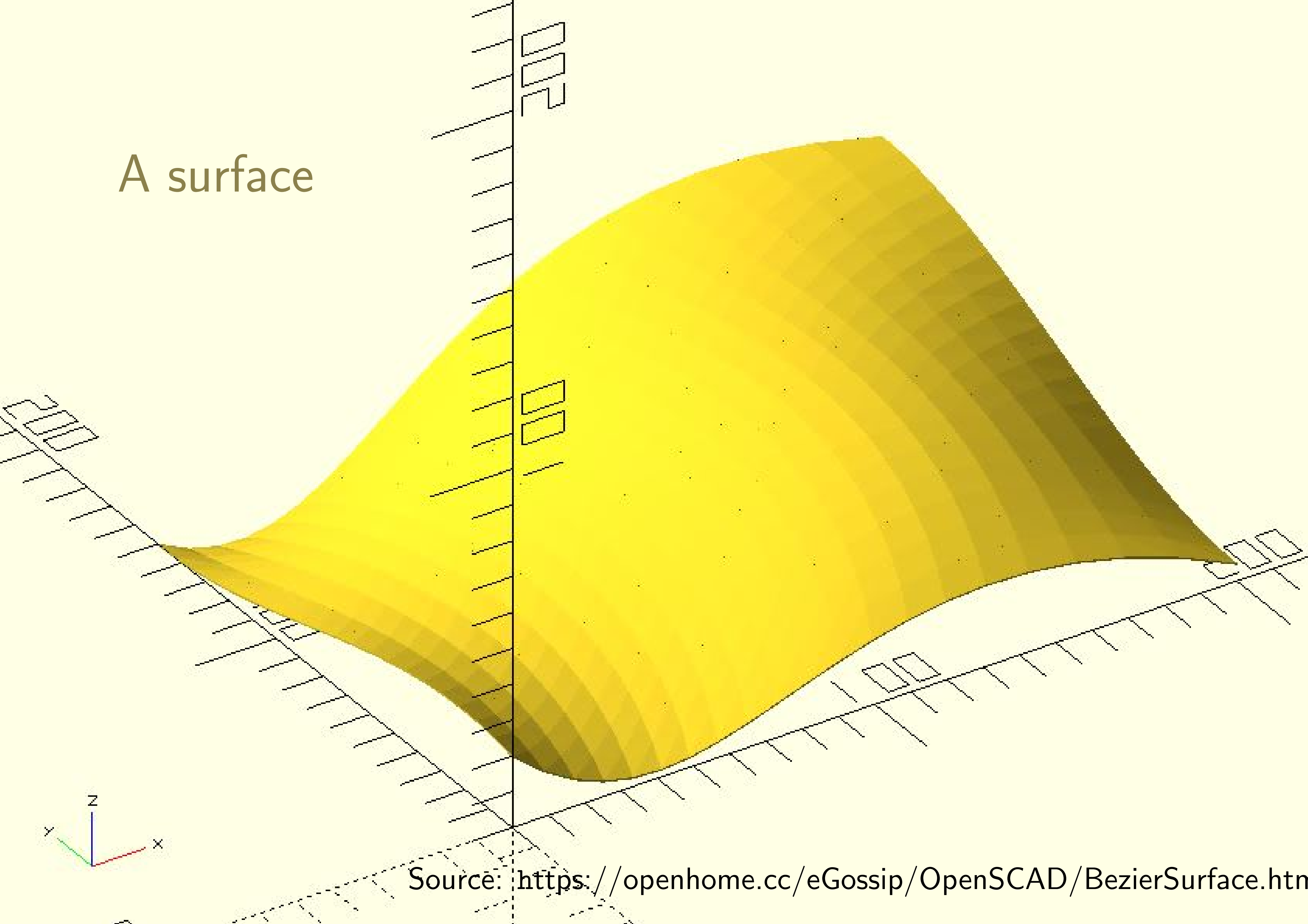
SURFACES

Rodrigo Silveira

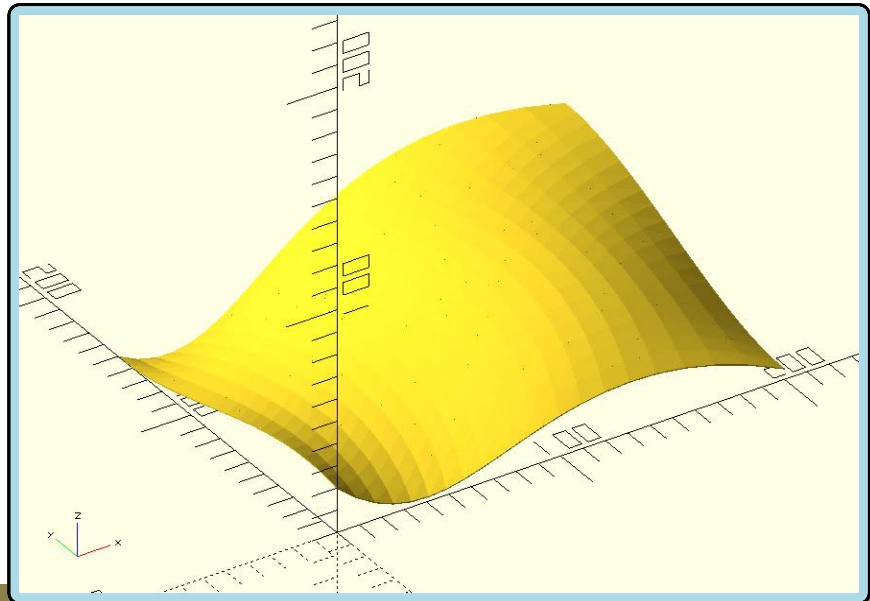
Curve and Surface Design
Facultat d'Informàtica de Barcelona
Universitat Politècnica de Catalunya

A surface

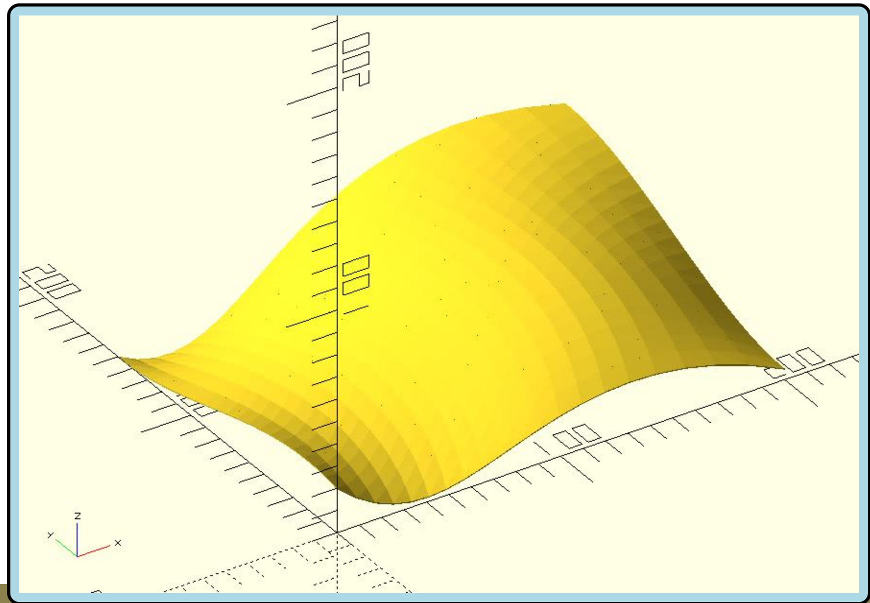


Source: https://openhome.cc/eGossip/OpenSCAD/BezierSurface.htm

PARAMETRIZING SURFACES



1) Explicit equation



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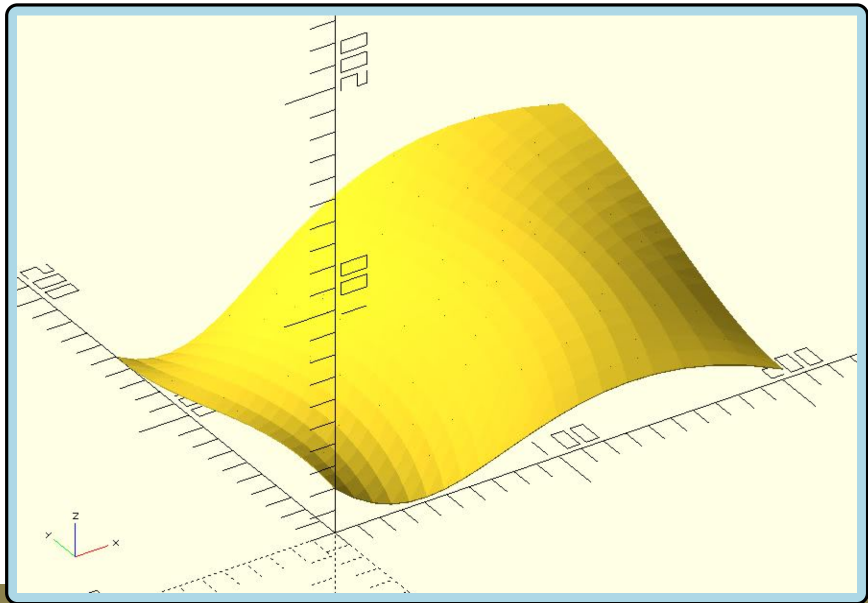
$z = f(x, y)$, for $x \in [x_0, x_1]$, $y \in [y_0, y_1]$, f continuous



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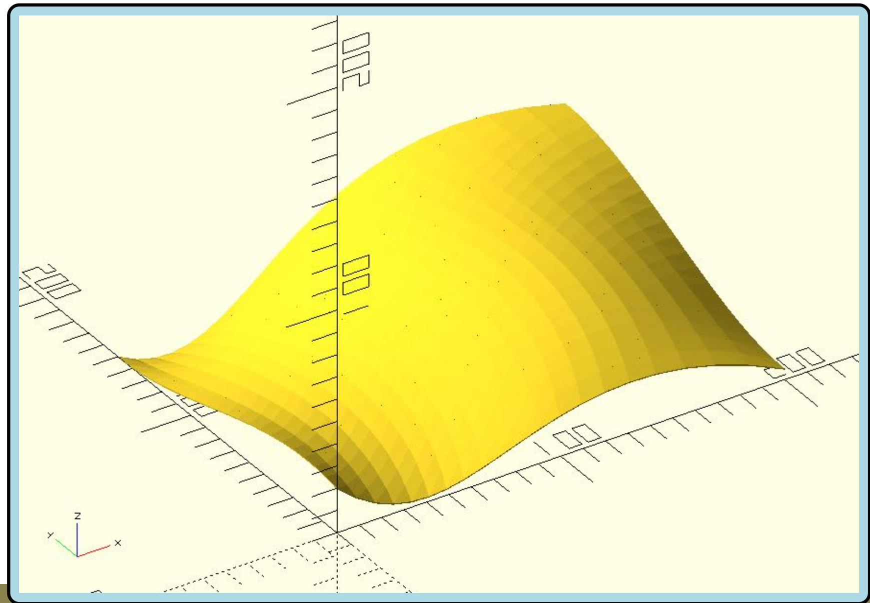
The surface is $S = \{(\mathbf{x}, \mathbf{y}, \mathbf{f}(\mathbf{x}, \mathbf{y}))/x \in [x_0, x_1], y \in [y_0, y_1]\}$



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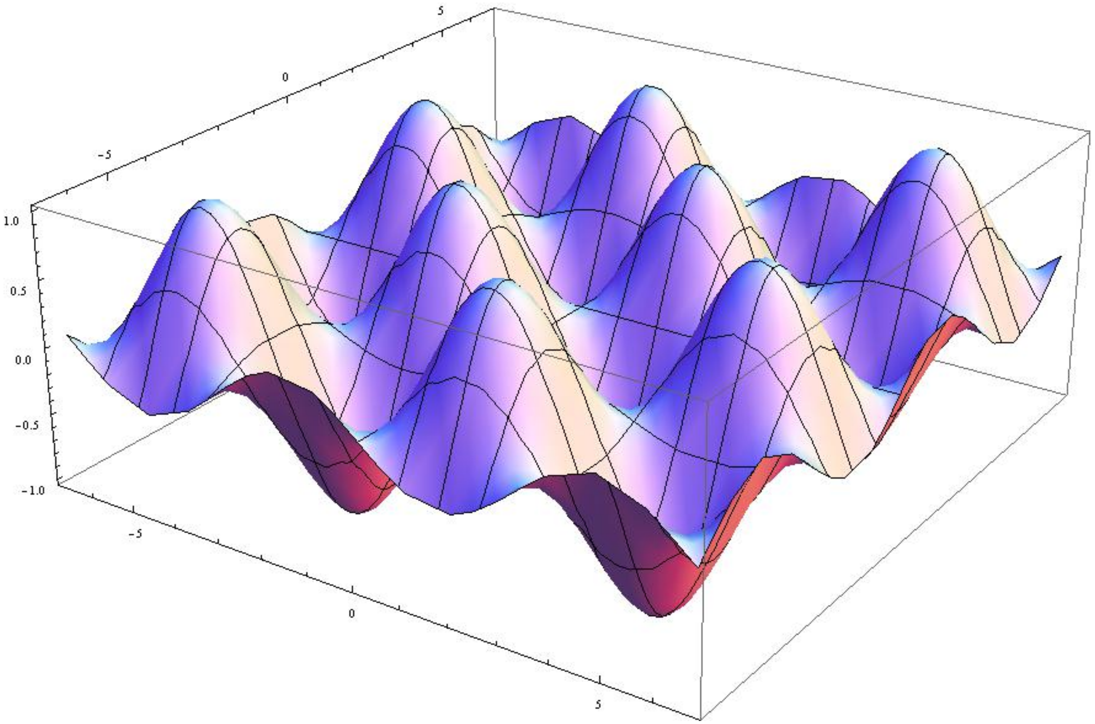
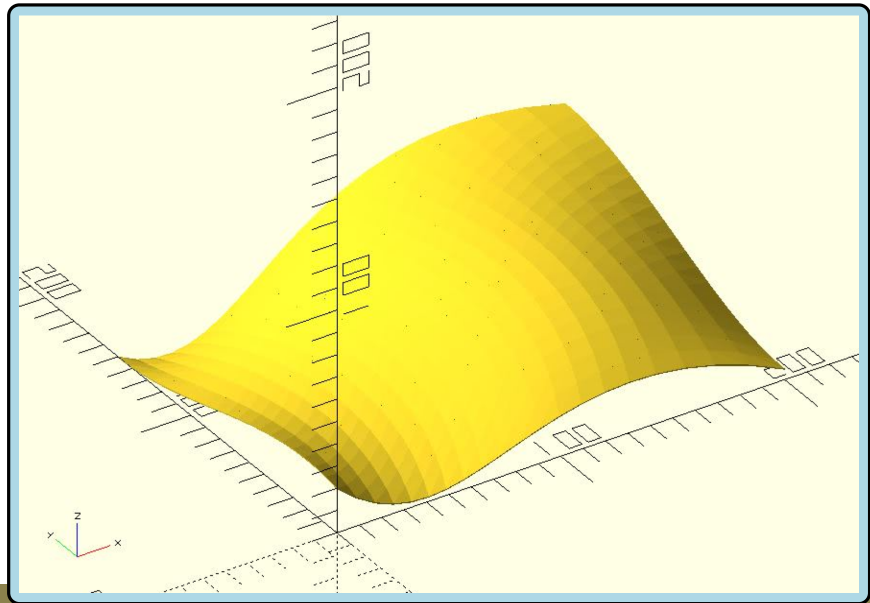
Example: $z = \sin x \sin y$





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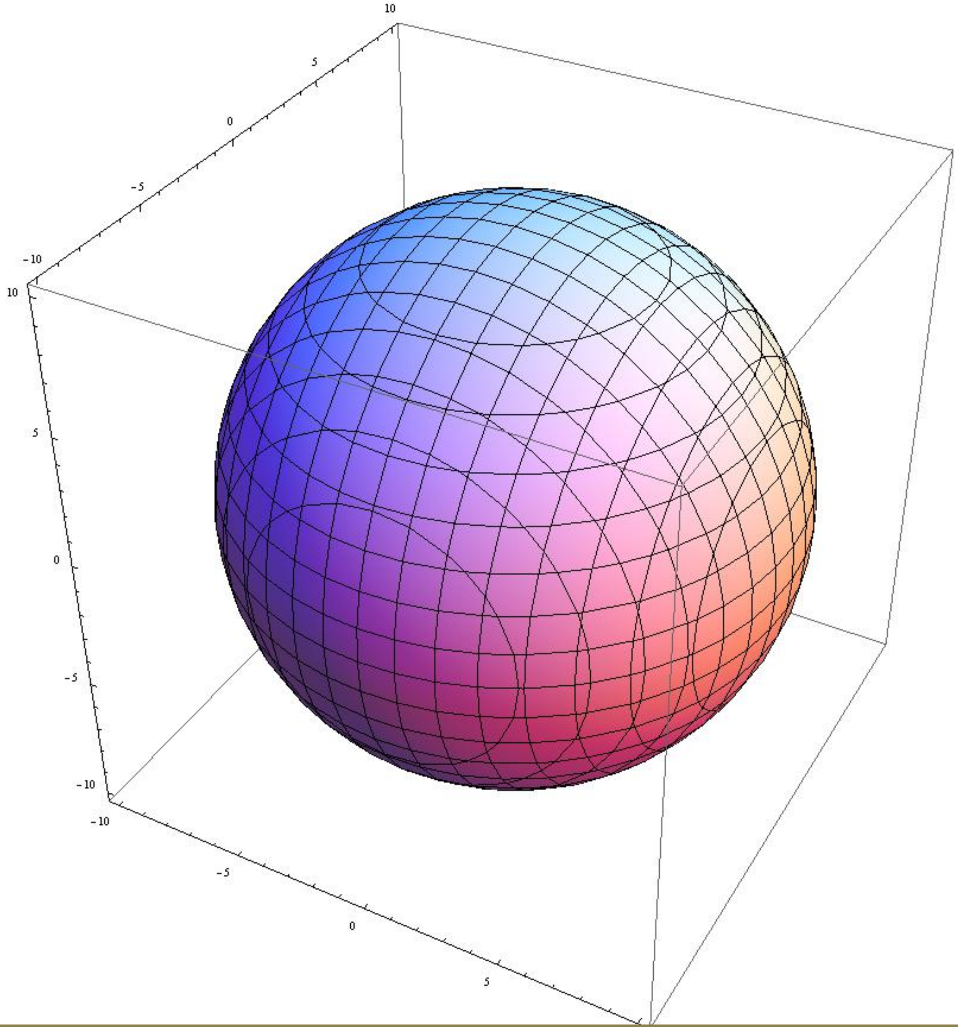

3) Parametric equation

$$\left.\begin{aligned} x &= x(u,v) \\ y &= y(u,v) \\ z &= z(u,v) \end{aligned}\right| \; u \in [u_0, u_1],\ v \in [v_0, v_1],\ x, y, z : \mathbb{R}^2 \to \mathbb{R} \text{ continuous}$$

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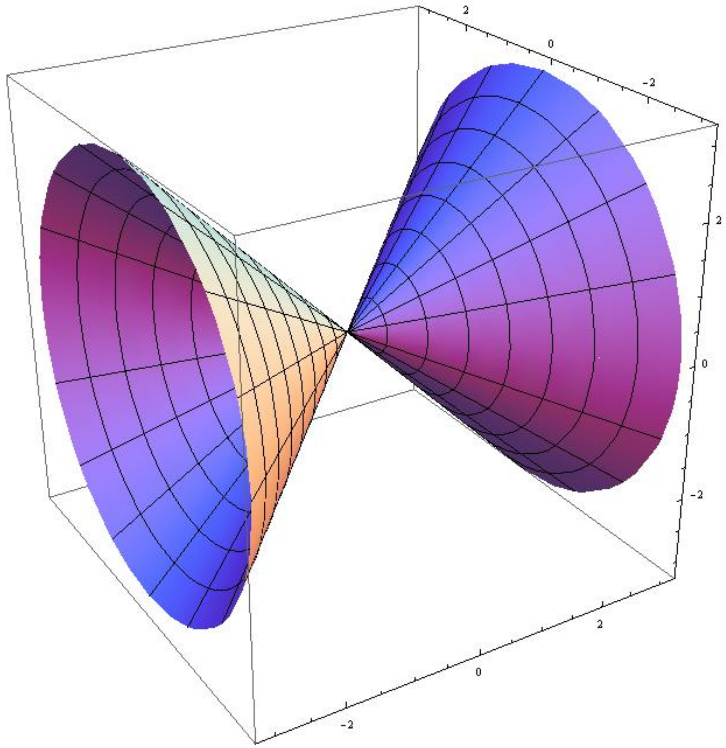

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the choice for surface design for CAD and Graphics

EXAMPLES OF SURFACES

Easier: surfaces based on curves

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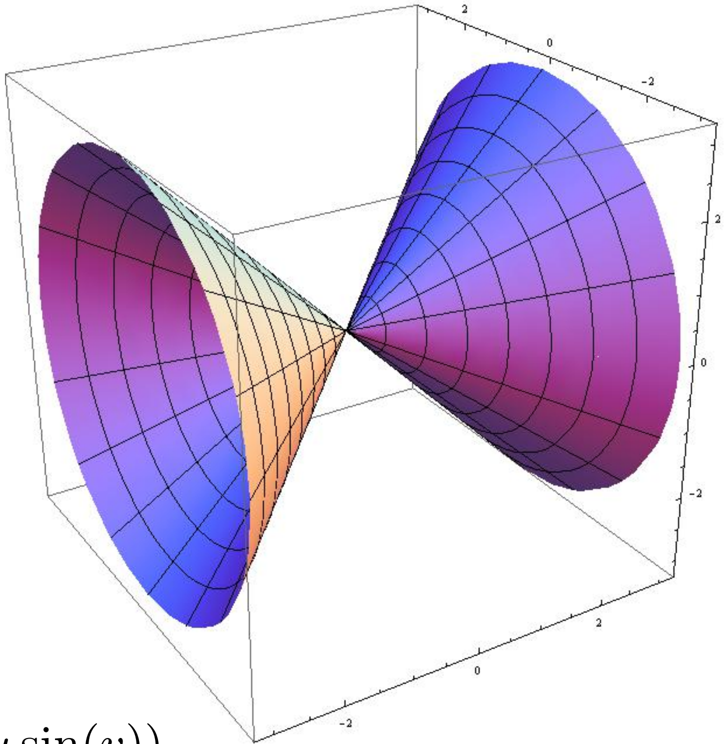

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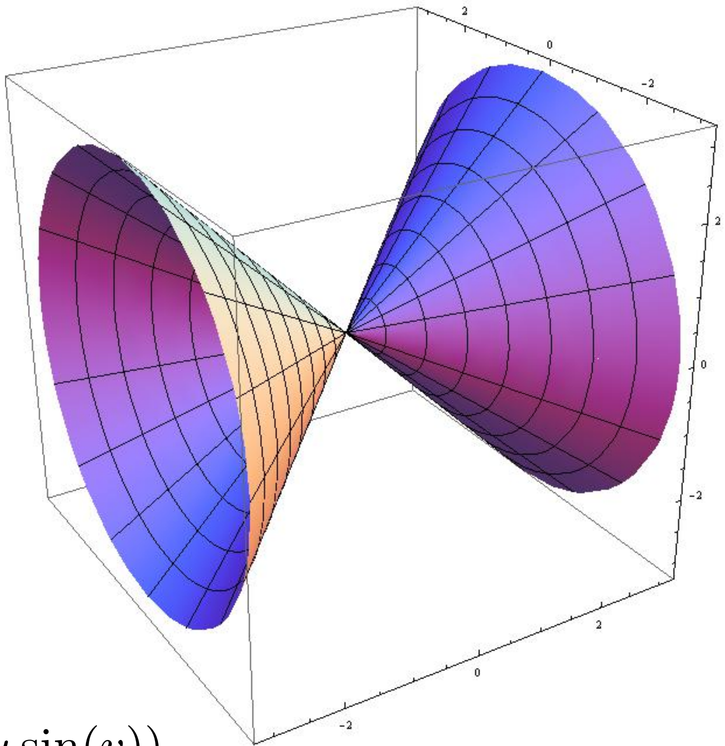

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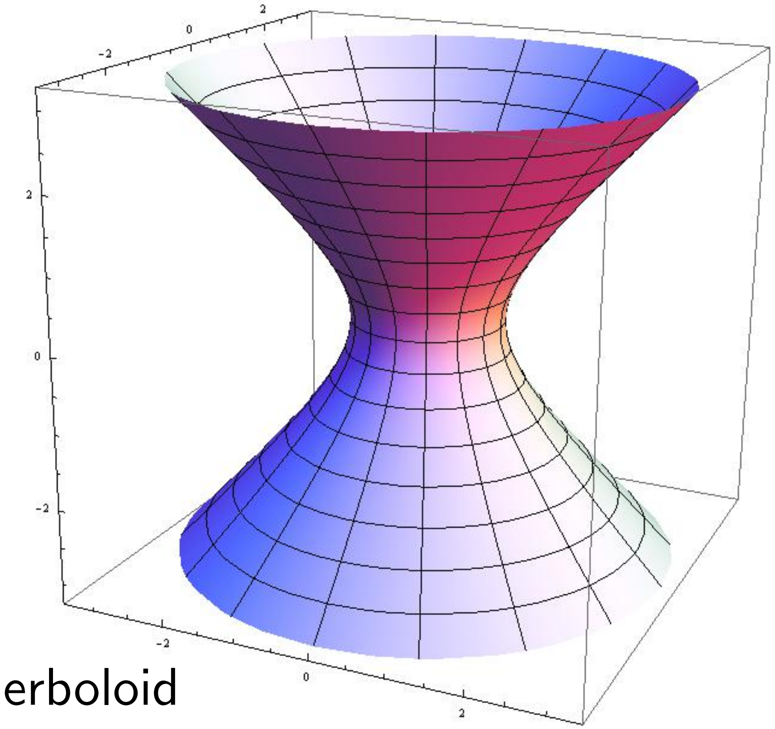

Hyperboloid

$(\sqrt{1+u^2}\cos v, \sqrt{1+u^2}\sin v, u)$

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Source: `http://math.arizona.edu/~models`

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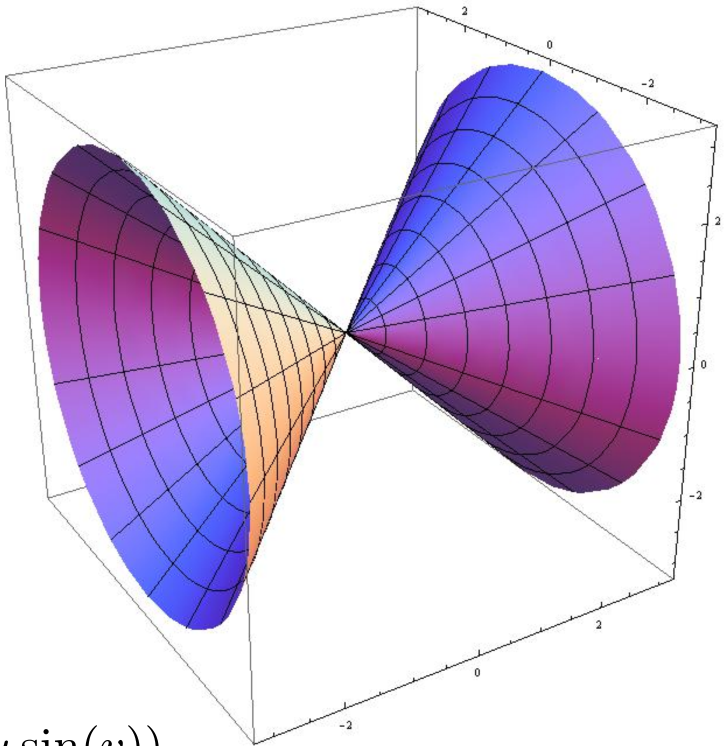

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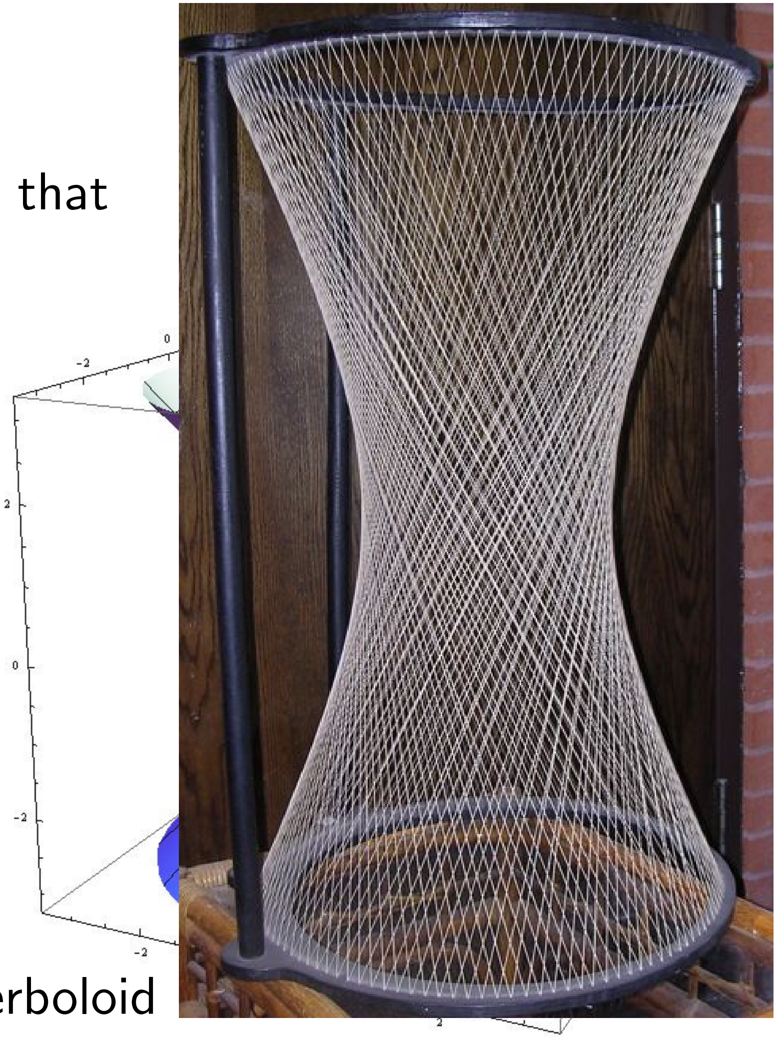

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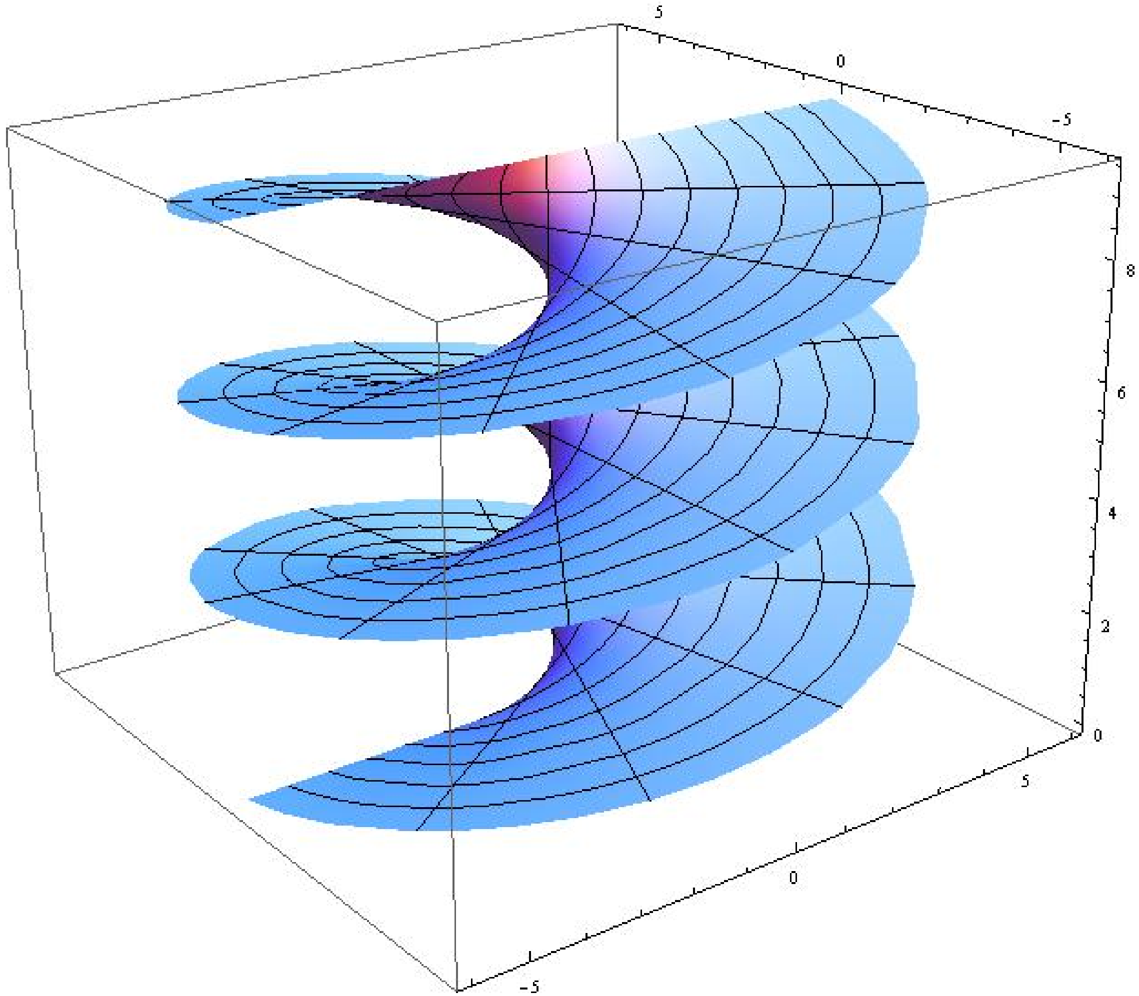
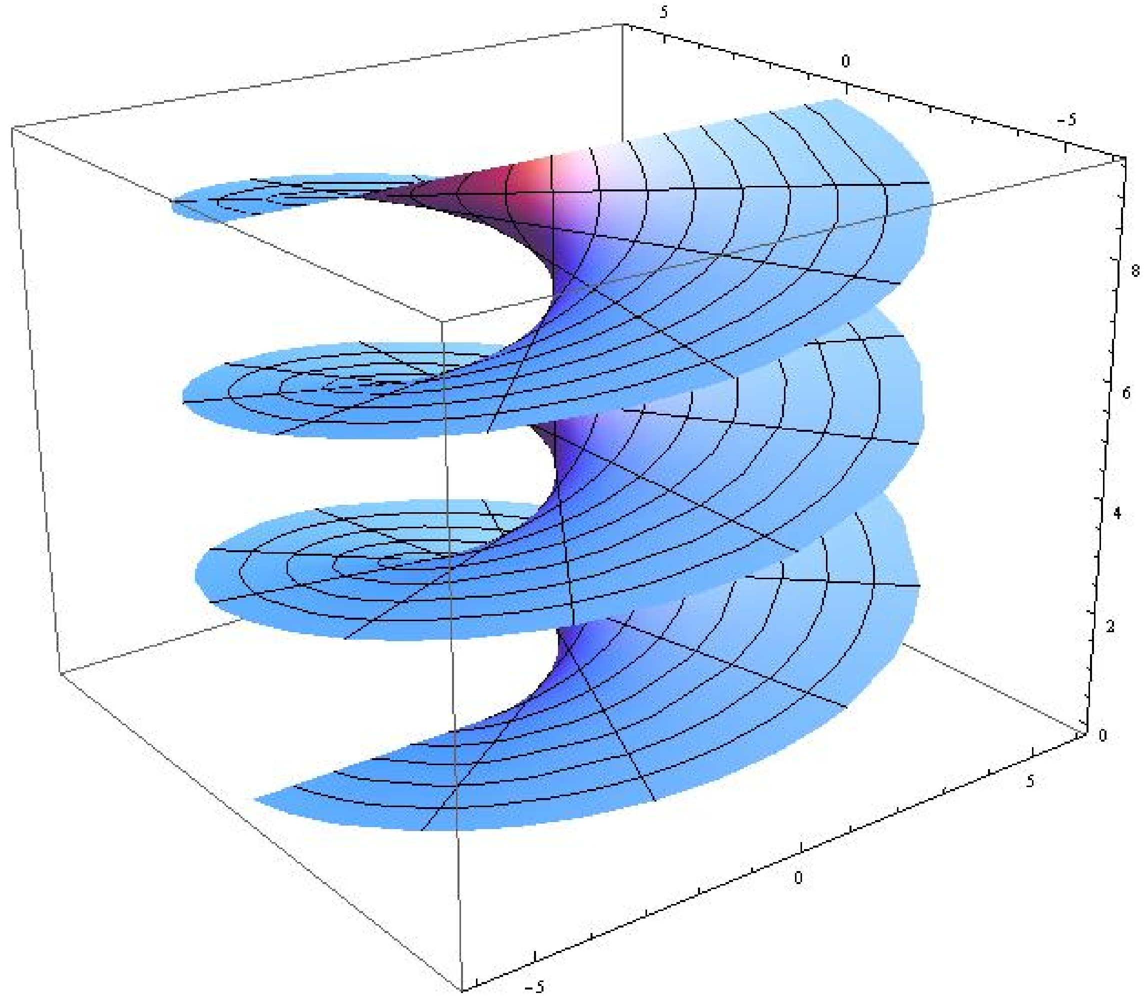


Ruled surface: helicoid

Consider a circular helix with axis $0z$

The *helicoid* associated is the set of all lines perpendicular to $0z$ that go through a point in $0z$ and one in the helix.



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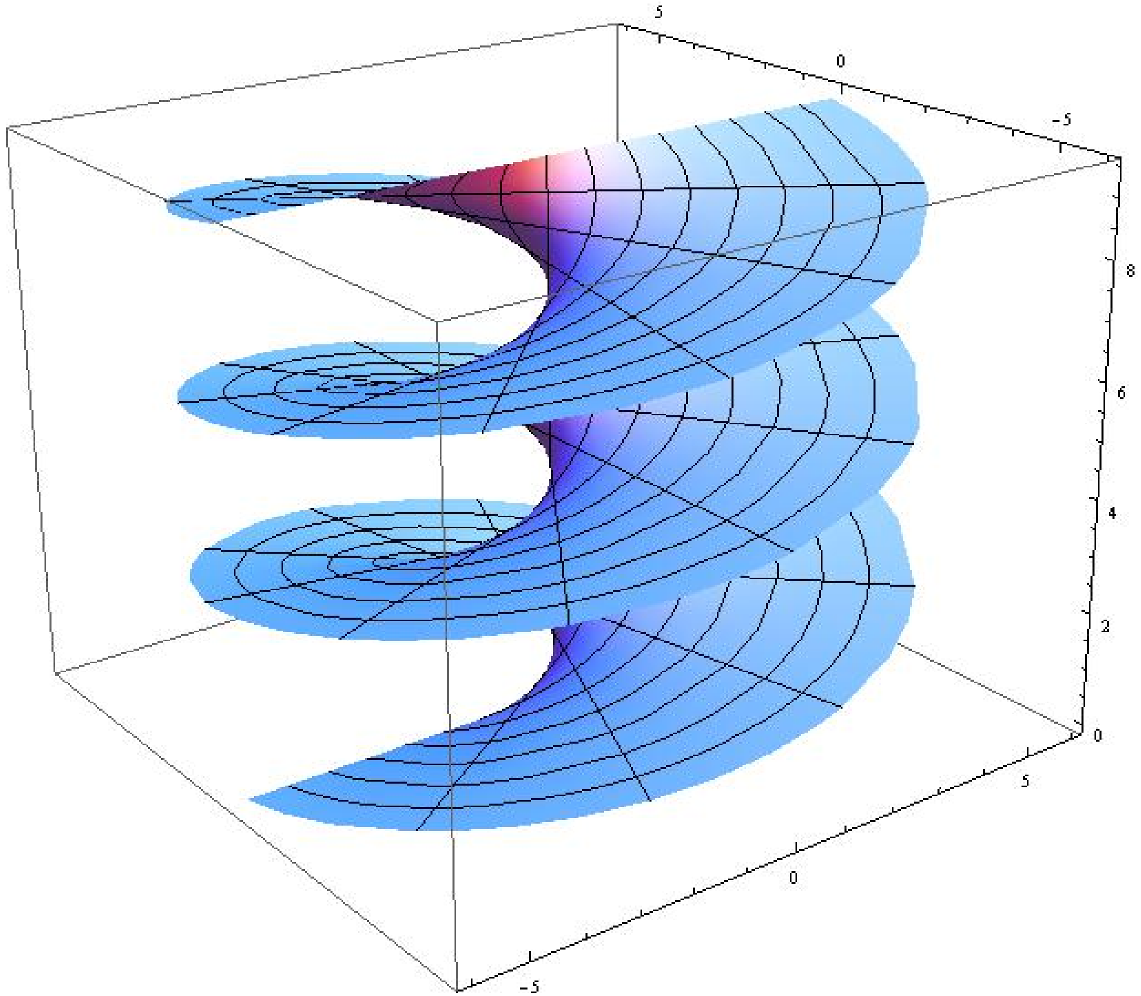

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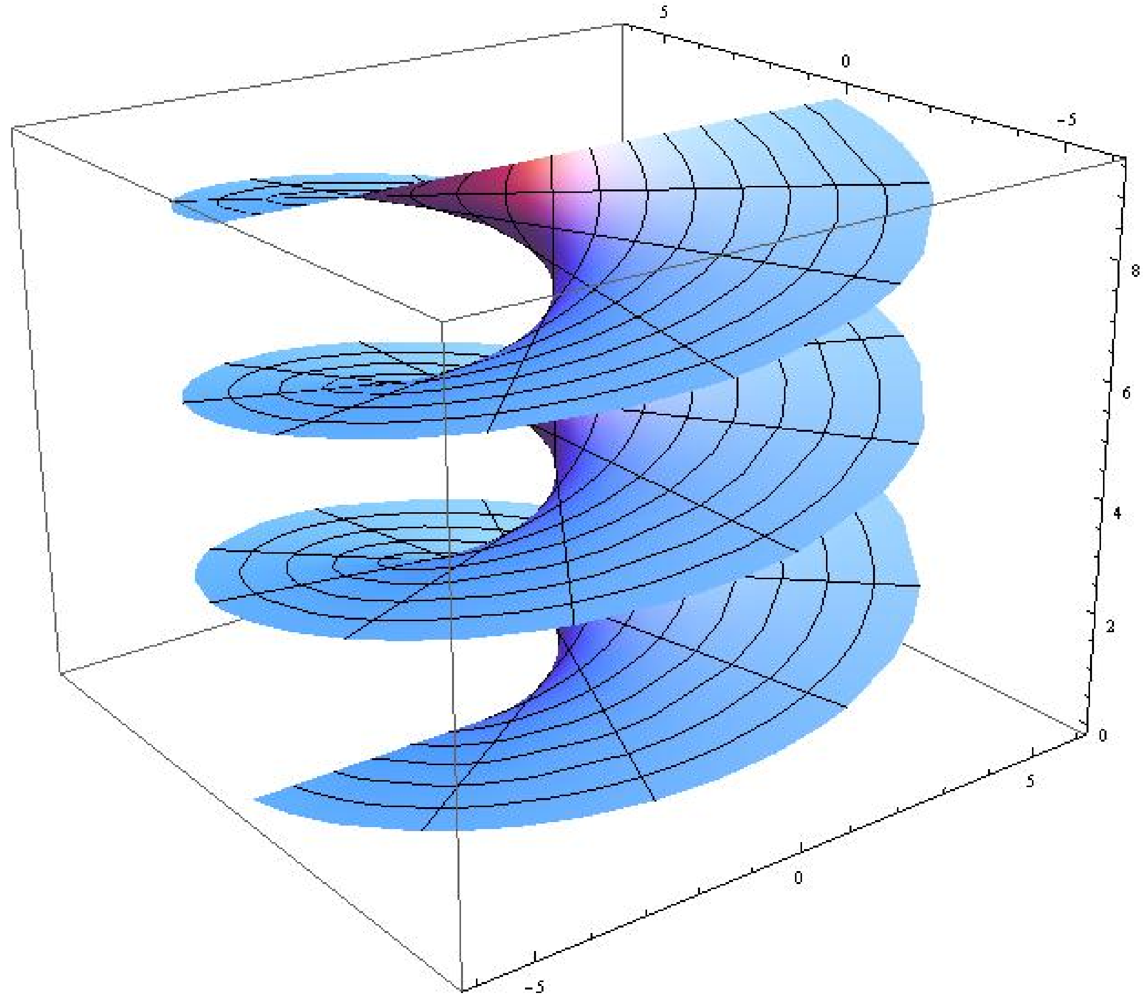

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A point on circular helix:
$P(t) = (a \cos t, a \sin t, bt)$

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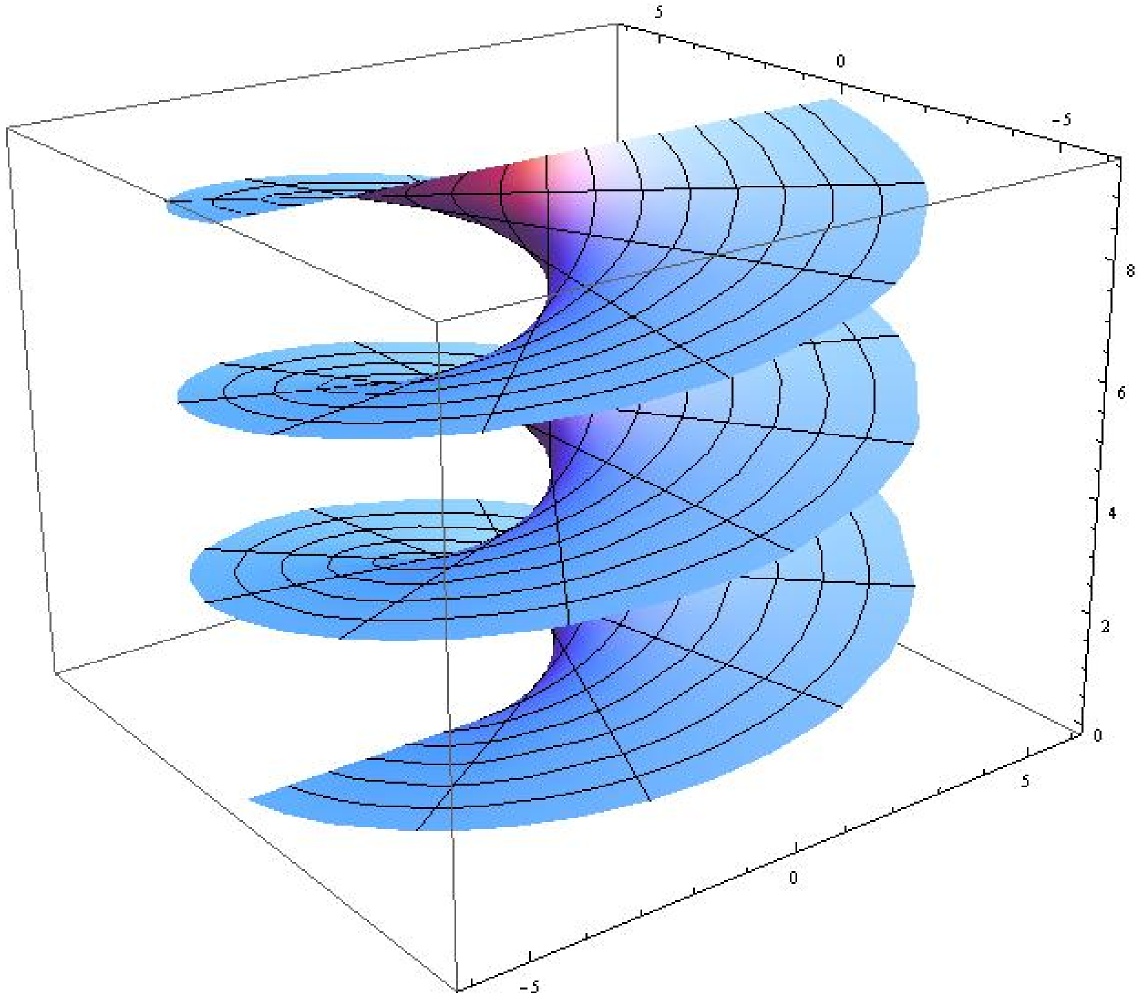

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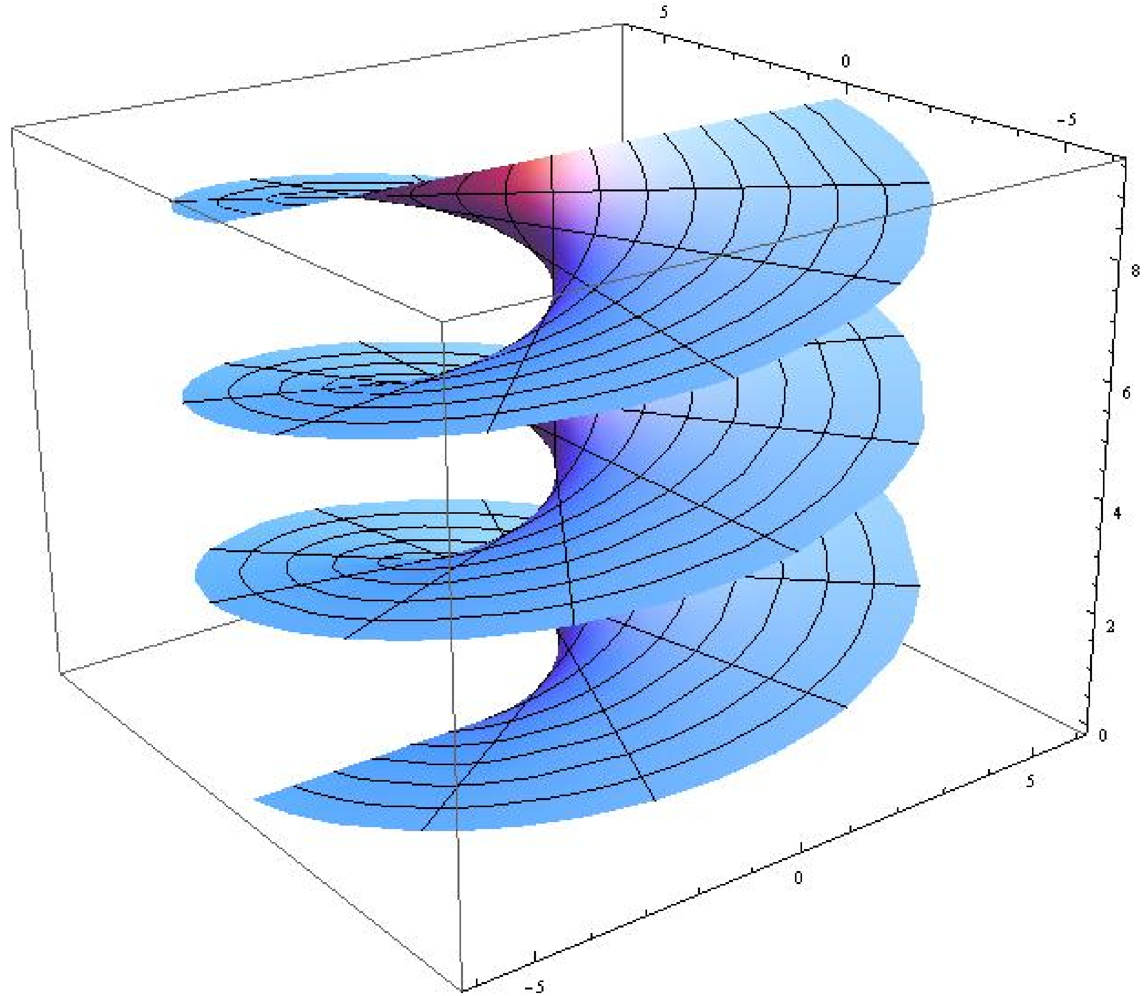

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Thus: $S(t, \lambda) = (1-\lambda)Q(t) + \lambda P(t)$
$= (1-\lambda)(0, 0, bt) + \lambda(a\cos t, a\sin t, bt)$
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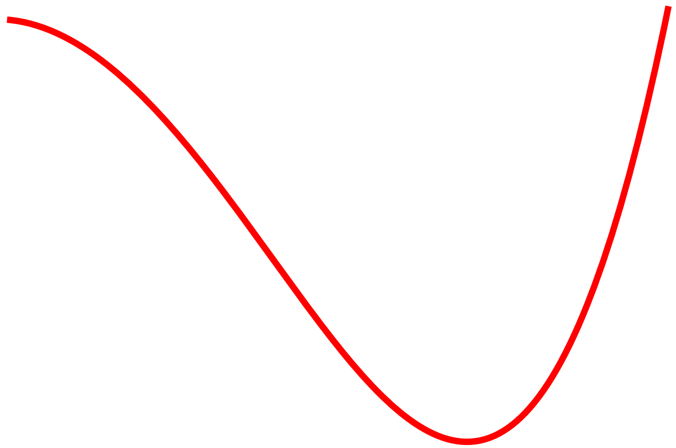

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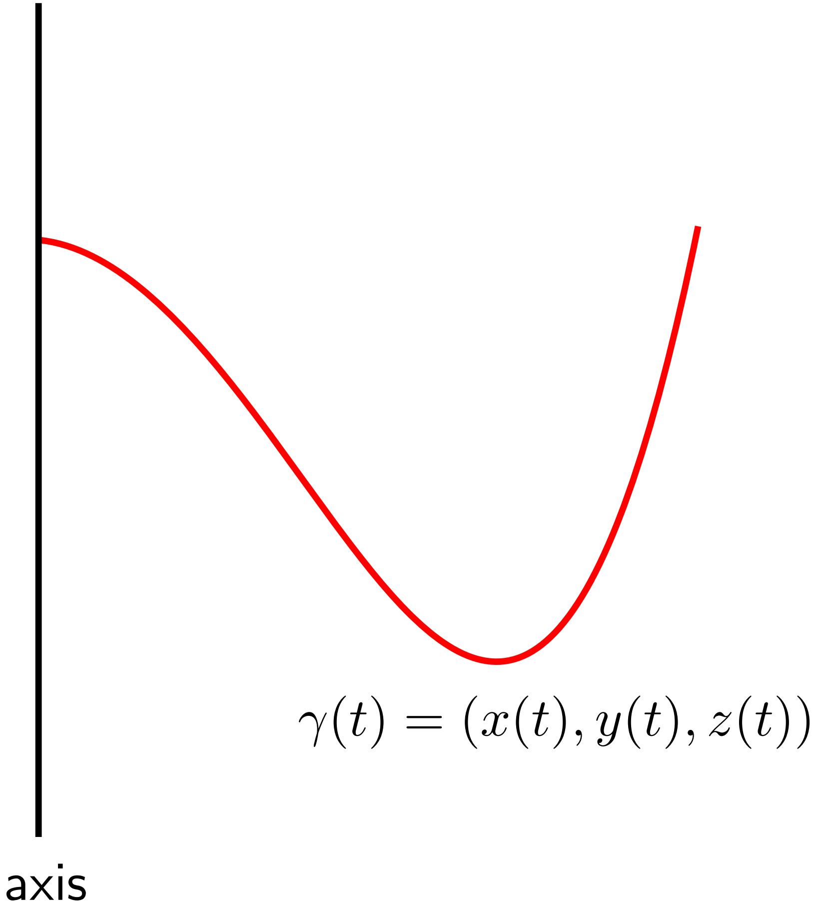

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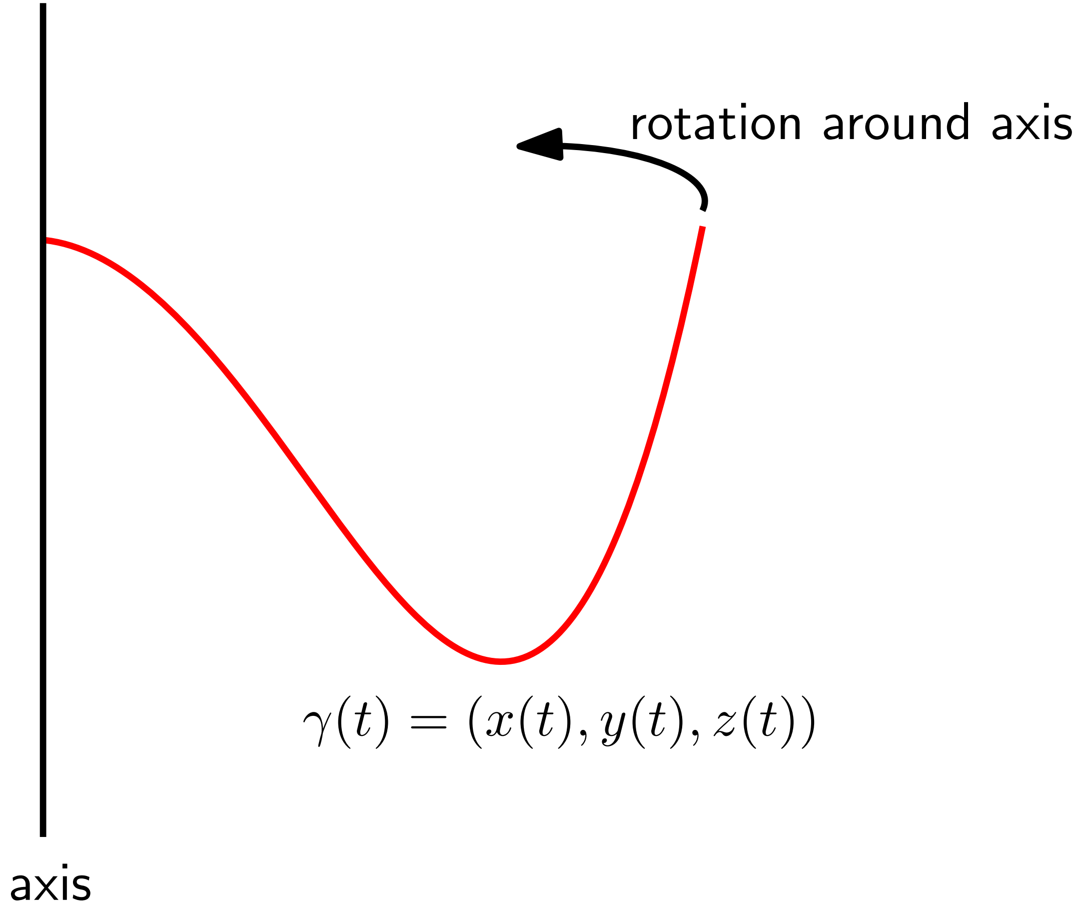

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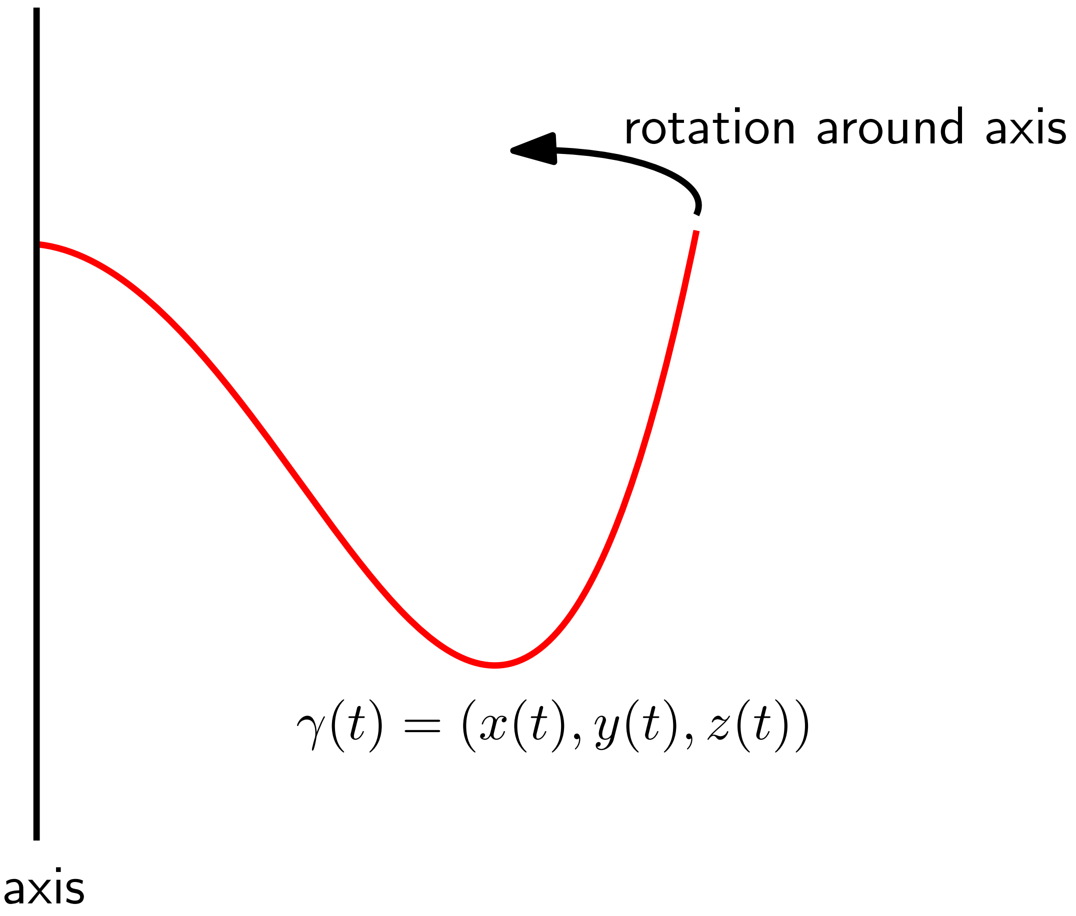

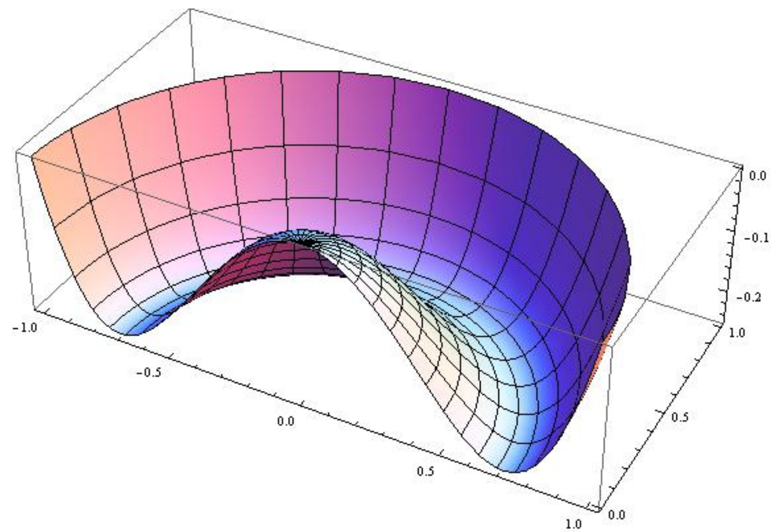

rotation from 0 to π

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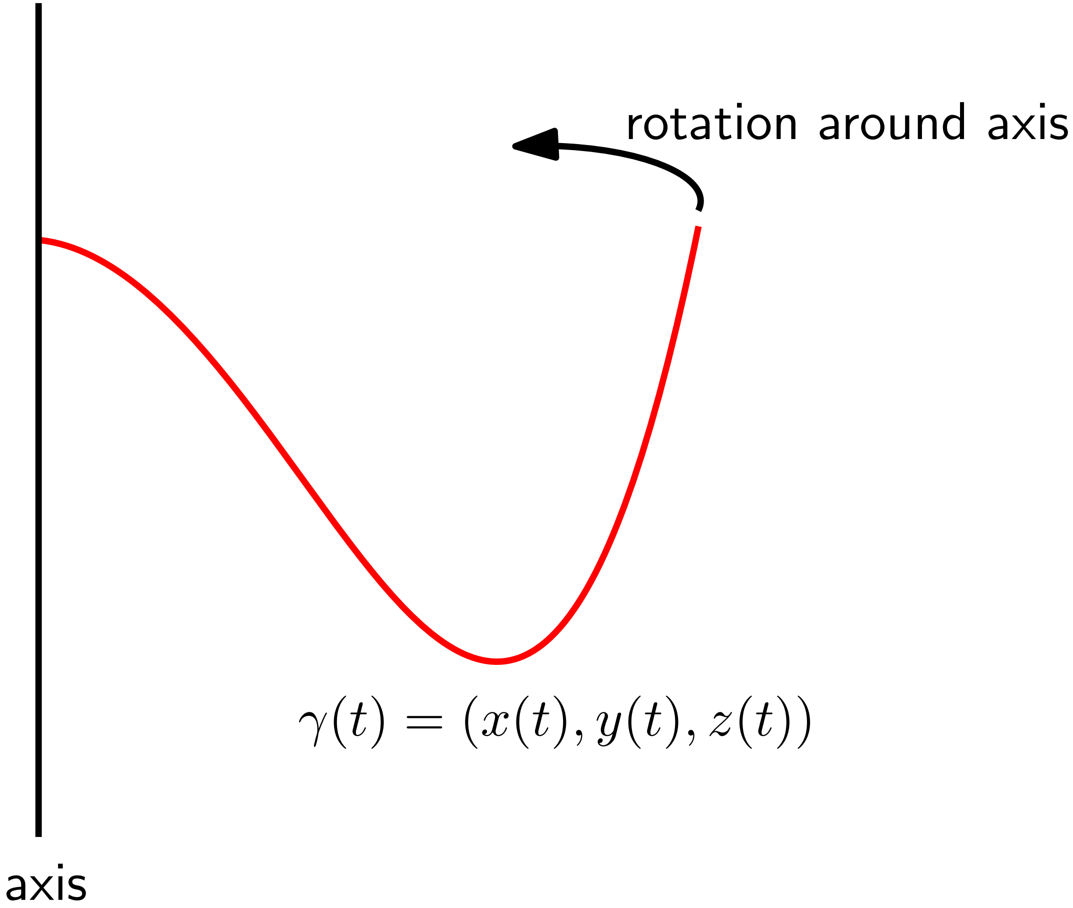

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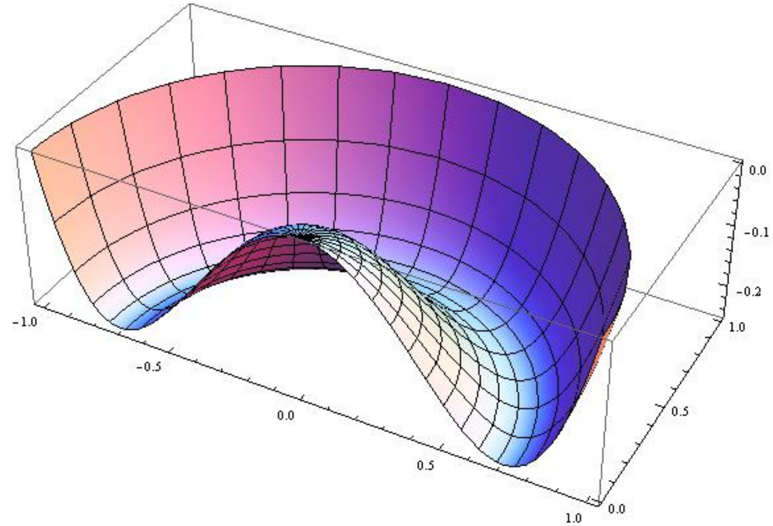

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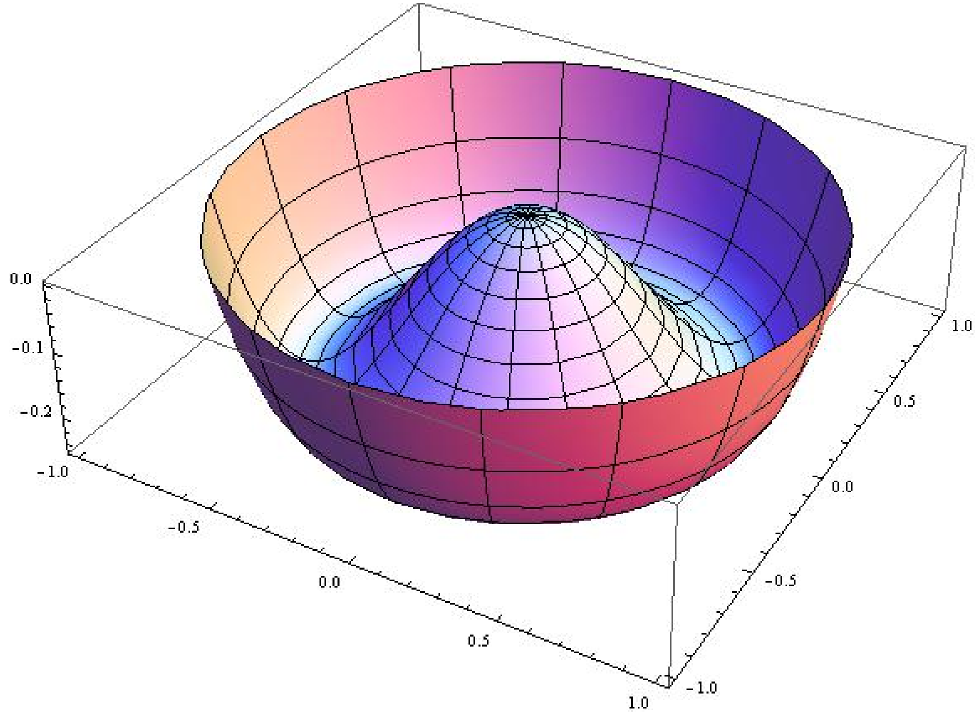

rotation from 0 to 2π

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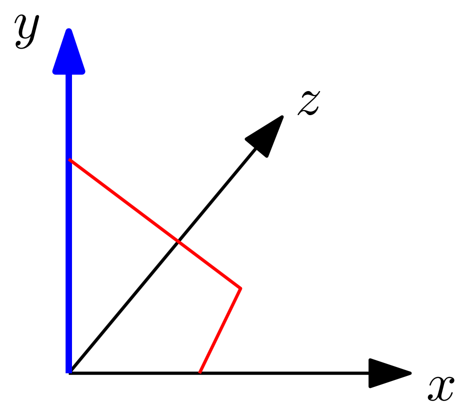
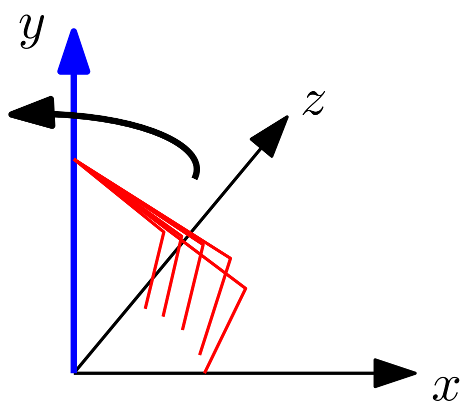


Surface created by rotating a curve (*generatrix*) around a line (*axis*)

Given a parametrization of the generatrix curve, say, in the xy-plane, so $P(t) = (x(t), y(t), 0)$, $t \in [0, 1]$, and an axis, say $0y$, we obtain the parametrization of the surface of revolution around tha axis as follows:

Surfaces of revolution



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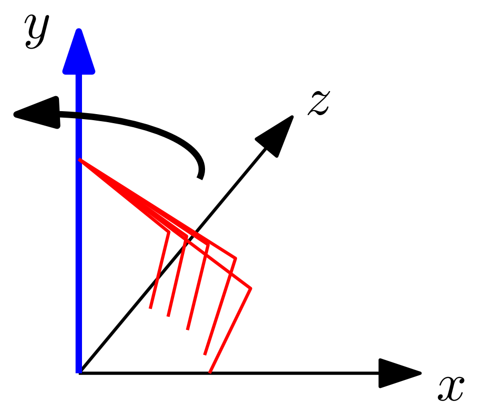

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$$T_y(w) = \begin{pmatrix} \cos w & 0 & \sin w \\ 0 & 1 & 0 \\ -\sin w & 0 & \cos w \end{pmatrix}$$

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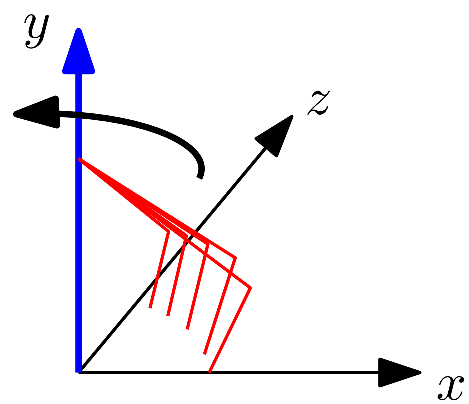

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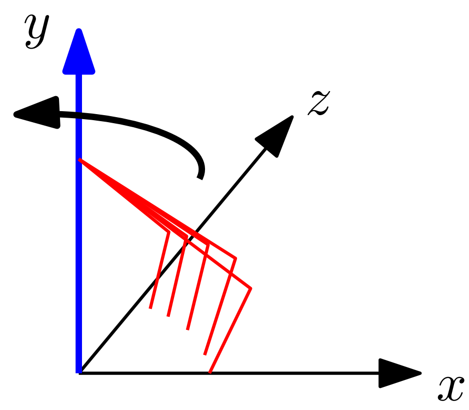

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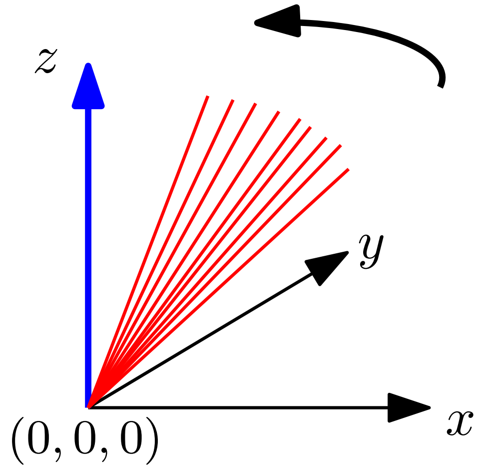
Example of surface of revolution: cone



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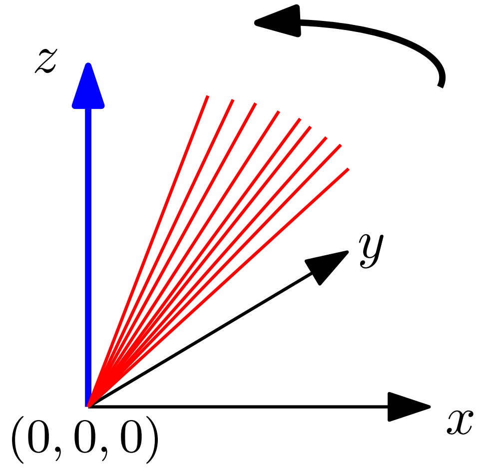

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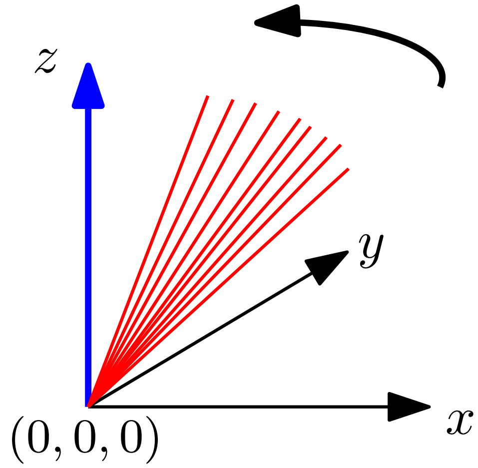

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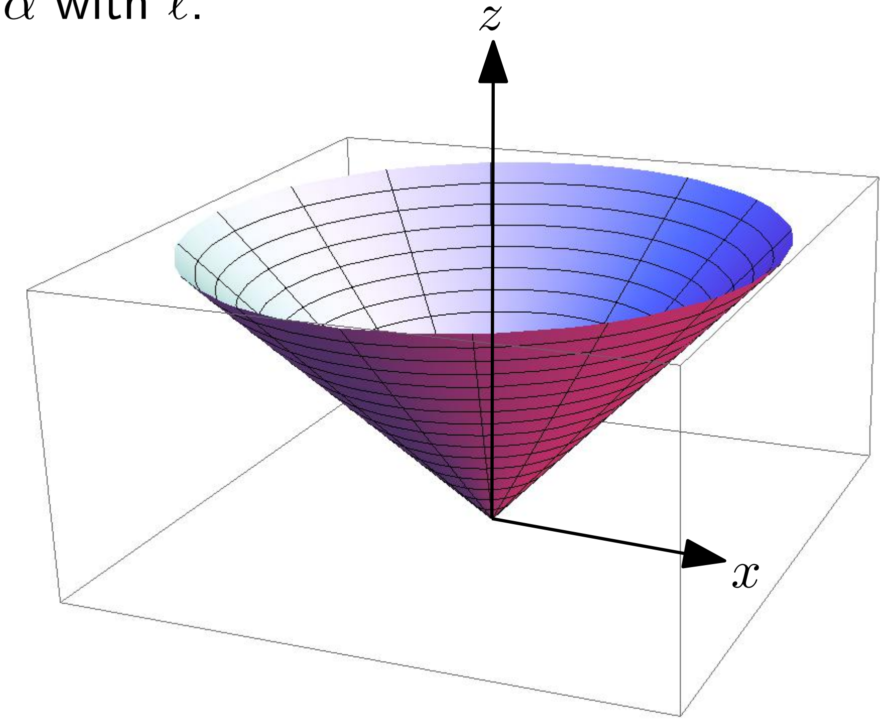

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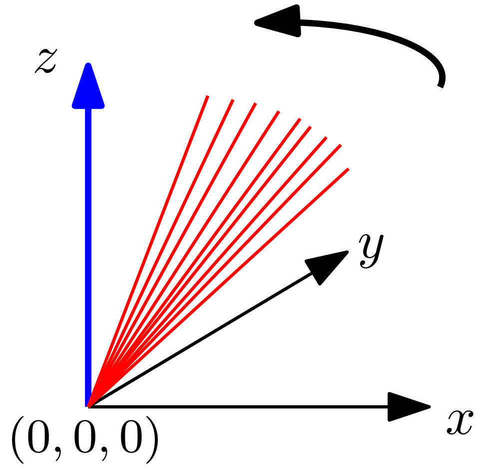

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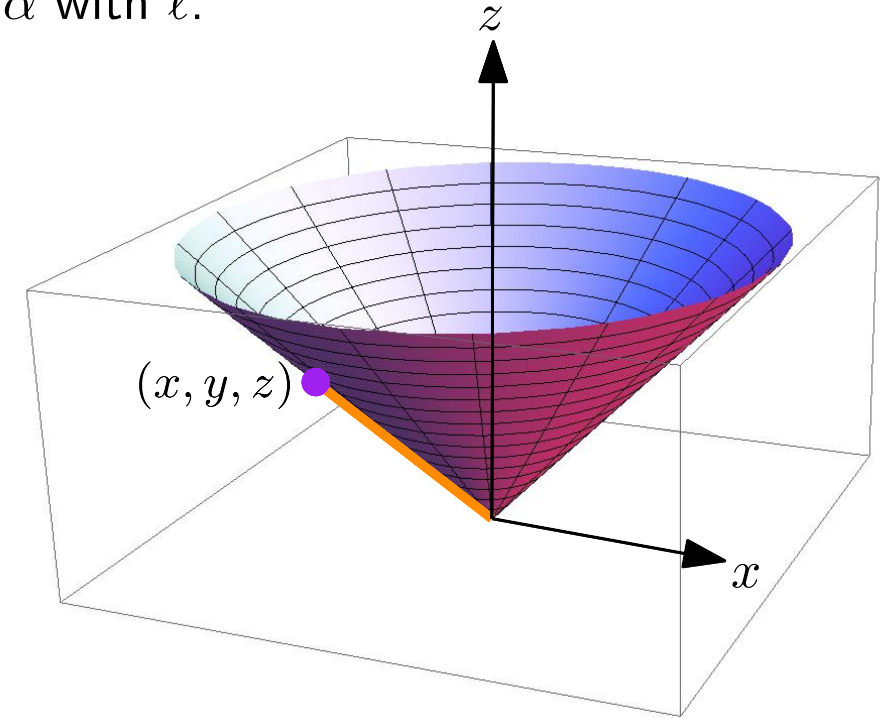
Example: apex is $(0, 0, 0)$, axis is $0z$, an equation is:



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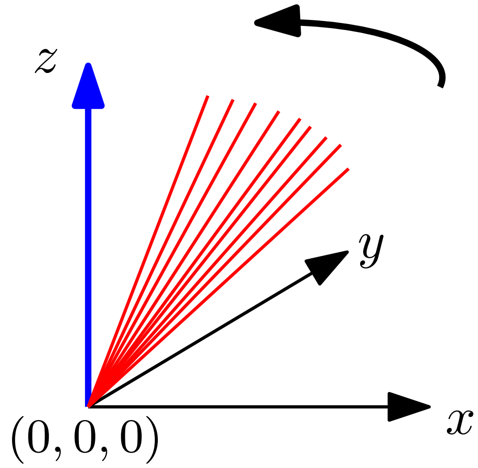

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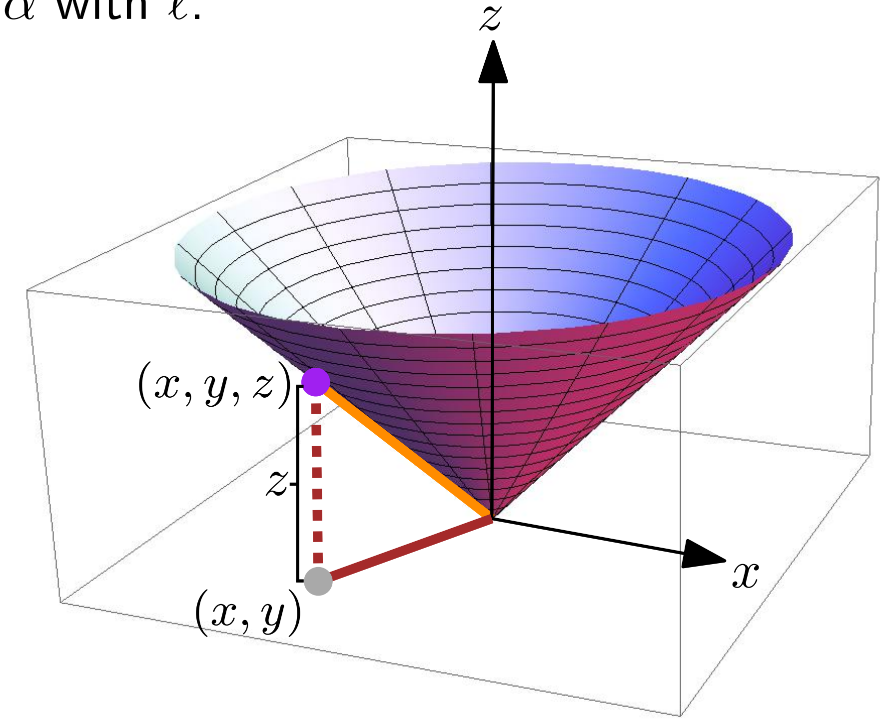
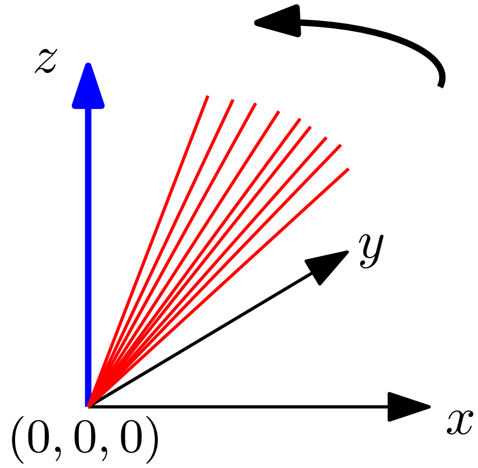


Warning: only half cone shown here

Example of surface of revolution: cone



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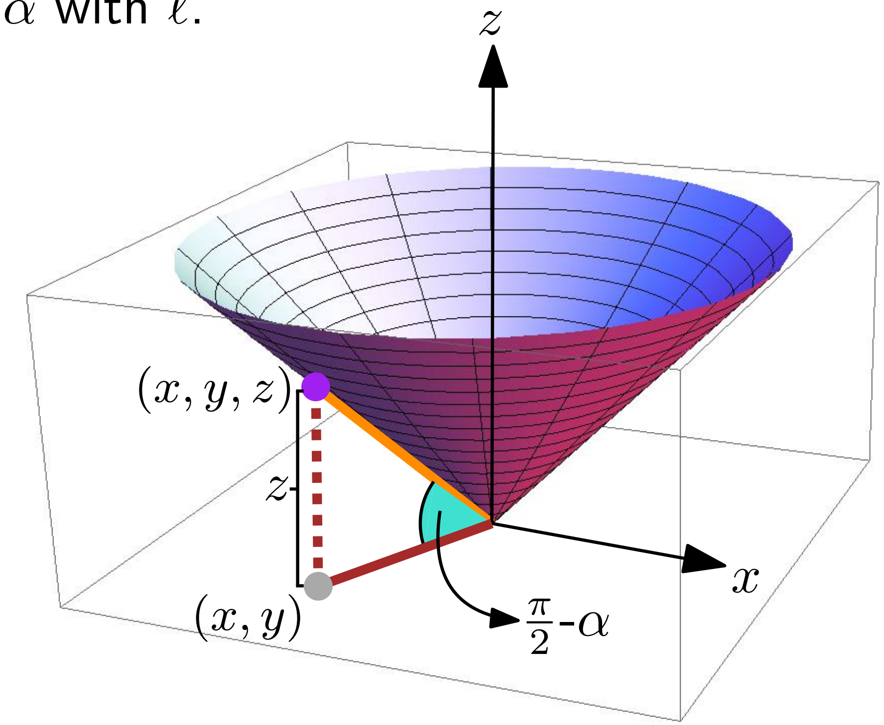

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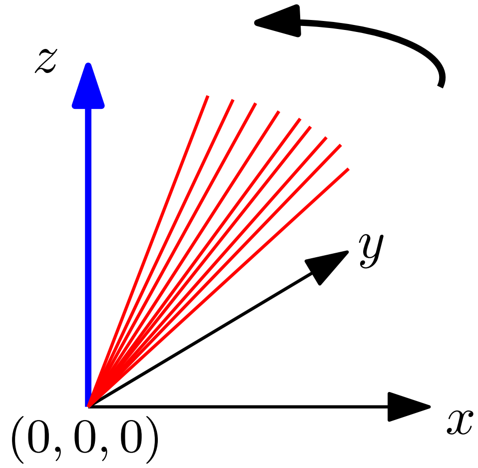

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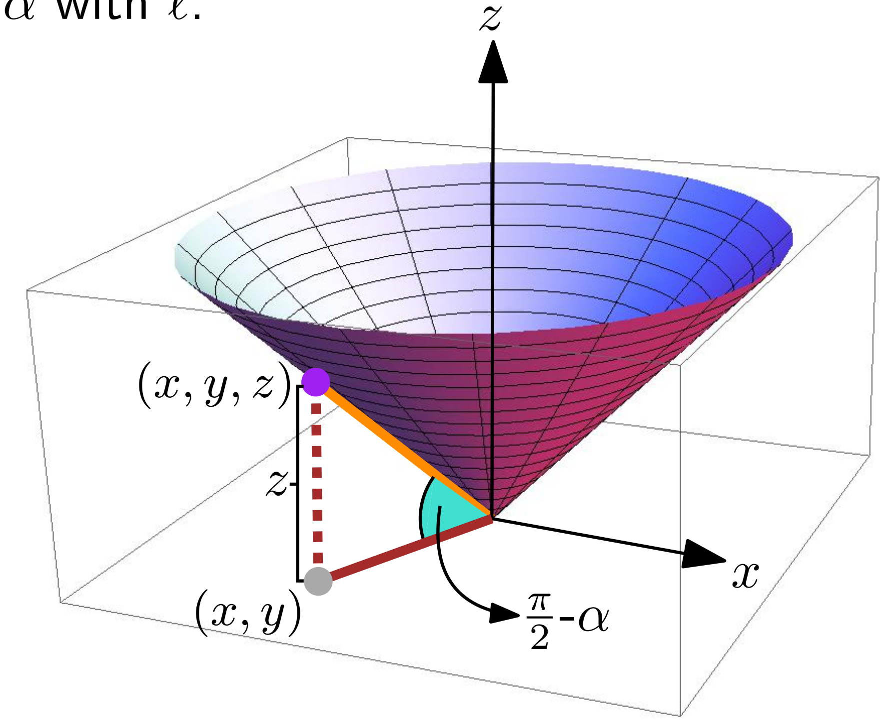

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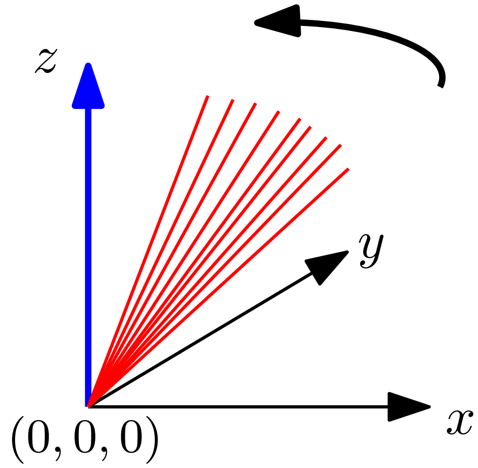

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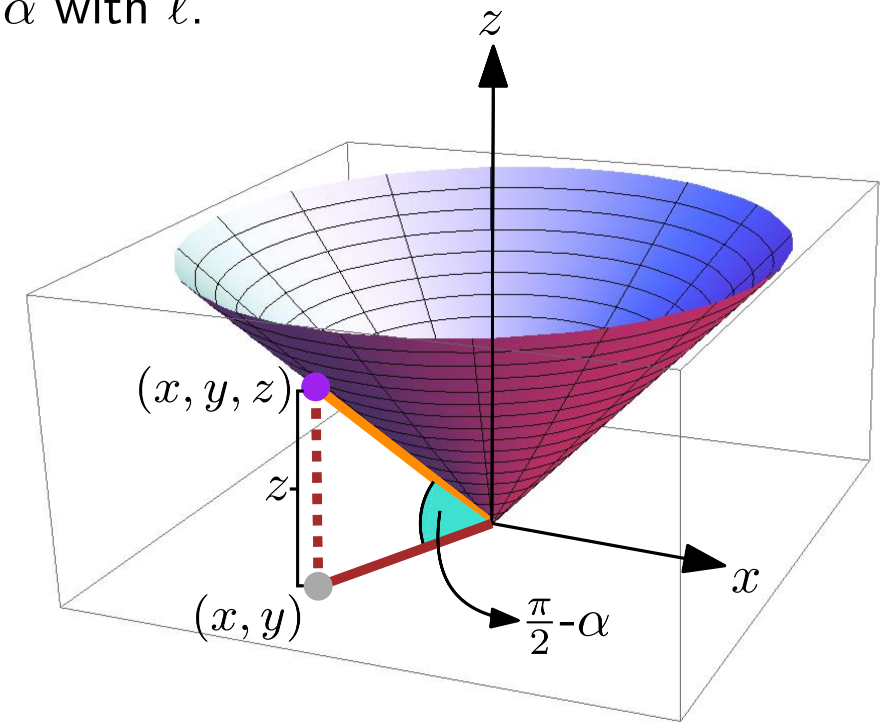

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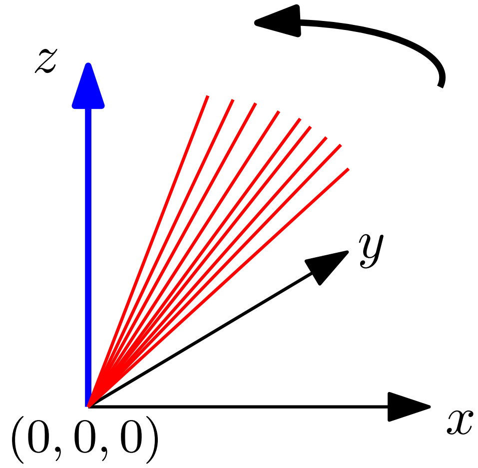

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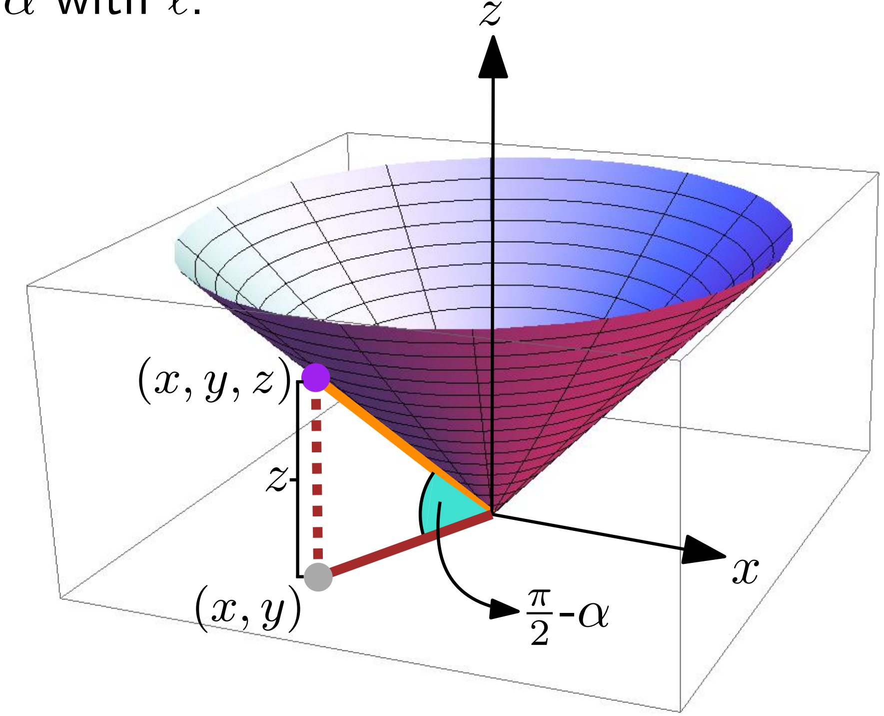

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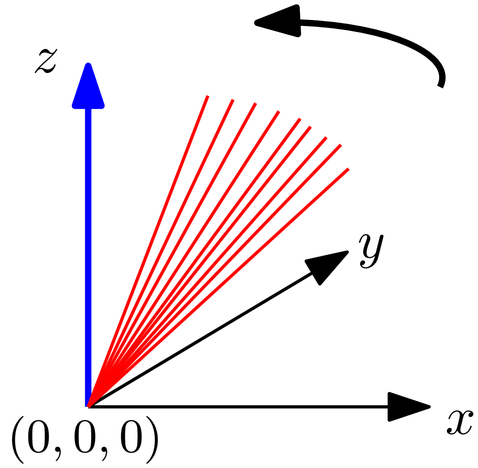

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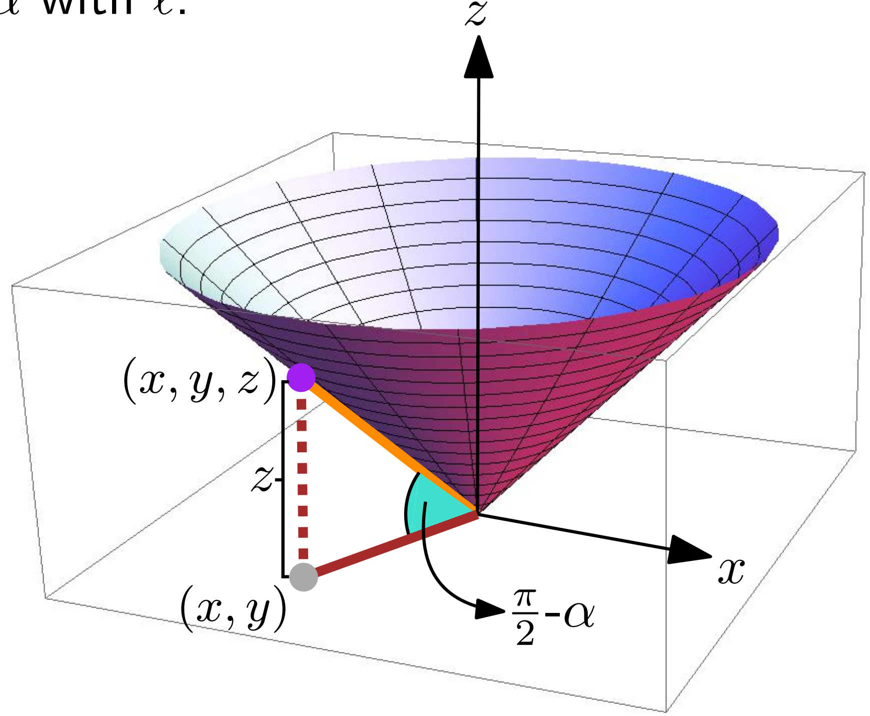
implicit equation of the cone



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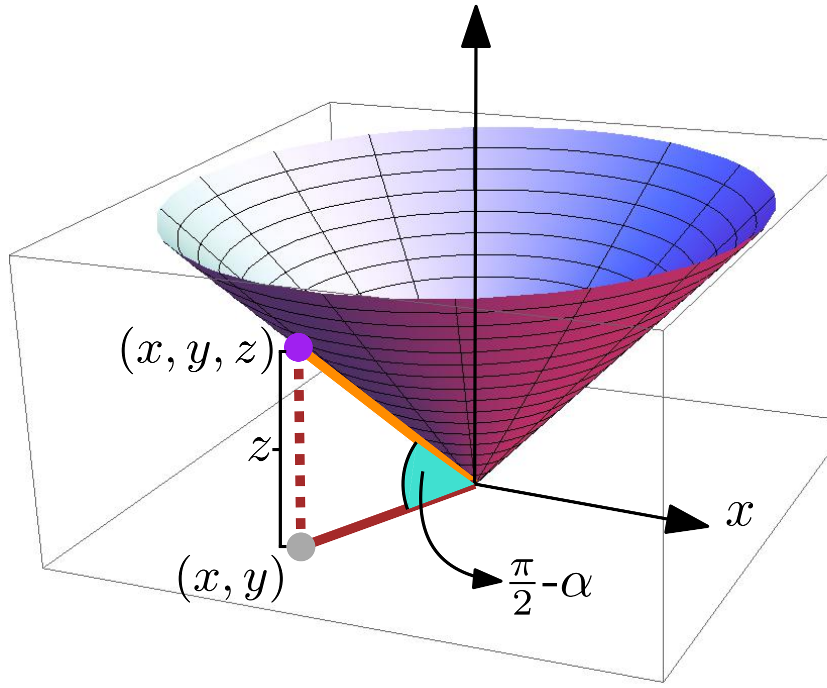
Example of surface of revolution: cone

Now as a surface of revolution...



Example of surface of revolution: cone

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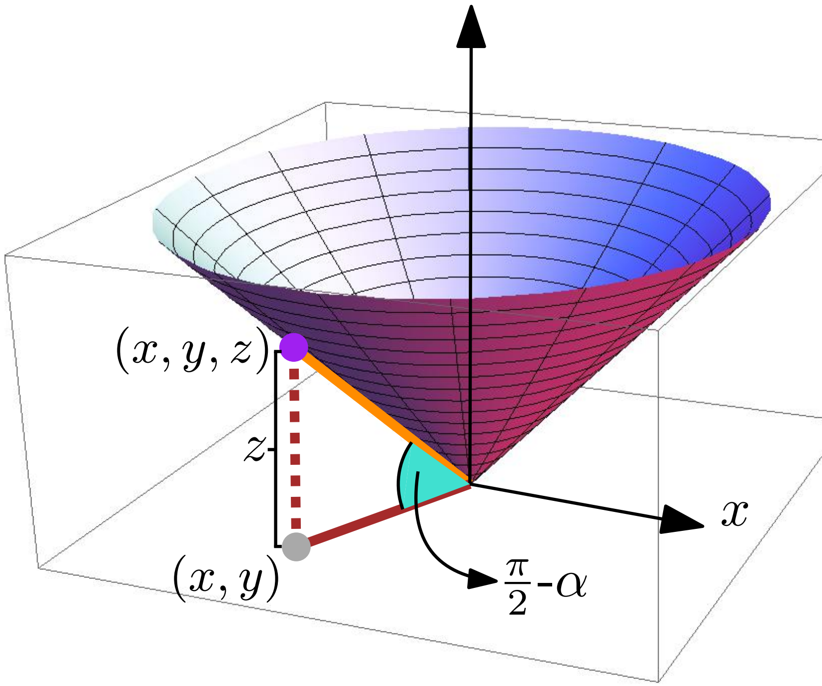

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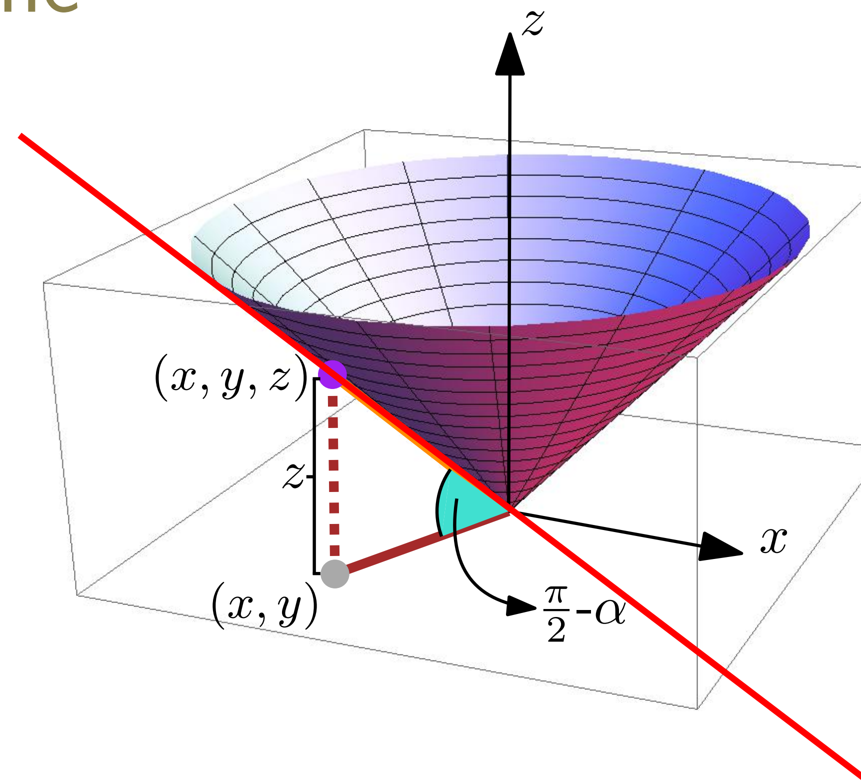

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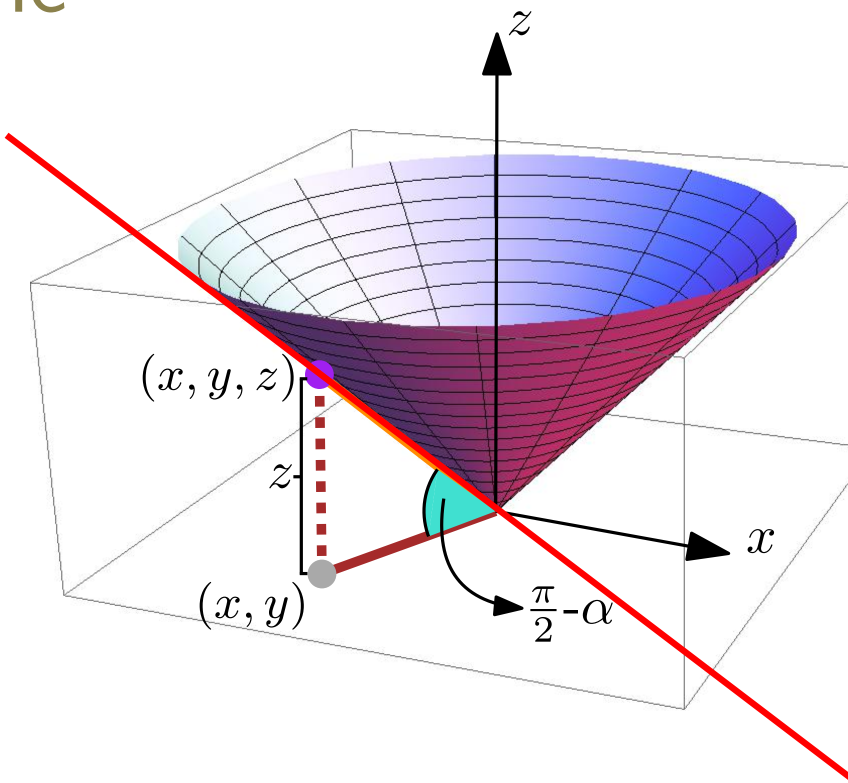

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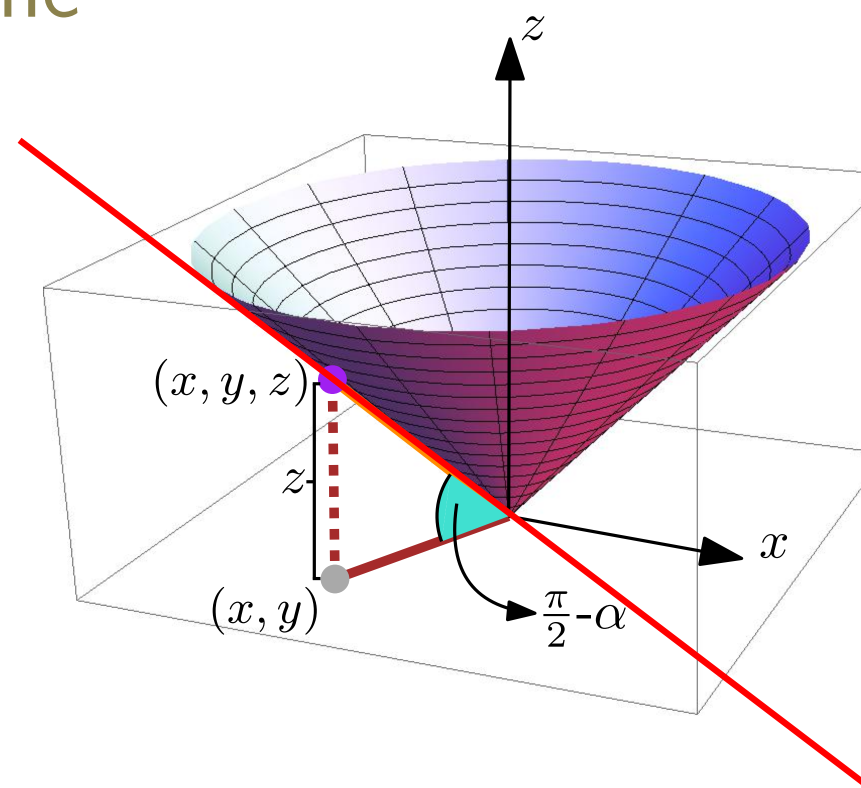

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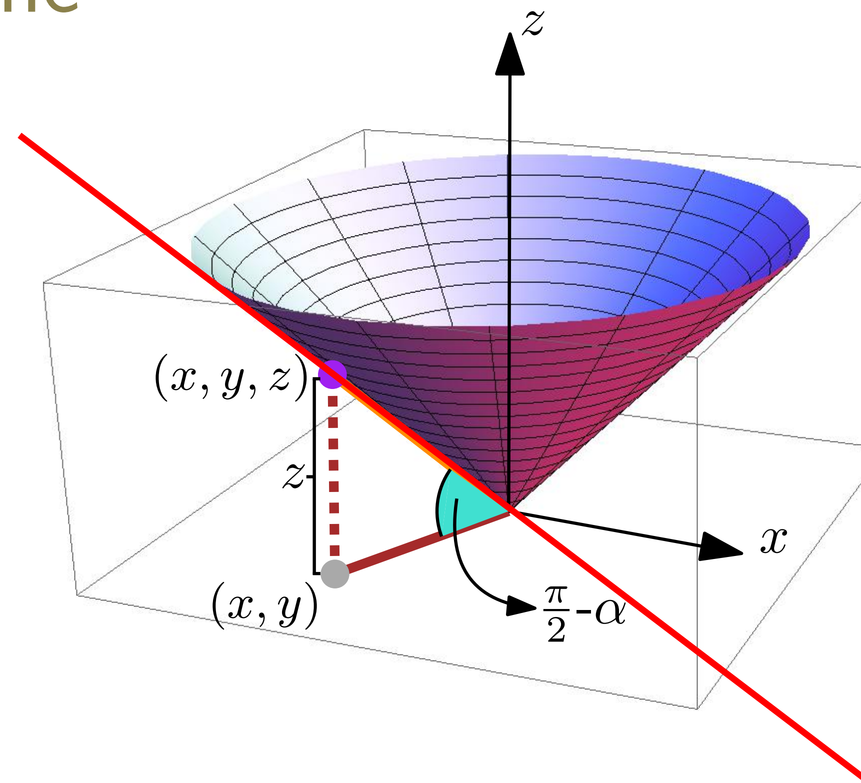

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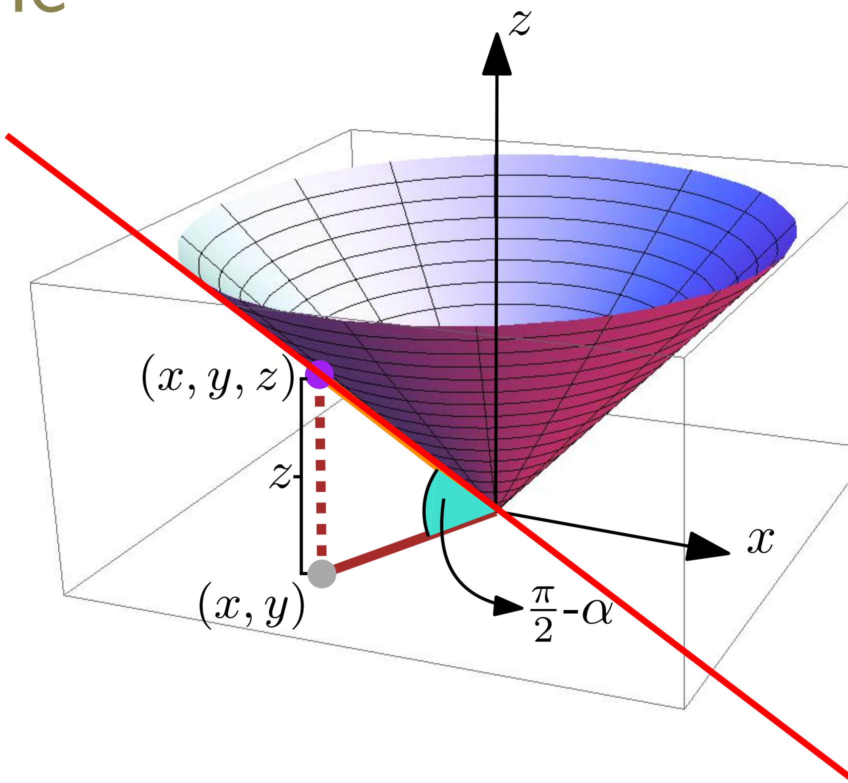

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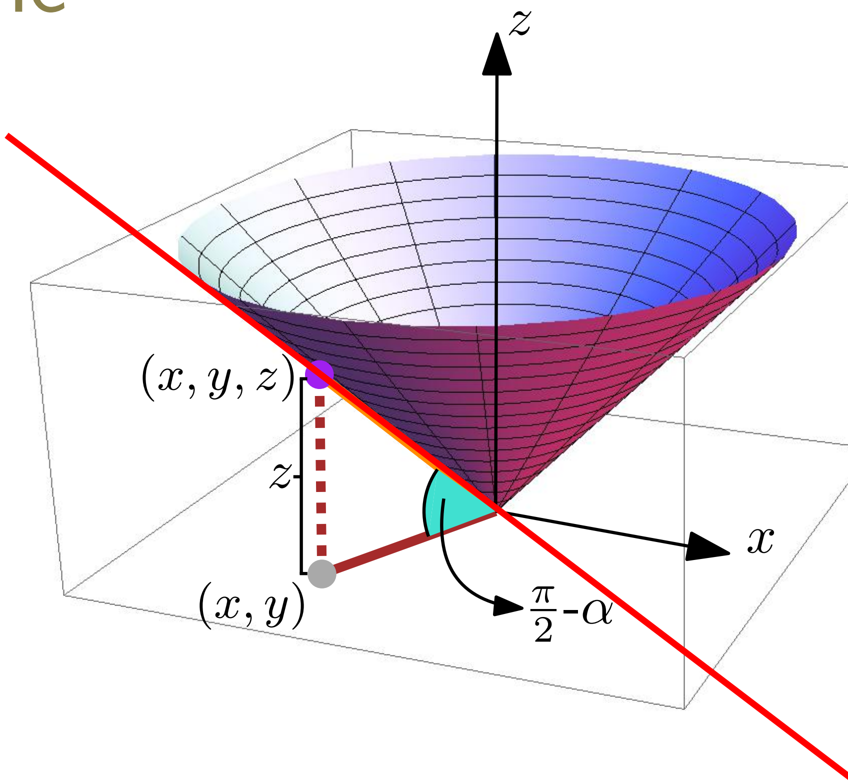

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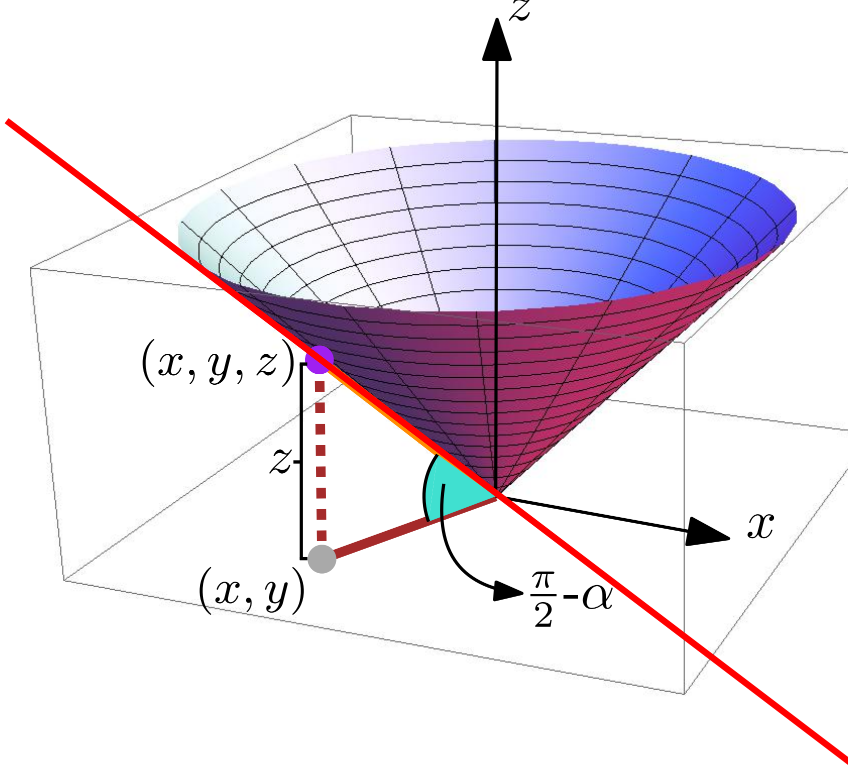

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parametric equation of the cone

Normal vector, tangent plane

Let S be a surface parametrized as
$S(u,v) = (x(u,v), y(u,v), z(u,v))$ for (u,v) in some domain

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If the functions x, y, z have partial derivatives, we consider $\begin{cases} \frac{\partial S}{\partial u} = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) \\ \frac{\partial S}{\partial v} = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) \end{cases}$

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A point P is called *regular* if $\frac{\partial S}{\partial u}(P)$ and $\frac{\partial S}{\partial v}(P)$:

- exist
- are continuous at P
- their cross product is not zero

If one of these conditions does not hold for P, it is called a *singular* point

A surface is called *regular* if all its points are regular

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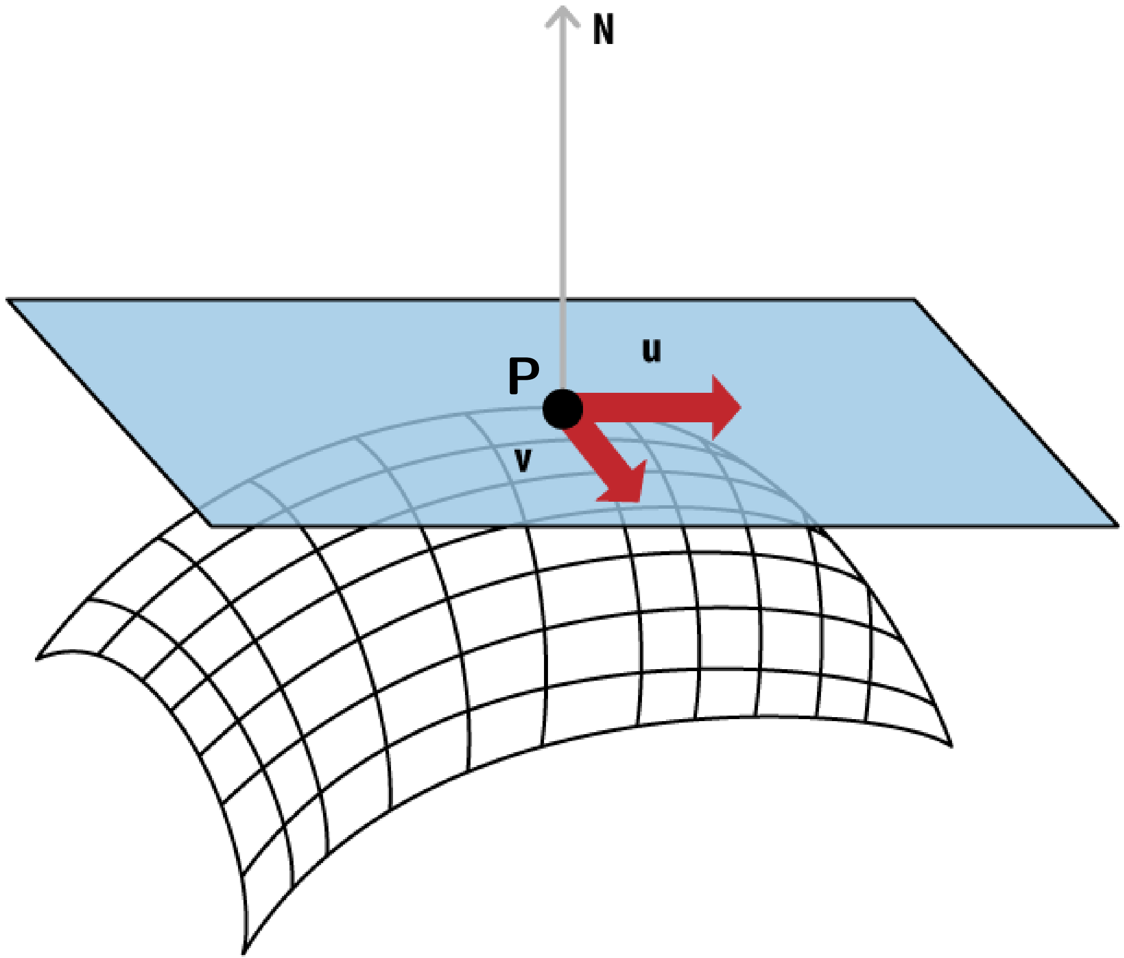

Figure from `oreilley.com`

Normal vector, tangent plane

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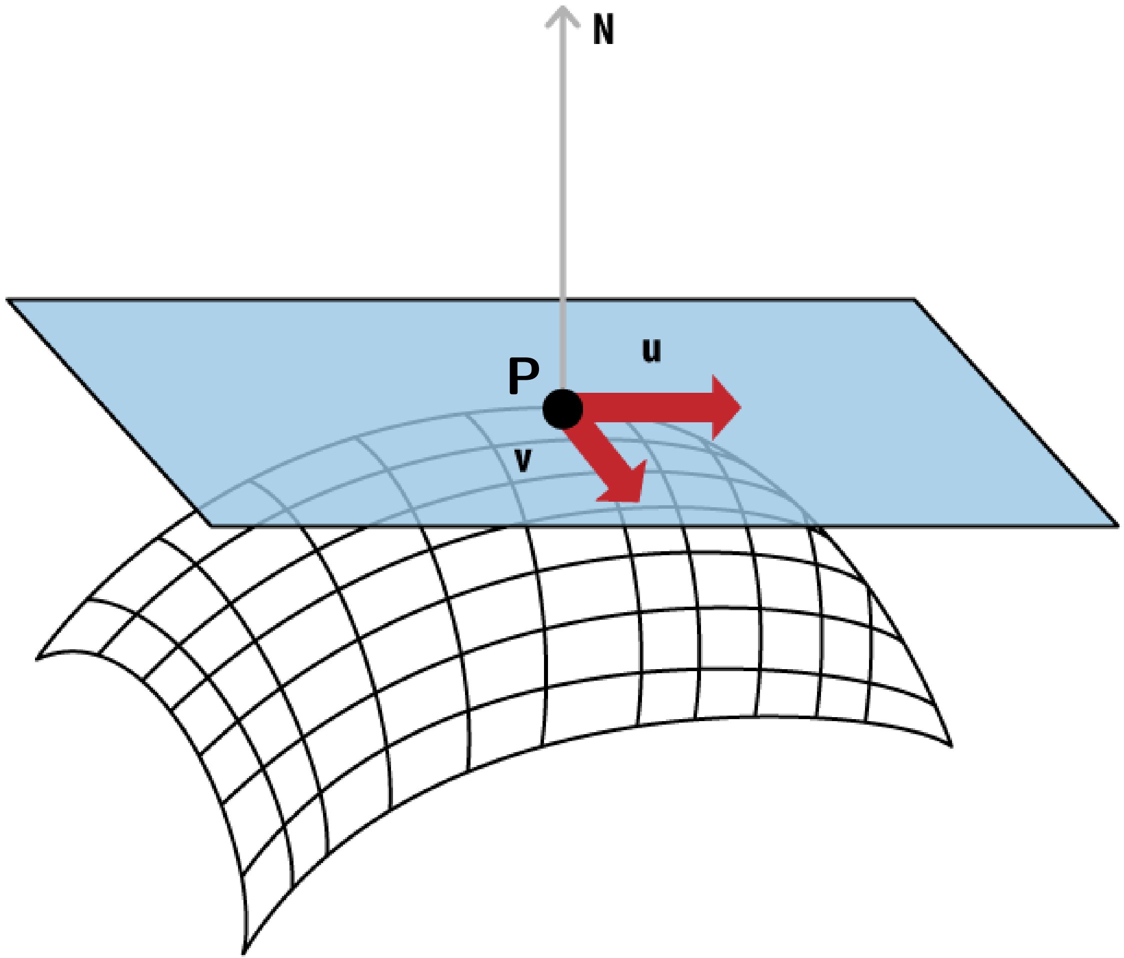
If nothing "goes wrong", they define the tangent plane at P



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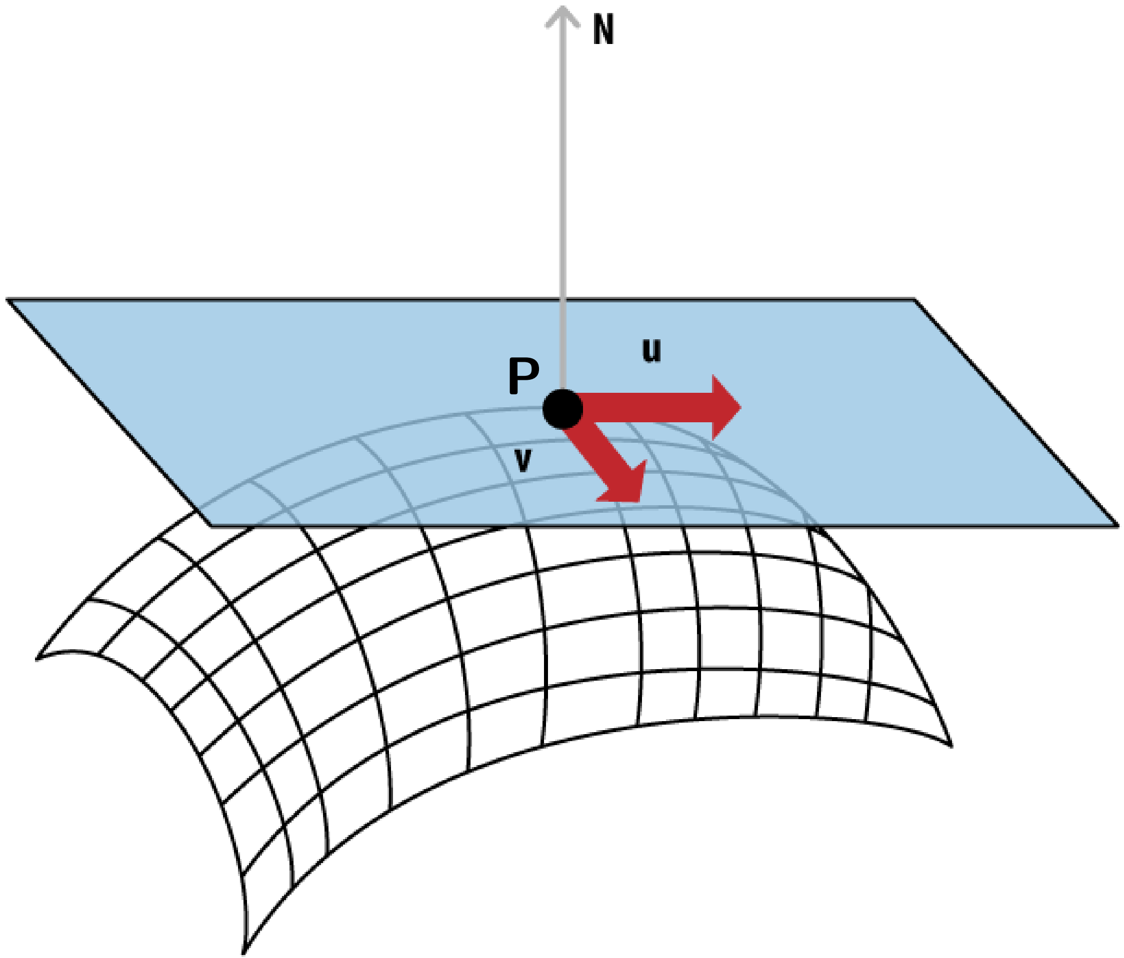
Exceptions:

- Surface does not have a tangent plane at P
- Parametrization is irregular at P



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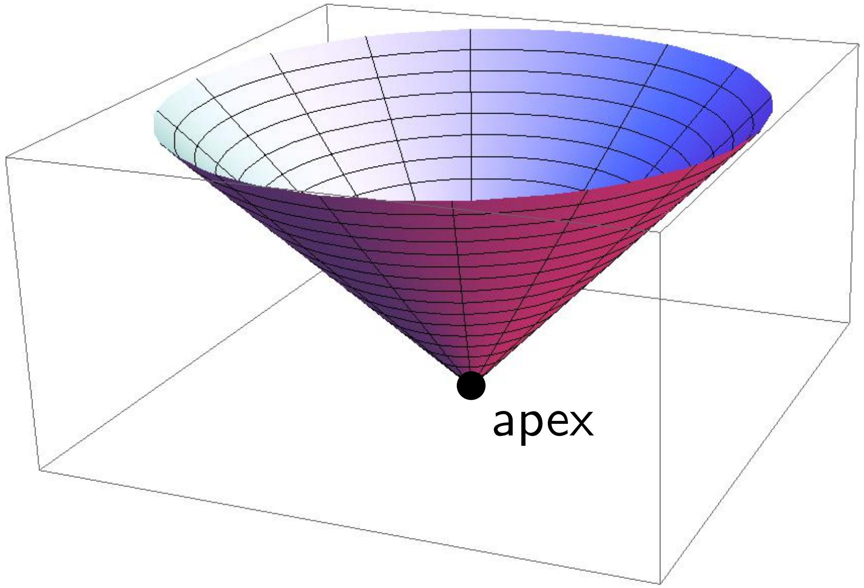
Normal vector, tangent plane: example

The apex of a cone is a **singular point**



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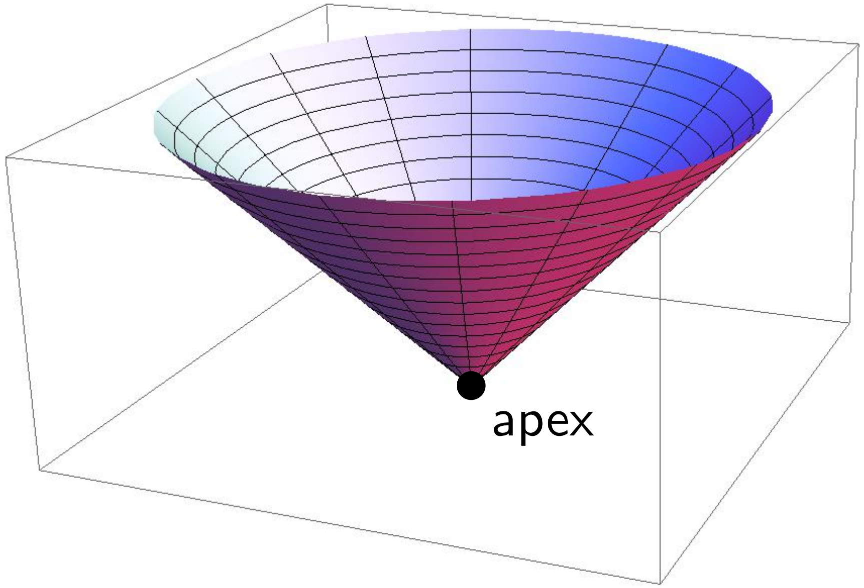

Consider the cone $x^2 + y^2 = z^2$, which can be parametrized as:

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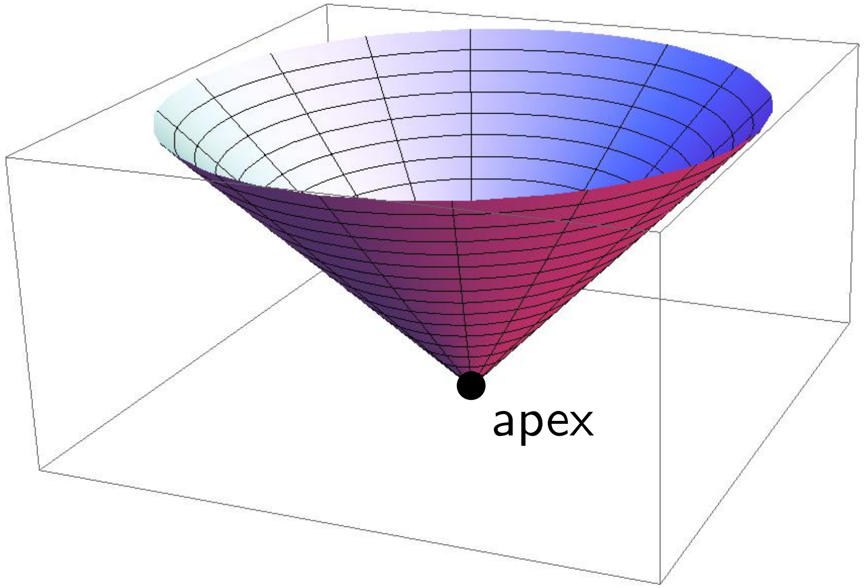

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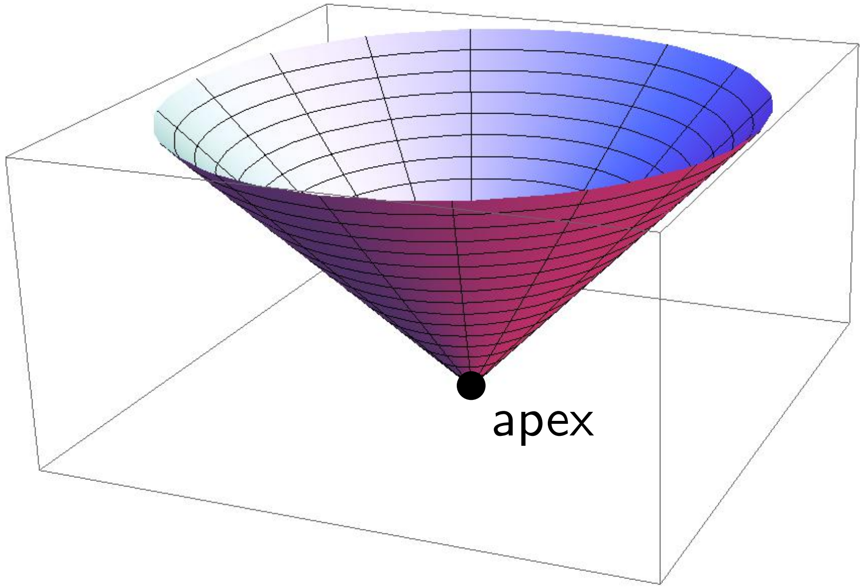

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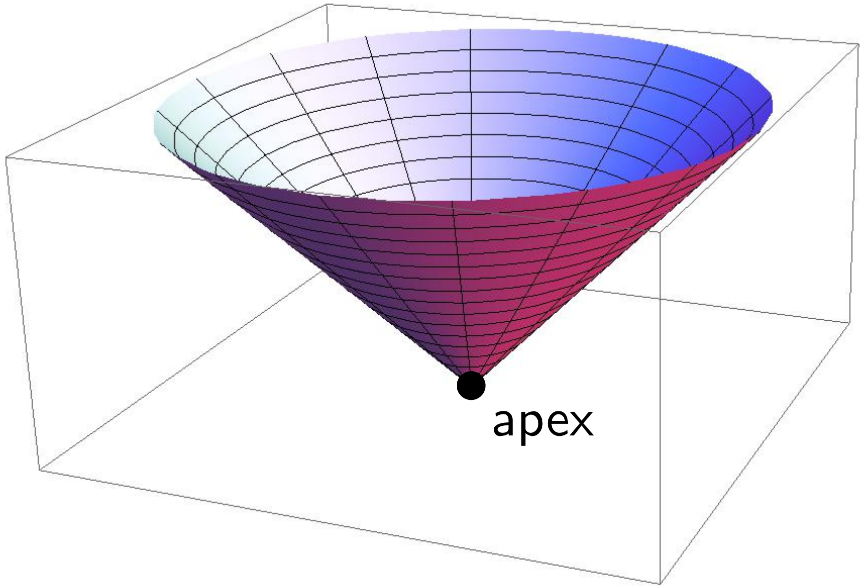

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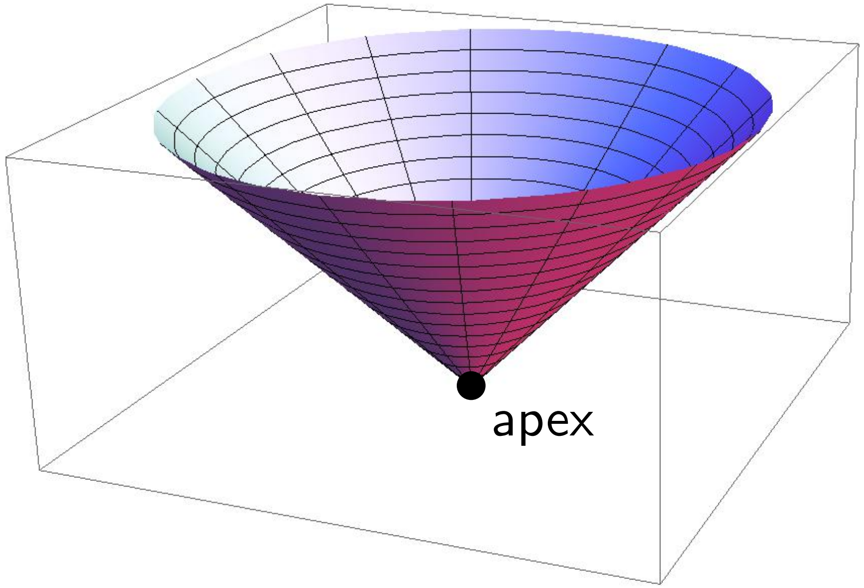

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For the apex, $P = (0,0,0)$, we get $\vec{N}(0,0) = 0$

(thus, the apex is a singular point)

Normal vector, tangent plane: example

Consider the north hemisphere of the unit sphere

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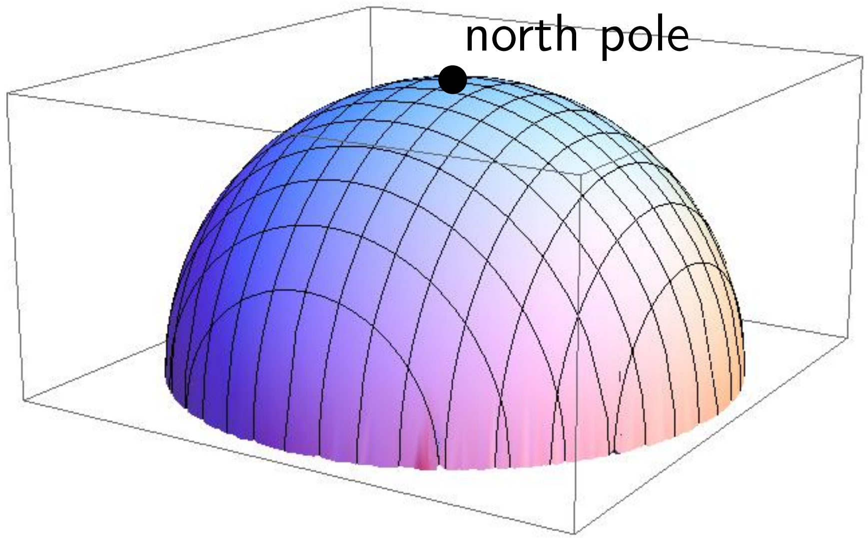

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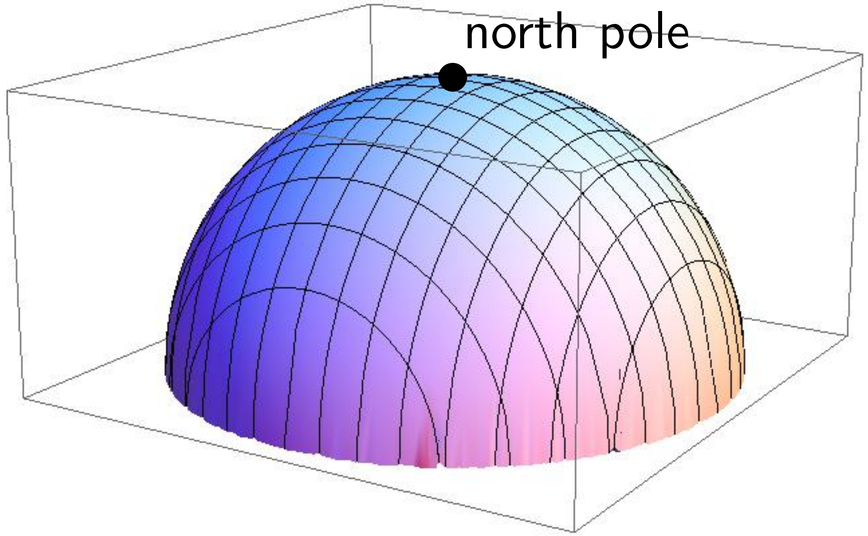

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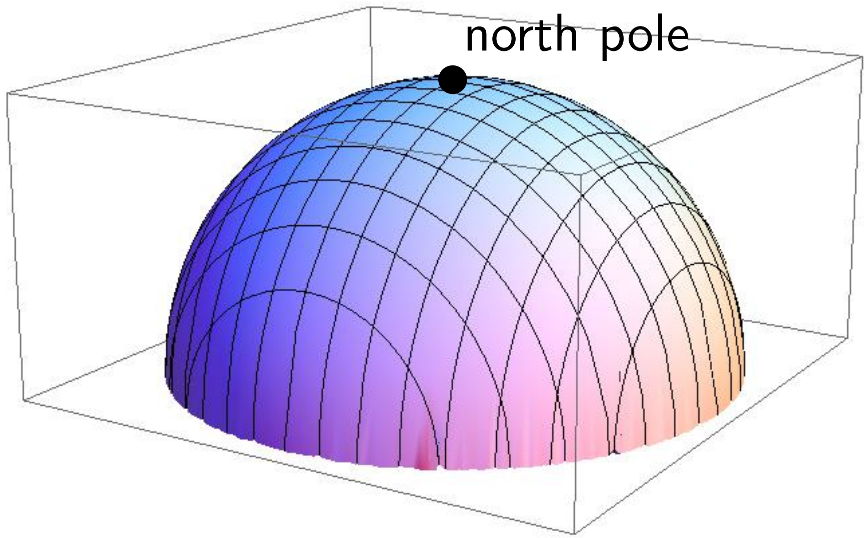

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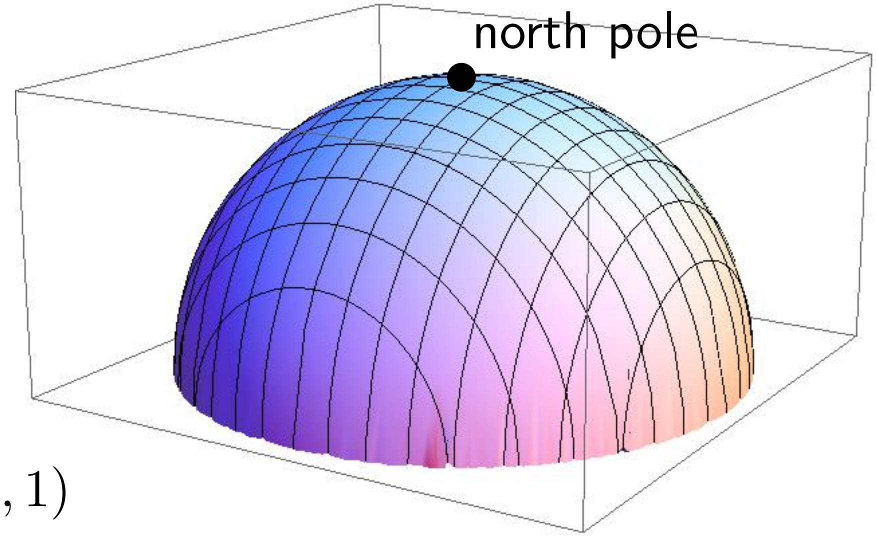

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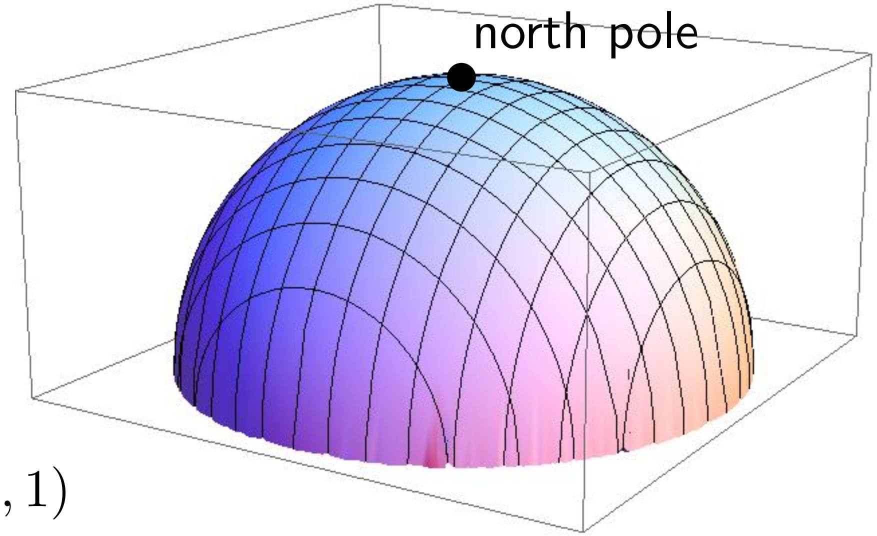

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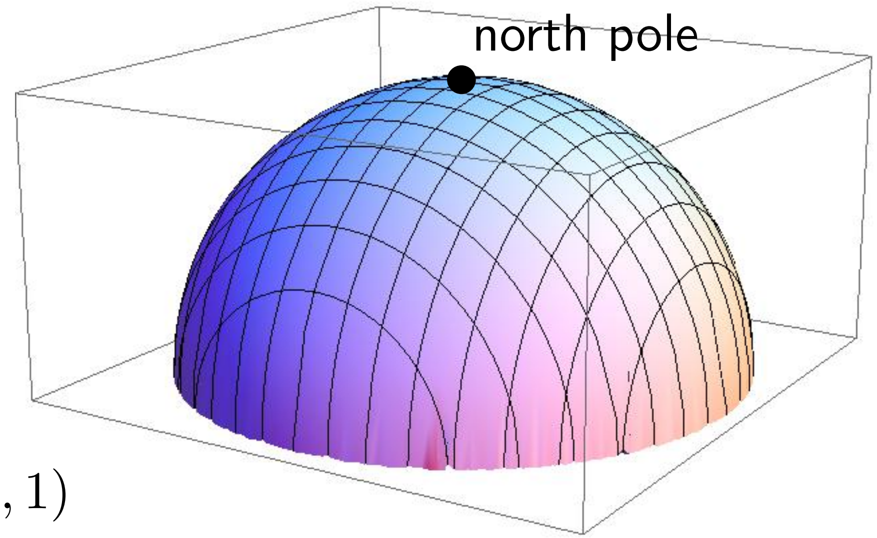

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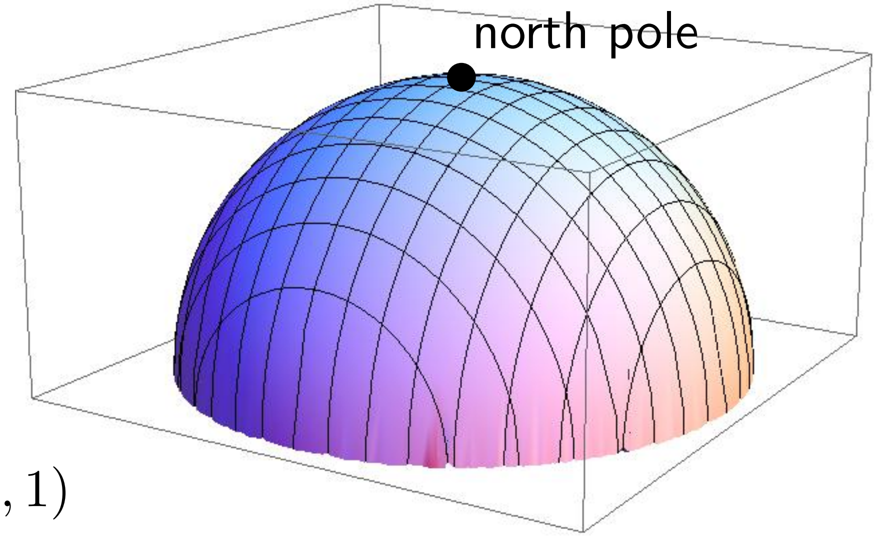

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$S(\theta, \varphi) = (\cos\varphi\cos\theta, \cos\varphi\sin\theta, \sin\varphi)$ for $\theta \in [0, 2\pi)$ and $\varphi \in [0, \pi/2)$

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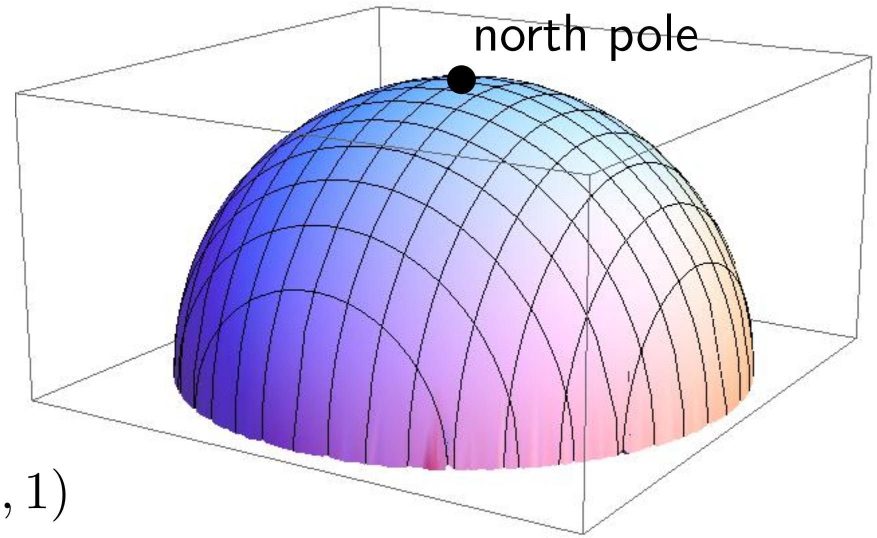

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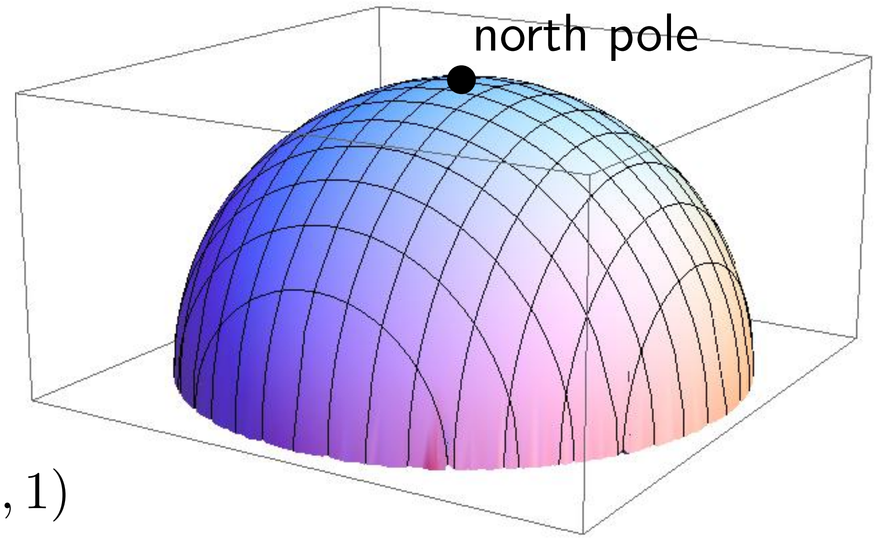

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north pole: $\varphi = \pi/2$ (any θ)$\Rightarrow$ $N(\theta, \pi/2) = (0,0,0)$

singular point