Type equation here.

Objective:

Consider an inverted pendulum on a moving platform, assuming $M = 2\ kg$, $m = 0.15\ kg$, $l =$ m.

a) Find the state-space model of the system in MATLAB if $x1 = \theta$, $x2 = \dot{\theta}$, $x3 = x$, $x4 = \dot{x}$, $y1 = x1 = \theta$ and $y2 = x3 = x$

b) Write down system transfer function in MATLAB.

c) Design a state feedback control with gain -K so that the closed-loop poles are located at $s = -4 + 2j$, $s = -4 - 2j$, $s = -10$, $s = -10$

d) Check Controllability and Observability of the system in MATLAB

e) Use MATLAB to plot the step response of the system.Show simulink model

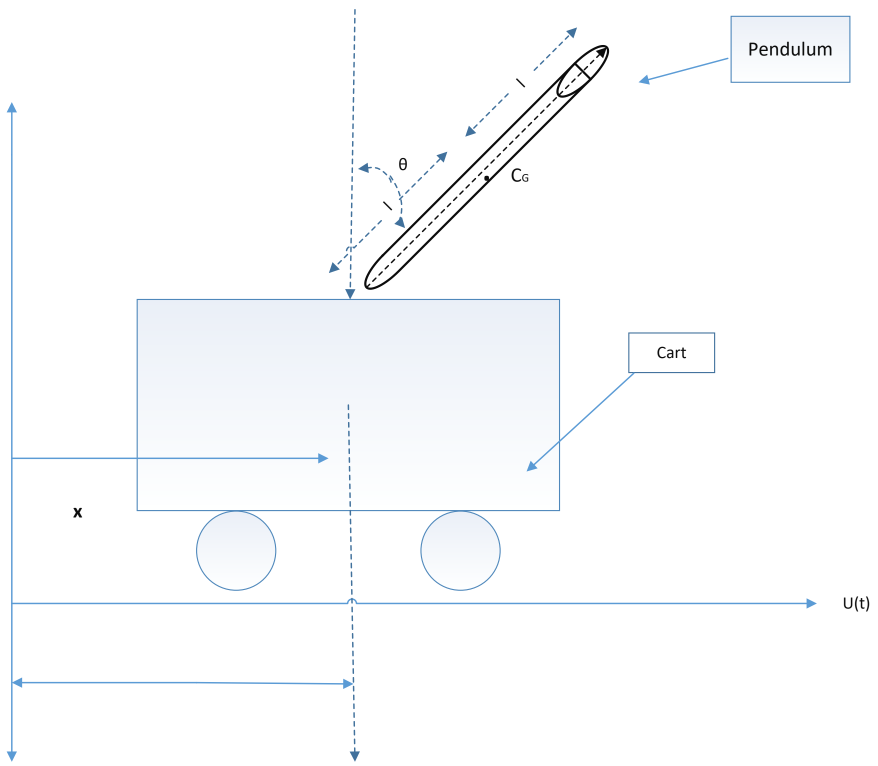


Fig.1 - Inverted Pendulum System

Cart: Horizontal position=x
Pendulum: Horizontal position=$x+lSin\theta$
Vertical position=$lCos\theta$

The Dynamic behavior of system is described by-
For Cart- Position and Velocity

For Pendulum- Angular position and angular Velocity

State –

$$x(t)=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}\theta\\\dot{\theta}\\x\\\dot{x}\end{bmatrix}$$

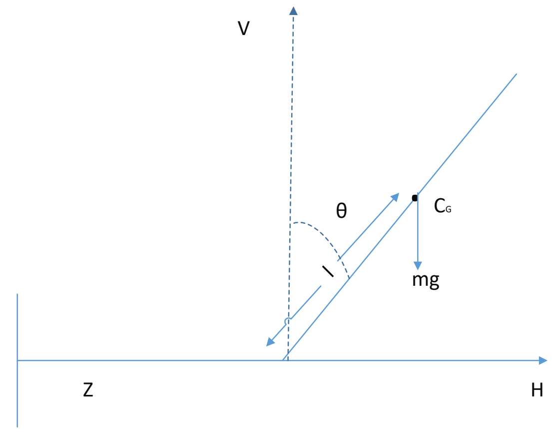


Fig.2-Free body diagram of Inverted Pendulum

M=mass of cart
m=mass of pendulum
J=moment of Inertia of pendulum with respect to center of gravity

$$J\frac{d^2\theta}{dt^2}=Vl\sin\theta-Hl\cos\theta \qquad \text{- (i)}$$

Forces in vertical direction:

$$V-mg=m\frac{d^2}{dt^2}(l\cos\theta) \qquad \text{- (ii)}$$

Forces in Horizontal direction:

$$H=m\frac{d^2}{dt^2}(\mathrm{x}+l\sin\theta) \qquad \text{- (iii)}$$

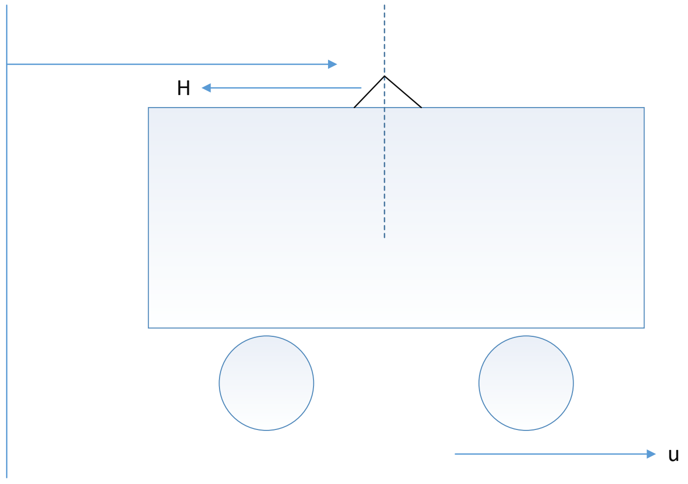


Fig.3-Free body diagram of Cart

Forces in cart in Horizontal direction:

$$u - H = M\frac{d^2x}{dt^2} \qquad \text{- (iv)}$$

Our objective is to keep pendulum upright so it seems θ is close to zero.

$$\sin\theta \simeq \theta \ , \quad \cos\theta \simeq 1$$

In view of approximation we get equation (i),(ii),(iii),(iv) becomes-

$$J\ddot{\theta} = Vl\theta - Hl \qquad \text{- (i)}$$

$$V - mg = m\frac{d}{dt}(l) \qquad \text{- (ii)}$$

$$H = m\frac{d^2}{dt^2}[x + l\theta] \ \Rightarrow \ H = m\ddot{x} + ml\ddot{\theta} \qquad \text{- (iii)}$$

$$u - H = M\ddot{x} \qquad \text{- (iv)}$$

Using iv equation-

$$u = H + M\ddot{x} \Rightarrow u = (m\ddot{x} + Ml\ddot{\theta}) + M\ddot{x}$$

$$u = (m + M)\ddot{x} + Ml\ddot{\theta} \qquad \text{- (v)}$$

Using i equation-

$$J\ddot{\theta} = Vl\theta - Hl$$

Since, $V = mg \ , \ H = m\ddot{x} + ml\ddot{\theta}$

$$J\ddot{\theta} = [mg]l\theta - [m\ddot{x} + ml\ddot{\theta}]l$$

$$J\ddot{\theta} = mgl\theta - m\ddot{x}l - ml^2\ddot{\theta}$$

$$(J + ml^2)\ddot{\theta} + ml\ddot{x} - mgl\theta = 0 \qquad \text{- (vi)}$$

Equation v and vi-

$$u = (\mathrm{m}+\mathrm{M})\ddot{x} + Ml\ddot{\theta} \qquad \text{- (v)}$$

$$(\mathrm{J}+ml^2)\ddot{\theta} + ml\ddot{x} - mgl\theta = 0 \qquad \text{- (vi)}$$

The equation can be rearranged as-

$$\ddot{\theta} = \frac{ml(M+m)g}{(M+m)J+mMl^2}\theta - \frac{ml}{(M+m)J+mMl^2}u$$

$$\ddot{x} = -\frac{m^2l^2g}{(M+m)J+mMl^2}\theta + \frac{(J+ml^2)}{(M+m)J+mMl^2}u$$

The system parameters are-

$$M = 2kg, m = 0.5kg, l = 1m$$
$$g = 9.81m/\sec^2$$
$$J = 0.2kg - m^2$$

For these parameters-

$$\ddot{\theta} = 8.175\theta - 0.33u$$
$$\ddot{x} = -3.27\theta + 0.466u$$

Choosing the states-

$$x_1 = \theta, x_2 = \dot{\theta}, x_3 = x, x_4 = \dot{x}$$

We obtain the state model for inverted pendulum on moving cart

$$\dot{x} = Ax + Bu$$

with $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 8.175 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3.27 & 0 & 0 & 0 \end{bmatrix}; \; B = \begin{bmatrix} 0 \\ -0.333 \\ 0 \\ 0.466 \end{bmatrix}$$

Transfer function model:

Now the transfer function of this system can be obtained using:

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

Type equation here.

Where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 8.175 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3.27 & 0 & 0 & 0 \end{bmatrix}$, $\quad B = \begin{bmatrix} 0 \\ -0.33 \\ 0 \\ 0.467 \end{bmatrix}$,

$$C = [1 \quad 0 \quad 0 \quad 0] \quad (for\ \theta), \quad D = 0$$

So now,

$$(sI - A)^{-1} = \left[s * \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 8.175 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -3.27 & 0 & 0 & 0 \end{pmatrix} \right]$$

$$= \begin{bmatrix} s & -1 & 0 & 0 \\ -8.175 & s & 0 & 0 \\ 0 & 0 & s & -1 \\ 3.27 & 0 & 0 & s \end{bmatrix}^{-1}$$

$$= \frac{1}{s^2(s^2-8.175)} \begin{bmatrix} s^3 & s^2 & 0 & 0 \\ -8.175s^2 & s^3 & 0 & 0 \\ -3.27s & -3.27 & s^3 - 8.175s & s^2 - 8.175 \\ -3.27s^2 & -3.27s & 0 & s^3 - 8.175s \end{bmatrix}$$

<u>So ,Transfer function with respect to angular displacment.</u>

$$\frac{\theta(s)}{U(s)} = C(sI - A)^{-1}B + D = \frac{-0.33s^2}{s^2-8.175}$$

<u>Similarly, Transfer function with respect to linear displacment</u>

$$\frac{x(s)}{U(s)} = \frac{0.467s^2-2.7386}{s^2(s^2-8.175)}$$

<u>Matlab code for finding transfer function</u>

```
%code for finding transfer function
clc;
clear all;
```

Type equation here.

```
% Initialising A,B,C,D matrices

A=[0       1    0    0;
   8.175   0    0    0;
    0      0    0    1;
   -3.27   0    0    0];
B= [  0;
    -0.33;
      0;
    0.467];
C1 = [1 0 0 0];              % for theta
C2=  [0 0 1 0];              % for x(displacement)
D =  0;

%Tranfer function with respect to Theta
[num1,den1]=ss2tf(A,B,C1,D);
sys1=tf(num1,den1)

% Transfer function with respect to x(Translational Displacement
[num2,den2]=ss2tf(A,B,C2,D);
sys2=tf(num2,den2)

% display the output
```

Matlab code for finding controllability and observability

```
%here we are checking the controllability and observability
clc;
clear all;
close all;
A=[0       1    0    0;
   8.175   0    0    0;
    0      0    0    1;
   -3.27   0    0    0];
B= [  0;
    -0.33;
      0;
    0.467];

%checking controllabilty
S=ctrb(A,B);
rank(S)

%checking observability for angular displacement(theta)
C1 = [1 0 0 0];
P1= obsv(A,C1);
```

```
rank(P1)

%%checking observability for linear displacement(x)
C2=  [0 0 1 0];
P2=obsv(A,C2);
rank(P2)
```

Result:

From the output we observe that the system is fully controllable and the system is not completely observable for angular displacement because the observability matrix is not full rank, and it is completely observable for linear displacement

Controller Design

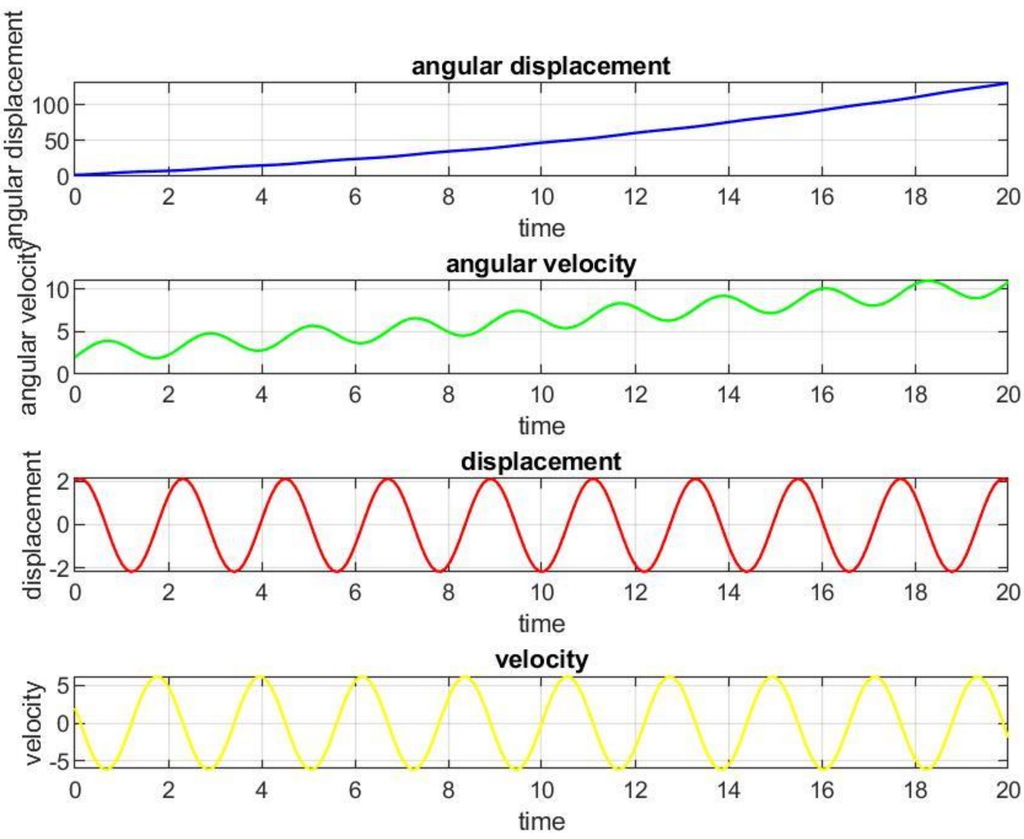
Matlab code for finding state feedback gain matrix

```
clear variables;
clc;
A=[0 1 0 0;8.175 0 0 0; 0 0 0 1; -3.27 0 0 0];
B= [0;-0.33; 0; 0.467];
C = [1 0 0 0; 0 1 0 0;0 0 1 0;0 0 0 1];
D = [0;0;0;0];
K = acker(A,B,[-4+2i -4-2i -10 -10])

Output:

 K=[-1906.73  -704.93   -730.29   -438.18]

Matlab code for the output of system without controller
  %INVERTED PENDULUM PLOT WITHOUT POLE PLACEMENT

clc;
close all;
clear all;
t=linspace(0,20,200);
x0=[2 2 2 2];
[t,y]=ode45('inverted_pend',t,x0);
f =[t,y];
subplot(4,1,1);
plot(t,y(:,1),'linewidth',1,'color','b');
title('angular displacement');
xlabel('time');
```

```
ylabel('angular displacement');
grid on;
hold on;
subplot(4,1,2);
plot(t,y(:,2),'lineWidth',1,'color','g');
title('angular velocity');
xlabel('time');
ylabel('angular velocity');
grid on;
hold on;
subplot(4,1,3);
plot(t,y(:,3),'linewidth',1,'color','r');
title('displacement');
xlabel('time');
ylabel('displacement');
grid on;
hold on;
subplot(4,1,4);
plot(t,y(:,4),'linewidth',1,'color','y');
title('velocity');
xlabel('time');
ylabel('velocity');
grid on;
hold on;


%STATE SPACE MODEL OF INVERTED PENDULUM

%calling function for state space model
function xdot=inverted_pend(t,x)
M=2;
m=0.5;
L=1;
g=-9.81;
J=0.2;
u=1;
xdot=zeros(4,1);
xdot(1)=x(2);
xdot(2)=(((m+M)*(m*g*L)*x(1))/((J+m*L^2)*(M+m)-(m^2*L^2)))-
((m*L*u)/((J+m*L^2)*(M+m)-(m^2*L^2)));
xdot(3)=x(4);
xdot(4)=(((J+m*L^2)*u)/((J+m*L^2)*(m+M)-(m^2)))-
(((m^2*L^2*g)*x(1))/((J+m*L^2)*(M+m)-(m^2*L^2)));
end
```

Output

Type equation here.



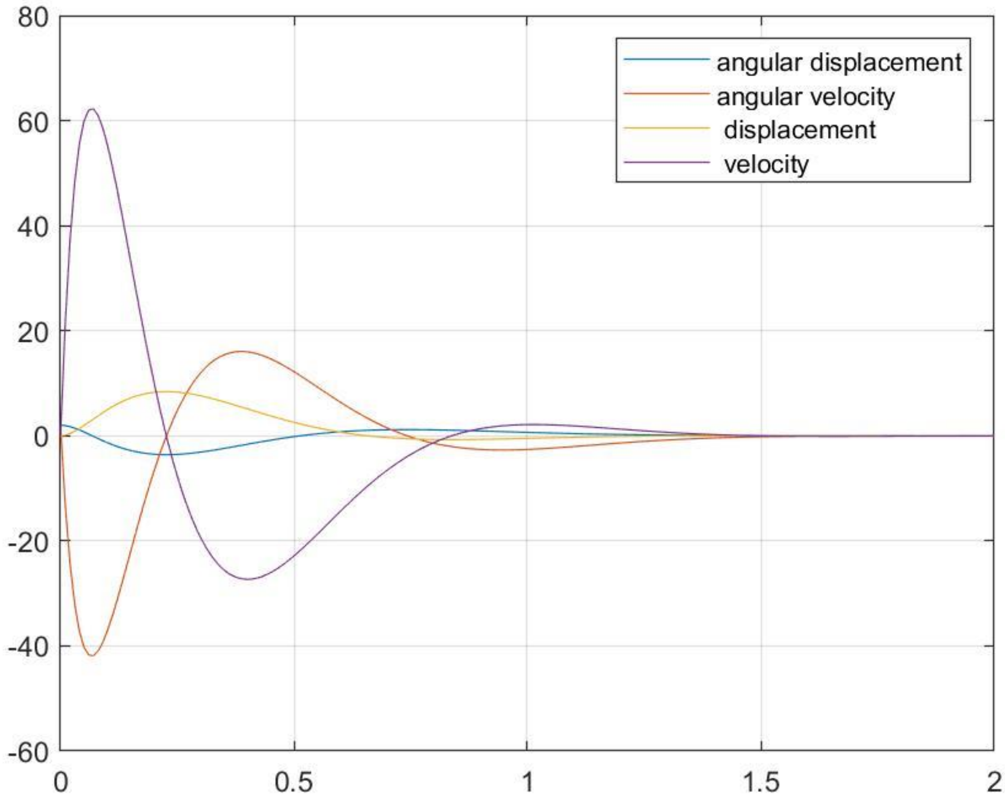
Matlab code for the output of the system with state feedback controller design

```
%INVERTED PENDULUM PLOT WITH POLE PLACEMENT

clc;

close all;
t=linspace(0,2,200);
x0=[2 2 0 0];
[t,x]=ode45('inverted2_pend',t,x0);
plot(t,x);
legend('angular displacement','angular velocity','
displacement',' velocity');
grid on;


%CALLING FUNCTION FOR POLE PLACEMNT

function xdot=inverted2_pend(t,x)
A=[0 1 0 0;8.175 0 0 0; 0 0 0 1; -3.27 0 0 0];
B= [0;-0.33; 0; 0.467];
k = acker(A,B,[-4+2i -4-2i -10 -10]);
e=eig(A)
```

Type equation here.

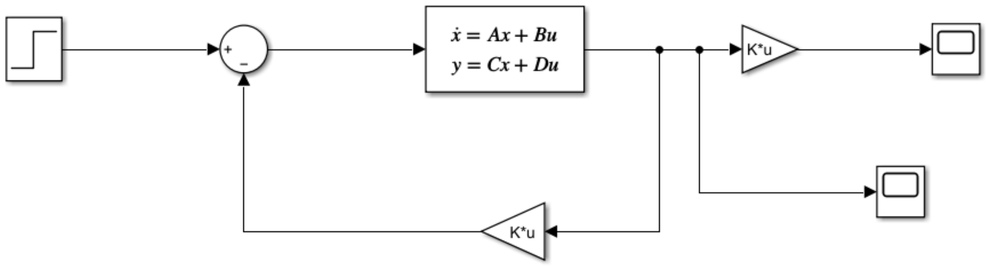
```
a=eig(A-(B*k))
u =-k*x;

M=2;
m=0.5;
L=1;
g=9.81;
J=0.2;
xdot=zeros(4,1);
xdot(1)=x(2);
xdot(2)=(((m+M)*(m*g*L)*x(1))/((J+m*L^2)*(M+m)-(m^2*L^2)))-
((m*L*u)/((J+m*L^2)*(M+m)-(m^2*L^2)));
xdot(3)=x(4);
xdot(4)=(((J+m*L^2)*u)/((J+m*L^2)*(m+M)-(m^2)))-
(((m^2*L^2*g)*x(1))/((J+m*L^2)*(M+m)-(m^2*L^2)));

end
```

Output:

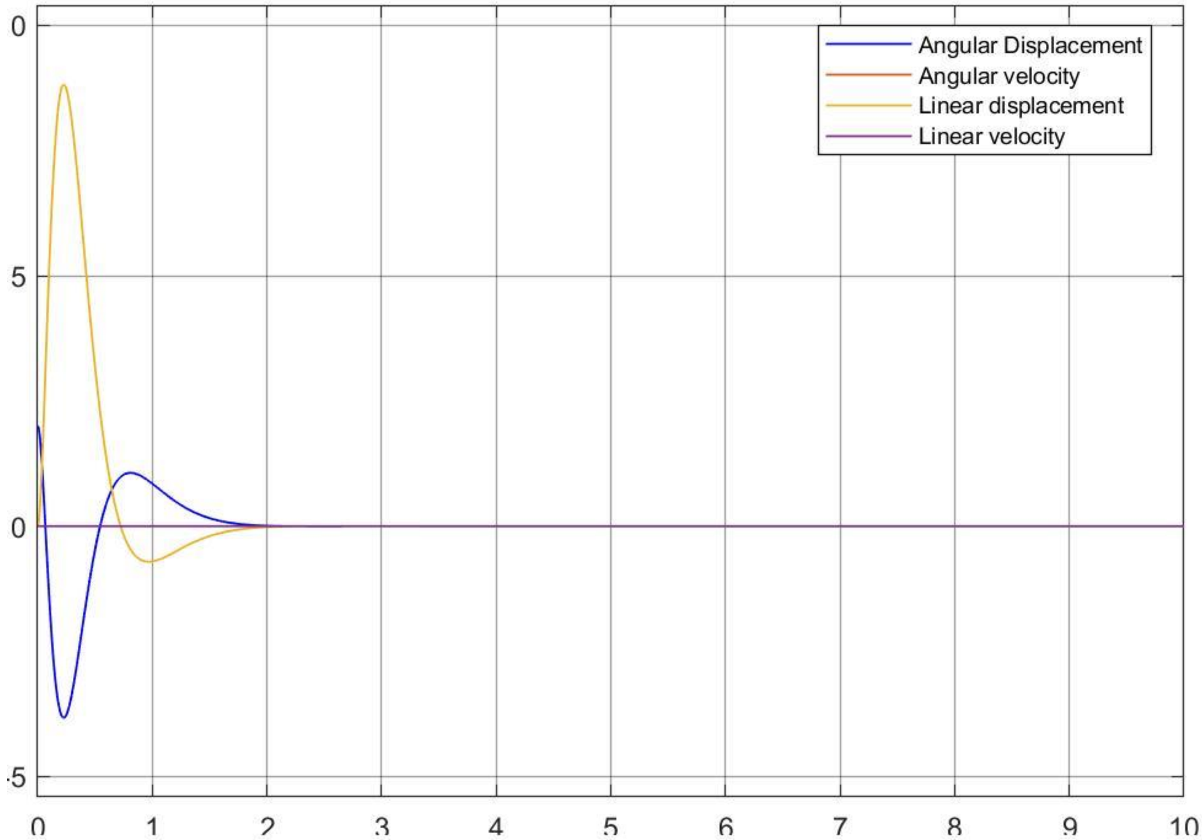


Type equation here.

Simulation Model for system with controller for step input

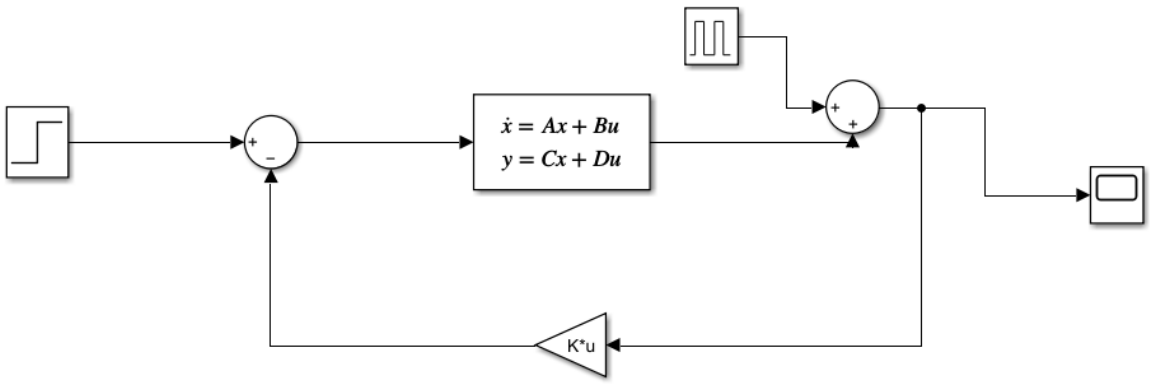


Output:

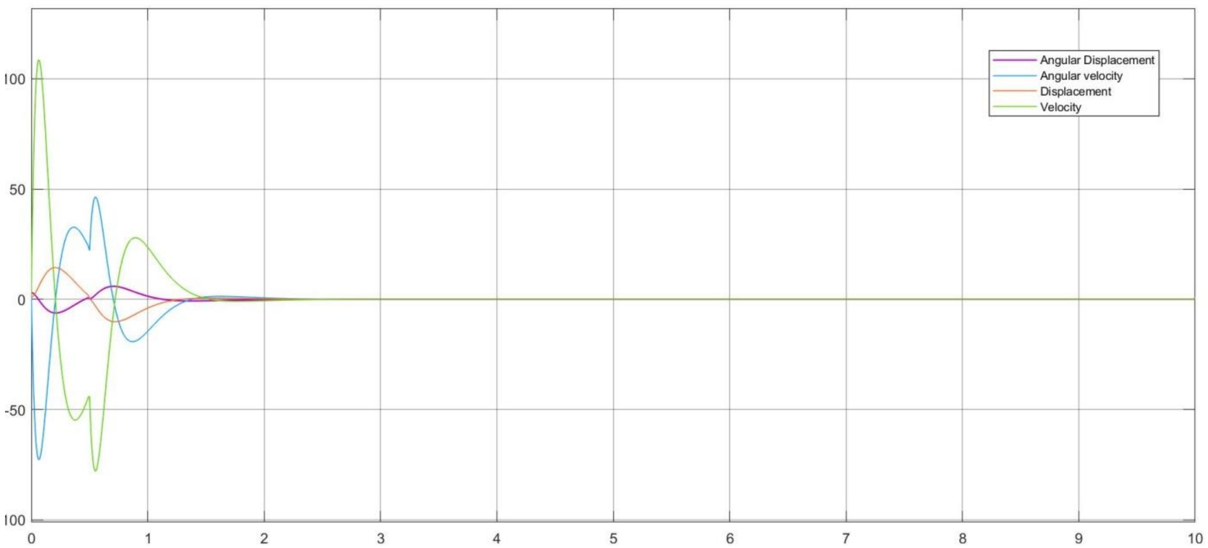


Type equation here.

Simulation Model for system with controller for step input with disturbance



Output:



Conclusion:

We have successfully placed the poles in the desired pole location and achieved stability, and our state feedback controller is strong enough to bear any disturbance.