BEZIER SURFACES

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INTRO TO BEZIER SURFACES

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Idea: use grid of control points

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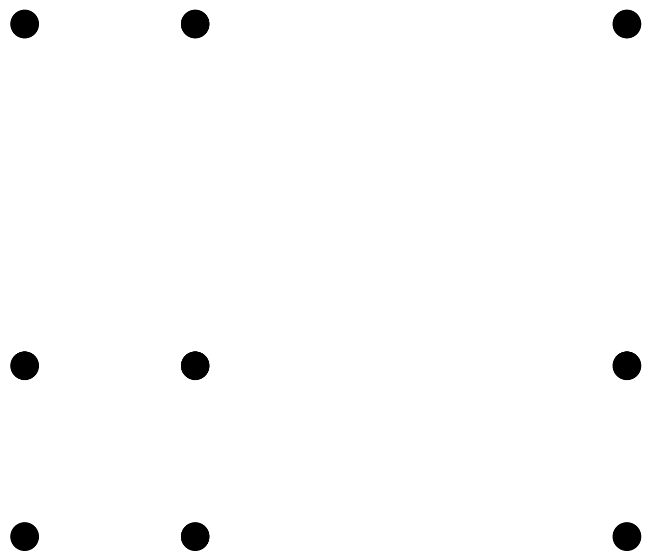

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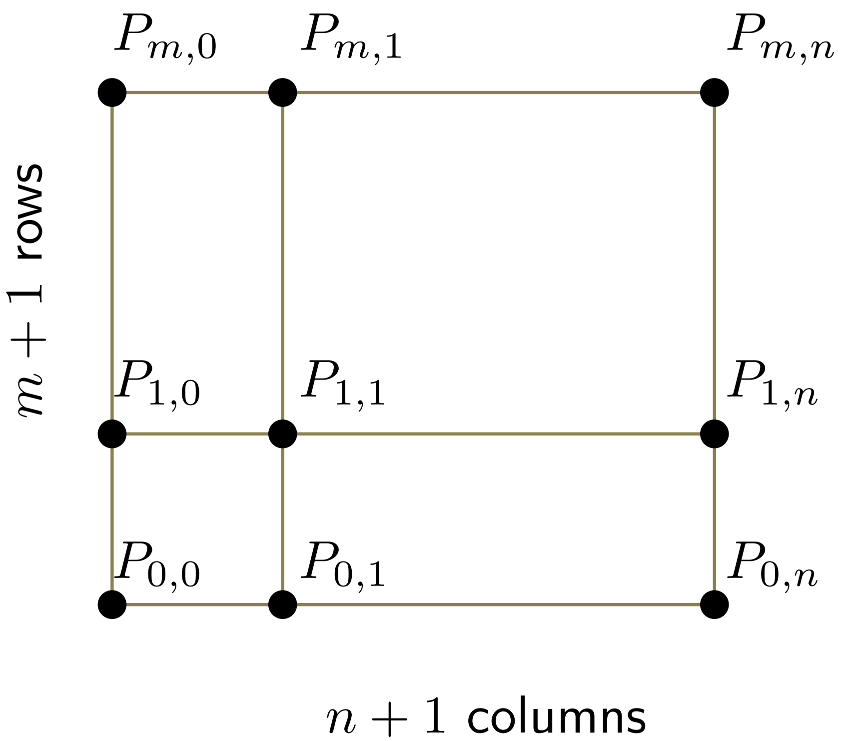

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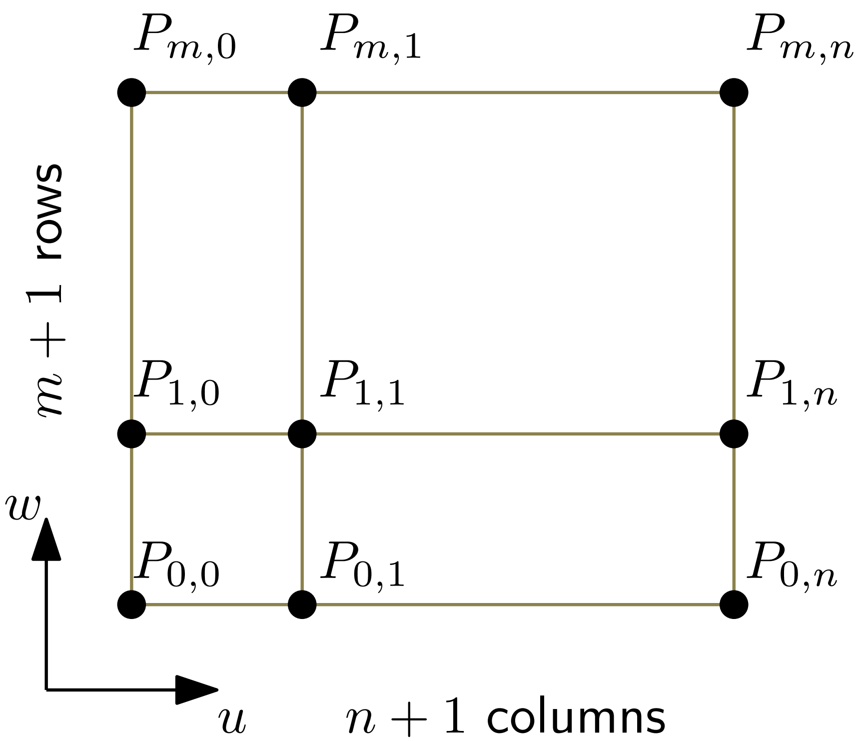

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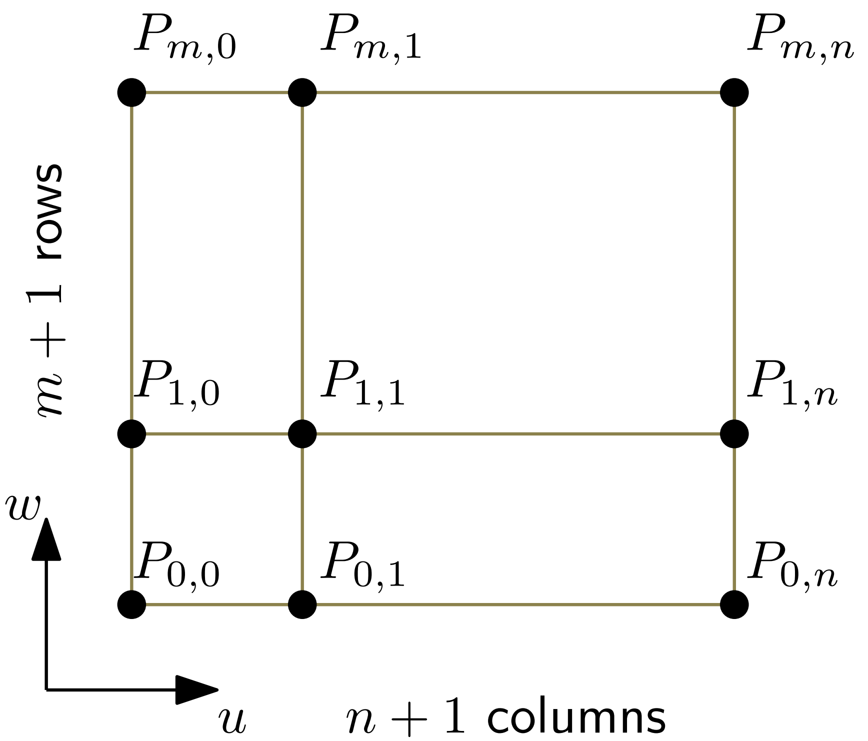


$$\boxed{P(u,w) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(w) P_{i,j}}$$

$$0 \leq u, w \leq 1$$

Note that the terms $B_{m,i}$ and $B_{n,j}(u)$ are the Bernstein polynomials, same as in Bézier curves

Idea: use grid of control points

Consider a grid of $(m+1)\times(n+1)$ control points arranged in a rectangular grid

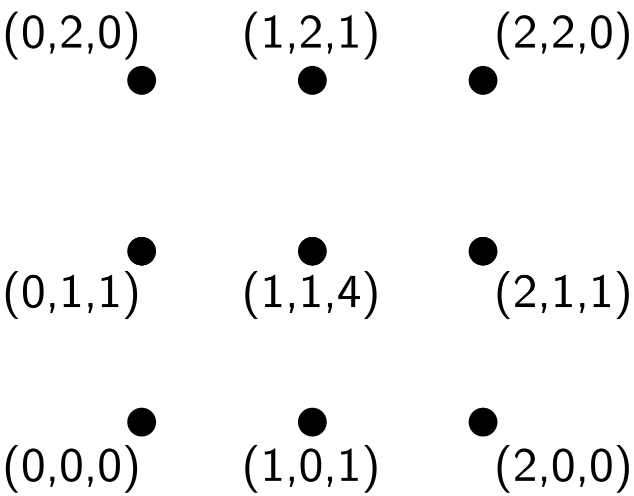
$$\boxed{P(u,w)=\sum_{i=0}^{m}\sum_{j=0}^{n} B_{m,i}(u)B_{n,j}(w)P_{i,j}} \qquad 0\le u,w\le 1$$

In matrix form:

$$P(u,w)=(B_{m,0}(u),B_{m,1}(u),\ldots,B_{m,m}(u))\begin{pmatrix} P_{0,0} & P_{0,1} & \ldots & P_{0,n}\\ P_{1,0} & P_{1,1} & \ldots & P_{1,n}\\ \vdots & \vdots & \ddots & \vdots\\ P_{m,0} & P_{m,1} & \ldots & P_{m,n}\end{pmatrix}\begin{pmatrix} B_{n,0}(w)\\ B_{n,1}(w)\\ \vdots\\ B_{n,n}(w)\end{pmatrix}$$

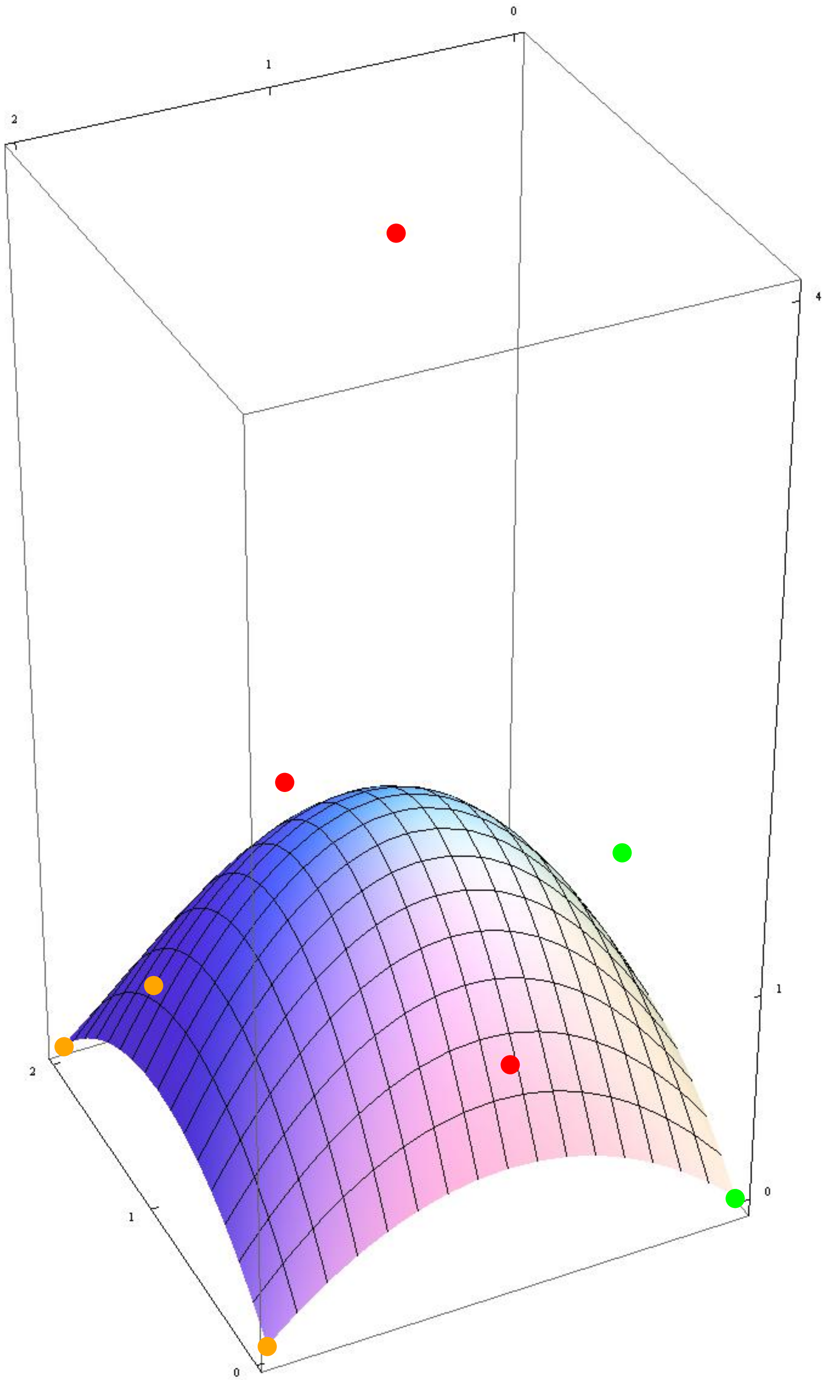
BEZIER SURFACES

Example



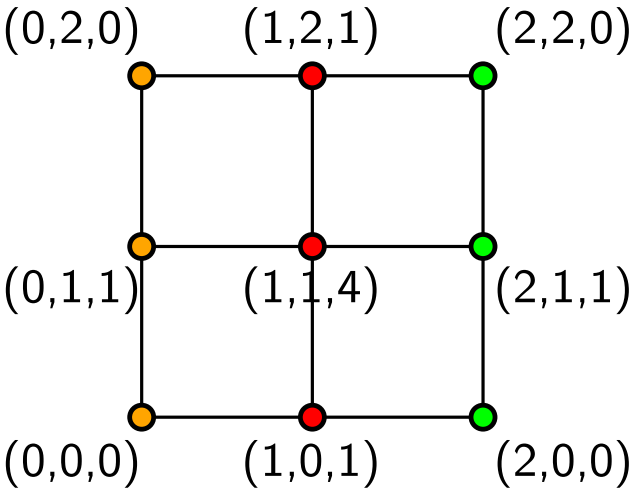
Example

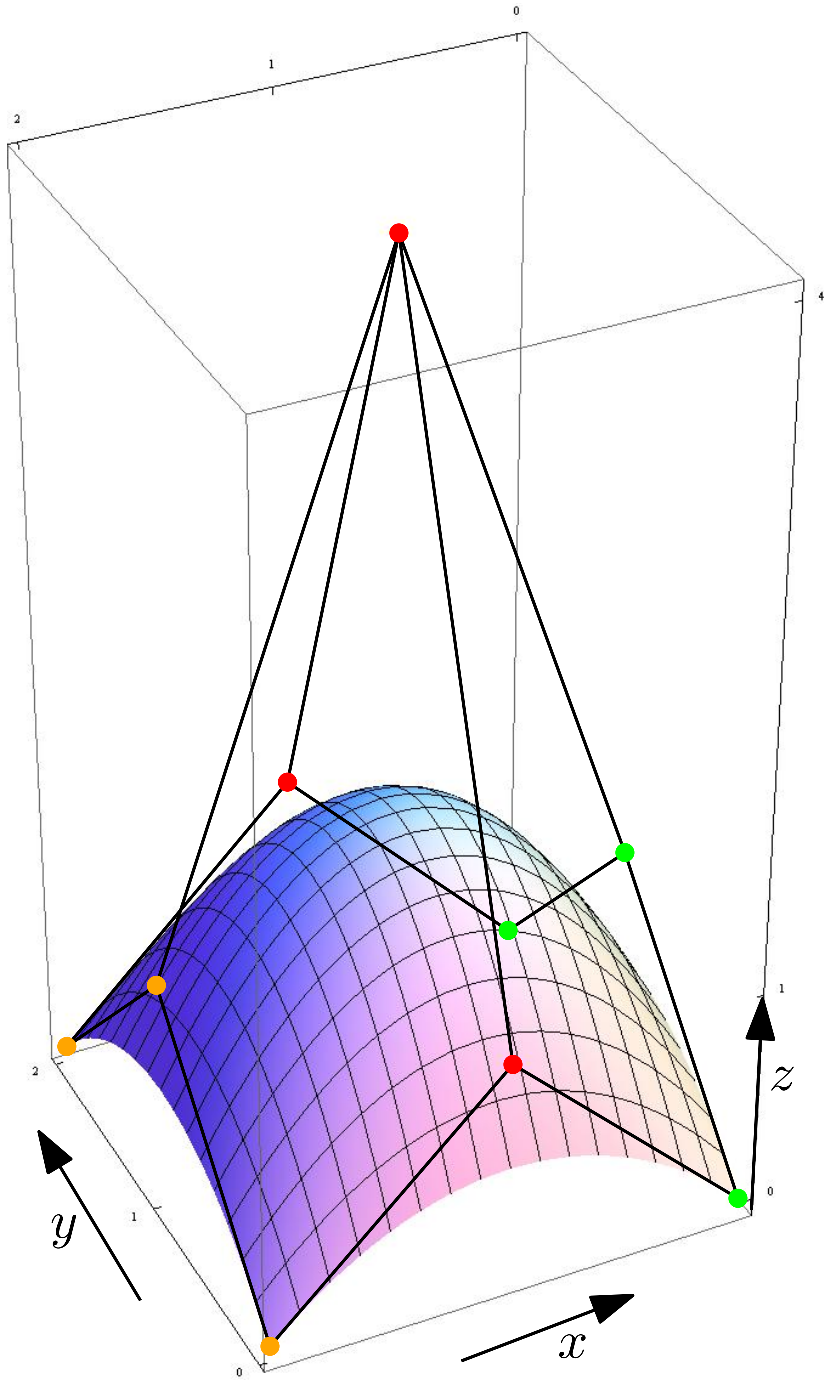
(0,2,0)	(1,2,1)	(2,2,0)
(0,1,1)	(1,1,4)	(2,1,1)
(0,0,0)	(1,0,1)	(2,0,0)



Biquadratic Bézier surface patch [Salomon, Fig 6.20]

Example





Biquadratic Bézier surface patch [Salomon, Fig 6.20]

Properties of Bézier surface (on rectangular grid)

$$\boxed{S(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) P_{i,j}}$$

$$0 \leq u, v \leq 1$$

Properties of Bézier surface (on rectangular grid)

- Endpoints (patch corners)

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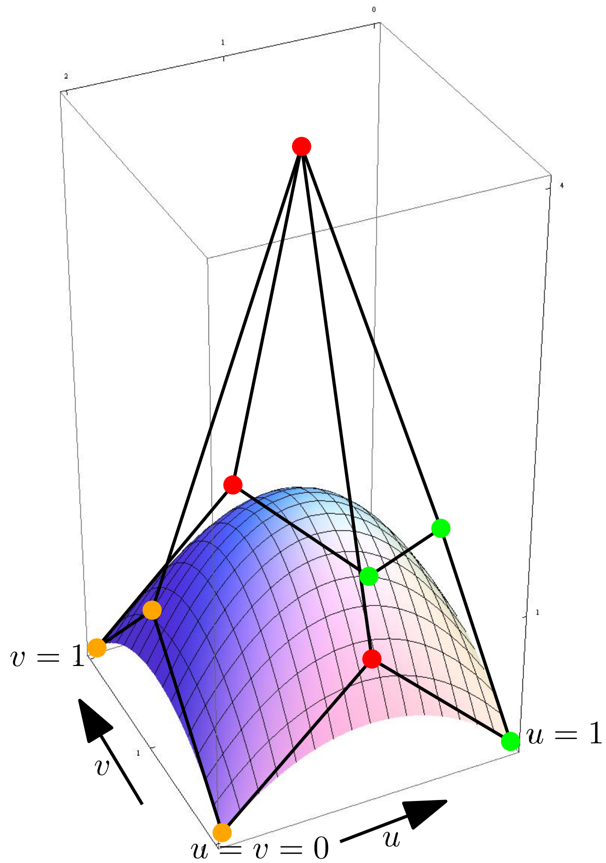

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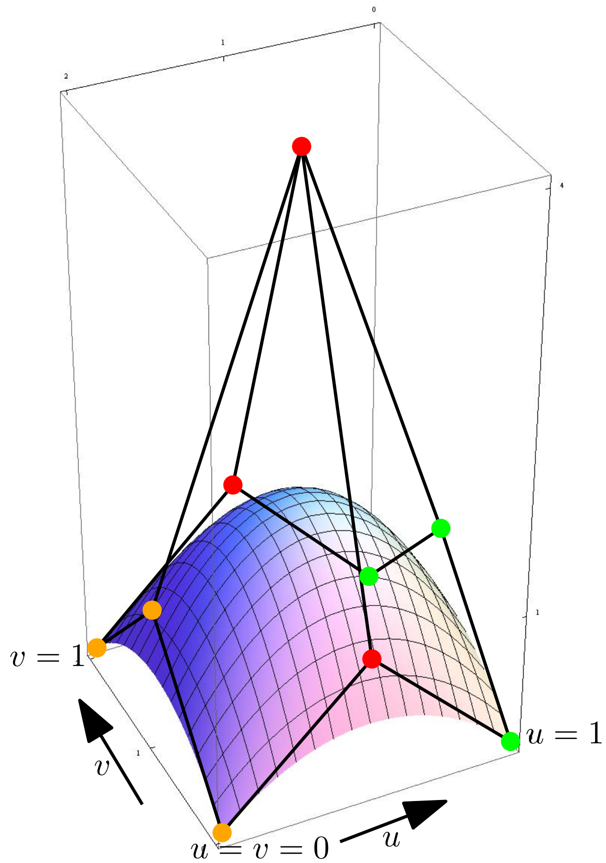

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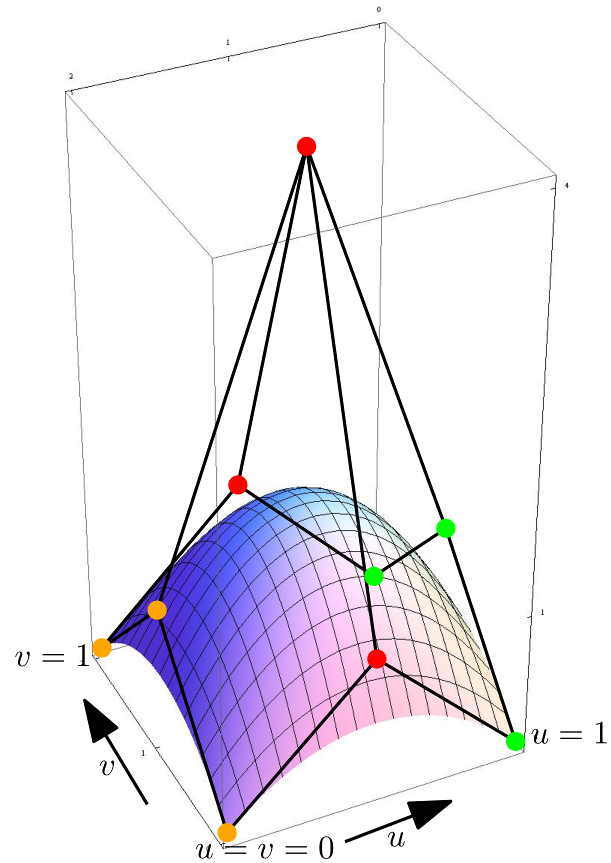
- Boundary curves



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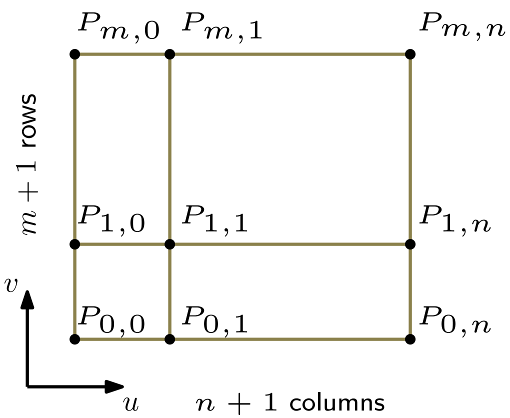
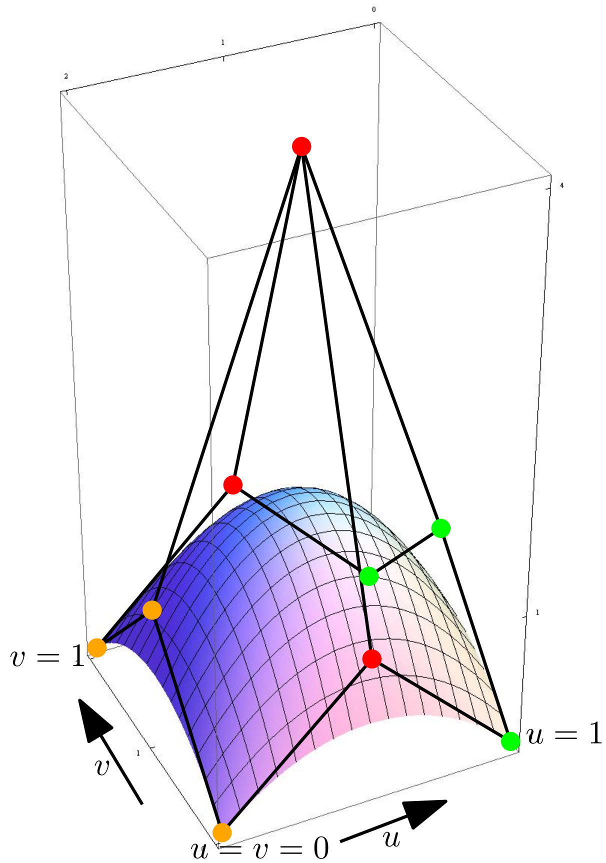
- Boundary curves

$S(0,v)$ is the Bézier curve defined by $P_{0,0}, P_{1,0}, \ldots, P_{n,0}$





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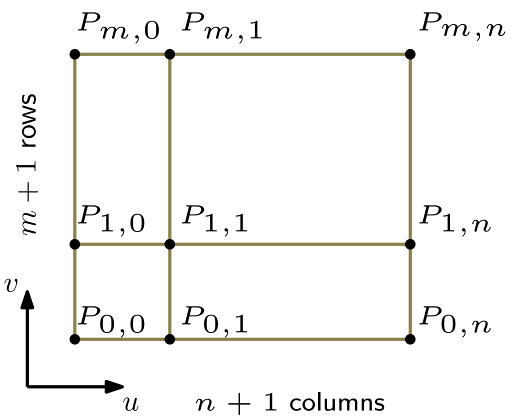

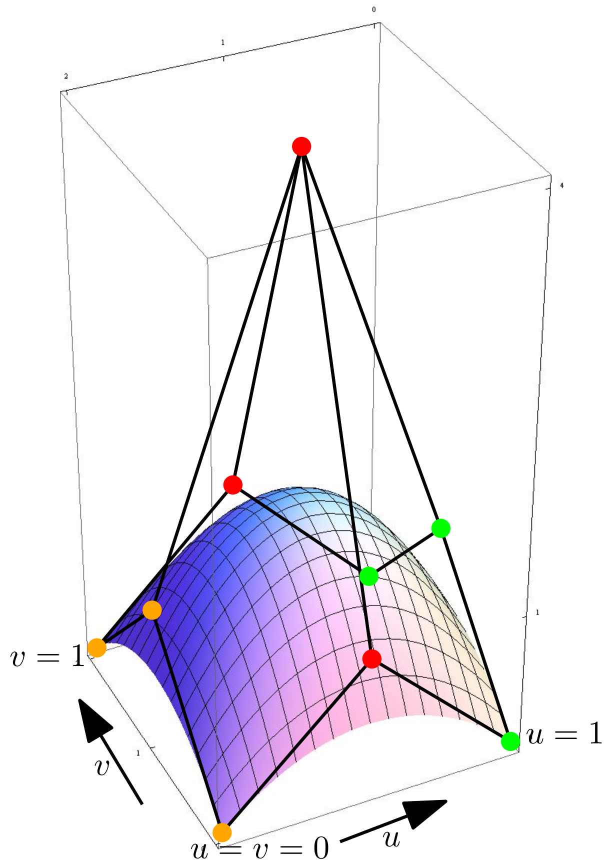

Properties of Bézier surface (on rectangular grid)

- Uniparametric curves
  (i.e., fixed u or fixed v)

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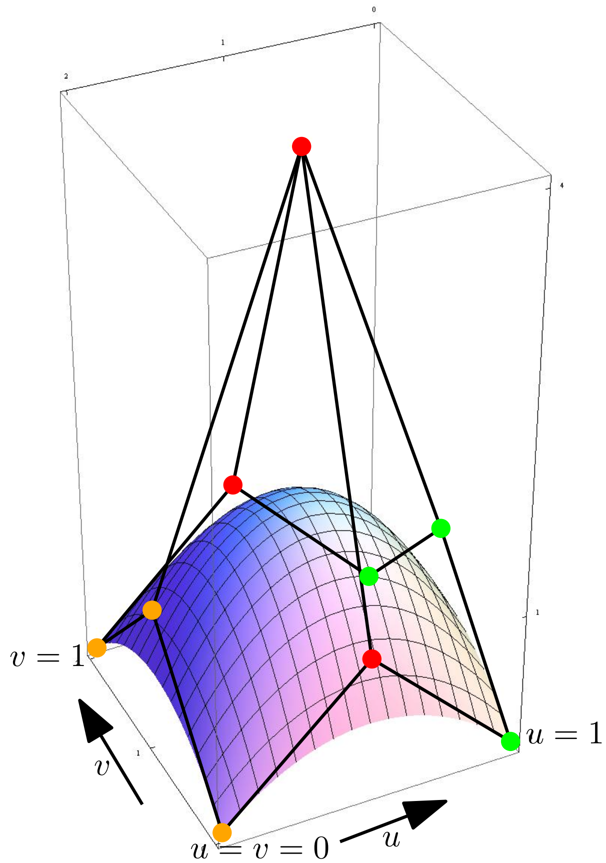

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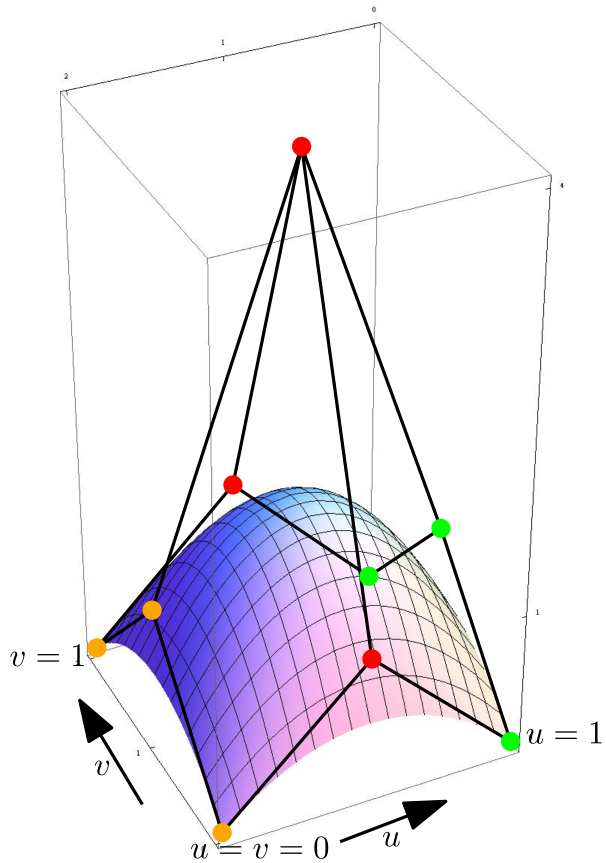
- Affine invariance

  $\sum_{i=0}^{m} \sum_{j=0}^{n} B_{m,i}(u) B_{n,j}(v) = 1$ for all (u,v)



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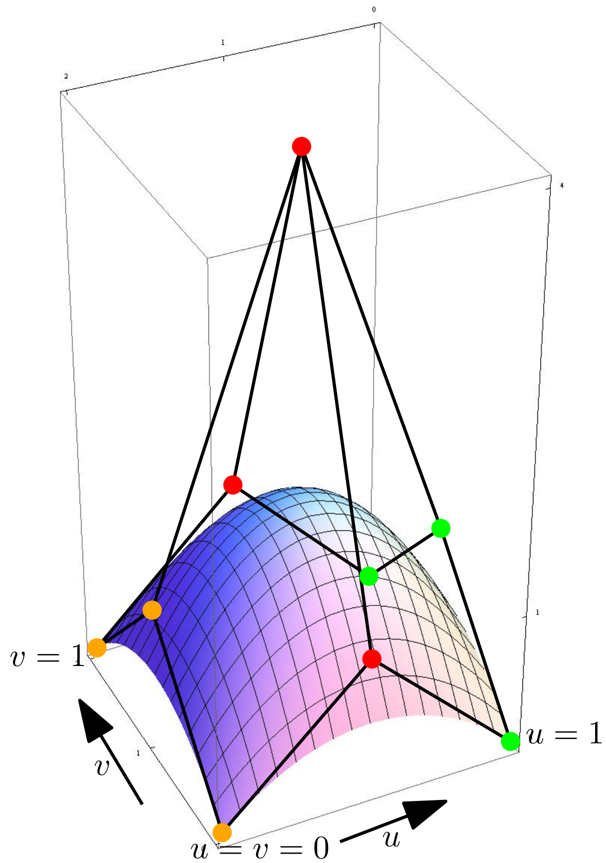

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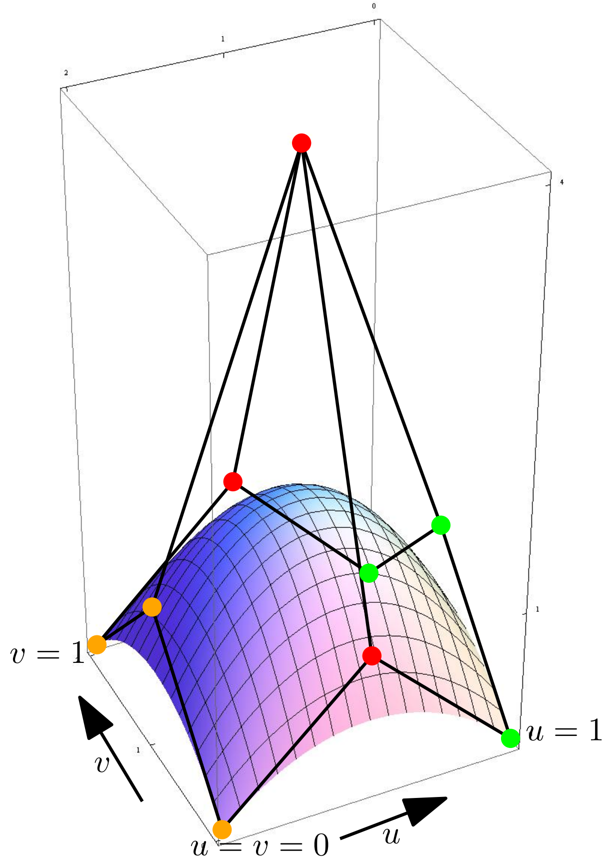
- No variation diminishing property



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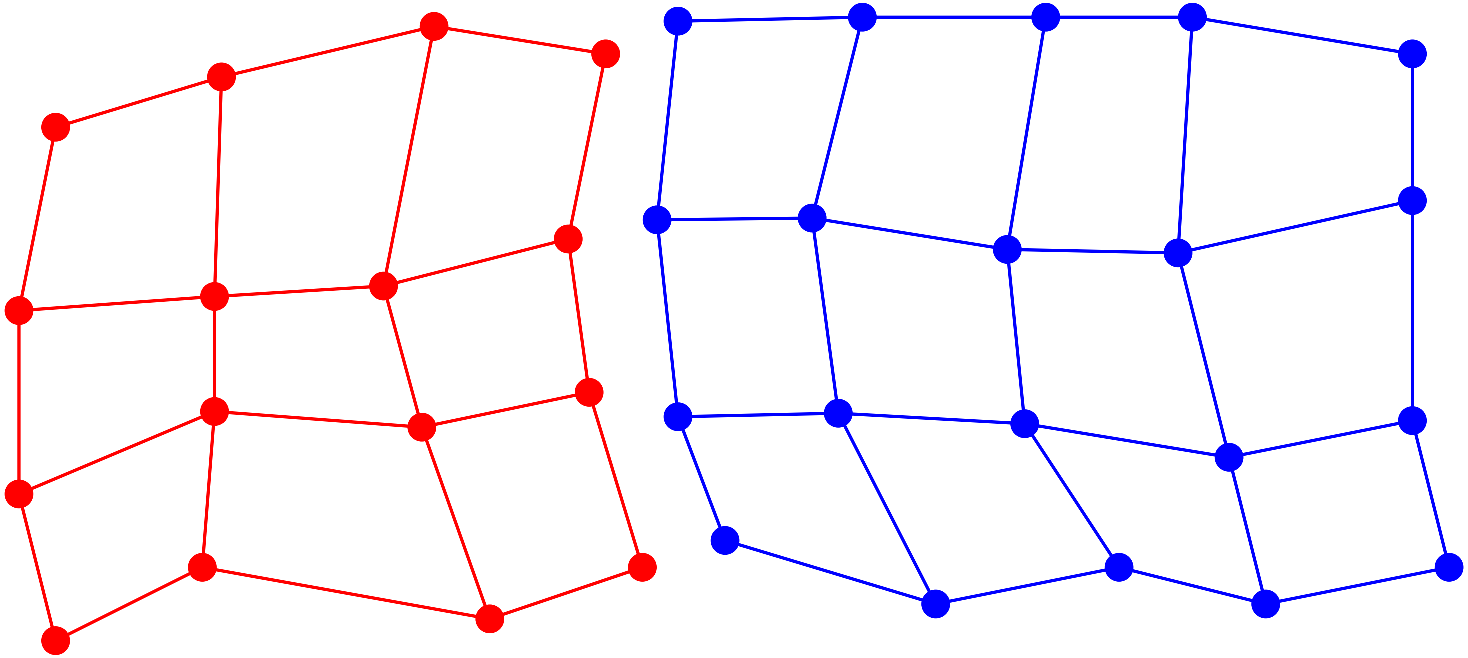

$P(u, v)$ $Q(u, v)$

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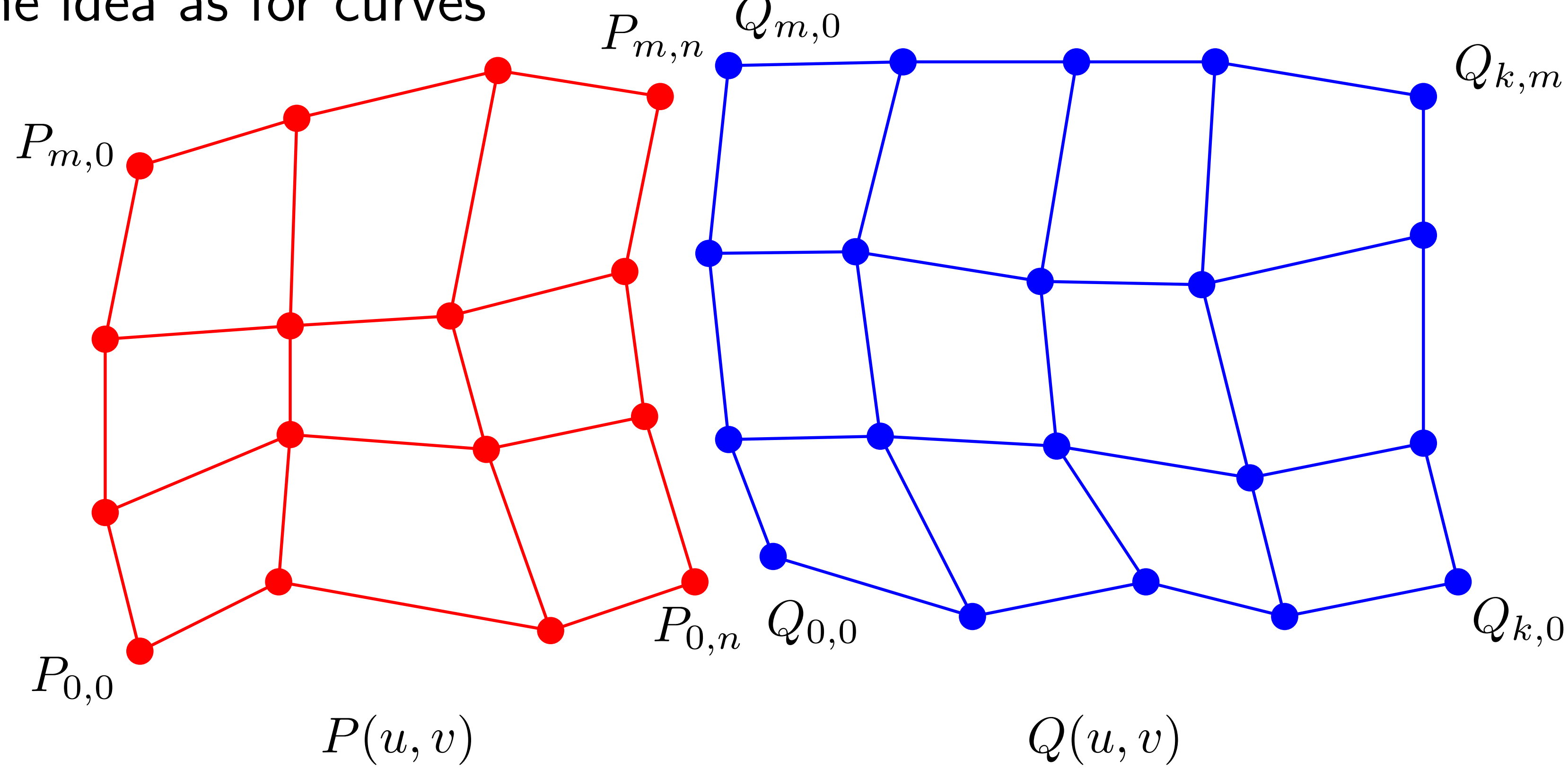

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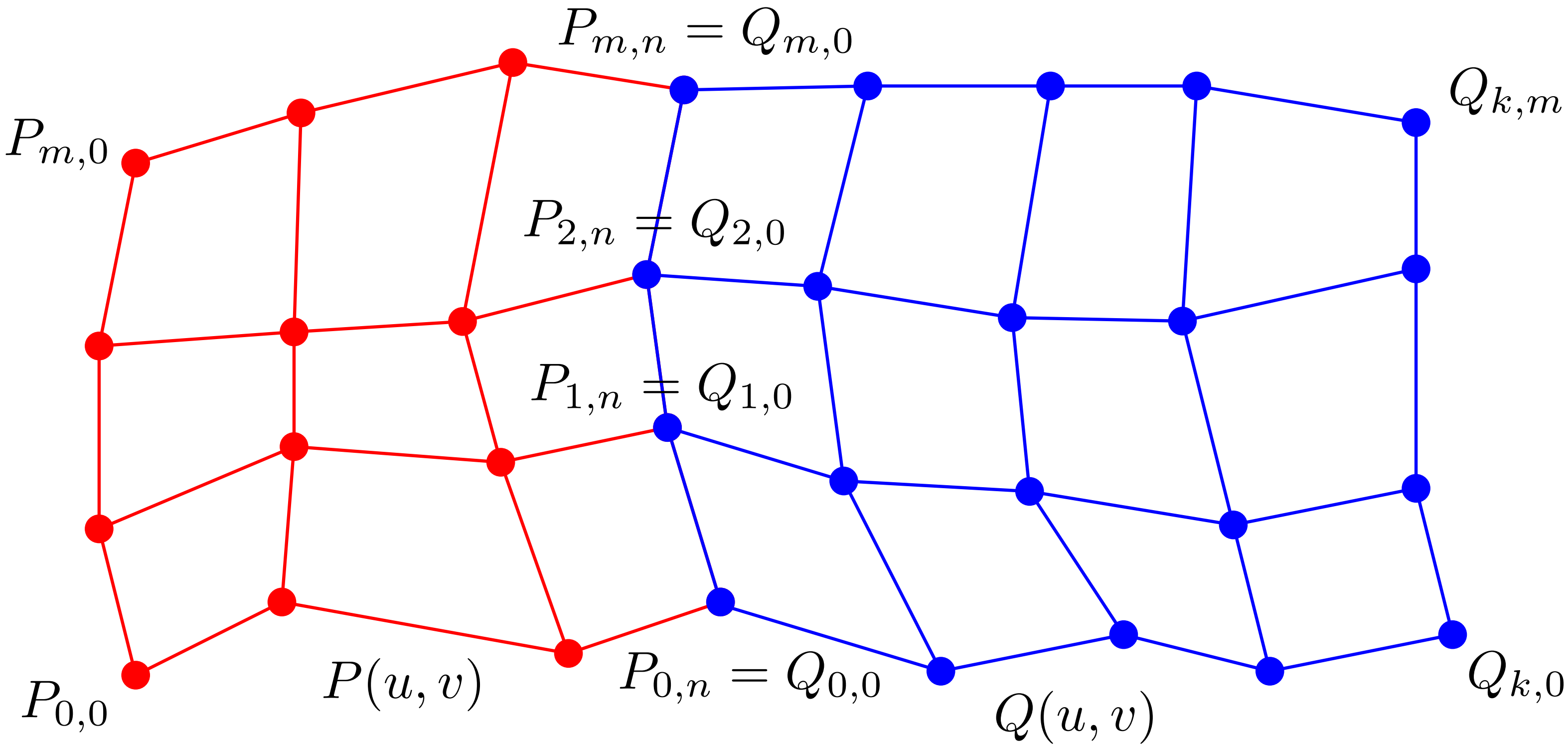

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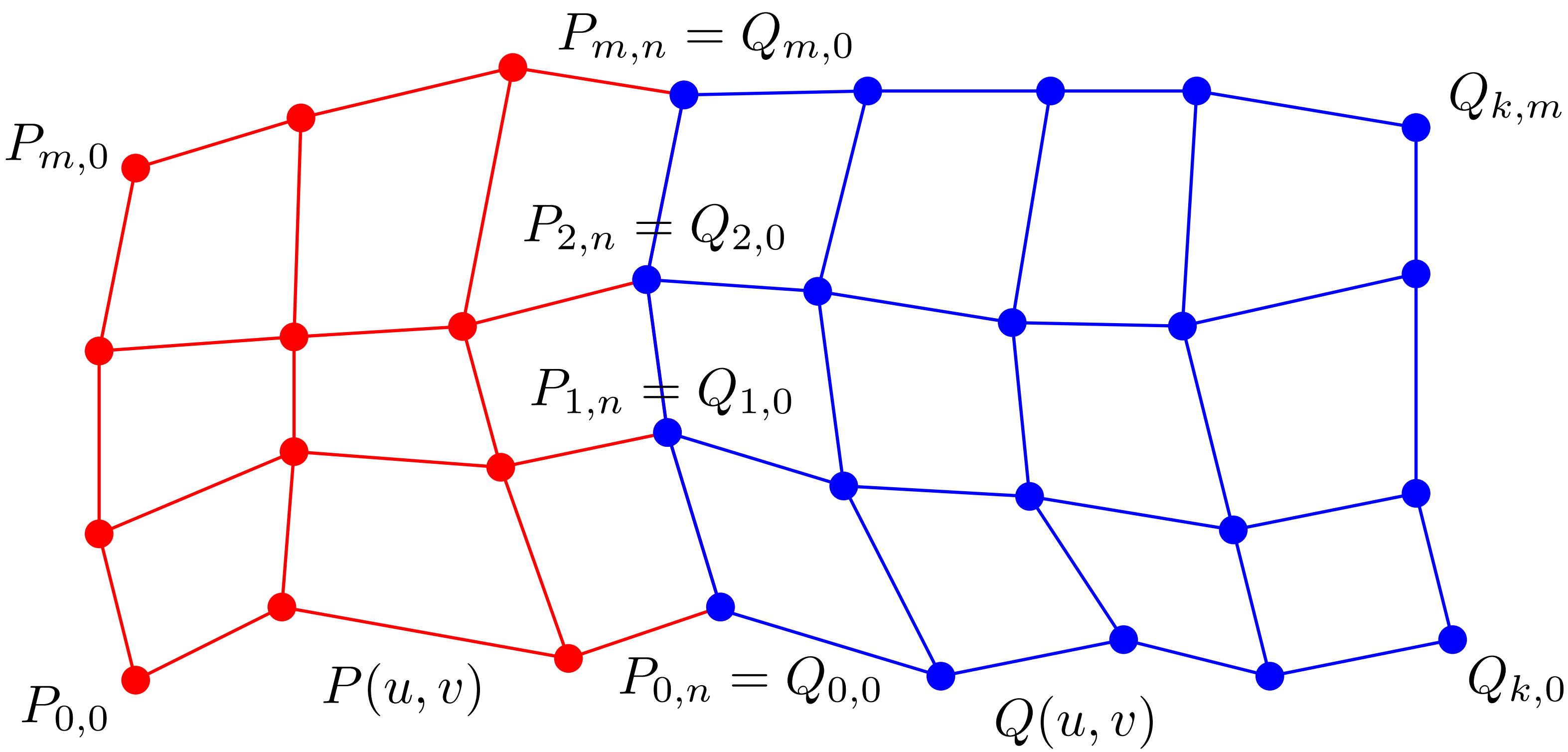

- Continuity (C^0-cont) $\qquad P_{i,n} = Q_{i,0} \quad \forall i = 0, \dots, m$

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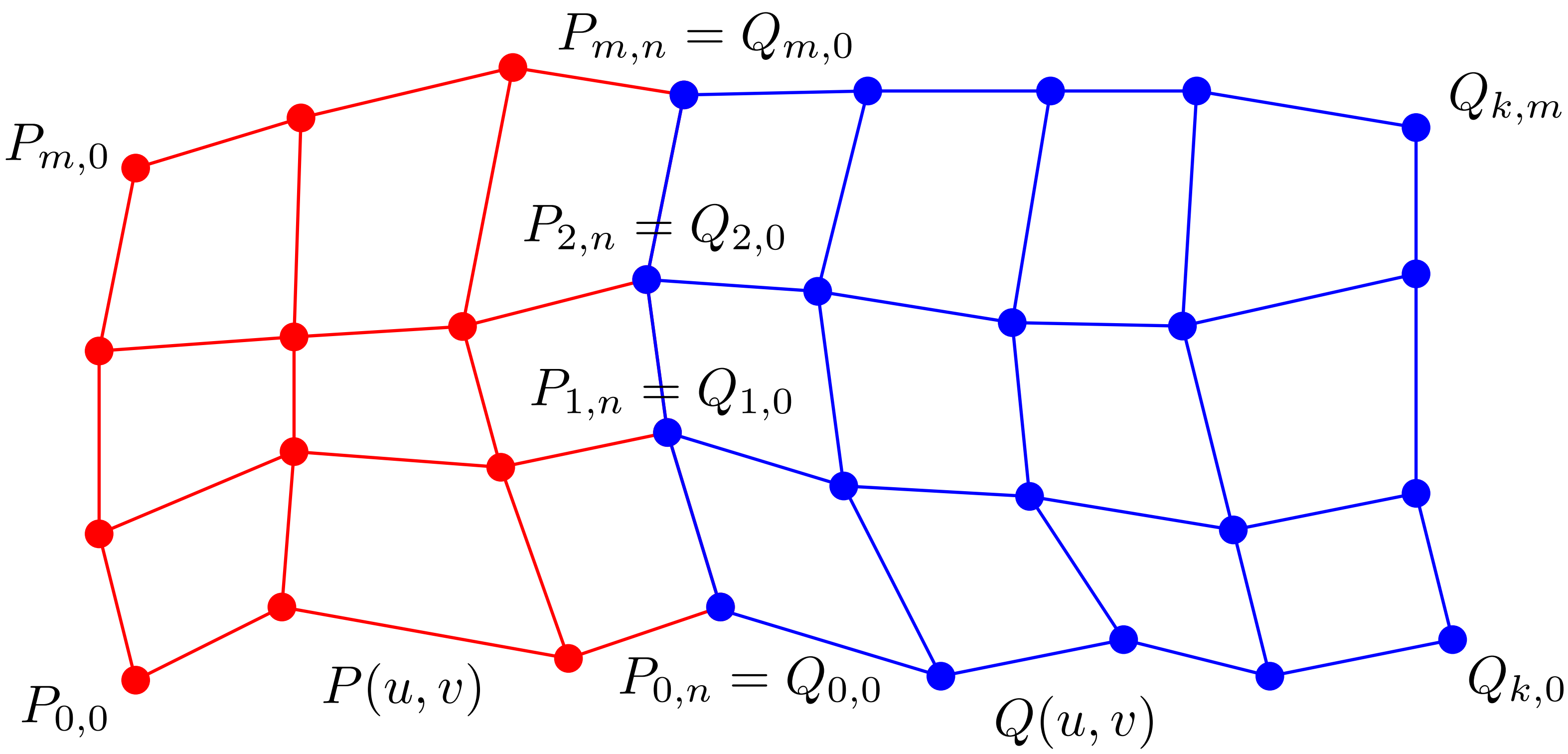

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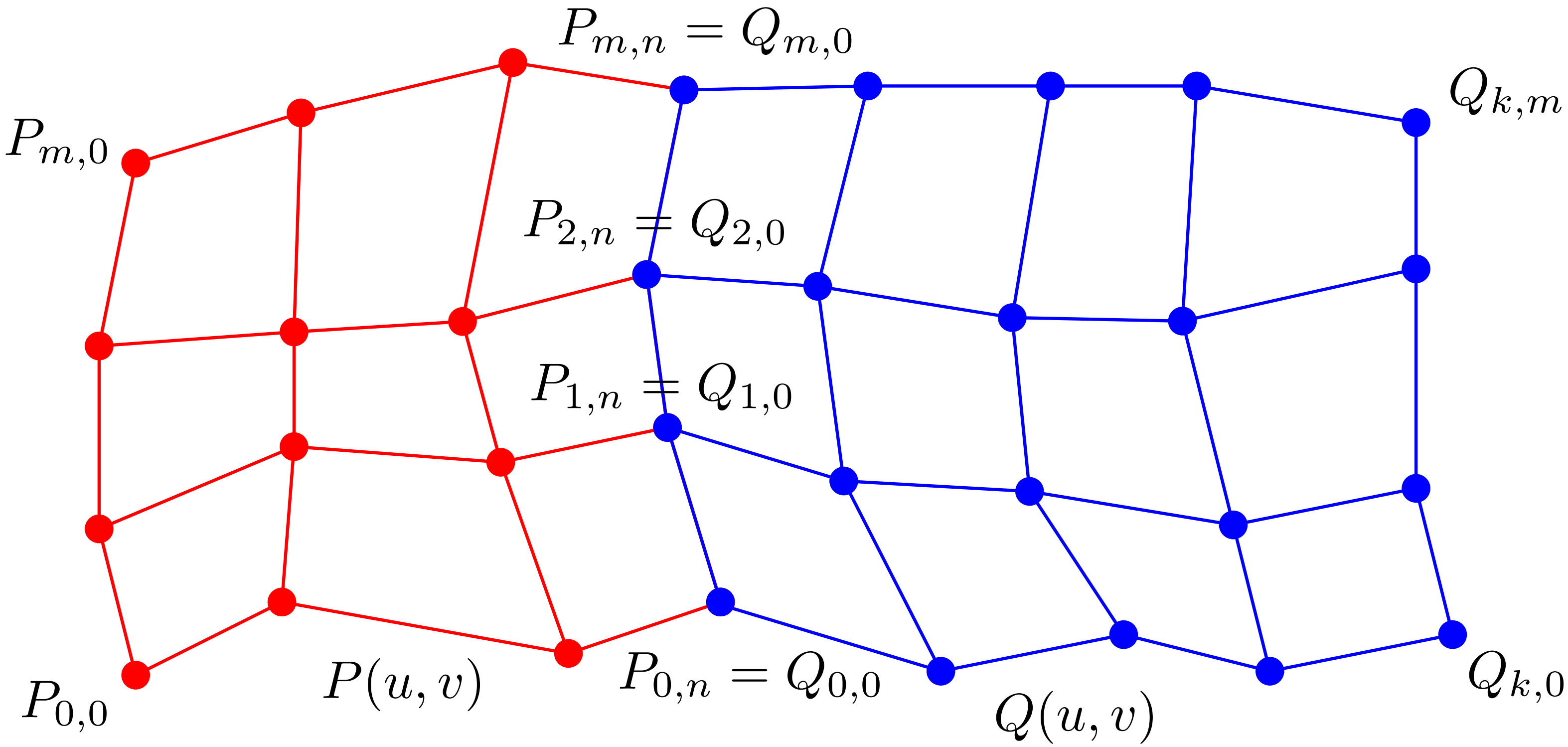

- Continuity (C^0-cont) $\quad P_{i,n} = Q_{i,0} \quad \forall i = 0, \ldots, m$
- Smoothness (C^1-cont) $\quad \left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0}$

CONNECTING BEZIER SURFACES

Smoothness condition (C^1-continuity)

$$\left.\frac{\partial P(u,v)}{\partial u}\right|_{u=1} = \left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0}$$

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Analogously,

$$\left.\frac{\partial Q(u,v)}{\partial u}\right|_{u=0} = k \sum_{i=0}^{m} \binom{m}{i} v^i (1-v)^{m-i} (Q_{i,1} - Q_{i,0})$$

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Therefore, the condition for C^1-continuity is: $n(P_{i,n} - P_{i,n-1}) = k(Q_{i,1} - Q_{i,0}) \;\; \forall i = 0,\ldots,m$

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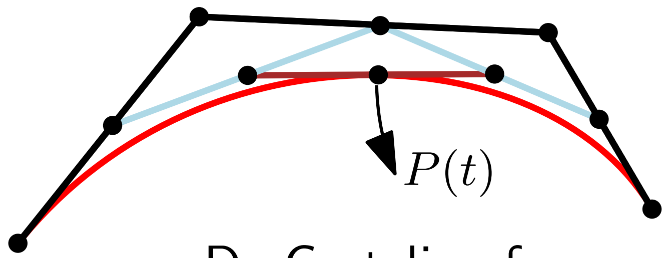
Therefore, the condition for C^1-continuity is: $\boxed{n(P_{i,n} - P_{i,n-1}) = k(Q_{i,1} - Q_{i,0}) \quad \forall i = 0,\dots,m}$

If we just want G^1-cont, it is enough with $P_{i,n} - P_{i,n-1} = \alpha(Q_{i,1} - Q_{i,0})$, for some $\alpha \neq 0 \in \mathbb{R}$

Applying De Casteljau to each dimension

- For surfaces: apply it in two phases (along u, and along v)



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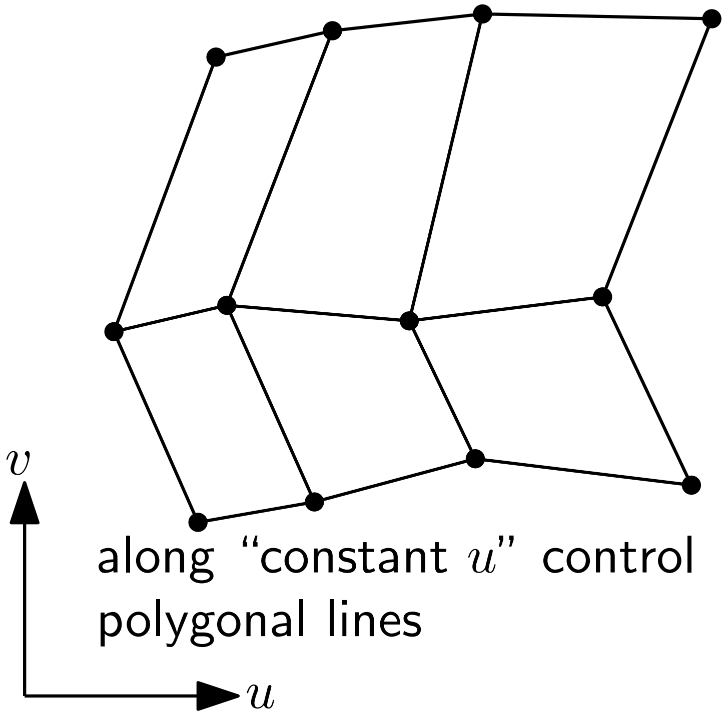
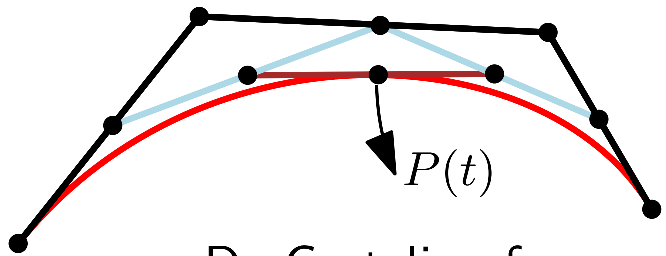


along "constant u" control polygonal lines



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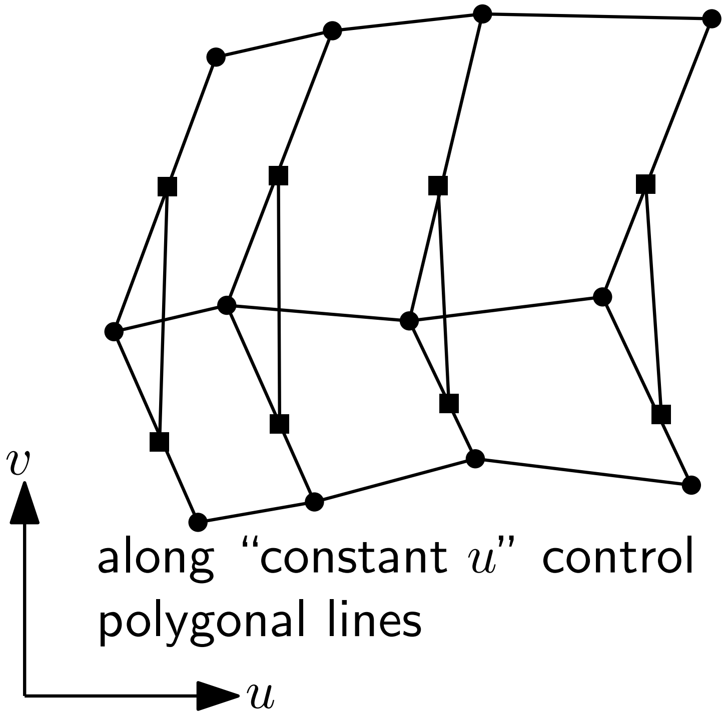

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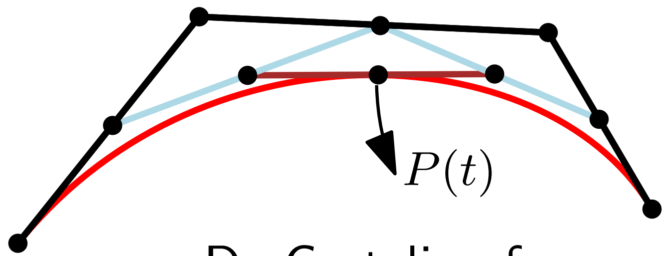

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Applying De Casteljau to each dimension

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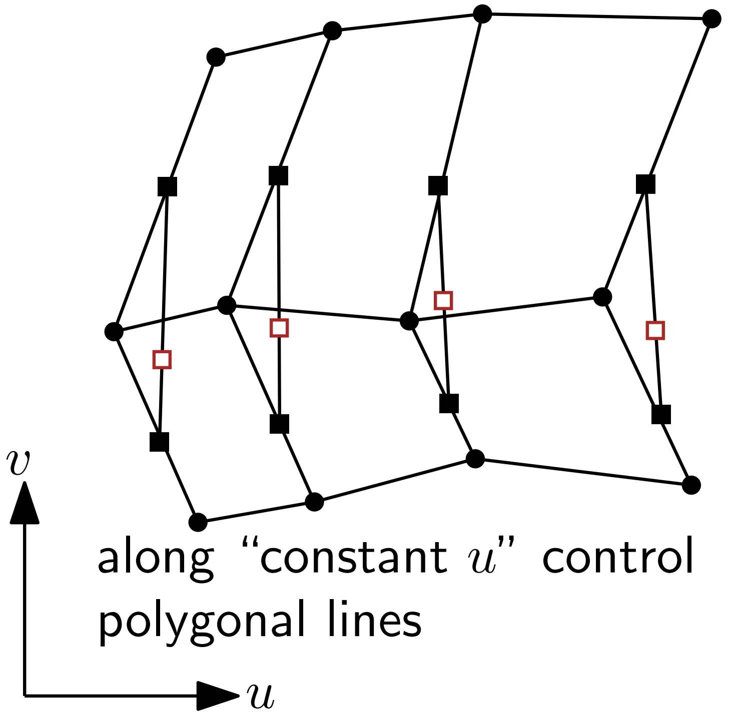
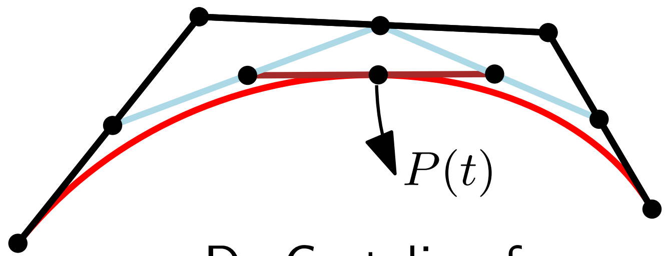


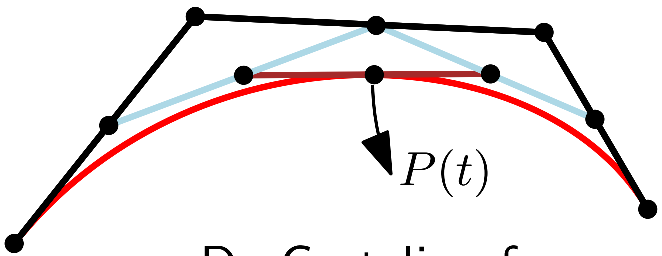
along "constant u" control polygonal lines



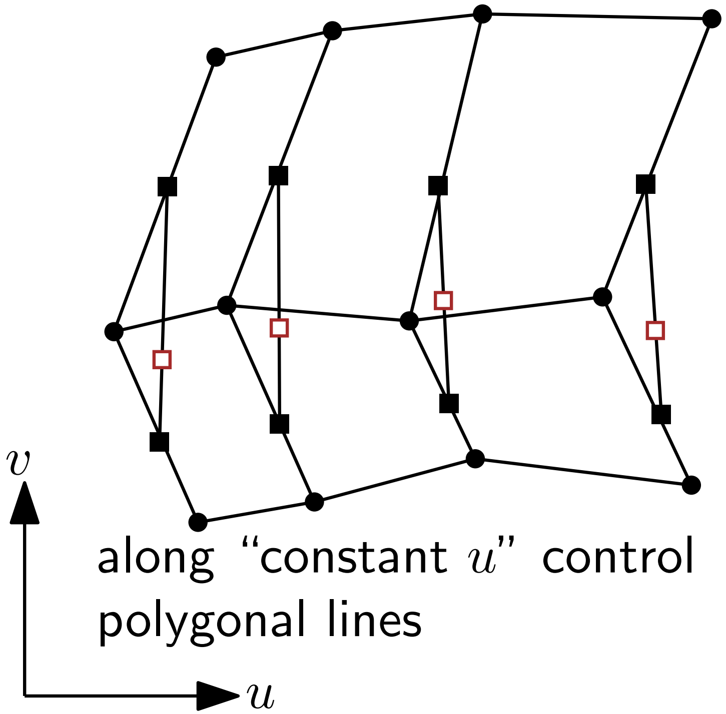
De Casteljau for curves

Applying De Casteljau to each dimension

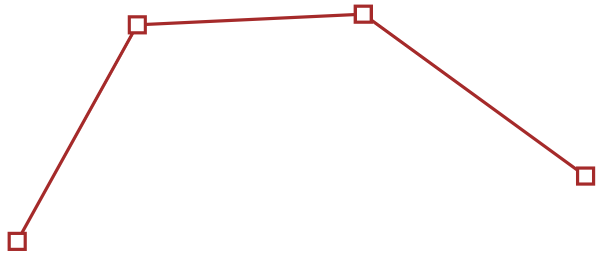
- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves



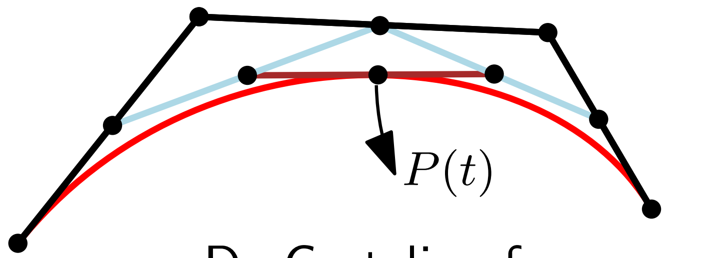
along "constant u" control polygonal lines



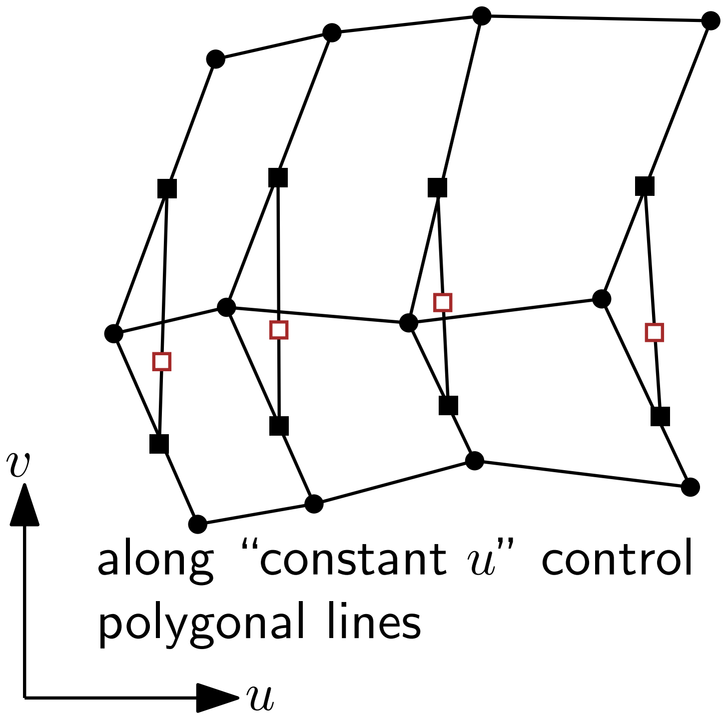
along one "constant v" polygon

Applying De Casteljau to each dimension

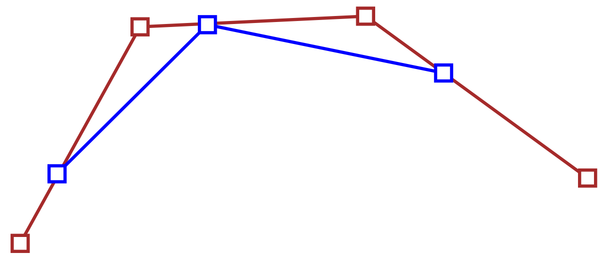
- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves



along "constant u" control polygonal lines



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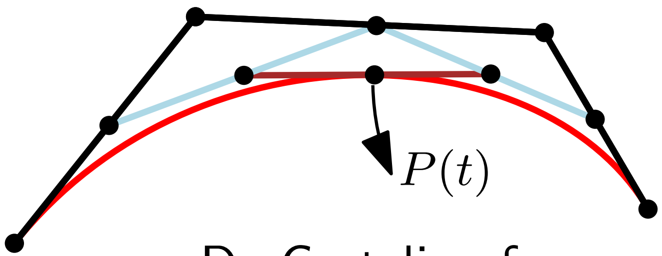
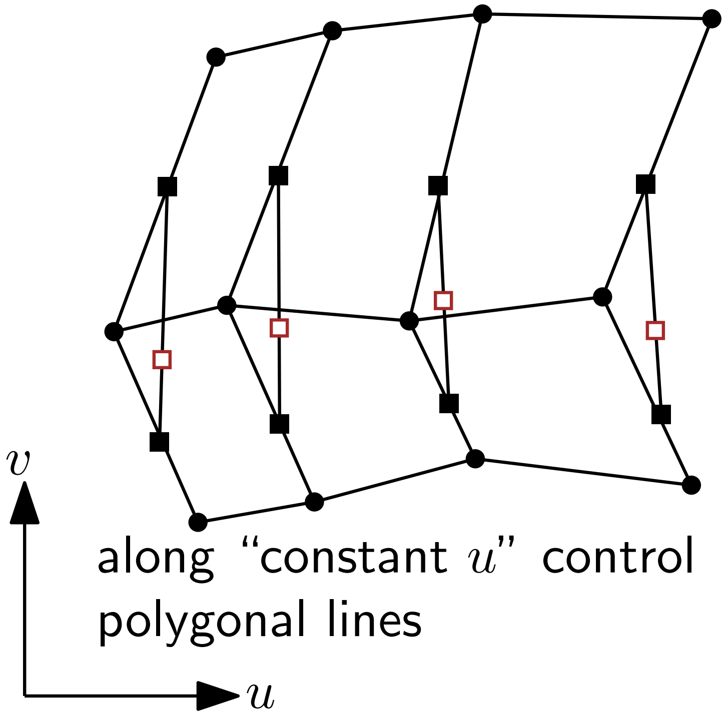


De Casteljau for curves



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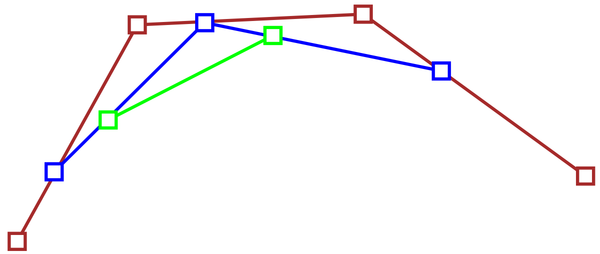

along one "constant v" polygon

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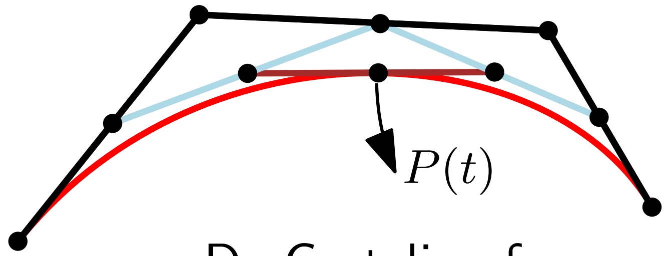
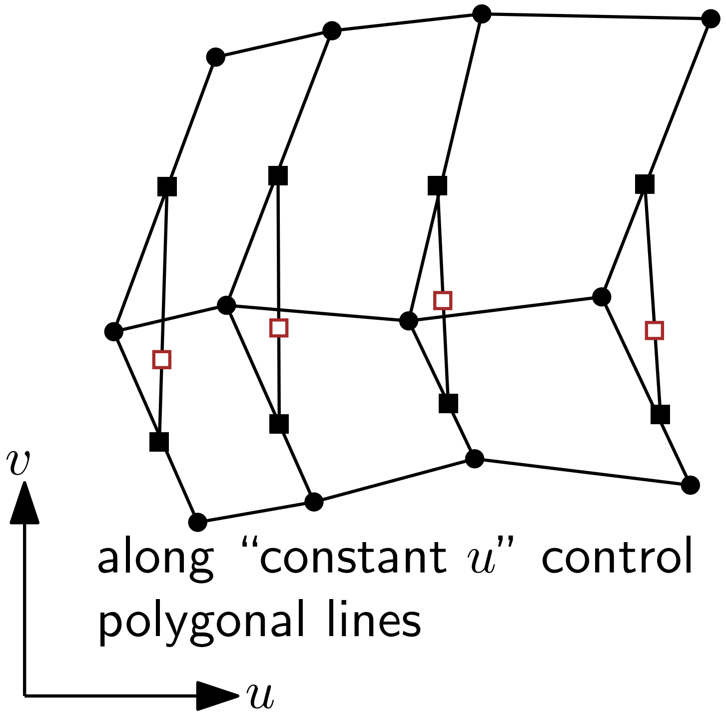
- For surfaces: apply it in two phases (along u, and along v)



De Casteljau for curves



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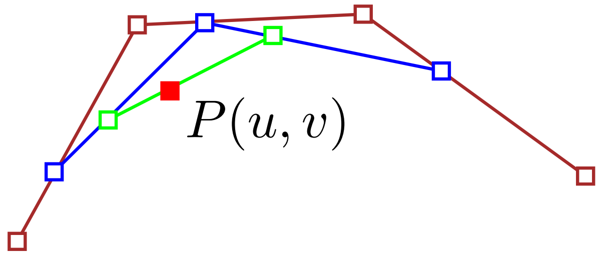

along one "constant v" polygon

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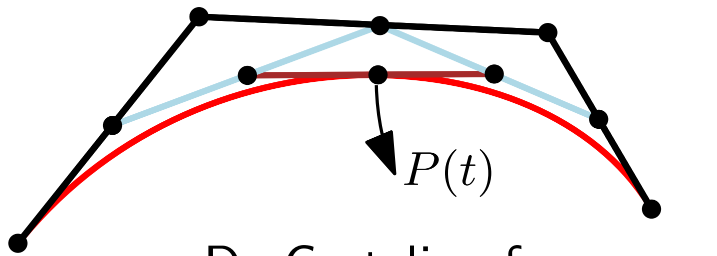
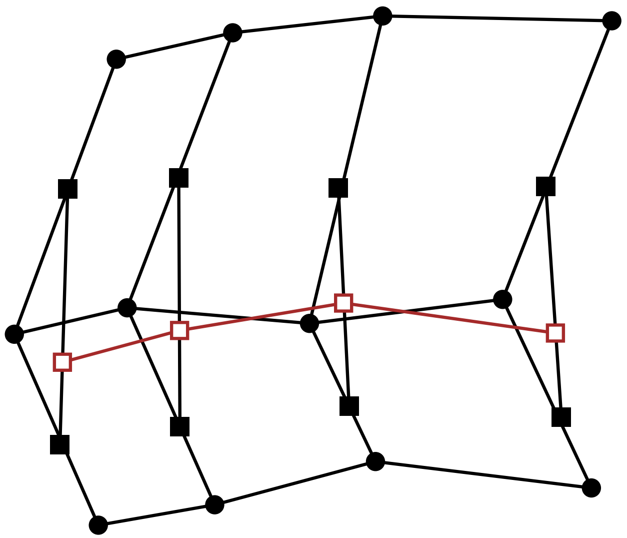
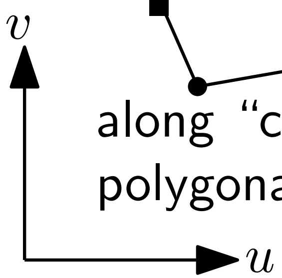
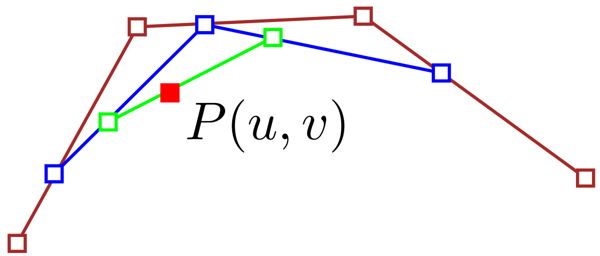


De Casteljau for curves



along "constant u" control polygonal lines



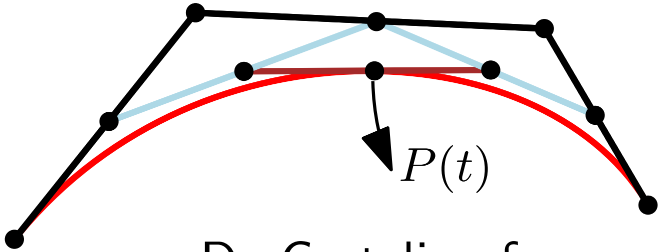


along one "constant v" polygon

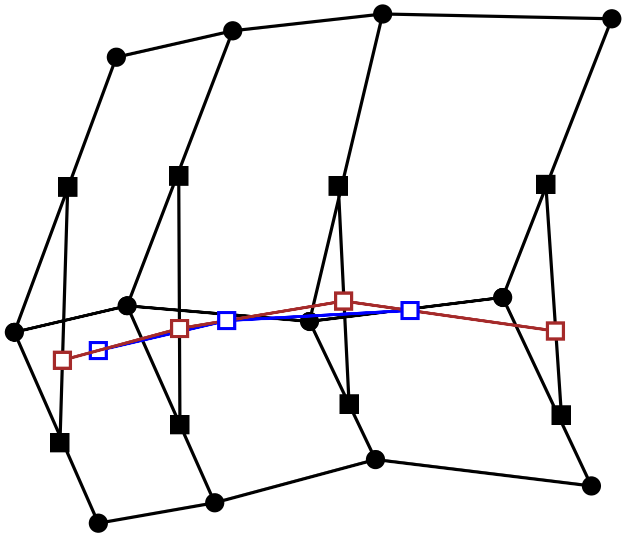
DE CASTELJAU'S ALGORITHM

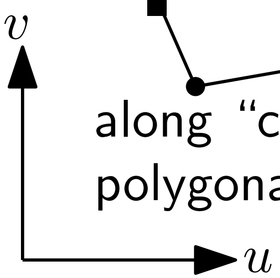
Applying De Casteljau to each dimension

- For surfaces: apply it in two phases (along u, and along v)

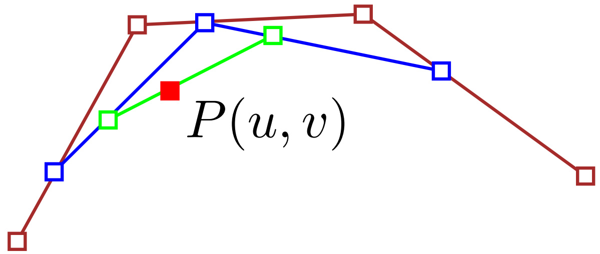


De Casteljau for curves





along "constant u" control polygonal lines



along one "constant v" polygon

Applying De Casteljau to each dimension

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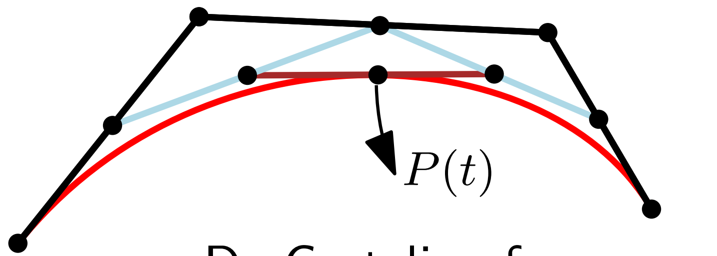
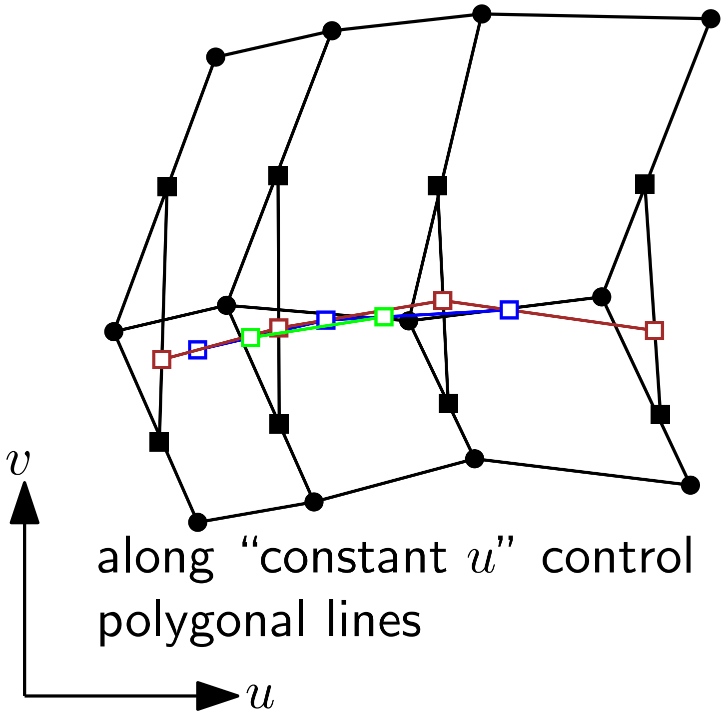
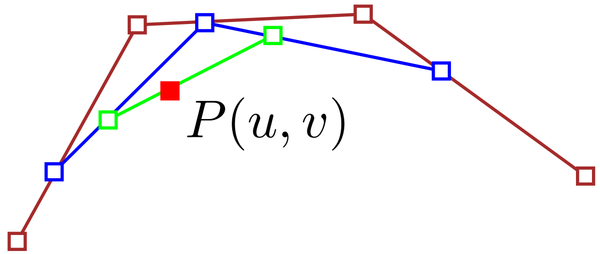


De Casteljau for curves



along "constant u" control polygonal lines



along one "constant v" polygon

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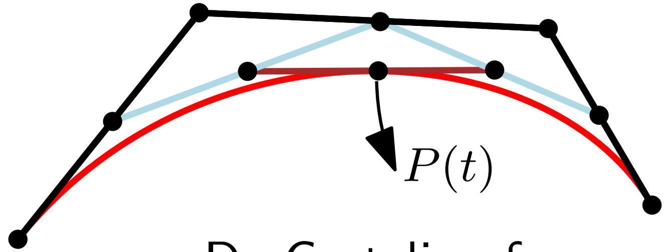
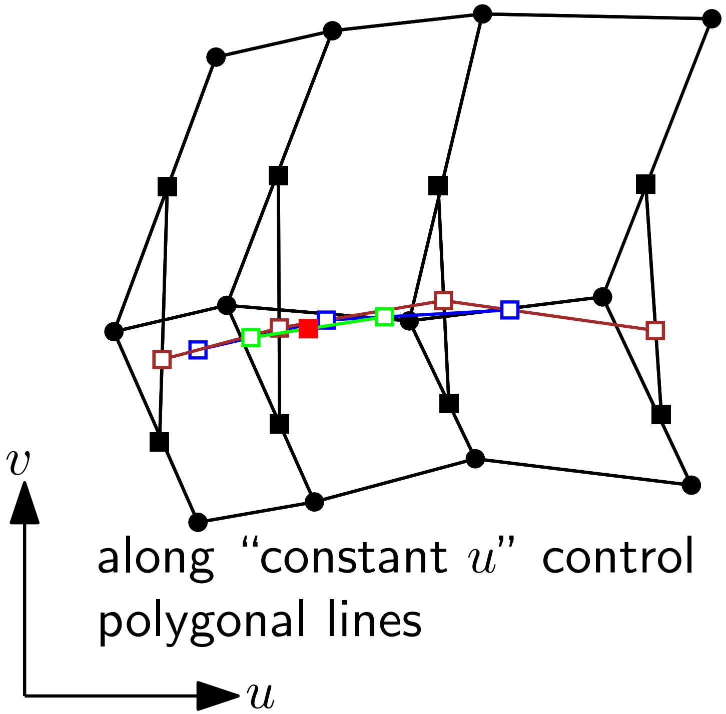
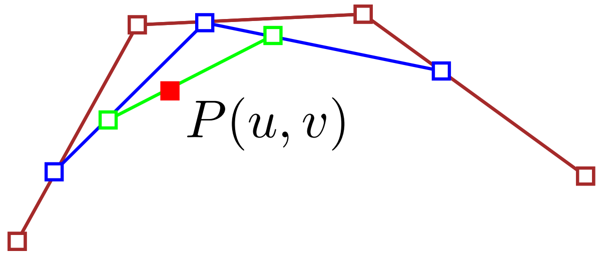


De Casteljau for curves



along "constant u" control polygonal lines



along one "constant v" polygon

- Applications of De Casteljau, e.g., to curve subdivision, also extend to surfaces

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

Interpolating Bézier surface patch

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Same approach as for curves:

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Same approach as for curves:

1) Select $m+1$ values u_k (for $k = 0, \ldots, m$)

Interpolating Bézier surface patch

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Same approach as for curves:

1) Select $m+1$ values u_k (for $k = 0, \ldots, m$)

2) Select $n+1$ values v_l (for $l = 0, \ldots, n$)

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

Same approach as for curves:

1) Select $m+1$ values u_k (for $k = 0, \ldots, m$)

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3) Require $S(u_k, v_l) = Q_{k,l}$ for all k, l

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Defines system of $(m+1) \times (n+1)$ equations that always has a solution

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Same approach as for curves:

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Defines system of $(m+1) \times (n+1)$ equations that always has a solution

$$S(u_k, v_l) = B^t_m(u_k) P B_n(v_l) = Q_{k,l}, \text{ for } k = 0, \ldots, m \text{ and } l = 0, \ldots, n$$

Interpolating Bézier surface patch

Problem: given $(m+1) \times (n+1)$ data points $Q_{k,l}$, compute a set of $(m+1) \times (n+1)$ control points $P_{i,j}$ such that the Bézier surface S defined by the points $P_{i,j}$ goes through the points $Q_{k,l}$

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Defines system of $(m+1) \times (n+1)$ equations that always has a solution

$$S(u_k, v_l) = \underbrace{B^t_m(u_k) P B_n(v_l)}_{\text{matricial form of Bézier surface formula}} = Q_{k,l}, \text{ for } k = 0, \ldots, m \text{ and } l = 0, \ldots, n$$

Interpolating Bézier surface patch

Example from [Salomon, page 232]:

Example: We choose $m = 3$ and $n = 2$. The system of equations becomes

$$\left[(1-u_k)^3, 3u_k(1-u_k)^2, 3u_k^2(1-u_k), u_k^3\right] \begin{bmatrix} \mathbf{P}_{00} & \mathbf{P}_{01} & \mathbf{P}_{02} \\ \mathbf{P}_{10} & \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{20} & \mathbf{P}_{21} & \mathbf{P}_{22} \\ \mathbf{P}_{30} & \mathbf{P}_{31} & \mathbf{P}_{32} \end{bmatrix} \begin{bmatrix} (1-w_l)^2 \\ 2w_l(1-w_l) \\ w_l^2 \end{bmatrix} = \mathbf{Q}_{kl}$$

Interpolating Bézier surface patch

Example from [Salomon, page 232]:

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12 equations, 12 unknowns

RATIONAL BÉZIER SURFACES

Rational rectangular Bézier surface patch

Rational rectangular Bézier surface patch

Definition anologous to the one for curves

Rational rectangular Bézier surface patch

Definition anologous to the one for curves

$$P(u,w) = \frac{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j} B_{m,i}(u) B_{n,j}(w) P_{i,j}}{\sum_{i=0}^{m}\sum_{j=0}^{n} w_{i,j} B_{m,i}(u) B_{n,j}(w)}$$

$0 \leq u, w \leq 1$
$w_{i,j} \in \mathbb{R}_{>0}$ for all i, j

If all weights are $w_{i,j} = 1$, it reduces to the ordinary Bézier surface

Surface patches don't need to be rectangular

$$\begin{array}{c} \mathbf{P}_{010} \\ \mathbf{P}_{001}\ \mathbf{P}_{100} \end{array} \qquad \begin{array}{c} \mathbf{P}_{020} \\ \mathbf{P}_{011}\ \mathbf{P}_{110} \\ \mathbf{P}_{002}\ \mathbf{P}_{101}\ \mathbf{P}_{200} \end{array} \qquad \begin{array}{c} \mathbf{P}_{030} \\ \mathbf{P}_{021}\ \mathbf{P}_{120} \\ \mathbf{P}_{012}\ \mathbf{P}_{111}\ \mathbf{P}_{210} \\ \mathbf{P}_{003}\ \mathbf{P}_{102}\ \mathbf{P}_{201}\ \mathbf{P}_{300} \end{array} \qquad \begin{array}{c} \mathbf{P}_{040} \\ \mathbf{P}_{031}\ \mathbf{P}_{130} \\ \mathbf{P}_{022}\ \mathbf{P}_{121}\ \mathbf{P}_{220} \\ \mathbf{P}_{013}\ \mathbf{P}_{112}\ \mathbf{P}_{211}\ \mathbf{P}_{310} \\ \mathbf{P}_{004}\ \mathbf{P}_{103}\ \mathbf{P}_{202}\ \mathbf{P}_{301}\ \mathbf{P}_{400} \end{array}$$

Surface patches don't need to be rectangular

Control points arranged as triangular array

$$
\begin{array}{c}
\mathbf{P}_{010} \\
\mathbf{P}_{001}\ \mathbf{P}_{100}
\end{array}
\quad
\begin{array}{c}
\mathbf{P}_{020} \\
\mathbf{P}_{011}\ \mathbf{P}_{110} \\
\mathbf{P}_{002}\ \mathbf{P}_{101}\ \mathbf{P}_{200}
\end{array}
\quad
\begin{array}{c}
\mathbf{P}_{030} \\
\mathbf{P}_{021}\ \mathbf{P}_{120} \\
\mathbf{P}_{012}\ \mathbf{P}_{111}\ \mathbf{P}_{210} \\
\mathbf{P}_{003}\ \mathbf{P}_{102}\ \mathbf{P}_{201}\ \mathbf{P}_{300}
\end{array}
\quad
\begin{array}{c}
\mathbf{P}_{040} \\
\mathbf{P}_{031}\ \mathbf{P}_{130} \\
\mathbf{P}_{022}\ \mathbf{P}_{121}\ \mathbf{P}_{220} \\
\mathbf{P}_{013}\ \mathbf{P}_{112}\ \mathbf{P}_{211}\ \mathbf{P}_{310} \\
\mathbf{P}_{004}\ \mathbf{P}_{103}\ \mathbf{P}_{202}\ \mathbf{P}_{301}\ \mathbf{P}_{400}
\end{array}
$$

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\begin{array}{c}
\mathbf{P}_{010} \\
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\end{array}
\qquad
\begin{array}{c}
\mathbf{P}_{020} \\
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\mathbf{P}_{002}\ \mathbf{P}_{101}\ \mathbf{P}_{200}
\end{array}
\qquad
\begin{array}{c}
\mathbf{P}_{030} \\
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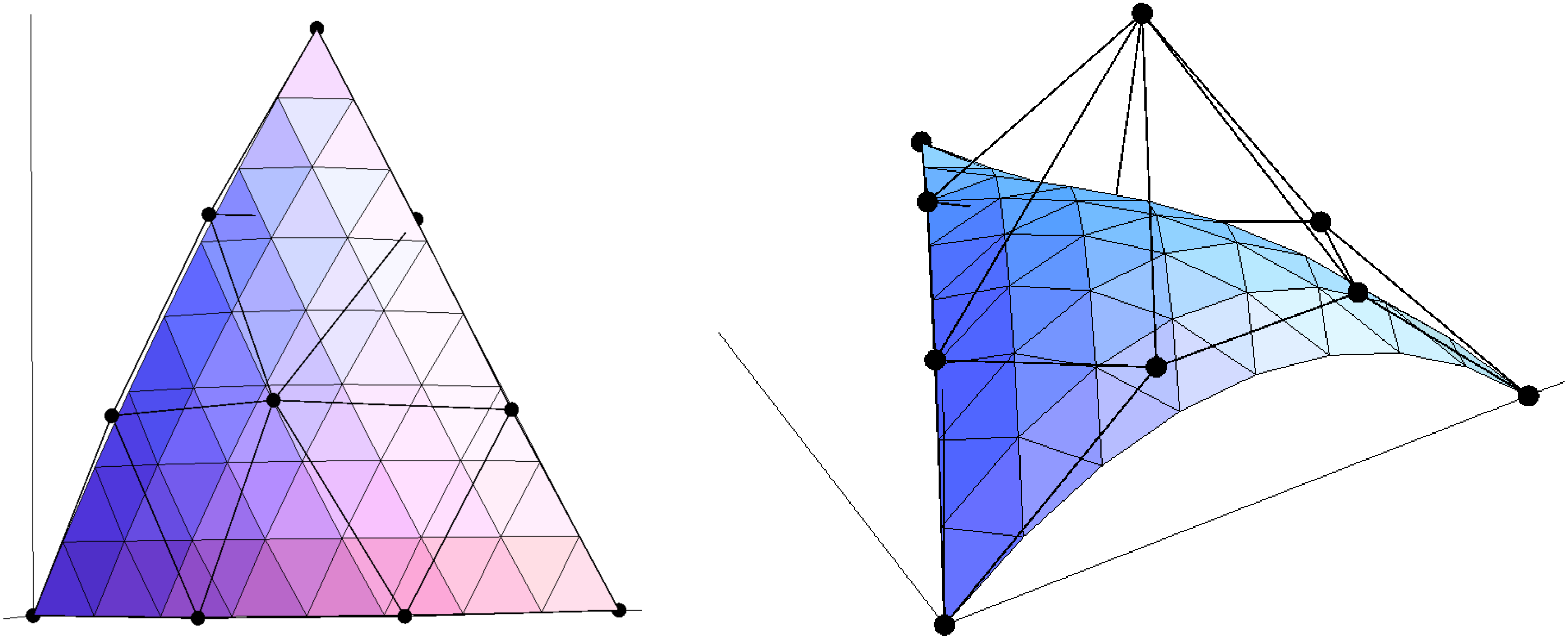
\end{array}
\qquad
\begin{array}{c}
\mathbf{P}_{040} \\
\mathbf{P}_{031}\ \mathbf{P}_{130} \\
\mathbf{P}_{022}\ \mathbf{P}_{121}\ \mathbf{P}_{220} \\
\mathbf{P}_{013}\ \mathbf{P}_{112}\ \mathbf{P}_{211}\ \mathbf{P}_{310} \\
\mathbf{P}_{004}\ \mathbf{P}_{103}\ \mathbf{P}_{202}\ \mathbf{P}_{301}\ \mathbf{P}_{400}
\end{array}
$$

Bézier formula needs version based on three variables

$$\mathbf{P}(u,v,w) = \sum_{i+j+k=n} \mathbf{P}_{ijk} \frac{n!}{i!\,j!\,k!} u^i v^j w^k = \sum_{i+j+k=n} \mathbf{P}_{ijk} B^n_{ijk}(u,v,w)$$

Example



$n = 3$