B-SPLINE CURVES

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Improving over Bézier curves

Bézier curves have some drawbacks:

- Degree is proportional to number of control points
- Does not offer true global control (at most "pseudo-local")
- C^2 continuity is not so easy to obtain for composite curves

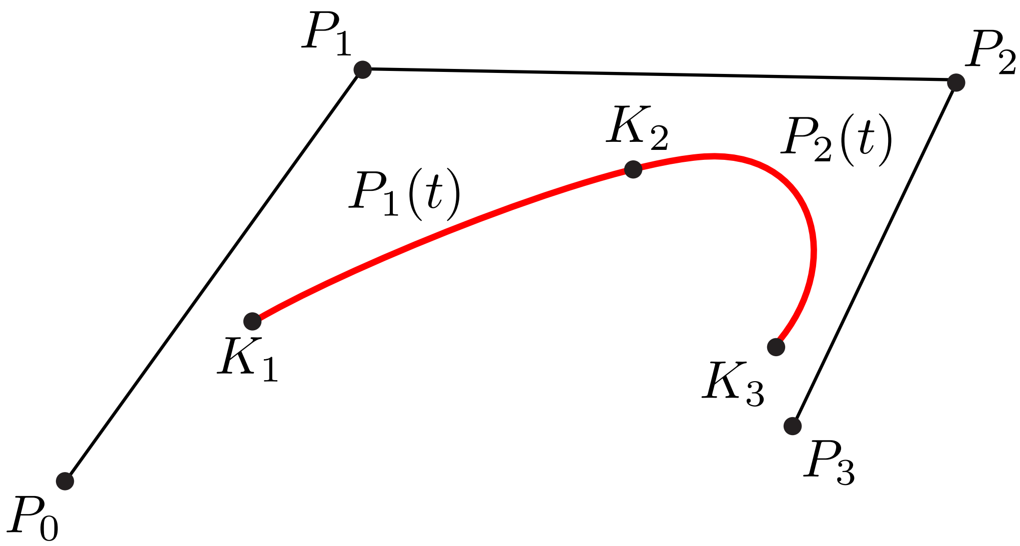
To overcome this: **B-splines**

- Developed by Riesenfeld and others in 1970s
- B-splines = Basis splines
- Several flavors: uniform, non-uniform, rational non-uniform (NURBs)...

Deriving the formula for the quadratic B-splines

Setting:

- Input: $n+1$ control points $P_0, \ldots, P_n$
- Output: **spline curve** where each segment $P_i(t)$ is a **quadratic** parametric polynomial based on P_{i-1},P_i and P_{i+1}



sketch of setting for $n = 3$ (not accurate!)

$$P_i(t) = (t^2, t, 1) \begin{pmatrix} & & \\ & ? & \\ & & \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}$$

$$= (t^2, t, 1) \begin{pmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}$$

Requirements:

1. $P_1(t)$ and $P_2(t)$ meet smoothly at common point
2. Affine combination of control points

Question: what is the matrix?

QUADRATIC UNIFORM B-SPLINES

Deriving the formula for the quadratic B-splines

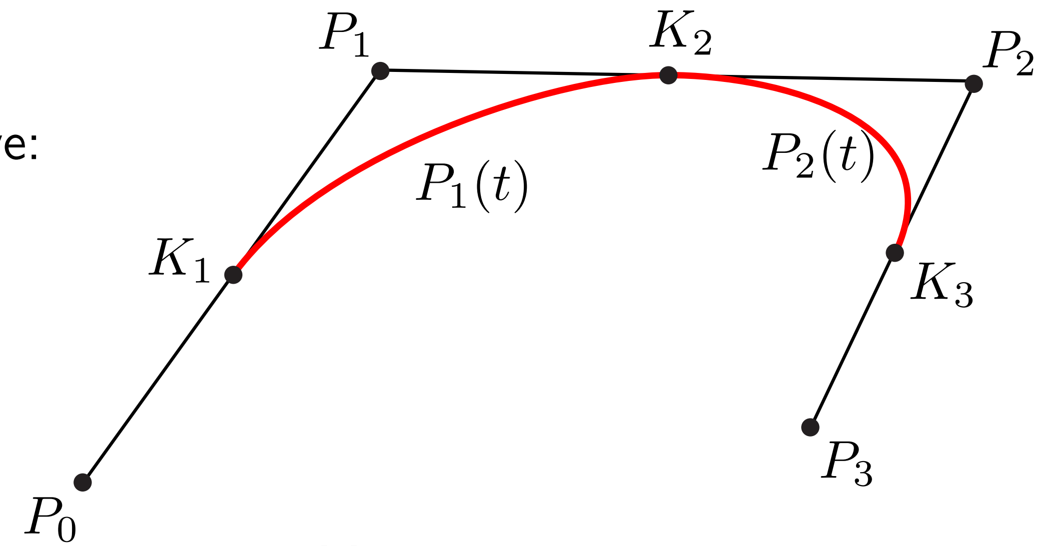
$$P_i(t) = \frac{1}{2}(t^2, t, 1)\begin{pmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \end{pmatrix}, i = 1, 2$$

$$= \frac{1}{2}(t^2 - 2t + 1)P_{i-1} + \frac{1}{2}(-2t^2 + 2t + 1)P_i + \frac{t^2}{2}P_{i+1}$$

- Start and endpoints: K_i and K_{i+1}

  Since $K_i = P_i(0)$ and $K_{i+1} = P_i(1)$, we have:

  $$K_i = P_i(0) = \tfrac{1}{2}(P_{i-1} + P_i)$$
  $$K_{i+1} = P_i(1) = \tfrac{1}{2}(P_i + P_{i+1})$$



More accurate picture

Question: what are the tangent vectors at the ends?

$$P_i'(0) = P_i - P_{i-1}$$
$$P_i'(1) = P_{i+1} - P_i$$

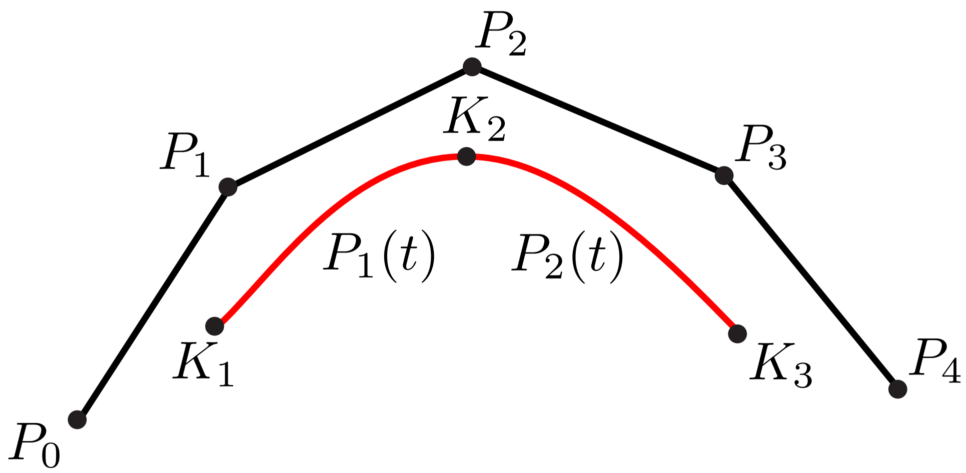
Example: use control points $\{(1,0), (1,1), (2,1), (2,0)\}$

Deriving the formula for the cubic B-splines

Setting:

- Input: $n+1$ control points $P_0, \ldots, P_n$
- Output: spline curve where each segment $P_i(t)$ is a **cubic** parametric polynomial based on P_{i-1},P_i, P_{i+1}, P_{i+2}

$$P_i(t) = (t^3, t^2, t, 1)M \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$



sketch of setting for $n = 4$ (not accurate!)

Requirements:

1. Consecutive segments meet with C^2-continuity
2. Entire curve is affine combination of control points

Deriving the formula for the cubic B-splines

$$P_i(t) = (t^3, t^2, t, 1) \begin{pmatrix} a_3 & b_3 & c_3 & d_3 \\ a_2 & b_2 & c_2 & d_2 \\ a_1 & b_1 & c_1 & d_1 \\ a_0 & b_0 & c_0 & d_0 \end{pmatrix} \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

Requirements:

1. Consecutive segments meet with C^2 continuity
2. Entire curve is affine combination of control points

Matrix derived in similar way:

- Equations for equal endpoints, equal derivatives and second derivatives at $t = 0$ and $t = 1$? 15 equations
- Equations for affine combination 4 equations

16 of them are independent $\Longrightarrow$ unique solution

Formula for the cubic B-splines

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1)\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$= \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1} + \frac{1}{6}(3t^3-6t^2+4)P_i + \frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1} + \frac{t^3}{6}P_{i+2}$$

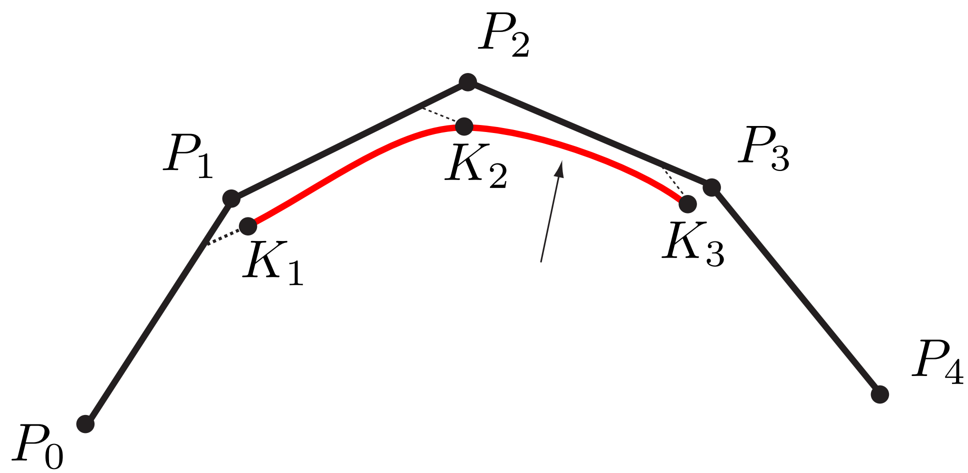
The two endpoints of each curve segment are

$$K_i = P_i(0) = \tfrac{1}{6}(P_{i-1} + 4P_i + P_{i+1})$$
$$K_{i+1} = P_i(1) = \tfrac{1}{6}(P_i + 4P_{i+1} + P_{i+2})$$

Geometrically, it makes more sense to rewrite as

$$K_i = \left(\tfrac{1}{6}P_{i-1} + \tfrac{5}{6}P_i\right) + \tfrac{1}{6}\left(P_{i+1} - P_i\right)$$
$$K_{i+1} = \left(\tfrac{1}{6}P_i + \tfrac{5}{6}P_{i+1}\right) + \tfrac{1}{6}\left(P_{i+2} - P_{i+1}\right)$$

Other geometric interpretations exist (e.g., $\frac{2}{3}$ rule)



Making the curve go from P_0 to P_n

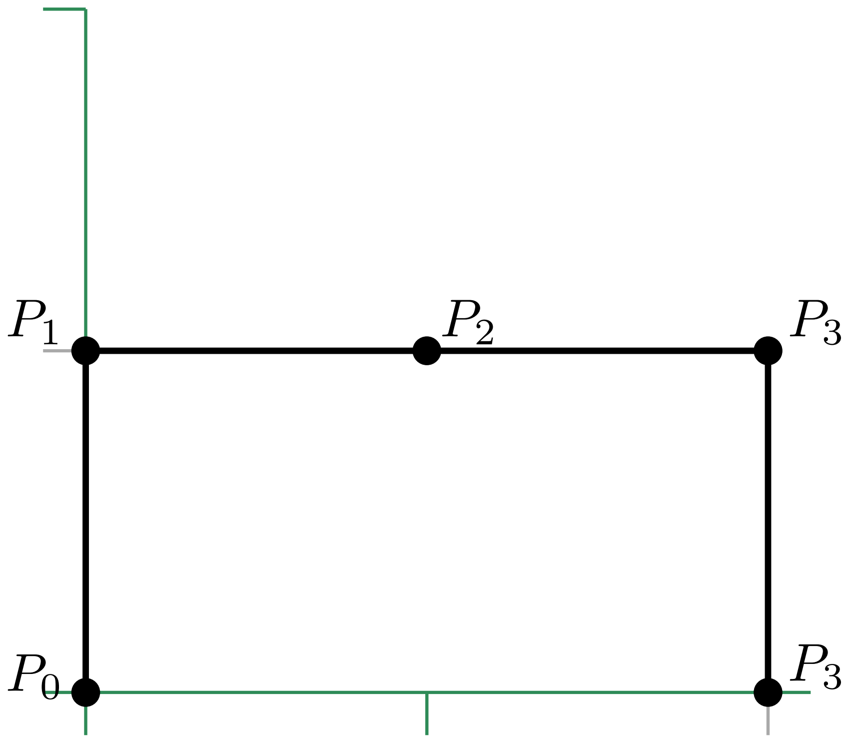
How can we force the curve go through P_0 and P_n?

Answer: add *dummy* control points P_{-1} and P_{n+1}

Question: what are P_{-1} and P_{n+1} points exactly?

$P_{-1} = 2P_0 - P_1$ and $P_{n+1} = 2P_n - P_{n-1}$

Example: use control points $\{(0,0),(0,1),(1,1),(2,1),(2,0)\}$.
(i) Draw the first segment.
(ii) Add P_{-1} to fix beginning of curve at P_0.



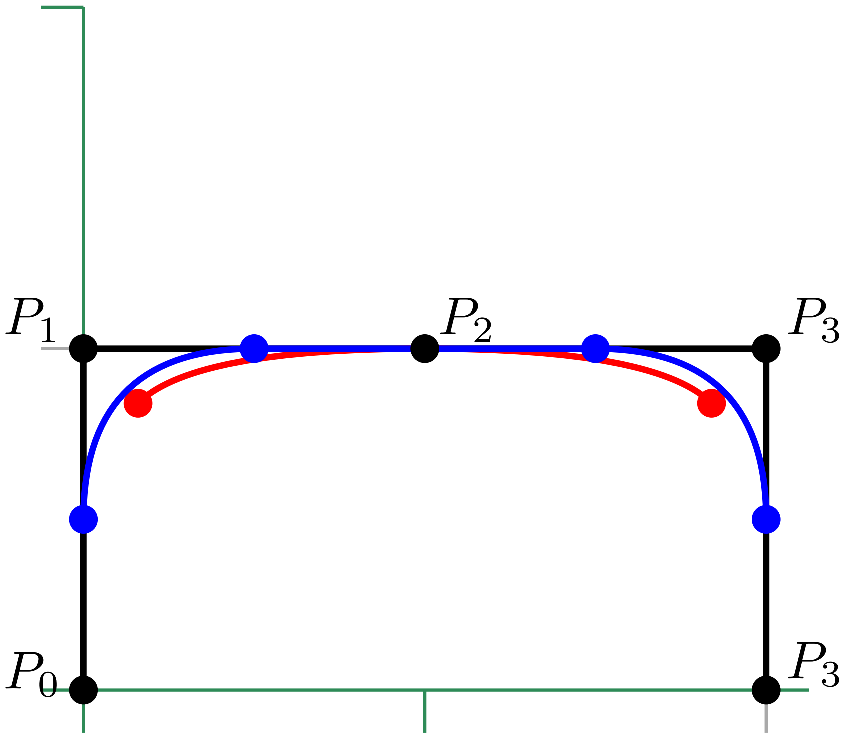
QUADRATIC VS. CUBIC

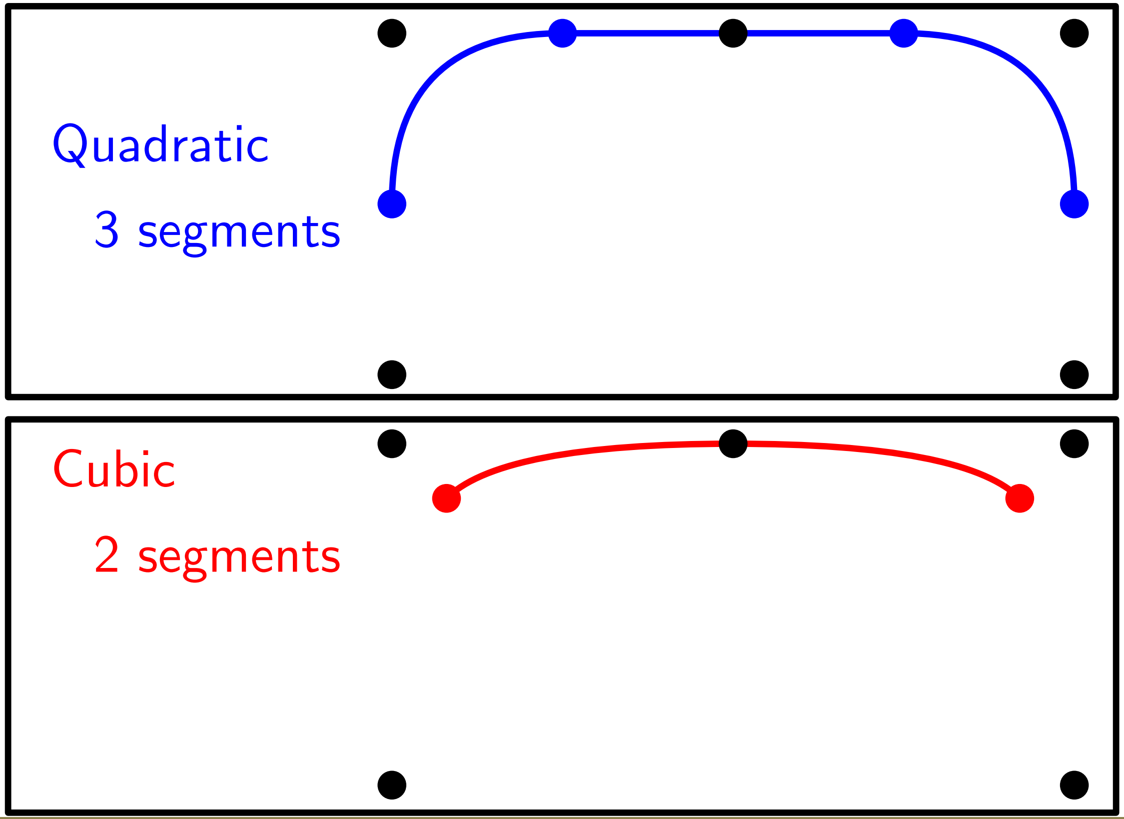
You can choose the order (=degree+1)

As opposed to Bézier curves, in B-splines the same $n+1$ points can be used to construct curve of order 2 (linear), 3 (quadratic), 4 (cubic) ...

Example: use control points $\{(0,0),(0,1),(1,1),(2,1),(2,0)\}$.

How would a quadratic B-spline curve look like?





Higher order, better fit

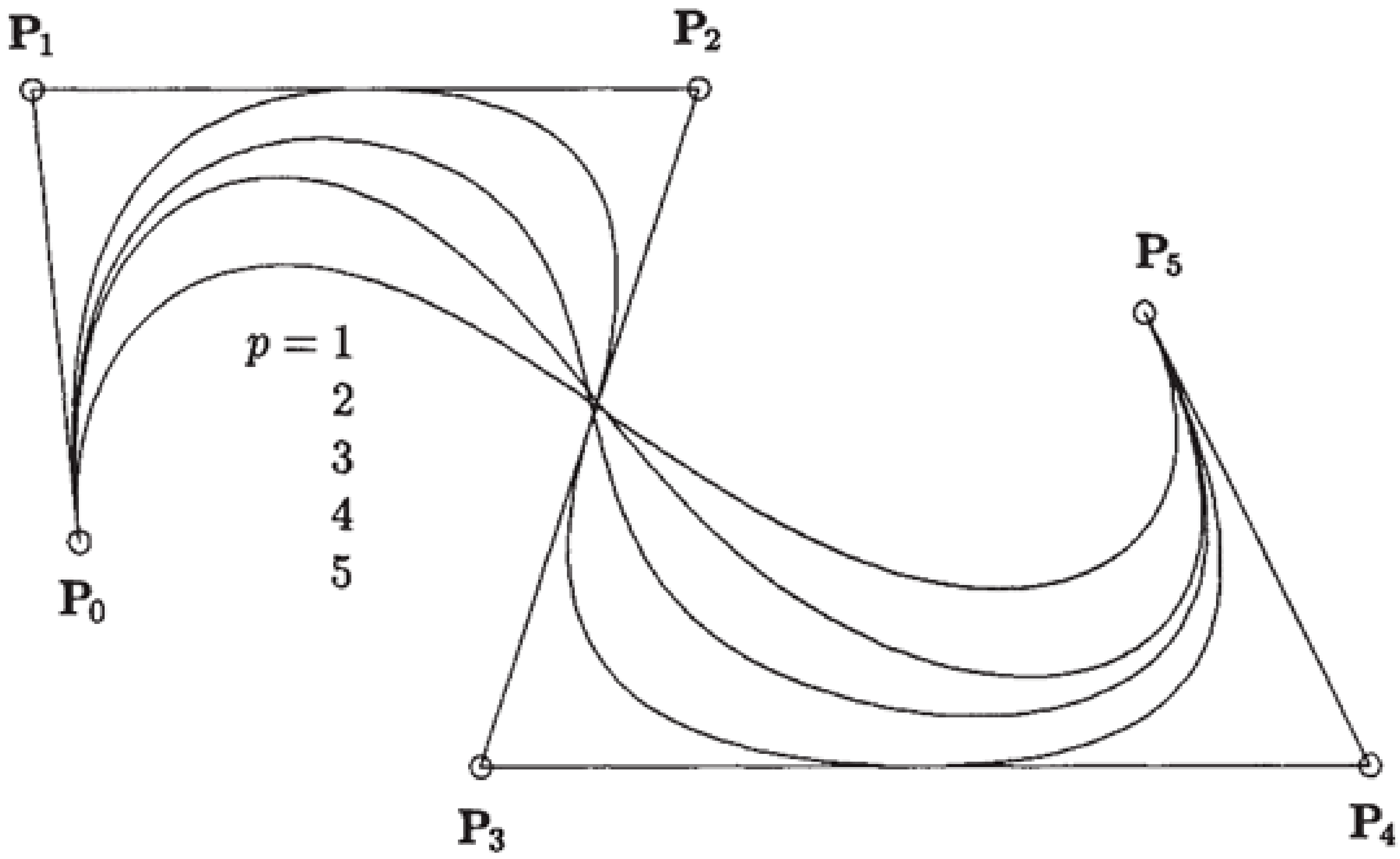


Figure 3.9. B-spline curves of different degree, using the same control polygon.

Where p is the degree of curve (i.e., $p = k - 1$) Figure from [Piegl and Tiller]

From a cubic B-Spline to a cubic Bézier curve

Given a cubic B-spline segment $P(t)$ based on control points $P_0, \cdots, P_3$, how can we find a Bézier curve $C(t)$ with the same shape?

$$P(t) = \frac{1}{6}(t^3, t^2, t, 1)\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$$

Matrix formulation of cubic B-spline

$$C(t) = (t^3, t^2, t, 1)\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\boxed{\begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \\ Q_3 \end{pmatrix}}$$

Matrix formulation of cubic Bézier

Unknowns

Question: what is the solution to the problem?

Solution: Solve $P(t) = C(t)$ for the unknowns $Q_0, \ldots, Q_3$

Result:

$$Q_0 = \tfrac{1}{6}(P_0 + 4P_1 + P_2),$$
$$Q_1 = \tfrac{1}{6}(4P_1 + 2P_2),$$
$$Q_2 = \tfrac{1}{6}(2P_1 + 4P_2),$$
$$Q_3 = \tfrac{1}{6}(P_1 + 4P_2 + P_3)$$

HIGHER ORDER B-SPLINE CURVES

B-splines of order higher than four

The formulas for quadratic and cubic uniform B-splines generalize to arbitrary order

$$P_i(t) = \frac{1}{6}(t^3, t^2, t, 1)\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}\begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ P_{i+2} \end{pmatrix}$$

$$P(t) = (t^n, \ldots, t^2, t, 1)M_n \begin{pmatrix} P_{i-1} \\ P_i \\ P_{i+1} \\ \vdots \\ P_{i+n-1} \end{pmatrix}$$

Matrix for
B-spline s

$$m_{ij} = \frac{1}{n!}\binom{n}{i}\sum_{k=j}^{n}(n-k)^i(-1)^{k-j}\binom{n+1}{k-j}$$

$$M_3 = \frac{1}{6}\begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{pmatrix}$$

$$M_4 = \frac{1}{24}\begin{pmatrix} -1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -6 & -6 & 6 & 0 \\ -4 & -12 & 12 & 4 & 0 \\ 1 & 11 & 11 & 1 & 0 \end{pmatrix}$$

$$M_5 = \frac{1}{120}\begin{pmatrix} -1 & 5 & -10 & 10 & -5 & 1 \\ 5 & -20 & 30 & -20 & 5 & 0 \\ -10 & 20 & 0 & -20 & 10 & 0 \\ 10 & 20 & -60 & 20 & 10 & 0 \\ -5 & -50 & 0 & 50 & 5 & 0 \\ 1 & 26 & 66 & 26 & 1 & 0 \end{pmatrix}$$

How to build an interpolating cubic B-spline curve

Unknowns

B-spline curve approximates control points. How can we make it interpolate?

B-spline curve: given $n+1$ given control points, produces curve through $n-1$ joints K_j

We want: given $n+1$ data points $K_0, \ldots, K_n$, produce $\boxed{n\text{-segment curve}}$ **through them**

To produce n segments, a cubic B-spline requires $\boxed{n+3 \text{ control points, } P_{-1}, \ldots, P_{n+1}}$

Recall formula of cubic B-spline:

$$P_i(t) = \frac{1}{6}(-t^3+3t^2-3t+1)P_{i-1}+\frac{1}{6}(3t^3-6t^2+4)P_i+\frac{1}{6}(-3t^3+3t^2+3t+1)P_{i+1}+\frac{t^3}{6}P_{i+2}$$

$$P_i(1) = \frac{1}{6}(P_i + 4P_{i+1} + P_{i+2}) = K_{i+1}$$

we require this → this gives one equation for each K_j ($n+1$ in total)

We need $n+3$ equations, so we need two more: we fix tangent vectors T_1 and T_n at ends

$$P'_0(0) = \frac{1}{2}(P_1 - P_{-1}) \qquad = T_1$$

$$P'_n(1) = \frac{1}{2}(P_{n+1} - P_{n-1}) \qquad = T_n$$

two more equations (T_1 and T_2 given by user)

System with $n+3$ unknowns and $n+3$ equations that (one can check) is nonsingular

An different way to look at B-splines

Recall Bézier curves: $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \quad t \in [0,1]$

We can do the same with B-splines! Approach based on using a *knot vector*

For now, we consider **cubic uniform B-splines** (order 4)

Example: 5 control points, so two cubic segments

cubic = order 4

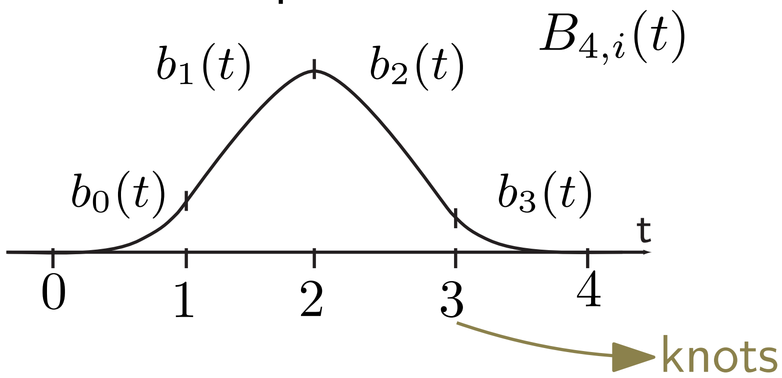
We need to find five weight (basis) functions $B_{4,0}(t), \ldots, B_{4,4}(t)$

- In this approach we assume each cubic segment is defined for one interval of one unit $[u, u+1]$
- Each integer value u is called a *knot* (the sequence of knot values is called *knot vector*)
- In *uniform* B-splines, knots are equally spaced, e.g., $0, 1, 2, \ldots$

in our case, since there are two segments, the parameter for the first one lives in $[0,1]$ and for the other in $[1,2]$

The cubic B-spline basis functions

Overall shape:



Each function should:

- Be a cubic polynomial
- Have it maximum near "its" control point
- Drop to zero away from "its" control point

We write the basis function as the union of **four parts** → each $b_i(t)$ is a cubic polynomial defined over $[0, 1]$

Conditions sought for the $b_i(t)$ functions:

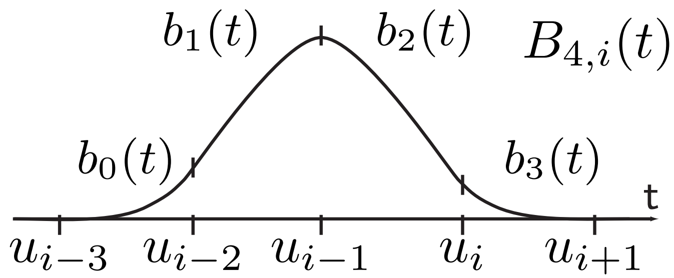
- Affine invariant
- C^2-continuous at three joints
- $b_0(t)$, $b_0'(t)$, $b_0''(t)$ should be zero at the start point $b_0(0)$
- $b_3(t)$, $b_3'(t)$, $b_3''(t)$ should be zero at the end point $b_3(1)$

How many equations on the coefficients of the $b_i(t)$ functions do we obtain from these conditions?

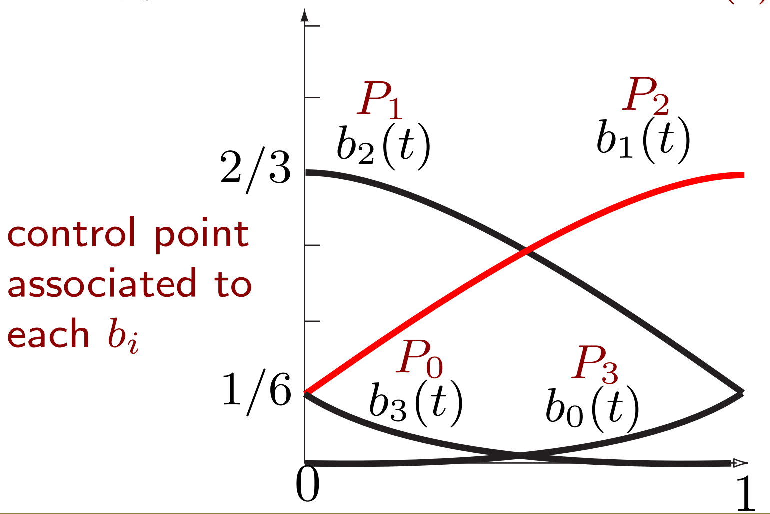
The cubic B-spline basis functions

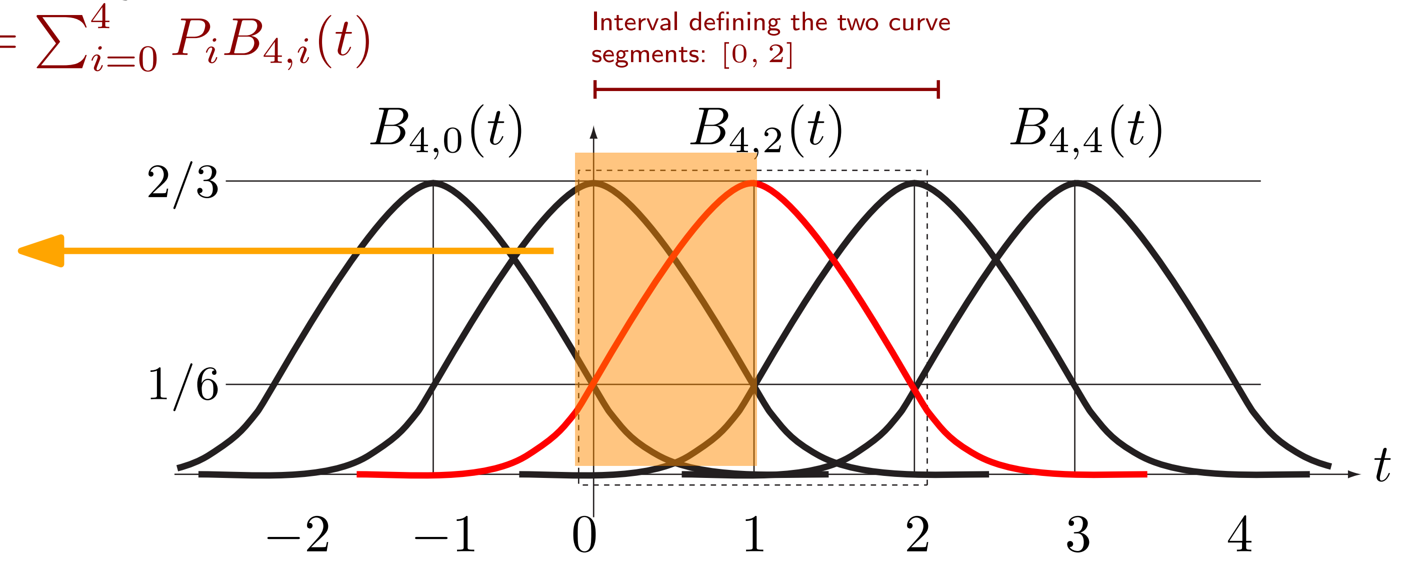
Solution to the equations:

$$N_{i,4}(t) = B_{4,i}(t) = \begin{cases} b_0(t) = \frac{1}{6}t^3 & u_{i-3} \le t \le u_{i-2} \\ b_1(t) = \frac{1}{6}(1 + 3t + 3t^2 - 3t^3) & u_{i-2} \le t \le u_{i-1} \\ b_2(t) = \frac{1}{6}(4 - 6t^2 + 3t^3) & u_{i-1} \le t \le u_i \\ b_3(t) = \frac{1}{6}(1 - 3t + 3t^2 - t^3) & u_i \le t \le u_{i+1} \end{cases}$$



Each control point P_i gets multiplied by a shifted copy of the basis function: $P(t) = \sum_{i=0}^{4} P_i B_{4,i}(t)$





KNOT VECTOR-BASED APPROACH

General B-spline basis functions

This can be generalized to $(n+1)$ control points and order k as follows:

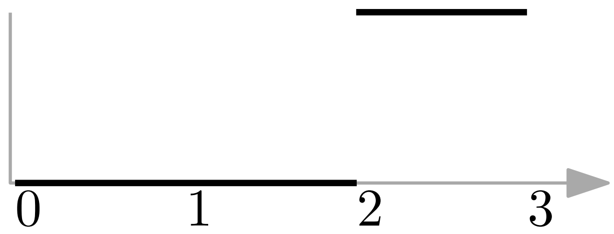
$$P(t) = \sum_{i=0}^{n} P_i \boxed{N_{i,k}(t)} \longrightarrow \text{B-spline basis function}$$

Given: control points $P_0, \ldots, P_n$, knots: $t_0 \leq t_1 \leq \cdots \leq t_{n+k}$, order: k

Note that # knots depends on # control points and order

Order 1 ($k = 1$, degree 0)

$$N_{i,1}(t) = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$



$N_{2,1}(t)$

t_is$= [0, 1, 2, 3, \ldots]$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

(taking $0/0$ as 0)

Examples of B-spline basis functions

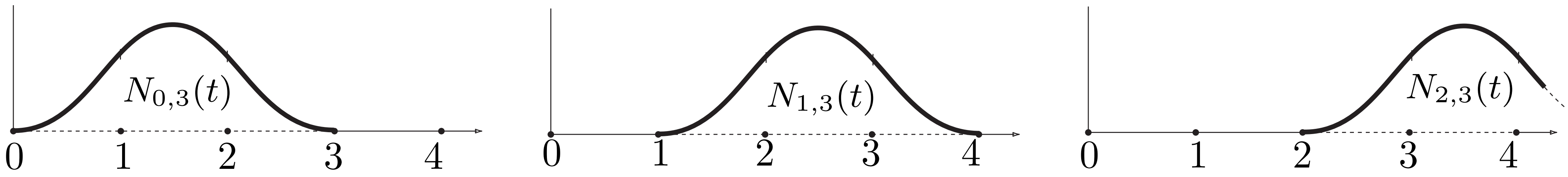
$$N_{i,1} = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

Order 2 ($k = 2$, degree 1)

Question: how do the basis functions look for order 2 ($k = 2$)? (for uniform knots)

Order 3 ($k = 3$, degree 2)



B-SPLINE BASIS FUNCTIONS

Some important properties

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

1) Shifted basis $N_{i,k}(t) = N_{0,k}(t - i)$

2) Local support $N_{i,k}(t) \neq 0$ only for $t \in [t_i, t_{i+k})$

3) Non-zero bases for fixed t

In any interval $[t_j, t_{j+1})$, at most k of the $N_{i,k}$ are non-zero: $N_{(j-k+1),k}, \ldots, N_{j,k}$

4) Non-negativity $N_{i,k}(t) \geq 0$ for all i, k, t

5) Affine invariance

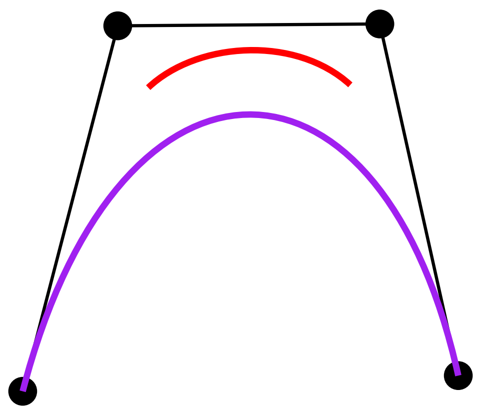
For any $t \in [t_j, t_{j+1})$, $\sum_{i=0}^{n} N_{i,k}(t) = 1$

6) Continuity

For uniform knots, the curve and its $k - 1$ derivatives are continuous
(Non-uniform B-Splines can have discontinuities at knot values!)

Open (or clampled) uniform B-Splines

Uniform knot vector except at ends: at the beginning and end knot values are repeated k times



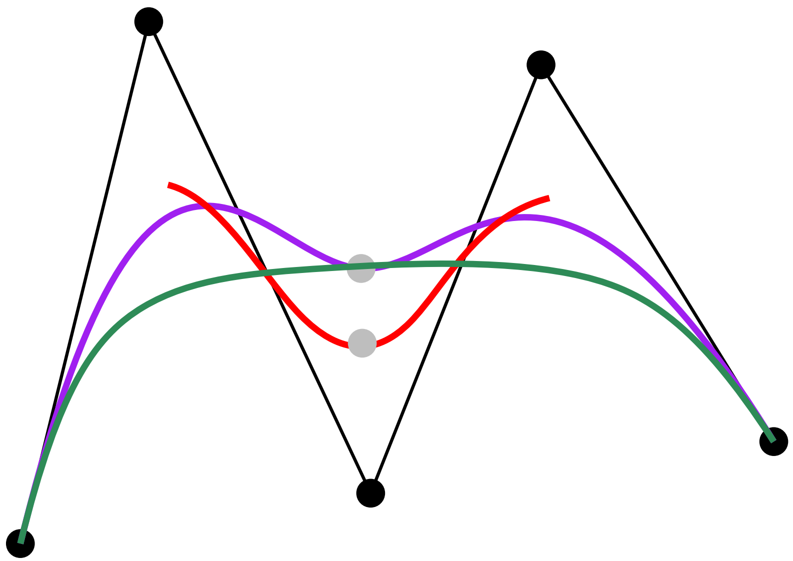
$n = 3$ (4 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7, 1)$

"open" knot vector: $(0, 0, 0, 0, 1, 1, 1, 1)$

Cubic Bézier curve! ⟶ Always the case when $k = n + 1$



$n = 4$ (5 control points)

$k = 4$ (cubic B-spline)

uniform knot vector: $(0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8, 1)$

"open" knot vector: $(0, 0, 0, 0, 0.5, 1, 1, 1, 1)$

degree-4 Bézier—knot vector: $(0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$

Open uniform B-spline curves always start at P_0 and end at P_n. Tangents are also like in Bézier curves

Example for quadratic open B-Splines

Example: compute basis functions for 5 control points ($n = 4$) and $k = 3$ (i.e., quadratic open B-splines)

Recall:

- knot vector: $(0, 0, 0, 1, 2, 3, 3, 3)$
- t goes from $t_{k-1} = t_2 = 0$ to $t_{n+1} = t_5 = 3$
- need to compute 5 bases: $N_{0,3}(t)$ to $N_{4,3}(t)$

$$N_{i,1} = \begin{cases} 1, & \text{if } t \in [t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases} \qquad N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

More examples

Bézier vs open B-Spline of order 3
where $n = 9$ and $k = 3$

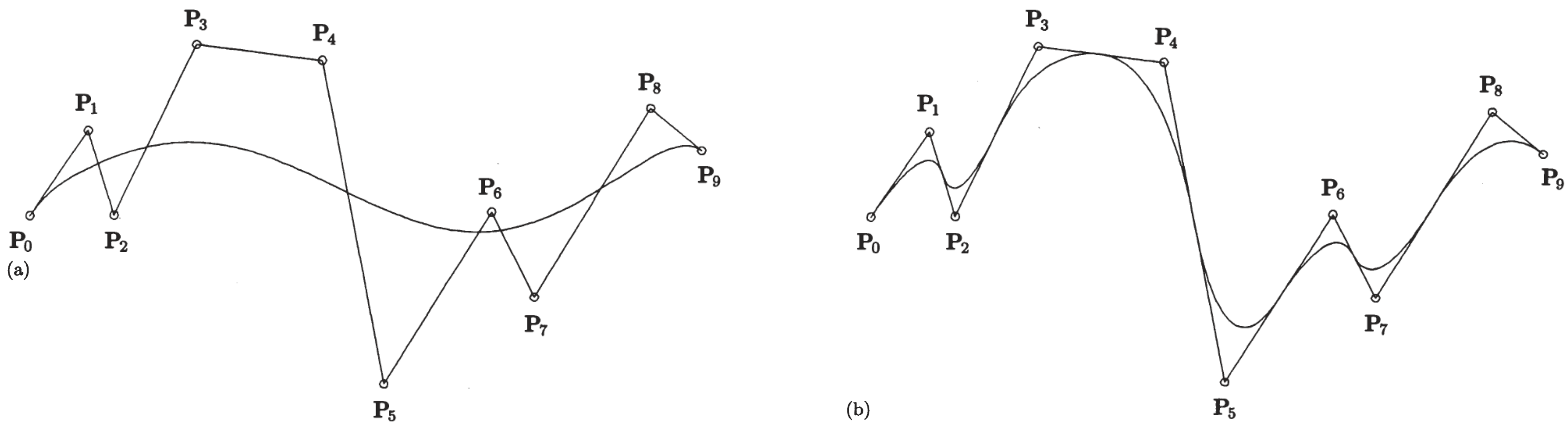


Figure 3.8. B-spline curves. (a) A ninth-degree Bézier curve on the knot vector $U = \{0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,1,1,1,1,1\}$; (b) a quadratic curve using the same control polygon defined on $U = \{0,0,0,\ ^1/_8,\ ^2/_8,\ ^3/_8,\ ^4/_8,\ ^5/_8,\ ^6/_8,\ ^7/_8, 1,1,1\}$.

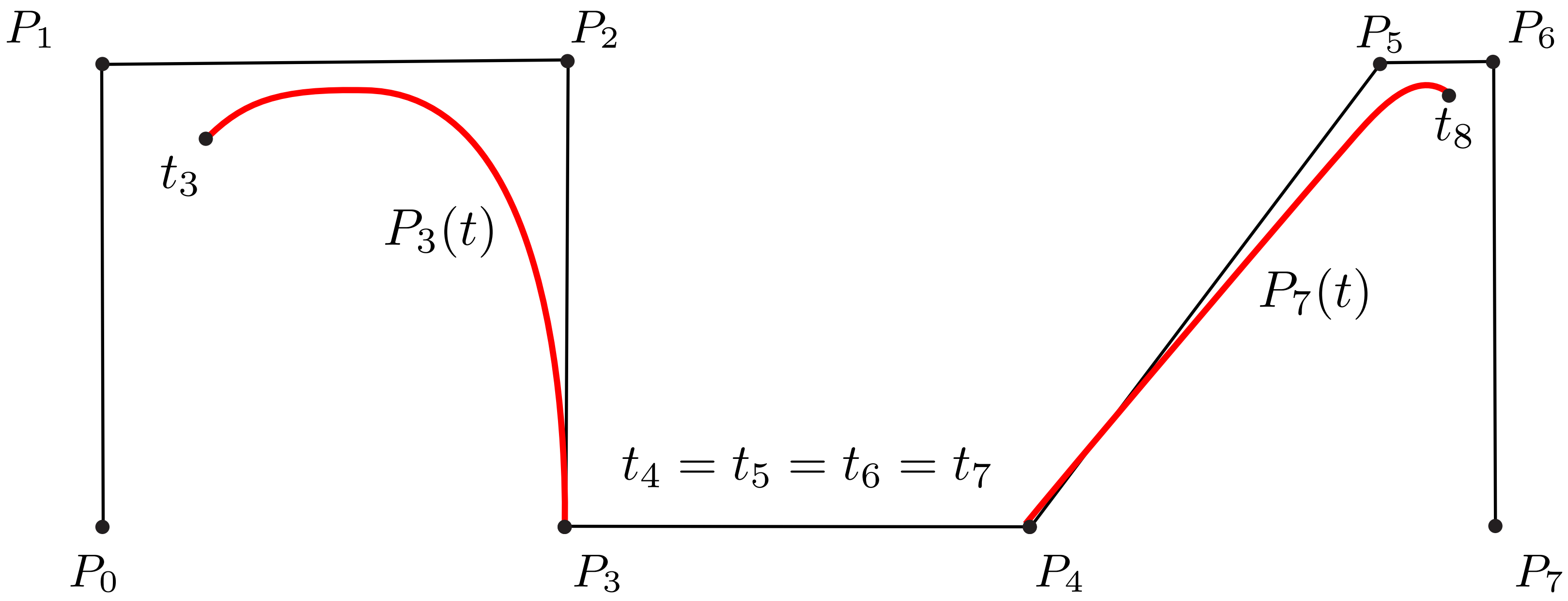
Figure from [Piegl and Tiller]

When knots are *not* equally spaced

ignoring the first and last k knots, in case of open b-splines

- Only restriction: non-decreasing knots
- Knots can have multiplicity larger than one

Effect of knot multiplicity for $k = 4$ (cubic)



Knot vector:
$(-3, -2, -1, 0, 1, 1, 1, 1, 2, 3, 4, 5)$

Understanding knot vectors

Open uniform B-splines interpolate the first and last control points due to the knot multiplicity

In general: continuity at the knots depends on multiplicity

$N_{i,k}(t)$ is $(k - m - 1)$ times continuously differentiable, where m is the multiplicity of the knot

(m=number of repetitions of knot value)

Examples:

- If all knots are different, a cubic $(k = 4)$ B-spline is C^2-continuous at every knot
- If a knot appears twice, the cubic B-spline will be only C^1-continuous there
- If a knot appears three times, the cubic B-spline will be only C^0-continuous there

See example is
http://geometrie.foretnik.net/files/NURBS-en.swf

Matrix form

Matrix-based expressions to compute non-uniform B-splines also exist

Cubic case $(k=4)$

$$N_{04}(u) = \begin{cases} \frac{u-u_0}{u_3-u_0}\cdot\frac{u-u_0}{u_2-u_0}\cdot\frac{u-u_0}{u_1-u_0} & \\ \frac{u-u_0}{u_3-u_0}\cdot\frac{u-u_0}{u_2-u_0}\cdot\frac{u_2-u}{u_2-u_1} & \\ \quad +\frac{u-u_0}{u_3-u_0}\cdot\frac{u_3-u}{u_3-u_1}\cdot\frac{u-u_1}{u_2-u_1} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u-u_1}{u_3-u_1}\cdot\frac{u-u_1}{u_2-u_1} & \\ \frac{u-u_0}{u_3-u_0}\cdot\frac{u_3-u}{u_3-u_1}\cdot\frac{u_3-u}{u_3-u_2} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u-u_1}{u_3-u_1}\cdot\frac{u_3-u}{u_3-u_2} & \\ \quad +\frac{u_4-u}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u-u_2}{u_3-u_2} & \\ \frac{u_4-u}{u_4-u_1}\cdot\frac{u_4-u}{u_4-u_2}\cdot\frac{u_4-u}{u_4-u_3} & \text{for } u\in[u_3,u_4), \\ 0 & \text{otherwise.} \end{cases}$$

We obtain:

$$\mathbf{P}(t) = (t^3, t^2, t, 1)\begin{pmatrix} -a & a+b+c & -b-c-d & d \\ 3a & -3a-3b & 3b & 0 \\ -3a & 3a-3e & 3e & 0 \\ a & 1-a-f & f & 0 \end{pmatrix}\begin{pmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix},$$

where

$$a = \frac{\Delta_3^2}{(\Delta_1+\Delta_2+\Delta_3)(\Delta_2+\Delta_3)}, \qquad d = \frac{\Delta_3^2}{(\Delta_3+\Delta_4+\Delta_5)(\Delta_3+\Delta_4)},$$

$$b = \frac{\Delta_3^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)}, \qquad e = \frac{\Delta_2\Delta_3}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)},$$

$$c = \frac{\Delta_3^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_3+\Delta_4)}, \qquad f = \frac{\Delta_2^2}{(\Delta_2+\Delta_3+\Delta_4)(\Delta_2+\Delta_3)}.$$

$N_{14}(u), N_{24}(u)$ and $N_{34}(u)$ are obtained by incrementing all the indices

take
$$\Delta_1 = u_2-u_1, \quad \Delta_2 = u_3-u_2, \quad \Delta_3 = u_4-u_3,$$
$$\Delta_4 = u_5-u_4, \quad \Delta_5 = u_6-u_5, \quad t = (u-u_3)/\Delta_3.$$

The most general parametric curve

Same idea as for rational Bézier: each control point P_i has a weight, $w_i \geq 0$. This gives even more flexibility to shape the curve

Advantages

- Invariant under projections
- It can represent conic curves exactly (e.g., segments of circles, ellipses, hyperbolas and parabolas)
- It is more general, so it includes as particular cases all other B-splines and Bézier curves

Recall: homogeneous coordinates

Any 2D point (x, y) is equivalent to a 3D point: (wx, wy, w)

→ the projection of 3D point (a, b, w) to 2D is $(a/w, b/w)$

Suppose your control points Q_i have one extra dimension $P_i \in \mathbb{R}^2 \to Q_i \in \mathbb{R}^3$

→ e.g., each point $P_i = (x_i, y_i)$, becomes $Q_i = (w_i x_i, w_i y_i, w_i)$, for some $w_i \geq 0$

Compute a 3D B-spline curve in the usual (non-rational) way: $P_{\text{nr}}(t) = \sum_{i=0}^{n} Q_i N_{i,k}(t)$

Now, we can project any point on the 3D curve $P_{\text{nr}}(t)$ to 2D:

→ we need to isolate the coefficients w_i multiplying each point and divide by them

$$P_{\text{r}}(t) = \frac{\sum_{i=0}^{n} w_i P_i N_{i,k}(t)}{\sum_{j=0}^{n} w_j N_{j,k}(t)} = \sum_{i=0}^{n} P_i \underbrace{\boxed{\left(\frac{w_i N_{i,k}(t)}{\sum_{j=0}^{n} w_j N_{j,k}(t)} \right)}}_{\text{rational blending functions}} = \sum_{i=0}^{n} P_i \underbrace{\boxed{R_{i,k}(t)}}_{\text{rational blending functions}}$$

Rational curves as curves in projective space

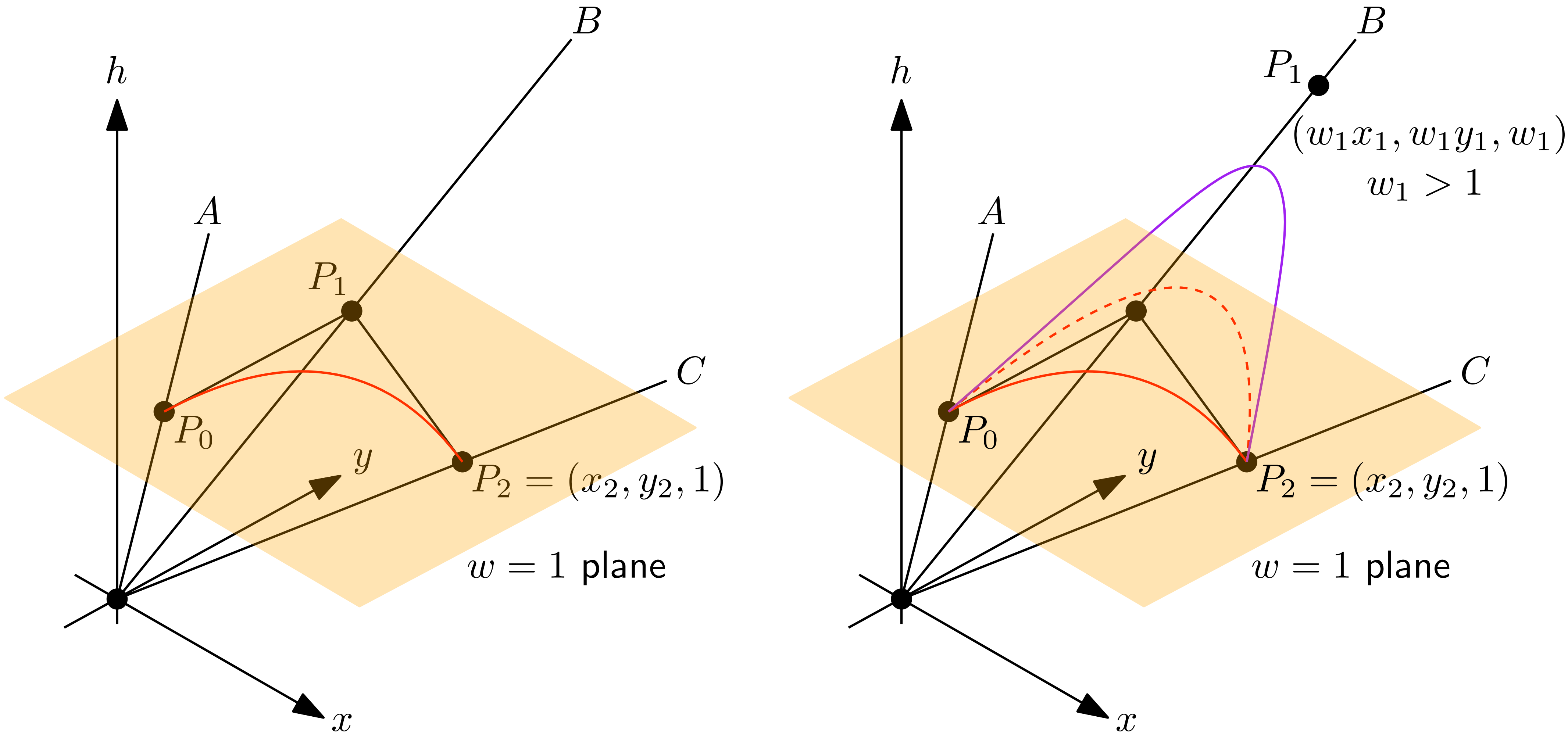


Figure adapted from book by Mortenson

Properties of rational basis functions and NURBS

Has most properties of the non-rational basis functions, plus a few more:

- Non-negativity, partition of unity, unimodality, local support (for control point position and weight), convex hull property, etc..
- Effect of changing weight: increasing weight moves curve closer to P_i

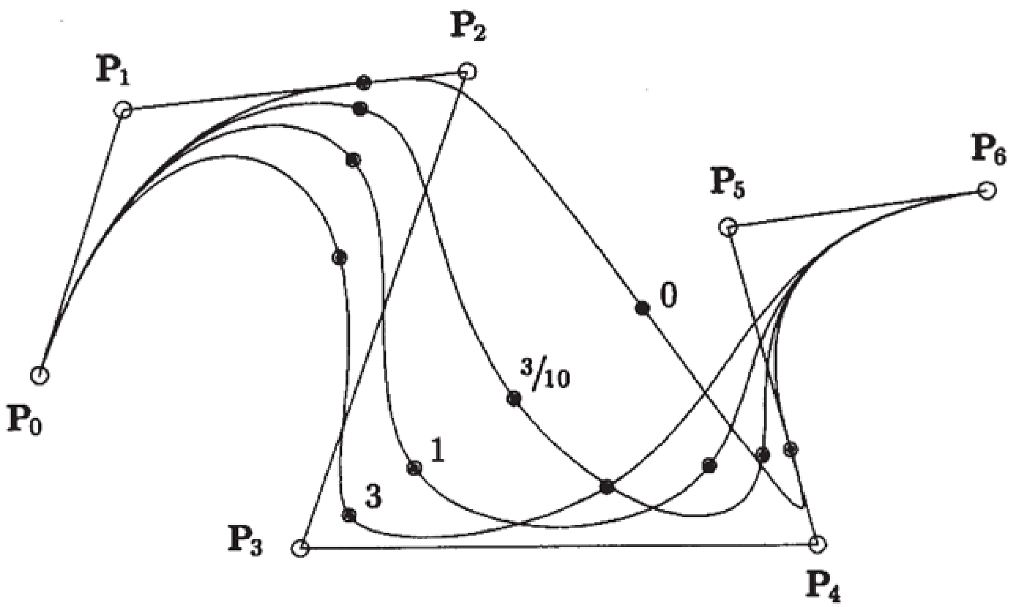


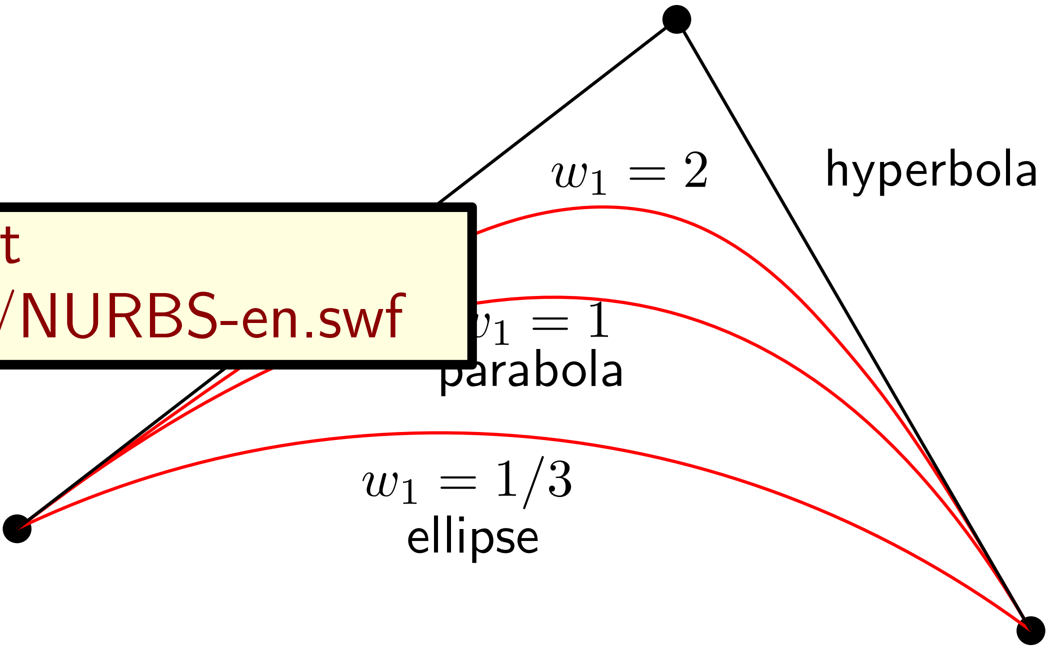
Figure 4.2. Rational cubic B-spline curves, with w_3 varying.

Figure from [Piegl and Tiller]

NON-UNIFORM RATIONAL B-SPLINES (NURBS)

Properties of rational basis functions and NURBS

- When all weights are the same, the curve becomes non-rational
- Curve is invariant under projective transformations
- Conic sections can be represented *exactly* (same as with rational Bézier curves)



Play with NURBS at
http://geometrie.foretnik.net/files/NURBS-en.swf