3
Polynomial Interpolation

Definition: A polynomial of degree n in x is the function

$$P_n(x) = \sum_{i=0}^{n} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

where a_i are the coefficients of the polynomial (in our case, they are real numbers). Note that there are $n+1$ coefficients.

Calculating a polynomial involves additions, multiplications, and exponentiations, but there are two methods that greatly simplify this calculation. They are the following:

1. Horner's rule. A degree-3 polynomial can be written in the form

$$P(x) = \big((a_3 x + a_2)x + a_1\big)x + a_0,$$

thereby eliminating all exponentiations.

2. Forward differences. This is one of Newton's many contributions to mathematics and it is described in some detail in Section 1.5.1. Only the first step requires multiplications. All other steps are performed with additions and assignments only.

Given a set of points, it is possible to construct a polynomial that when plotted passes through the points. When fully computed and displayed, such a polynomial becomes a curve that's referred to as a *polynomial interpolation* of the points. The first part of this chapter discusses methods for polynomial interpolation and shows their limitations. The second part extends the discussion to a two-dimensional grid of points, and shows how to compute a two-parameter polynomial that passes through the points. When fully computed and displayed, such a polynomial becomes a surface. The methods described here apply the algebra of polynomials to the geometry of curves and surfaces, but this application is limited, because high-degree polynomials tend to oscillate. Section 1.5, and especially Exercise 1.20 show why this is so. Still, there are cases where high-degree polynomials are useful.

This chapter starts with a simple example where four points are given and a cubic polynomial that passes through them is derived from first principles. Following this, the Lagrange and Newton polynomial interpolation methods are introduced. The chapter continues with a description of several simple surface algorithms based on polynomials. It concludes with the Coons and Gordon surfaces, which also employ polynomials.

3.1 Four Points

Four points (two-dimensional or three-dimensional) $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ are given. We are looking for a PC curve that passes through these points and has the form

$$\mathbf{P}(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + \mathbf{d} = (t^3, t^2, t, 1)(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})^T = \mathbf{T}(t)\mathbf{A} \quad \text{for} \quad 0 \le t \le 1, \tag{3.1}$$

where each of the four coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ is a pair (or a triplet), $\mathbf{T}(t)$ is the row vector $(t^3, t^2, t, 1)$, and $\mathbf{A}$ is the column vector $(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})^T$. The only unknowns are $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$.

Since the four points can be located anywhere, we cannot assume anything about their positions and we make the general assumption that $\mathbf{P}_1$ and $\mathbf{P}_4$ are the two endpoints $\mathbf{P}(0)$ and $\mathbf{P}(1)$ of the curve, and that $\mathbf{P}_2$ and $\mathbf{P}_3$ are the two interior points $\mathbf{P}(1/3)$ and $\mathbf{P}(2/3)$. (Having no information about the locations of the points, the best we can do is to use equi-distant values of the parameter t.) We therefore write the four equations $\mathbf{P}(0) = \mathbf{P}_1$, $\mathbf{P}(1/3) = \mathbf{P}_2$, $\mathbf{P}(2/3) = \mathbf{P}_3$, and $\mathbf{P}(1) = \mathbf{P}_4$, or explicitly

$$\begin{aligned}
\mathbf{a}(0)^3 + \mathbf{b}(0)^2 + \mathbf{c}(0) + \mathbf{d} &= \mathbf{P}_1,\\
\mathbf{a}(1/3)^3 + \mathbf{b}(1/3)^2 + \mathbf{c}(1/3) + \mathbf{d} &= \mathbf{P}_2,\\
\mathbf{a}(2/3)^3 + \mathbf{b}(2/3)^2 + \mathbf{c}(2/3) + \mathbf{d} &= \mathbf{P}_3,\\
\mathbf{a}(1)^3 + \mathbf{b}(1)^2 + \mathbf{c}(1) + \mathbf{d} &= \mathbf{P}_4.
\end{aligned} \tag{3.2}$$

The solutions of this system of equations are

$$\begin{aligned}
\mathbf{a} &= -(9/2)\mathbf{P}_1 + (27/2)\mathbf{P}_2 - (27/2)\mathbf{P}_3 + (9/2)\mathbf{P}_4,\\
\mathbf{b} &= 9\mathbf{P}_1 - (45/2)\mathbf{P}_2 + 18\mathbf{P}_3 - (9/2)\mathbf{P}_4,\\
\mathbf{c} &= -(11/2)\mathbf{P}_1 + 9\mathbf{P}_2 - (9/2)\mathbf{P}_3 + \mathbf{P}_4,\\
\mathbf{d} &= \mathbf{P}_1.
\end{aligned} \tag{3.3}$$

Substituting these solutions into Equation (3.1) gives

$$\begin{aligned}
\mathbf{P}(t) =& \big(-(9/2)\mathbf{P}_1 + (27/2)\mathbf{P}_2 - (27/2)\mathbf{P}_3 + (9/2)\mathbf{P}_4\big)t^3\\
&+ \big(9\mathbf{P}_1 - (45/2)\mathbf{P}_2 + 18\mathbf{P}_3 - (9/2)\mathbf{P}_4\big)t^2\\
&+ \big(-(11/2)\mathbf{P}_1 + 9\mathbf{P}_2 - (9/2)\mathbf{P}_3 + \mathbf{P}_4\big)t + \mathbf{P}_1.
\end{aligned}$$

After rearranging, this becomes

$$\begin{aligned}\mathbf{P}(t) =&(-4.5t^3+9t^2-5.5t+1)\mathbf{P}_1+(13.5t^3-22.5t^2+9t)\mathbf{P}_2\\&+(-13.5t^3+18t^2-4.5t)\mathbf{P}_3+(4.5t^3-4.5t^2+t)\mathbf{P}_4\\=&G_1(t)\mathbf{P}_1+G_2(t)\mathbf{P}_2+G_3(t)\mathbf{P}_3+G_4(t)\mathbf{P}_4\\=&\mathbf{G}(t)\mathbf{P},\end{aligned} \tag{3.4}$$

where the four functions $G_i(t)$ are cubic polynomials in t

$$\begin{aligned}G_1(t)&=(-4.5t^3+9t^2-5.5t+1), & G_3(t)&=(-13.5t^3+18t^2-4.5t),\\ G_2(t)&=(13.5t^3-22.5t^2+9t), & G_4(t)&=(4.5t^3-4.5t^2+t),\end{aligned} \tag{3.5}$$

$\mathbf{P}$ is the column $(\mathbf{P}_1,\mathbf{P}_2,\mathbf{P}_3,\mathbf{P}_4)^T$ and $\mathbf{G}(t)$ is the row $(G_1(t),G_2(t),G_3(t),G_4(t))$ (see also Exercise 3.8 for a different approach to this polynomial).

The functions $G_i(t)$ are called blending functions because they represent any point on the curve as a blend of the four given points. Note that they are barycentric (they should be, since they blend points, and this is shown in the next paragraph). We can also write

$$G_1(t)=(t^3,t^2,t,1)(-4.5,9,-5.5,1)^T$$

and similarly for $G_2(t)$, $G_3(t)$, and $G_4(t)$. The curve can now be expressed as

$$\mathbf{P}(t)=\mathbf{G}(t)\mathbf{P}=(t^3,t^2,t,1)\begin{bmatrix}-4.5 & 13.5 & -13.5 & 4.5\\ 9.0 & -22.5 & 18 & -4.5\\ -5.5 & 9.0 & -4.5 & 1.0\\ 1.0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{P}_1\\ \mathbf{P}_2\\ \mathbf{P}_3\\ \mathbf{P}_4\end{bmatrix}=\mathbf{T}(t)\,\mathbf{N}\,\mathbf{P}. \tag{3.6}$$

Matrix $\mathbf{N}$ is called the basis matrix and $\mathbf{P}$ is the geometry vector. Equation (3.1) tells us that $\mathbf{P}(t)=\mathbf{T}(t)\,\mathbf{A}$, so we conclude that $\mathbf{A}=\mathbf{N}\,\mathbf{P}$.

The four functions $G_i(t)$ are barycentric because of the nature of Equation (3.2), not because of the special choice of the four t values. To see why this is so, we write Equation (3.2) for four different, arbitrary values t_1, t_2, t_3, and t_4 (they have to be different, otherwise two or more equations would be contradictory).

$$\begin{aligned}at_1^3+bt_1^2+ct_1+d&=P_1,\\ at_2^3+bt_2^2+ct_2+d&=P_2,\\ at_3^3+bt_3^2+ct_3+d&=P_3,\\ at_4^3+bt_4^2+ct_4+d&=P_4,\end{aligned} \tag{3.7}$$

(where we treat the four values P_i as numbers, not points, and as a result, a, b, c, and d are also numbers). The solutions are of the form

$$\begin{aligned}a&=c_{11}P_1+c_{12}P_2+c_{13}P_3+c_{14}P_4,\\ b&=c_{21}P_1+c_{22}P_2+c_{23}P_3+c_{24}P_4,\\ c&=c_{31}P_1+c_{32}P_2+c_{33}P_3+c_{34}P_4,\\ d&=c_{41}P_1+c_{42}P_2+c_{43}P_3+c_{44}P_4.\end{aligned} \tag{3.8}$$

Comparing Equation (3.8) to Equations (3.3) and (3.5) shows that the four functions $G_i(t)$ can be expressed in terms of the c_{ij} in the form

$$G_i(t) = (c_{1i}t^3 + c_{2i}t^2 + c_{3i}t + c_{4i}). \tag{3.9}$$

The point is that the 16 coefficients c_{ij} do not depend on the four values P_i. They are the same for any choice of the P_i. As a special case, we now select $P_1 = P_2 = P_3 = P_4 = 1$ which reduces Equation (3.8) to

$$\begin{aligned} at_1^3 + bt_1^2 + ct_1 + d = 1, &\quad at_2^3 + bt_2^2 + ct_2 + d = 1,\\ at_3^3 + bt_3^2 + ct_3 + d = 1, &\quad at_4^3 + bt_4^2 + ct_4 + d = 1. \end{aligned}$$

Because the four values t_i are arbitrary, the four equations above can be written as the single equation $at^3 + bt^2 + ct + d = 1$, that holds for any t. Its solutions must therefore be $a = b = c = 0$ and $d = 1$.

Thus, we conclude that when all four values P_i are 1, a must be zero. In general, $a = c_{11}P_1 + c_{12}P_2 + c_{13}P_3 + c_{14}P_4$, which implies that $c_{11} + c_{12} + c_{13} + c_{14}$ must be zero. Similar arguments show that $c_{21} + c_{22} + c_{23} + c_{24} = 0$, $c_{31} + c_{32} + c_{33} + c_{34} = 0$, and $c_{41} + c_{42} + c_{43} + c_{44} = 1$. These relations, combined with Equation (3.9), show that the four $G_i(t)$ are barycentric.

To calculate the curve, we only need to calculate the four quantities **a**, **b**, **c**, and **d** (that constitute vector **A**), and write Equation (3.1) using the numerical values of **a**, **b**, **c**, and **d**.

Example: (This example is in two dimensions, each of the four points $\mathbf{P}_i$ along with the four coefficients $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ form a pair. For three-dimensional curves the method is the same except that triplets are used instead of pairs.) Given the four two-dimensional points $\mathbf{P}_1 = (0,0)$, $\mathbf{P}_2 = (1,0)$, $\mathbf{P}_3 = (1,1)$, and $\mathbf{P}_4 = (0,1)$, we set up the equation

$$\begin{pmatrix} \mathbf{a}\\ \mathbf{b}\\ \mathbf{c}\\ \mathbf{d} \end{pmatrix} = \mathbf{A} = \mathbf{NP} = \begin{pmatrix} -4.5 & 13.5 & -13.5 & 4.5\\ 9.0 & -22.5 & 18 & -4.5\\ -5.5 & 9.0 & -4.5 & 1.0\\ 1.0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (0,0)\\ (1,0)\\ (1,1)\\ (0,1) \end{pmatrix}.$$

Its solutions are

$$\begin{aligned} \mathbf{a} &= -4.5(0,0) + 13.5(1,0) - 13.5(1,1) + 4.5(0,1) = (0,-9),\\ \mathbf{b} &= 19(0,0) - 22.5(1,0) + 18(1,1) - 4.5(0,1) = (-4.5, 13.5),\\ \mathbf{c} &= -5.5(0,0) + 9(1,0) - 4.5(1,1) + 1(0,1) = (4.5,-3.5),\\ \mathbf{d} &= 1(0,0) - 0(1,0) + 0(1,1) - 0(0,1) = (0,0). \end{aligned}$$

So the curve $\mathbf{P}(t)$ that passes through the given points is

$$\mathbf{P}(t) = \mathbf{T}(t)\,\mathbf{A} = (0,-9)t^3 + (-4.5, 13.5)t^2 + (4.5,-3.5)t.$$

It is now easy to calculate and verify that $\mathbf{P}(0) = (0,0) = \mathbf{P}_1$, and

$$\begin{aligned} \mathbf{P}(1/3) &= (0,-9)(1/27) + (-4.5,13.5)(1/9) + (4.5,-3.5)(1/3) = (1,0) = \mathbf{P}_2,\\ \mathbf{P}(1) &= (0,-9)1^3 + (-4.5,13.5)1^2 + (4.5,-3.5)1 = (0,1) = \mathbf{P}_4. \end{aligned}$$

◇ **Exercise 3.1:** Calculate $\mathbf{P}(2/3)$ and verify that it equals $\mathbf{P}_3$.

◇ **Exercise 3.2:** Imagine the circular arc of radius 1 in the first quadrant (a quarter circle). Write the coordinates of the four points that are equally spaced on this arc. Use the coordinates to calculate a PC approximating this arc. Calculate point $\mathbf{P}(1/2)$. How far does it deviate from the midpoint of the true quarter circle?

◇ **Exercise 3.3:** Calculate the PC that passes through the four points $\mathbf{P}_1$ through $\mathbf{P}_4$ assuming that only the three relative coordinates $\Delta_1 = \mathbf{P}_2 - \mathbf{P}_1$, $\Delta_2 = \mathbf{P}_3 - \mathbf{P}_2$, and $\Delta_3 = \mathbf{P}_4 - \mathbf{P}_3$ are given. Show a numeric example.

The main advantage of this method is its simplicity. Given the four points, it is easy to calculate the PC that passes through them. This, however, is also the reason for the downside of the method. It produces only *one* PC that passes through four given points. If that PC does not have the required shape, there is nothing the user can do. This simple curve method is not interactive.

Even though this method is not very useful for curve drawing, it may be useful for interpolation. Given two points $\mathbf{P}_1$ and $\mathbf{P}_2$, we know that the point midway between them is their average, $(\mathbf{P}_1+\mathbf{P}_2)/2$. A natural question is: given four points $\mathbf{P}_1$ through $\mathbf{P}_4$, what point is located midway between them? We can answer this question by calculating the average, $(\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3+\mathbf{P}_4)/4$, but this weighted sum assigns the same weight to each of the four points. If we want to assign more weight to the interior points $\mathbf{P}_2$ and $\mathbf{P}_3$, we can calculate the PC that passes through the points and compute $\mathbf{P}(0.5)$ from Equation (3.6). The result is

$$\mathbf{P}(0.5) = -0.0625\mathbf{P}_1 + 0.5625\mathbf{P}_2 + 0.5625\mathbf{P}_3 - 0.0625\mathbf{P}_4.$$

This is a weighted sum that assigns more weight to the interior points. Notice that the weights are barycentric. Exercise 3.13 provides a hint as to why the two extreme weights are negative. This method can be extended to a two-dimensional grid of points (Section 3.6.1).

A precisian professor had the habit of saying: "...quartic polynomial $ax^4 + bx^3 + cx^2 + dx + e$, where e need not be the base of the natural logarithms."
—J. E. Littlewood, *A Mathematician's Miscellany*

◇ **Exercise 3.4:** The preceding method makes sense if the four points are (approximately) equally spaced along the curve. If they are not, the following approach may be taken. Instead of using 1/3 and 2/3 as the intermediate values, the user may specify values α and β, both in the interval $(0, 1)$, such that $\mathbf{P}_2 = \mathbf{P}(\alpha)$ and $\mathbf{P}_3 = \mathbf{P}(\beta)$. Generalize Equation (3.6) such that it depends on α and β.

3.2 The Lagrange Polynomial

The preceding section shows how a cubic interpolating polynomial can be derived for a set of four given points. This section discusses the Lagrange polynomial, a general approach to the problem of polynomial interpolation.

Given the $n+1$ data points $\mathbf{P}_0 = (x_0, y_0)$, $\mathbf{P}_1 = (x_1, y_1), \ldots, \mathbf{P}_n = (x_n, y_n)$, the problem is to find a function $y = f(x)$ that will pass through all of them. We first try an expression of the form $y = \sum_{i=0}^{n} y_i L_i^n(x)$. This is a weighted sum of the individual y_i coordinates where the weights depend on the x_i coordinates. This sum will pass through the points if

$$L_i^n(x) = \begin{cases} 1, & x = x_i, \\ 0, & \text{otherwise.} \end{cases}$$

A good mathematician can easily guess that such functions are given by

$$L_i^n(x) = \frac{\Pi_{j\neq i}(x - x_j)}{\Pi_{j\neq i}(x_i - x_j)} = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

(Note that $(x - x_i)$ is missing from the numerator and $(x_i - x_i)$ is missing from the denominator.) The function $y = \sum_{i=0}^{n} y_i L_i^n(x)$ is called the Lagrange polynomial because it was originally developed by Lagrange [Lagrange 77] and it is a polynomial of degree n. It is denoted by LP.

Horner's rule and the method of forward differences make polynomials very desirable to use. In practice, however, polynomials are used in parametric form as illustrated in Section 1.5, since any explicit function $y = f(x)$ is limited in the shapes of curves it can generate (note that the explicit form $y = \sum_{i=0}^{n} y_i L_i^n(x)$ of the LP cannot be calculated if two of the $n+1$ given data points have the same x coordinate).

The LP has two properties that make it impractical for interactive curve design, it is of a high degree and it is unique.

1. Writing $P_n(x) = 0$ creates an equation of degree n in x. It has n solutions (some may be complex numbers), so when plotted as a curve it intercepts the x axis n times. For large n, such a curve may be loose because it tends to oscillate wildly. In practice, we normally prefer tight curves.

2. It is easy to show that the LP is unique (see below). There are infinitely many curves that pass through any given set of points and the one we are looking for may not be the LP. Any useful, practical mathematical method for curve design should make it easy for the designer to change the shape of the curve by varying the values of parameters.

It's easy to show that there is only one polynomial of degree n that passes through any given set of $n+1$ points.

A root of the polynomial $P_n(x)$ is a value x_r such that $P_n(x_r) = 0$. A polynomial $P_n(x)$ can have at most n distinct roots (unless it is the zero polynomial). Suppose that there is another polynomial $Q_n(x)$ that passes through the same $n+1$ data points. At the points, we would have $P_n(x_i) = Q_n(x_i) = y_i$ or $(P_n - Q_n)(x_i) = 0$. The difference $(P_n - Q_n)$ is a polynomial whose degree must be $\leq n$, so it cannot have more than n distinct roots. On the other hand, this difference is 0 at the $n+1$ data points, so it has

$n+1$ roots. We conclude that it must be the zero polynomial, which implies that $P_n(x)$ and $Q_n(x)$ are identical.

This uniqueness theorem can also be employed to show that the Lagrange weights $L_i^n(x)$ are barycentric. Given a function $f(x)$, select $n+1$ distinct values x_0 through x_n, and consider the $n+1$ *support* points $(x_0, f(x_0))$ through $(x_n, f(x_n))$. The uniqueness theorem states that there is a unique polynomial $p(x)$ of degree n or less that passes through the points, i.e., $p(x_k) = f(x_k)$ for $k = 0, 1, \ldots, n$. We say that this polynomial interpolates the points. Now consider the constant function $f(x) \equiv 1$. The Lagrange polynomial that interpolates its points is

$$\mathrm{LP}(x) = \sum_{i=0}^{n} y_i L_i^n(x) = \sum_{i=0}^{n} 1 \times L_i^n(x) = \sum_{i=0}^{n} L_i^n(x).$$

On the other hand, $\mathrm{LP}(x)$ must be identical to 1, because $\mathrm{LP}(x_k) = f(x_k)$ and $f(x_k) = 1$ for any point x_k. Thus, we conclude that $\sum_{i=0}^{n} L_i^n(x) = 1$ for any x.

Because of these two properties, we conclude that a practical curve design method should be based on polynomials of low degree and should depend on parameters that control the shape of the curve. Such methods are discussed in the chapters that follow. Still, polynomial interpolation may be useful in special situations, which is why it is discussed in the remainder of this chapter.

◇ **Exercise 3.5:** Calculate the LP between the two points $\mathbf{P}_0 = (x_0, y_0)$ and $\mathbf{P}_1 = (x_1, y_1)$. What kind of a curve is it?

> I have another method not yet communicated... a convenient, rapid and general solution of this problem, *To draw a geometrical curve which shall pass through any number of given points...* These things are done at once geometrically with no calculation intervening... Though at first glance it looks unmanageable, yet the matter turns out otherwise. For it ranks among the most beautiful of all that I could wish to solve.
> (Isaac Newton in a letter to Henry Oldenburg, October 24, 1676, quoted in [Turnbull 59], vol. II, p 188.)
>
> —James Gleick, *Isaac Newton* (2003).

The LP can also be expressed in parametric form. Given the $n+1$ data points $\mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_n$, we need to construct a polynomial $\mathbf{P}(t)$ that passes through all of them, such that $\mathbf{P}(t_0) = \mathbf{P}_0$, $\mathbf{P}(t_1) = \mathbf{P}_1$, ..., $\mathbf{P}(t_n) = \mathbf{P}_n$, where $t_0 = 0$, $t_n = 1$, and t_1 through t_{n-1} are certain values between 0 and 1 (the t_i are called *knot* values). The LP has the form $\mathbf{P}(t) = \sum_{i=0}^{n} \mathbf{P}_i L_i^n(t)$. This is a weighted sum of the individual points where the weights (or basis functions) are given by

$$L_i^n(t) = \frac{\Pi_{j\neq i}^n (t - t_j)}{\Pi_{j\neq i}^n (t_i - t_j)}. \tag{3.10}$$

Note that $\sum_{i=0}^{n} L_i^n(t) = 1$, so these weights are barycentric.

◇ **Exercise 3.6:** Calculate the parametric LP between the two general points $\mathbf{P}_0$ and $\mathbf{P}_1$.

◇ **Exercise 3.7:** Calculate the parametric LP for the three points $\mathbf{P}_0 = (0,0)$, $\mathbf{P}_1 = (0,1)$, and $\mathbf{P}_2 = (1,1)$.

◇ **Exercise 3.8:** Calculate the parametric LP for the four equally-spaced points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, and $\mathbf{P}_4$ and show that it is identical to the interpolating PC given by Equation (3.4).

The parametric LP is also mentioned on page 109, in connection with Gordon surfaces.

The LP has another disadvantage. If the resulting curve is not satisfactory, the user may want to fine-tune it by adding one more point. However, all the basis functions $L_i^n(t)$ will have to be recalculated in such a case, since they also depend on the points, not only on the knot values. This disadvantage makes the LP slow to use in practice, which is why the Newton polynomial (Section 3.3) is sometimes used instead.

3.2.1 The Quadratic Lagrange Polynomial

Equation (3.10) can easily be employed to obtain the Lagrange polynomial for three points $\mathbf{P}_0$, $\mathbf{P}_1$, and $\mathbf{P}_2$. The weights in this case are

$$\begin{aligned}
L_0^2(t) &= \frac{\prod_{j\neq 0}^2 (t-t_j)}{\prod_{j\neq 0}^2 (t_0-t_j)} = \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)},\\
L_1^2(t) &= \frac{\prod_{j\neq 1}^2 (t-t_j)}{\prod_{j\neq 1}^2 (t_1-t_j)} = \frac{(t-t_0)(t-t_2)}{(t_1-t_0)(t_1-t_2)},\\
L_2^2(t) &= \frac{\prod_{j\neq 2}^2 (t-t_j)}{\prod_{j\neq 2}^2 (t_2-t_j)} = \frac{(t-t_0)(t-t_1)}{(t_2-t_0)(t_2-t_1)},
\end{aligned} \tag{3.11}$$

and the polynomial $\mathbf{P}_2(t) = \sum_{i=0}^2 \mathbf{P}_i L_i^2(t)$ is easy to calculate once the values of t_0, t_1, and t_2 have been determined.

The Uniform Quadratic Lagrange Polynomial is obtained when $t_0 = 0$, $t_1 = 1$, and $t_2 = 2$. (See discussion of uniform and nonuniform parametric curves in Section 1.4.1.) Equation (3.11) yields

$$\begin{aligned}
\mathbf{P}_{2u}(t) &= \frac{t^2-3t+2}{2}\mathbf{P}_0 - (t^2-2t)\mathbf{P}_1 + \frac{t^2-t}{2}\mathbf{P}_2\\
&= (t^2, t, 1)\begin{pmatrix} 1/2 & -1 & 1/2\\ -3/2 & 2 & -1/2\\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}.
\end{aligned} \tag{3.12}$$

The sums of three rows of the matrix of Equation (3.12) are (from top to bottom) 0, 0, and 1, showing that the three basis functions are barycentric, as they should be.

The Nonuniform Quadratic Lagrange Polynomial is obtained when $t_0 = 0$, $t_1 = t_0+\Delta_0 = \Delta_0$, and $t_2 = t_1+\Delta_1 = \Delta_0+\Delta_1$ for some positive Δ_0 and Δ_1. Equation (3.11) gives

$$L_0^2(t) = \frac{(t-\Delta_0)(t-\Delta_0-\Delta_1)}{(-\Delta_0)(-\Delta_0-\Delta_1)},\ L_1^2(t) = \frac{(t-0)(t-\Delta_0-\Delta_1)}{\Delta_0(-\Delta_1)},\ L_2^2(t) = \frac{(t-0)(t-\Delta_0)}{(\Delta_0+\Delta_1)\Delta_1},$$

and the nonuniform polynomial is

$$\mathbf{P}_{2nu}(t) = (t^2, t, 1)\begin{bmatrix} \dfrac{1}{\Delta_0(\Delta_0+\Delta_1)} & -\dfrac{1}{\Delta_0\Delta_1} & \dfrac{1}{(\Delta_0+\Delta_1)\Delta_1} \\ \dfrac{-1}{\Delta_0+\Delta_1} - \dfrac{1}{\Delta_0} & \dfrac{1}{\Delta_0} + \dfrac{1}{\Delta_1} & -\dfrac{1}{\Delta_1} + \dfrac{1}{\Delta_0+\Delta_1} \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{bmatrix}. \tag{3.13}$$

For $\Delta_0 = \Delta_1 = 1$, Equation (3.13) reduces to the uniform polynomial, Equation (3.12). For $\Delta_0 = \Delta_1 = 1/2$, the parameter t varies in the "standard" range $[0, 1]$ and Equation (3.13) becomes

$$\mathbf{P}_{2std}(t) = (t^2, t, 1)\begin{pmatrix} 2 & -4 & 2 \\ -3 & 4 & -1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix}\mathbf{P}_0\\ \mathbf{P}_1\\ \mathbf{P}_2\end{pmatrix}. \tag{3.14}$$

(Notice that the three rows again sum to 0, 0, and 1, to produce three barycentric basis functions.) In most cases, Δ_0 and Δ_1 should be set to the chord lengths $|\mathbf{P}_1 - \mathbf{P}_0|$ and $|\mathbf{P}_2 - \mathbf{P}_1|$, respectively.

◇ **Exercise 3.9:** Use Cartesian product to generalize Equation (3.14) to a surface patch that passes through nine given points.

Example: The three points $\mathbf{P}_0 = (1, 0)$, $\mathbf{P}_1 = (1.3, .5)$, and $\mathbf{P}_2 = (4, 0)$ are given. The uniform LP is obtained when $\Delta_0 = \Delta_1 = 1$ and it equals

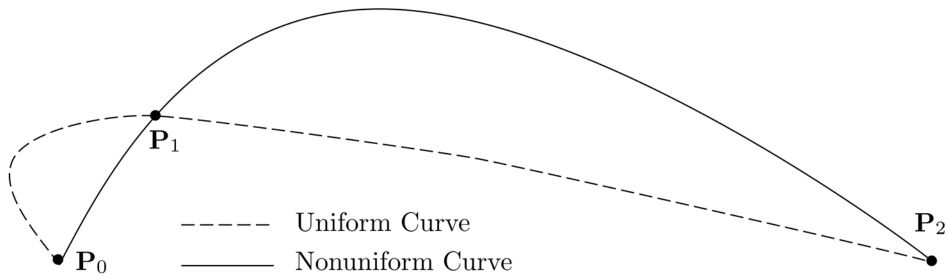
$$\mathbf{P}_{2u}(t) = \bigl(1 - 0.9t + 1.2t^2, 0.5(2 - t)t\bigr).$$

Many nonuniform polynomials are possible. We select the one that's obtained when the Δ values are the chord lengths between the points. In our case, they are $\Delta_0 = |\mathbf{P}_1 - \mathbf{P}_0| \approx 0.583$ and $\Delta_1 = |\mathbf{P}_2 - \mathbf{P}_1| \approx 2.75$. This polynomial is

$$\mathbf{P}_{2nu}(t) = (1 + 0.433t + 0.14t^2, 1.04t - 0.312t^2).$$

These uniform and nonuniform polynomials are shown in Figure 3.1. The figure illustrates how the nonuniform curve based on the chord lengths between the points is tighter (features smaller overall curvature). Such a curve is generally considered a better interpolation of the three points.

Figure 3.2 shows three examples of nonuniform Lagrange polynomials that pass through the three points $\mathbf{P}_0 = (1, 1)$, $\mathbf{P}_1 = (2, 2)$, and $\mathbf{P}_2 = (4, 0)$. The value of Δ_0 is 1.414, the chord length between $\mathbf{P}_0$ and $\mathbf{P}_1$. The chord length between $\mathbf{P}_1$ and $\mathbf{P}_2$ is 2.83 and Δ_1 is first assigned this value, then half this value, and finally twice it. The three




```
(* 3-point Lagrange polynomial (uniform and nonunif) *)
Clear[T,H,B,d0,d1];
d0=1; d1=1;
T={t^2,t,1};
H={{1/(d0(d0+d1)),-1/(d0 d1),1/(d1(d0+d1))},
{-1/(d0+d1)-1/d0,1/d0+1/d1,-1/d1+1/(d0+d1)},{1,0,0}};
B={{1,0},{1.3,.5},{4,0}};
Simplify[T.H.B];
C1=ParametricPlot[T.H.B,{t,0,d0+d1},PlotRange->All, Compiled->False,
 PlotStyle->AbsoluteDashing[{2,2}], DisplayFunction->Identity];
d0=.583; d1=2.75;
H={{1/(d0(d0+d1)),-1/(d0 d1),1/(d1(d0+d1))},
{-1/(d0+d1)-1/d0,1/d0+1/d1,-1/d1+1/(d0+d1)},{1,0,0}};
Simplify[T.H.B];
C2=ParametricPlot[T.H.B,{t,0,d0+d1},PlotRange->All, Compiled->False,
 DisplayFunction->Identity];
Show[C1, C2, AspectRatio->Automatic, DefaultFont->{"cmr10", 10},
 DisplayFunction->$DisplayFunction];
```

Figure 3.1: Three-Point Lagrange Polynomials.

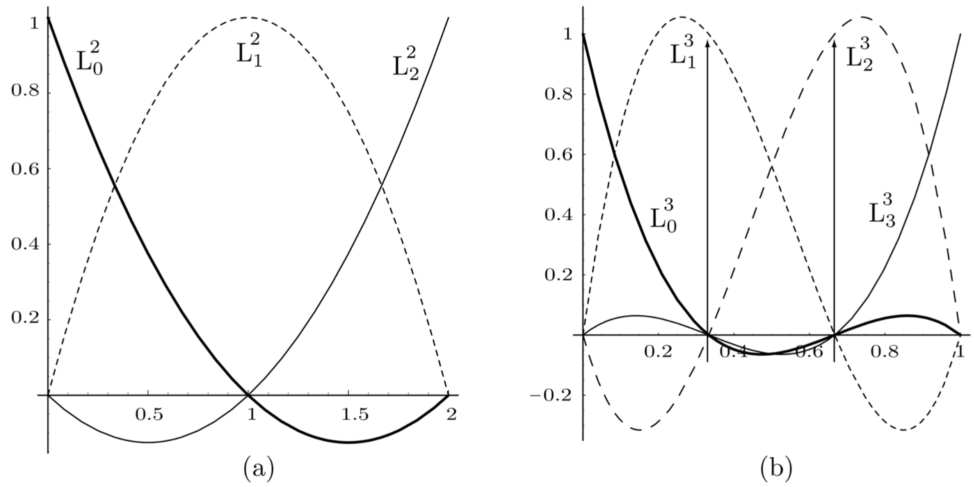
resulting curves illustrate how the Lagrange polynomial can be reshaped by modifying the Δ_i parameters. The three polynomials in this case are

$$
\begin{gathered}
(1+0.354231t+0.249634t^2, 1+1.76716t-0.749608t^2),\\
(1+0.70738t-0.000117766t^2, 1+1.1783t-0.333159t^2),\\
(1+0.777945t-0.0500221t^2, 1+0.919208t-0.149925t^2).
\end{gathered}
$$

3.2.2 The Cubic Lagrange Polynomial

Equation (3.10) is now applied to the cubic Lagrange polynomial that interpolates the four points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$. The weights in this case are

$$
\begin{aligned}
L_0^3(t) &= \frac{\prod_{j\neq 0}^3 (t-t_j)}{\prod_{j\neq 0}^3 (t_0-t_j)} = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)},\\
L_1^3(t) &= \frac{\prod_{j\neq 1}^3 (t-t_j)}{\prod_{j\neq 1}^3 (t_1-t_j)} = \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)},\\
L_2^3(t) &= \frac{\prod_{j\neq 2}^3 (t-t_j)}{\prod_{j\neq 2}^3 (t_2-t_j)} = \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_2-t_0)(t_2-t_1)(t_2-t_3)},
\end{aligned}
$$




```
(* Plot quadratic and cubic Lagrange basis functions *)
lagq={t^2,t,1}.{{1/2,-1,1/2},{-3/2,2,-1/2},{1,0,0}};
Plot[{lagq[[1]],lagq[[2]],lagq[[3]]},{t,0,2},
 PlotRange->All, AspectRatio->Automatic, DefaultFont->{"cmr10", 10}];
lagc={t^3,t^2,t,1}.{{-9/2,27/2,-27/2,9/2},{9,-45/2,18,-9/2},
 2{-11/2,9,-9/2,1},{1,0,0,0}}
Plot[{lagc[[1]],lagc[[2]],lagc[[3]],lagc[[4]]},{t,0,1},
 PlotRange->All, AspectRatio->Automatic, DefaultFont->{"cmr10", 10}];
```

Figure 3.3: (a) Quadratic and (b) Cubic Lagrange Basis Functions.

appropriate mathematical software and use code similar to that of Figure 3.3 to plot the basis functions for various values of Δ_i.

It should be noted that the basis functions of the Bézier curve (Section 6.2) are more intuitive and provide easier control of the shape of the curve, which is why Lagrange interpolation is not popular and is used in special cases only.

3.2.3 Barycentric Lagrange Interpolation

Given the $n+1$ data points $\mathbf{P}_0 = (x_0, y_0)$ through $\mathbf{P}_n = (x_n, y_n)$, the explicit (nonparametric) Lagrange polynomial that interpolates them is $\text{LP}(x) = \sum_{i=0}^{n} y_i L_i^n(x)$, where

$$L_i^n(x) = \frac{\Pi_{j\neq i}^n (x - x_j)}{\Pi_{j\neq i}^n (x_i - x_j)} = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}.$$

This representation of the Lagrange polynomial has the following disadvantages:

1. The denominator of $L_i^n(x)$ requires n subtractions and $n-1$ multiplications, for a total of $O(n)$ operations. The denominators of the $n+1$ weights therefore require $O(n^2)$ operations. The numerators also require $O(n^2)$ operations, but have to be recomputed for each value of x.

2. Adding a new point $\mathbf{P}_{n+1}$ requires the computation of a new weight $L_{n+1}^{n+1}(x)$