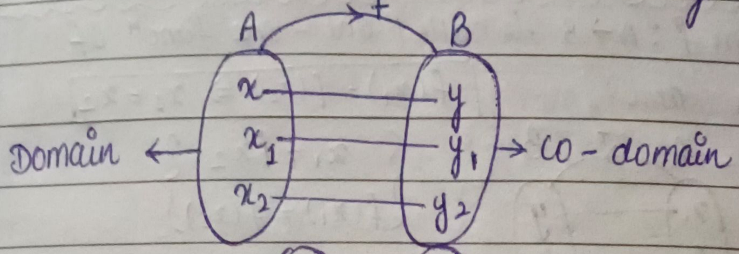
21/12/23

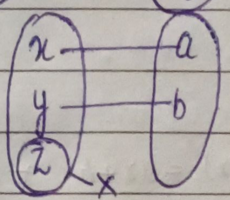
Discrete structure & Theory of logics

Unit-02

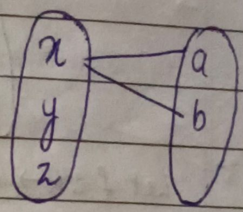
Functions

Function → Let A and B are two non-empty sets. A function $f: A \to B$ is a rule/ mapping / relation from A to B such that each element of A is related to exactly one element of set B.





x - not a function

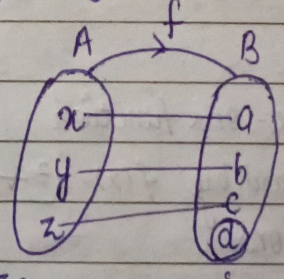


x - not a function

If $x \in A \longrightarrow f(x) = y \in B$

pre-image of y   image of x

* **Range of a function :-**



This is a function

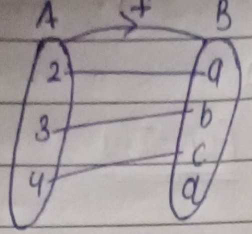
Let $f: A \to B$ is a function then range of f is denoted by $R(f)$ or $Range(f)$ or $f(A)$ and defined by

$$f(A) \text{ or } R(f) = \{ f(x) : x \in A \}$$

Note :- ① $f(A)$ or $R(f) \subseteq B$

② Range of a function is a set of images.

example :- Let $A = \{2, 3, 4\}$ & $B = \{a, b, c, d\}$



$R(f) = \{a, b, c\}$

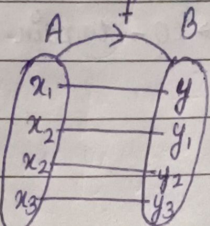
Classification of function :-

① One-one function / injective function / one to one function :-

A function $f: A \to B$ is called one-one func^n if

$x_1, x_2 \in A$ and $\boxed{f(x_1) = f(x_2) \Rightarrow x_1 = x_2}$

$\left\{ \begin{array}{l} x_1 \neq x_2 \\ f(x_1) \neq f(x_2) \end{array} \right\}$



example :- Let $f: R \to R$ such that $f(x) = ax + b \quad \forall x \in R$

Let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$\Rightarrow ax_1 + \cancel{b} = ax_2 + \cancel{b}$

$\Rightarrow \cancel{a}x_1 = \cancel{a}x_2$

$\Rightarrow \boxed{x_1 = x_2}$

Hence it is one-one function

example :- Let $f: R \to R$ defined by $f(x) = x^2 \ \forall x \in R$

$\therefore f$ is not one-one

Let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$x_1^2 = x_2^2$

It is not one-one function

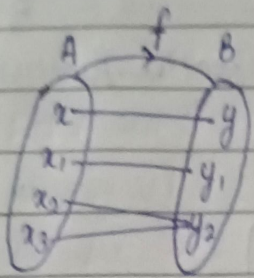
$f(-2) = 4 = f(2)$

$f(-2) = f(2)$

$\boxed{-2 \neq 2}$

(2) Many one function :-

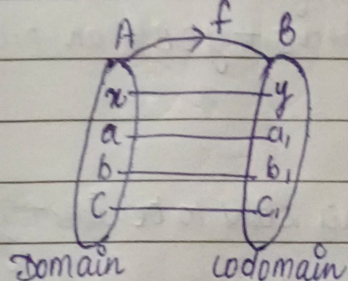
A function $f: A \to B$ is many-one function if $x_1, x_2 \in A$ and $f(x_1) = f(x_2) \Rightarrow \boxed{x_1 \neq x_2}$



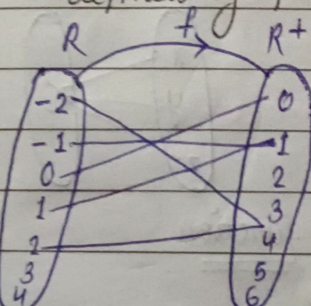
(3) surjective function or onto function :-

A function $f: A \to B$ is called onto function if $\forall\, y \in B$ $\exists$ an element $x \in A$ such that $\boxed{f(x) = y}$.

ie each element of codomain have have pre-image (at least one).



example:- $f: R \to R^+$ defined by $f(x) = x^2 \ \forall\, x \in R$

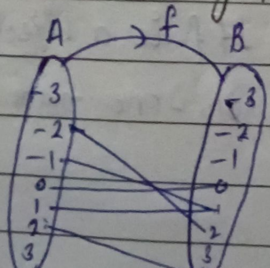


$f(x) = x^2$

$\therefore$ This function is onto function becoz $\forall\, y \in R^+$ $\exists$ an element $x \in A$.

example:-

Let $f: R \to R$ defined by $f(x) = x^2 \ \forall\, x \in R$



This is not an onto function becoz $\forall\, y \in R$ $\nexists$ an element $x \in A$.

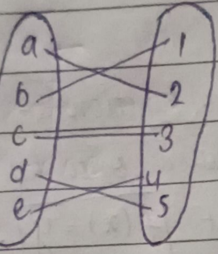
(4) **Into function :-**

A function $f: A \to B$ is called into function if $\exists$ at least one element $b \in B$ which has no pre image under f.

(5) **Bijective function :-**

A function is said to be bijective function when the function is both one-one & onto.

example :-



→ This function is one-one becoz $\forall x \in A$ $\exists$ an only one element or image in $y \in B$.

→ This function is onto becoz $\forall y \in B$ $\exists$ an pre-image in $x \in A$.

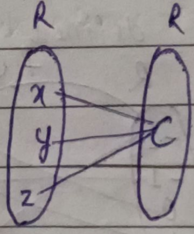
* **Identity function :-**

A function $f: A \to A$ is said to be identity function if $\boxed{f(x) = x}$ $\forall x \in A$.

* **constant function :-**

Let $f: R \to R$

$f(x) = c \quad \forall x \in R$



Inverse function

or

invertible function

or one-one onto function,

A function $f: A \to B$ is called invertible function if f is one-one & onto i.e. iff is a bijective function.

Note : (1) If $f: A \to B$ is one-one onto then inverse function $f^{-1}: B \to A$

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(2) $f(x)=y$ then $f^{-1}(y)=x$, If $f: A \to B$ is one-one onto function then f^{-1} is unique function.

(3) $(f^{-1})^{-1} = f$

Ques Let $f: R \to R$ is defined by $f(x) = ax+b$ where $a, b, x \in R$ $a \neq 0$ show that f is invertible and find the inverse of f.

Ans given $f: R \to R$

$f(x) = ax+b$

for one-one Let $x_1, x_2 \in R$

$f(x_1) = f(x_2)$

$ax_1 + \cancel{b} = ax_2 + \cancel{b}$

$ax_1 = ax_2 \Rightarrow \boxed{x_1 = x_2}$

Onto

$\forall$ $y \in R$ $\exists$ an element in $x \in R$ such that

$f(x) = y$

$= ax+b = y$

$\Rightarrow ax = y-b \Rightarrow \boxed{x = \frac{y-b}{a} \in R}$

$\therefore \forall y \in R$ $\exists$ an element in $x = \frac{y-b}{a} \in R$

$f(x) = f\left(\frac{y-b}{a}\right)$

$= \cancel{a}\left(\frac{y-b}{\cancel{a}}\right) + b$

$= y - \cancel{b} + \cancel{b} = y$

such that $\boxed{f(x) = y}$ $\therefore f$ is onto

Hence f is invertible.

$f^{-1}(y) = x$

$\boxed{f^{-1}(y) = \frac{y-b}{a}}$

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Ques Show that the mapping $f : R \to R$

$f(x) = \frac{1}{x}, x \neq 0$ and $x \in R$

Prove that f is invertible and find f^{-1}.

Ans For one - one :-

$f(x_1) = f(x_2)$

$\frac{1}{x_1} = \frac{1}{x_2}$

$\boxed{x_2 = x_1}$

$\therefore f$ is one - one

for onto :-

$f(x) = y$

$\frac{1}{x} = y$

$\boxed{\frac{1}{y} = x}$

$\boxed{x = \frac{1}{y} \in R}$ $y \neq 0$

$\therefore f$ is onto $\forall y \in R \ \exists$ an element $x = \frac{1}{y} \in R$

$\therefore f$ is bijective + hence invertible.

$f^{-1}(y) = x$

$\boxed{f^{-1}(y) = \frac{1}{y}}$

Ques Is the function $f : N \to N$

deffined by $f(x) = x^2, x \in N$ is invertible.

Soln One - One :- $f(x_1) = f(x_2)$, $x_1, x_2 \in N$

$x_1^2 = x_2^2$

$\boxed{x_1 = x_2}$ $\therefore f$ is one - one

Onto :- ~~f(x) = y~~ $f(x) = y$

~~fx~~

$x^2 = y$

$x = \sqrt{y} \notin N$

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$\therefore$ f is not onto $\therefore$ f is not bijection & hence f is not invertible.

Ques find the inverse of then $f(x) = \frac{2x}{x-1}$

Sol^n Let $f(x) = y$

$y = \frac{2x}{x-1} \Rightarrow y(x-1) = 2x$

$yx - y = 2x \Rightarrow yx - 2x = y$

$x(y-2) = y \Rightarrow \boxed{x = \frac{y}{y-2}}$

$f^{-1}(y) = x$

$\boxed{f^{-1}(y) = \frac{y}{y-2}}$

* Operations on functions :-

Let f and g are two functions and domain of $f = D_f$
domain of $g = D_g$. Then

① sum of functions $\rightarrow (f+g)(x) = f(x) + g(x)$, $\boxed{D_{f+g} = D_f \cap D_g}$

② Difference of functions $\rightarrow (f-g)(x) = f(x) - g(x)$, $\boxed{D_{f-g} = D_f \cap D_g}$

③ Product of functions $\rightarrow (f \cdot g)(x) = f(x) \cdot g(x)$, $\boxed{D_{f \cdot g} = D_f \cap D_g}$

④ Quotient of functions $\rightarrow \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$

$D_{\frac{f}{g}} = (D_f \cap D_g) - \{x : g(x) = 0\}$

Ques consider the following functions :-

$f(x) = 9x - 5$ & $g(x) = 4x + 1$

Sol^n (i) $(f+g)(x) = f(x) + g(x)$

$= 9x - 5 + 4x + 1 = 13x - 4$

(ii) $(f-g)(x) = f(x) - g(x) = 9x - 5 - 4x - 1 = 5x - 6$

(iii) $(fg)(x) = f(g)x = (9x-5)(4x+1) = 36x^2 + 9x - 20x - 5$

$= 36x^2 - 11x - 5$

(iv) $(f/g)(x) = \frac{f(x)}{g(x)} = \frac{9x-5}{4x+1}$, $x \neq -\frac{1}{4}$

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Ques $f: Z \to Z \quad f(x) = x^2$

Let $f(x) = f(y)$

$x^2 = y^2$

$x^2 - y^2 = 0 \Rightarrow (x+y)(x-y) = 0$

$\boxed{x = y}$ & $\boxed{x = -y}$

Not a one-one function

composite function

Let A, B & C be three non-empty sets and

$f: A \to B$ & $g: B \to C$

The composition of f and g is a function denoted by

$gof: A \to C$

and is defined by

$gof(x) = g(f(x)) \quad \forall x \in A$

Ques Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, $C = \{5, 6, 7\}$

Define $f: A \to B$ where $f = \{(1,a), (2,a), (3,b)\}$

& $g: B \to C$ where $g = \{(a,5), (b,7)\}$

find the composition of f & g.

Soln (1) $(gof)(1) = g(f(1)) = g(a) = 5$

(2) $(gof)(2) = g(a) = 5$

(3) $(gof)(3) = g(b) = 7$

(4) ~~fog(5) = f(g(5)) =~~

$gof = \{(1,5), (2,5), (3,7)\}$

Ques Let $f: R \to R$ and $g: R \to R$ where R be the set of the real numbers.

find (i) gof (ii) fog (iii) fof (iv) gog

where $f(x) = x^2$ & $g(x) = x+4$.

Soln $f: R \to R$ & $g: R \to R \quad f(x) = x^2 \quad g(x) = x+4$

(i) $gof(x) = g(f(x)) = g(x^2) = x^2 + 4$

(ii) $fog(x) = f(g(x)) = f(x+4) = (x+4)^2 = x^2 + 16 + 8x$

(iii) $fof(x) = f(f(x)) = f(x^2) = (x^2)^2 = x^4$

(iv) $gog(x) = g(g(x)) = g(x+4) = x+4+4$

$= x+8$

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Theorem :- Let $f : A \to B$ & $g : B \to C$ be one-one & onto function. Then the composite function

$g \circ f : A \to C$ is one-one onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Ans Since $f : A \to B$ is one-one & onto & $g : B \to C$ is also one-one & onto then f^{-1} & g^{-1} exists and both are one-one & onto.

$g \circ f : A \to C$ is $g(f(x))$ $\forall x \in A$

One-one :-

Let x_1 & $x_2 \in A$

then y_1 & $y_2 \in B$ such that

$y_1 = f(x_1)$ & $y_2 = f(x_2)$.

Now, $g \circ f(x_1) = g \circ f(x_2)$

$g(f(x_1)) = g(f(x_2))$

$g(y_1) = g(y_2)$ [g is one-one]

$y_1 = y_2$

$\Rightarrow f(x_1) = f(x_2)$ [f is one-one]

$\Rightarrow \boxed{x_1 = x_2}$

$\therefore$ $g \circ f$ is one-one

Let $z \in C$. Since g is onto then $\exists$ an element $y \in B$ such that $\boxed{g(y) = z}$

Again f is onto then $\exists$ an element $x \in A$ such that

$\boxed{f(x) = y}$

$g \circ f(x) = g(f(x))$

$= g(y)$

$= z$

$\therefore$ $\forall z \in C$ $\exists$ an element $x \in A$ such that

$g \circ f(x) = z$

Hence $g \circ f$ is one-one onto.

Now, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ $\quad f^{-1}(y) = x$, $g^{-1}(z) = y$

L.H.S. $(g \circ f)^{-1}(z) = \boxed{x}$ $\quad (g \circ f)^{-1}(z) = x$

R.H.S. $(f^{-1} \circ g^{-1})(z) = f^{-1}(g^{-1}(z))$

$= f^{-1}(y) = \boxed{x}$

Hence prove!!

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Ques Let $f: R \to R$ be a function defined by $f(x) = px + q$ $\forall x \in R$. Also $fof = I_R$.
find the value of $p+q$.

Ans
$(fof)(x) = I_R(x)$ $\quad [\because I_R(x) = x]$

$f(f(x)) = x$ $\quad gof(x) = g(f(x))$

$f(px+q) = x$

$p(px+q) + q = x \Rightarrow p^2x + pq + q = x$

$\Rightarrow p^2x + qp + q - x = 0$

$\Rightarrow x(p^2 - 1) + pq + q = 0$

$p^2 - 1 = 0$ & $pq + q = 0$

$p^2 = 1 \Rightarrow \boxed{p = \pm 1}$ $\quad q(p+1) = 0$

$\boxed{q = 0,\ p = -1}$

$\therefore$ i) $p = 1$ & $q = 0$

ii) $p = -1$ & q is any real number

Boolean Algebra

or

Axioms of booleans Algebra

Let B be any non-empty set with at least two distinct element 0 & 1.

Then algebraic structure $(B, +, \cdot)$ is called a Boolean algebra if B satisfies the following properties:-

① **Closure law**:- $\forall a, b \in B$

$a + b \in B$ & $a \cdot b \in B$

② **commutative law**:- $\forall a, b \in B$

$a + b = b + a$ & $a \cdot b = b \cdot a$

③ **Distributive Law**:-

(i) $a \cdot (b + c) = a \cdot b + a \cdot c$

(ii) $(b + c) \cdot a = b \cdot a + c \cdot a$

(iii) $a + (b \cdot c) = a + b \cdot a + c$

(iv) $(b \cdot c) + a = b + a \cdot c + a$

④ Identity Law :- $\forall a \in B$

(i) $a+0 = a = 0+a$ (ii) $1 \cdot a = a = a \cdot 1$

additive identity Multiplicative identity

⑤ complement law :-

$\forall a \in B \; \exists \; a' \in B$ such that

(i) $a+a' = a'+a = 1$ (ii) $a \cdot a' = a' \cdot a = 0$

The boolean algebra is denoted by $(B, +, \cdot)$

Ques Show that the structure $(B, +, \cdot, ')$ is a Boolean Algebra where $B = \{0, 1\}$ and '+' and '.' are two binary operation and ''' a uniary operation on B, defined by the following table :

+	0	1
0	0	1
1	1	1

·	0	1
0	0	0
1	0	1

a	a'
0	1
1	0

Soln ① closure law :- Since in the '+' and '.' table, all elements belong to the set B.

$\Rightarrow$ B is closed under '+' & '.'.

$\forall a, b \in B \Rightarrow a+b \in B \;\&\; a \cdot b \in B$

② commutative law :- First two table are symmetrical in rows & column i.e. $0+1 = 1+0$ & $0 \cdot 1 = 1 \cdot 0$ Hence both operations '+' & '.' are commutative.

③ Distributive law :- Since $0, 1 \in B$, we have

$a+(b \cdot c) = (a+b) \cdot (a+c)$ — ①

$a \cdot (b+c) = a \cdot b + a \cdot c$ — ②

Let $a = 0, b = 1, c = 0$

$0+(1 \cdot 0) = (0+1) \cdot (0+0)$ (from ①)

$0+0 = 1 \cdot 0$

$\boxed{0 = 0}$

$0 \cdot (1+0) = 0 \cdot 1 + 0 \cdot 0$ (from ②)

$0 \cdot 1 = 0+0$

$\boxed{0 = 0}$

$\therefore$ Distributive law hold $\forall a, b \in B$

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(4) Identity Law :-

(i) from table '+' 0 is the additive identity
(ii) from table '.' 1 is the multiplicative identity

(5) complement law :-

$\forall a \in B \; \exists$ an element $a' \in B$

i.e. $\boxed{0' = 1}$ & $\boxed{1' = 0}$

Here $(B, +, \cdot, ', 0, 1)$ is a Boolean algebra.

Definition 2 :-

Let B be a non-empty set with atleast two distinctive element 0 & 1.

Then an algebraic structure $(B, \vee, \wedge)$ is called a boolean algebra if it satisfies the following properties :-

(1) closure law :- $\forall a, b \in B$

$a \vee b \in B$ & $a \wedge b \in B$

(2) commutative law :- $\forall a, b \in B$

(i) $a \vee b = b \vee a$ (ii) $a \wedge b = b \wedge a$

(3) Distributive law :- $\forall a, b, c \in B$

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(4) Identity Law :-

$\exists$ two elements 0 & 1 in B such that

(i) $a \vee 0 = a$ (ii) $a \wedge 1 = a$

(5) complement law :-

$\forall a \in B \; \exists$ an element $a' \in B$ such that

(i) $a \vee a' = 1$ (ii) $a \wedge a' = 0$

This Boolean algebra is denoted by $(B, \vee, \wedge, ', 0, 1)$.

Note :- A complemented distributive ~~law~~ lattice is called a Boolean algebra.

Ques Let B be the set of all +ve divisors of 30 and the operations $\wedge$ & $\vee$ on B are defined as follows:-

$a \vee b = LCM(a, b)$

$a \wedge b = HCF(a, b)$

Prove that ~~(B, ∧, ∨~~ $(B, \vee, \wedge, ')$ is a Boolean algebra.

Soln

$B = \{1, 2, 3, 5, 6, 10, 15, 30\}$

V	1	2	3	5	6	10	15	30
1	1	2	3	5	6	10	15	30
2	2	2	~~2~~6	10	6	10	30	30
3	3	6	3	15	6	30	15	30
5	5	10	15	5	30	10	15	30
6	6	6	6	30	6	30	30 ~~15~~	30
10	10	10	30	10	30	10	30	30
15	15	30	15	15	30	30	15	30
30	30	30	30	30	30	30	30	30

2 | 2, 30
15 | 1, 15

∧	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

$\left\{ \begin{array}{l} a \vee a' = 30 \\ a \wedge a' = 1 \end{array} \right\}$

a	a'
1	30
2	15
3	10
5	6
6	5

a	a'
10	3
15	2
30	1

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① closure law :- since all the elements in the composition table belongs to set B.
$\Rightarrow$ B satisfy closure law under $\wedge$ & $\vee$
i.e. $\forall a, b \in B \Rightarrow a \vee b \in B$ & $a \wedge b \in B$

② commutative law :- since rows & column in the composition tables are identical.
Then commutative law hold for $\vee$ & $\wedge$
i.e. (i) $a \vee b = b \vee a$ (ii) $a \wedge b = b \wedge a$
$\forall a, b \in B$.

③ Distributive law :- $a, b, c \in B$
(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
Let $a = 3, b = 5, c = 10$
(i) $3 \vee (5 \wedge 10) = 3 \vee 5 = 15$ & $(3 \vee 5) \wedge (3 \vee 10) = 15$
(ii) $3 \wedge (5 \vee 10) = 3 \wedge 10 = 30$ & $(3 \wedge 5) \vee (3 \wedge 10) = 30$

④ Identity Law :-
$\forall a \in B$ $\exists$ $1 \in B$ & $30 \in B$ such that
$a \vee 1 = a$ & $a \wedge 30 = a$
$\therefore$ 1 is identity element for $\vee$
& 30 is identity element for $\wedge$.

⑤ complement law :-
$\forall a \in B$ $\exists$ an element $a' \in B$ such that
$a \vee a' = 30$ & $a \wedge a' = 1$
Hence $(B, \vee, \wedge, ')$ is an Boolean algebra.

Definition 3 :-
An algebric structure $(B, \cup, \cap)$ is called Boolean algebra, if B satisfy the following properties.
① closure law :- $\forall A_1, A_2 \in B$
(i) $A_1 \cup A_2 \in B$ (ii) $A_1 \cap A_2 \in B$

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(2) Commutative law :- $\forall A_1, A_2 \in B$
(i) $A_1 \cup A_2 = A_2 \cup A_1$ (ii) $A_1 \cap A_2 = A_2 \cap A_1$

(3) Distributive law :- $\forall A_1, A_2, A_3 \in B$
(i) $A_1 \cup (A_2 \cap A_3) = (A_1 \cup A_2) \cap (A_1 \cup A_3)$
(ii) $A_1 \cap (A_2 \cup A_3) = (A_1 \cap A_2) \cup (A_1 \cap A_3)$

(4) Identity Law :- $\forall A \in B$ $\exists$ two elements ϕ & U in B such that
(i) $A \cup \phi = A$ (ii) $A \cap U = A$

(5) Complement law :- $\forall A \in B$ $\exists$ an element $A' \in B$ such that
(i) $A \cup A' = U$ (ii) $A \cap A' = \phi$
It is denoted by $(B, \cup, \cap, ')$.

Ques Let S be a family of sets which is closed under '$\cup$' & '$\cap$' and complement '$'$', then prove that $(S, \cup, \cap, ')$ is a Boolean Algebra.

ex :- Show that a boolean algebra cannot have three elements.

Sol^n Let $(B, +, \cdot, ')$ be a Boolean algebra where $B = \{0, 1, a\}$.
Since 0 & 1 are the identity elements with respect to the binary operations '+' & '$\cdot$' respectively.
Also $\boxed{0' = 1}$ & $\boxed{1' = 0}$
Also we know that complement of any element is unique. Then a' (complement of a) does not exist in B. Hence $(B, +, \cdot, ')$ cannot be a boolean algebra when B contains three elements.

Theorems of Boolean Algebra
or
Axioms and Theorem of Boolean algebra

Theorem 1 :-
(1) Idempotent law :- For every element of a boolean algebra B :- (i) $a + a = a$ (ii) $a \cdot a = a$

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Proof :- (1) To prove that $\boxed{a+a=a}$

$R.H.S. = a = a+0$ [$\because a+0=a$] Identity

$= a+a.a'$ [by complement law $a.a'=0$]

$= (a+a).(a+a')$ [by distributive law]

$= (a+a).1$ [$\because a+a'=1$]

$= (a+a)$ [$\because a.1=a$] Identity law

(2) $a.a=a$

$R.H.S. = a \cancel{.1} = a.1$ [by identity law]

$= a.(a+a')$ [by complement law]

$= a.a+a.a'$ [by distributive law]

$= a.a+0$ [$\because a.a'=0$]

$= \boxed{a.a}$ [$a.0=a$] (identity law)

Theorem 2 :- (i) $a+1=1$ (ii) $a.0=0$

(i) $a+1=1$

$L.H.S. = a+1$

$= a+(a+a')$ [$\because$ by complement law]

$= (a+a)+a'$ $\cancel{(a+a)*(a*a')}$ [$\because$ by ~~distributive~~ Associative law]

$= \cancel{(a+a)+a}$ [by idempotent law]

$= a+a'$ [by complement law]

$= 1$

(ii) $a.0=0$

$L.H.S. = a.0$

$= a.(a.a')$ [by complement law]

$= (a.a).a'$ [by idempotent law]

$= a.a'$ [by complement law]

$= 0$

Theorem 3:- **Absorption law :-**

(i) $a+(a\cdot b)=a$ (ii) $a\cdot(a+b)=a$

(i) $a+(a\cdot b)=a$

L.H.S. $a+(a\cdot b)$

$= a\cdot 1 + a\cdot b$ (By identity law)

$= a\cdot(1+b)$ (by distributive law)

$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$

$= a\cdot 1$ $[\because a+1=1]$

$= a$ = R.H.S. [By identity law]

(ii) $a\cdot(a+b)=a$

L.H.S. $a\cdot(a+b)$

$=(a+0)\cdot(a+b)$ (By identity law)

$= a+(0\cdot b)$ (By distributive law)

$= a+0$ $(\because 0\cdot b = 0)$

$= a$ = R.H.S. [by identity law]

Theorem 4:- **Associative law :-**

The binary operation '+' & '.' both obey the associative laws in boolean algebra B.

(i) $a+(b+c)=(a+b)+c$

(ii) $a\cdot(b\cdot c)=(a\cdot b)\cdot c$

(i) $a+(b+c)=(a+b)+c$

L.H.S. $a+(b+c)$

$=[a+(b+c)]\cdot 1$ [By identity law]

$=[a+(b+c)]\cdot(c+c')$ [By complement law]

$=[a+(b+c)]\cdot c+[a+(b+c)]\cdot c'$ [By distributive law]

$= c\cdot[a+(b+c)]+c'\cdot[a+(b+c)]$ [By commutative law]

$=\{c\cdot a + c\cdot(b+c)\}+\{c'\cdot a + c'\cdot(b+c)\}$

(By distributive law)

$=\{c\cdot a + c\}+\{c'\cdot a + c'\cdot(b+c)\}$

using commutative & absorption law

$=\{c+c\cdot a\}+\{c'\cdot a+c'\cdot(b+c)\}$

$= c+\{c'\cdot a+c'\cdot(b+c)\}$

(using commutative & absorption law)

$= c+[c'.a+c'.b+c'.c]$ [using distributive law]

$= c+[c'.a+c'.b+c.c']$ [using commutative law]

$= c+[c'.a+c'.b+\overset{0}{}]$ [using complement law $c.c'=0$]

$= c+[c'.a+c'.b]$ [using identity law]

$= c+[c'.(a+b)]$ [using distributive law]

$= (c+c').[c+(a+b)]$ [using distributive law]

$= 1.[c+(a+b)]$ [using complement law]

$= \boxed{c+(a+b)}$ [using identity law]

Hence proved!!

(ii) ~~$a.(b.c) = (a.b).c$~~

~~Ans. $[a.(b.c)]+1[$~~

$(a.b).c = a.(b.c)$

$= a.(b.c)+0$ [by identity law]

$= [a.(b.c)]+(c.c')$ [by complement law]

$= \{[a.(b.c)]+c\}.\{[a.(b.c)]+c'\}$ [by distributive law]

$= \{c+\{a.(b.c)\}\}.\{c'+[a.(b.c)]\}$ (by commutative law)

$= \{(c+a).[c+(b.c)]\}.\{(c'+a).[c'+(b.c)]\}$ [by distributive

$= \{(c+a).c\}.\{(c'+a).[c'+(b.c)]\}$ [using commutative & absorption]

$= \{c.(c+a)\}.\{(c'+a).[c'+(b.c)]\}$ [using commutative law]

$= c.\{(c'+a).[c'+(b.c)]\}$ [using absorption law]

$= c.\{(c'+a).[(c'+b).(c'+c)]$ [using distributive law]

$= c \cdot \{(c'+a) \cdot [(c'+b) \cdot 1]\}$ [using complement & commutative law]

$= c \cdot \{(c'+a) \cdot (c'+b)\}$ [using identity law]

$= c \cdot \{c' + (a \cdot b)\}$ [using distributive law]

$= c \cdot c' + [c \cdot (a \cdot b)]$ [using distributive law]

$= 0 + [c \cdot (a \cdot b)]$ [using complement law]

$= c \cdot (a \cdot b)$ [using identity law & commutative law]

$= \boxed{(a \cdot b) \cdot c}$ [using commutative law]

Hence proved!!

Theorem 5:- In a Boolean algebra B, the identity elements are complementary to each other.

For $0, 1 \in B$ we have

(i) $0' = 1$ (ii) $1' = 0$

(i) To prove :- $\boxed{0' = 1}$

L.H.S. $= 0'$

$= 0' + 0$ [by identity law] [$a + 0 = a$]

$= 1$ [by complement law] [$a + a' = 1$]

(ii) To prove :- $1' = 0$

L.H.S. $= 1'$

$= 1' \cdot 1$ [by identity law] [$a \cdot 1 = a$]

$= 0$ [by complement law] [$a \cdot a' = 0$]

Theorem 6:- Prove that the identity elements in a Boolean algebra are unique.

Sol^n (i) To prove that identity 0 is unique for +

Suppose 0 and 0_1 are two identity elements of boolean algebra B for '+'.

If 0 is identity element then $0_1 + 0 = 0_1$

If 0_1 is identity element then $0 + 0_1 = 0$

But $0_1 + 0 = 0 + 0_1$ (commutative law)

$\therefore \boxed{0_1 = 0}$

Hence, identity element 0 is unique.

(iii) To prove identity element 1 is unique for '.'

Let 1 & 1_1 are two identity elements of boolean algebra B for '.'.

If 1 is identity element, then $1_1 . 1 = 1_1$

If 1_1 is identity element, then $1 . 1_1 = 1$

But $1_1 . 1 = 1 . 1_1$ (by commutative law)

$\therefore \boxed{1_1 = 1}$

Hence identity element 1 is unique.

Theorem 7:- The complement of each element of boolean algebra B is unique.

Sol^n Let $a \in B$ be any element, suppose x & y are two complement of a.

Then $a + x = x + a = 1$ & $a + y = y + a = 1$ — ①

& $a . x = x . a = 0$ & $a . y = y . a = 0$

Now $x = x . 1$ [by identity law]

$= x . (a + y)$ [from ①]

$= x . a + x . y$ [By distributive law]

$= 0 + x . y$ [$x . a = 0$ from ①]

$= a . y + x . y$ [$\because a . y = 0$]

$= (a + x) . y$ [by distributive law]

$= 1 . y$ [from ①]

$= y$ [by identity law]

$\Rightarrow \boxed{x = y}$

Hence the complement of each element is unique.

Theorem 8:- Involution Law:-

For each element a of boolean algebra B

$\boxed{(a')' = a}$

Sol^n Let complement of $a \in B$ be $a' \in B$, then

$a + a' = 1$ & $a . a' = 0$ — ①

* Involution law can also be proved by using V.A.

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But operation '+' & '.' in B are commutative

$$a'+a=1 \;\&\; a'.a=0 \quad -(2)$$

eqⁿ (2) shows that complement of $a' \in B$ is $a \in B$. But complement of every element of Boolean algebra is unique. Hence

complement of $a' = a$

$$\Rightarrow \boxed{(a')' = a}$$

Hence proved!!

Imp. De Morgan's law :-

→ For any two elements a and b of boolean algebra B, we have

(i) $(a+b)' = a'.b'$

(ii) $(a.b)' = a'+b' \quad \forall a,b \in B$

(i) To prove :- $(a+b)' = a'.b'$

$\Rightarrow (a+b) + a'.b' = 1 \quad -(1)$

& $(a+b).(a'.b') = 0 \quad -(2)$

To show that complement of $(a+b)$ is $a'.b'$.

Now, $(a+b)+(a'.b) = 1$

$= (a+b) + (a'.b')$ (by distributive law)

$= ((a+b)+a').((a+b)+b')$

$= (a+(b+a')).(a+(b+b'))$ (by associative law)

$= (a+(b+a')).(a+1)$ $(b+b'=1)$

$= (a+(a'+b)).(a+1)$ (by commutative law)

$= [(a+a')+b)].(a+1)$ [by ~~commu~~ associative law]

$= [1+b].(a+1)$ ~~[by identity law]~~

$= 1.1$ $[a+1=1]$

$= \boxed{1}$

Now, $(a+b).(a'.b') = 0$

$= (a+b).(a'.b')$

$= (a'.b').(a+b)$ (by commutative law)

$= [(a'.b').a] + [(a'.b').b]$ (by distributive law)

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$\Rightarrow (a'.b').a + a'.(b'.b)$ (by associative law)

$\Rightarrow (a'.b').a + a'.0$ ($b'.b = 0$ by complement law)

$= (a'.b').a + 0$ ($a'.0 = a'$ by identity law)

$= (b'.a').a + 0$ (by commutative law)

$= b'.(a'.a) + 0$ (by associative law)

$= b'.0 + 0$ ($a'.a = 0$ by complement law)

$= 0 + 0$

$= \boxed{0}$

Hence $(a+b)' = a'.b'$

(ii) To prove that $(a.b)' = a'+b'$

Here we have to show that the complement of $a.b$ is $a'+b'$, for which it is sufficient to prove :-

$$(a.b) + (a'+b') = 1 \quad \text{—①}$$

$$\& \quad (a.b).(a'+b') = 0 \quad \text{—②}$$

(i) $(a.b)+(a'+b') = 1$

$= (a.b)+(a'+b')$

$= (a'+b') + (a.b)$ (by commutative law)

$= [(a'+b')+a)].[(a'+b')+b]$ (by distributive law)

$= [(b'+a')+a].[a'+(b'+b)]$ [by commutative law & associative law]

$= [b'+(a'+a)].[a'+1]$

$= [b'+1].[a'+1]$ [by associative law & complement law]

$= 1.1$

$= \boxed{1}$ [by complement law]

ii) $(a.b).(a'+b') = 0$

$= (a.b).(a'+b')$

$= (a'+b').(a.b)$ (by commutative law)

$= [(a'+b').a].[(a'+b').b]$ [by distributive law]

$= [(b'+a').a].[a'+(b'.b)]$ [by commutative & associative law]

$= [(b'+(a'.a)].[a'+(b'.b)]$ [by associative law]

$= [b'+0] \cdot [a'+0]$ [by complement law]

$= 0 \cdot 0$

$= \boxed{0}$

Hence $(a \cdot b)' = a'+b'$ Ans

eg: If B is a Boolean algebra, then prove that the following statements are equivalent.

(i) $a \cdot b' = 0$ (ii) $a+b=b$ (iii) $a'+b=1$

Soln To prove (i) $\Rightarrow$ (ii) i.e.

If $\boxed{a \cdot b' = 0}$ then $\boxed{a+b=b}$

L.H.S. $\Rightarrow a+b = b$

$= a+b = (a+b) \cdot 1$ (by identity law)

$= (a+b) \cdot (b+b')$ (by complement law)

$= [(a+b) \cdot b] + [(a+b) \cdot b']$ [by distributive law]

$= a \cdot b + b \cdot b + a \cdot b' + b \cdot b'$ [Idempotent law $b \cdot b = b$]

$= a \cdot b + b + 0 + 0$ & [complement law $b \cdot b' = 0$]

$= a \cdot b + b$

$=$ ~~(Absorption law)~~ give $\boxed{a \cdot b' = 0}$

$= b \cdot a + b \cdot 1$ [by commutative & identity law]

$= b \cdot (a+1)$ (by distributive law)

$= b \cdot 1$ [$\because a+1=1$]

$= \boxed{b}$ [by identity law]

To prove (ii) $\Rightarrow$ (iii) i.e.

If $\boxed{a+b=b}$ then $\boxed{a'+b=1}$

L.H.S. $= a'+b = a'+(a+b)$ (given $\boxed{a+b=b}$)

$= (a'+a)+b$ [by associative law]

$= 1+b$ (by complement law]

$= 1 =$ R.H.S. [$\because 1+b=1$]

To prove (iii) $\Rightarrow$ (i) i.e. If $\boxed{a'+b=1}$ then $\boxed{a \cdot b' = 0}$

~~L.H.S. =~~ $\Rightarrow (a'+b)' = 1'$

$\Rightarrow (a')' \cdot b' = 0$ (by De-Morgan's law)

$= a \cdot b' = \boxed{0}$ (by involution law)

Hence the statements are equivalent.

Note → $\wedge$ or $\cdot$ → Meet or conjunction
$\vee$ or $+$ → Join or disjunction

Boolean function or boolean polynomial →

An expression obtained by the application of binary operation (+ or $\vee$) and ($\cdot$ or $\wedge$) and uniary operation (') on finite number of elements of boolean algebra $(B, +, \cdot, ')$ is called a Boolean function or Boolean ~~algebra~~ polynomial.

→ All the boolean expressions regardless of three forms can be converted into either of two standard forms :-

1. Sum of products
2. Products of sum

1. Standard sum of products :-

eg :- $AB + B(CD + EF)$

$AB + BCD + BEF$

Ques $AB + ABC + CB$

$AB(C+\bar{C}) + ABC + (\bar{A}+A)CB$

$= ABC + AB\bar{C} + ABC + \bar{A}BC + ABC$

$= ABC + \bar{A}BC + AB\bar{C}$

2. Product of sum :-

eg :- $(A+B)\cdot(B+C)$

Ques $(A+B)\cdot(A+B\cdot\bar{B})$

$= (A+B)\cdot(A+\bar{B}\cdot B)$

$= (A+B)\cdot(A+B)\cdot(A+\bar{B})$

$= (A+B)\cdot(A+\bar{B})$

* Canonical form of boolean expression :-

Variable		Minterms
A	B	
0	0	$\bar{A}\bar{B} = m_0$
0	1	$\bar{A}B = m_1$
1	0	$A\bar{B} = m_2$
1	1	$AB = m_3$

⇒ Maxterm is the sum of various different variables in which each variables occurs atleast once, the o/p result of maxterms function is 0. It is represented by M, to represent the function as product of maxterms or product of sums.

Variables		Maxterms
A	B	Term representation
0	0	$\bar{A} + \bar{B} = M_0$
0	1	$\bar{A} + B = M_1$
1	0	$A + \bar{B} = M_2$
1	1	$A + B = M_3$

A	B	C	Minterms	Maxterms
0	0	0	$\bar{A}\bar{B}\bar{C} = m_0$	$A + B + C = M_0$
0	0	1	$\bar{A}\bar{B}C = m_1$	$A + B + \bar{C} = M_1$
0	1	0	$\bar{A}B\bar{C} = m_2$	$A + \bar{B} + C = M_2$
0	1	1	$\bar{A}BC = m_3$	$A + \bar{B} + \bar{C} = M_3$
1	0	0	$A\bar{B}\bar{C} = m_4$	$\bar{A} + B + C = M_4$
1	0	1	$A\bar{B}C = m_5$	$\bar{A} + B + \bar{C} = M_5$
1	1	0	$AB\bar{C} = m_6$	$\bar{A} + \bar{B} + C = M_6$
1	1	1	$ABC = m_7$	$\bar{A} + \bar{B} + \bar{C} = M_7$

A	B	C	D	Minterms	Maxterms
0	0	0	0	$\bar{A}\bar{B}\bar{C}\bar{D} = m_0$	$A+B+C+D = M_0$
0	0	0	1	$\bar{A}\bar{B}\bar{C}D = m_1$	$A+B+C+\bar{D} = M_1$
0	0	1	0	$\bar{A}\bar{B}C\bar{D} = m_2$	$A+B+\bar{C}+D = M_2$
0	0	1	1	$\bar{A}\bar{B}CD = m_3$	$A+B+\bar{C}+\bar{D} = M_3$
0	1	0	0	$\bar{A}B\bar{C}\bar{D} = m_4$	$A+\bar{B}+C+D = M_4$
0	1	0	1	$\bar{A}B\bar{C}D = m_5$	$A+\bar{B}+C+\bar{D} = M_5$
0	1	1	0	$\bar{A}BC\bar{D} = m_6$	$A+\bar{B}+\bar{C}+D = M_6$
0	1	1	1	$\bar{A}BCD = m_7$	$A+\bar{B}+\bar{C}+\bar{D} = M_7$
1	0	0	0	$A\bar{B}\bar{C}\bar{D} = m_8$	$\bar{A}+B+C+D = M_8$
1	0	0	1	$A\bar{B}\bar{C}D = m_9$	$\bar{A}+B+C+\bar{D} = M_9$
1	0	1	0	$A\bar{B}C\bar{D} = m_{10}$	$\bar{A}+B+\bar{C}+D = M_{10}$
1	0	1	1	$A\bar{B}CD = m_{11}$	$\bar{A}+B+\bar{C}+\bar{D} = M_{11}$
1	1	0	0	$AB\bar{C}\bar{D} = m_{12}$	$\bar{A}+\bar{B}+C+D = M_{12}$
1	1	0	1	$AB\bar{C}D = m_{13}$	$\bar{A}+\bar{B}+C+\bar{D} = M_{13}$
1	1	1	0	$ABC\bar{D} = m_{14}$	$\bar{A}+\bar{B}+\bar{C}+D = M_{14}$
1	1	1	1	$ABCD = m_{15}$	$\bar{A}+\bar{B}+\bar{C}+\bar{D} = M_{15}$

Ques Express $f(A,B,C,D)$ in minterm or SOP form.

$$F = A+BC$$
$$= A(B+\bar{B})(C+\bar{C}) + (A+\bar{A})BC$$
$$= AB + A\bar{B}(C+\bar{C}) + ABC + \bar{A}BC$$
$$= ABC + AB\bar{C} + A\bar{B}C + A\bar{B}\bar{C} + ABC + \bar{A}BC$$
$$= ABC + AB\bar{C} + A\bar{B}C + \bar{A}BC + A\bar{B}\bar{C}$$
$$F = m_7 + m_6 + m_5 + m_3 + m_4$$
$$F = \Sigma(m_7, m_6, m_5, m_3, m_4)$$
$$= \Sigma(7, 6, 5, 3, 4)$$

Ques

$$\cancel{F = (A+B)\cdot(A+\bar{B}+\bar{C})}$$
$$\cancel{= (A+B)+(C\cdot\bar{C})\cdot(A+\bar{B}+\bar{C})}$$
$$\cancel{= A+B(C\cdot\bar{C})\cdot(A+\bar{B}+\bar{C})}$$
$$\cancel{= A+(BC\cdot B\bar{C})\cdot(A+\bar{B}+\bar{C})}$$
$$\cancel{= A+BC}$$

Ques $F = (A+B)\cdot(A+\bar{B}+\bar{C})$

$= (A+B)+(C\cdot\bar{C})\cdot(A+\bar{B}+\bar{C})$

$= (A+B+C)\cdot(A+B+\bar{C})\cdot(A+\bar{B}+\bar{C})$

$= M_0 \cdot M_1 \cdot M_3$

$= \pi(0,1,3)$ Ans

Ques Find the sum of product and product of sums equations from the given truth table?

A	B	C	Output
0	0	0	$0 \rightarrow A+B+C$
0	0	1	$1 \rightarrow \bar{A}\bar{B}C$
0	1	0	$1 \rightarrow \bar{A}B\bar{C}$
0	1	1	$0 \rightarrow A+\bar{B}+\bar{C}$
1	0	0	$1 \rightarrow A\bar{B}\bar{C}$
1	0	1	$0 \rightarrow \bar{A}+B+\bar{C}$
1	1	0	$1 \rightarrow AB\bar{C}$
1	1	1	$0 \rightarrow \bar{A}+\bar{B}+\bar{C}$

$SOP = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}\bar{C} + AB\bar{C} = \Sigma(1,2,4,6)$

$POS = (A+B+C)\cdot(A+\bar{B}+\bar{C})\cdot(\bar{A}+B+\bar{C})\cdot(\bar{A}+\bar{B}+\bar{C}) = \pi(0,3,5,7)$

Ques $F = (A+B+C)\cdot(A+B)\cdot(B+C+D)$

$= [A+B+C+(D\cdot\bar{D})]\cdot(A+B+(C\cdot\bar{C})+(D\cdot\bar{D}))\cdot[(A\cdot\bar{A})+B+C+D]$

$= [A+B+(C+D)\cdot(C+\bar{D})]\cdot[(A+B+C)\cdot(A+B+\bar{C})+(D\cdot\bar{D})]\cdot$
$[(A\cdot\bar{A})+B+C+D]$

$= (A+B+C+D)\cdot(A+B+C+\bar{D})\cdot(A+B+C+D)\cdot(A+B+C+\bar{D})\cdot$
$(A+B+\bar{C}+D)\cdot(A+B+\bar{C}+\bar{D})\cdot(A+B+C+D)\cdot(\bar{A}+B+C+D)$

$= (A+B+C+D)\cdot(A+B+C+\bar{D})\cdot(A+B+\bar{C}+\bar{D})\cdot(\bar{A}+B+C+D)\cdot$
$(A+B+\bar{C}+D)$

$= M_0, M_1, M_8, M_2, M_3$

$\pi(0,1,2,3,8)$ Ans

Important Questions

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Ques 1 Find the number from 1 to 1000 that are not divisible by 2 or 3 or 5 or 7.

Ques 2 Prove by principle of mathematical induction : $n^4 - 4n^2$ is divisible by 3 $\forall$ $n \geq 2$?

c) Let A = {2, 4, 5, 7, 8} = B, a R b if and only if $a+b \leq 12$ find relation matrix ?

answer

Ans 1

A = ~~Integ~~ Integers divisible by 2 = $\left\lfloor \frac{1000}{2} \right\rfloor = 500$

B = Integers divisible by 3 = $\left\lfloor \frac{1000}{3} \right\rfloor = 333$

C = Integers divisible by 5 = $\left\lfloor \frac{1000}{5} \right\rfloor = 200$

D = Integers divisibly by 7 = $\left\lfloor \frac{1000}{7} \right\rfloor = 142$

$A \cap B = \left\lfloor \frac{1000}{6} \right\rfloor = 166$ $\quad A \cap D = \left\lfloor \frac{1000}{14} \right\rfloor = 71$

$A \cap C = \left\lfloor \frac{1000}{10} \right\rfloor = 100$ $\quad B \cap D = \left\lfloor \frac{1000}{21} \right\rfloor = 47$

$B \cap C = \left\lfloor \frac{1000}{15} \right\rfloor = 66$ $\quad C \cap D = \left\lfloor \frac{1000}{35} \right\rfloor = 28$

$A \cap B \cap C = \left\lfloor \frac{1000}{30} \right\rfloor = 33$ $\quad B \cap C \cap D = \left\lfloor \frac{1000}{105} \right\rfloor = 9$

$A \cap B \cap D = \left\lfloor \frac{1000}{42} \right\rfloor = 23$ ~~BA~~

$A \cap C \cap D = \left\lfloor \frac{1000}{70} \right\rfloor = 14$

$A \cap B \cap C \cap D = \left\lfloor \frac{1000}{210} \right\rfloor = 4$

(i) Integers divisible by 2 or 3 or 5 or 7 = $|A \cup B \cup C \cup D|$
$= |A|+|B|+|C|+|D| - |A\cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$
$= 500 + 333 + 200 + 142 - 166 - 100 - 66 - 71 - 47 - 28 + 33 + 23 + 14 + 9 - 4$
$= 772$

(ii) Integers divisibly by 2, 3, 5 or 7 = $1000 - 772$
$= \boxed{228}$ This includes 1000

for exclusive = $228 - 1 = \boxed{227}$ Ans

Mathematical induction

Let $P(n)$ be a statement ~~involving~~ involving the natural number. To prove that $P(n)$ is true for natural number $n \geq 1$. We proceed as follows:

(i) verify $P(n)$ for $n=1$ i.e. $P(n)$ is true for $n=1$.
(ii) suppose the result $P(n)$ is also true for $n=k>1$ (i.e. for $P(k)$)

(iii) using (i) & (ii), prove that $P(k+1)$ is true.
This is known as the first principle of mathematical induction.

Que Prove by induction method :-
$$1+2+3+4+\dots+n = \frac{n(n+1)}{2} \quad \forall n \in N$$

Soln Given $P(n) \rightarrow 1+2+3+4+\dots n = \frac{n(n+1)}{2}$ —①

For $n=1$ L.H.S. $= \boxed{1}$

R.H.S. $= \frac{1(2)}{2} = \boxed{1}$

$\Rightarrow$ L.H.S = R.H.S.

$\therefore$ eqn ① for $n=1$

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* Assume eqn ① is true for $n = k$

$$1+2+3 \dots + k = \frac{k(k+1)}{2} \quad - ②$$

* Now for $n = k+1$

$$L.H.S. = 1+2+3+\dots+k+k+1$$
$$= (1+2+3+\dots+k)+(k+1)$$
$$= \frac{k(k+1)}{2} + k+1 \quad (\text{from } ②)$$
$$= (k+1)\left(\frac{k}{2}+1\right)$$
$$= \frac{(k+1)(k+2)}{2} = R.H.S.$$

∴ eqn ① is also true for $n = k+1$

Hence ① is true for all values of $n \in N$.

Ques $1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}, n \geq 1$

Let $P(n) = 1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6} \quad - ①$

for $n = 1$

$$L.H.S = 1^2 = 1$$
$$R.H.S = \frac{1(2)(3)}{6} = 1$$

L.H.S = R.H.S

∴ ① is true for $n = 1$

* suppose eqn ① is true for $n = k$

$$1^2+2^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6} \quad - ②$$

* Now for $n = k+1$

$$L.H.S. = 1^2+2^2+\dots+k^2+(k+1)^2$$
$$= (1^2+2^2+\dots+k^2)+(k+1)^2$$
$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

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$= (K+1)\left[\frac{K(2K+1)}{6} + K+1\right]$

$= (K+1)\left[\frac{K(2K+1)+6(K+1)}{6}\right]$

$= \frac{(K+1)}{6}[2K^2+K+6K+6] \Rightarrow \frac{(K+1)}{6}[2K^2+7K+6]$

$= \frac{(K+1)}{6}[\cancel{2K} \; 2K(K+2)+3(K+2)]$

$= \frac{(K+1)(K+2)(2K+3)}{6}$

= R.H.S. for $n = K+1$

Hence ① is true for $\forall\, n \in N$.

Ques $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} \geq \sqrt{n}$

Ans Let $P(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}$ — ①

* For $n = 1$

L.H.S. $= \frac{1}{\sqrt{1}} = 1$

R.H.S. $= \sqrt{1} = 1$

L.H.S = R.H.S.

∴ ① is true for $n = 1$

* Assume the result for $n = k$.

$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{K}} \geq \sqrt{K}$ - ②

* ~~step~~ for $n = K+1$

L.H.S. $= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{K}} \cancel{\geq \sqrt{K+1}} + \frac{1}{\sqrt{K+1}}$

$= \left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{K}}\right) + \frac{1}{\sqrt{K+1}} \cancel{\geq \sqrt{K+1}}$

$\geq \sqrt{K} + \frac{1}{\sqrt{K+1}} \cancel{\geq \frac{\sqrt{K}(\sqrt{K+1})+1}{\sqrt{K+1}}}$

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$$\Rightarrow \sqrt{k} + \frac{1}{\sqrt{k+1}+\sqrt{k}}$$

$$\Rightarrow \sqrt{k} + \frac{1}{\sqrt{k+1}+\sqrt{k}} \times \frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k+1}-\sqrt{k}}$$

$$\Rightarrow \sqrt{k} + \frac{\sqrt{k+1}-\sqrt{k}}{k+1-k}$$

$$\Rightarrow \sqrt{k} + \frac{\sqrt{k+1}-\sqrt{k}}{1}$$

$\Rightarrow \sqrt{k+1} = R.H.S.$

$\therefore$ eqn (1) is true for all $n \in N$.

Que Show that n^3+2n is divisible by 3 for $n \geq 1$.

Ans Let $P(n) = n^3+2n$, $n \in N$ — (1)

For $n=1$

$P(1) = 1^3 + 2(1) = 1+2 = 3$

$P(1)$ is divisible by 3

* Assum $n=k$ is also divisible by 3

$= k^3 + 2(k)$ — (2)

* For $n = k+1$

$P(k+1) = (k+1)^3 + 2(k+1)$ $\qquad a^3+b^3+3a^2b+3ab^2$

$= k^3+1+3k^2+3k+2k+2$

$= k^3 + 3 + 5k + 3k^2$

$= \underbrace{(k^3+2k)}_{\text{is divisible by 3 from (2)}} + 3k^2 + 3k + 3$

$3k^2$ is a multiple of 3 hence is divisible by 3

$3k$ is also multiple of 3 hence is divisible by 3

3 is also multiple of 3 hence is divisible by 3

$\therefore$ adding of $(k^3+2k)+3k^2+3k+3$ is divisible by 3

Hence eqn (1) is true $\forall$ all $n \in N$.

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Ques Prove by induction, n^2+n is even number for all natural number n.

soln $P(n) = n^2+n$ — ①

for $n=1$ $P(1) = 1^2+1 = 2 = $ even

$\therefore P(1)$ is true for $n=1$

* Assume for $n=k$ i.e.

$P(k) = k^2+k$ — ② is a even number

* for $n=k+1$

$P(k) = (k+1)^2 + k+1$

$= k^2+1+2k+k+1$

$= k^2+3k+2$

$= k^2+2k+1k+2$

$= k(k+2)+1(k+2)$

$= (k+1)(k+2)$

$= k^2+3k+2$

$= k^2+k+2k+2$

$= \underbrace{k^2+k}+2(k+1)$

it is even from ②

$2(k+1)$ is a multiple of 2 hence it is a even number.

so, the addition of $(k^2+k)+2(k+1)$ is also a even number

Hence, eqn ① is even number $\forall n \in N$.

Ques Prove that $7^{2n}+2^{3n-3}\cdot 3^{n-1}$ is divisible by 25 $\forall n \in N$.

or is a multiple of 25 $\forall n \in N$.

Ans $P(n) = 7^{2n}+2^{3n-3}\cdot 3^{n-1}$

$\Rightarrow 49^n + 8^{n-1}\cdot 3^{n-1} \Rightarrow 49^n + 24^{n-1}$ — ①

* for $\boxed{n=1}$

$\Rightarrow 49+1 \Rightarrow \boxed{50 = P(1)}$

$P(1)$ is divisible by 25.

* Assume for $n=k$,

$P(k) = 49^k + 24^{k-1}$ — ②

K-1+K

$P(K)$ is divisible by 25.

$$\frac{49^K+24^{K-1}}{25} = m$$

$$49^K+24^{K-1} = 25m, \; m \in Z \quad —③$$

* Now for $n=K+1$,

$$P(K+1) = 49^{K+1}+24^K$$
$$= 49^K \cdot 49 + 24^K \quad (\text{from } ③)$$
$$= (25m-24^{K-1}) \cdot 49 + 24^K$$
$$= 25m \cdot 49 - 49 \cdot 24^{K-1} + 24^K$$
$$= 25m \cdot 49 - 24^{K-1}(49-24)$$
$$= 25m \cdot 49 - 24^{K-1}(25)$$
$$= 25[49m - 24^{K-1}]$$

Hence it is a multiple of 25 ∴ is divisible by 25.

⇒ Hence eqⁿ ① is multiple of 25 or is divisible by 25.

Ques Show that for any integer n,

$11^{n+2}+12^{2n+1}$ is divisible by 133.

Solⁿ $P(n) = 11^{n+2}+12^{2n+1} \quad —①$

* For $\boxed{n=1}$ is

$$P(1) = 11^3+12^3$$
$$= 1331+1728$$
$$= 3059$$
$$= 133 \times 23 \quad (\text{Multiple of } 133)$$

$P(1)$ is divisible by 133.

* Assume for $n=K$ eqⁿ ① is true

$$P(K) = 11^{K+2}+12^{2K+1}$$
$$P(K) = 133m$$
$$P(K) = 11^{K+2}+12^{2K+1} = 133m \quad —②$$

* Now for $\boxed{n=K+1}$

$$P(K+1) = 11^{K+3}+12^{2K+3}$$
$$= 11(11^{K+2})+12^{2K+3}$$

$= 11(133m - 12^{2k+1}) + 12^{2k+3}$ (from ②)

$= 11 \times 133m - 11 \times 12^{2k+1} + 144 \times 12^{2k+1}$

$= 11 \times 133m + 133\, 12^{2k+1}$

$= 133(11m + 12^{2k+1})$

Hence it is multiple of 133 & is divisible by 133

Hence eqn ① is divisible by 133.

Ques 2 $\quad P(n) = n^4 - 4n^2$ — ①

* for $n = 1$

$P(1) = 1 - 4 = -3$ ∴ it is divisible by 3

* Assume for $n = k$ is divisible by 3

$P(k) = k^4 - 4k^2$ — ②

* for $n = k+1$

$$P(k+1) = (k+1)^4 - 4(k+1)^2$$

$$= (k+1)^2(k+1)^2 - 4(k+1)^2$$

$$= (k^2+1+2k)(k^2+2k+1) - 4(k^2+1+2k)$$

$$= k^4 + 2k^3 + k^2 + k^2 + 2k + 1 + 2k^3 + 4k + 3k - (4k^2 + 4 + 8k)$$

$$= k^4 - 4k^2 + 4k^3 + 6k^2 - 4k - 3$$

$$= k^4 - 4k^2 + 3(2k^2 - 1) + 4k^3 - 4k$$

$$= \underbrace{k^4 - 4k^2}_{\text{from ②}} + \underbrace{3(2k^2-1)}_{\text{divisible by 3}} + 4\underbrace{(k-1)(k)(k+1)}_{\text{divisible by 3}}$$

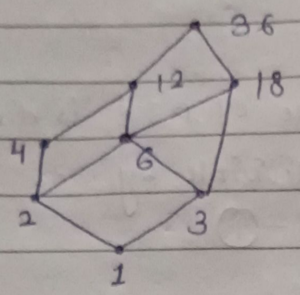
Hence eqn ① is divisible by 3.

Ans 3:-

	2	4	5	7	8
2	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	0
7	1	1	1	0	0
8	1	1	0	0	0

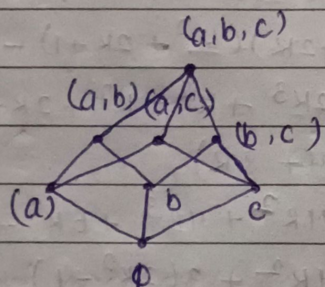
Ques Draw the hasse diagram of D_{36} ?

$D_{36} = \{1, 2, 3, 4, 6, 12, 18, 36\}$



Ques Draw the hasse diagram of $\{P\{a,b,c\}, \subseteq\}$?

$P\{a,b,c\} = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$

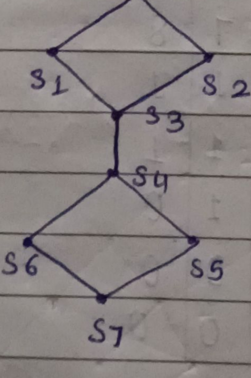


Ques Draw the hasse diagram of the poset $(L, \subseteq)$ where L is

$L = \{S_0, S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$

$S_0 = \{a,b,c,d,e,f\}$, $S_1 = \{a,b,c,d,e\}$, $S_2 = \{a,b,c,e,f$

$S_3 = \{a,b,c,e\}$, $S_4 = \{a,b,c\}$, $S_5 = \{a,b\}$, $S_6 = \{a,c\}$, S_7



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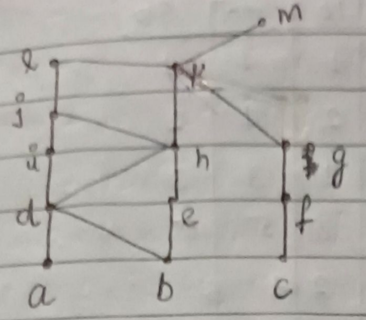
Ques



(i) Maximal elements = $\{l, m\}$

(ii) Minimal elements = $\{a, b, c\}$

(iii) find the greatest element if it exist = Does not exist

(iv) find the least element if it exist = Does not exist

(v) find the upper bound of $\{a, b, c\}$ = $\{l, K, m\}$

(vi) find the least upper bound of $\{a, b, c\}$ = K

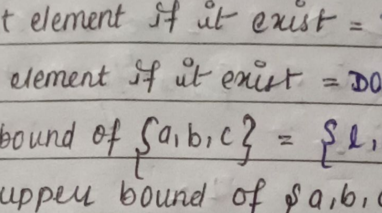
(vii) find the lower bound of $\{f, g, h\}$ = Does not exist

(viii) find the least lower bound and greatest lower bound of $\{f, g, h\}$ = Does not exist

(ix) find the lower bound of $\{K, m\}$ = $\{a, b, c, d, e, f, g, h, K\}$

(x) find the greatest lower bound of $\{K, m\}$ = K.

Ques Find the maximal & minimal elements from the hasse diagram :-



Maximal elements = $\{25, 20, 12\}$, Minimal elements = $\{5, 2\}$

Ques $(D_{30}, |)$ is a lattice

$a \vee b = \text{lcm}\{a, b\}$ & $a \wedge b = \{a, b\}$ gcd or hcf

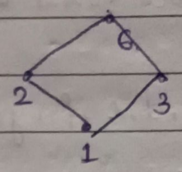
$\vee$	1	2	3	5	6	10	15	30
1	1	2	3	5	6	10	15	30
2	2	2	6	10	6	10	30	30
3	3	6	3	15	6	30	15	30
5	5	10	15	5	30	10	15	30
6	6	6	6	30	6	30	30	30
10	10	10	30	10	30	10	30	30
15	15	30	15	15	30	30	15	30
30	30	30	30	30	30	30	30	30

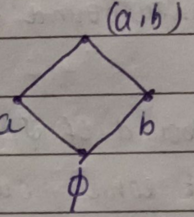
∧	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

Ques Let L_1 be the lattice D_6 & L_2 be the lattice defined as $(P(S), \subseteq)$ where S is $\{a, b\}$ justify that the two lattice is isomorphic?

L_1 $\{1, 2, 3, 6\}$

L_2 $\{\phi, \{a\}, \{b\}, \{a, b\}\}$



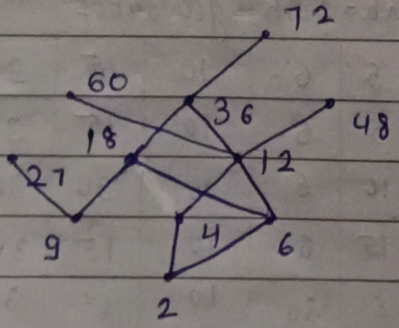


→ Same number of greatest & least elements, have identical hasse diagram, both have same no. of maximal & minimal elements, have same number of edges.

Distributive lattice :- at most one element

complement lattice :- at least one element

Ques $\{2, 4, 6, 9, 12, 18, 27, 36, 48, 60, 72\}$



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Ques a) find the maximal elements = {72, 60, 27, 48}
b) find the minimal elements = {2, 9}
c) Is there a greatest element ? NO
d) Is there a least element ? NO
e) Find all ~~upper~~ upper bound of {2, 9} ? {36, 72, 18}
f) Find the lower bound of {2, 9} ? Does not exist