BÉZIER CURVES

Rodrigo Silveira

Curve and Surface Design
Facultat d'Informàtica de Barcelona
Universitat Politècnica de Catalunya

Interpolation or... approximation!

Previous curve design methods based on **interpolation**

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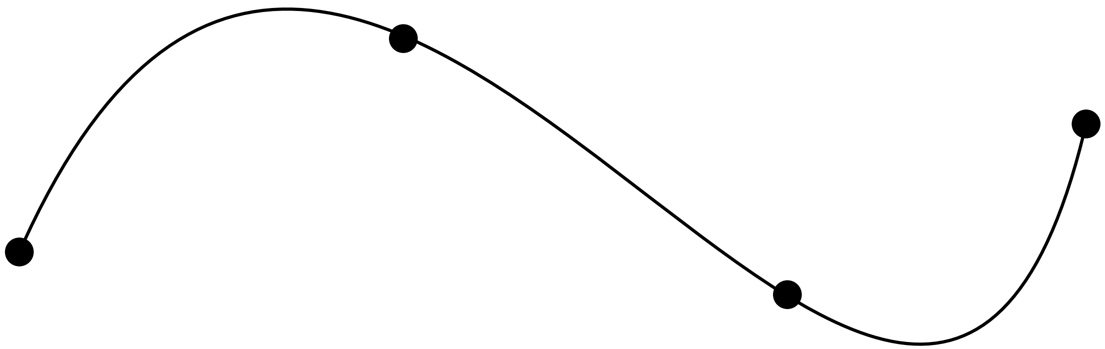

Interpolating curve

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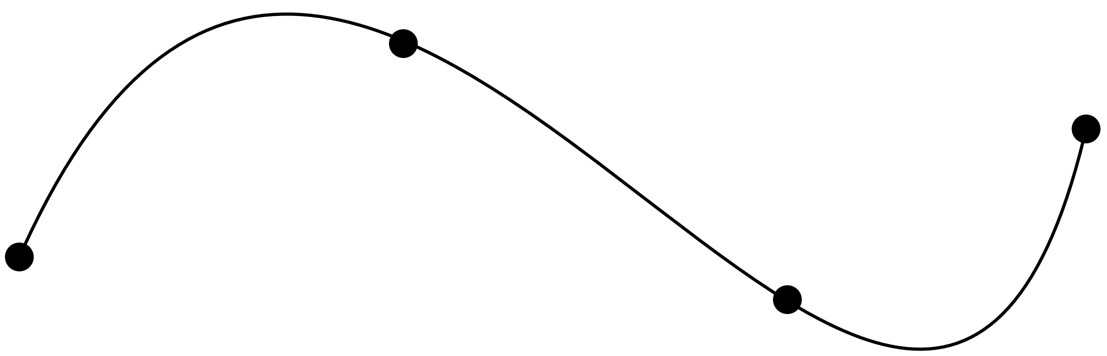
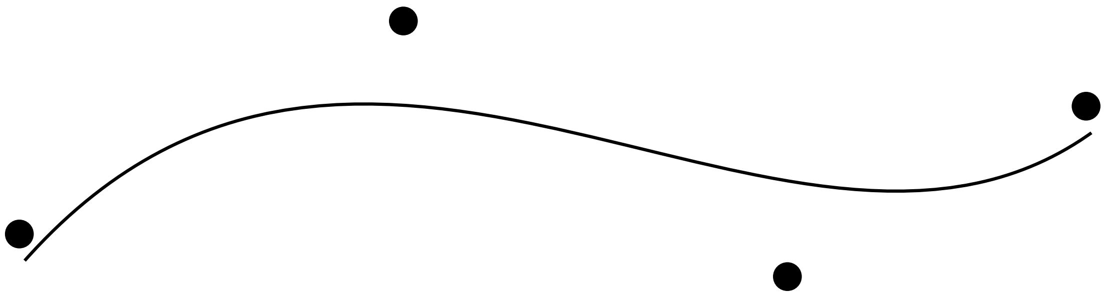


Interpolating curve

Curve passes exactly through given points



Approximating curve

Curve passes near the given points

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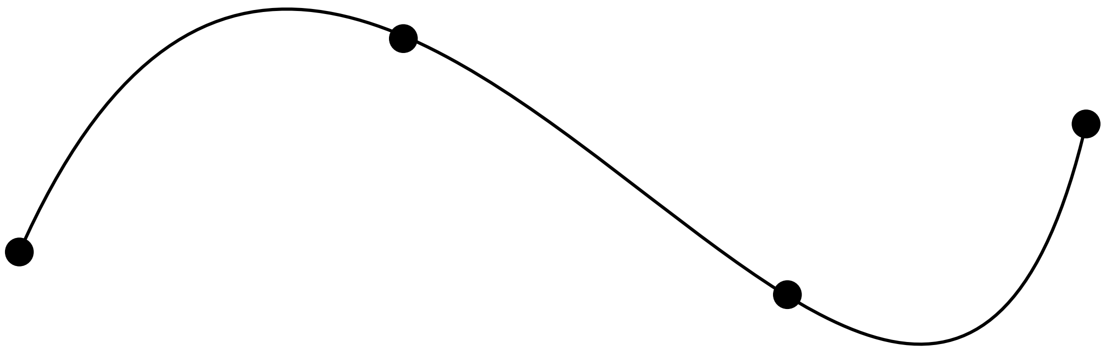

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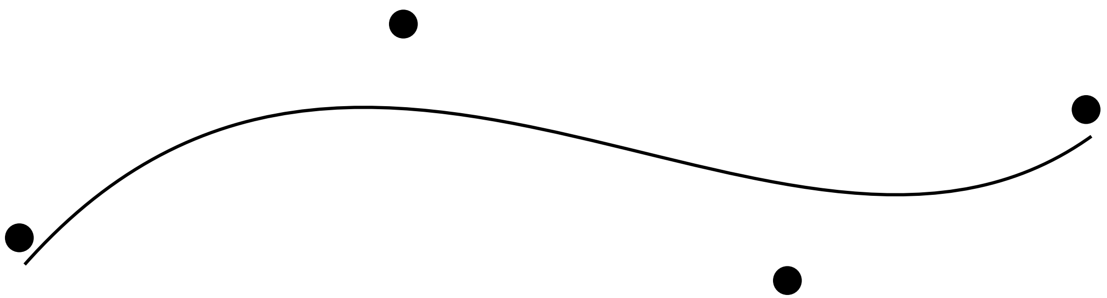

Approximating curve

Curve passes near the given points

What's wrong with interpolation? Curve change when moving points is unpredictable

Approximating curves can provide better "shape control"

Bézier curves

Named after Pierre Bézier (1910-1999)

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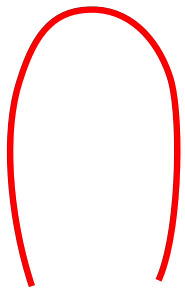
Bézier curve

- Parametric ($P(t)$)
- Polynomial
- Based on *control points*

Some examples of Bézier curves



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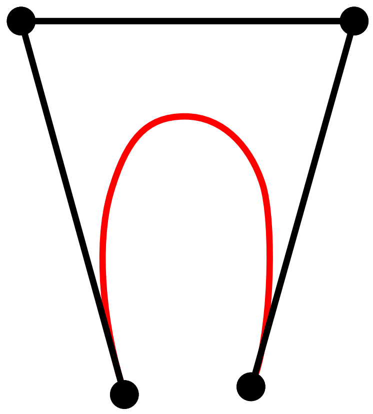

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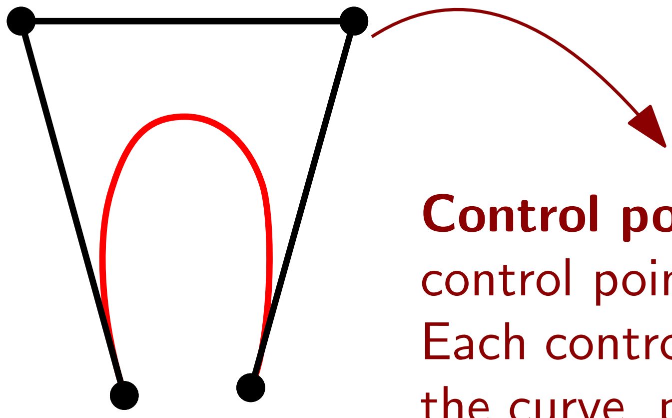
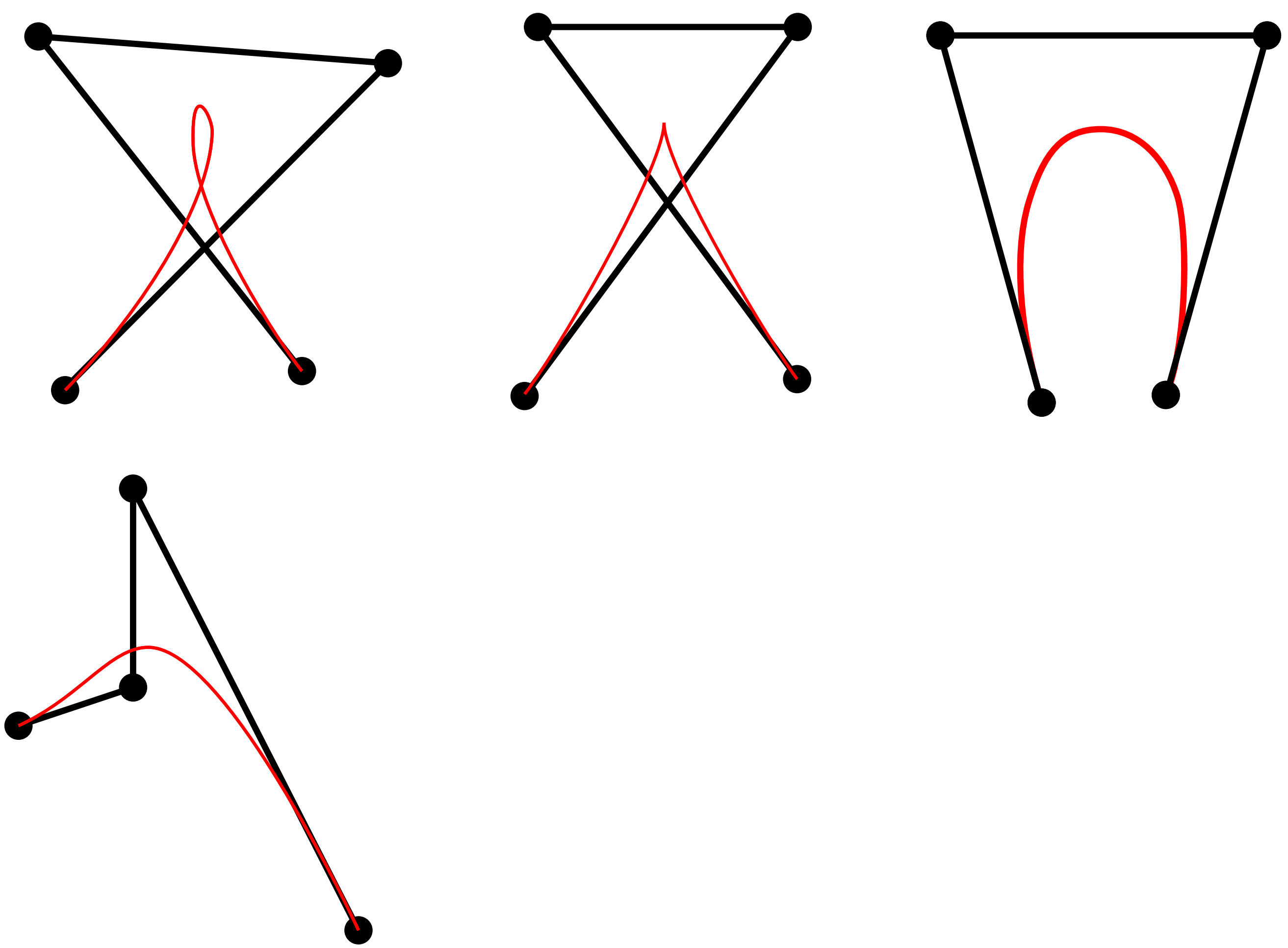


Control polygonal line, made of control points.
Each control point exerts a pull on the curve, pulling it towards itself

Some examples of Bézier curves



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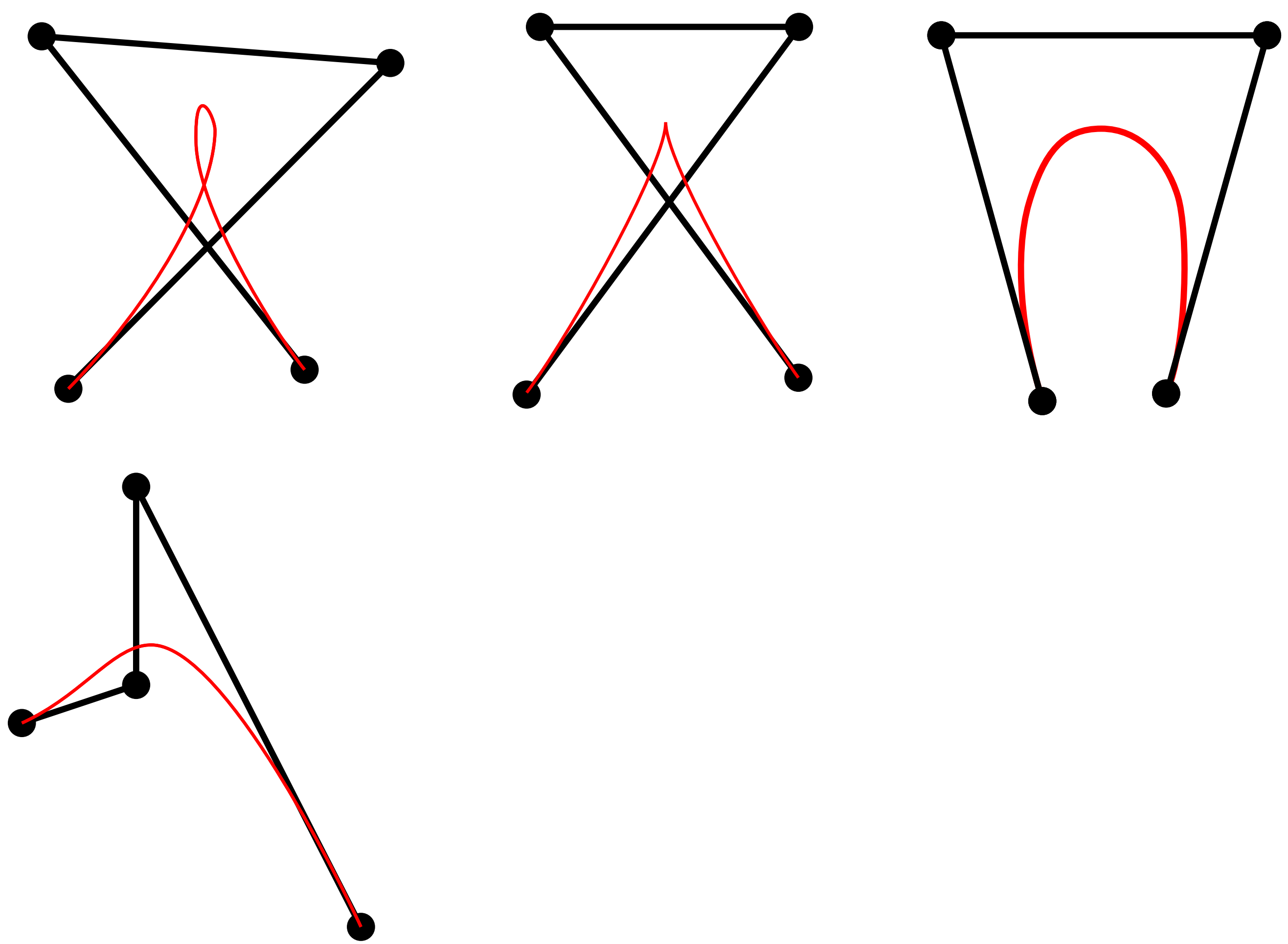

Each curve here is a polynomial, of degree....

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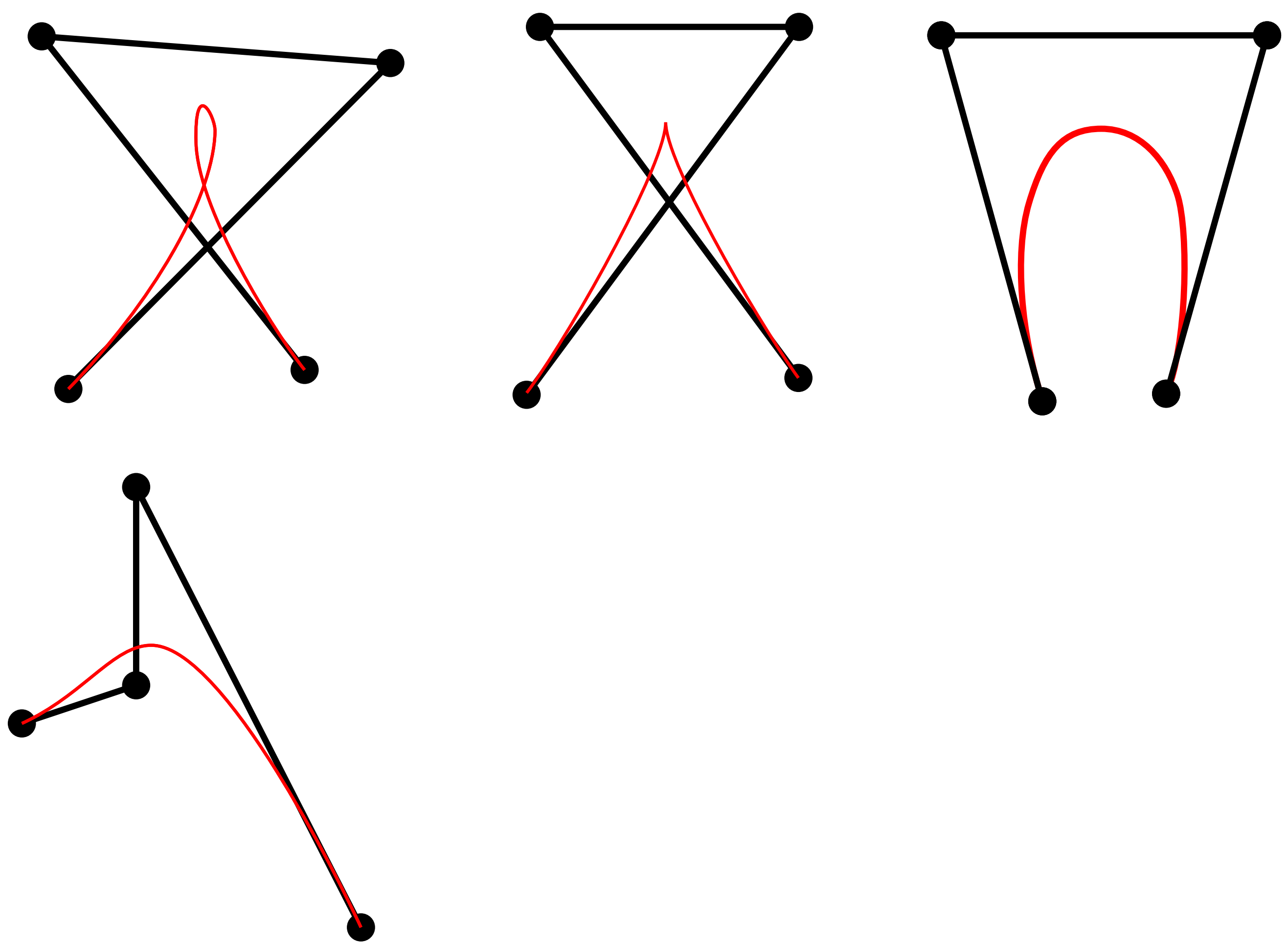

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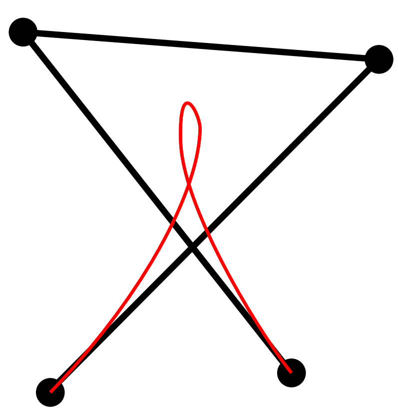
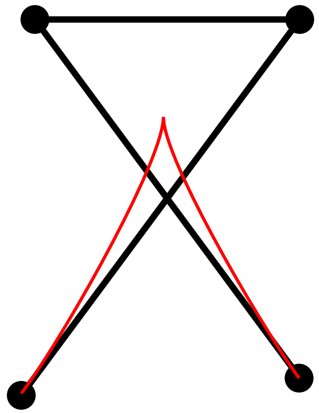
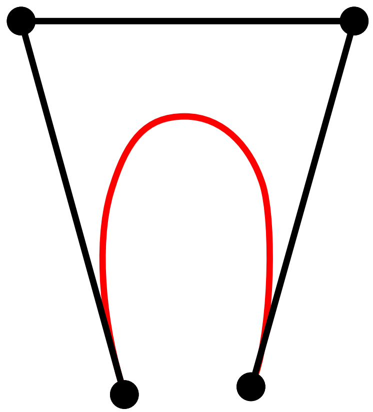
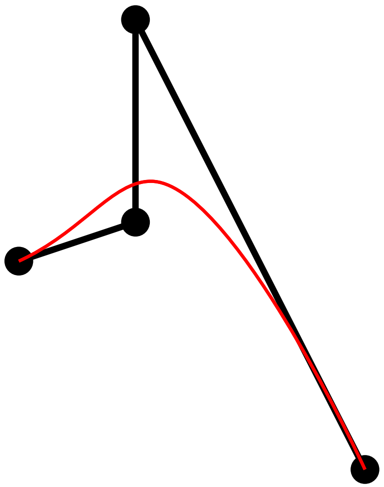
INTRODUCTION TO BÉZIER CURVES

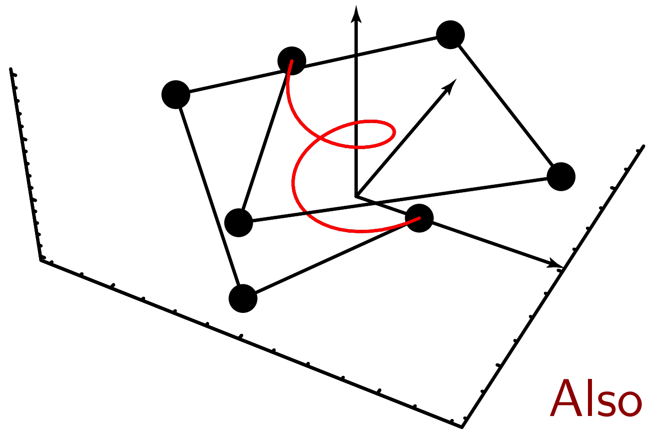
Some examples of Bézier curves











Also works in 3D!

What is a Bézier curve?

General form

$$P(t) = \sum_{i=0}^{n} P_i f_i(t) \qquad\qquad t \in [0, 1]$$

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- Similar to 2), for higher order derivatives: $P^{(k)}(0)$ should depend on $P_0, \ldots, P_k$ only (e.g., $P''(0)$ should depend only on P_0, P_1, and P_2)

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- The basis functions must be symmetric with respect to t and $(1-t)$ (so reversing the parameter and the order of control points gives the same curve)
- Control point weights are barycentric: shape independent from coordinate system. That is: $P(t)$ is an affine combination of control points, so curve is invariant under affinities

Basis functions

The family a functions used are **Bernstein polynomials**

$$f_i(t) = B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

recall that
$0 \leq i \leq n$,
$\binom{n}{i} = \frac{n!}{i!(n-i)!}$, $0! = 1$
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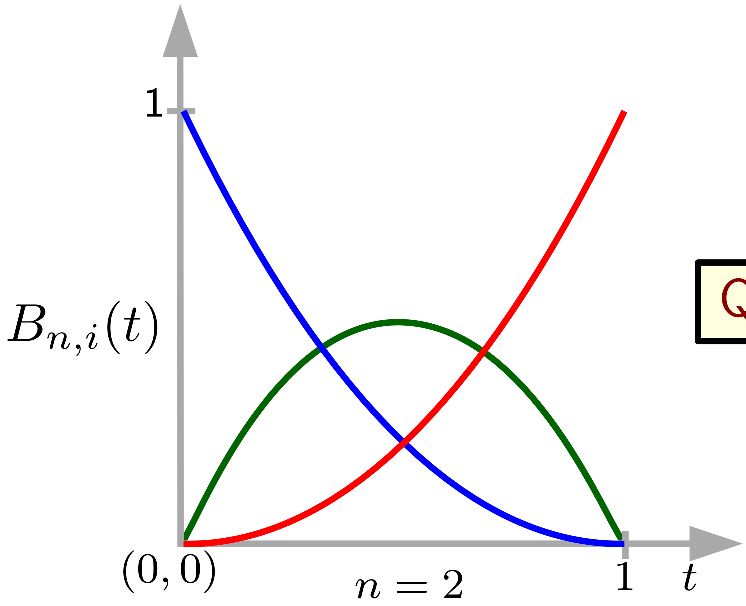

Question: Which basis function is which?

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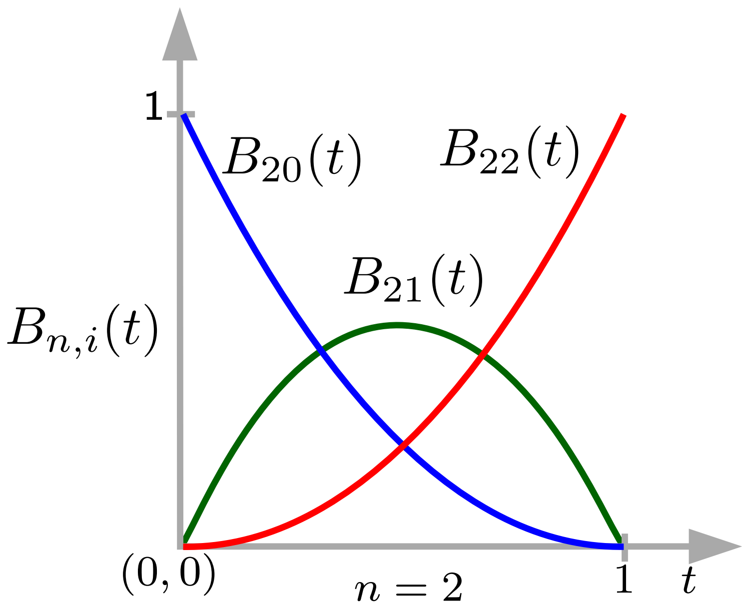

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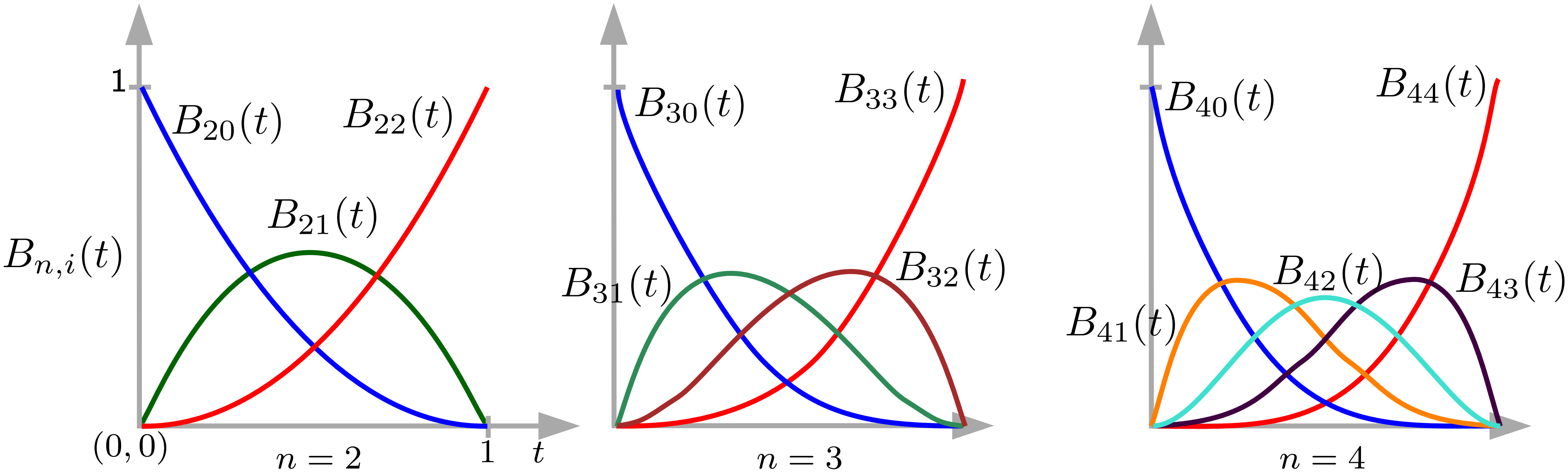

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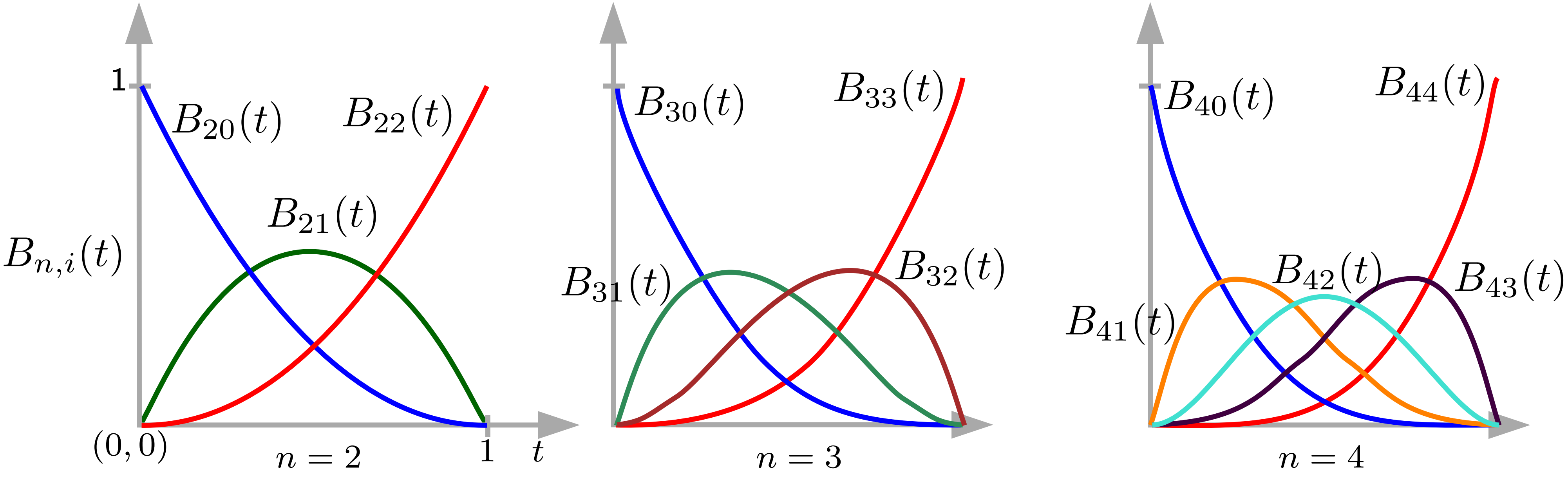

The Bézier curve becomes $$P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) \qquad t \in [0, 1]$$

Example: degree-2 Bézier curve

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For $n = 2$, we have $B_{n,i}(t) = \binom{2}{i} t^i (1-t)^{2-i}$, for $0 \leq i \leq 2$

So, for $n = 2$, these are the three Bernstein polynomials:

- $B_{2,0}(t) = \binom{2}{0} t^0 (1-t)^{2-0} = (1-t)^2$
- $B_{2,1}(t) = \binom{2}{1} t^1 (1-t)^{2-1} = 2t(1-t)$
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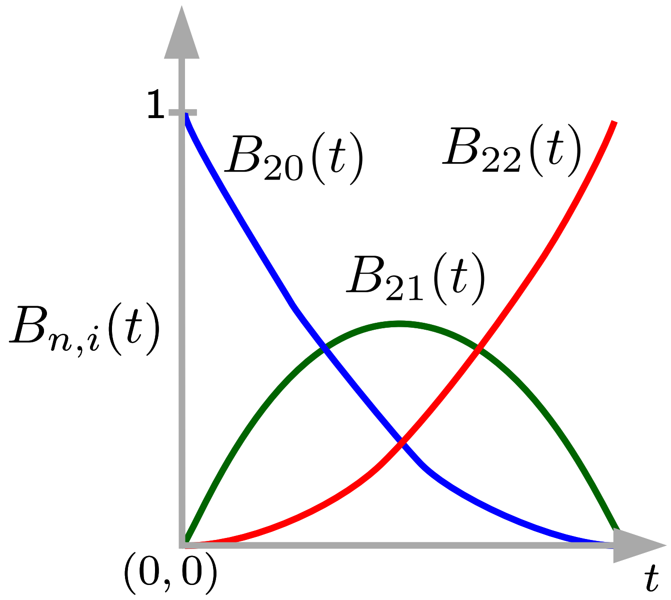
Example?

Question: Does this curve satisfy the properties in the previous slide?

Properties of Bézier curves

1. Endpoint interpolation
2. Symmetry
3. Affine invariance
4. Invariance under affine parameter transformations
5. Convex hull property
6. Pseudolocal control
7. Variation-diminishing property



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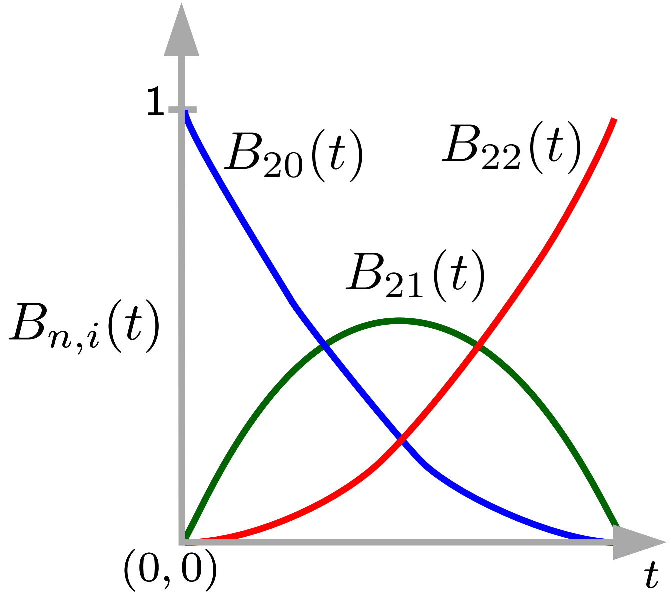

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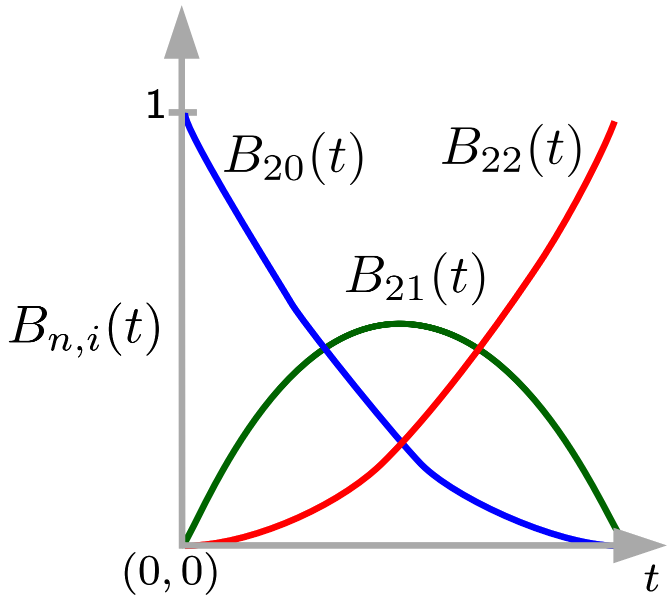
Properties of Bézier curves

3. Affine invariance

Applying an affine transformation to the curve is the same as applying the transformation to the control points
More precisely: $f(P(t)) = \sum_{i=0}^{n} f(P_i) B_{n,i}(t)$, for any affine map f, i.e., $f(v) = Av + W$



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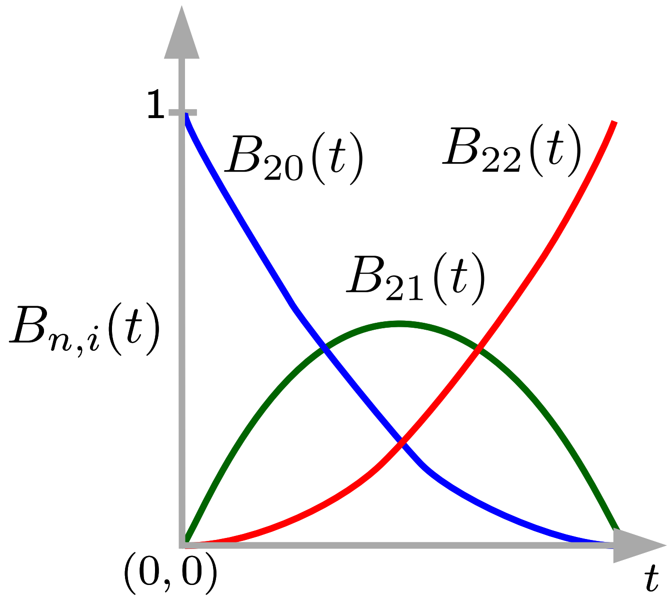
Why is that? Observe that $\sum_{i=0}^{n} B_{n,i}(t) = 1$

This follows from binomial theorem:
$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$, with $a = 1$ and $b = (1-t)$

Affine maps are precisely the maps that leave affine combinations invariant, so same applies to Bézier curves!



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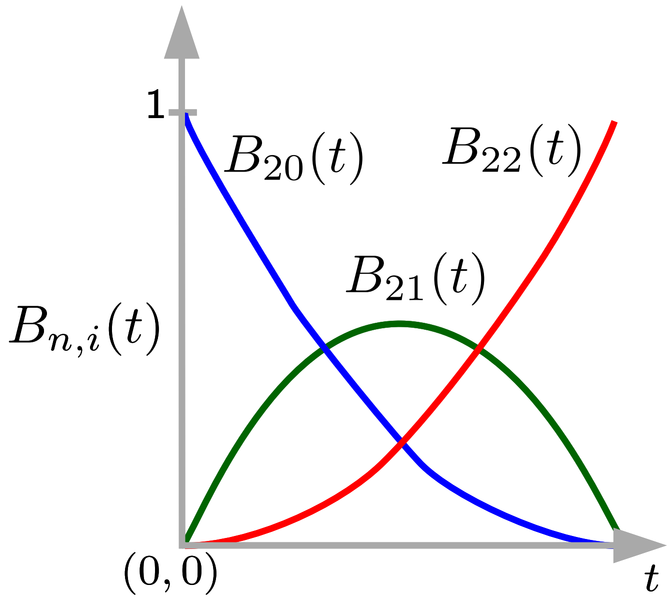

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4. Invariance under affine parameter transformations

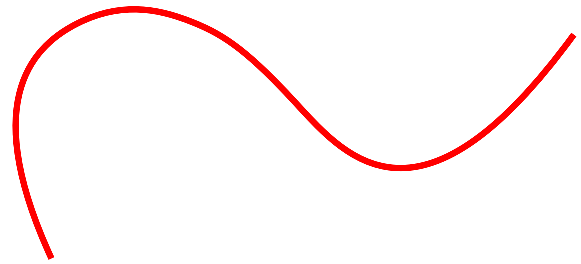
That is: $\sum_{i=0}^{n} P_i B_{n,i}(t) = \sum_{i=0}^{n} P_i B_{n,i}(\frac{u-a}{b-a})$

Practical consecuence: it is easy to have a curve defined over $[a, b]$ instead of $[0, 1]$

Properties of Bézier curves

5. Convex hull property

The curve lies inside the convex hull of the control points



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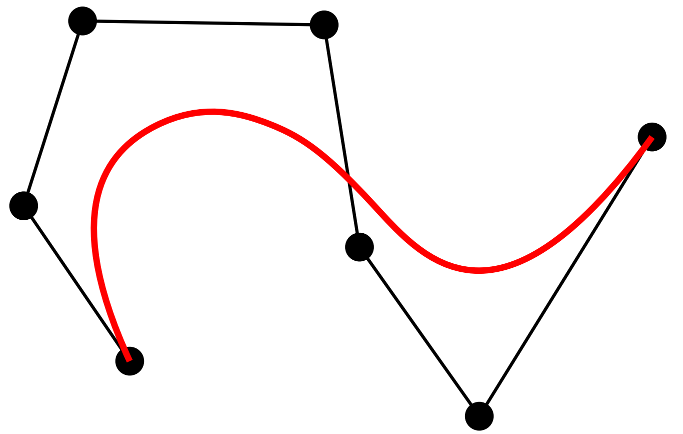

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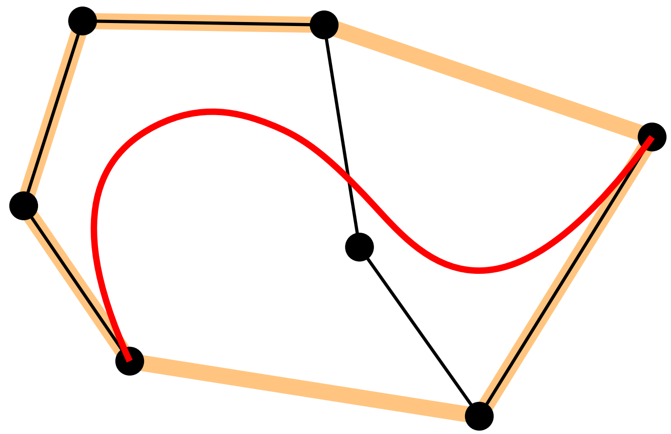
Why is this property true?



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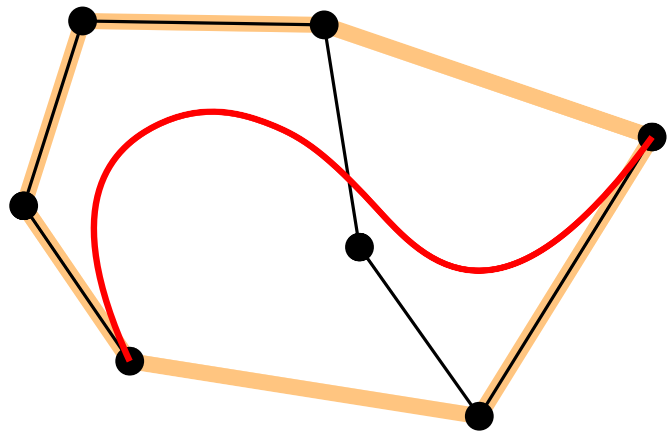

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$P(t)$ is a **convex combination** of the control points.

The convex hull of the a set of points S is **exactly** the set of all convex combinations of points in S, thus all points in the curve belong to the convex hull.

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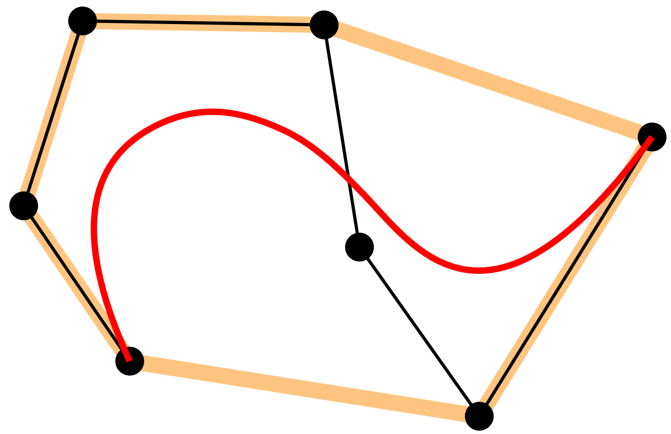

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Question: What does this say about collinear control points?

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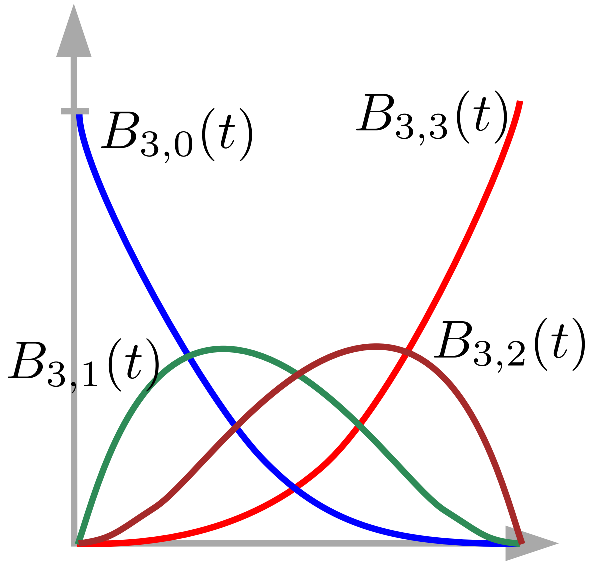
6. "Pseudolocal" control

Question: When does a control point influence the curve most?



Properties of Bézier curves

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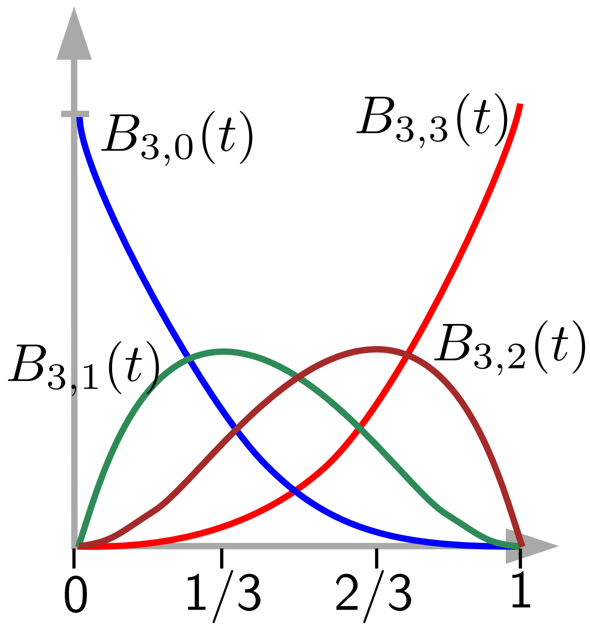

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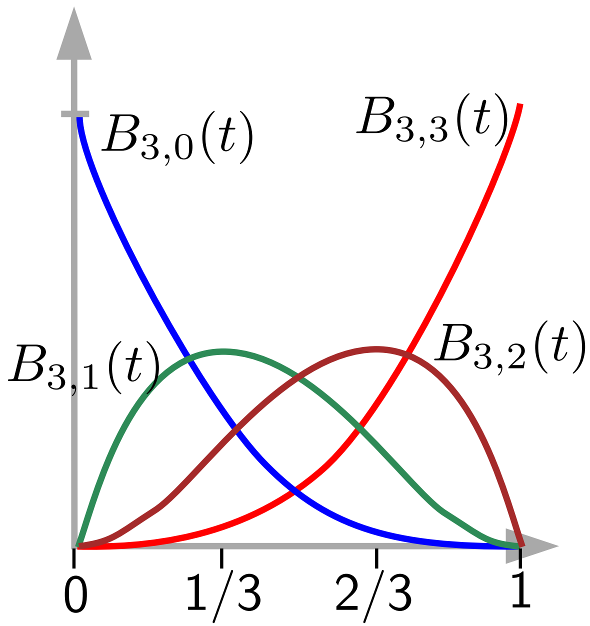

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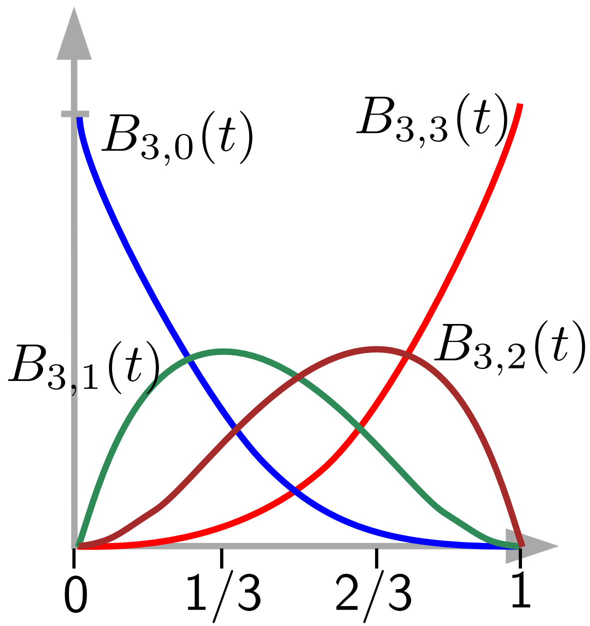


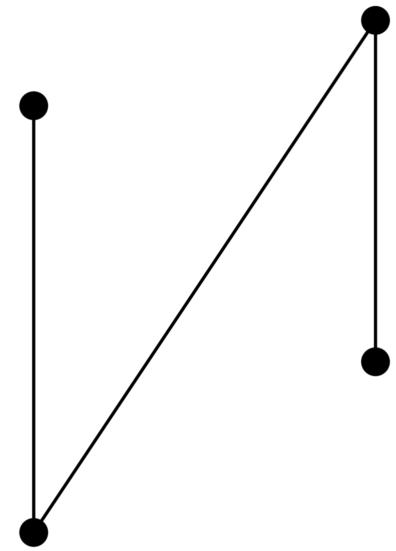
local maxima of $B_{n,i}$s

Question: What happens to $P(t)$ if P_k is moved by a vector (α, β)?

Properties of Bézier curves

7. Variation-diminishing property

The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygonal line



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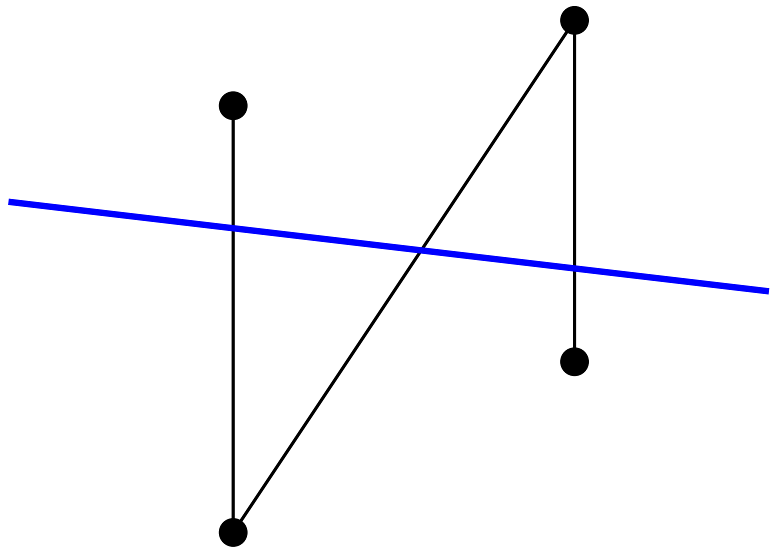

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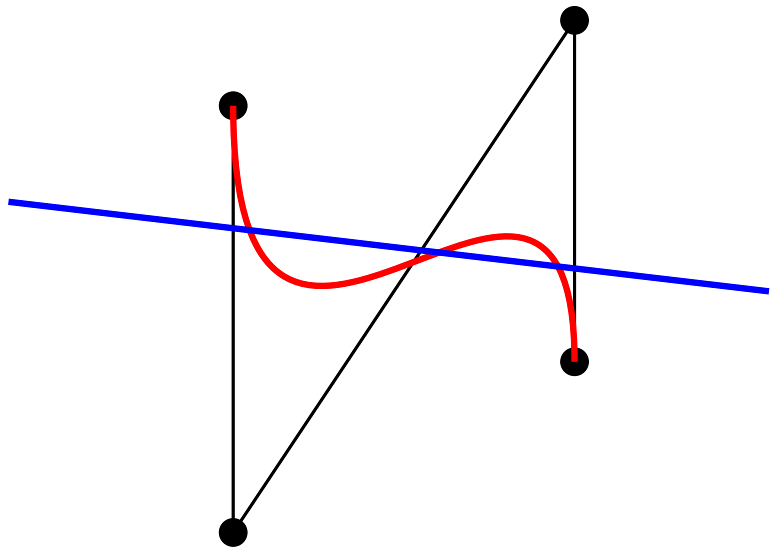

Properties of Bézier curves

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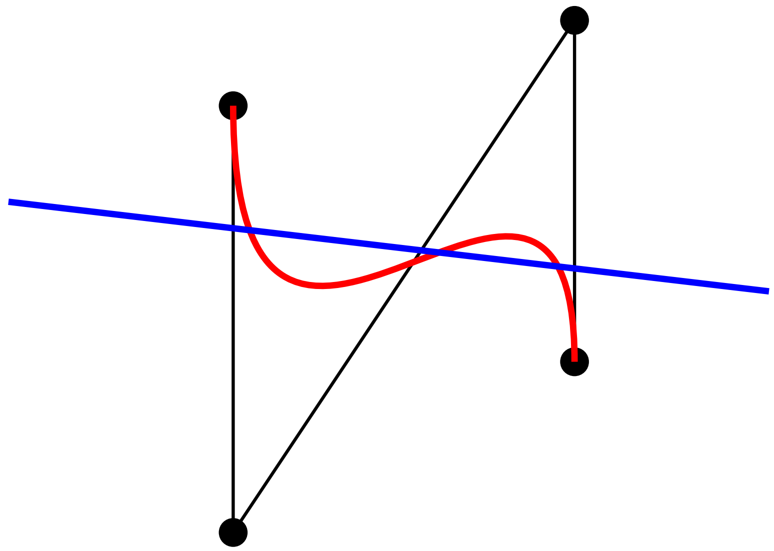
The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygonal line

This means that, to some extent, the curve imitates the shape and is not “rougher” than the corresponding control polygon,

One consequence: if the control polygon is convex, then the Bézier curve is also convex

Proof? Later, after looking at **degree elevation**



Connecting two curves

- In practice, one should avoid high-degree Bézier curves
- Better use many low-degree curves (they give local control)
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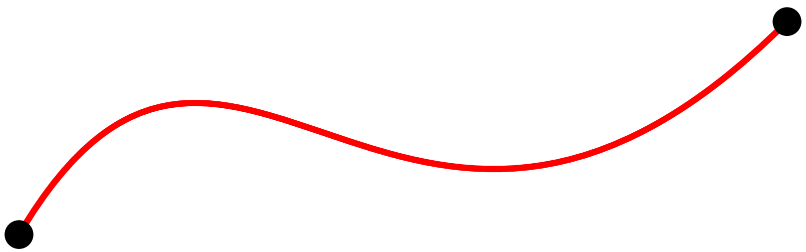

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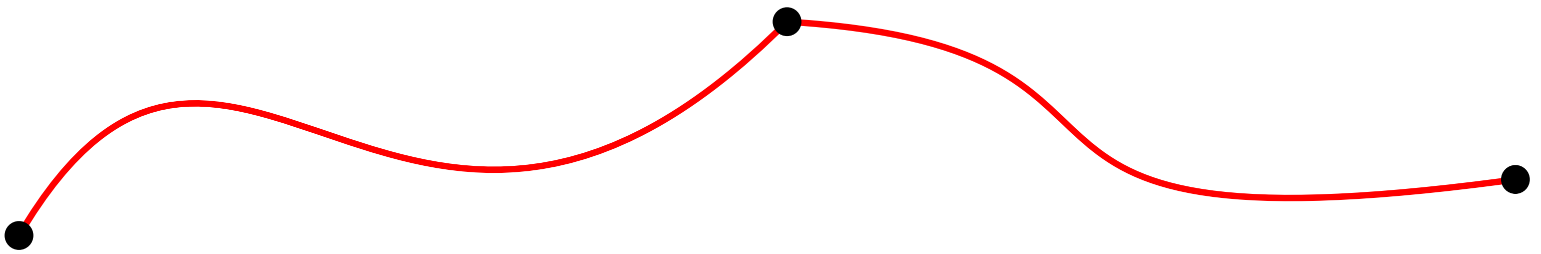

COMPOSITE BÉZIER CURVES

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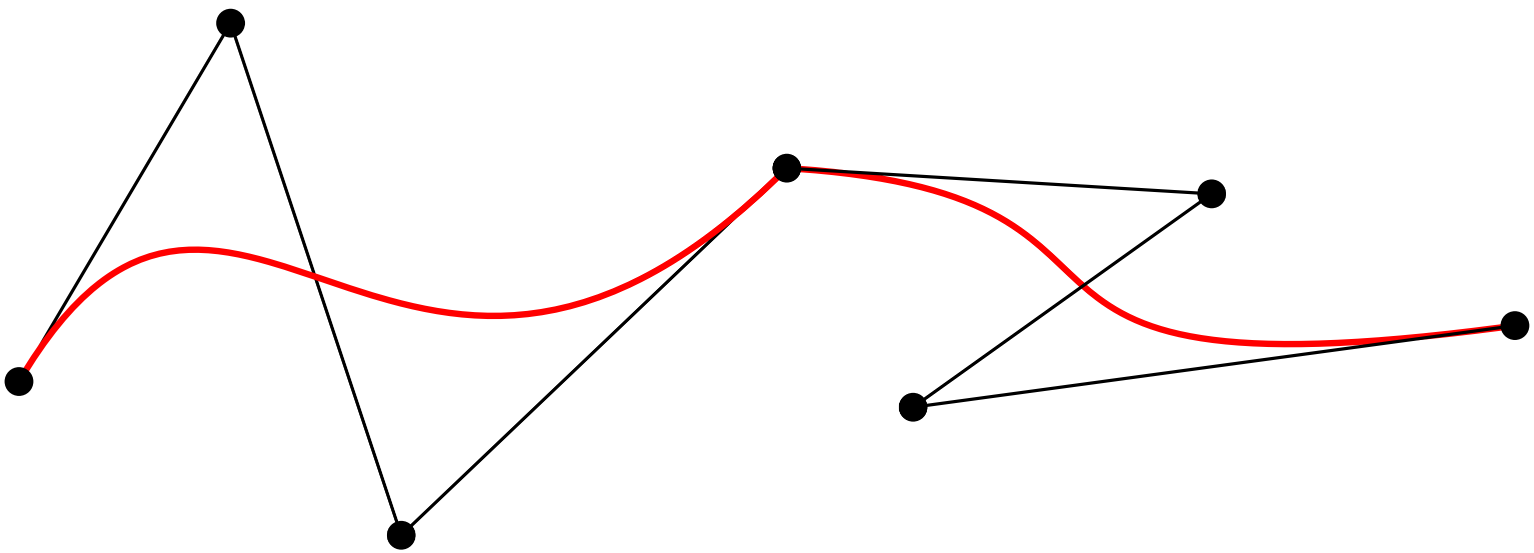

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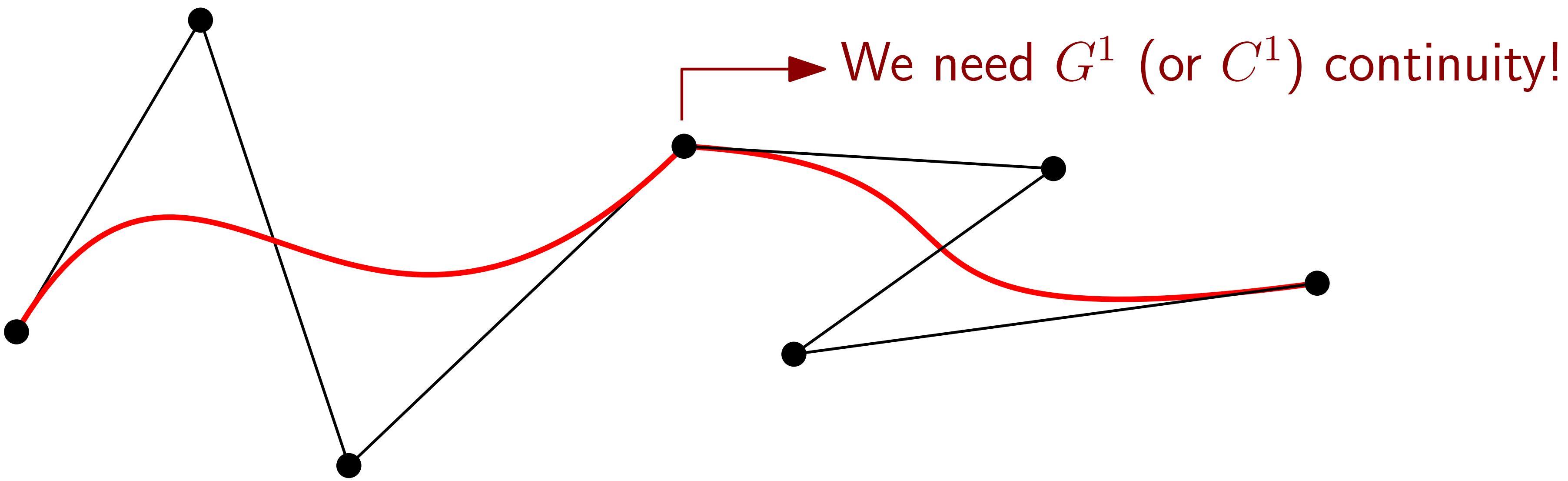

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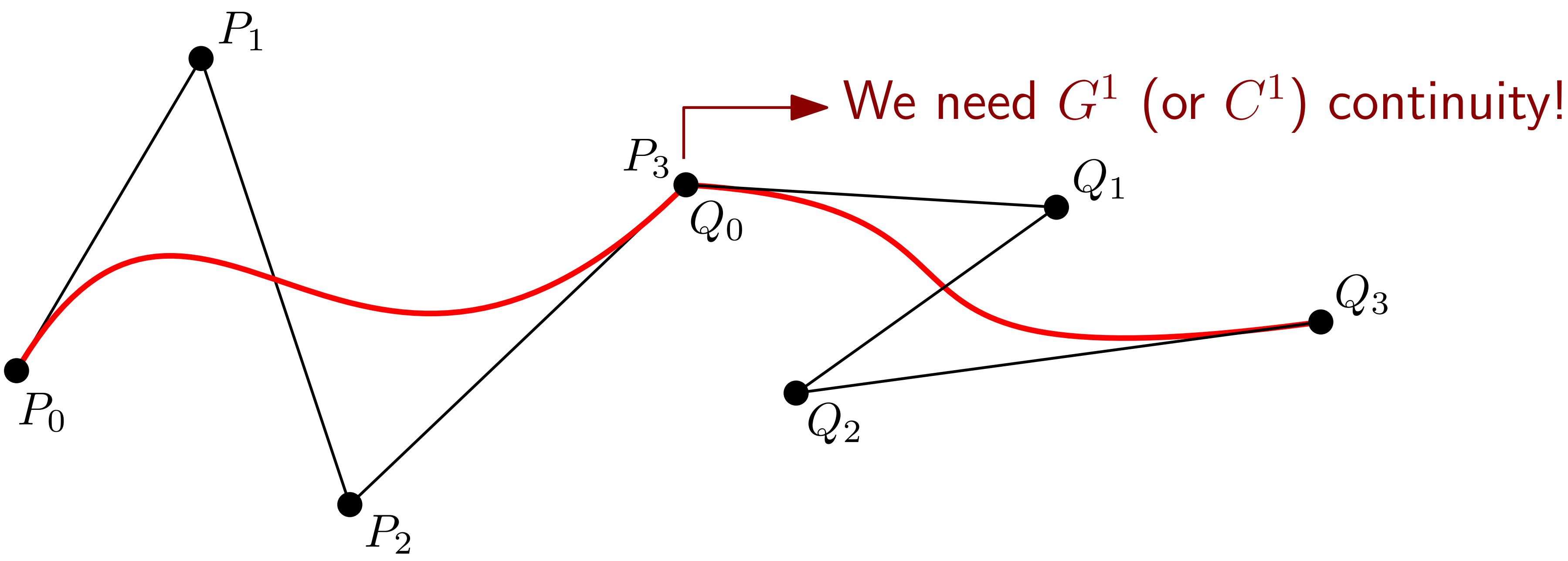

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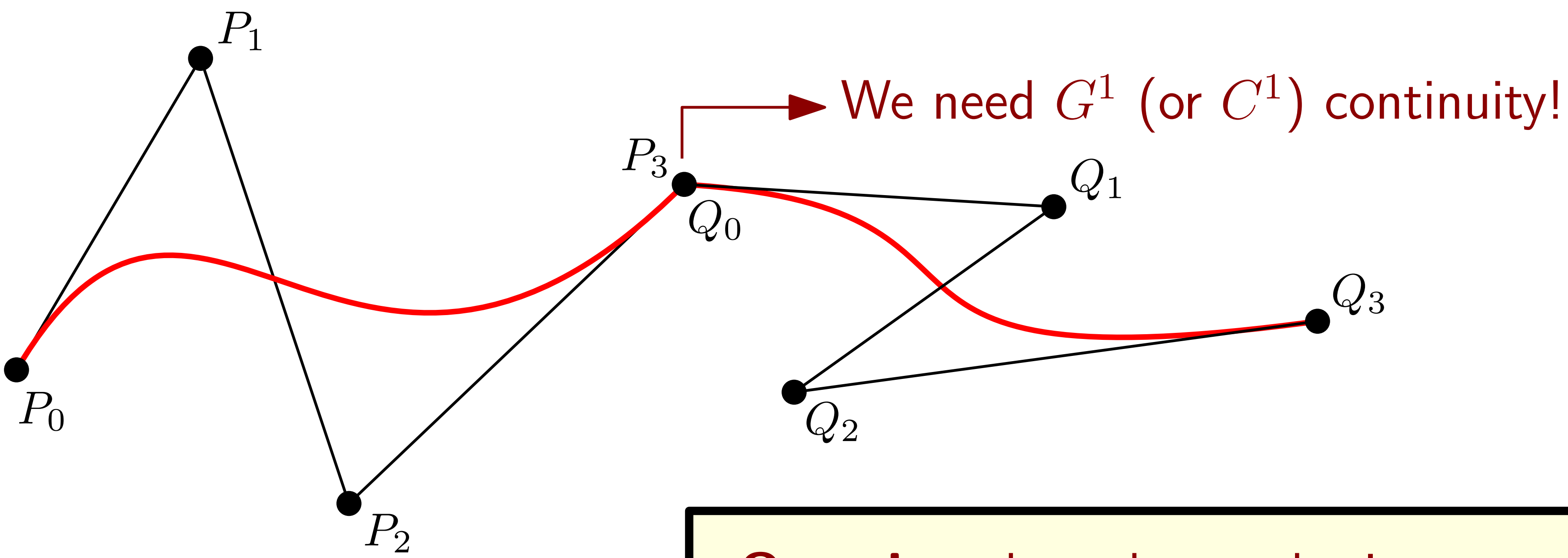
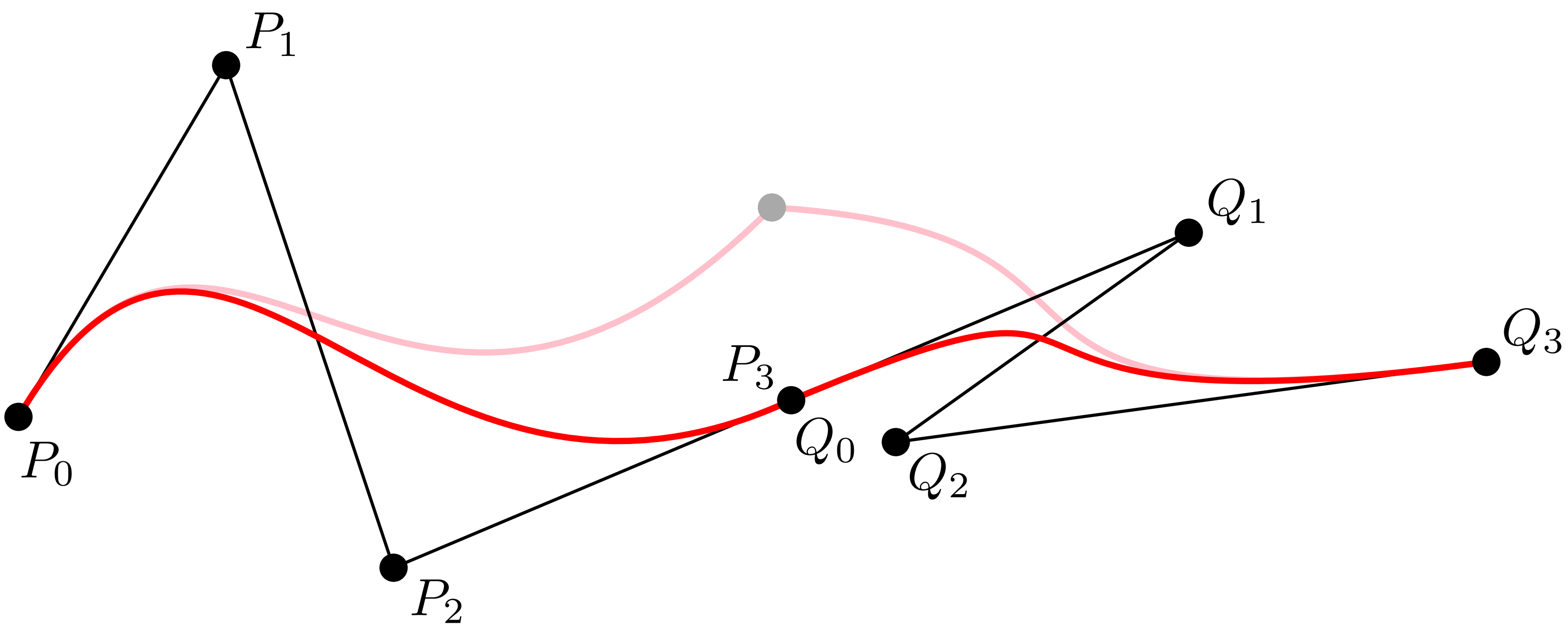


Question: how do we obtain a smooth joint?

Connecting two curves



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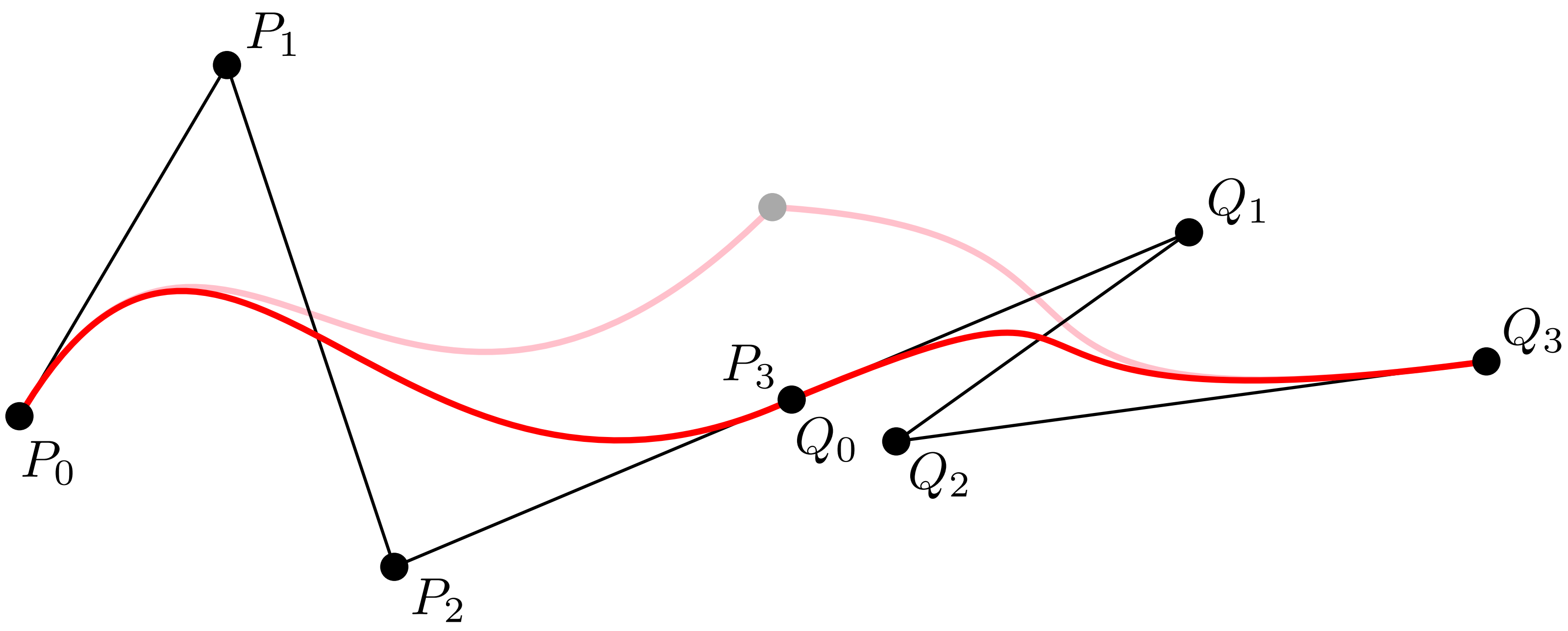

In general, for a curve P with $(n+1)$ control points and Q with $(m+1)$, the C^1-continuity condition is

$$Q_0 = P_n = \frac{m}{m+n}Q_1 + \frac{n}{m+n}P_{n-1}$$

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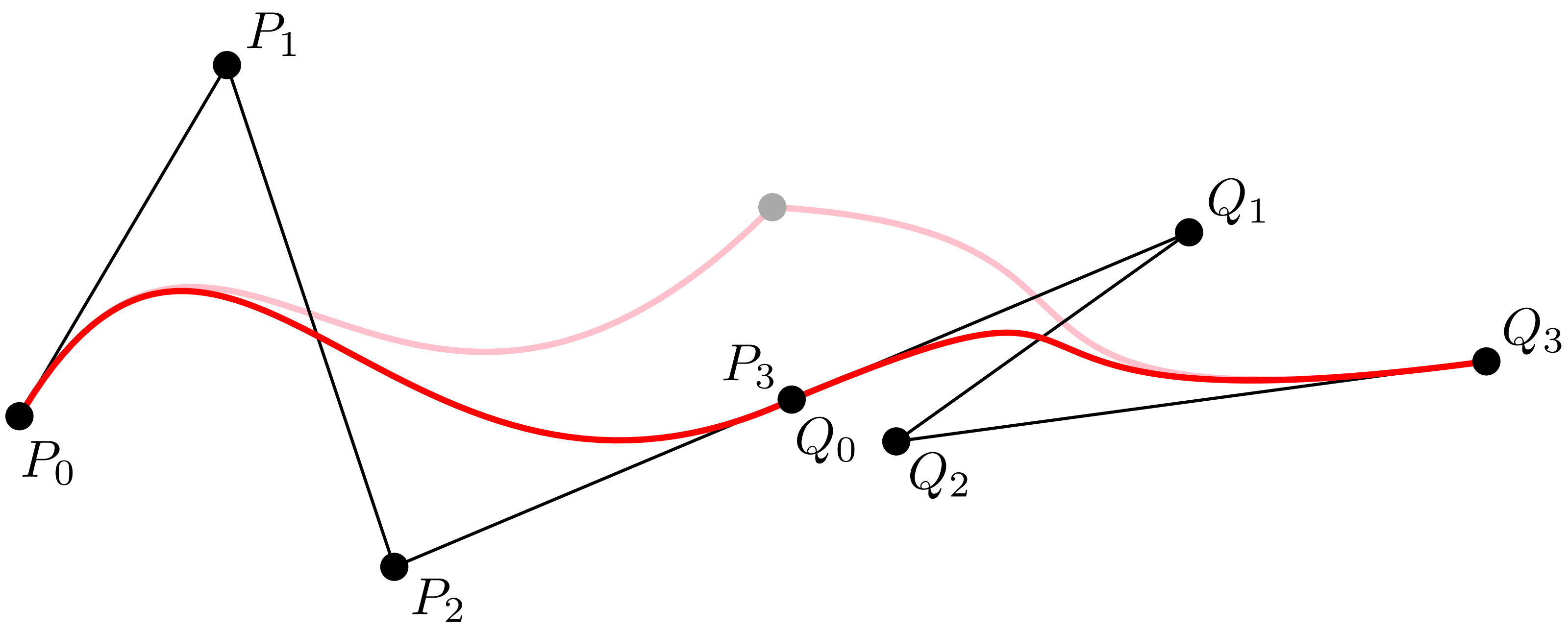

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Question: how can you obtain higher-degree continuity?

EXAMPLE: FONT DESIGN

Guess how are the fonts you use designed?

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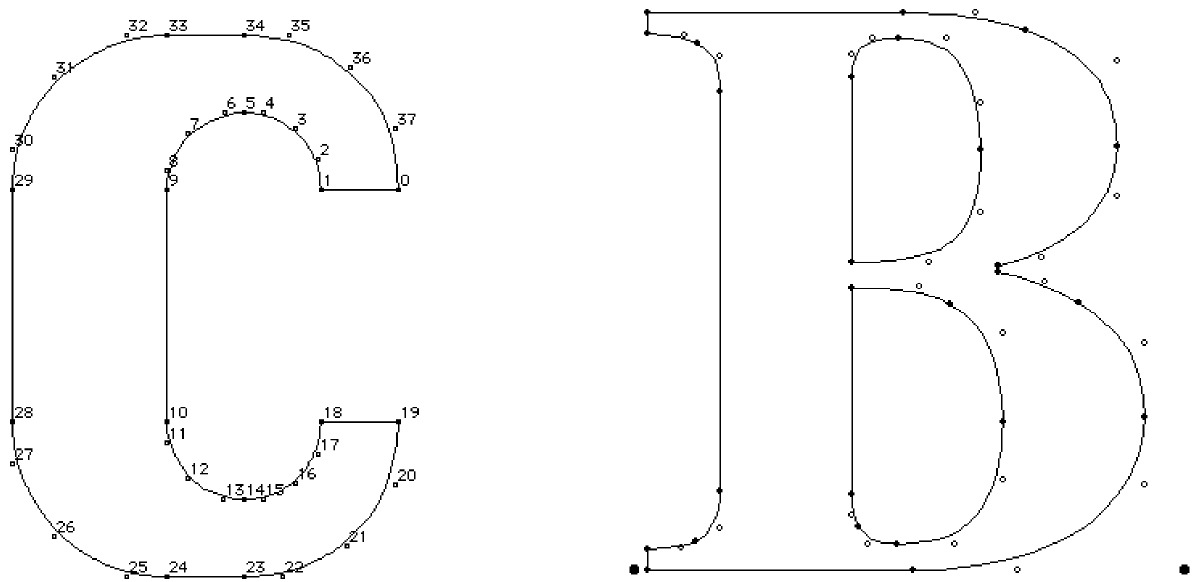

Glyphs of two characters in a true-type font

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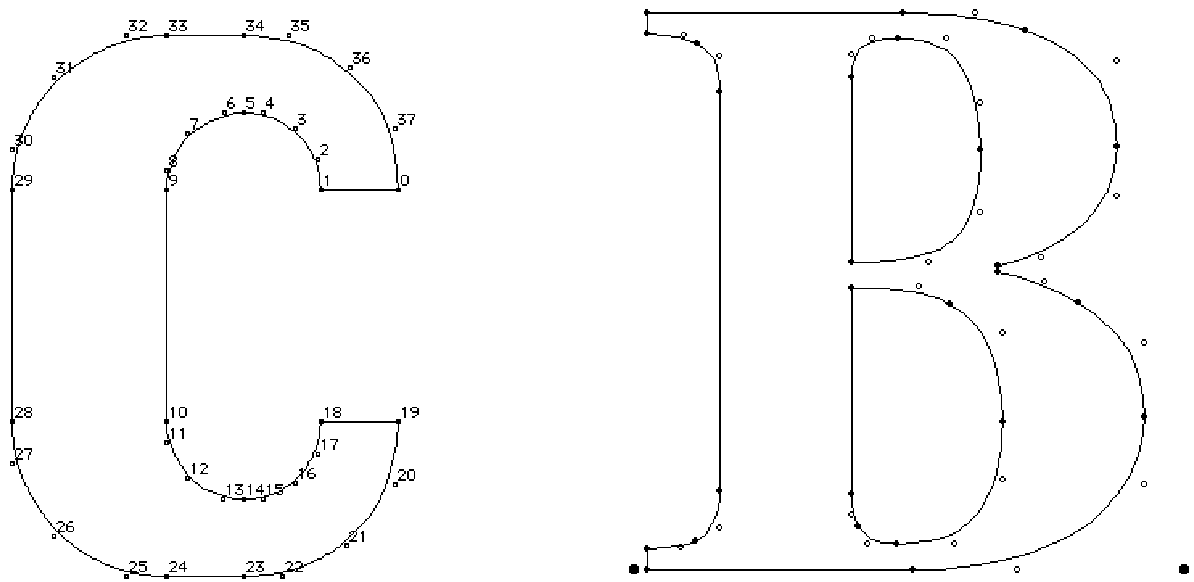
Question: Can you convert between these types of fonts?



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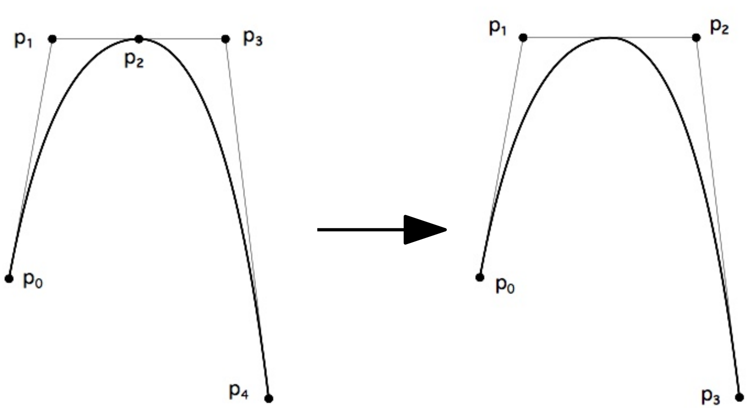

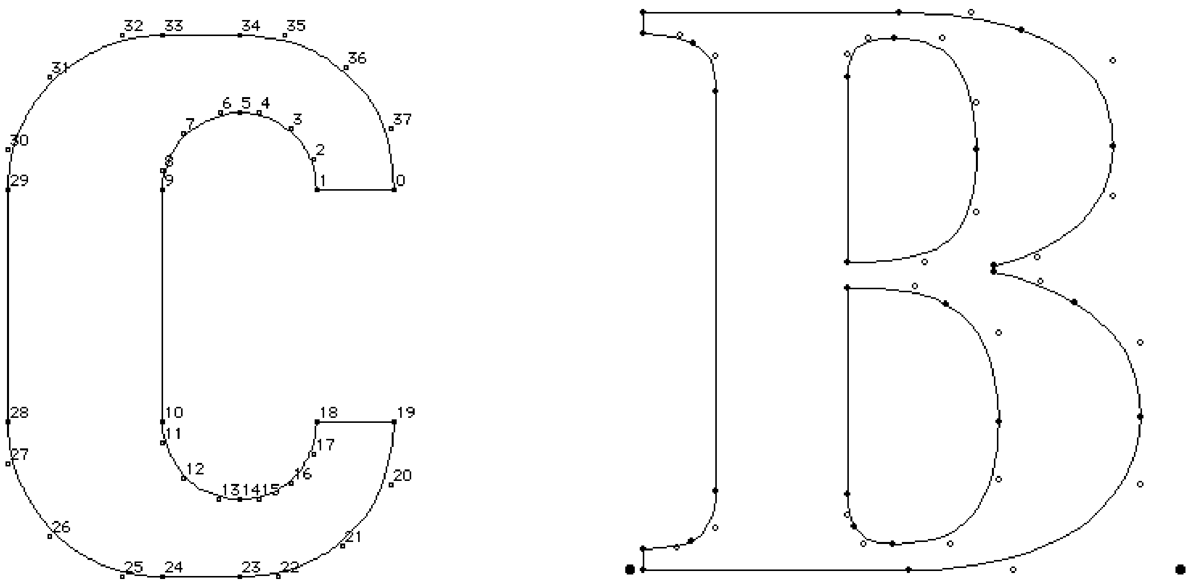

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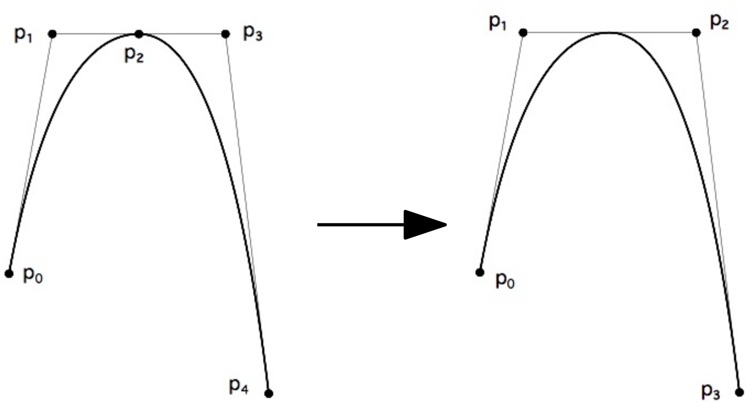

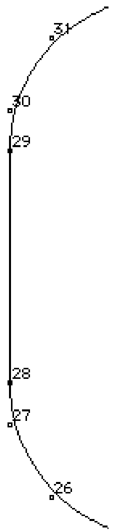

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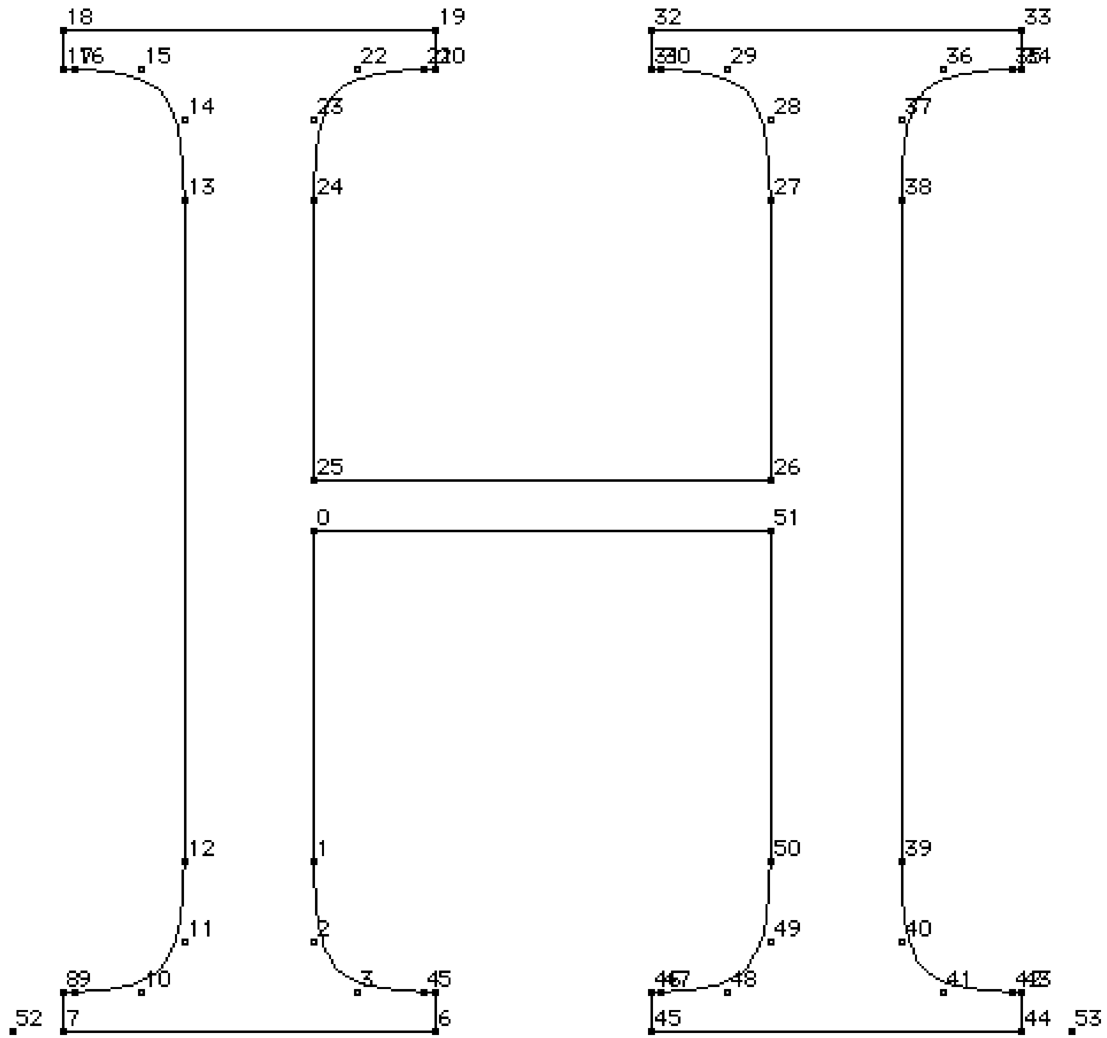

Difference between points on-curve and off-curve

An alternative approach to Bézier curves

De Casteljau (Citroën) followed a different approach based on **repeated linear interpolation**

BÉZIER CURVES AS LINEAR INTERPOLATION

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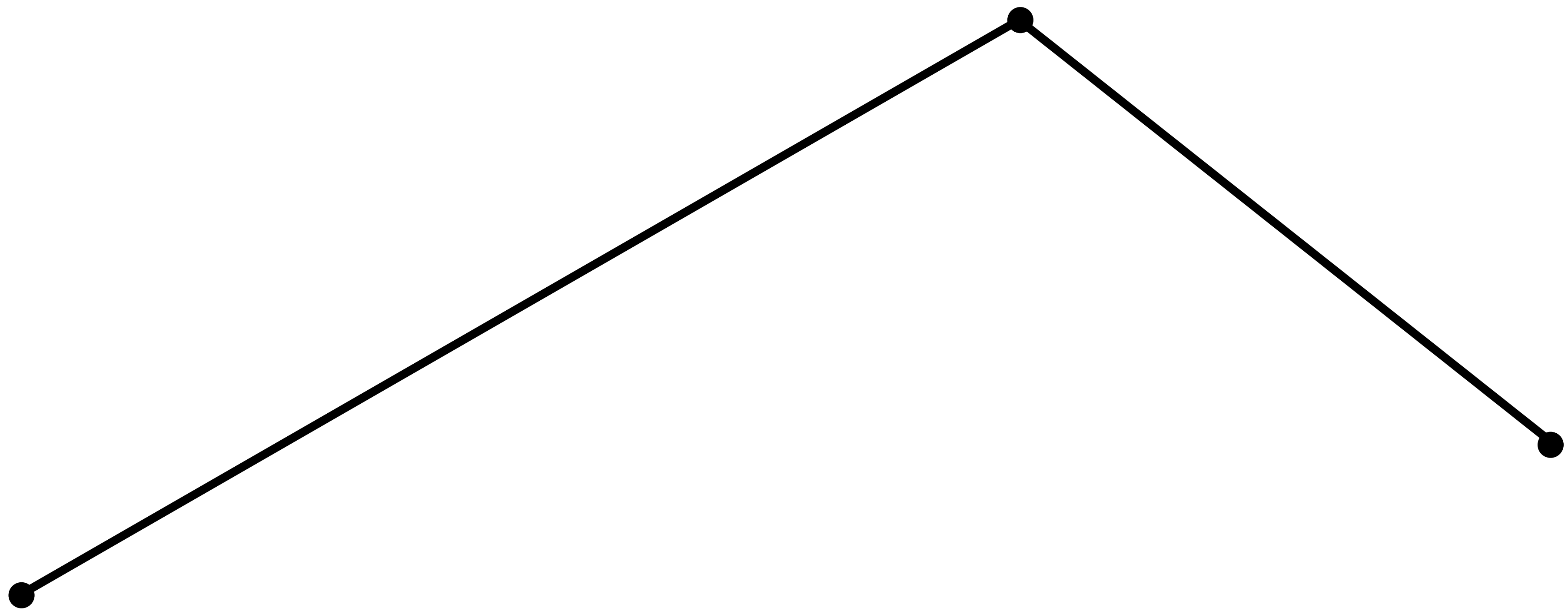

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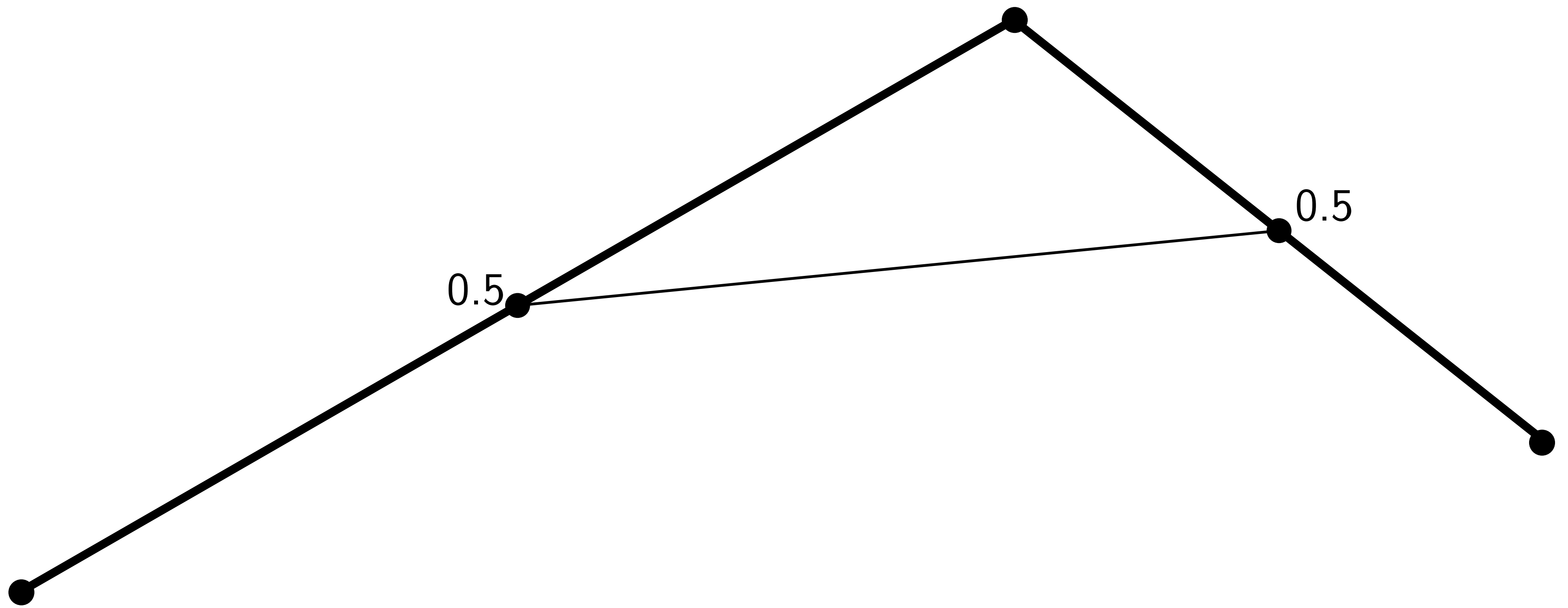

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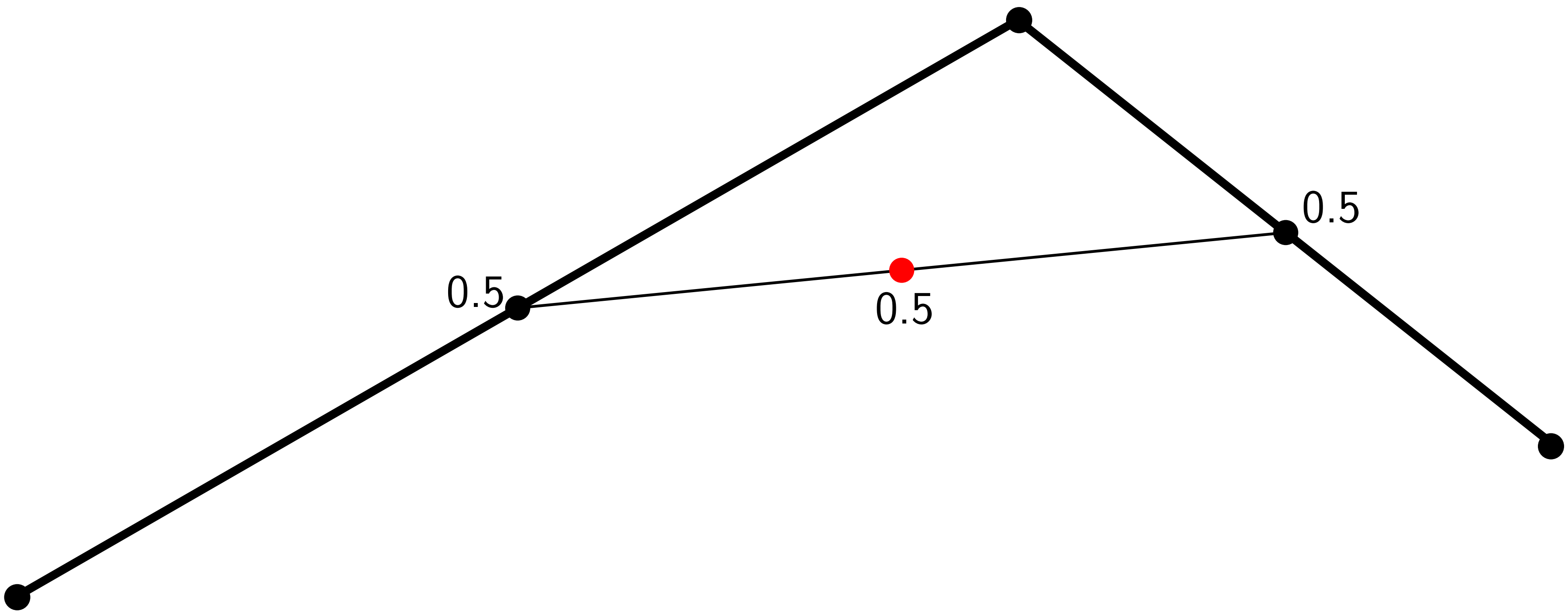

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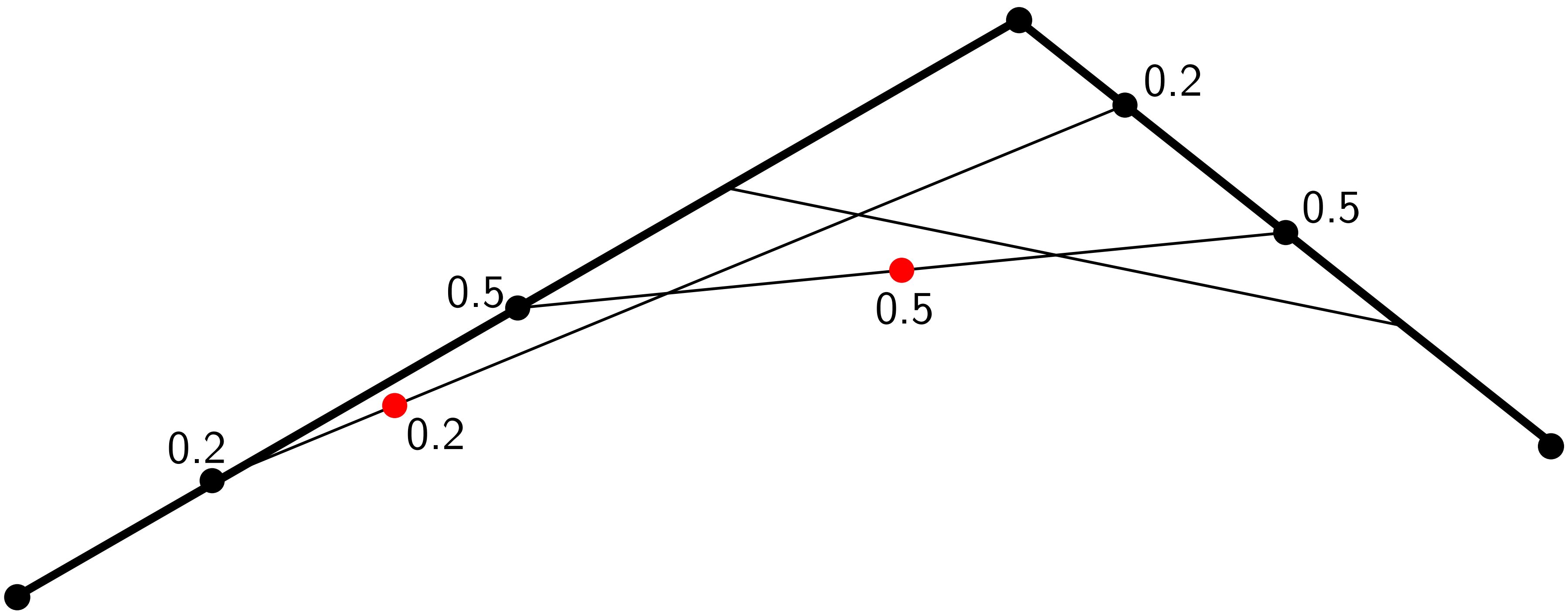

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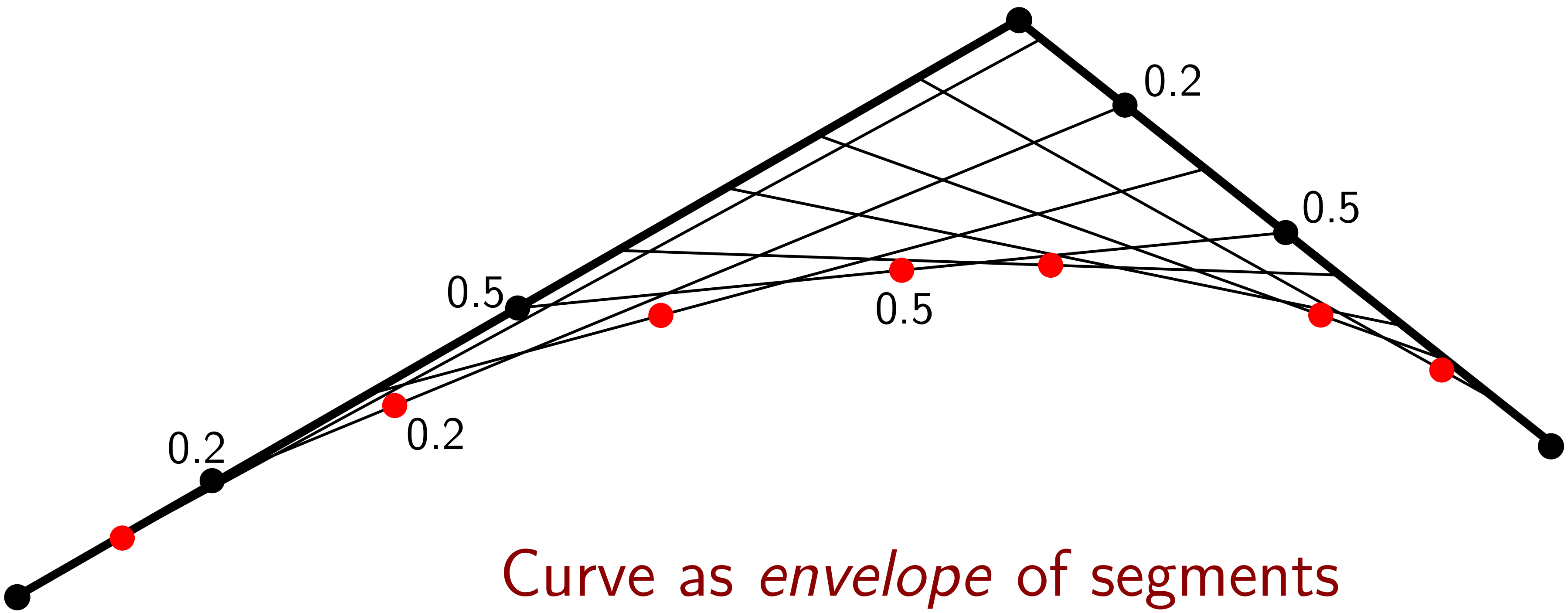

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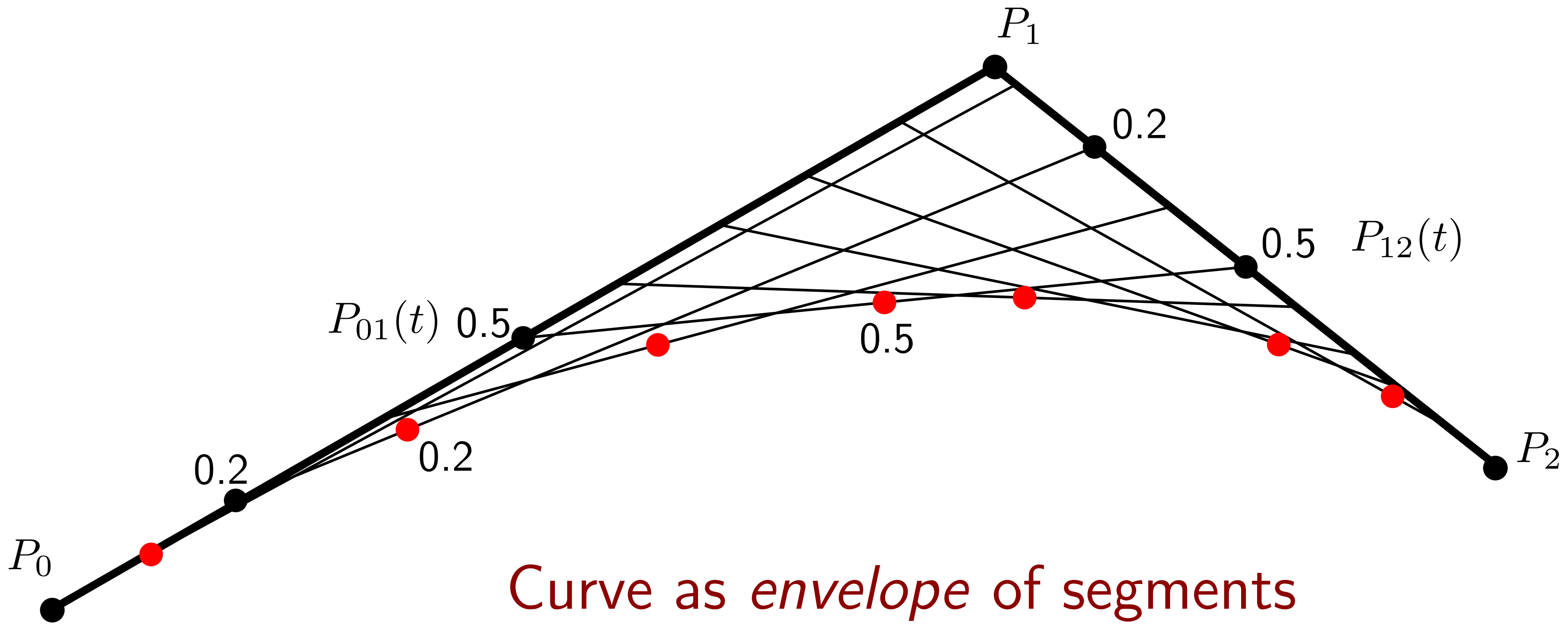

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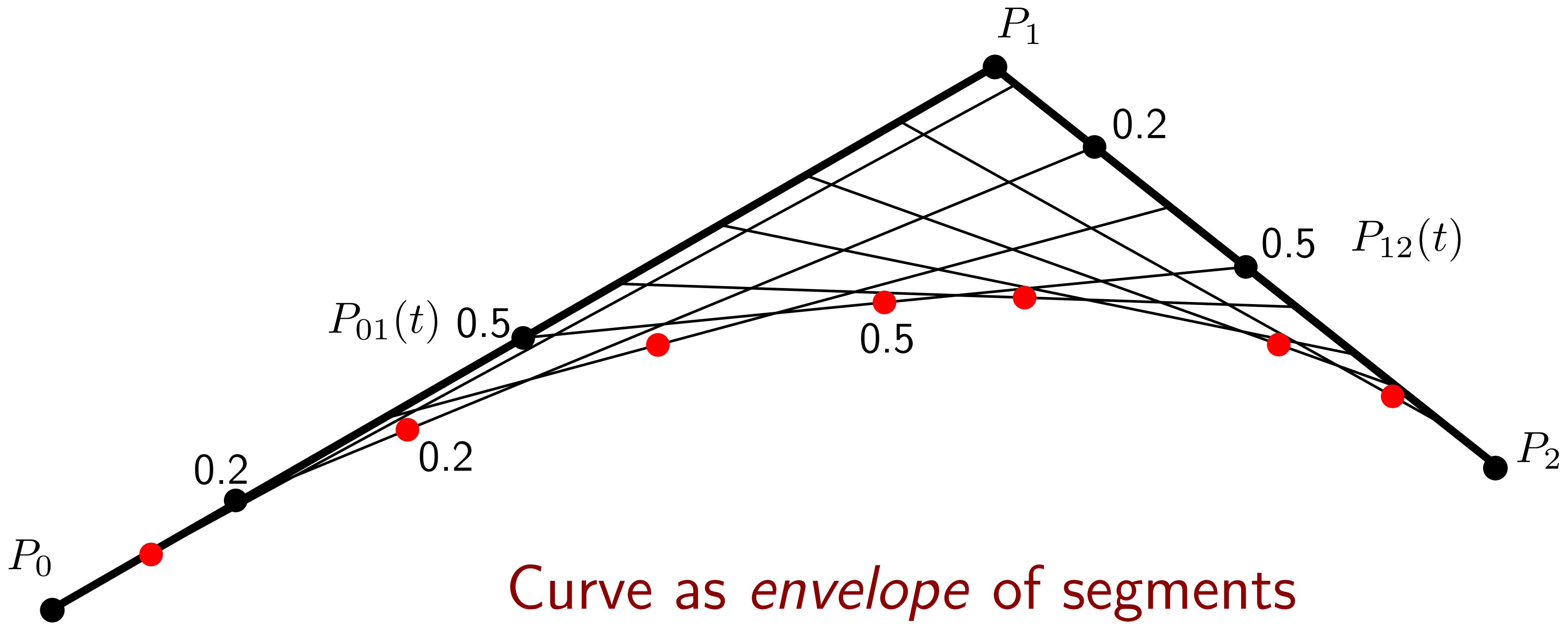

Curve as *envelope* of segments

Question: What is the expression of this envelope, as a function of t?

De Casteljau's algorithm

For 3 points P_0, P_1, P_2, and $0 \leq t \leq 1$, we have:

$$\begin{aligned} P_{01}(t) &= (1-t)P_0 + tP_1 \\ P_{12}(t) &= (1-t)P_1 + tP_2 \\ P(t) = P_{012}(t) &= (1-t)P_{01}(t) + tP_{12}(t) \end{aligned}$$

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Recursive / geometric construction method

The final curve is given by $P(t) = P_{0\ldots n}(t)$

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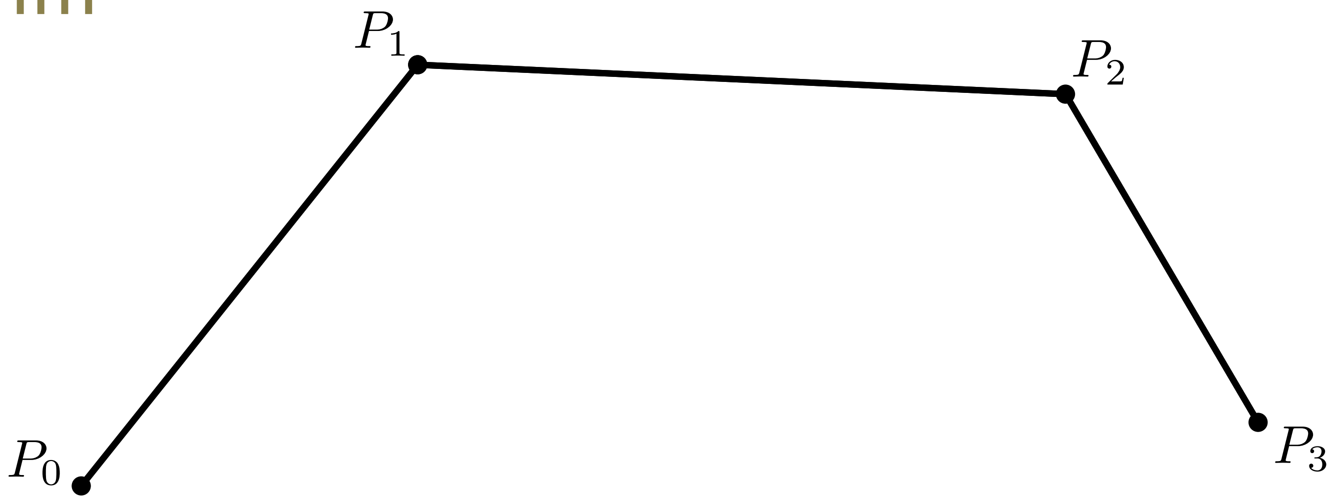
Example for $n = 3$ and $t = 1/2$

$$P(t) = P_{0123}(t)$$



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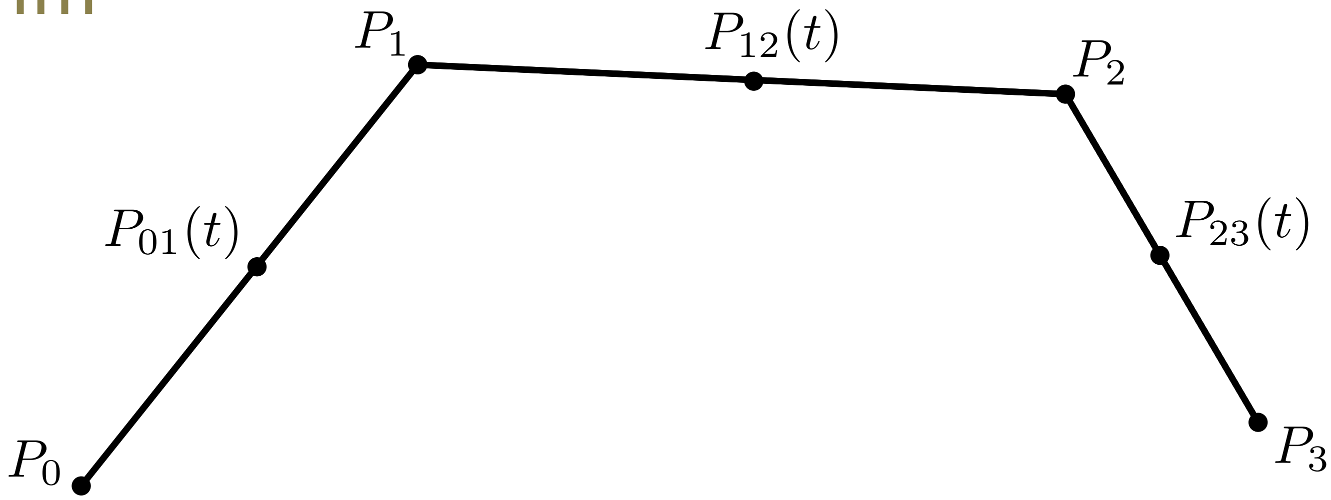

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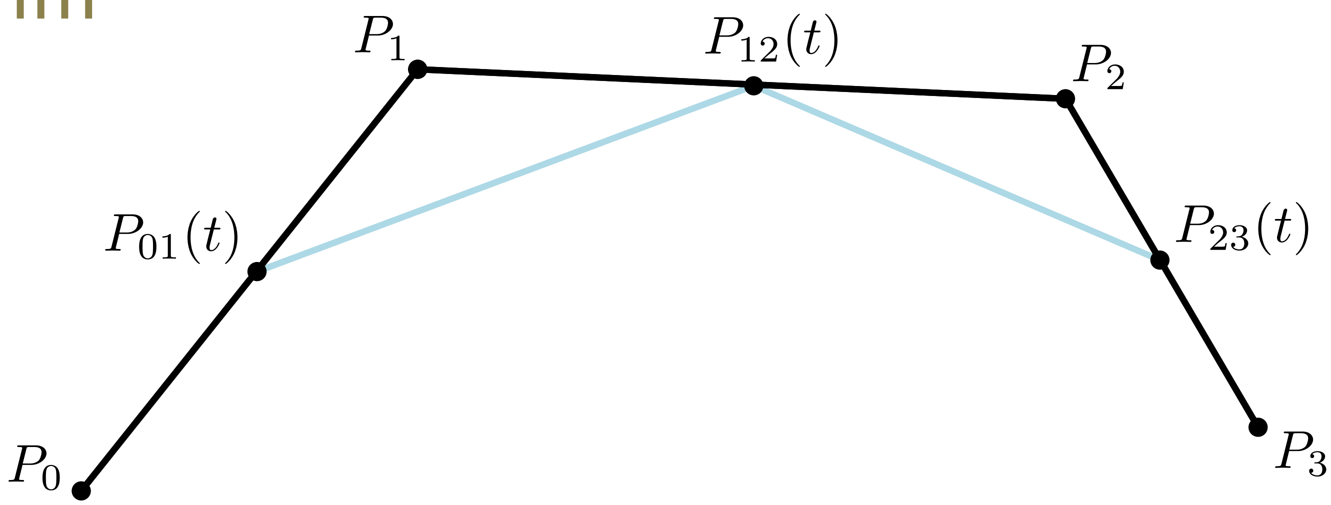

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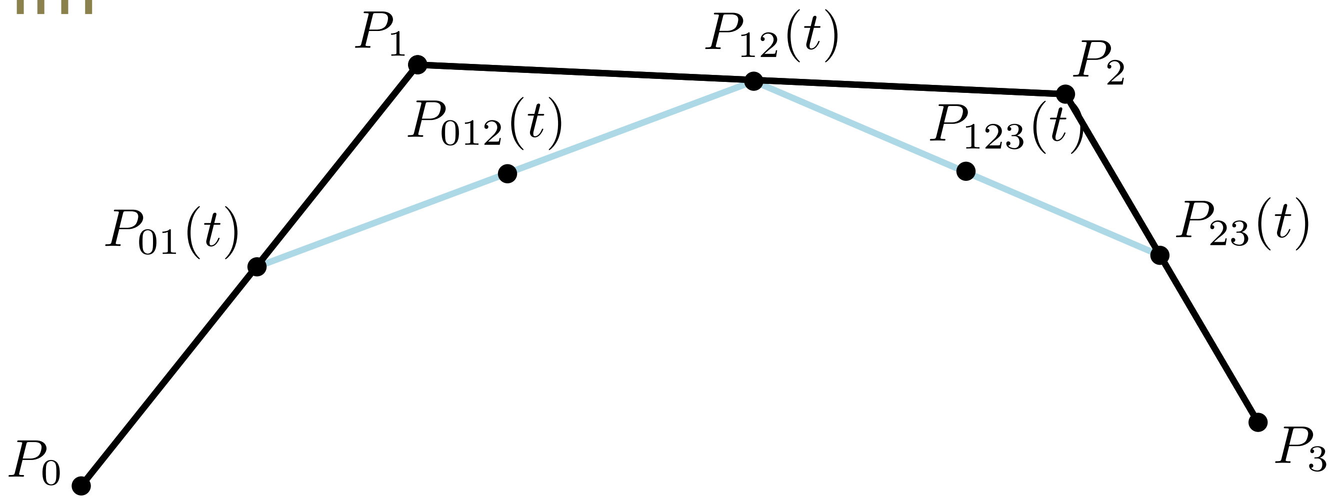

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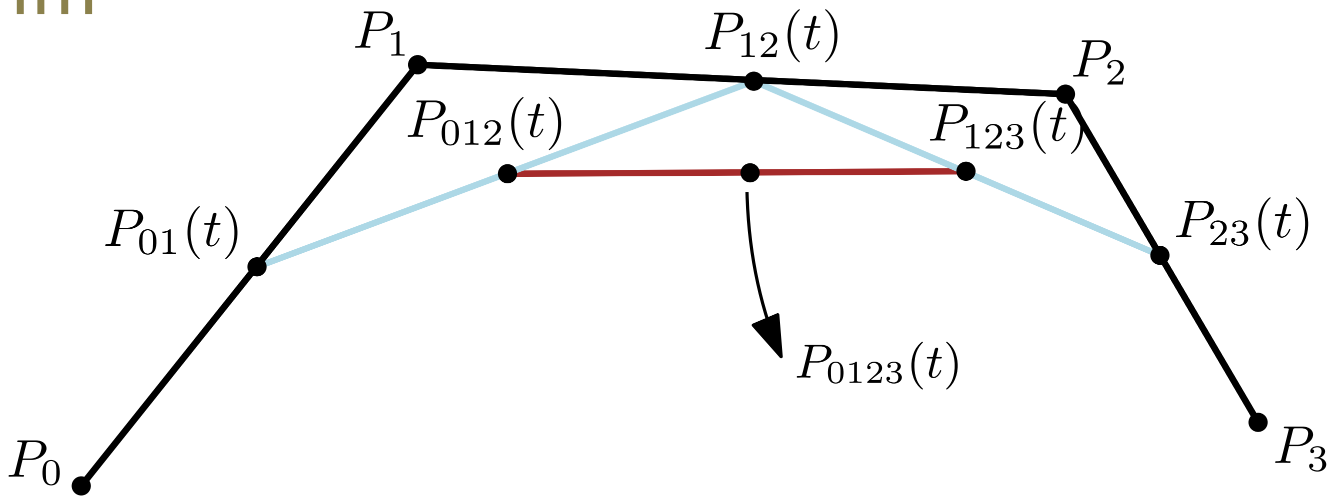

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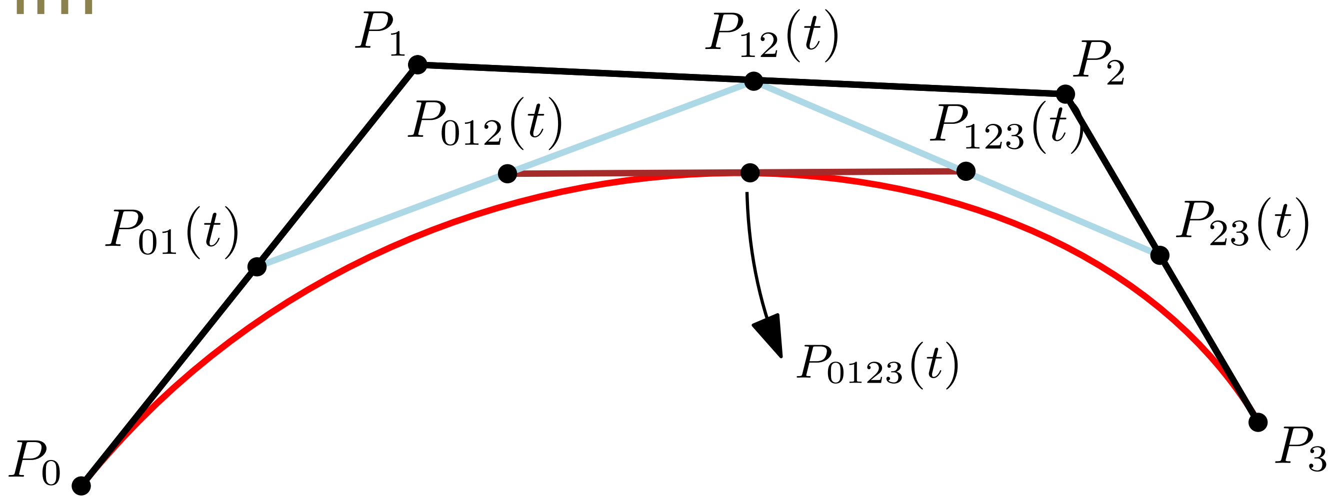

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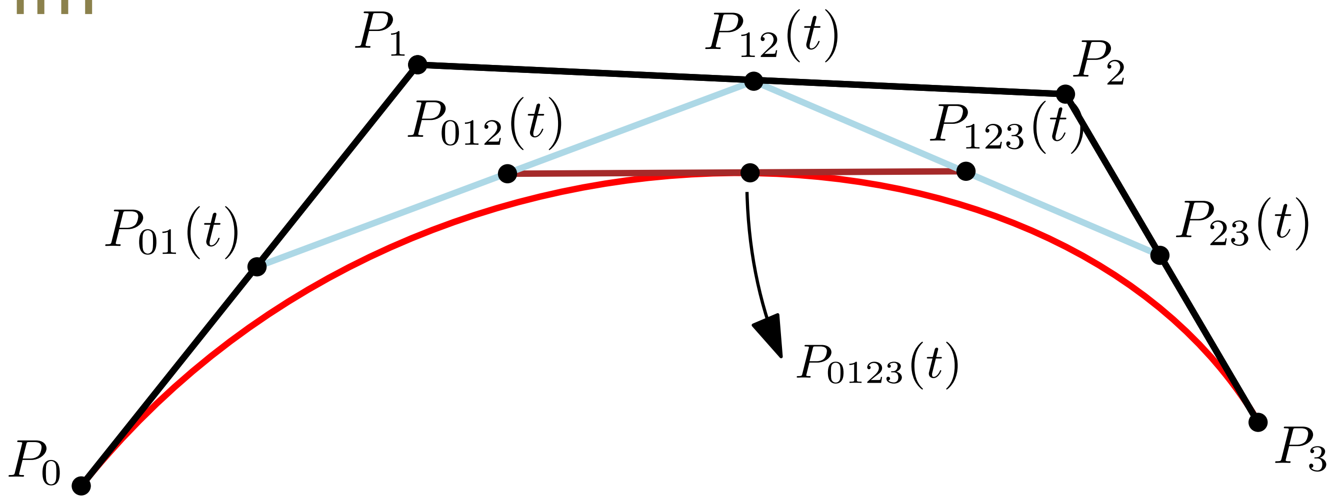

Implementation of the algorithm

How to evaluate $P(1/2)$?

BÉZIER CURVES AS LINEAR INTERPOLATION

De Casteljau's algorithm

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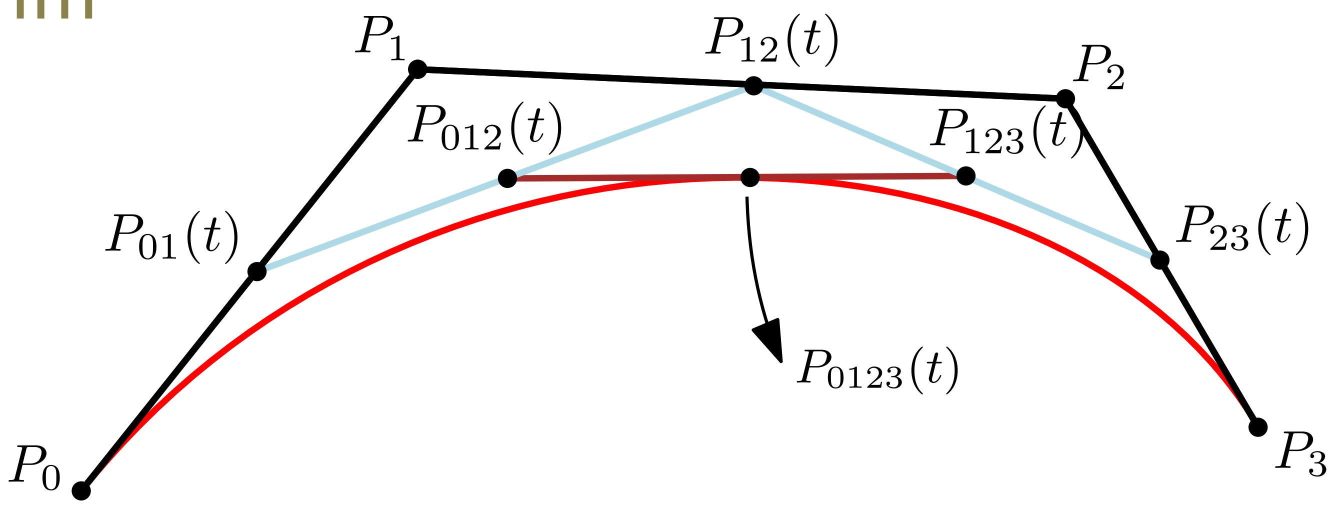

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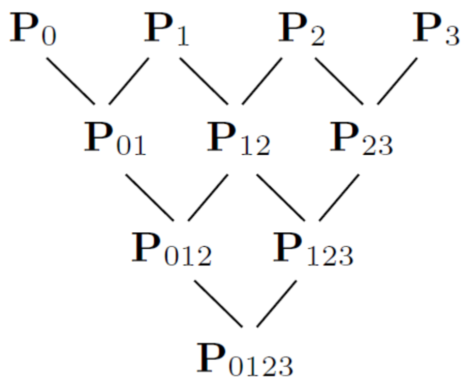
How to evaluate $P(1/2)$?

Step	Points constructed	#points
1	$\mathbf{P}_{01}\ \mathbf{P}_{12}\ \mathbf{P}_{23} \ldots \mathbf{P}_{n-1,n}$	n
2	$\mathbf{P}_{012}\ \mathbf{P}_{123}\ \mathbf{P}_{234} \ldots \mathbf{P}_{n-2,n-1,n}$	$n-1$
3	$\mathbf{P}_{0123}\ \mathbf{P}_{1234}\ \mathbf{P}_{2345} \ldots \mathbf{P}_{n-3,n-2,n-1,n}$	$n-2$
$\vdots$	$\vdots$	$\vdots$
n	$\mathbf{P}_{0123\ldots n}$	



BÉZIER CURVES AS LINEAR INTERPOLATION

De Casteljau's algorithm

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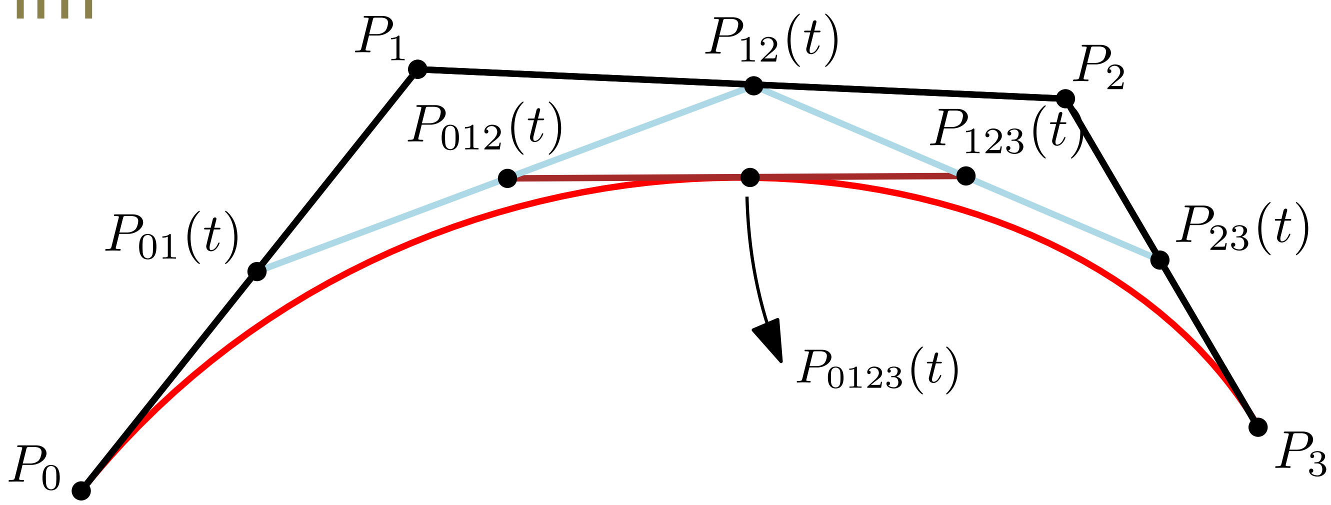

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$\vdots$	$\vdots$	$\vdots$
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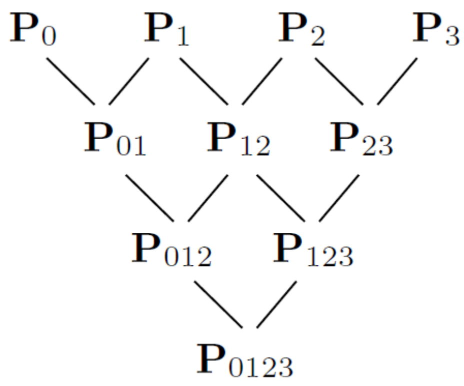
How many points computated in total?



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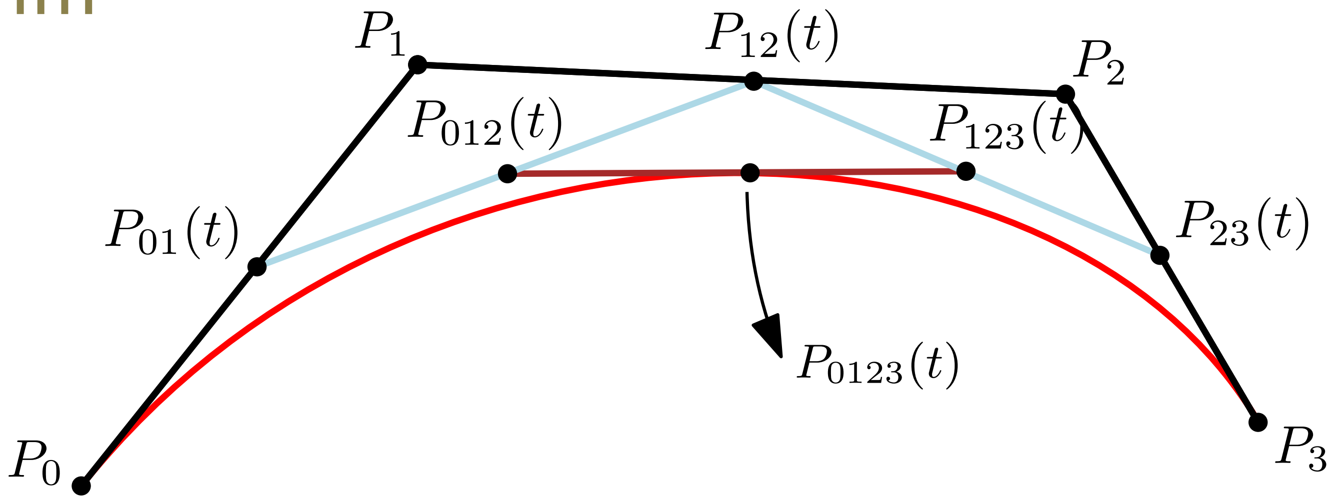

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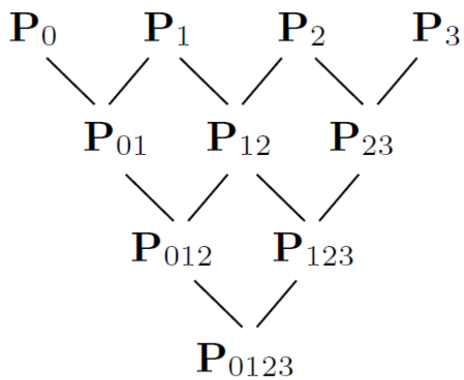
How to evaluate $P(1/2)$?

Step	Points constructed	#points
1	$\mathbf{P}_{01}\ \mathbf{P}_{12}\ \mathbf{P}_{23} \ldots \mathbf{P}_{n-1,n}$	n
2	$\mathbf{P}_{012}\ \mathbf{P}_{123}\ \mathbf{P}_{234} \ldots \mathbf{P}_{n-2,n-1,n}$	$n-1$
3	$\mathbf{P}_{0123}\ \mathbf{P}_{1234}\ \mathbf{P}_{2345} \ldots \mathbf{P}_{n-3,n-2,n-1,n}$	$n-2$
$\vdots$	$\vdots$	$\vdots$
n	$\mathbf{P}_{0123\ldots n}$	



How many points computated in total?

$$n + (n-1) + (n-2) + \cdots + 2 + 1 = n(n+1)/2$$

De Casteljau's algorithm

Note: to generate one point on the curve, $\approx n^2/2$ computations is quite a lot...

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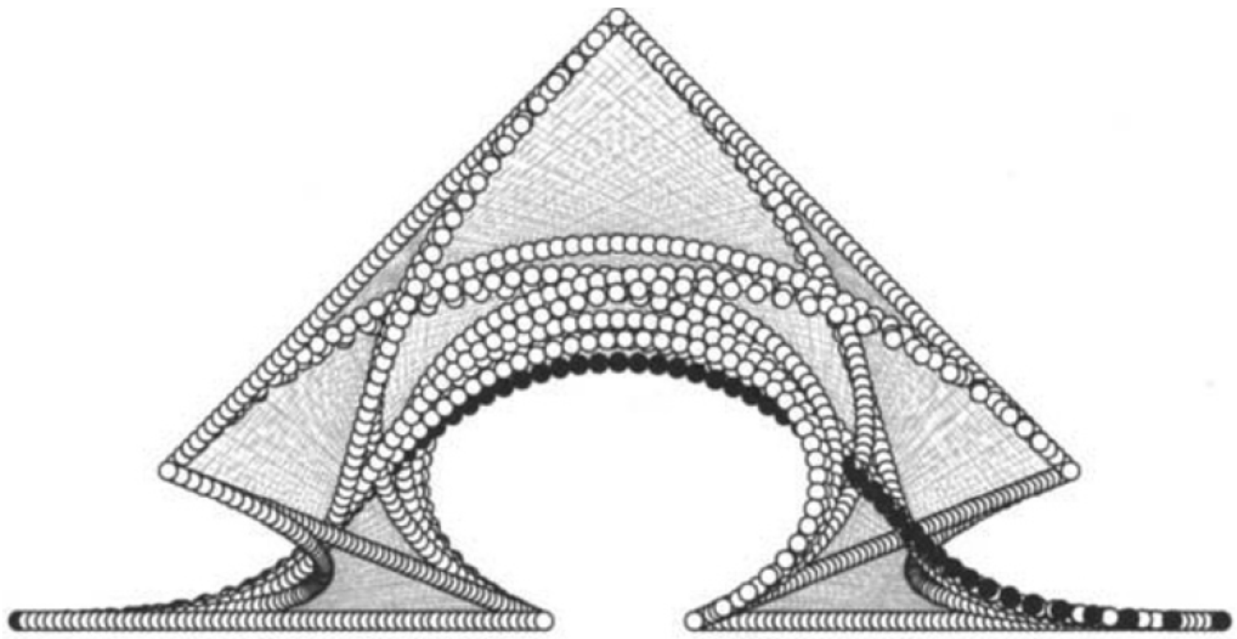

Figure 4.3 The de Casteljau algorithm: 60 points are computed on a degree six curve; all intermedediate points $\mathbf{b}_i^r$ are shown.

Figure from book by Farin (page 47)

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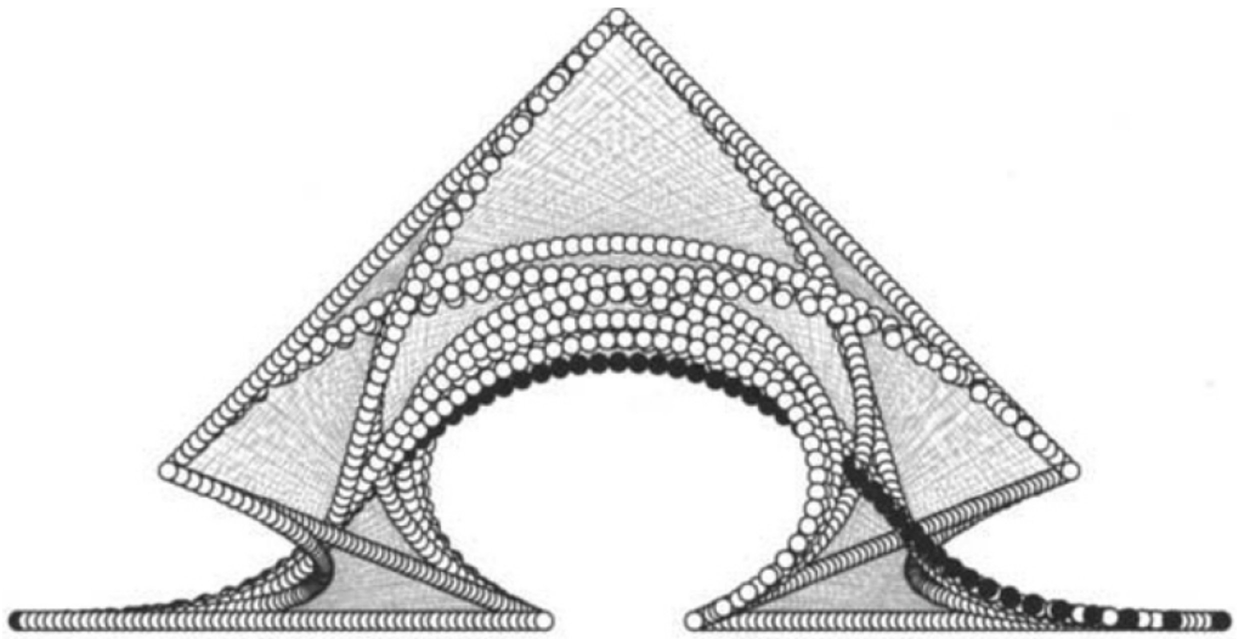

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Question for later: Is the computation based on Bernstein polynomials faster?

BÉZIER CURVES AS LINEAR INTERPOLATION

Using De Casteljau's to subdivide a curve

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(We need this when we need more flexibility to design the curve)

Goal: increase number of points, but preserve shape of curve

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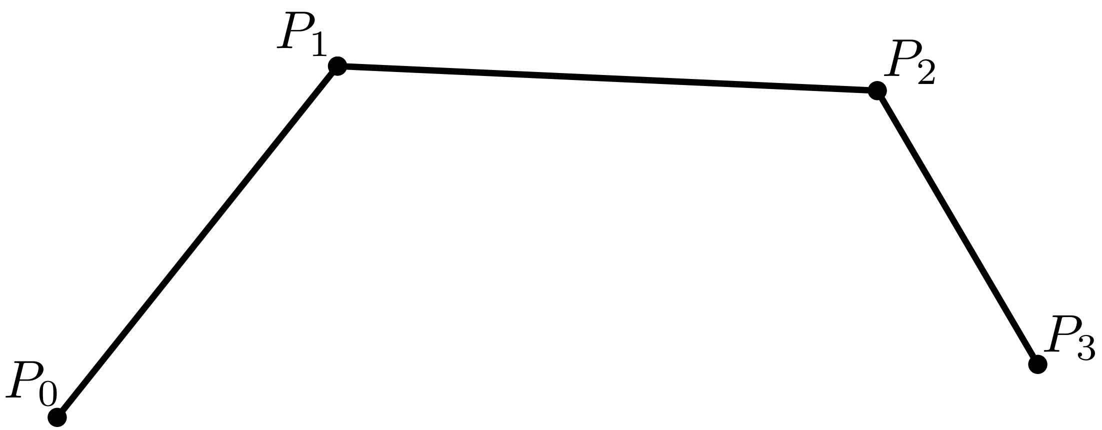

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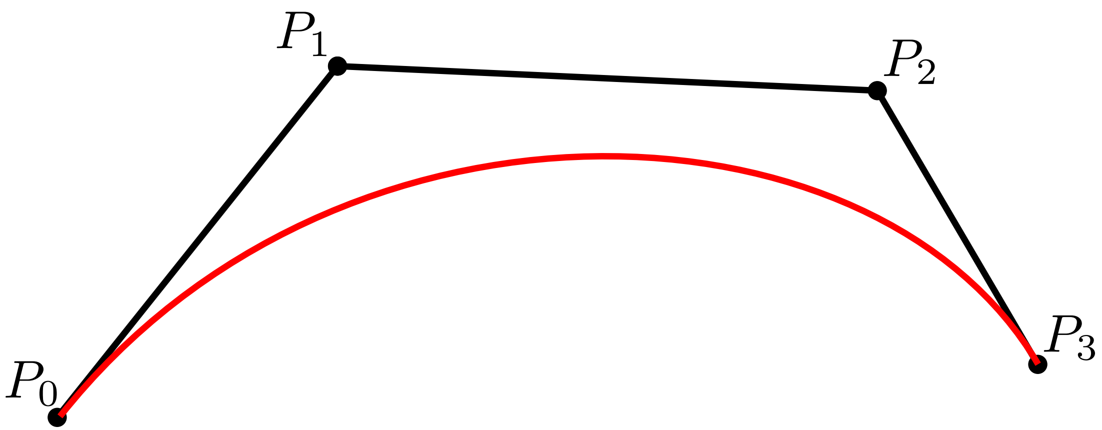

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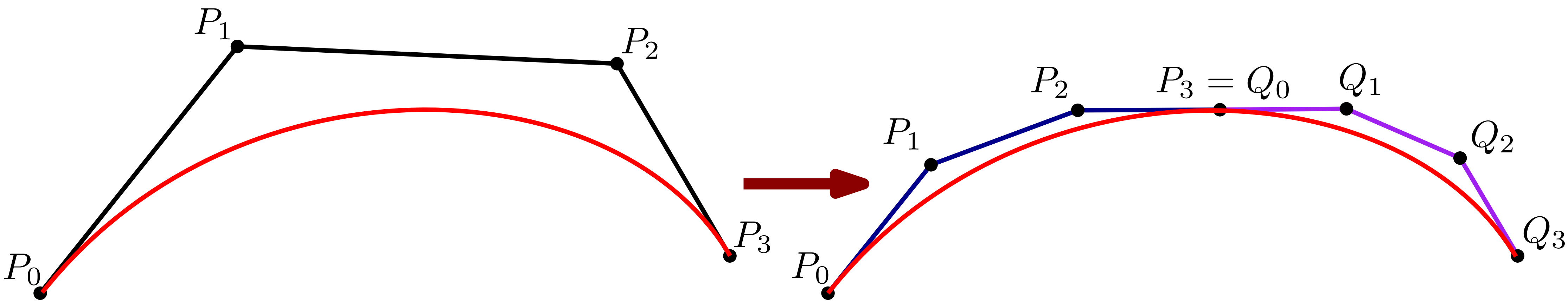

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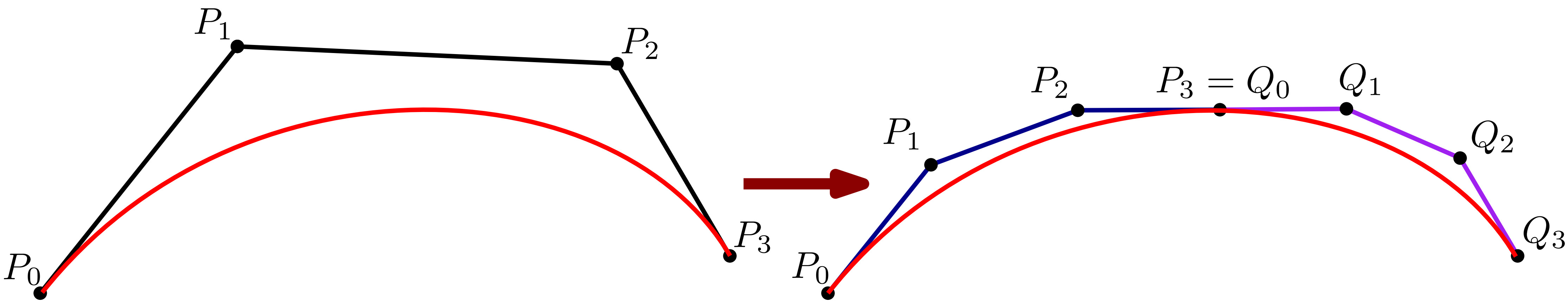

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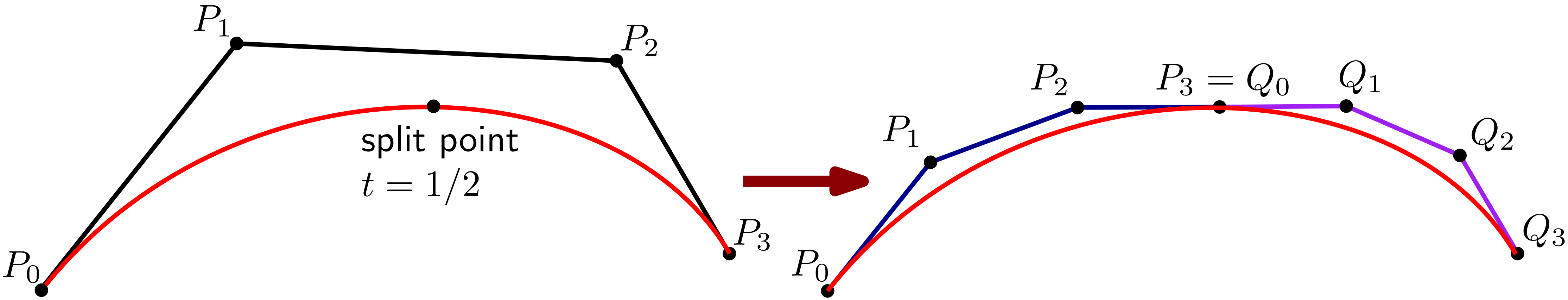

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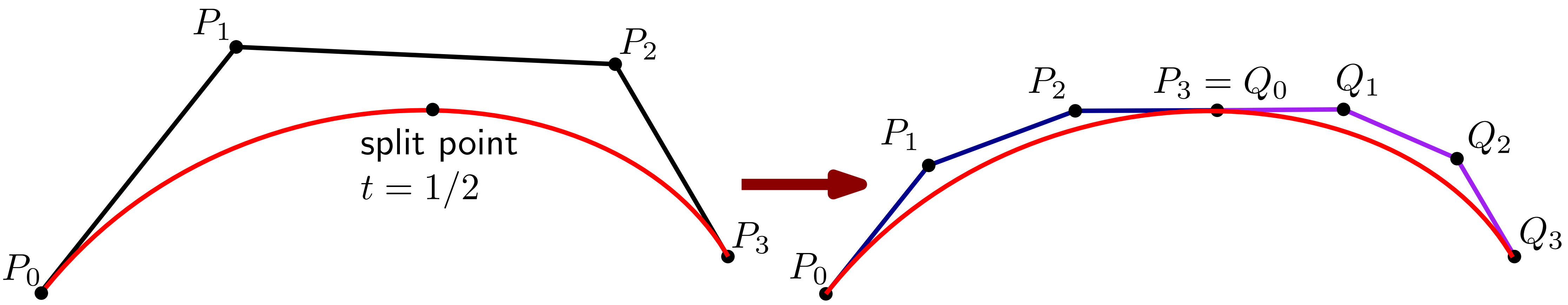

Recall the De Casteljau algorithm ($t = 1/2$):

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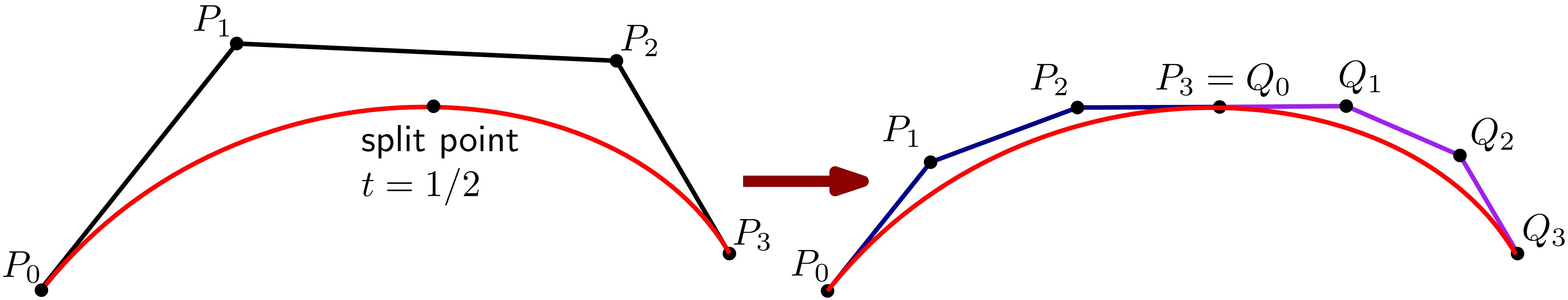

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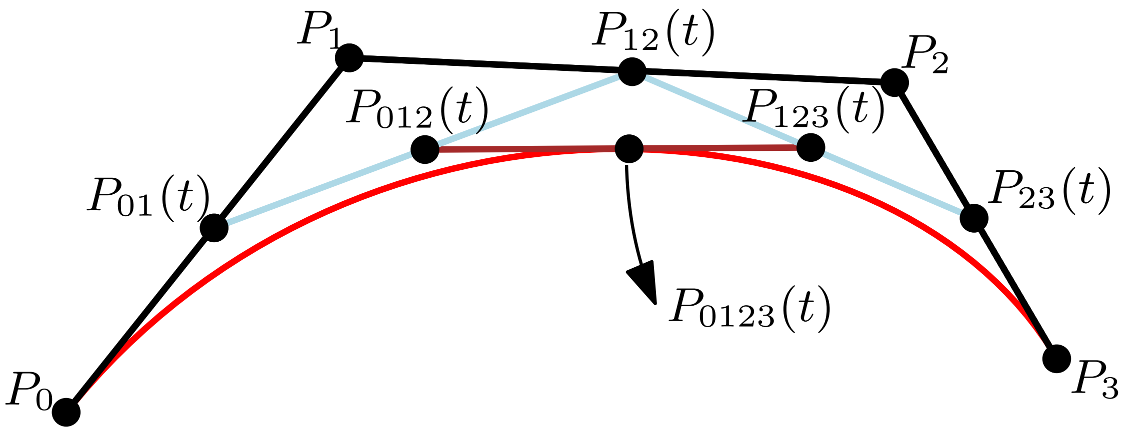

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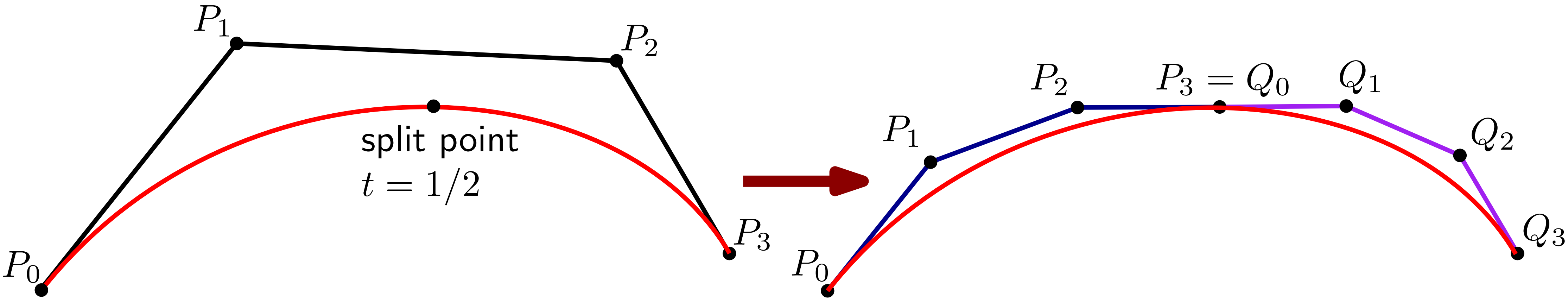

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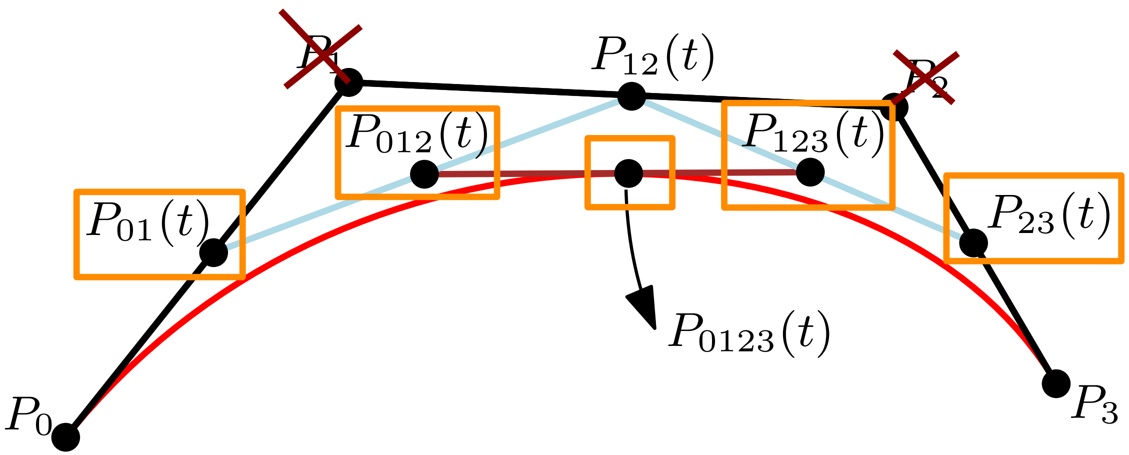

In general, the subdivision is done by:

- Discarding interior control points $P_1, \ldots, P_{n-1}$
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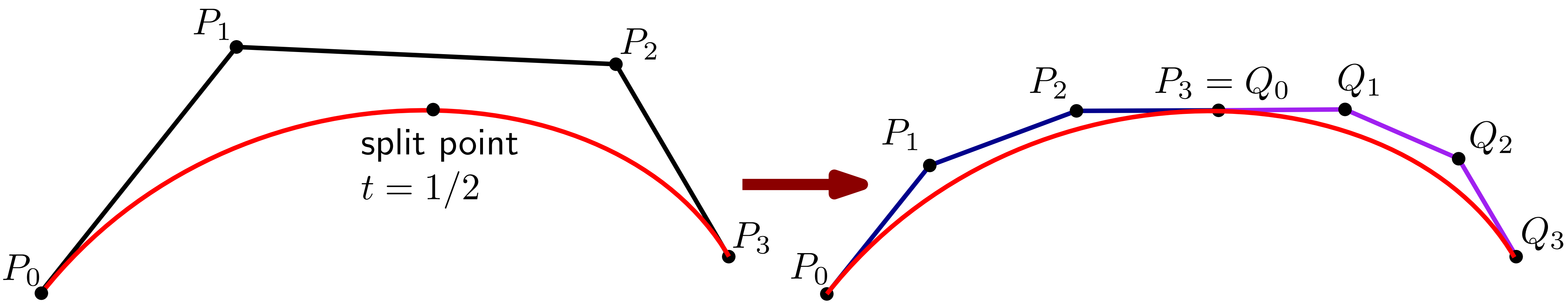

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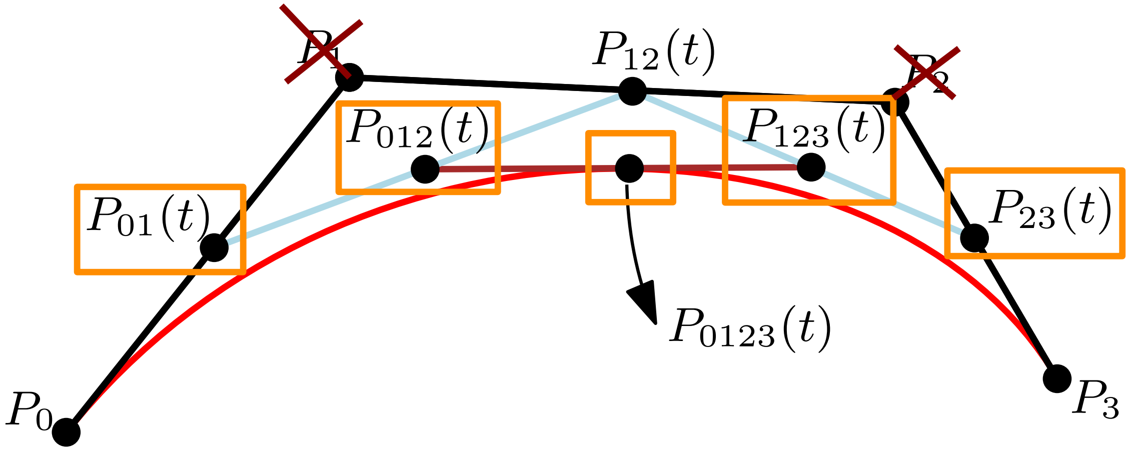

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This method is also useful for **clipping**

Computation of a Bézier curve

Recall definition

$B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$

recall that $0 \leq i \leq n$, $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, and $0! = 1$

$P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t)$

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$\sum_{i=0}^{n} \text{Table}_1[i,n] \times \text{Table}_2[t,i] \times \text{Table}_2[1-t, n-i] \times P_i$

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This can also be stored in a table, and reused for other points

Even faster: forward differences

Idea: find a method to "jump" from one point in $P(t)$ to the next one ,$P(t+\Delta)$, using only a few computations for each jump

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Goal: find quantity dP such that $P(t+\Delta) = P(t) + dP$

If dP would exist, then we could do:

$$
\begin{aligned}
P(0) &= P_0 \\
P(0+\Delta) &= P(0) + dP = P_0 + dP \\
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Infinite series

But becomes finite if $P(t)$ has constant degree!

Forward differences for cubic Bézier curve

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For a cubic Bézier curve, we have

$P(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$
or, equivalently,
$P(t) = at^3 + bt^2 + ct + d$
where:
$a = 3(P_1 - P_2) - P_0 + P_3$, $b = 3(P_0 + P_2) - 6P_1$, $c = 3(P_1 - P_0)$, $d = P_0$

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Problem: dP depends on t!

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procedure FastCubicBézierSketch
    Precalculate certain quantities
    P ← P_0
    for t = 0 to 1 step Δ do
        Draw point P
        P ← P + dP
        dP ← dP + ddP
        ddP ← ddP + dddP
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We need to figure out values for dP, ddP and $dddP$

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degree-1 polynomial on t

Forward differences for cubic Bézier curve

$dP(t) = 3at^2\Delta + 2bt\Delta + c\Delta + 3at\Delta^2 + b\Delta^2 + a\Delta^3$ degree-2 polynomial on t

Idea: use the same technique again, to get polynomial of degree 1...

...then do it again, to obtain polynomial of degree 0, i.e., constant!

procedure FASTCUBICBÉZIERSKETCH
  Precalculate certain quantities
  $P \leftarrow P_0$
  for $t = 0$ to 1 step Δ **do**
    Draw point P
    $P \leftarrow P + dP$
    $dP \leftarrow dP + ddP$
    $ddP \leftarrow ddP + dddP$

We need to figure out values for dP, ddP and $dddP$

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degree-1 polynomial on t

One more time: let's compute $dddP(t)$

$$dddP(t) = ddP'(t)\Delta = 6a\Delta^3$$

Forward differences for cubic Bézier curve

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procedure FastCubicBézierSketch
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degree-1 polynomial on t

One more time: let's compute $dddP(t)$

$dddP(t) = ddP'(t)\Delta = 6a\Delta^3$ $dddP$ is a constant! (does not depend on t)

Forward differences for cubic Bézier curve

Final code, trying to reuse computations as much as possible

procedure $\textsc{FastCubicBézier}$
- $Q_1 \leftarrow 3\Delta$
- $Q_2 \leftarrow Q_1 \cdot \Delta$ $\quad\triangleright\ 3\Delta^2$
- $Q_3 \leftarrow \Delta^3$
- $Q_4 \leftarrow 2Q_2$ $\quad\triangleright\ 6\Delta^2$
- $Q_5 \leftarrow 6Q_3$ $\quad\triangleright\ 6\Delta^3$
- $Q_6 \leftarrow P_0 - 2P_1 + P_2$
- $Q_7 \leftarrow 3(P_1 - P_2) - P_0 + P_3$ $\quad\triangleright\ a$
- $P \leftarrow P_0$
- $dP \leftarrow (P_1 - P_0)Q_1 + Q_6 \cdot Q_2 + Q_7 \cdot Q_3$
- $ddP \leftarrow Q_6 \cdot Q_4 + Q_7 \cdot Q_5$
- $dddP \leftarrow Q_7 \cdot Q_5$
- **for** $t = 0$ to 1 step Δ **do**
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The reduction in # of operations is huge: Ignoring the initialization, 3 sums for each evaluation of t

Matrix formulation

Bézier curves are often expressed in matrix form

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Note: matrix notation is very practical, but not necessarily very efficient, or numerically stable

Question: how many sums/products to evaluate $P(t)$?

Another way to increase number of points

- Recall: curve subdivision took a degree-n curve and produced two curves of degree-n ($2n + 1$ control points in total)
- Alternative: add points (increase degree) while preserving curve

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original curve		new curve
$P_n(t)$	$\rightarrow$	$P_{n+1}(t)$
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DEGREE ELEVATION

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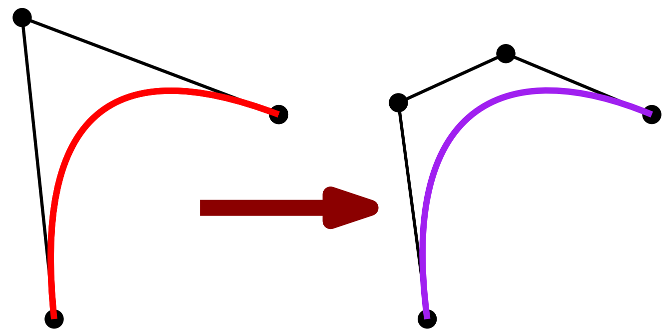

DEGREE ELEVATION

Adding one more point

original curve

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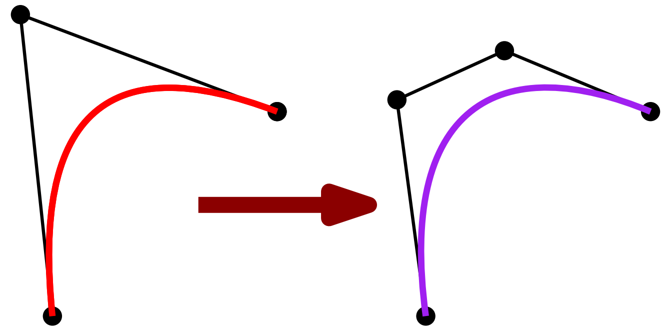

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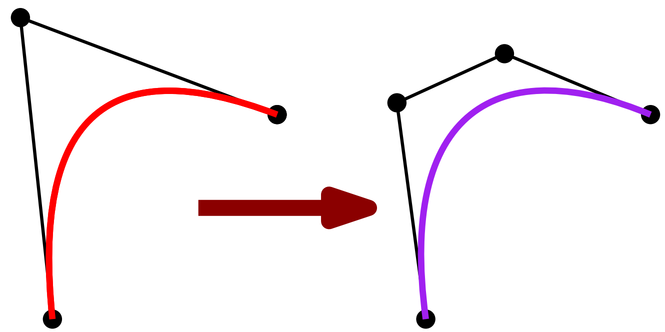

Producing control points for $P_{n+1}(t)$

Adding one more point

original curve

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new curve

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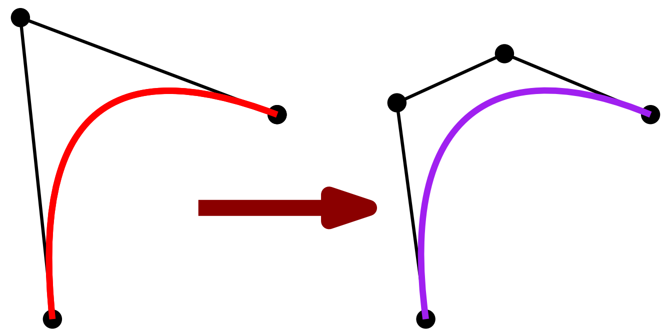

Producing control points for $P_{n+1}(t)$

With some basic algebraic tricks one can write $P(t)$ as an $(n+1)$-degree Bézier curve

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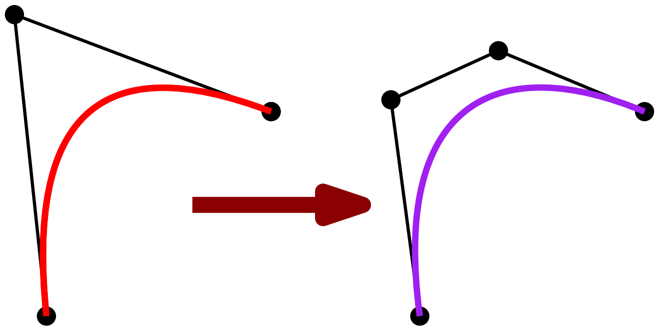

Producing control points for $P_{n+1}(t)$

> With some basic algebraic tricks one can write $P(t)$ as an $(n+1)$-degree Bézier curve

Start from trivial identity $P(t) = (t + (1-t))P(t) = tP(t) + (1-t)P(t)$

Use that $P(t) = \sum_{i=0}^{n} P_i B_{n,i}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i$, and extract coefficientes of new $(n+2)$ control points

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original curve

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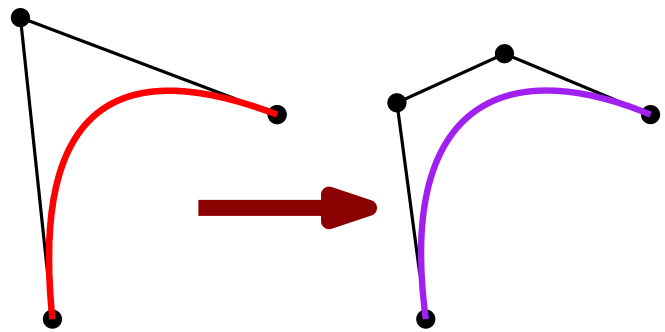

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Result:

$$P(t) = tP(t) + (1-t)P(t) = \sum_{i=0}^{n+1} \binom{n+1}{i} t^i (1-t)^{n+1-i} \left(\frac{i}{n+1} P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i\right)$$

Adding one more point

original curve

$P_n(t)$

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new curve

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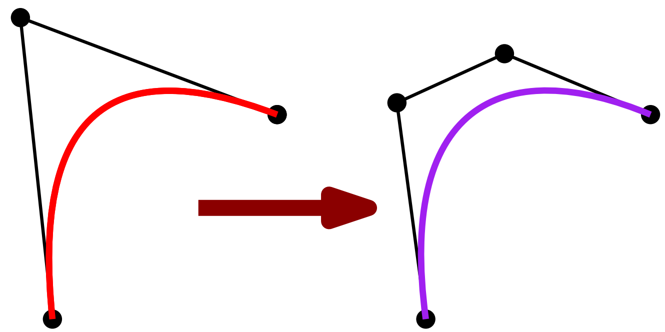

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new control points

Bézier curve of degree $(n+1)$!

$Q_i = \alpha_i P_{i-1} + (1-\alpha_i) P_i$

Note: here we assume $P_{-1} = 0$ and $P_{n+1} = 0$

Summary

The expression obtained for $P_{n+1}(t)$ is:

$$P_{n+1}(t) = \sum_{i=0}^{n+1} \binom{n+1}{i} t^i (1-t)^{n+1-i} \left(\frac{i}{n+1} P_{i-1} + \left(1 - \frac{i}{n+1}\right) P_i \right)$$

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Example

degree-4 curve (5 control points)

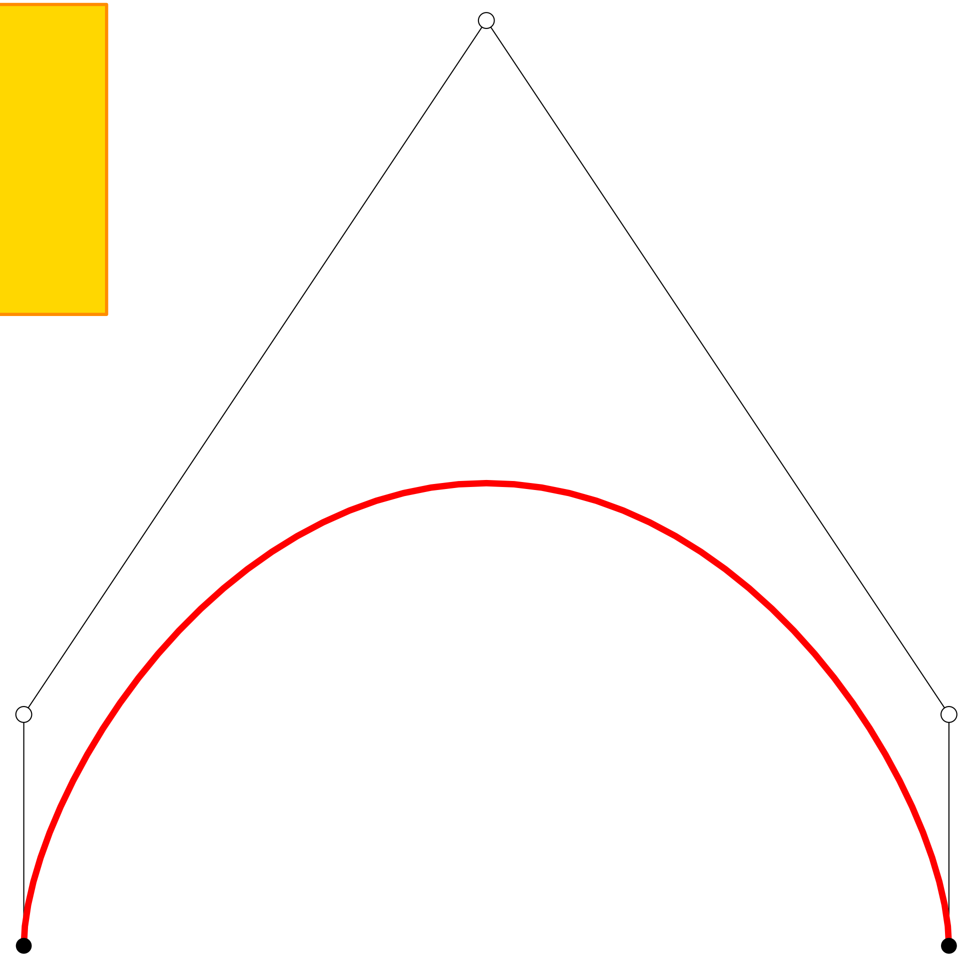

Figure from [G. Farin and D. Hansford, *The Essentials of CAGD*, AK Peters, 2000].

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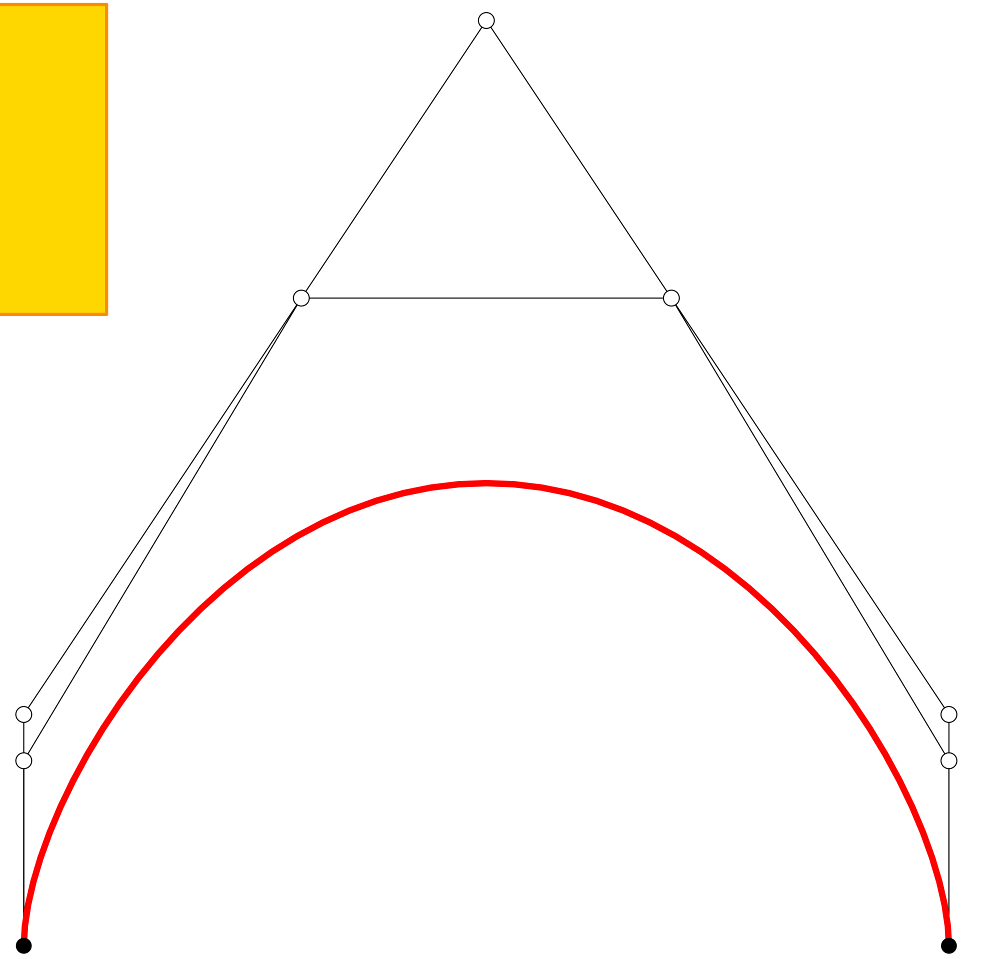

Example

degree-4 curve (5 control points)

degree-5 curve (6 control points)

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The expression obtained for $P_{n+1}(t)$ is:

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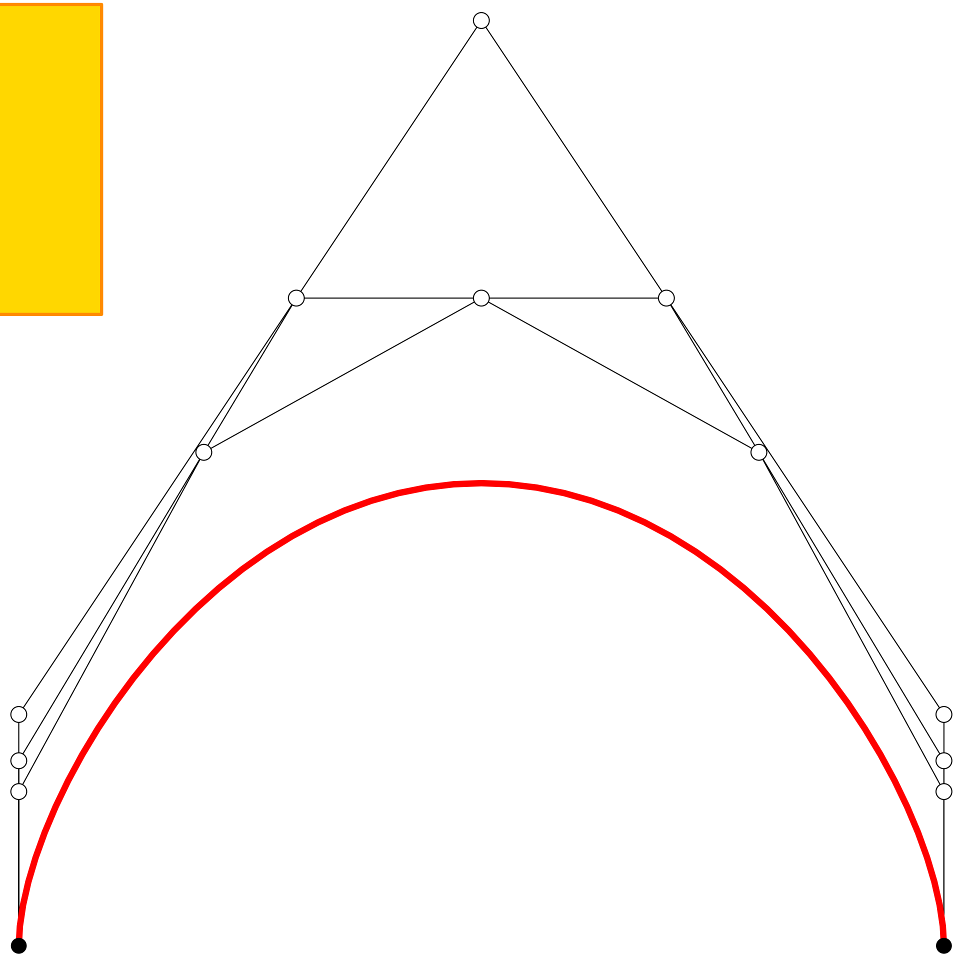
Example

degree-4 curve (5 control points)

degree-5 curve (6 control points)

degree-6 curve (7 control points)



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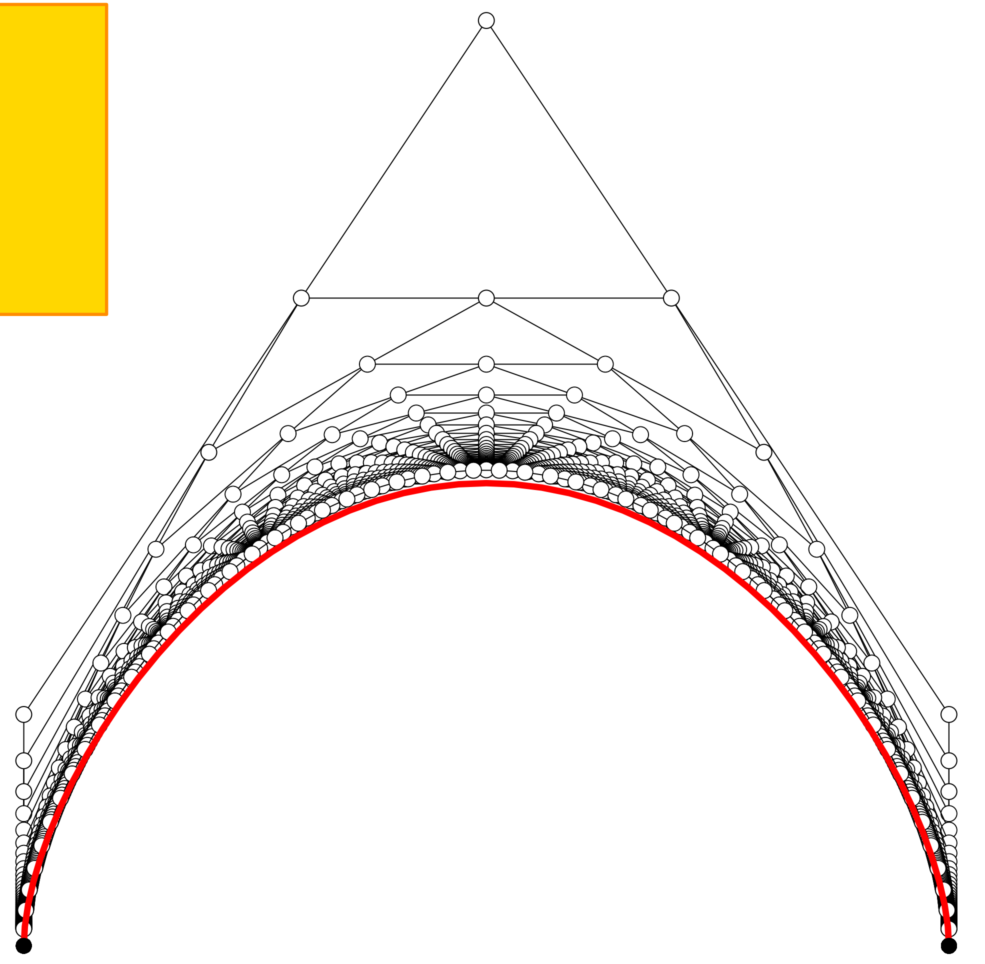
degree-6 curve (7 control points)

degree-n curve ($n+1$ control points)



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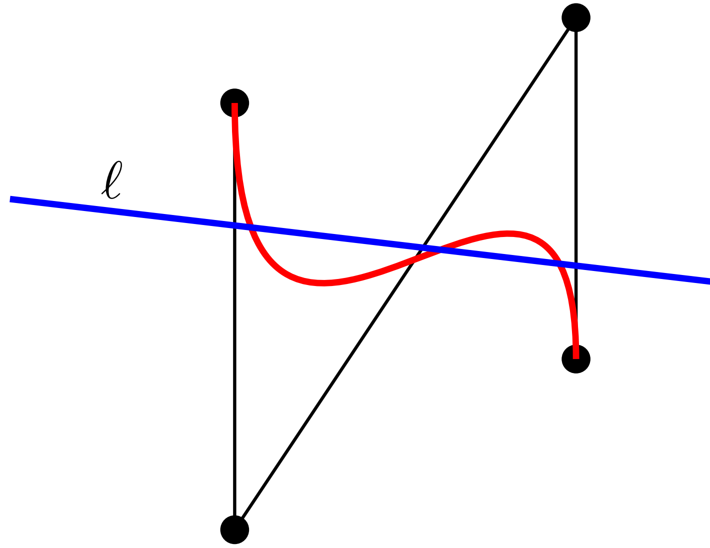
Back to the variation dimishing property

Recall the property: The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygon



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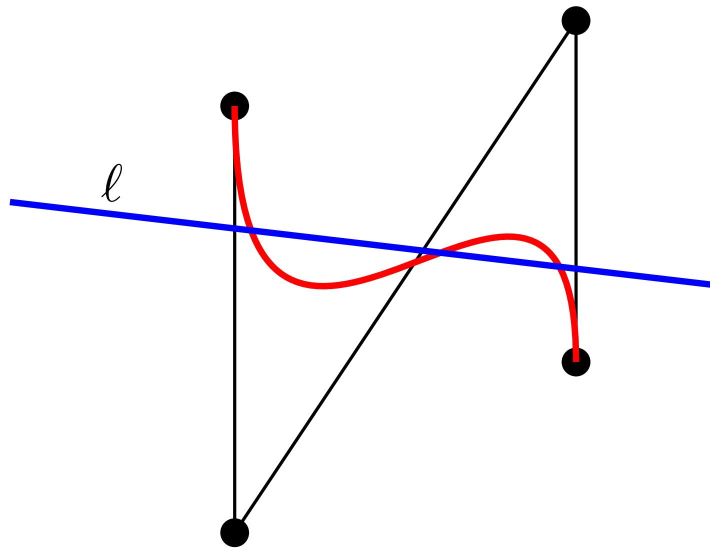
Proof sketch using degree elevation



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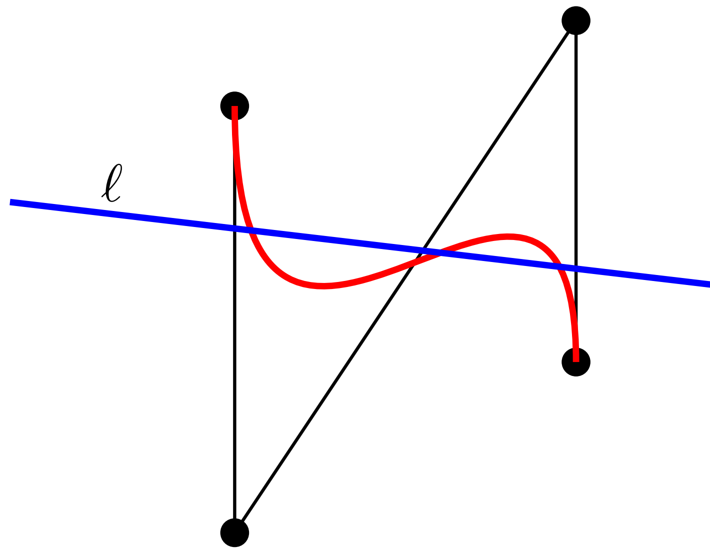
Recall: linear interpolation is variation diminishing



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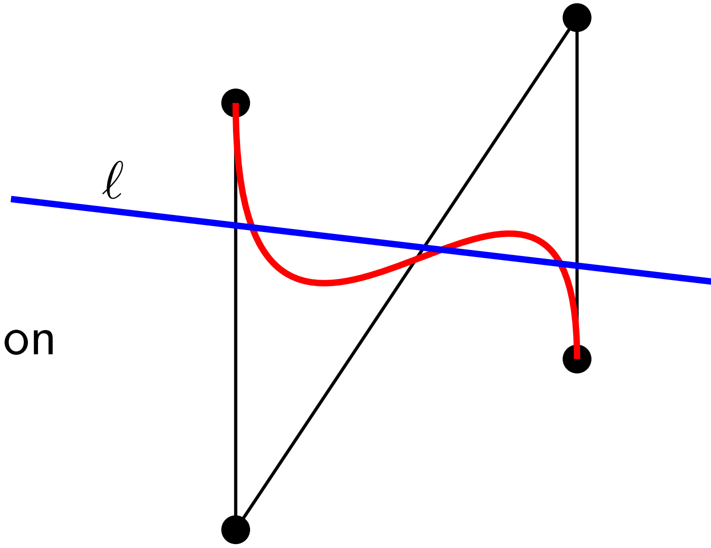
Observation 2: degree elevation is an instance of linear interpolation



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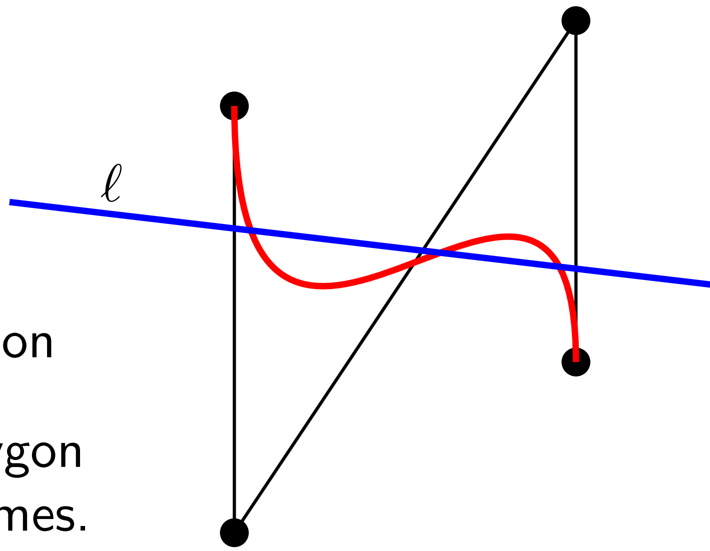
Recall the property: The number of intersections of any line with a Bézier curve is at most the number of intersections of the line with the control polygon



Proof sketch using degree elevation

Recall: linear interpolation is variation diminishing

Observation 2: degree elevation is an instance of linear interpolation

Let R_0 be the control polygon of $P(t)$, let R_1 be the control polygon after increasing the degree by one, and R_k after increasing it k times.

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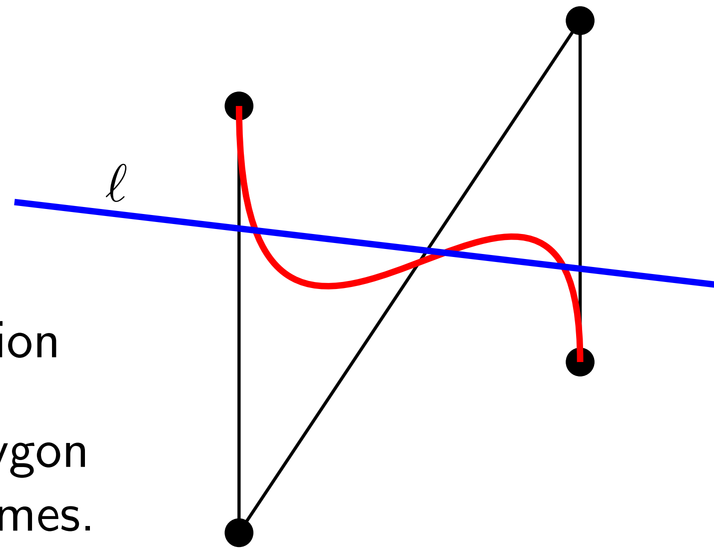

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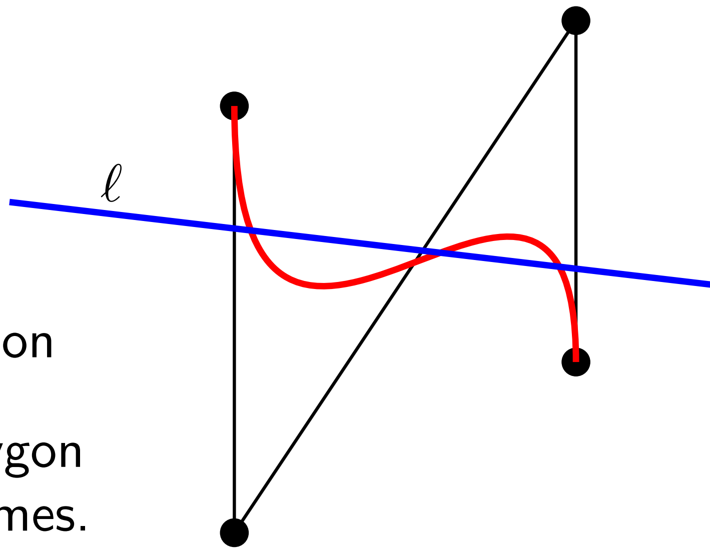

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INTERPOLATION WITH BÉZIER CURVES

Goal: find Bézier curve that interpolates given points

- In general, the Bézier curve does not interpolate its control points
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We are free to choose the values of the t_is!

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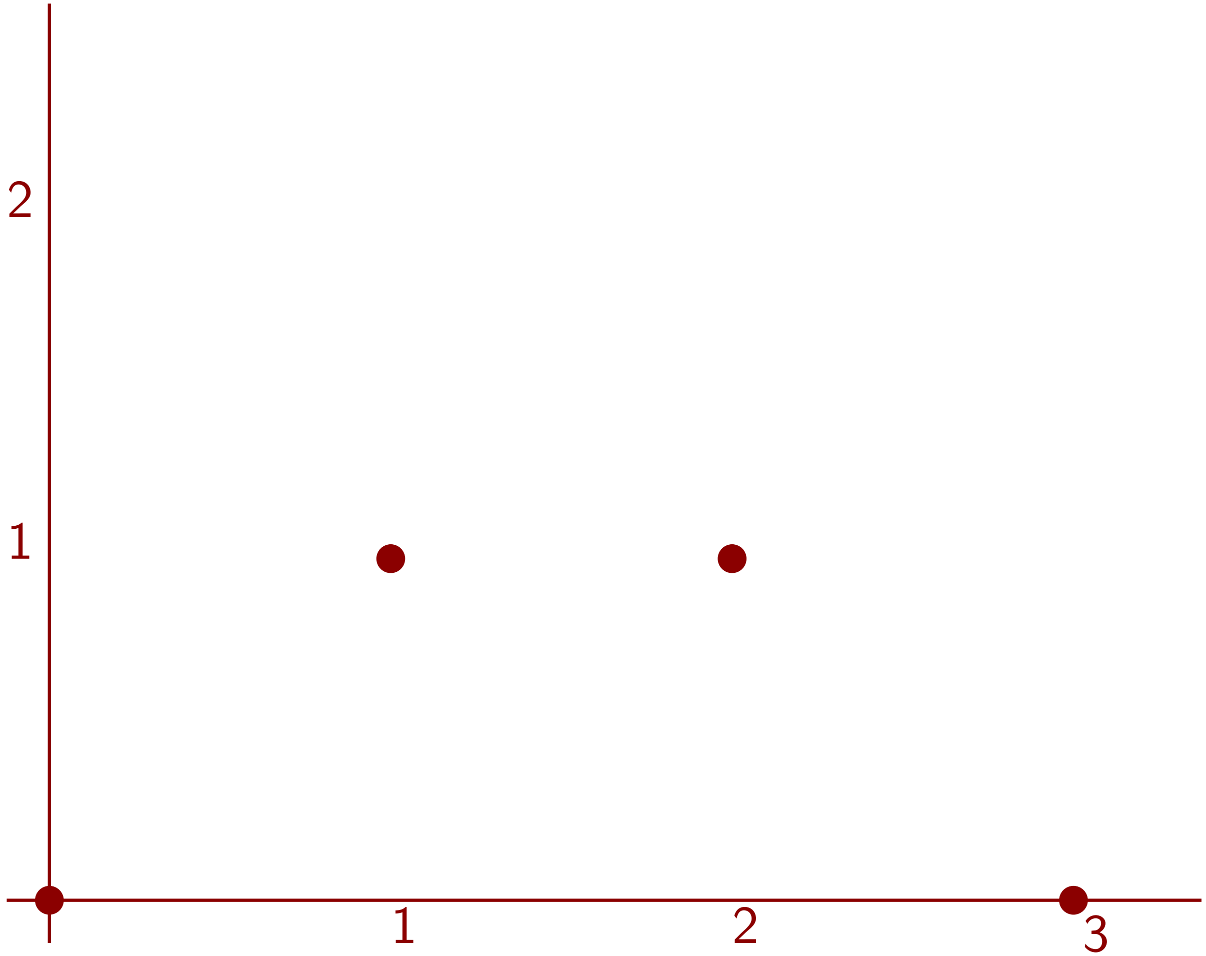

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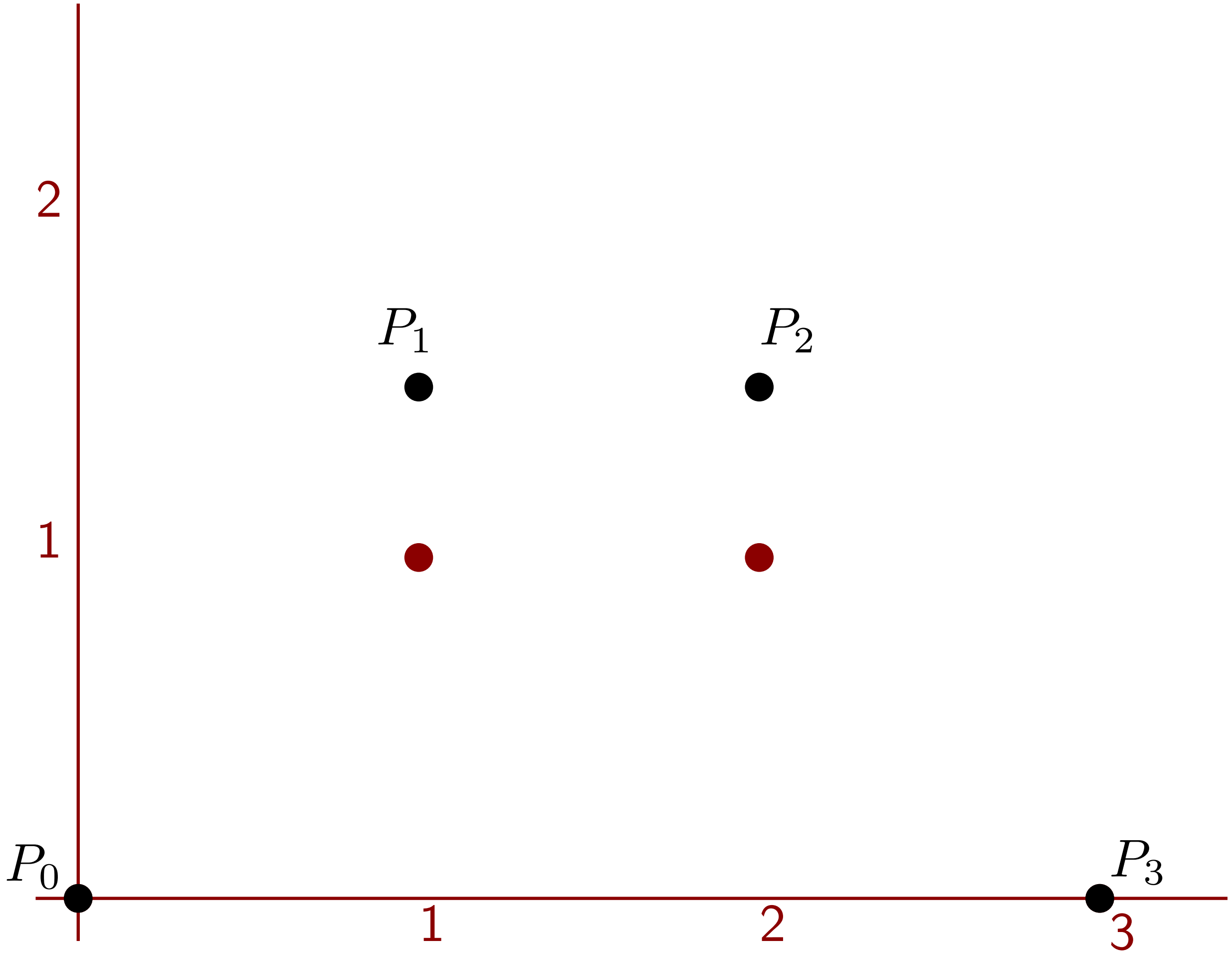

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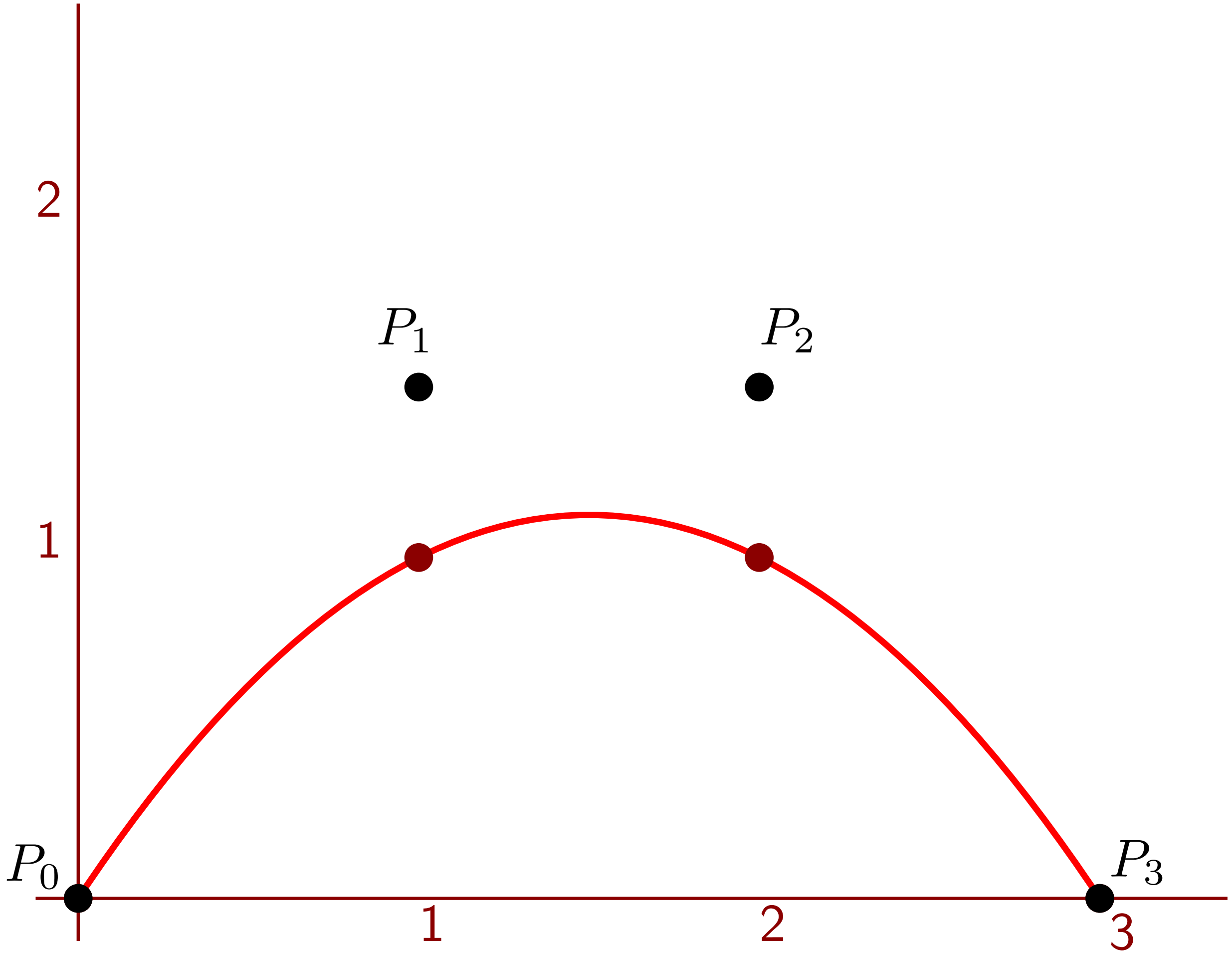

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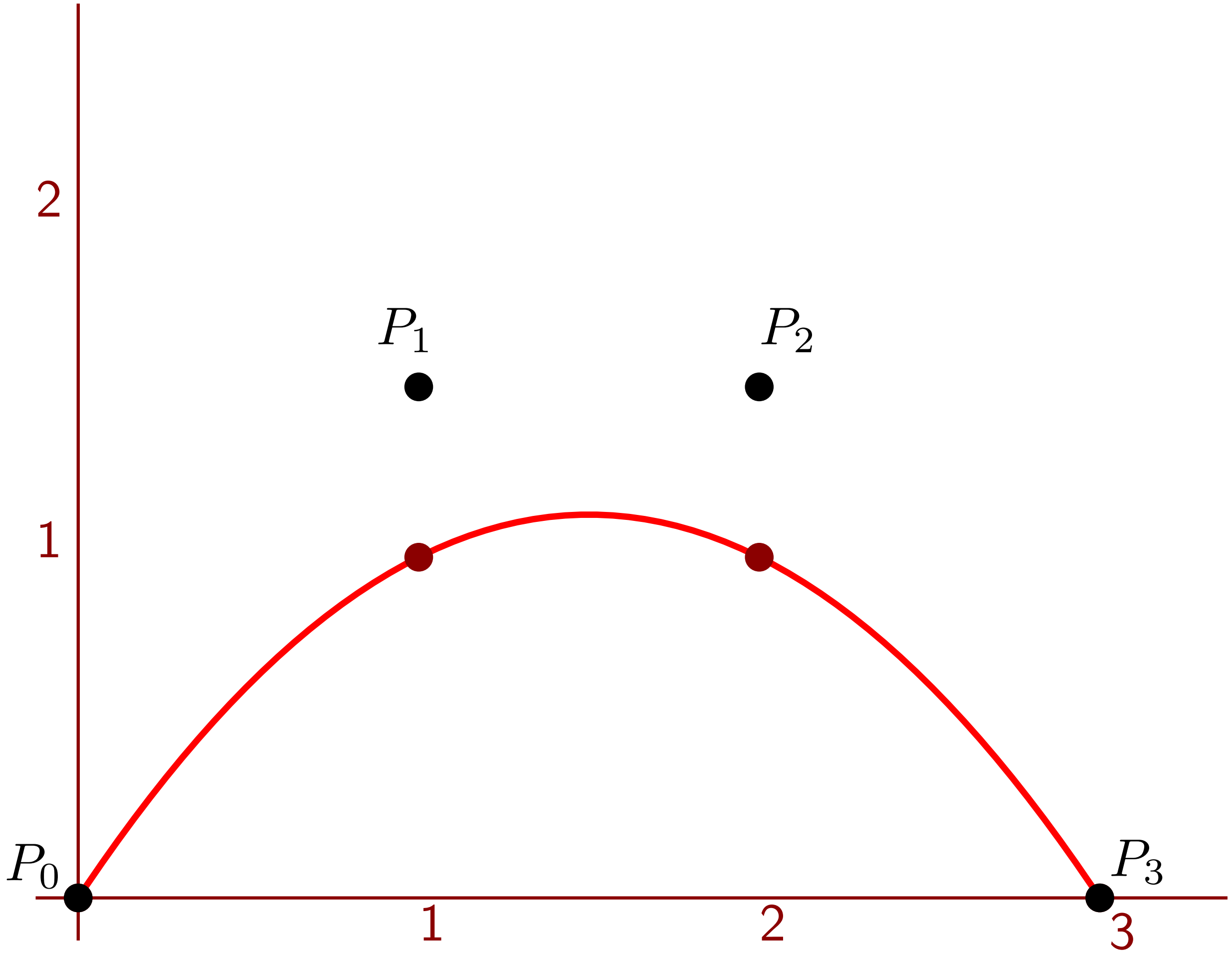

This is only one way to interpolate with Bézier curves, others are possible

Rational Bézier curves

Rational Bézier curves

Each control point has a weight, giving more flexibility to shape the curve

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Bézier curve

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Bézier curve — Rational Bézier curve

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Advantages: why complicate things so much?

- Invariant under projections
- It can represent conic curves (impossible with Hermite or Bézier curves) (e.g., segments of circles, ellipses, hyperbolas and parabolas)

Understanding rational Bézier curves

Effect of the weights

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Understanding rational Bézier curves

Effect of the weights

$$P(t) = \frac{\sum_{i=0}^{n} w_i P_i B_{n,i}(t)}{\sum_{j=0}^{n} w_j B_{n,j}(t)} = \sum_{i=0}^{n} P_i \left(\frac{w_i B_{n,i}(t)}{\sum_{j=0}^{n} w_j B_{n,j}(t)} \right)$$

- If $w_i > 1$, the curve gets closer to P_i
- If $w_i < 1$, the curve moves away from P_i

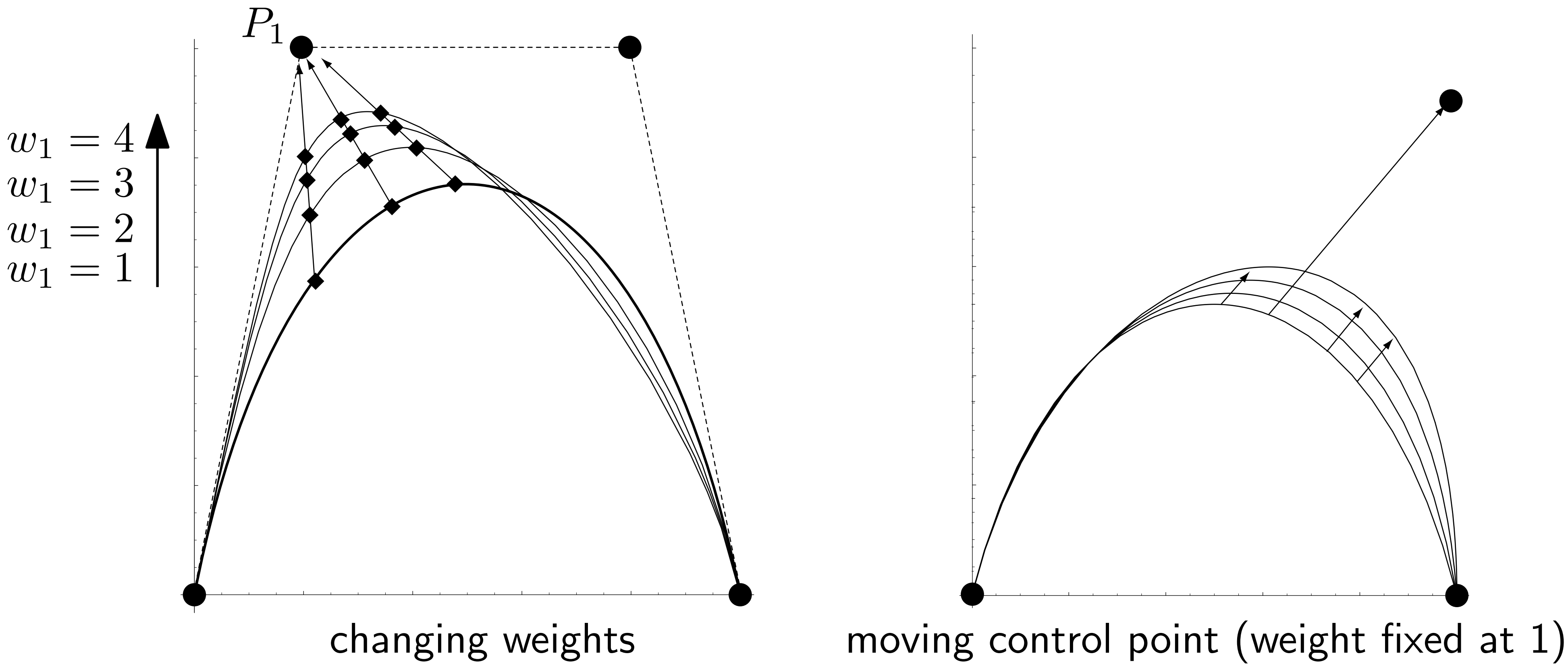

Figure from book by Salomon (page 219)

Rational Bézier curves as curves in projective space

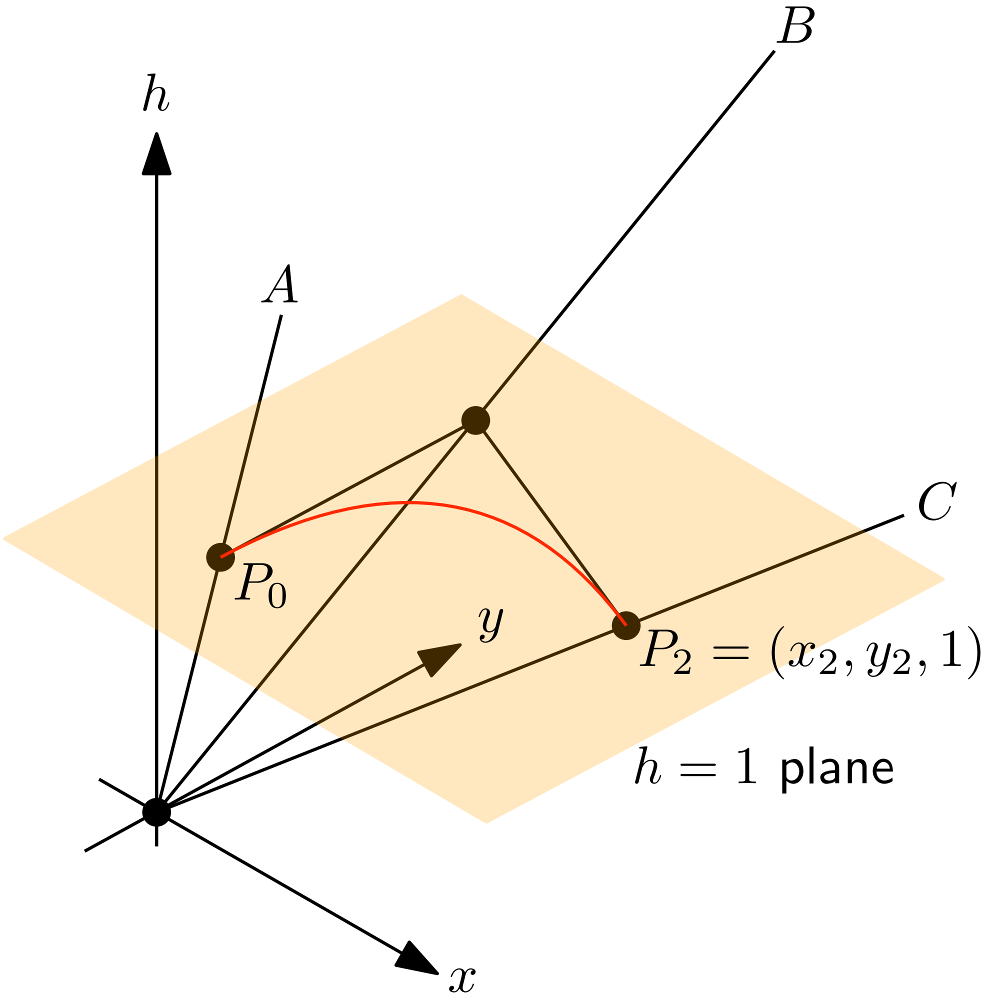

Figure adapted from book by Mortenson

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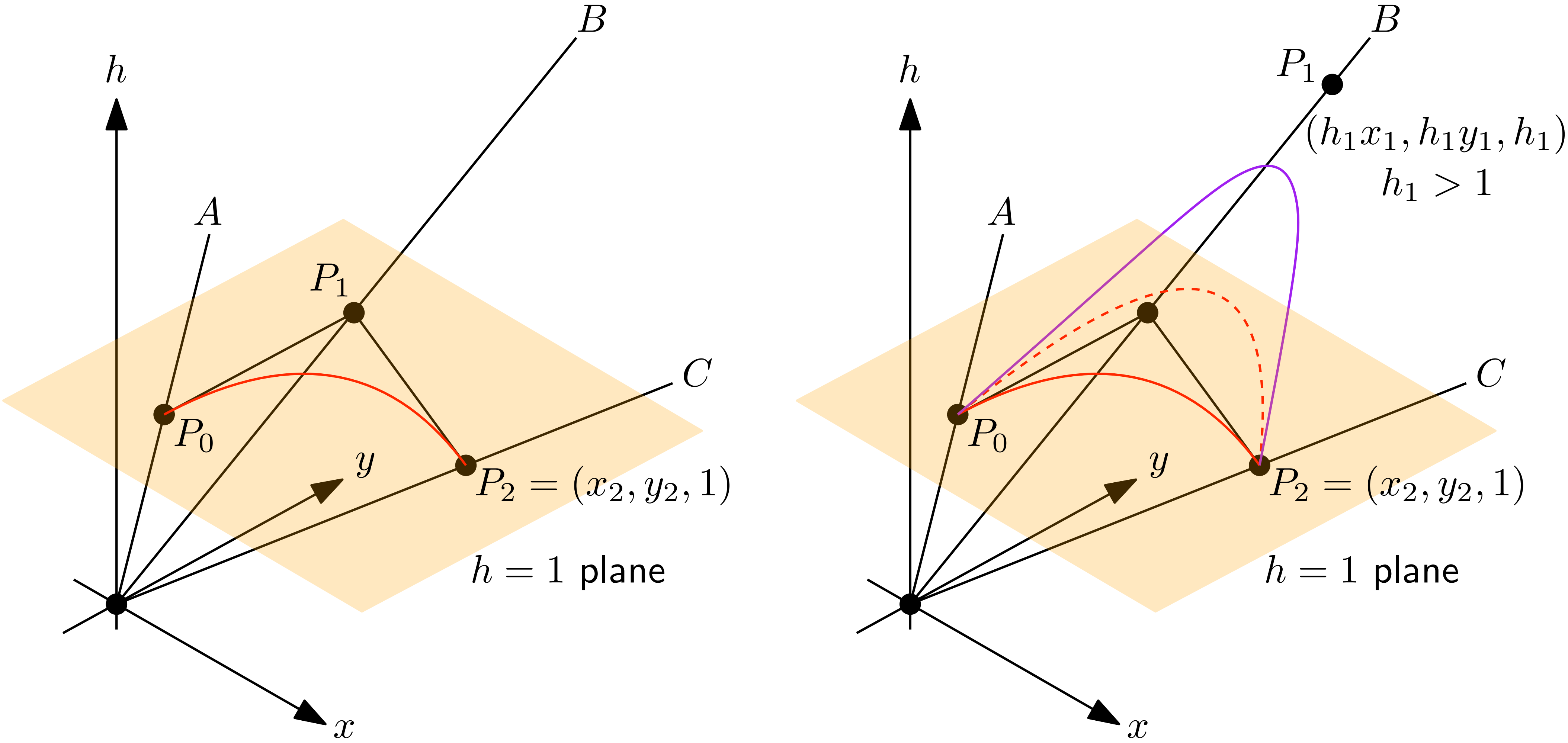

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Theorem Consider a conic curve $C(t)$. Then there exist weights w_0, w_1, w_2 and control points P_0, P_1, P_2 such that

$$C(t) = \frac{w_0 P_0 B_{2,0}(t) + w_1 P_1 B_{2,1}(t) + w_2 P_2 B_{2,2}(t)}{w_0 B_{2,0}(t) + w_1 B_{2,1}(t) + w_2 B_{2,2}(t)}$$

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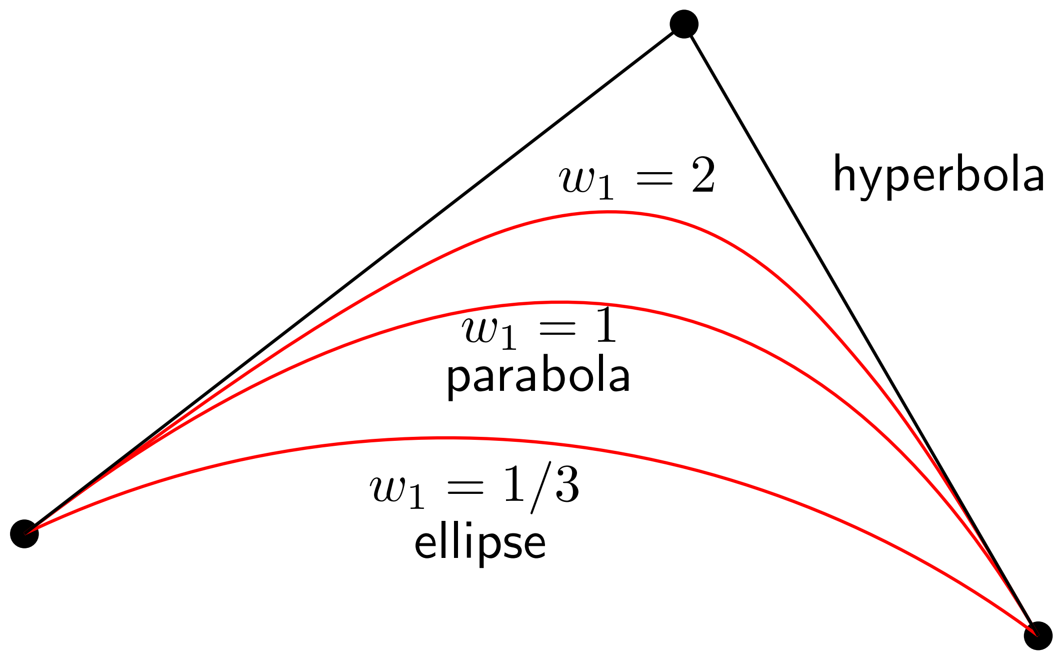
Example

Take $w_0 = w_2 = 1$ and let $s = \frac{w_1}{1+w_1}$

- $s = 1/2$ produces a **parabolic** arc
- $s < 1/2$ produces an **elliptic** arc
- $s > 1/2$ produces a **hyperbolic** arc

for any three non-colinear control points



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