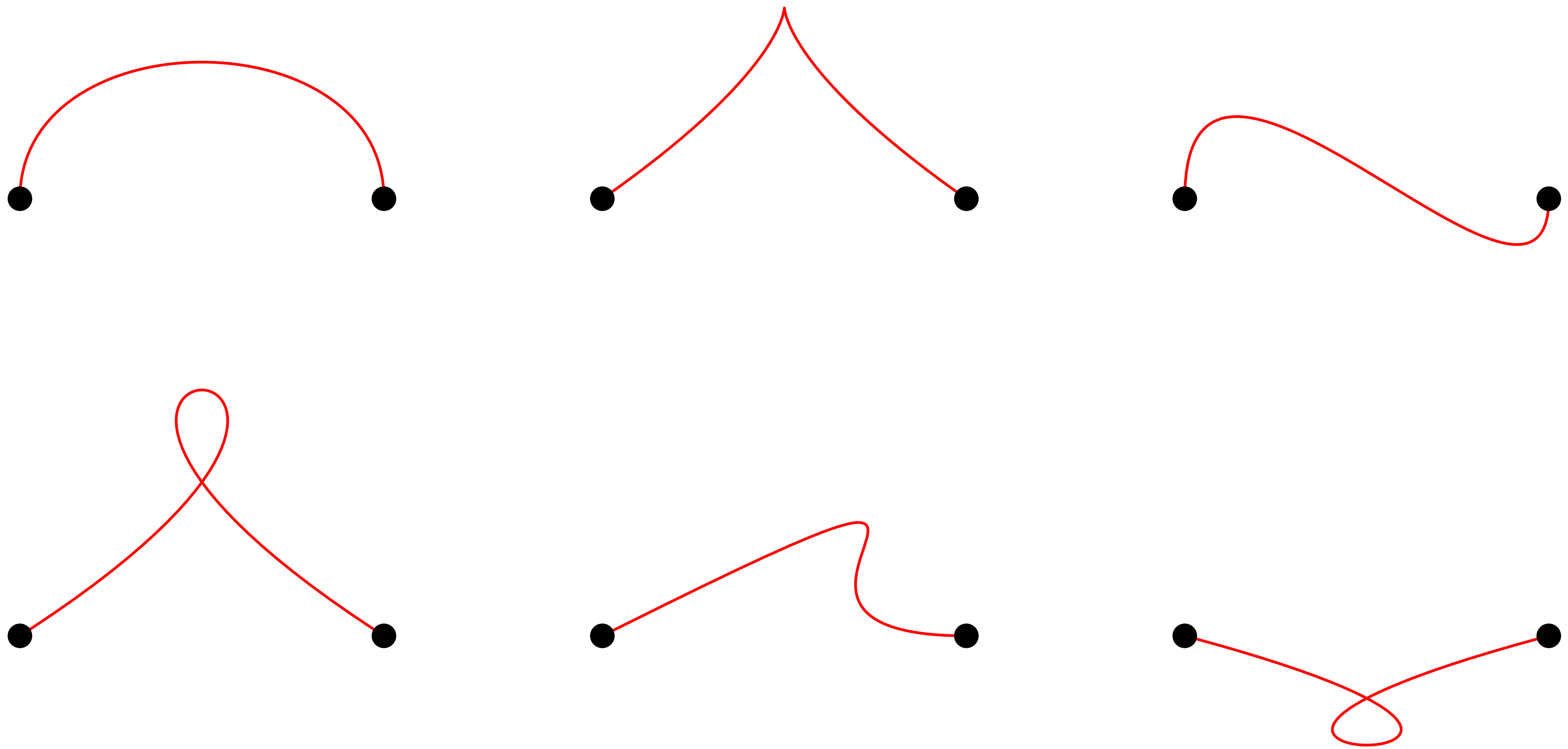

Can you guess how the tangent vectors look like?

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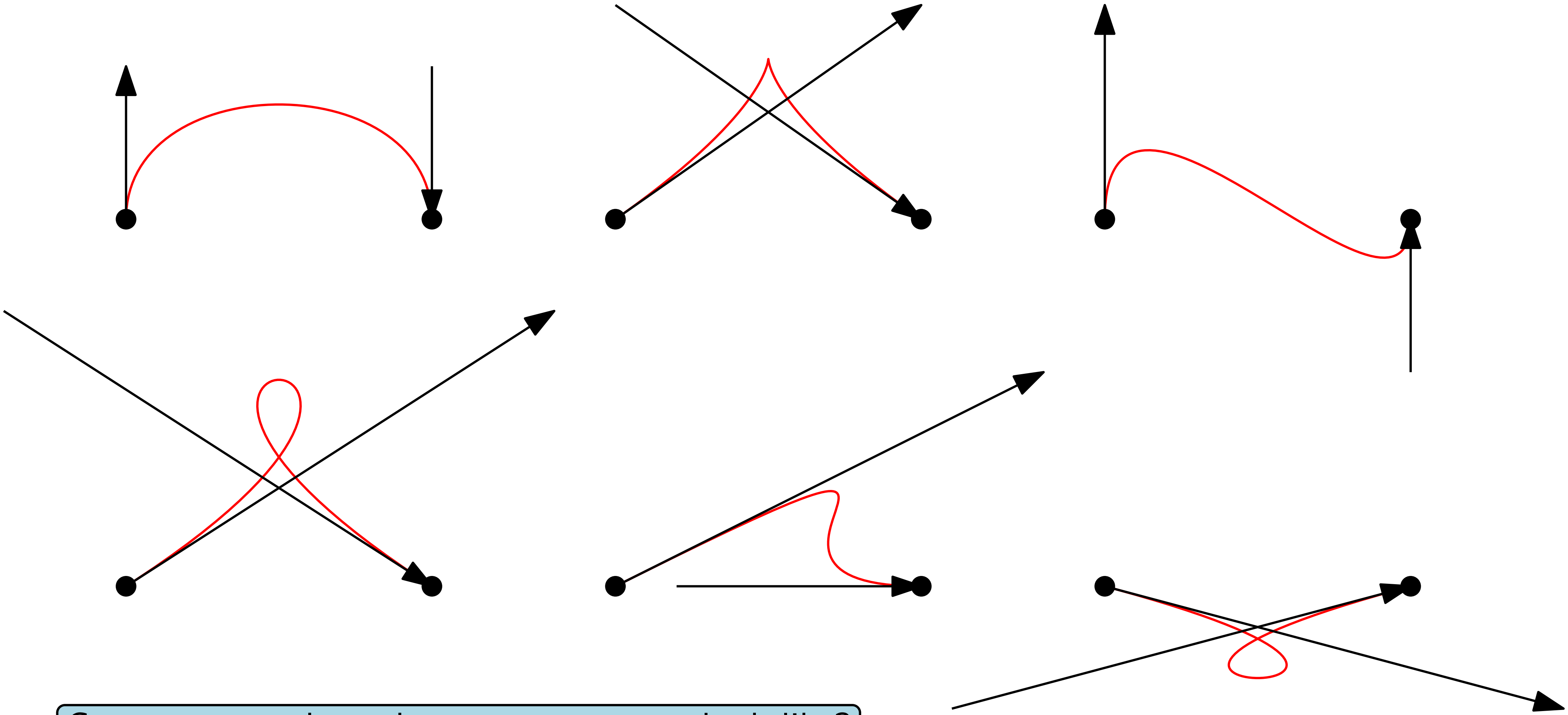

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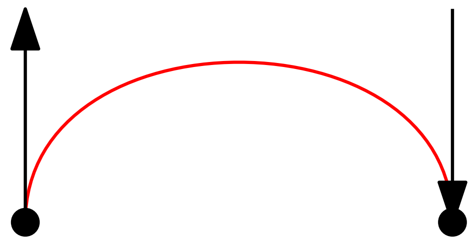
Cubic Hermite interpolation



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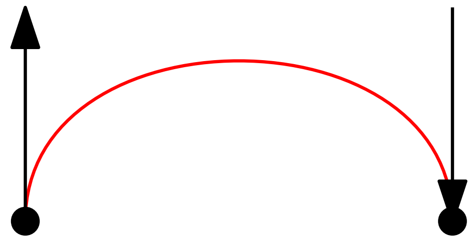

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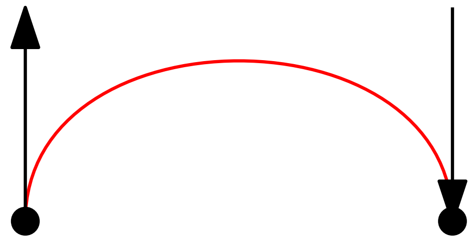

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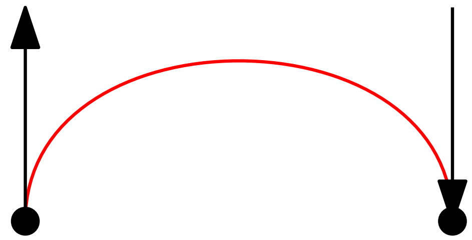

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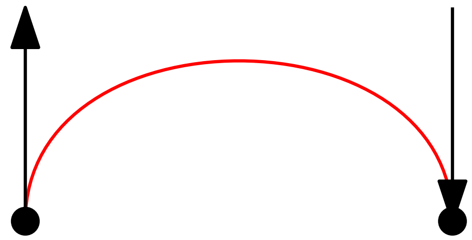

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Proof. First we prove uniqueness, as we did with Lagrange interpolation.

Secondly, we prove that it exists, by deducing an expression for it.

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Let γ and δ be two curves that satisfy the constraints above, and consider a third curve r defined as $r(t) = \gamma(t) - \delta(t)$. Clearly, $r(t)$ is a polynomial of degree at most three. Since $r(0) = 0$ and $r(1) = 0$, we can write it as $r(t) = at(t-1)(t-t_0)$, for two unknown values a and t_0.

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Now consider $r'(t)$. We know that $r'(t) = \gamma'(t) - \delta'(t)$, so we have that $r'(0) = 0$ and $r'(1) = 0$. We can also write $r'(t)$ as follows

$$r'(t) = at(t-1) + at(t-t_0) + a(t-1)(t-t_0)$$

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Therefore, since $r'(0) = 0$, we have $0 = a(-1)(-t_0) = at_0$

Similarly, since $r'(1) = 0$, we have $0 = a(1)(1-t_0) = a(1-t_0)$

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iff a is 0, thus $r(t) = 0$ for all t, and therefore $\gamma(t) = \delta(t)$

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- $P_0 = \gamma(0) = D$, thus $D = P_0$
- $\overrightarrow{v_0} = \gamma'(0) = C$, thus $C = \overrightarrow{v_0}$
- $P_1 = \gamma(1) = A + B + C + D$, thus $B = P_1 - P_0 - \overrightarrow{v_0} - A$
- $\overrightarrow{v_1} = \gamma'(1) = 3A + 2B + C = 3A + 2(P_1 - P_0 - \overrightarrow{v_0} - A) + \overrightarrow{v_0} = A + 2P_1 - 2P_0 - \overrightarrow{v_0}$

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Exercise: verify that $\gamma(t)$ satisfies the four constraints of the theorem

Cubic Hermite blending functions

The concept of blending functions is fundamental for many curve design methods

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These are the four blending functions in Hermite interpolation:

$F_1(t) = 2t^3 - 3t^2 + 1$ $\quad$ $F_3(t) = t^3 - 2t^2 + t$

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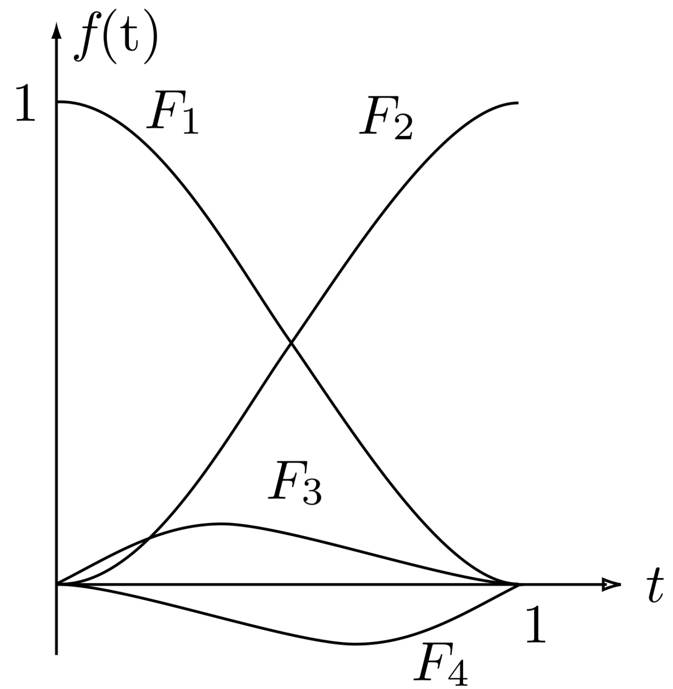

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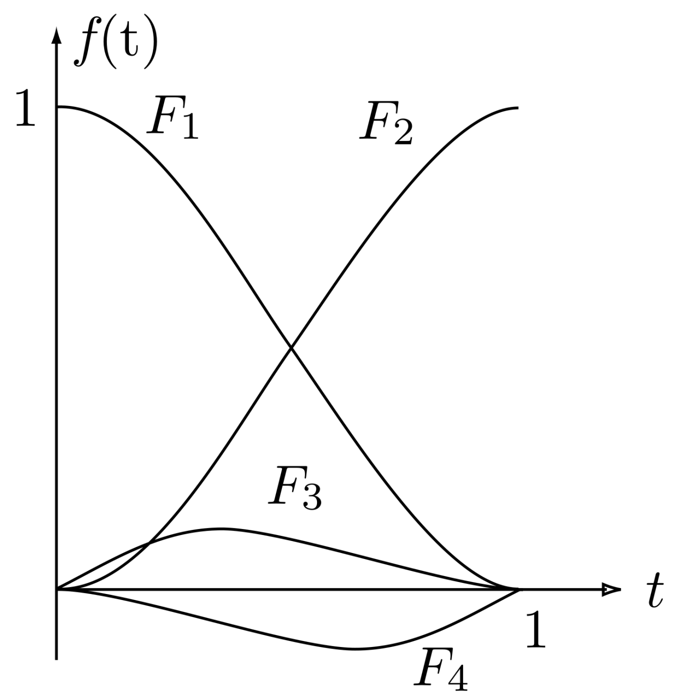

The functions control the weights of $P_0, P_1, \vec{v_0}, \vec{v_1}$:

- For $t = 0$, $F_1(t) = 1$, and all others are 0: this makes the curve start at P_0. Similarly, at $t = 1$, $F_2(t) = 1$, and all others are 0, so the curve ends at P_1.
- $F_3(t)$ has a less clear behavior: for small values of t, it has little effect (the curve stays close to P_0). For t around $1/3$, $F_3(t)$ has its maximum influence, pulling the curve in direction $\vec{v_0}$. For larger t, $F_3(t)$ again has almost no effect.
- $F_4(t)$ behaves in a symmetric way to $F_3(t)$.

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Consider $f(x) = Ax + W$. Recall only linear part of f applies to a vector, i.e., $f(\overrightarrow{u}) = A\overrightarrow{u}$

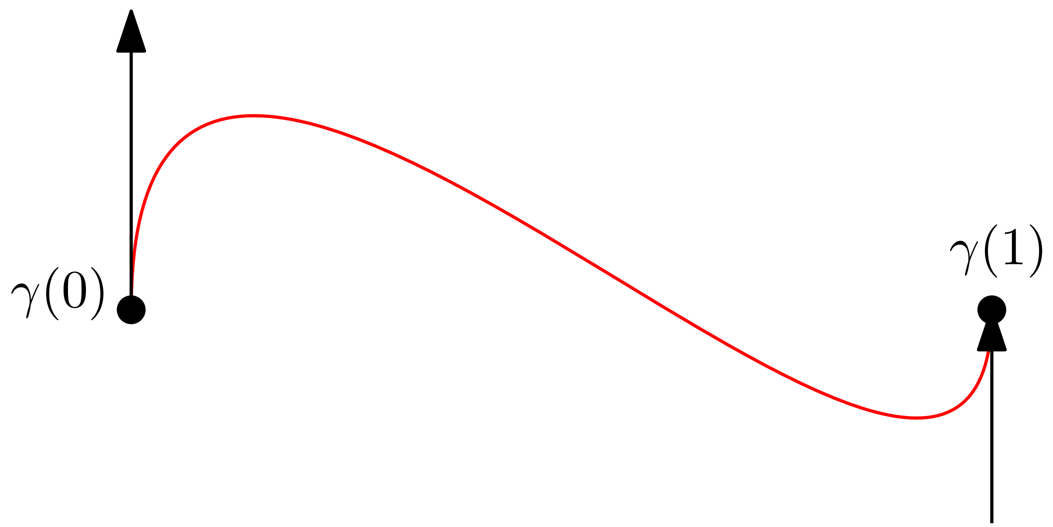
$\gamma(t) = a(t)P_0 + (1 - a(t))P_1 + \alpha(t)\overrightarrow{v_0} + \beta(t)\overrightarrow{v_1}$, for some functions $a(t), \alpha(t), \beta(t)$

$$\begin{aligned}
f(\gamma(t)) &= A(a(t)P_0 + (1 - a(t))P_1 + \alpha(t)\overrightarrow{v_0} + \beta(t)\overrightarrow{v_1}) + W \\
&= a(t)AP_0 + (1 - a(t))AP_1 + \alpha(t)A\overrightarrow{v_0} + \beta(t)A\overrightarrow{v_1} + W \quad \text{(using } W = (a(t) + (1 - a(t))W\text{)} \\
&= a(t)AP_0 + (1 - a(t))AP_1 + \alpha(t)A\overrightarrow{v_0} + (a(t) + (1 - a(t))W + \beta(t)A\overrightarrow{v_1} \\
&= a(t)(AP_0 + W) + (1 - a(t))(AP_1 + W) + \alpha(t)A\overrightarrow{v_0} + \beta(t)A\overrightarrow{v_1} \\
&= a(t)f(P_0) + (1 - a(t))f(P_1) + \alpha(t)f(\overrightarrow{v_0}) + \beta(t)f(\overrightarrow{v_1}) \qquad \therefore \text{ it is affine invariant!}
\end{aligned}$$

Clipping Hermite curves

Clipping is a basic operation with curves: extract a continuous part of an Hermite curve $\gamma(t)$ into a new Hermite curve

- $\gamma(t)$ is parametrized in $[0, 1]$



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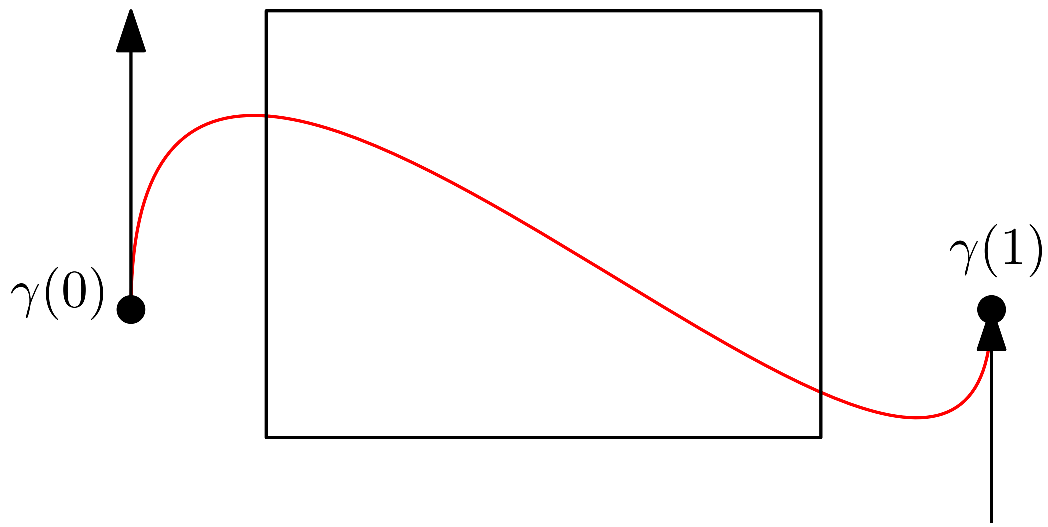

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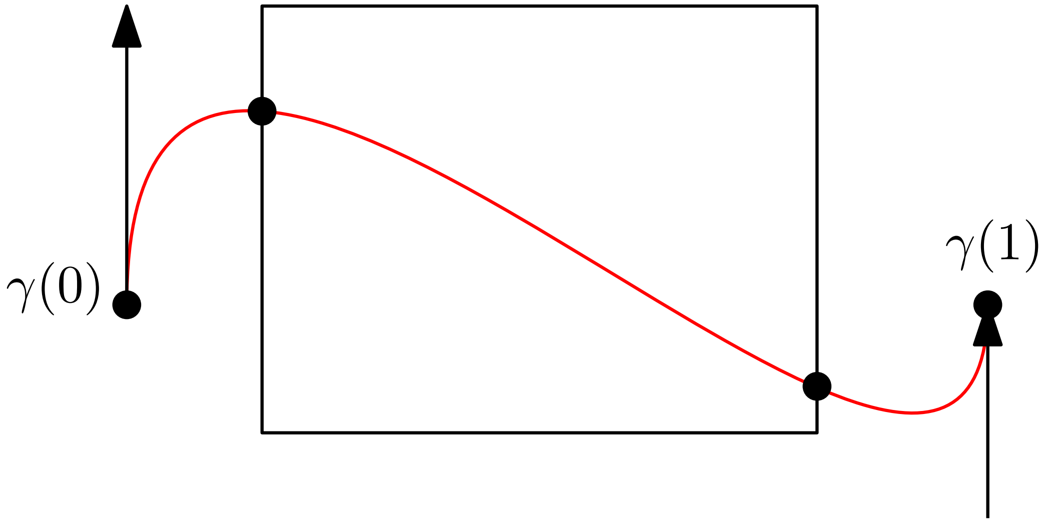

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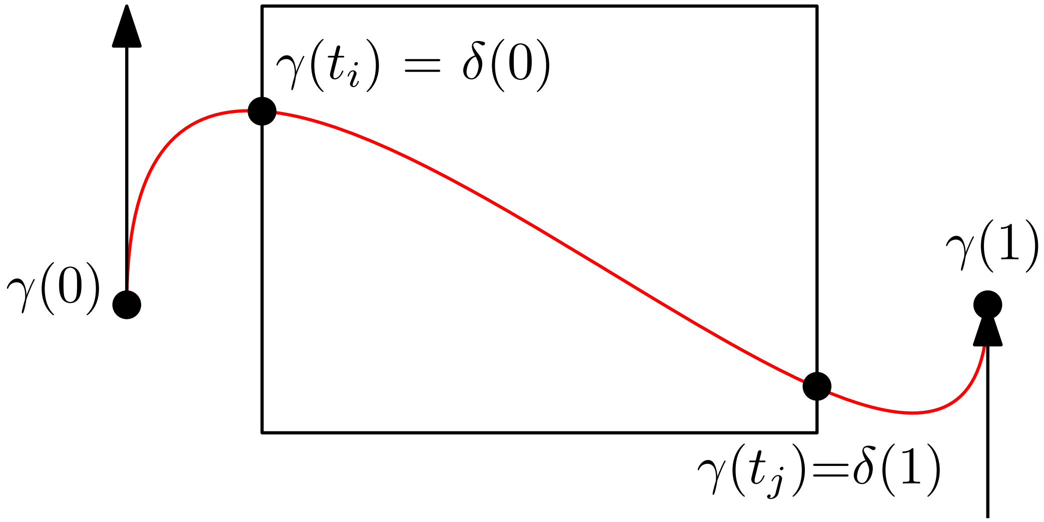

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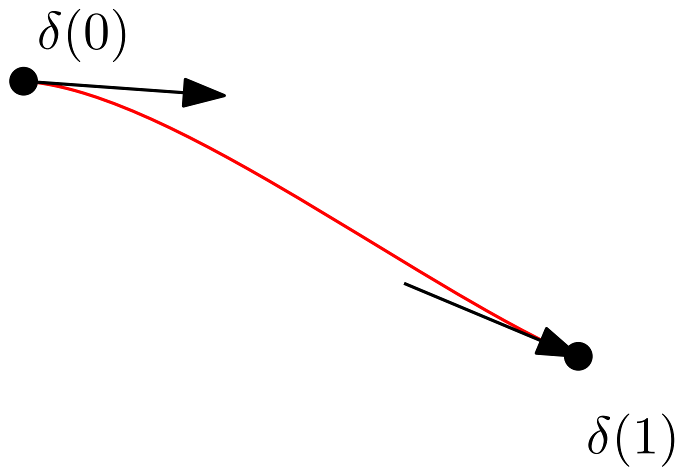

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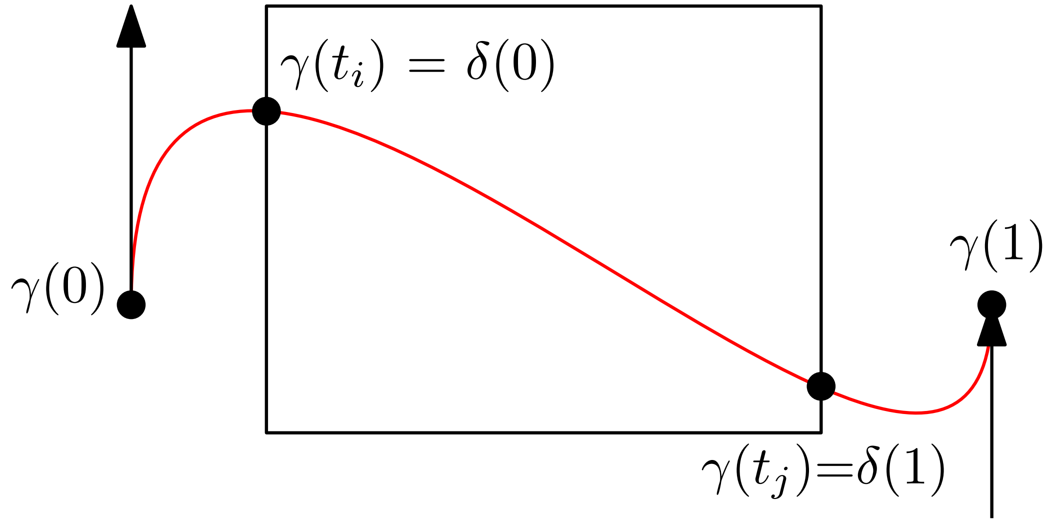

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We need to find the two points and two vectors that define δ

We need to reparametrize that portion of γ:

$$[0, 1] \longleftrightarrow [t_i, t_j]$$

$$s \longleftrightarrow t(s) = t_i + s(t_j - t_i)$$

and make sure that:

$\delta(0) = \gamma(t_i)$, $\delta'(0) = \gamma'(t_i)$

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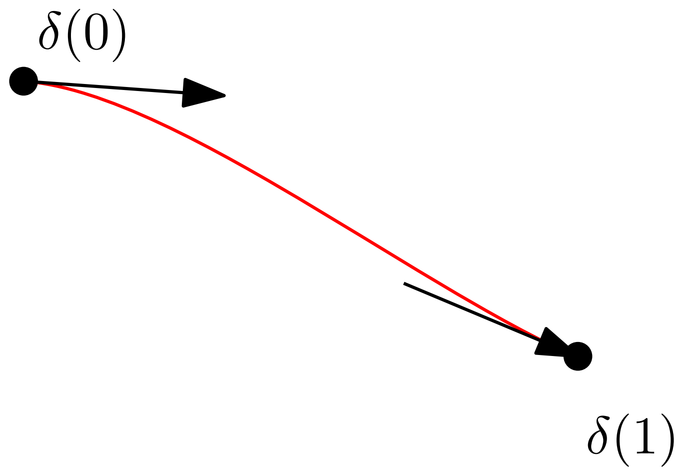

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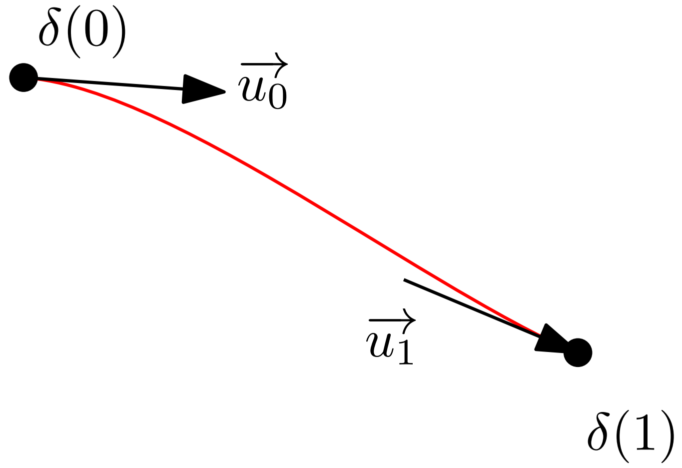

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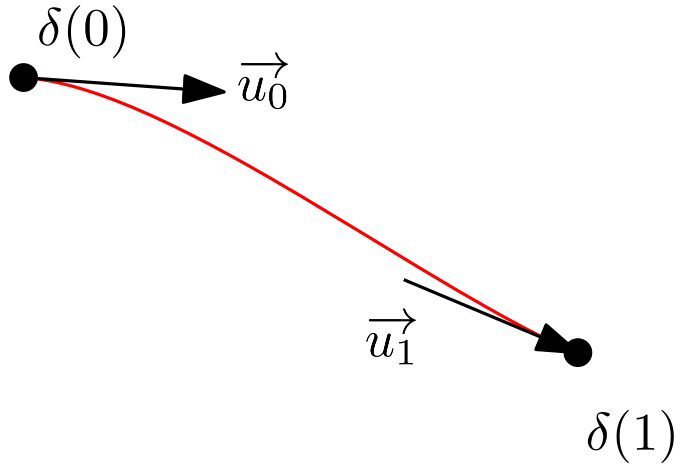

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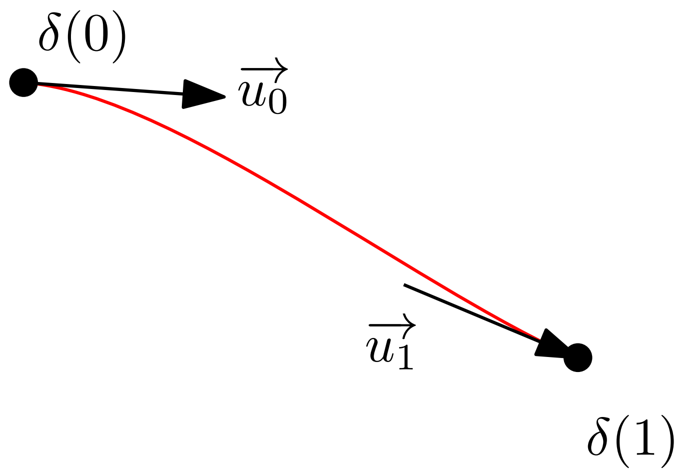

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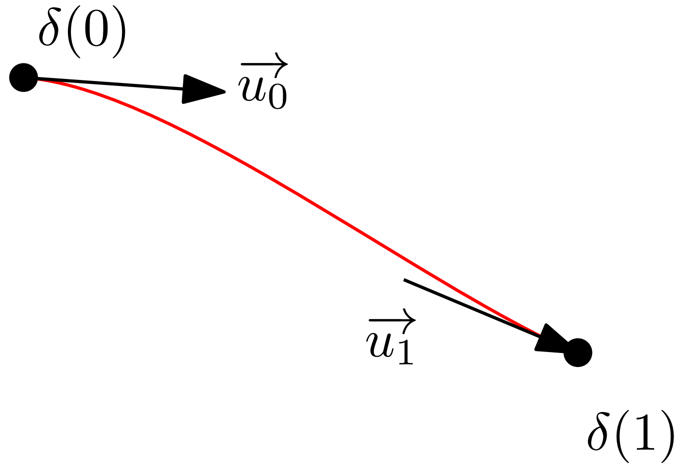

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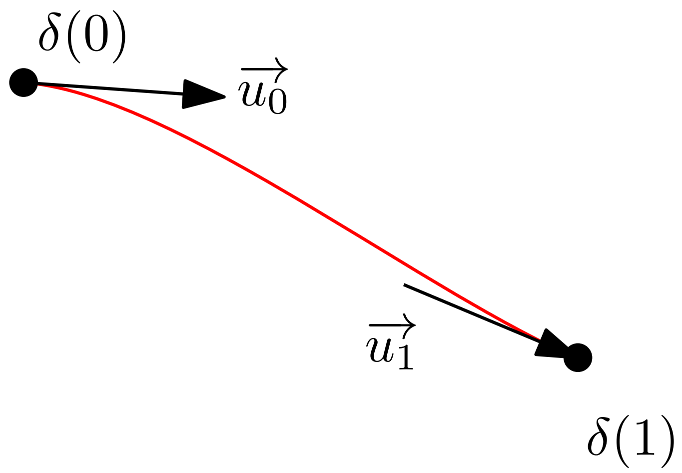

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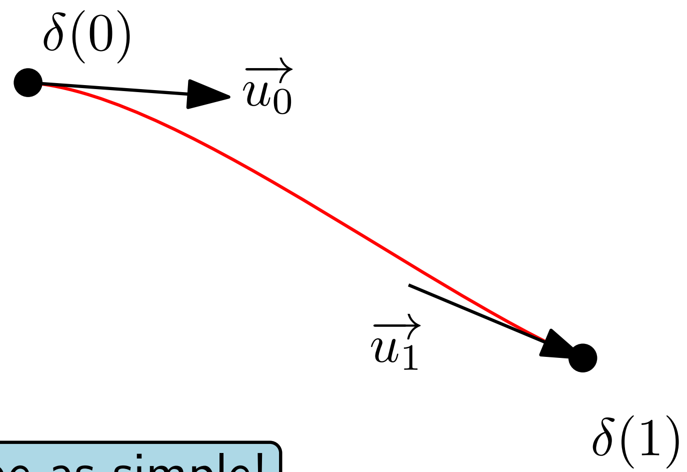

Notice that clipping in a Lagrange polynomial would not be as simple!

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is equivalent to:

$$\gamma(t) = (t^3, t^2, t, 1)\begin{pmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ \overrightarrow{v_0} \\ \overrightarrow{v_1} \end{pmatrix}$$

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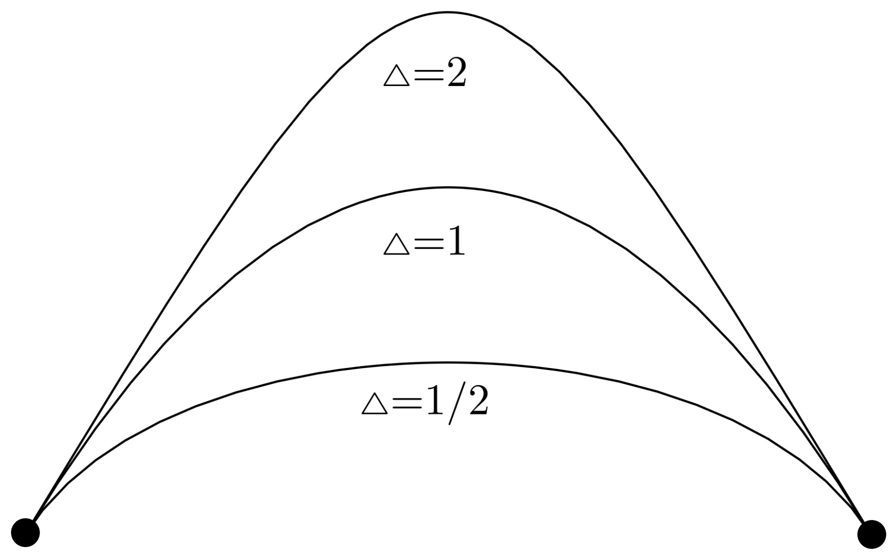

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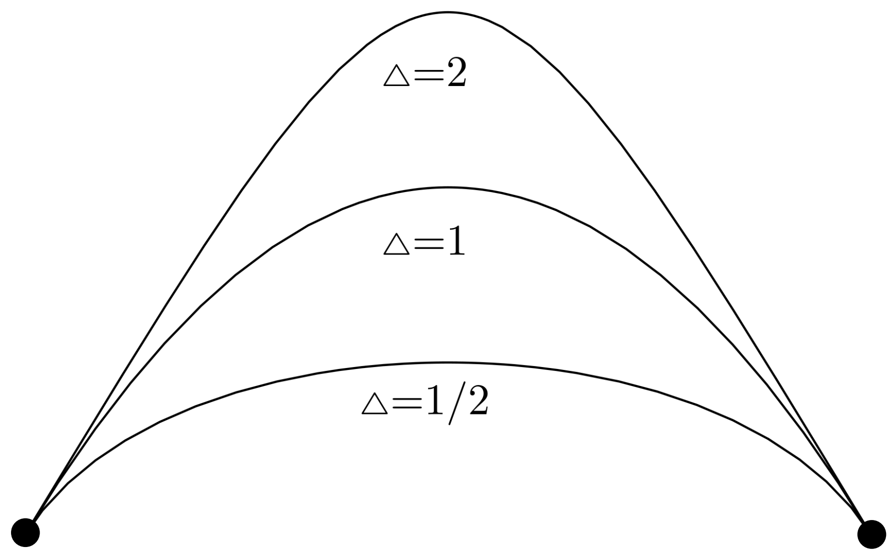

The smaller the Δ, the higher the tension in the curve

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- For degree-5 we can use two endpoints, two tangent vectors, and two second derivative vectors (i.e., principal normal vectors) at the endpoints
- In general, for degree $2k+1$ we can use two endpoints and the first k derivatives at each of them $(2k+2$ items)

  The formulas for such polynomials can be derived as we did for degree 3

  However, higher degree Hermite polynomials are not of much use in practice!