CURVES: BASIC REPRESENTATION

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Mathematical representations of curves (in 2D)

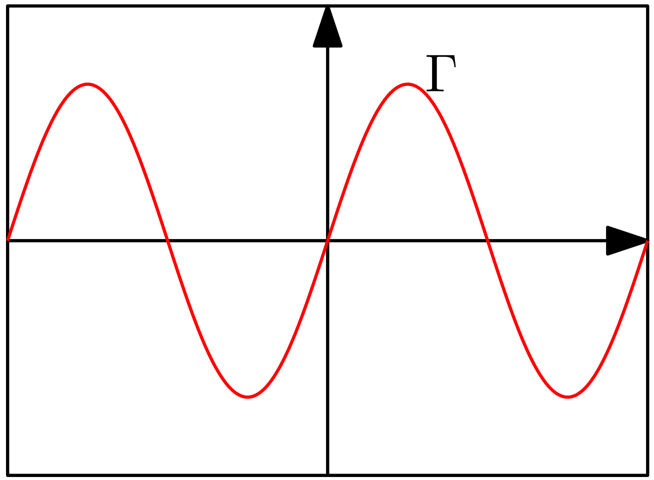
Explicit equation

If $I \subseteq \mathbb{R}$ is any interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ is a continuous function, then the curve is

$$\Gamma = \{(x, f(x)) : x \in I\}$$

Example: $y = \sin x$

$\Gamma = \{(x, \sin x) : x \in \mathbb{R}\}$



Good:

- Easy to **discretize**
- Testing $P \in \Gamma$ easy
- Unique equation for each curve

Not so good:

- Doesn't extend easily to 3D
- Not all curves have one!

E.g., the unit circle, $x^2 + y^2 = 1$

HOW TO REPRESENT A CURVE?

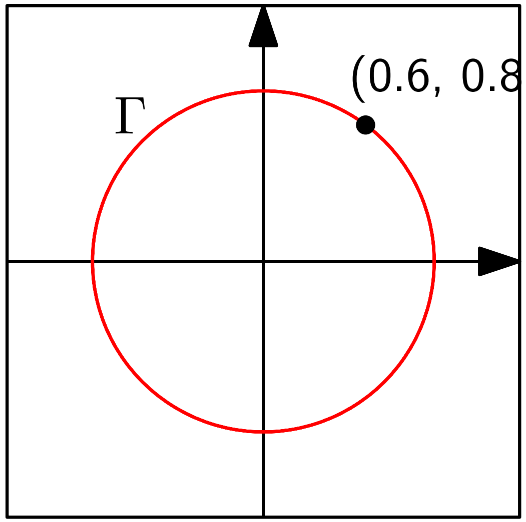
Implicit equation

For some function $F : \mathbb{R}^2 \to \mathbb{R}$, the curve is

$$\Gamma = \{(x, y) : F(x, y) = 0\}$$

Example: $x^2 + y^2 = 1$

$\Gamma = \{(x, y) : x^2 + y^2 - 1 = 0\}$



Good:

- All curves have one
- Testing $P \in \Gamma$ easy
- Unique equation (up to scaling)

Not so good:

- Doesn't extend easily to 3D
- Not so easy to discretize

HOW TO REPRESENT A CURVE?

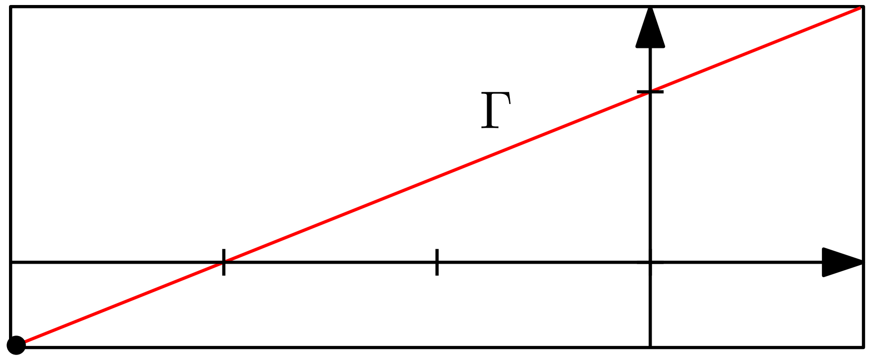
Parametric equation

Parametrized by a continuous function $\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^2$ that gives the curve

$$\Gamma = \{(x, y) : \exists t \in I \text{ such that } (x, y) = \gamma(t)\}$$

Example: $\gamma(t) = (2t, t+1)$, for $t \in \mathbb{R}$

$$\Gamma = \{(x, y) : \exists t \in \mathbb{R} \text{ such that } (x, y) = (2t, t+1)\}$$



(-3,-0.5)$= (2 \cdot (-1.5), (-1.5) + 1)$ ($t=-1.5$)

Good:

- All curves have one
- Easy to discretize
- Easy to extend to 3D

Not so good:

- Testing $P \in \Gamma$ not so easy
- Equation is not unique

In curve and surface design, we always use parametric equations!

EXAMPLES OF PARAMETRIC CURVES

1) Lines, halflines, and line segments

in 2D and 3D

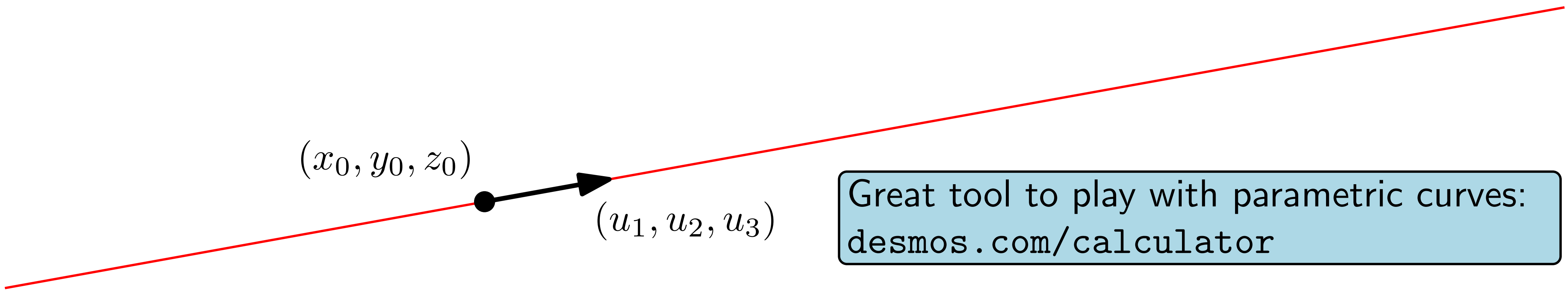
Example in 2D: $\gamma(t) = (2t, t+1)$, for $t \in \mathbb{R}$

In 3D, with supporting point (x_0, y_0, z_0) and direction vector (u_1, u_2, u_3)

$$\left.\begin{aligned} x(\lambda) &= x_0 + \lambda u_1 \\ y(\lambda) &= y_0 + \lambda u_2 \\ z(\lambda) &= z_0 + \lambda u_3 \end{aligned}\right| \quad \gamma(\lambda) = (x_0 + \lambda u_1, y_0 + \lambda u_2, z_0 + \lambda u_3) \quad \begin{array}{ll} \lambda \in \mathbb{R} & \text{(line)} \\ \lambda \in \mathbb{R}^+ & \text{(halfline)} \\ \lambda \in [a, b] & \text{(segment)} \end{array}$$

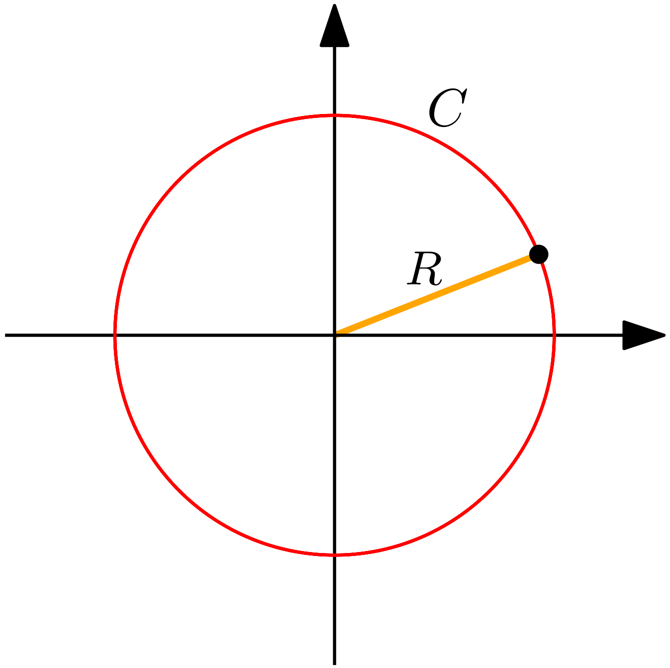
Is this equation unique?

No! Different supporting points and direction vectors give different equations



Great tool to play with parametric curves:
`desmos.com/calculator`

2) Circles (2D)



Consider a circle C with radius R centered at $(0,0)$

$$C = \{(x,y) \in \mathbb{R}^2 : d((x,y),(0,0)) = R\}$$
$$= \{(x,y) \in \mathbb{R}^2 : \sqrt{x^2+y^2} = R\}$$
$$= \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = R^2\}$$

($x^2+y^2-R^2=0$ is the implicit equation)

... in Cartesian coordinates. What about in Polar coordinates?

$C = \{(r,\theta) \in \mathbb{R}^+ \times \mathbb{R} : r = R\}$ $\rightarrow$ in Polar coordinates, the equation is just $r = R$

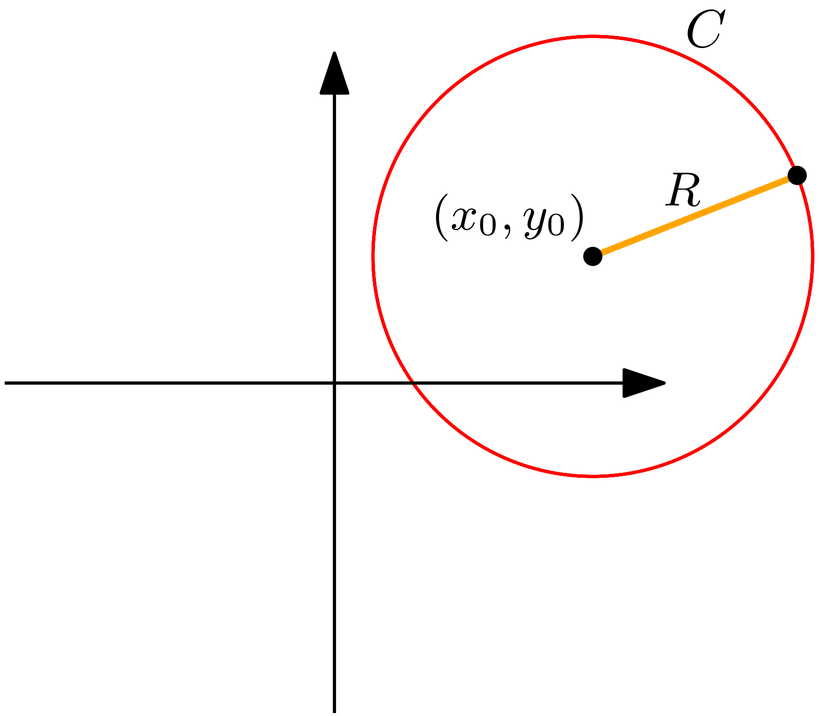
Recall: to convert from Polar to Cartesian coordinates, we do $(r,\theta) \rightarrow (r\cos\theta, r\sin\theta)$

Therefore, circle C can be re-parametrized in Cartesian coordinates as follows:

$$C = \{(r,\theta) \in \mathbb{R}^+ \times \mathbb{R} : r = R\} = \{(R,\theta), \theta \in \mathbb{R}\} = \boxed{\{(R\cos\theta, R\sin\theta), \theta \in \mathbb{R}\}}$$

In fact, $\theta \in [0, 2\pi)$ is enough

2) Circles (2D) (cont'd)



$\gamma(\theta) = (R\cos\theta, R\sin\theta) \quad \theta \in [0, 2\pi)$

- What if the circle has center (x_0, y_0)?

$$(x - x_0)^2 + (y - y_0)^2 = R^2$$

or

$$\gamma(\theta) = (x_0 + R\cos\theta, y_0 + R\sin\theta) \quad \theta \in [0, 2\pi)$$

- Are these parametrizations unique?

$\gamma'(\alpha) = (x_0 + R\cos 2\alpha, y_0 + R\sin 2\alpha)$ $\boxed{\alpha \in [0, \pi)}$ $\rightarrow$ it is very important to specify the domain of the function!

$\gamma(\theta)$ and $\gamma'(\alpha)$ describe the same curves!

- Parametrizations of the same curve can be very different!

$(\cos\theta, \sin\theta)$ for $\theta \in [0, \pi/2]$ $\quad$ $(\cos 2\alpha, \sin 2\alpha)$ for $\alpha \in [0, \pi/4]$ $\quad$ $\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ for $t \in [0, 1]$

3) Ellipses (2D)

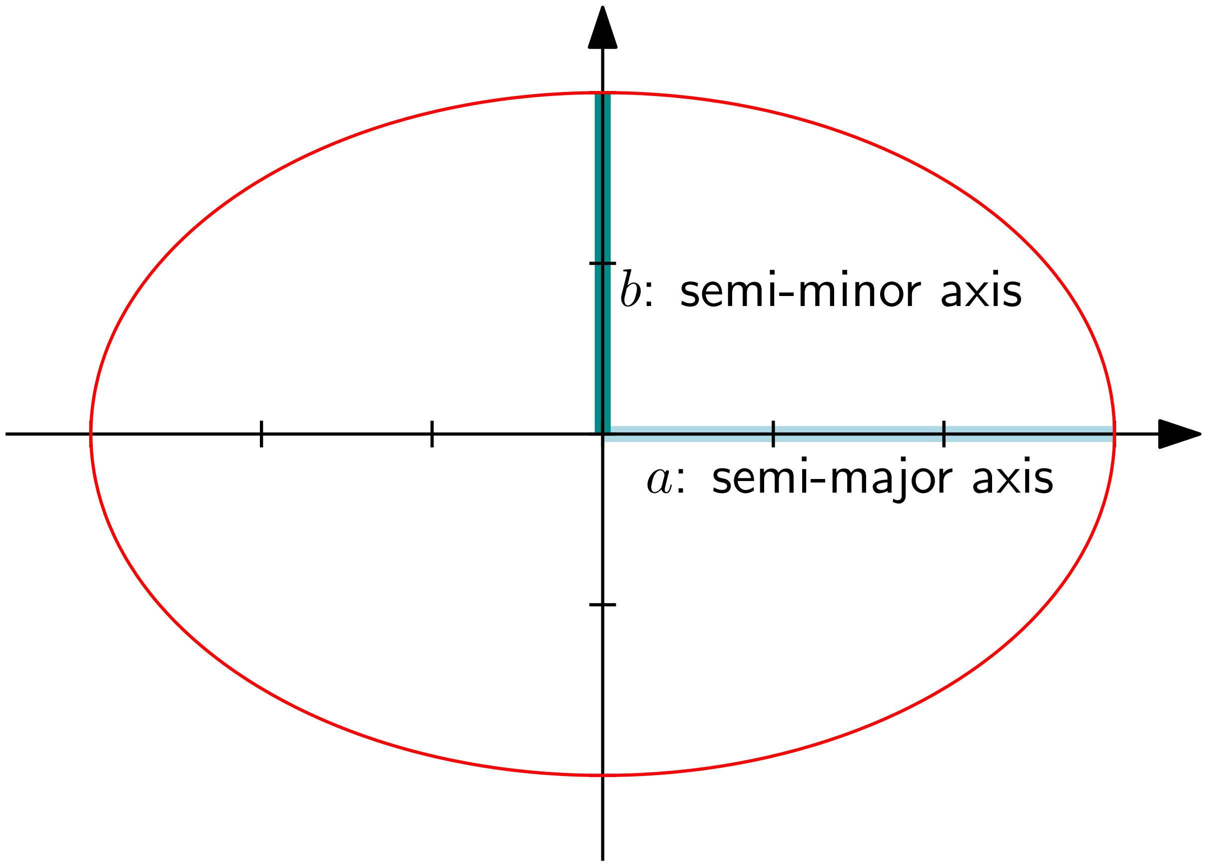
Consider an ellipse centered at $(0,0)$ with semi-axes a and b (at the x and y axis, resp.)

An ellipse can be seen as a *stretched* circle

$\gamma(\theta) = (a\cos\theta, b\sin\theta) \qquad \theta \in [0, 2\pi)$

(a circle when $a = b$)

Its implicit equation is $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$



in the example, $a = 3$ and $b = 2$

4) Archimedean spiral

Spiral where any ray from the origin intersects the spiral at equally-spaced points

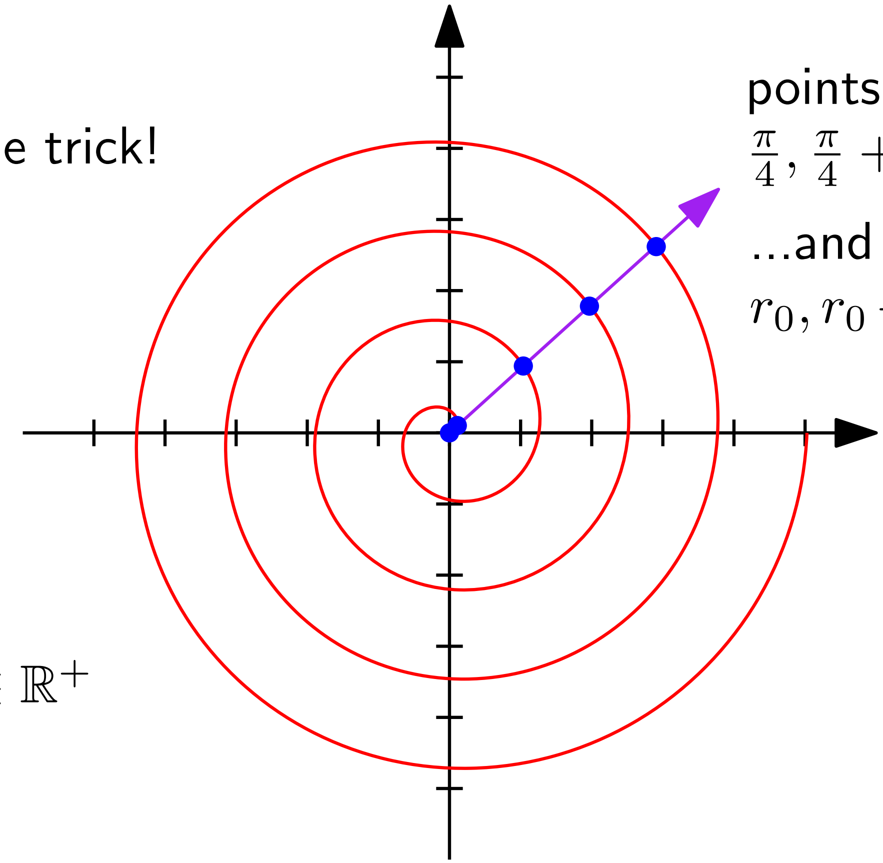
How can we find a parametrization?

Let's think in Polar coordinates

$r = a\theta$, for some fixed $a > 0$, does the trick!

In the example: $(0.2\theta, \theta)$, for $\theta \in \mathbb{R}^+$

(thus $d = 0.2 \cdot 2\pi \approx 1.25$)



points with $\theta =$ $\frac{\pi}{4}, \frac{\pi}{4} + 2\pi, \frac{\pi}{4} + 4\pi, \ldots$

...and radii: $r_0, r_0 + d, r_0 + 2d, \ldots$

In Cartesian coordinates...

$\gamma(\theta) = ((a\theta)\cos\theta, (a\theta)\sin\theta)$, for $\theta \in \mathbb{R}^+$

5) Logarithmic spiral

Distances between points intersected by the same ray grow exponentially

Appears a lot in nature, and is relation to Fibonacci numbers

How can we find a parametrization?

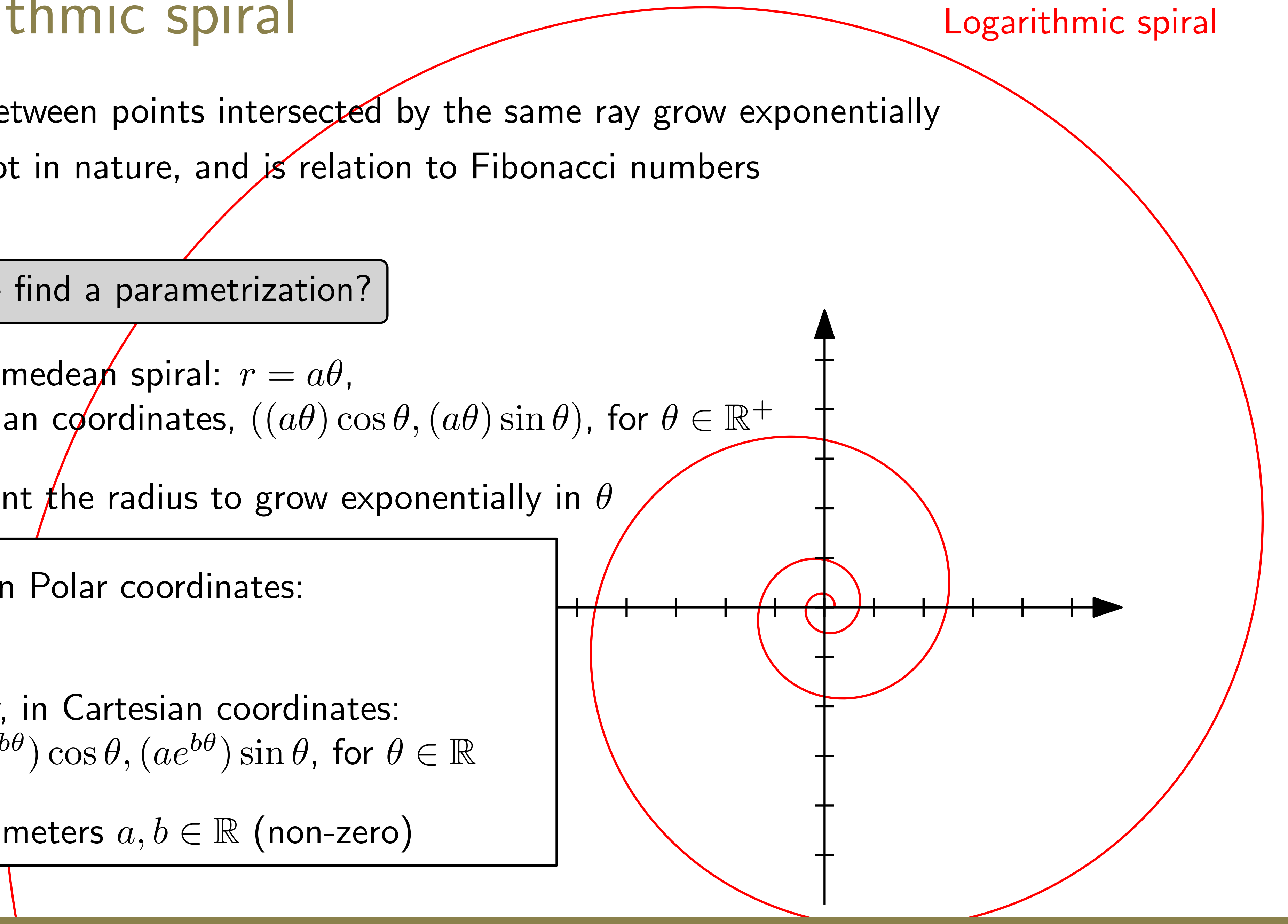
Recall Archimedean spiral: $r = a\theta$,
or in Cartesian coordinates, $((a\theta)\cos\theta, (a\theta)\sin\theta)$, for $\theta \in \mathbb{R}^+$

Now, we want the radius to grow exponentially in θ

In general, in Polar coordinates:
$r = ae^{b\theta}$

Equivalently, in Cartesian coordinates:
$\gamma(\theta) = ((ae^{b\theta})\cos\theta, (ae^{b\theta})\sin\theta$, for $\theta \in \mathbb{R}$

for two parameters $a, b \in \mathbb{R}$ (non-zero)



5) Logarithmic spiral (cont'd)

- They appear in nature

- They fascinated some famous mathematicians

Jacob Bernoulli
(Basel, 1654-1705)

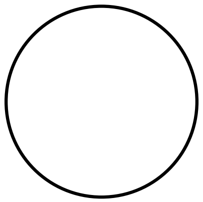
By mistake, an Archimedean spiral was carved instead!

6) Helices (3D)

Helix: a curve that goes around a central tube or cone shape in the form of a spiral (From Cambridge Dictionary)

Circular (or elliptical) helix

Base: circle



Parametric formula:

$(a\cos t, a\sin t, bt)$, $t \in \mathbb{R}$

a: radius
b: pitch
($b > 0$ right-handed,
$b < 0$ left-handed)

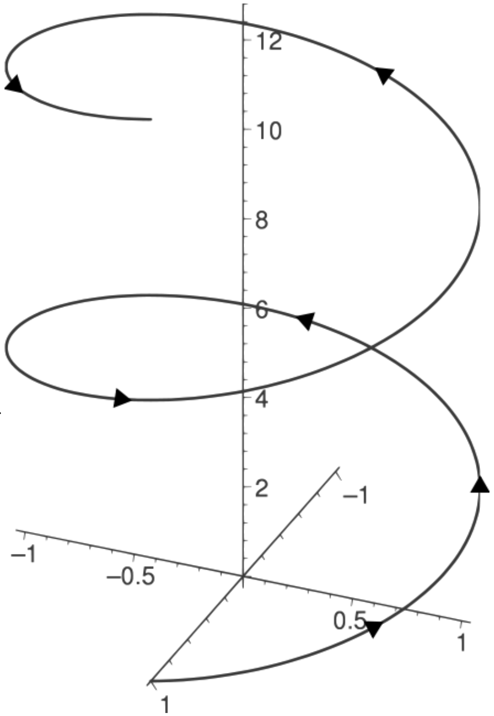
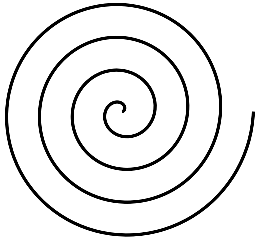


Image source: Wikipedia

Spiral helix

Base: Archimedean spiral



Formula:

Lab assignment

Image source: Wikipedia

7) Cardioid

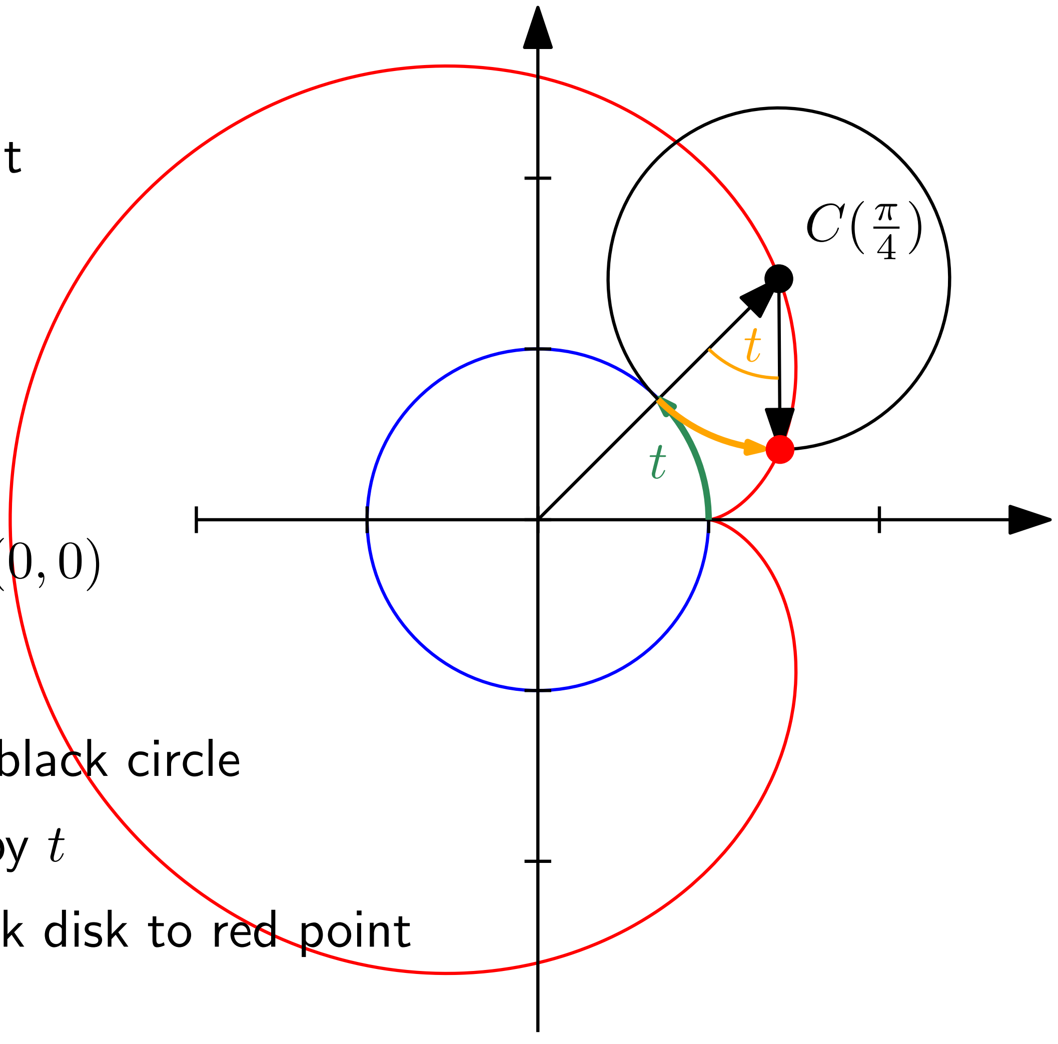
Curve traced by a point on the black circle as it rolls around the blue circle

How can we parametrize it?

Assume, without loss of generality:
circles of radius 1, and blue circle centered at $(0, 0)$

Possible approach:

1) Find parametric equation $C(t)$ of center of black circle
2) Observe that red point rotates w.r.t. $C(t)$ by t
3) Find equation for vector from center of black disk to red point
4) Compute equation of red point $\gamma(t)$



7) Cardioid: solution

Approach:

1) Find parametric equation $C(t)$ of center of black circle $C(t) = (2\cos t, 2\sin t) \quad t \in [0, 2\pi)$

2) Observe that red point rotates by t, counterclockwise

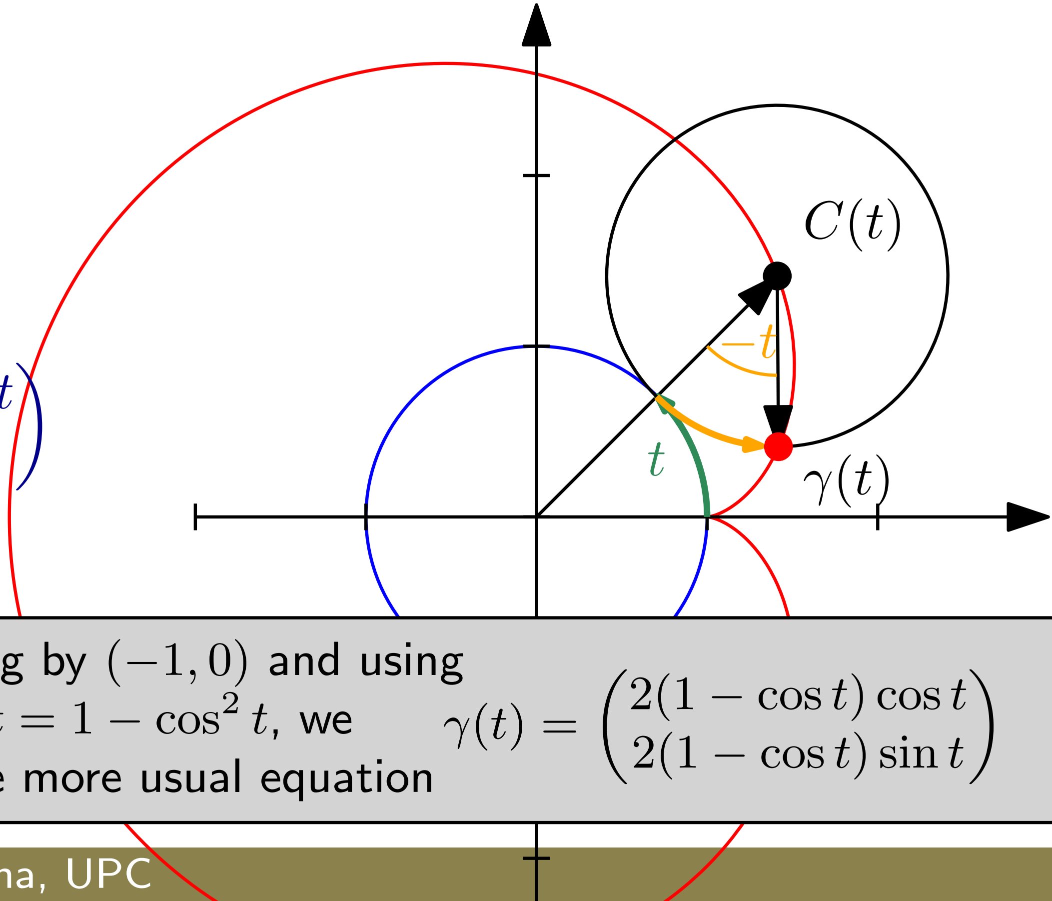
3) Find equation for vector from center of black disk to red point

4) Compute equation of red point $\gamma(t)$

3) We want the vector from $C(t)$ to $\gamma(t)$

It is -half the vector from the origin to $C(t)$, rotated by t, i.e., $-\frac{C(t)}{2}$ rotated by t

$$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix} = \begin{pmatrix} \sin^2 t - \cos^2 t \\ -2\sin t \cos t \end{pmatrix}$$



4) $\gamma(t)$ is $C(t)$ plus the vector computed in 3)

$$\gamma(t) = \begin{pmatrix} 2\cos t + \sin^2 t - \cos^2 t \\ 2\sin t - 2\sin t \cos t \end{pmatrix} \quad t \in [0, 2\pi)$$

Translating by $(-1, 0)$ and using that $\sin^2 t = 1 - \cos^2 t$, we obtain the more usual equation $\gamma(t) = \begin{pmatrix} 2(1-\cos t)\cos t \\ 2(1-\cos t)\sin t \end{pmatrix}$

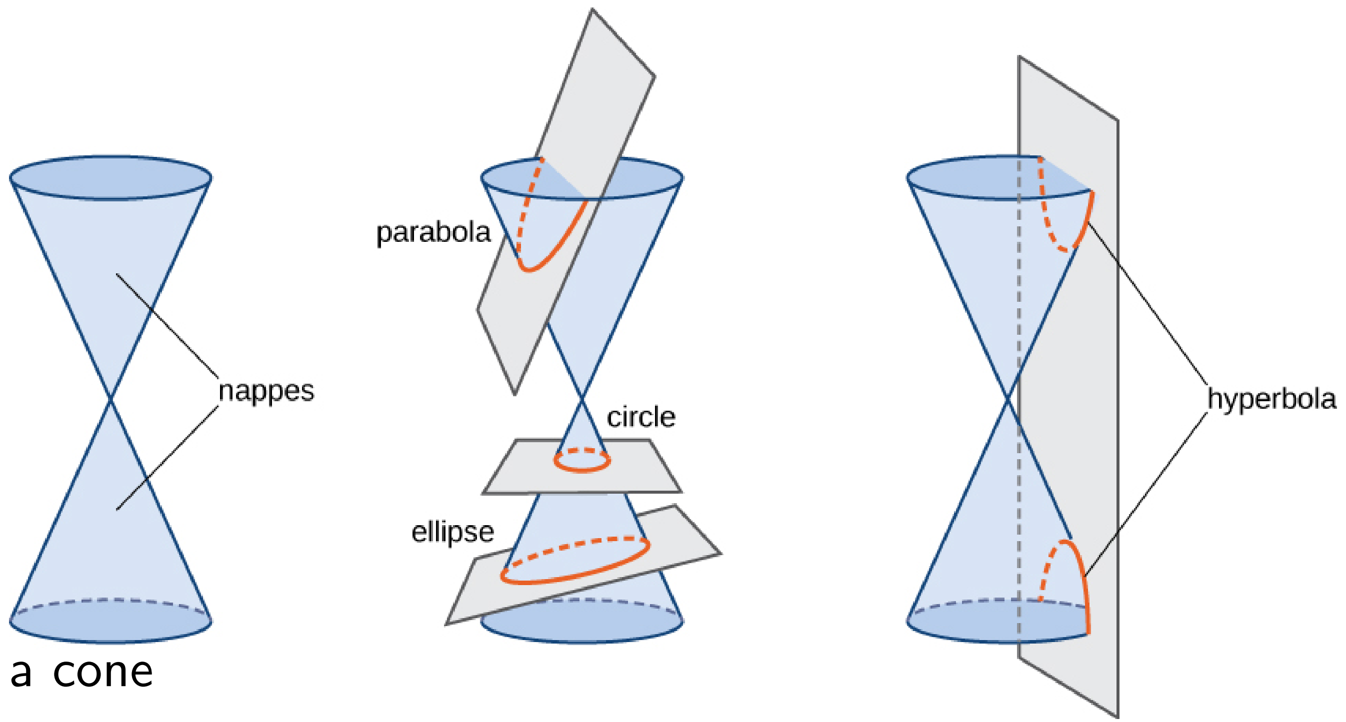
Conics: ellipses, hyperbolas and parabolas

Curves that result from intersecting a cone with a plane

- They appear all around us (planet trajectories are ellipses, antennas use parabolic shapes, etc.)

- They are easy to deal with (plane curves given by low-degree polynomial)

- They have several useful properties (e.g., reflection, focal, and excentricity properties)

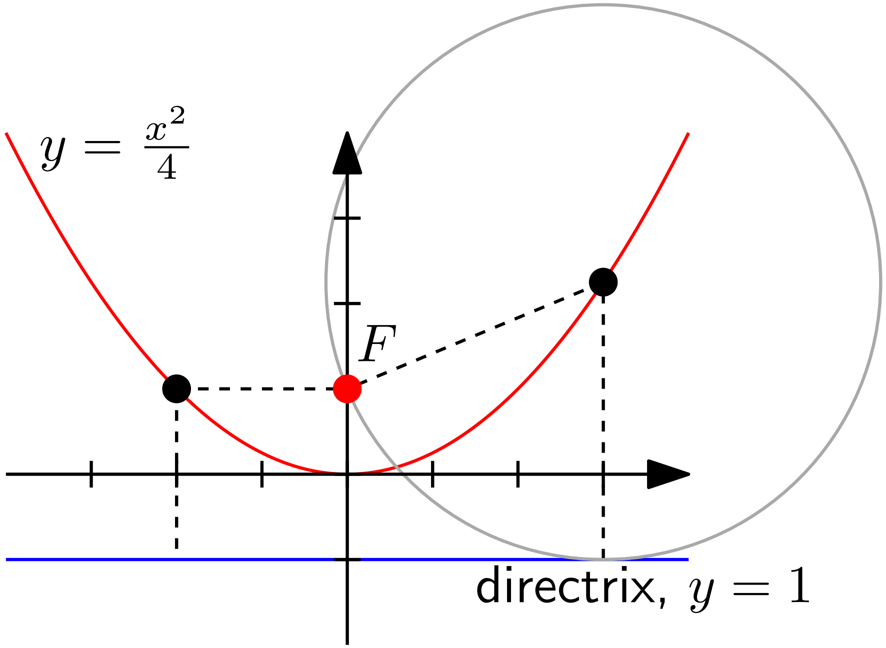


CONIC SECTIONS

Parabolas

Set of points in the plane that are at equal distance from a fixed line (directrix) and a fixed point F (focus)

Image source: Wikipedia



When the focus is $(0, p)$ and the directrix $y = -p$:

Explicit equation: $y = \frac{x^2}{4p}$

Parametric equation: $(t, \frac{t^2}{4p})$, for $t \in \mathbb{R}$

Ellipses

Set of points in the plane the **sum** of whose distances from two fixed *focus points* F_1 and F_2 is constant.

The three (orange, brown, green) pairs of segments have the same total length (6)

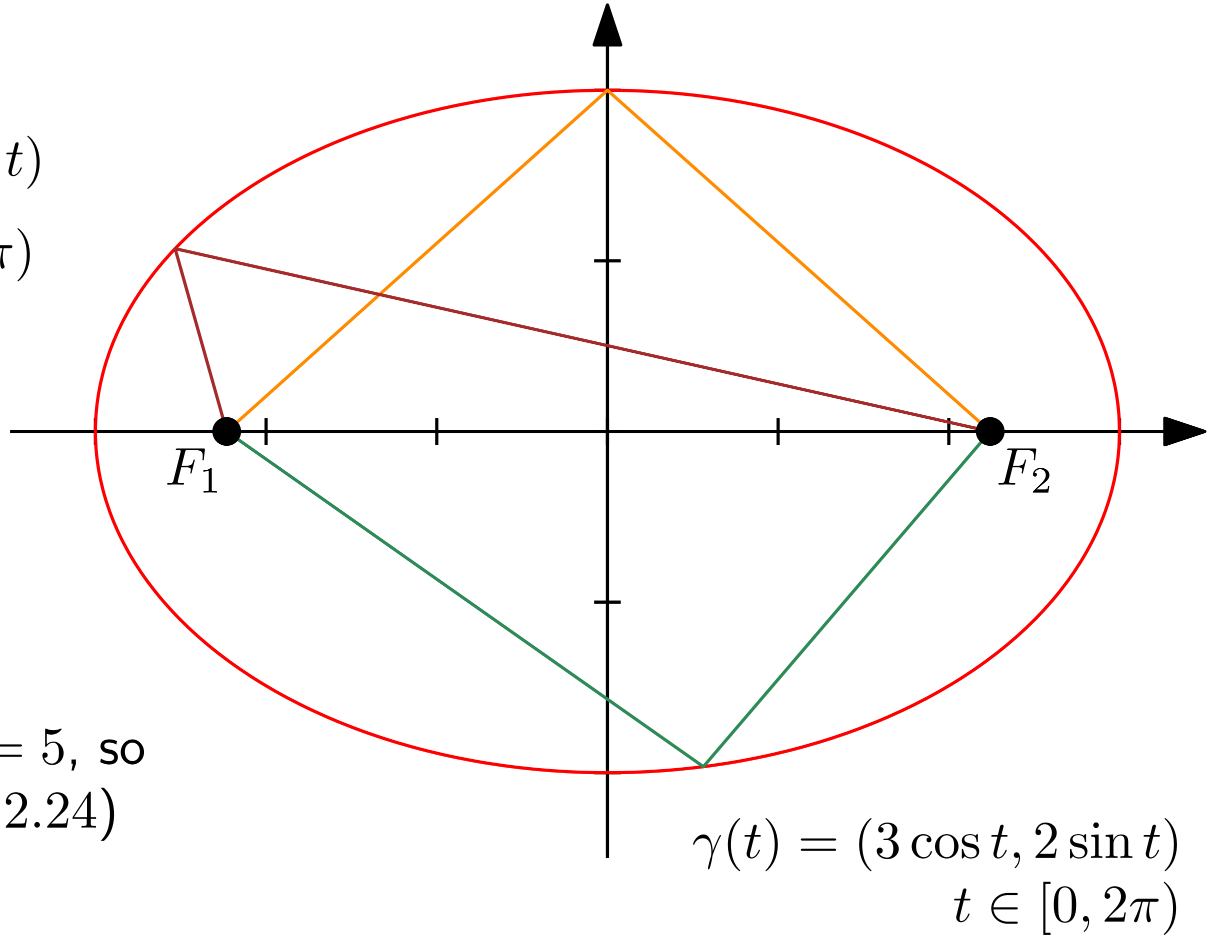
Implicit equation: $(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$

Parametric equation: $\gamma(t) = (a\cos t, b\sin t)$

for $t \in [0, 2\pi)$

In general, the focus points are $(\pm c, 0)$, where $c^2 = a^2 - b^2$ (assuming $a \geq b$)

In the example, $a = 3, b = 2$, $c^2 = 9 - 4 = 5$, so the foci are $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$ ($\sqrt{5} \approx 2.24$)



Hyperbolas

Set of points in the plane the **difference** of whose distances from two fixed *focus points* F_1 and F_2 is constant.

Implicit equation: $(\frac{x}{a})^2 - (\frac{y}{b})^2 = 1$

A hyperbola is composed of two curves (*branches*), with foci $(\pm c, 0)$, for $c^2 = b^2 + a^2$, and vertices $(\pm a, 0)$

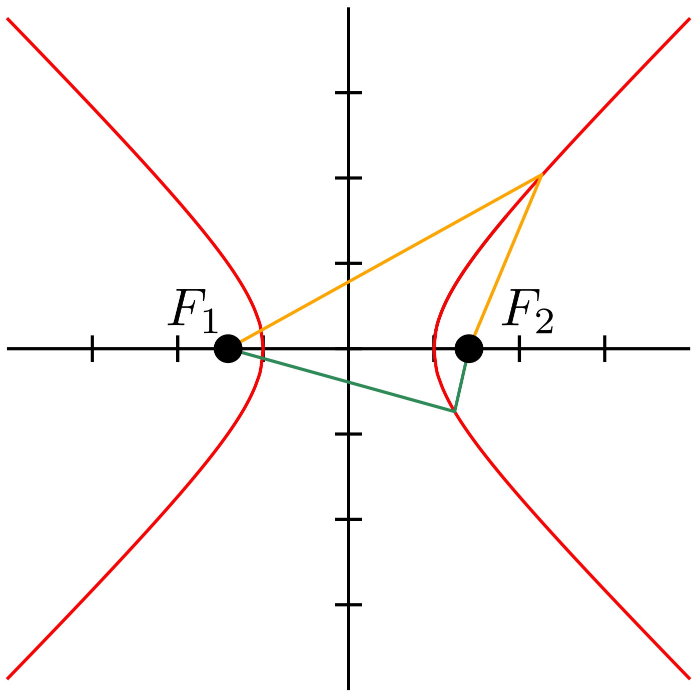
The difference between the lengths of each of the two same-color segments is the same $(\pm 2a)$

Parametric equation: $(\pm a \sec t, b \tan t)$ for $t \in (-\pi/2, \pi/2)$

Equivalently:

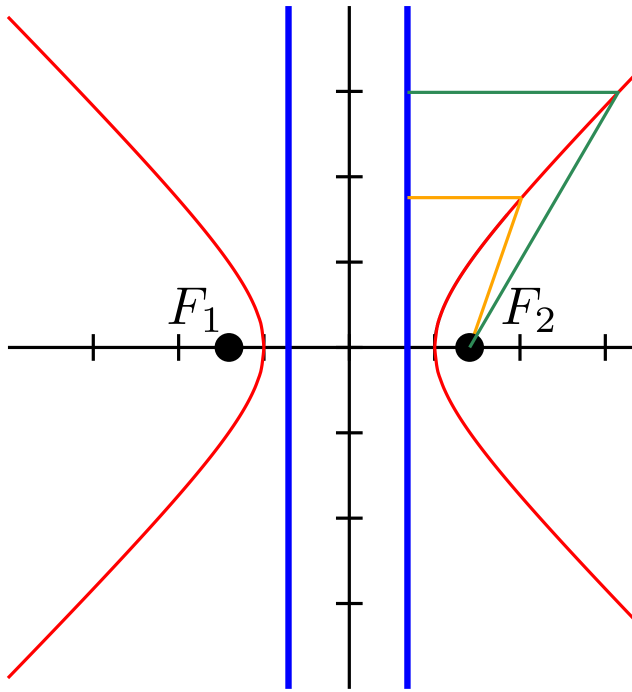
$(\pm \frac{a}{\cos t}, b \tan t)$ for $t \in (-\pi/2, \pi/2)$

$(\pm a \cosh t, b \sinh t)$ for $t \in \mathbb{R}$



Example with $a = b = 1$
$F_1 = (-\sqrt{2}, 0)$, $F_2 = (\sqrt{2}, 0)$

General formula



Conics can be defined in a unified way:

A conic is the locus of all the points P that satisfy the following:

- The distance from P to a fixed point F (focus) is *proportional* to the distance from P to a fixed line D (directrix)

That is: $\{P \in \mathbb{R}^2 : d(P, F) = \lambda \cdot d(P, D)\}$, for some fixed $\lambda > 0$

In particular:

- If $\lambda = 1$, we obtain a parabola
- If $\lambda < 1$, we obtain an ellipse (a circle is the limit case where $\lambda \to 0$)
- If $\lambda > 1$, we obtain a hyperbola

For the particular case where D is the y-axis and $P = (k, 0)$, this gives the general formula:

$\frac{\sqrt{(x-k)^2+y^2}}{|x|} = \lambda$ or equiv. $(1 - \lambda^2)x^2 - 2kx + y^2 + k^2 = 0$

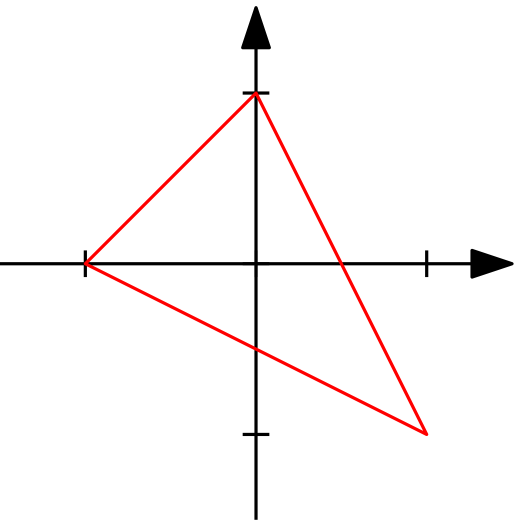
See [Salomon, pg 364] for a classification of conics

Intrinsec properties of curves

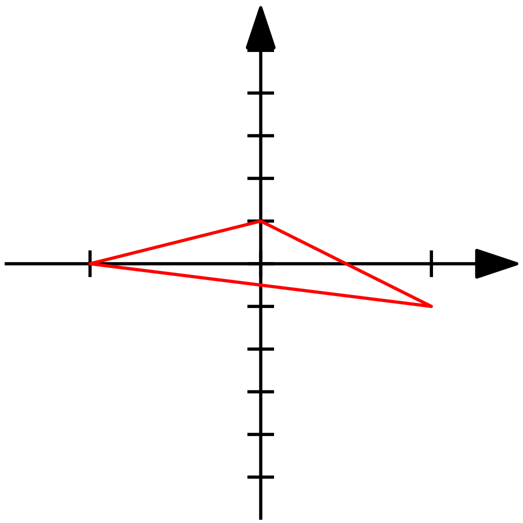
Properties that depend only on the *shape* of the curve, and not on the coordinate system

Examples

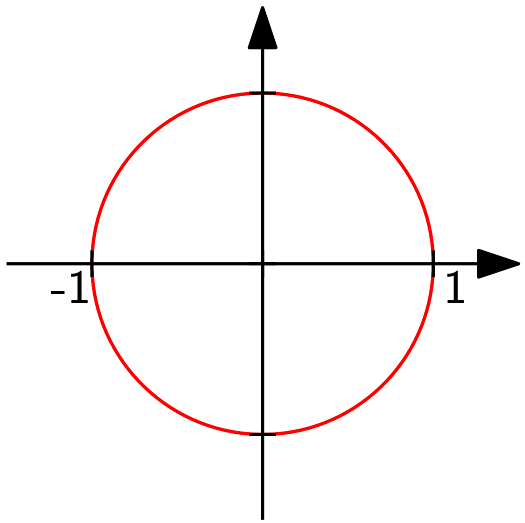
- *Smoothness* is an intrinsec property
- The fact that rectangles have four equal angles, is intrinsec
- Area and length of a curve, however, are not (they are *extrinsic* properties)



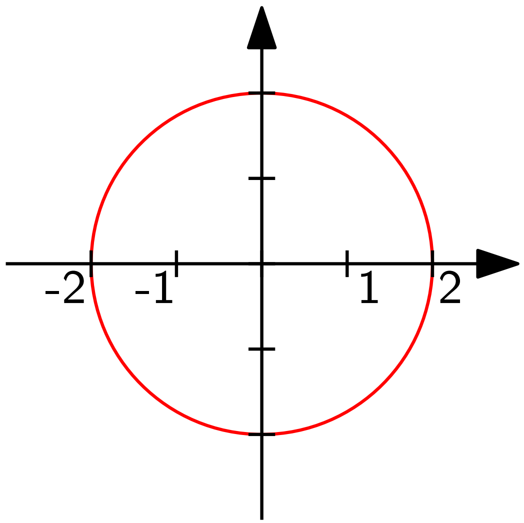
Perimeter: ≈ 5.89



Perimeter: ≈ 16.66



Perimeter: $2\pi r = 2\pi$



Perimeter: $2\pi r = 2\pi \cdot 2 = 4\pi$

Tangent vector

Intrinsec property

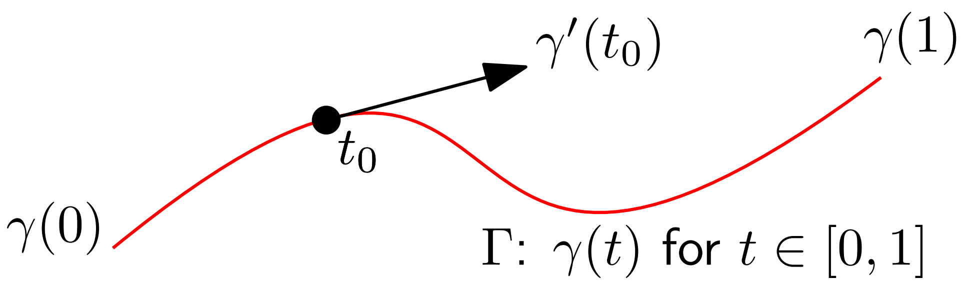
Local property: varies from point to point

Let Γ be a curve parametrized by $\gamma : [a, b] \to \mathbb{R}^d$, such that:

- γ is differentiable in t
- $\gamma'(t) \neq 0$

$\gamma'(t)$ is the *tangent vector* of Γ at point $\gamma(t)$

In other words, if Γ is parametrized as $(x(t), y(t), z(t))$, for $t \in [a, b]$, then the tangent vector of Γ at $t_0 \in [a, b]$ is $(x'(t_0), y'(t_0), z'(t))$ as long as it exists and does not vanish



Tangent vector: examples

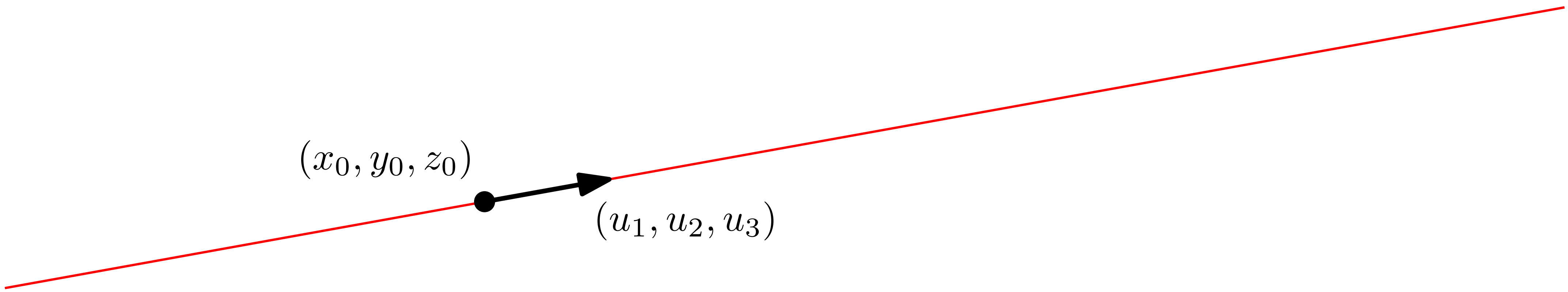
1) Line

In 3D, with supporting point (x_0, y_0, z_0) and direction vector (u_1, u_2, u_3)

$$\gamma(t) = (x_0 + tu_1, y_0 + tu_2, z_0 + tu_3) \qquad \text{for } t \in \mathbb{R}$$

Tangent vector

$$\gamma'(t) = (u_1, u_2, u_3)$$ Constant! And equal to direction vector (as expected)



Tangent vector: examples

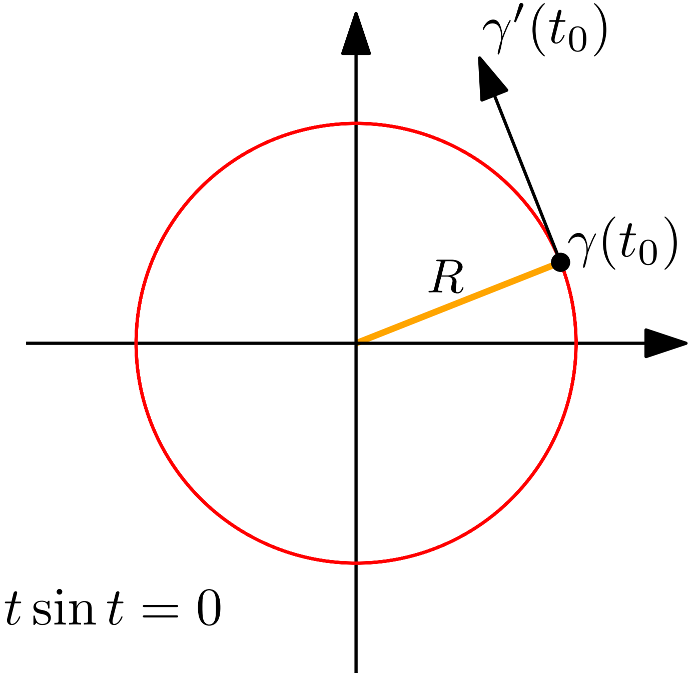
2) Circle

$\gamma(t) = (R\cos t, R\sin t)$ for $t \in [0, 2\pi)$

Tangent vector

$$\gamma'(t) = (-R\sin t, R\cos t)$$

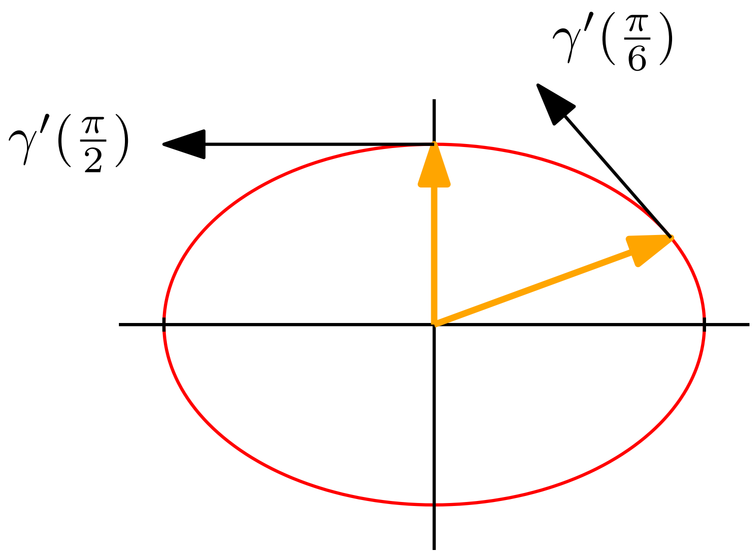
Observe that $\gamma(t) \perp \gamma'(t)$: $\quad \gamma(t)\gamma'(t) = -R^2\cos t\sin t + R^2\cos t\sin t = 0$



3) Ellipse

$\gamma(t) = (a\cos t, b\sin t)$ for $t \in [0, 2\pi)$

$\gamma'(t) = (-a\sin t, b\cos t)$ for $t \in [0, 2\pi)$



Observe that $\gamma(t)$ and $\gamma'(t)$ are not always perpendicular

Tangent vector: examples

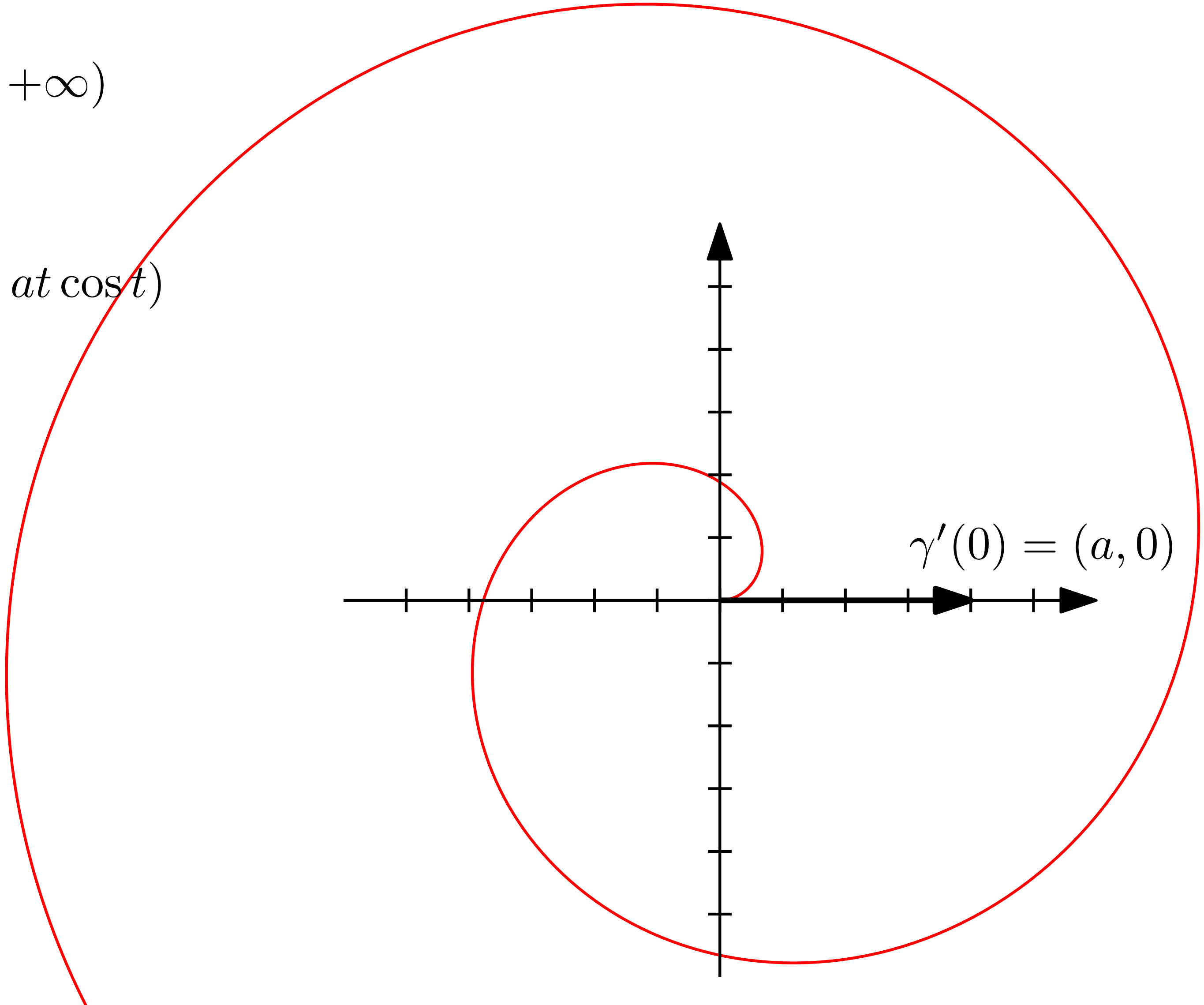
4) Archimedean spiral

$\gamma(t) = (at\cos t, bt\sin t)$ for $t \in [0, +\infty)$

Tangent vector

$$\gamma'(t) = (a\cos t - at\sin t, a\sin t + at\cos t)$$

What is $\gamma'(0)$?

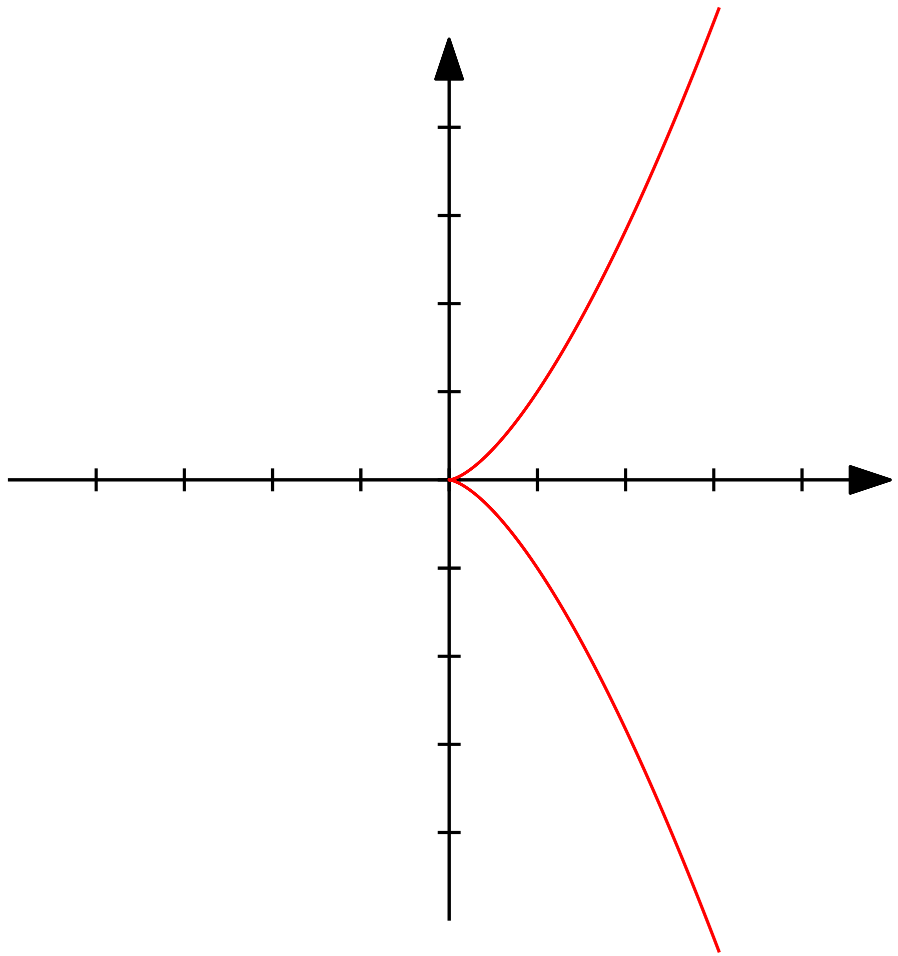


Tangent vector: examples

5) An example where $\gamma'(t) = (0,0)$

$\gamma(t) = (t^2, t^3)$ for $t \in \mathbb{R}$

$\gamma'(t) = (2t, 3t^2)$ for $t \in \mathbb{R}$



$\gamma'(-1) = (-2, 3)$

$\gamma'(-0.5) = (-1, 0.75)$

$\gamma'(-0.1) = (-0.2, 0.03)$

$\gamma'(0) = (0, 0)$

tangent vector vanishes at $t = 0$

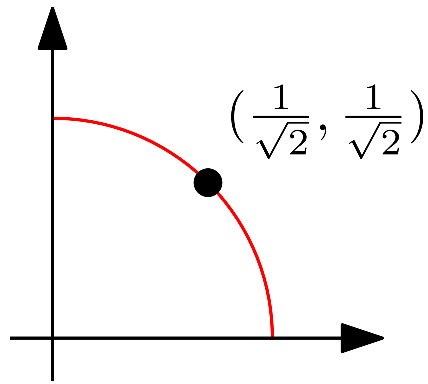
Tangent vector: effect of parametrizations

Consider different parametrizations of the same curve, for example:

$(\cos\theta, \sin\theta)$ for $\theta \in [0, \pi/2]$	$(\cos 2\alpha, \sin 2\alpha)$ for $\alpha \in [0, \pi/4]$	$\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right)$ for $t \in [0, 1]$

Same curve, with $\theta = 2\alpha$ and $t = \tan\alpha = \tan\frac{\theta}{2}$

$\rightarrow$ different parametrizations can be seen as different **speeds**



Consider the point $P = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(it belongs to the curve, for $\theta = \frac{\pi}{4}$, $\delta = \frac{\pi}{8}$ and $t = \sqrt{2} - 1$)

Tangent vector at P in each parametrization:

$\gamma_1'(\theta) = (-\sin\theta, \cos\theta)$	$\gamma_2'(\theta) = (-2\sin 2\theta, 2\cos 2\theta)$	$\gamma_3'(t) = \left(\frac{-4t}{(1+t^2)^2}, \frac{2(1-t^2)}{(1+t^2)^2}\right)$
$\gamma_1'(\frac{\pi}{4}) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\gamma_2'(\frac{\pi}{8}) = (-\sqrt{2}, \sqrt{2})$	$\gamma_2'(\sqrt{2} - 1) \approx (-0.035, 0.035)$

The three vectors are proportional!

This is always the case, as long as the parametrizations are "nice" (i.e., they are always differentiable, and the tangent never vanishes)

Unit tangent vector

Often only the direction of the tangent vector matters, so we can use the unit tangent vector:

$$T(t) = \frac{\gamma'(t)}{||\gamma'(t)||}$$

If $\gamma(t)$ is a parametrization such that $||\gamma(t)|| = 1$ for all t, then we say that γ is a *unit-speed parametrization* of Γ

Example: $\gamma(\theta) = (\cos\theta, \sin\theta)$ for $\theta \in [0, \pi/2]$

$$\gamma'(\theta) = (-\sin\theta, \cos\theta)$$

$$||\gamma_1'(\theta)|| = \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = \sqrt{\sin^2\theta + \cos^2\theta} = 1$$

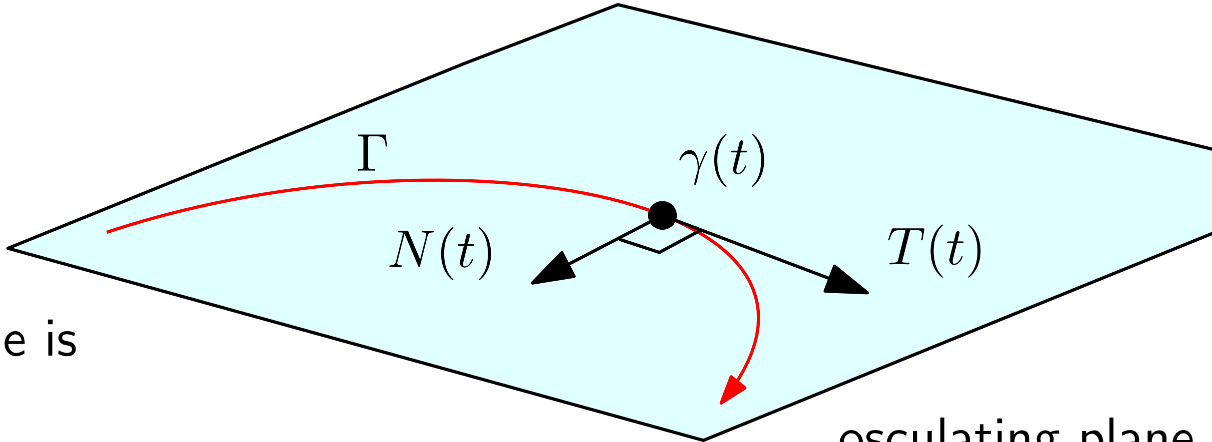
$\rightarrow \gamma(\theta)$ is a unit-speed parametrization

Normal vector and curvature

Another important vector associated with a curve Γ is the *principal normal vector* $N(t)$

If $\gamma : I \to \mathbb{R}^d$ is a unit-speed parametrization of a curve Γ, then:

$N(t) = \gamma''(t)$ (If not unit-speed, the formula is more complicated, see [Salomon, pg.28])



- $N(t)$ is normal to $T(t)$
- $N(t)$, $T(t)$ and $\gamma(T)$ define the *osculating plane*
- If Γ is a plane curve, its osculating plane is the plane of the curve
- The length of $N(t)$, $\kappa(t) = ||N(t)||$ is the *curvature* of Γ at point $\gamma(t)$
- $1/\kappa(t)$ is the radius of the circle "most similar" to Γ at $\gamma(t)$

Exercise: compute the curvature of $\gamma(t) = (\cos t, \sin t)$