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Unit-01

Number system

Natural numbers :-

$N = \{1, 2, 3 \ldots\}$

Whole numbers :-

$W = \{0, 1, 2, 3 \ldots\}$

Integers :- Z or $I = \{-2, -1, 0, 1, 2, 3\}$

Rational numbers :-

$$Q = \{\frac{p}{q} ;\ p, q \in Z, q \neq 0\}$$

Irrational numbers :- $Q_i = \{\frac{p}{q} \rightarrow$ non-terminating & non-repeating)

Real numbers :-

All numbers fraction, natural, whole etc.

Complex numbers :- $C = \{x + iy ;\ x \in R \ \& \ y \in R\}$

Sets Theory

Set → It is a well-defined collection of distinct objects is called a set.

The objects in a set are called members or elements.

We denote a set by capital letters A, B, C etc. & the elements of a set are denoted by small letters.

Representation :-

(1) **Roster or tabular form :-**

In this form, all the elements are listed within braces {} and are seperated by commas.

eg :- B = {Set of all even +ve integer < 10

$B = \{2, 4, 6, 8\}$

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(2) Set builder ~~or~~ or rule method :-

In this form, we list the property satisfied by all the elements of the set.
It is written as $\{x : x \text{ satisfy the property}\}$

eg:- (1) $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$A = \{x : x \in N < 9\}$
(2) $C = \{x + iy : x, y \in R\}$
(3) $Q = \{p/q : p, q \in Z \;\&\; q \neq 0\}$

Types of sets

(1) Empty set or null set or void set :-
A set containing no element at all. It is denoted by ϕ or $\{\}$.
(*) Note :- If a set has at least one element then that set is ~~ton~~ known as non-empty set.
(2) Singleton set :- A set containing exactly one element belongs to a set.
(*) * Cardinality of a set :-

The number of distinct elements containing in a set A. It is denoted by $n(A)$ or $|A|$.
$X = \{y, x, z, t\}$
$n(A) = 4$
(3) Finite and infinite set :-
Set is said to be finite if it consists of only finite number of elements otherwise it is called infinite set.
(4) ~~Equivalent~~ Equal set :-
Two sets A & B are equal if they have exactly same elements & we write $A = B$.

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(5) Equivalent set :-
Two finite sets A & B are said to be equivalent if $n(A) = n(B)$.

Subsets & power sets

* Subsets :- A set A is said to be subset of B, if every element of A is also an element of B. & we write $A \subseteq B$
* Superset :- If $A \subseteq B$ then B is a superset of A. Also write $B \supseteq A$
* Proper subset :- If $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B & we write $A \subset B$.

let,
eg :- $A = \{2, 3, 5\}$
$B = \{2, 3, 5, 7, 9\}$
$A \subset B$ (proper subset)

$A = \{2, 1, 3\}$
$B = \{2, 3, 1\}$
$A \subseteq B$ (subset)
and B is a superset.

Note (1) For every set A, we have $A \subseteq A$
(2) since ϕ has no element, therefore ϕ is a subset of every set.

Intervals as subsets of R → let $a, b \in R$ & $a < b$
(i) closed interval $[a, b] = \{x \in R : a \leq x \leq b\}$
(ii) open interval $(a, b) = \{x \in R : a < x < b\}$
(iii) $[a, b) = \{x \in R ; a \leq x < b\}$
(iv) $(a, b] = \{x \in R ; a < x \leq b\}$

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Power set

The set of all subset of A (given set) is called power set of A and is denoted by P(A).

Note :- If $\boxed{n(A)=m}$ then $\boxed{n[P(A)] = 2^m}$

eg: Write down all possible subsets of

(i) $A = \{2,3\}$

(ii) $B = \{a,b\}$

(iii) $C = \{a,b,c\}$

Soln (i) $A = \{\phi, \{2\}, \{3\}, \{2,3\}\}$

$n(A) = 2$, $n[P(A)] = 2^2 = 4$

(ii) $B = \{\phi, \{a\}, \{b\}, \{a,b\}\}$ $n(A) = 2$, $n[P(A)] = 4$

(iii) $C = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

$n(C) = 3$, $2^3 = n(P(A)) = 8$

eg :- Two finite sets have m & n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of m & n.

Soln Let A & B are two given sets such that $n(A) = m$ & $n(B) = n$

If a set A has m elements then total no. of subsets of A is 2^m.

$$\left.\begin{aligned} n[P(A)] &= 2^m \\ \& \; n[P(B)] &= 2^n \end{aligned}\right\} - (2)$$

Given $n[P(A)] = n[P(B)] + 56$

$$2^m = 2^n + 56$$

$$2^m - 2^n = 56$$

$$2^n[2^{m-n} - 1] = 56$$

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$= 2^n[2^{m-n} - 1] = 2^3 \times 7$

comparing

$2^n = 2^3 \Rightarrow \boxed{n=3}$

$[2^{m-n} - 1] = 7$

$2^{m-n} = 8 \Rightarrow 2^{m-3} = 8$

$2^{m-3} = 2^3$

$m - 3 = 3 \Rightarrow \boxed{m=6}$

$\therefore \boxed{m=6}, \boxed{n=3}$

* Operation on sets :-

① Union of set :- The union of two sets A and B, denoted by $A \cup B$, is the set of all these elements which are either in A or in B or in both A and B.

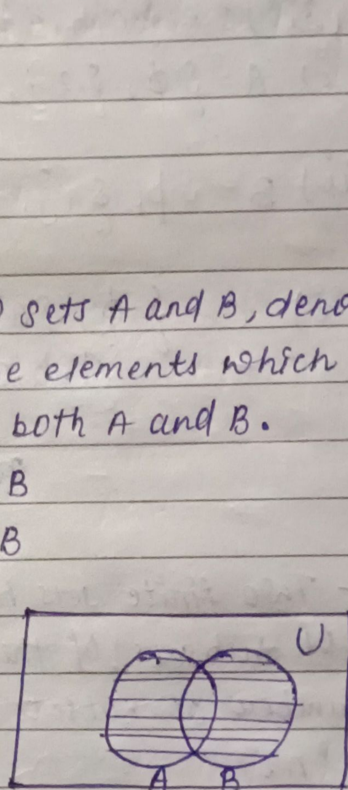
Note → (i) $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

(ii) $x \notin A \cup B \Rightarrow x \notin A$ or $x \notin B$

eg :- Let $A = \{2, 3, 4\}$

$B = \{1, 5, 4, 6\}$

$A \cup B = \{1, 2, 3, 4, 5, 6\}$



eg :- $A = \{x : x$ is a prime no. and less than $10\}$

$A = \{2, 3, 5, 7\}$

$B = \{x : x \in N, x$ is a factor of $8\}$

$B = \{1, 2, 4, 8\}$

Soln $A \cup B = \{1, 2, 3, 4, 5, 7, 8\}$

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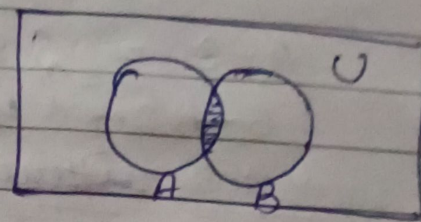
(2) **Intersection of sets** :- The intersection of two sets A and B denoted by $A \cap B$, is the set of all those elements which are common in b/w A and B.

Note :- 1) $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

2) $x \notin A \cap B \Rightarrow x \notin A$ or $x \notin B$



eg :- $A = \{1, 3, 5, 7, 9\}$

$B = \{2, 3, 5, 7, 11, 13\}$

$A \cap B = \{3, 5, 7\}$

Ques If $A_i = \{0, i\}$ where $i \in Z$, the set of integer, find

(i) $A_1 \cup A_2$

(ii) $A_3 \cap A_4$

(iii) $\bigcup_{i=5}^{10} A_i$

$A_1 = \{0, 1\}$

$A_2 = \{0, 2\}$

$A_3 = \{0, 3\}$

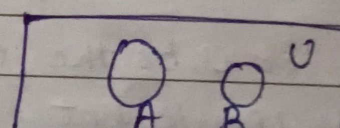
$A_4 = \{0, 4\}$

Soln (i) $\{0, 1, 2\}$

(ii) $\{0\}$

(iii) $\{0, 5, 6, 7, 8, 9, 10\}$

(3) **Disjoint sets** :- Two sets A and B are said to be disjoint if $A \cap B = \phi$



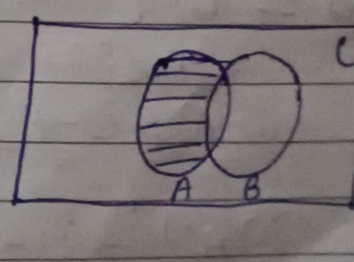
eg :- $A = \{1, 2, 3\}$ & $B = \{5, 6, 7\}$

$A \cap B = \phi$

A & B are disjoint sets.

(4) **Difference of sets** :- For any sets A and B, their difference (A − B) is defined as

$A - B = \{x : x \in A \text{ and } x \notin B\}$



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eg:- If $A = \{x : x \in N, x \text{ is a factor of } 6\}$
$B = \{x : x \in N, x \text{ is a factor of } 8\}$

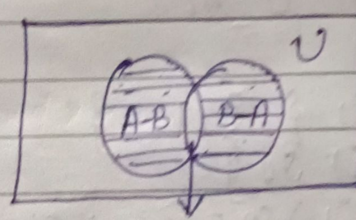
$$A = \{1, 2, 3, 6\}, \quad B = \{1, 2, 4, 8\}$$

$A - B = \{3, 6\}$ $\quad\quad$ $A \Delta B = \{3, 6, 4, 8\}$
$B - A = \{4, 8\}$

(5) Symmetric difference of sets → The symmetric difference of two sets A & B is denoted by $\boxed{A \Delta B}$ and defined by

$$A \Delta B = (A - B) \cup (B - A)$$

$$A \Delta B = (A \cup B) - (A \cap B)$$

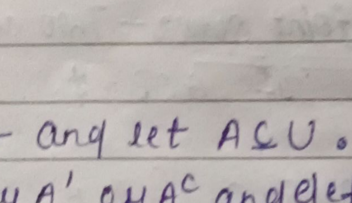


A∩B

(6) Complement of a set :-

Let U be the universal set and let $A \subseteq U$. Then, the complement of A denoted by A' or A^c and defined as

$$A' \text{ or } A^c = U - A$$



Ques If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$
$A = \{2, 4, 6, 8\}$

(i) $A' = U - A = \{1, 3, 5, 7\}$

(ii) $(A')' = U - A'$
$= \{2, 4, 6, 8\} = A$

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General Identities on sets

(1) Idempotent law :-

(i) $A \cup A = A$ (ii) $A \cap A = A$

(2) commutative law :-

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

(3) Associative law :-

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

(4) Distributive Law :-

(1) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(2) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(5) De Morgan Law :-

(1) $(A \cup B)' = A^c \cap B^c$ or $(A \cup B)' = A' \cap B'$

(2) $(A \cap B)' = A^c \cup B^c$ or $A' \cup B'$

(6) Identity Law :-

(1) $A \cup \phi = A$ (3) $A \cap \phi = \phi$

(2) $A \cup U = U$ (4) $A \cap U = A$

(7) complement law :-

(1) $A \cup A^c = U$

(2) $A \cap A^c = \phi$

(3) $U^c = \phi$

(4) $\phi^c = U$

(8) Involution Law :-

$(A^c)^c = A$

(9) Absorption law :-

$A \cup (A \cap B) = A$

$A \cap (A \cup B) = A$

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Proof :-

$\left\{ \begin{array}{l} A = B \\ A \subset B \\ B \subset A \end{array} \right.$ If $A \subset B$ and $B \subset A$ then $A = B$

① Idempotent law :-

→ (i) $A \cup A = A$

Let $x \in A \cup A \Rightarrow x \in A$ or $x \in A$

$\Rightarrow x \in A$

$\Rightarrow A \cup A \subset A$ — ①

$x \in A \cup A = x \in A$

$A \cup A \subset A$

Again $y \in A \Rightarrow y \in A$ or $y \in A$

$\Rightarrow y \in (A \cup A)$

$A \subset A \cup A$ — ②

from ① & ②

$\boxed{A \cup A = A}$

→ (ii) $A \cap A = A$

let $x \in A \cap A \Rightarrow x \in A$ and $x \in A$

$\Rightarrow x \in A$

$= A \cap A \subset A$ — ①

Let $y \in A \Rightarrow y \in A$ and $y \in A$

$\Rightarrow y \in A \cap A$

$= A \subset A \cap A$ — ②

from ① & ②

$\boxed{A \cap A = A}$

② Commutative law :-

(i) $A \cup B = B \cup A$

Let $x \in A \cup B = \{x : x \in A \text{ or } x \in B\}$

$= \{x : x \in B \text{ or } x \in A\}$

$= B \cup A$

(ii) $A \cap B = B \cap A$

Let $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$= \{x : x \in B \text{ and } x \in A\}$

$= B \cap A$

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③ Associative law:-

(i) $(A \cup B) \cup C = A \cup (B \cup C)$

$x \in (A \cup B) \cup C = \{x : x \in A \cup B \text{ or } x \in C\}$

$= \{x : (x \in A \text{ or } x \in B) \text{ or } x \in C\}$

$= \{x : x \in A \text{ or } (x \in B \text{ or } x \in C)\}$

$= A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

$(A \cap B) \cap C = \{x : x \in A \cap B \text{ and } x \in C\}$

$= \{x : (x \in A \text{ and } x \in B) \text{ and } x \in C\}$

$= \{x : x \in A \text{ and } (x \in B \text{ and } x \in C)\}$

$= A \cap (B \cap C)$

~~④ Distributive Laws:-~~

④ De Morgan's law :-

(i) $(A \cup B)' = A' \cap B'$

Let $x \in (A \cup B)' = \{x \notin A \cup B\}$

$= x \notin A \text{ and } x \notin B$

$= x \in A' \text{ and } x \in B'$

$= x \in A' \cap B'$

$= (A \cup B)' \subset A' \cap B'$ — ①

Let $y \in A' \cap B' = \{y \in A' \text{ and } y \in B'$

$= y \notin A \text{ and } y \notin B$

$= y \notin A \cup B$

$= y \in (A \cup B)'$.

$A' \cap B' \subset (A \cup B)'$ — ②

from ① & ②

$(A \cup B)' = A' \cap B'$

Hence proved

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(ii) $(A \cap B)' = A' \cup B'$

Let $x \in (A \cap B)' = \{x \notin A \cap B$

$= x \notin A$ or $x \notin B$

$= x \in A'$ or $x \in B'$

$= x \in A' \cup B'$

$(A \cap B)' \subset A' \cup B'$ — (1)

Let $y (A' \cup B') = y \in A'$ or $y \in B'$

$= y \notin A$ or $y \in B$

$= y \notin A \cap B$

$= y \in (A \cap B)'$

$A' \cup B' = (A \cap B)'$ — (2)

from (1) & (2)

$(A \cap B)' = A' \cup B'$

Hence prove!!

(5) Distributive Law :-

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let $x \in A \cap (B \cup C) = x \in A$ and $x \in B \cup C$

$= x \in A$ and $(x \in B$ or $x \in C)$

$= \cancel{x \in A \text{ and } x \in B)}$

$= (x \in A$ and $x \in A)$ and $(x \in B$ or $x \in C)$

$= (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$= x \in A \cap B$ or $A \cap C$

$= x \in (A \cap B) \cup (A \cap C)$

$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ — (1)

Let $y \in (A \cap B) \cup (A \cap C) =$

$= y \in A \cap B$ or $y \in A \cap C$

$= (y \in A$ and $y \in B)$ or $(y \in A$ and $y \in C)$

$= (y \in A$ or $y \in A)$ and $(y \in B$ or $y \in C)$

$= y \in A \cap y \in B \cup C$

$= y \in A \cap (B \cup C)$ — (2)

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$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Hence proved!!

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C) = x \in A$ or $x \in B \cap C$

$= x \in A$ or $(x \in B$ and $x \in C)$

$= (x \in A$ ~~and~~ or $x \in A)$ ~~or~~ and $(x \in B$ ~~and~~ or $x \in C)$

$= (x \in A$ ~~and~~ or $x \in B)$ ~~or~~ and $(x \in A$ ~~and~~ or $x \in C)$

$= \cancel{(A \cup B) \cup (A \cup C)}\ (A \cup B) \cap (A \cup C)$

$A \cup (B \cap C) \subset (A \cup B) \cup (A \cup C)$ — ①

Let $y \in (A \cup B) \cap (A \cup C) = y \in A \cup B$ ~~and~~ and $y \in A \cup C$

$= (y \in A$ ~~and~~ or $y \in B)$ ~~or~~ and $(y \in A$ and $y \in C)$

$= (y \in A$ ~~or~~ ~~or~~ ~~or~~ and $y \in A)$ ~~and~~ or $(y \in B$ ~~or~~ and $y \in C)$

$= (y \in A) \cup (B \cap C)$

$= A \cup (B \cap C)$ — ②

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Hence proved!!

Ques For any set A and B, show that

$A - B = A \cap B'$ $\quad A - B$

or $A - B = A \cap B'$

~~Ans~~ Sol^n let $x \in A - B = x \in A$ and $x \notin B$

$= x \in A$ and $x \in B'$

$= x \in A \cancel{\cup} \cap B'$

$A - B \subset A \cap B'$ — ①

$y \in A \cap B' = y \in A$ and $y \in B'$

$= y \in A$ and $y \notin B$

$= y \in A - B$

$A \cancel{A} \cap B' \subset y \in A - B$ — ②

$\boxed{A - B = A \cap B'}$ Hence proved!

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Ques $A-(A\cap B) = A-B$

$A-(A\cap B)$ $\quad A-B = A\cap B'$

$= A\cap(A\cap B)'$

$= A\cap(A'\cup B')$ [De Morgan's Law)

$= (A\cap A')\cup(A\cap B')$

$= \phi\cup(A\cap B')$

$= A\cap B'$

$= A-B$ $[A\cap B' = A-B]$

Hence prove!!

Ques $A-(B\cap C) = (A-B)\cup(A-C)$

L.H.S. $A-(B\cap C)$

$= A \cap (B\cap C)'$

$= \cancel{A\cap B}\ A\cap(B'\cup C')$

$= (A\cap B')\cup(A\cap C')$

$= (A-B)\cup(A-C)$

Hence prove!!

Ques (i) $(A\cap B)\cup(A-B) = A$ $\quad A\cap(B\cup C)$

$(A\cap B)\cup(A\cap C)$

L.H.S.

$(A\cap B)\cup(A-B) = (A\cap B)\cup(A\cap B')$

$= A\cap(B\cup B')$ [Distributive Law]

$= A\cap U$

$= \boxed{A}$ Ans

(ii) $A\cup(B-A) = A\cup B$

$A\cup(B\cap A') = (A\cup B)\cap A\cup A'$

$= (A\cup B)\cap U$

$= \boxed{A\cup B} = $ R.H.S. Ans

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Ques $A-(B\cup C) = (A-B)\cap(A-C)$

L.H.S. $= A\cap(B\cup C)'$

$= A\cap(B'\cap C')$

$= (A\cap B')\cap(A\cap C')$

$= (A-B)\cap(A-C)$

Hence prove!!

Ques $A\subseteq B$ then $B^C \subseteq A^C$

Let $A\subseteq B$

$x\in A \Rightarrow x\in B$

$x\notin B \Rightarrow x\notin A$

$x\in B' \Rightarrow x\in A'$

$\boxed{B^C \subseteq A^C}$ Ans

* Ordered pairs :- It consists of two ~~pairs~~ elements such that one of them is designated as first member and other as second member. If p is the first element and q is the second element, the ordered pair is written as (p, q)

* Cartesian product of two sets :-

The cartesian product of two sets A and B is the set of all ordered pairs whose first member belong to the set A and second member to the set B and is denoted as $A\times B$.

$A\times B = \{(x, y) : x\in A \text{ and } y\in B\}$

Note :- $n(A\times B) = n\times m$

eg :- $A = \{1, 2\}$

$B = \{3, 4, 5\}$

$A\times B = \{(1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$

$n(A\times B) = n(A)\times n(B)$

$= 2\times 3 = 6$

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Ques $P = \{a, b, c\}$ & $P = \{a, b, c\}$

$= \{(a,b), (a,a), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

$= 3 \times 3 = 9$

* Relation or binary Relation →

Let A and B be non-empty sets, then any subset R of the cartesian product $A \times B$ is called a relation from A to B is donated by R.

∴ R is relation from $A \to B$.

Ques Let $A = \{1, 2, 5\}$ and $B = \{2, 4\}$ be two given sets. Find out the relation from A to B defined by "less than".

Ans $A = \{1, 2, 5\}$

$B = \{2, 4\}$

$A \times B = \{(1,2), (1,4), (2,2), (2,4), (5,2), (5,4)\}$

$R = \{(1,2), (1,4), (2,4)\}$

Domain R = The set of first coordinate of every element of R.

Range R = The set of second " " " " of R.

$D = \{(1, 2)\}$, $R = \{2, 4\}$

Ques Total number of distinct binary relation from $A \to B$

$2^{m \cdot n}$

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* Identity Relation :-

The identity relation I_A on a set A is defined as
$I_A = \{(x,y) : x, y \in A \text{ and } x = y\} = \{(x,x) : x \in A\}$

Ques $A = \{a, b, c\}$
$I_A = \{(a,a), (b,b), (c,c)\}$

* Complement of a relation :-

Consider a relation R from $A \to B$. The complement of relation R denoted by $\overline{R}$ or R' is a relation from $A \to B$ such that R' or $\overline{R} = \{(a,b) : (a,b) \notin R\}$

Ques Let R be relation from $x \to y$, then
$X = \{1, 2, 3\}$
$Y = \{8, 9\}$
$R = \{(1,8), (2,8), (1,9), (3,9)\}$ find the $\overline{R}$ or R'

Ans $X \times Y = \{(1,8), (1,9), (2,8), (2,9), (3,8), (3,9)\}$

$R = \{(1,8), (2,8), (1,9), (3,9)\}$
$\overline{R} = \{(2,9), (3,8)\}$ Ans

* Inverse relation :-

$A \to B = R$
$R^{-1} = B \to A$

$R^{-1} = \{(b,a) : (b,a) \in R^{-1} \text{ and } (a,b) \in R\}$

Ques R^{-1} of R on A defined by "x+y divisible by 2".
For $A = \{1,2,3\}$

Soln $A = \{1,2,3\}$
$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$
$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$
$R^{-1} = \{(1,1), (3,1), (2,2), (3,3), (1,3)\}$ Ans

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* Intersection and union of relation →

If R and S are two relations then

$R \cup S = \{ \}$

$R \cap S = \{ \}$

eg :- Let $R_1 = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3)\}$

$R_2 = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1)\}$

$R_1 \cup R_2 = \{(1,1), (2,2), (3,3), (4,4), (3,4), (4,3), (1,2), (2,1)\}$

$R_1 \cap R_2 = \{(1,1), (2,2), (3,3), (4,4)\}$

* Properties of relation :-

① Reflexive relative :-

A relation R on a set A is reflexive relation if $(a,a) \in R \; \forall \; a \in A$

ex :- $A = \{a,b\}$

$R = \{(a,a), (b,b), (a,b)\}$

It is a reflexive relation

eg :- $A = \{a,b\}$

$R = \{(a,a), (b,a)\}$

$(b,b) \notin R$ It is not a reflexive relation

② Irreflexive relation :-

$(a,a) \notin R$

Let $A = \{(1,2)\}$ and

$R = \{(1,2), (2,1)\}$

It is irreflexive relation

(3) Non-reflexive relation :-
A relation R on a set A is non-reflexive if R is neither reflexive nor irreflexive.

(4) Symmetric Relation :- If R is a relation in the set A, then R is called symmetric relation.
if "a is related to b then b is also related to a".
i.e. $(a,b) \in R$ then $(b,a) \in R$
for symmetric relation $R = R^{-1}$

Ques $A = \{2,4,5,6\}$
$R_1 = \{(2,4),(4,2),(4,5),(5,4),(6,6)\}$
It is symmetric relation

(5) Antisymmetric Relation :-
A relation R is said to be antisymmetric if $(a,b) \in R$ and $(b,a) \in R \Rightarrow \boxed{a=b}$

(6) Asymmetric Relation :- A relation R on set A is asymmetric if $(a,b) \in R$ & $(b,a) \notin R$

$A = \{1,2,3\}$
$R = \{(1,2),(1,1),(2,3),(3,1)\}$
Asymmetric Relation

(7) Transitive relation :- The relation R on set A is called transitive if $(a,b) \in R$ $(b,c) \in R$ and $(a,c) \in R$

Ques If $A = \{1,3,5\}$
$R = \{(1,3),(1,5),(3,5)\}$
It is a Transitive relation

Ques $A = \{3,4,5\}$
$R = \{(3,4),(4,3),(5,4),(5,3)\}$
Not a transitive relation

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Equality of Relation
or
Equivalence Relation

⇒ Let A be non-empty set and R be any relation defined on A. The R is said to be equivalence if it is transitive, reflexive & symmetric.

* **Equivalence classes :-**

Consider an equivalence relation R on a set A.
The equivalence class of an element $a \in A$ is the set of elements of A to which element a is related.
It is denoted by $[a]$ or $\bar{a}$.

$$[a] = \{ b \in A : a R b \}$$

Ques Let $A = \{0, 1, 2, 3, 4\}$
Show that

$$R = \{ (0,0), (0,4), (1,1), (1,3), (2,2), (3,1), (3,3), (4,0), (4,4) \}$$

→ Reflexive relation ✓
→ Symmetric relation ✓
→ Transitive relation ✓

It is an equivalence relation

equivalence classes :-

$[0] = \{0, 4\}$
$[1] = \{1, 3\}$
$[2] = \{2\}$
$[3] = \{1, 3\}$
$[4] = \{0, 4\}$

Ques $X = \{1, 2, 3, 4, 5, 6, 7\}$
$R = \{(x, y) : (x-y)$ is divisible by 3$\}$

Sol^n
for* Reflexive $= \{(1,1), (2,2) (3,3), (4,4), (5,5) (6,6)\}$
$R = (x-x)$ is divisble by 3
$(x, x) \in R$
$= xRx \ \forall \ x \in X$
R is reflexive

(2) Symmetric :-

Let $x, y \in X$ and $(x, y) \in R$
$\Rightarrow (x-y)$ is divisible by 3.
$= (x-y) = 3n_1 \ (n_1 \in I)$
$\Rightarrow (y-x) = -3n_1$
$= \frac{(y-x)}{3} = -n_1$

$\frac{x-y}{3} = n_1$

$\frac{x-y}{3} = n_1$

$(y-x)$ is divisible by 3
$\Rightarrow (y, x) \in R$
$(x, y) \in R \Rightarrow (y, x) \in R$

(3) Transitive :- Let $x, y, z \in X$
$(x, y) \in R$ and $(y, z) \in R$
$\Rightarrow (x-y)$ is divisible by 3 & $(y-z)$ is divisible by 3
$\Rightarrow x-y = 3n_1$ & $y-z = 3n_2$, n_1 & $n_2 \in I$
$\Rightarrow (x-y) + (y-z) = 3(n_1+n_2)$
$= (x-z) = 3(n_1+n_2)$
$= (x-z) = 3n_3$
$= \frac{x-z}{3} = n_3$

$(x-z)$ is divisible by 3
$(x; z) \in R$ Hence $(x, y) \in R$ & $(y, z) \in R$
$\Rightarrow (x, z) \in R$.

Hence It is equivalence relation

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Ques Let ~~R~~ A be the set of all integers and a relation R is defined as

$R = \{(x,y) : x \equiv y \pmod m,\ m \text{ divide } (x-y)\}$

where m is a +ve integer.

Prove that R is equivalence relation.

Show :- $x_1 \equiv y_1$ & $x_2 \equiv y_2$ then $x_1 + x_2 \equiv y_1 + y_2$

Soln $R = \{(x,y) : x \equiv y \pmod m, (x-y) \text{ divisible by } m\}$

(1) Reflexive →

since $(x-x)$ is divisible by m

$\Rightarrow x \equiv x \pmod m$

$\Rightarrow xRx \ \forall x \in A$

R is reflexive

(2) Symmetric →

If $x, y \in A$

Let $(x,y) \in R \Rightarrow x \equiv y \pmod m$

$\Rightarrow (x-y)$ is divisible by m

$\Rightarrow (y-x)$ is also divisible by m

$\Rightarrow y \equiv x \pmod m$

$(y,x) \in R$

$(x,y) \in R \Rightarrow (y,x) \in R$ ∴ R is symmetric

(3) Transitive :-

If $x, y, z \in A$

$(x,y) \in R$ & $(y,z) \in R$

$\Rightarrow (x \equiv y \pmod{m})$ & $(y-z) \pmod m$

$\Rightarrow x-y$ divisible by m & $(y-z)$ divisible by m

$=$ $(x-y) + (y-z)$ is divisible by m

$(x-z)$ is divisible by m

$x \equiv z \pmod m$

$(x,z) \in R$

$(x,y) \in R$ & $(y,z) \in R \Rightarrow (x,z) \in R$

It is an equivalence relation.

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Ques $A = R \times R$ (R be the set of real numbers)

$(a,b) \; R \; (c,d) \Rightarrow a^2+b^2 = c^2+d^2$

verify (A, R) is an equivalence relation

Ans

① Reflexive :- $\forall (a, a) \in A = R \times R$

$(a,a) \in A = R \times R$

$a^2+a^2 = a^2+a^2$

$(a,a) \; R \; (a,a) \; \forall \; (a,a) \in A$

R is reflexive.

② Symmetric :- Let $(a,b) \; R \; (c,d)$.

$a^2+b^2 = c^2+d^2$

$= c^2+d^2 = a^2+b^2$

$= (c,d) \; R \; (a,b)$

$(a,b) \; R \; (c,d) = (c,d) \; R \; (a,b)$

③ Transitive :- Let $(a,b) \; R \; (c,d)$ & $(c,d) \; R \; (e,f)$

$a^2+b^2 = c^2+d^2$ & $c^2+d^2 = e^2+f^2$

$\Rightarrow a^2+b^2 = e^2+f^2$

$\Rightarrow (a,b) \; R \; (e,f)$

$(a,b) \; R \; (c,d)$ & $(c,d) \; R \; (e,f)$

$(a,b) \; R \; (e,f)$

Hence, it is an equivalence relation

Ques $A = \{1,2,3,4,5,6,7,8,9\}$

and let n be the relation on $A \times A$ defined as

$(a,b) \, n \, (c,d)$ if $a + d = b + c$

(i) n is a equivalence relation

(ii) Find $[(2,5)]$, the equivalence class of $(2,5)$

Soln Let $R = n$

① Reflexive :-

$a+a = a+b$

$(a+b) \, n \, (a+b)$

$(a,b) \; R \; (a,b) \; \forall \; (a,b) \in R$

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$(2,5)\ R\ (5,8)$
$2+8=5+5$

Date. $(2,5)$

② Symmetric :

$(2,5)\ R$
$a+d=b+c$
$2+5=5+2$
$10=10$

$2+4=5+1$
$6=6$

Let $(a,b)\ R\ (c,d)$
$= (a,b) \sim (c,d)$
$= a+d=b+c$
$= b+c=a+d$
$(b,a) \sim (c,d)$
$(c,d)\ R\ (a,b)$
R is symmetric

③ Transitive :-
Let $(a,b)\ R\ (c,d)$ & $(c,d)\ R\ (e,f)$
$a+d=b+c$ & $c+f=d+e$
Adding ① & ②
$a+d+c+f=b+c+d+e$
$a+f=b+e$
$(a,b)\ R\ (e,f)$
R is Transitive

$a+d=b+c$
$(a,b)\ (c,d)$
$(2,5)=2,5$
$2+5$

⇒ R is a equivalence relation on A Ans
$A\times A = ○$ $A\times A\{($
equivalence class of $(2,5)$ is
$[(2,5)] = \{(2,5)\ (1,4), (3,6), (4,7), (5,8), (6,9)\}$
(a,b) (c,d) (c,d)

Ques $N=\{1,2,3 \ldots\}$
$N\times N$ as follows : (a,b) is related to (c,d)
$ad=bc$
R is equivalence or not.

Soln Reflexive :

~~Let $a \in N$~~
~~$a.b = b.a$~~
~~$(a,b)\ R\ (a,b)$~~

Let $a \in N$
$a.a = a.a$
a is reflexive

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(2) Symmetric :-

Let $(a,b)\ R\ (c,d)$

$ad = bc \Rightarrow cb = da$

$\Rightarrow (c,d)\ R\ (a,b)$

R is symmetric

(3) Transitive :-

Let $(a,b)\ R\ (c,d)$ & $(c,d)\ R\ (e,f)$

$\Rightarrow ad = bc$ & $cf = de$

$\Rightarrow a\cancel{d}\cancel{c}f = b\cancel{c}\cancel{d}e$

$\Rightarrow af = be$

$= (a,b)\ R\ (e,f)$

It is an equivalence relation

Ques Let R be a binary ~~ref~~ relation defined as

$R = \{(a,b) \in R^2 : (a-b) \leq 3\}$

Determine whether R is an equivalence relation.

Soln (1) Reflexive :-

$a - a \leq 3$ (True)

$\Rightarrow (a,a) \in R$

$\Rightarrow aRa\ \forall\ a \in R$

It is a reflexive relation.

(2) Symmetric :-

Let $(a,b) \in R$

$= a - b \leq 3$

$= b - a \nleq 3$

It is not symmetric relation.

(3) Transitive :-

Let $(a,b) \in R$ & $(b,c) \in R$

$a - b \leq 3$ & $b - c \leq 3$

$(a-b) + (b-c) \leq 3 + 3$

$a - c \leq 6$

$(a,c) \notin R$

$(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \notin R$

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Ques S be the set of all points in a plane. Let R be a relation such that for any two points a & b, $(a,b) \in R$ if b is within two centimeter from a. Show that R is not an equivalence relation.

Soln

$(a,b) \in R$ if b is within two centimeter from a

$\Rightarrow aRb$ if $|a-b| < 2$

① Reflexive :-

$(a-a) < 2$

$(a,a) \in R \ \forall a \in R$

② Symmetric :-

$|(a-b)| < 2$

$-|b-a| < 2$

$= |b-a| < 2$

$= bRa$

③ Transitive :-

aRb & bRc

$|a-b| < 2$ & $|b-c| < 2$

$|a-b+b-c| < 2+2$

$|a-\cancel{b}+\cancel{b}-c| < 4$

$|a-c| < 4$

It is not a transitive relation

$\therefore$ It is not an equivalence relation.

Theorem 1 :- If R_1 & R_2 are two equivalence relation on a set A, then prove that $R_1 \cap R_2$ is also an equivalence relation on A.

Soln Since R_1 & R_2 are equivalence relation $\therefore$ R_1 & R_2 are symmetric, reflexive & transitive.

① Reflexive :- Let $x \in A$ then $(x,x) \in R_1$ (R_1 is reflexive)

& $(x,x) \in R_2$ (R_2 is reflexive)

$= (x,x) \in R_1 \cap R_2 \ \forall x \in A$

(2) Symmetric :- Let $x, y \in A$

$(x,y) \in R_1 \Rightarrow (y,x) \in R_1$ } R_1 & R_2 are
$(x,y) \in R_2 = (y,x) \in R_2$ } symmetric

$\therefore (x,y) \in R_1 \cap R_2 \Rightarrow (y,x) \in R_1 \cap R_2$

(3) Transitive :- Let $x, y, z \in A$

$(x,y) \in R_1$ & $(y,z) \in R_1 \Rightarrow (x,z) \in R_1$ } R_1 & R_2 are
$(x,y) \in R_2$ & $(y,z) \in R_2 \Rightarrow (x,z) \in R_2$ } Transitive

$(x,y) \in R_1 \cap R_2$ & $(y,z) \in R_1 \cap R_2 \Rightarrow (x,z) \in R_1 \cap R_2$
$R_1 \cap R_2$ is transitive.

$\therefore R_1 \cap R_2$ is also an equivalence relation.

→ The union of two equivalence relation on a set is not necessarily an equivalence relation.

Let $A = \{a, b, c\}$ & equivalence relation
$R = \{(a,a), (b,b), (c,c), (a,b), (b,a)\}$
& $S = \{(a,a), (b,b), (c,c), (b,c), (c,b)\}$
$R \cup S$ is not transitive.
as $(a,b) \in R \cup S$ and $(b,c) \in R \cup S$
but $(a,c) \notin R \cup S$

Theorem :- 2 If R is an equivalence relation on A, then prove that R^{-1} is also an equivalence relation on A.

Sol^n R is an equivalence relation
then
R is reflexive, symmetric & transitive.

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(1) Reflexive :- Let $x \in A$

since R is reflexive $\Rightarrow (x,x) \in R$

$\Rightarrow (x,x) \in R^{-1} \ \forall x \in A$

(2) symmetric :- Let $x, y \in R$

$(x,y) \in R \Rightarrow (y,x) \in R$

$(y,x) \in R^{-1} \Rightarrow (x,y) \in R^{-1}$

$\therefore (x,y) \in R^{-1} \Rightarrow (y,x) \in R^{-1}$

(3) Transitive :- Let $x, y, z \in A$

$(x,y) \in R$ & $(y,z) \in R \Rightarrow (x,z) \in R$

$(y,x) \in R^{-1}$ & $(z,y) \in R^{-1} \Rightarrow (z,x) \in R^{-1}$

$\therefore (z,y) \in R^{-1}$ & $(y,x) \in R^{-1} \Rightarrow (z,x) \in R^{-1}$

R^{-1} is transitive.

R^{-1} is an equivalence relation.

Composite Relation

Let A, B and C are three non-empty sets.
Suppose
R be a relation from A to B
and S be a relation from B to C. Then
the composite relation of the R & S is a relation from
A to C and denoted by SOR

$\therefore (a,b) \in R$ & $(b,c) \in S \Rightarrow (a,c) \in SOR$

$\therefore ROR = R^2, \quad R^2OR = R^3$

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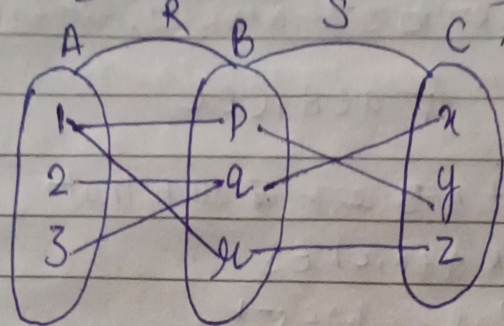
Ques $A = \{1, 2, 3\}$

$B = \{p, q, r\}$

$C = \{x, y, z\}$

$R = \{(1, p), (1, r), (2, q), (3, q)\}$

$S = \{(p, y), (q, x), (r, z)\}$



SoR : S ↗ R

RoS ↗ S

SoR

RoS

compute (SoR)

$\{(1, y), (1, z), (2, x), (3, x)\}$

Ques $R = \{(1, 2), (3, 4), (2, 2)\}$ and

$S = \{(4, 2), (2, 5), (3, 1), (1, 3)\}$

find SoR, RoS, RoR

Ro(SoR), (RoS)oR

SoR

RoS

(i) SoR

$= \{(1, 5), (3, 2), (2, 5)\}$

(ii) RoS

$= \{(4, 2), (3, 2), (1, 4)\}$

(iii) $RoR = R^2$

$= \{(1, 2), (2, 2)\}$

(iv) Ro(SoR) →

$SoR = \{(1, 5), (3, 2), (2, 5)\}$

$R = \{(1, 2), (3, 4), (2, 2)\}$

$Ro(SoR) = \{(3, 2)\}$

(v) (RoS)oR → $R = \{(1, 2), (3, 4), (2, 2)\}$

$RoS = \{(4, 2), (3, 2), (1, 4)\}$

$(RoS)oR = \{(3, 2)\}$

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* **Reversal law in composite relation :-**

Theorem → Let R be a relation from the set A to the set B and S be a relation from the set B to C then

$$(S o R)^{-1} = R^{-1} o S^{-1}$$

Sol^n Let $(c,a) \in (S o R)^{-1}$ $\forall\, a \in A, c \in C$

$(a,c) \in S o R$

$\Rightarrow \exists$ an element $b \in B$ ~~set~~ $\exists$ such that

$(a,b) \in R$ & $(b,c) \in S$

$\Rightarrow (b,a) \in R^{-1}$ & $(c,b) \in S^{-1}$

$\Rightarrow (c,b) \in S^{-1}$ & $(b,a) \in R^{-1}$

$= (c,a) \in R^{-1} o S^{-1}$

$\therefore (c,a) \in (S o R)^{-1} \Rightarrow (c,a) \in R^{-1} o S^{-1}$

$$\boxed{(S o R)^{-1} = R^{-1} o S^{-1}}$$

Order of relation

Partial order relation — Total order relation

① **Partial order Relation :-** symmetric, reflexive & antisymmetric.

POSET or partial order set :- The set A together with partial order relation R on the set A i.e. (A,R) is called partial order set.

Ques show that the relation
$R = \{(x,y) : x \geq y\}$ when $x, y \in I^+$
is a partial order relation.

Sol^n

① **Reflexive :-** ~~$x \in A$~~ $x \in I^+$
since $x \geq x \Rightarrow (x,x) \in R$
$\therefore$ R is reflexive

② **Antisymmetric :-** $x \in I^+$ & $y \in I^+$
$\Rightarrow (x,y) \in R$ & $(y,x) \in R$
$\Rightarrow x \geq y$ ~~ara~~ & $y \geq x$
$\Rightarrow x = y$
$(x,y) \in R$ & $(y,x) \in R \Rightarrow \boxed{x = y}$

③ **Transitive :-**

Let $(x,y) \in R$ & $(y,z) \in R$
$\Rightarrow x \geq y$ & $y \geq z$
$\Rightarrow x \geq z$
$\Rightarrow (x,z) \in R$
$\therefore (x,y) \in R$ & $(y,z) \in R$
$\Rightarrow (x,z) \in R$
R is a poset on I^+

② **Total order relation :-**
Consider the relation R on the set A
If $\forall a, b \in A$ we have
either $(a,b) \in R$
or $(b,a) \in R$
or $a = b$, then R is called
total order relation on R.

Ques Show that the relation '<' (less than) defined on N, the set of +ve integers is neither an equivalence relation nor partially ordered relation but is a total order relation.

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~~A~~ Sol^n Let $a, b \in N$

$R = \{(a,b) : a < b\}$

Reflexive :- since $a \nless a$

$(a,a) \notin R \Rightarrow a \not R\, a$

R is not reflexive

$\therefore$ The relation is not reflexive $\therefore$ it is neither equivalence relation nor partialorder relation.

But as

$\forall\, a, b \in N$, we have

either $a < b$

or $a > b$

or $a = b$

so, the relation is a total order relation.

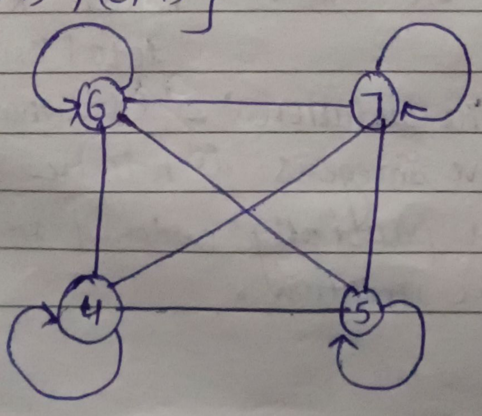
Partial order set (POSET) & Hasse Diagram

Hasse Diagram :- A graphical representation of a partial order relation in which all arrow heads are understood to be pointing upward is known as Hasse Diagram.

Ques consider the set

$A = \{4, 5, 6, 7\}$ Let R be the relation $\leq$ on A.

$R = \{(4,4), (5,5), (6,6), (7,7), (4,5), (4,6), (4,7), (5,6), (5,7), (6,7)\}$



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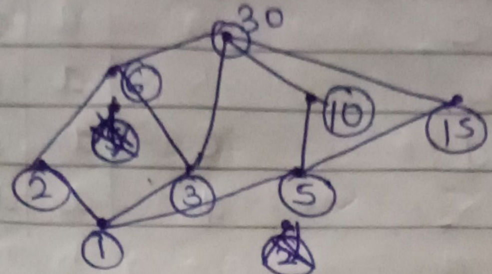
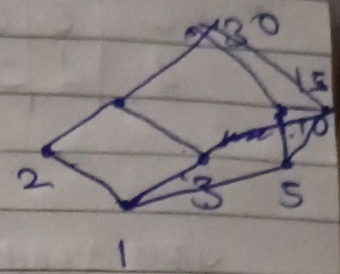
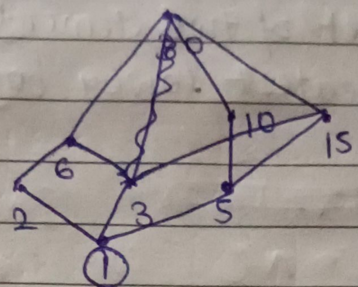
$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$







→ Trivial number are the number divisible by 30 but non-trivial numbers are the numbers after removing 1 and number itself.

Lattice

a partial order set or POSET (L, R) is a lattice

$\forall a, b \in L$

$\sup\{a,b\}$ and $\inf\{a,b\}$ exists in L

lub = least upper bound = supremum

glb = greatest ~~upper~~ lower bound = infimum

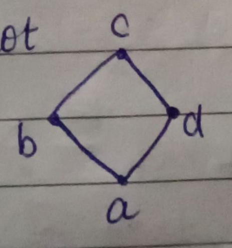
① $a \vee b$ = a join b = $\sup\{a,b\}$ = lub $\{a,b\}$

② $a \wedge b$ = a meet b = $\inf\{a,b\}$ = glb $\{a,b\}$

$a \vee b = a \cup b = a + b$

$a \wedge b = a \cap b = a \cdot b$

Ques Determine whether the following hasse diagram represent lattice or not.



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lub Table = supremum

(1)

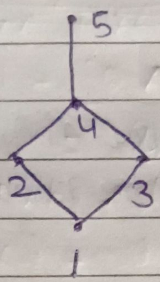
∨	a	b	c	d
a	a	b	c	d
b	b	b	c	c
c	c	c	c	c
d	d	c	c	d

glb Table

∧	a	b	c	d
a	a	a	a	a
b	a	b	b	a
c	a	b	c	d
d	a	a	d	d

Since each subset of two elements has least upper bound and a greatest lower bound. so this is a lattice.

Ques



lub

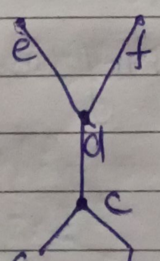
∨	1	2	3	4	5
1	1	2	3	4	5
2	2	2	4	4	5
3	3	4	3	4	5
4	4	4	4	4	5
5	5	5	5	5	5

glb

∧	1	2	3	4	5
1	1	1	1	1	1
2	1	2	1	2	2
3	1	1	3	3	3
4	1	2	3	4	4
5	1	2	3	4	5

It is a lattice.

Ques



→ Join lub – supremum

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∨	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	–
f	f	f	f	f	–	f

No upper bound exist of $e \vee f = \text{lub}\{e, f\}$
Therefore it is not a lattice

Properties of lattice

The general notation of lattice is $(L, \leq)$.

① Idempotent Law :-
(i) $a \vee a = a$
(ii) $a \wedge a = a$

② commutative Law :-
(i) $a \vee b = b \vee a$
(ii) $a \wedge b = b \wedge a$

③ Associative Law :-
(i) ~~$a \vee (a \wedge b)$~~ $(a \vee b) \vee c = a \vee (b \vee c)$
(ii) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

④ Absorption law :-
(i) $a \vee (a \wedge b) = a$
(ii) $a \wedge (a \vee b) = a$

Distributive Lattice :- A lattice $(L, \leq)$ is called distributive Lattice if for any $a, b, c \in L$
(i) $a \vee (b \wedge c) = (a \vee b) \wedge (b \vee c)$
(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

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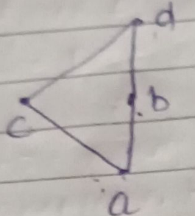
If the lattice L does not satisfies the above properties then it is called a distributive lattice.

Ques



$a \vee (b \wedge c) = a \vee a = a$

$a \wedge (b \vee c) = a \wedge d = a$

~~$a \vee (b \wedge c) = a \wedge (b \vee c) = a$~~

$(a \vee b) \wedge (a \vee c) = b \wedge c$

$= a$

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$(a \wedge b) \vee (a \wedge c) = a \vee a$

$= a$

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

c b d

$c \wedge (b \vee d)$

$(c \wedge b) \vee (c \wedge d)$

↑ Join

$c \wedge (b \vee d) =$

$c \wedge d = c$

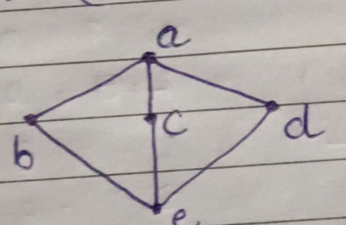
$(c \wedge b) \vee (c \wedge d)$

↓ meet

$a \vee c = c$

$a' = c$

Ques (i)



(i) $b \wedge (c \vee d) = (b \wedge c) \vee (b \wedge d)$

$b \wedge a = e \vee e$

$b \neq e$

It is not distributive Lattice

Theorem:- Let $a, b, c \in L$ where $(L, \leq)$ is a distributive lattice.

Then $\left.\begin{array}{l} a \vee b = a \vee c \\ a \wedge b = a \wedge c \end{array}\right\} \Rightarrow \boxed{b = c}$

Proof :-

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$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ — ①

$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ — ②

$b = b \vee (b \wedge a)$ (by absorption Law)

$\cancel{= b} = b \vee (a \wedge b)$ (by commutative Law)

$= b \vee (a \wedge c)$ (given)

$= (b \vee a) \wedge (b \vee c)$ (by distributive Law)

$= (a \vee b) \wedge (c \vee b)$ (by commutative Law)

$= (a \vee c) \wedge (c \vee b)$

$= \cancel{a \vee (c \wedge b)}$

$= (c \vee a) \wedge (c \vee b)$

$= c \vee (a \wedge b)$ (by distributive Law)

$= c \vee (a \wedge c)$ (given)

$= c \vee (c \wedge a)$

$= c$

$\boxed{b = c}$

Bounded Lattice

A lattice $(L, \leq)$ is called a bounded lattice if it has a greatest element 1 and a least element 0.

Properties :

(1) $a \vee 1 = 1$

(2) $a \wedge 1 = a$

(3) $a \vee 0 = a$

(4) $a \wedge 0 = 0$

Every finite lattice is bounded

Complemented Lattice

complement of an element in a lattice →

Let $(L, \leq)$ is a bounded lattice with least element 0 and greatest element 1 is called complemented of $a \in L$ if

$$\boxed{\begin{array}{l} a \vee a' = 1 \\ a \wedge a' = 0 \end{array}}$$

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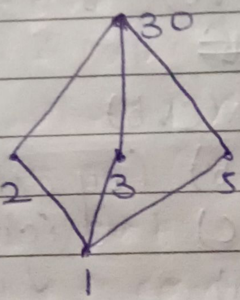
Note :- (1) If it is not necessary that every element $a \in L$ has a complement.

(2) An element $a \in L$ have more than one complement

(3) $1' = 0$ & $0' = 1$

Ques $A = \{1, 2, 3, 5, 30\}$ and $a \le b$ iff a divided b. Then find the complement of 2.



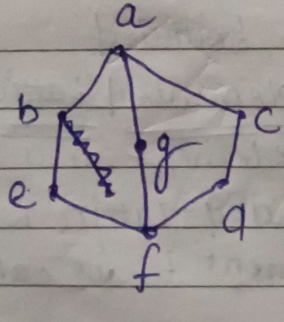
Sol^n Since it has greatest element $= 30$
least element $= 1$

$a \vee a' = 30$
$a \wedge a' = 1$
$2 \vee 3 = 30$ $\quad\quad 2 = 3'$
$2 \wedge 3 = 1$
$2 \vee 5 = 30$ & $2 \wedge 5 = 1$

Hence 2 has two complement 3 & 5

Ques How many complements does the element 'e' have? given all.

Sol^n



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least elemnt = f
greatest element = a
given lattice is bounded.

$\boxed{\begin{array}{l} e \vee g = a \\ e \wedge g = f \end{array}}$ & $\boxed{\begin{array}{l} e \vee d = a \\ e \wedge d = f \end{array}}$

e has 2 complement g and d.

$\left\{ \boxed{\begin{array}{l} e' = d \\ e' = g \end{array}} \right\}$

Theorem :- (i) $(a')' = a$
(ii) $(a \vee b)' = a' \wedge b'$
(iii) $(a \wedge b)' = a' \vee b'$
(iv) $a \vee (a' \wedge b) = a \vee b$
(v) $a \wedge (a' \vee b) = a \wedge b$

Proof :- $(a')' = a$
If complement of $a \in L$ is $a' \in L$ i.e. $a' = a$
then $a \vee a' = 1$, $a \wedge a' = 0$
$a' \vee a = 1$ & $a' \wedge a = 0$
This shows that the complement of a' is a
i.e $(a')' = a$

(ii) $(a \vee b)' = a' \wedge b'$
$\left.\begin{array}{l} (a \vee b) \vee (a' \wedge b') = 1 \\ (a \vee b) \wedge (a' \wedge b') = 0 \end{array}\right\}$ complement
$(a \vee b) \vee (a' \wedge b') = [(a \vee b) \vee a'] \wedge [(a \vee b) \vee b']$
(by distributive law)
$= [a \vee (b \vee a')] \wedge [a \vee (b \vee b')]$ [Ass. Law]
$= [a \vee (a' \vee b)] \wedge [a \vee 1]$ [comm. law]
$= [(a \vee a') \vee b] \wedge 1$
$= (1 \vee b) \wedge 1$
$= 1 \wedge 1 = \boxed{1}$

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4 $(a \vee b) \wedge (a' \wedge b')$

$= (a \wedge (a' \wedge b') \vee (b \wedge (a' \wedge b'))$ (Distr. law)

$= (a \wedge a') \wedge \cancel{()} b') \vee (b \wedge (b' \wedge a'))]$

$= \{(0 \wedge b') \vee ((b \wedge b') \wedge a')$

$= (0 \wedge b') \vee (0 \wedge a')$

$= 0 \cancel{\vee} 0$

$= 0$

$\boxed{(a \vee b') = a' \wedge b'}$

(iv) $a \vee (a' \wedge b) = a \vee b$

$a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b)$

$= 1 \wedge (a \vee b)$

$= (a \vee b)$

(v) $a \wedge (a' \vee b) = a \wedge b$

$a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$ (Distri. Law)

$= 0 \vee (a \wedge b)$

$= a \wedge b$

Hence prove!!

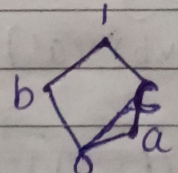
Modular Lattice

$a \vee (b \wedge c)$

A lattice $(L, \leq)$ is said to be Modular lattice if $\forall a, b, c \in L$

$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Ques Show that the pentagonal lattice given is not modular.



Soln $a \vee (b \wedge c) = (a \vee b) \wedge c$

$a \vee (b \wedge c)$

$= (a \vee b) \wedge c$

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for $a \leq c$

$a = b$, $b = 1$, $c = c$

$b \vee (1 \wedge c) = b \vee c = 1$

$(b \vee 1) \wedge c = 1 \wedge c = c$

$b \vee (1 \wedge c) \neq (b \vee 1) \wedge c$

It is not a modular lattice

Theorem Prove that every distributive lattice is modular.

Let $(L, \leq)$ is a distributive lattice then

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad -(1)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \quad -(2)$$

$$a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$$

Let $a \leq c$ then $a \vee c = c \quad -(3)$

from (1)

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge c$$

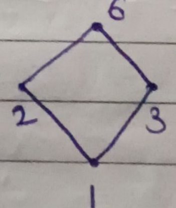
Hence proved!!

Converse is not true means every modular lattice may or not be distributive.

Complete Lattice →

A lattice $(L, \leq)$ is complete if every non-empty subset of L has a lub (sup) and glb (inf) in L

eg :-



(consider D_6, 1)

$D_6 = \{1, 2, 3, 6\}$

~~Subset $\{\phi, \{1\}, \{2\}, \{1,2\}, \{2,3\}\}$~~

It is a complete lattice as each element has glb & lub in this lattice.

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Theorem:

* Prove that every finite lattice is a complete lattice.

Ans

Let L be a finite lattice & S be any non-empty subset of L

Then S is also finite.

Let

$S = \{a, a_1 \ldots, a_n\}$

Then lub $= \{a_1 a_2 a_3 \ldots a_n\} = a_1 \vee a_2 =$ lub
$=$ Sub $S \in L$

glb $= \{a_1, a_2, a_3 \ldots a_n\} = a_1 \wedge a_2 =$ glb
$=$ ~~Sub~~ Inf $S \in L$

$\therefore$ It is a complete lattice.

Ques show that every complete Lattice is a bounded lattice.

Ans Let L be a complete lattice.

Then every non-empty subset of L has lub & glb

$\Rightarrow$ L itself has lub and glb.

$\Rightarrow$ L is bounded.