Matrix Computations in Data Mining

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We discuss a few areas of data mining where matrix computations plays an important role:

- Text mining (information retrieval).
- Pattern recognition (classification of handwritten digits, face recognition).
- Google's PageRank algorithm for web search engine.

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Doc. 2:	G_{ij} is nonzero if there is a link from web page j to i.
Doc. 3:	The Google matrix G is used to rank all web pages.
Doc. 4:	The ranking is done by solving a matrix eigenvalue problem.
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The key words or terms are colored in blue. The set of terms is called Dictionary. Counting the frequency of terms in each document we obtain a term-document matrix.

Term-document matrix

Term	Doc. 1	Doc. 2	Doc. 3	Doc. 4	Doc. 5
eigenvalue	0	0	0	1	0
England	0	0	0	0	1
FIFA	0	0	0	0	1
Google	1	0	1	0	0
Internet	1	0	0	0	0
link	0	1	0	0	0
matrix	1	0	1	1	0
page	0	1	1	0	0
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Each document is represented by a column of the term-document matrix $A \in \mathbb{R}^{10\times 5}$ which is a vector in $\mathbb{R}^{10}$.

Query vector

Suppose that we want to find all documents that are relevant to the query ranking of web pages. This is represented by a query vector, constructed in the same way as the term-document matrix, using the same dictionary:

$$\mathbf{v} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^{10}.$$

Mathematical Problem

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The task is to extract useful information from large and often unstructured sets of data. Hence the methods used must be efficient and specially designed for large problems.

Query matching

Query matching is the process of finding all documents that are relevant to a particular query $\mathbf{v}$. The cosine distance measure is often used to return relevant documents:

$$\cos\theta_j := \frac{\mathbf{v}^T A e_j}{\|\mathbf{v}\|_2 \|A e_j\|_2} > \mathrm{tol}$$

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But at the same time there is a risk that more documents that are not relevant are also returned.

Latent semantic indexing

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Consider SVD $A = U\Sigma V^T$ and the rank-k approximation $A_k := U_k \Sigma_k V_k^T$ so that $\|A - A_k\|_F / \|A\|_F$ small.

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Set $\mathbf{v}_k := U_k^T \mathbf{v}$ and $D_k := \Sigma_k V_k^T$. Compute the cosines

$$\cos\theta_j := \frac{\mathbf{v}_k^T D_k(:, j)}{\|\mathbf{v}_k\|_2 \|D_k(; , j)\|_2}.$$

Example

Consider the term-document matrix A and the query ("ranking of web pages") vector $\mathbf{v}$. Then the cosines measures of the query and the original data are given by

$$[0, 0.6667, 0.7746, 0.3333, 0.3333]^T$$

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Now consider the LSI method with rank-2 approximation. The cosines measures of the projected query and the projected data are given by

$$[0.7857,\ 0.8332,\ 0.9670,\ 0.4873,\ 0.1819]^T.$$

Note that Doc. 1 is deemed highly relevant!

Clustering

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Idea: Given a column partitioning of a term-document matrix A into k clusters

$$A := [A_1, \ldots, A_k] \text{ with } A_j \in \mathbb{R}^{m \times n_j},$$

compute the centroid $\mathbf{c}_j := \frac{1}{n_j} A_j e^{(j)}$ and $C := [\mathbf{c}_1, \ldots, \mathbf{c}_k]$, where $e^{(j)} := [1, 1, \cdots, 1]^T \in \mathbb{R}^{n_j}$.

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Example: Training set of US Postal Service database contains 7291 handwritten digits and test set contains 2007 digits.

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Example:

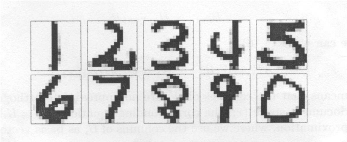
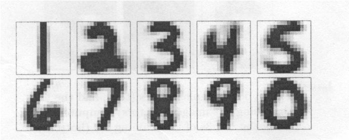


Figure: Handwritten digits from the US Postal service database.

Classification of handwritten digits

The mean digits (centroids) of the training set are given below.



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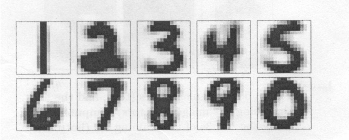

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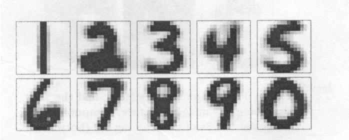

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It follows that a majority of the digits are well written (badly written digits would demonstrate as diffuse means).

Hence a simple algorithm that computes the distance from each unknown digit to the mean digits is likely to work well.

Classification of handwritten digits

A simple classification algorithm

- Training. Given the training set, compute the mean digit (centroid) of all digits of one kind.
- Classification. For each digit in the test set, compute the distance to all ten mean digits, and classify as the closest.

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The success rate of this algorithm for the training set (US Postal Service) is around 75%, which is not good enough.

The reason is that the algorithm does not use any information about the variation of the digits of one kind. This variation can be modelled using the SVD.

SVD based classification of handwritten digits

Let $A \in \mathbb{R}^{m \times n}$, $m = 256$, be the matrix consisting of all the training digits of one kind, say 3. The column space $R(A)$ cannot be expected to have a large dimension because the subspaces of the different kinds of digits would intersect.

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An orthonormal basis of $R(A)$ can be computed using the SVD

$$A = \sum_{j=1}^{k} \sigma_j \mathbf{u}_j \mathbf{v}_j^T.$$

Also A can be approximated by a low rank matrix.

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We expect that the first singular image (image corresponding to the first singular vector) to look like 3 and the remaining singular images should represent the dominating variations of the training set around the first singular image.

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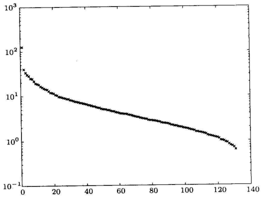

Figure: Singular values computed using 131 images of 3 from training set.

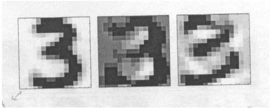

Figure: The first three singular images of 3.

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Let U_k be k-singular basis. Given a test digit $\mathbf{t}$

$$\text{solve } \mathbf{t}_k := \operatorname{argmin}_{\mathbf{x}} \|\mathbf{t} - U_k\mathbf{x}\|_2$$

and compute the residual

$$r_k := \min_{\mathbf{x}} \|\mathbf{t} - U_k\mathbf{x}\|_2$$

for the 10 digits $k = 0, 1, \ldots, 9$.

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Then $\mathbf{t}_k = U_k^T \mathbf{t}$ and $r_k = \|(I - U_k U_k^T)\mathbf{t})$.
r_k depends on the number of basis elements in k-singular basis.

Example

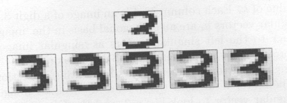

Figure: Unknown digit (nice 3) and 1, 3, 5, 7, and 9 basis elements in 3-basis.

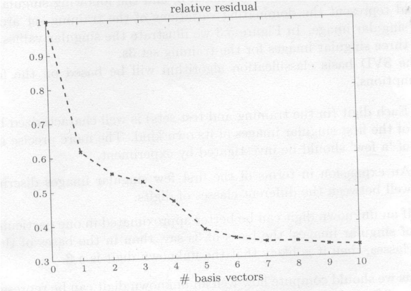

Figure: Relative residuals: $r_k/\|\mathbf{t}\|_2$.

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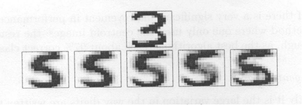

Figure: Unknown digit (nice 3) and $1, 3, 5, 7$, and 9 basis elements in 5-basis.

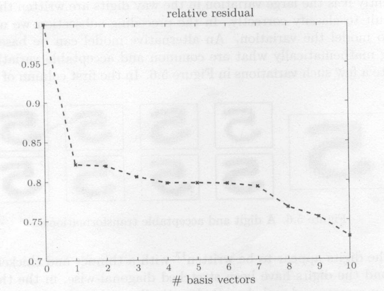

Figure: Relative residuals $r_k/\|\mathbf{t}\|_2$.

SVD based classification of handwritten digits

An SVD based classification algorithm

- Training: For the training set of known digits, compute the SVD of each class of digits and use k basis vectors for each class.
- Classification: For a given test digit, compute its relative residual in all ten bases. If one residual is significantly smaller than all the others, classify as that. Else give up.

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Test results for the US Postal Service data set with 7291 training digits and 2007 test digits. Result for each class.

# basis vector	1	2	4	6	8	10
correct (%)	80	86	90	90.5	92	93

Face recognition

Figure: Library of twelve Presidents of the United States.

We store these images as vectors or matrices and apply PCA (Principal Component Analysis) to process the images.

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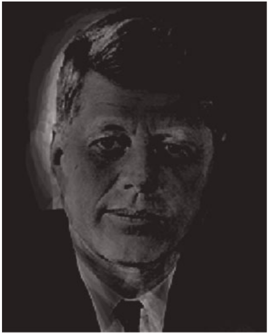
Consider the average image $\mathbf{p} := Ae/12$, where $e := [1, 1, \ldots, 1]^T \in \mathbb{R}^{12}$. Then $A := P - \mathbf{p}e^T$ is the mean-subtracted image and $C := AA^T$ is the covariance matrix.

Face recognition

(a) label 1



(b) label 2

Figure: (a) The average image of twelve U.S. Presidents and (b) the average image subtracted from the image of President Kennedy .

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Since $A^T A$ and C have the same nonzero eigenvalues, we solve $A^T Au = \lambda u$. If $\lambda \neq 0$ then $AA^T(Au) = \lambda(Au)$ and $Au \neq 0$. Hence $v := Au$ is an eigenvector of C corresponding to λ.

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Eigenvectors of C are eigenfaces. These eigenfaces are obtained from eigenvectors of a 12×12 matrix $A^T A$.

Face recognition

Figure: A half dozen eigenfaces of the library of twelve U.S. Presidents.

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This allows us to approximate a large collection of images, like 5,000, with only a small subset of eigenfaces.

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Project the mean-subtracted image $\mathbf{v}$ on the space spanned by the eigenfaces $\mathbf{u}_j$. .

$$\mathbf{v}_{proj} := \sum_{j=1}^{12} (\mathbf{u}_j^T \mathbf{v})\mathbf{u}_j.$$

Then adding the average presidential image we obtain a recognizable face (or an approximation) $\mathbf{v}_{proj} + \mathbf{p}$.

Face recognition

(a) label 1

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Figure: (a) An altered image of President Kennedy and (b) the image recreated using the six eigenfaces.

Face recognition

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One can use a large library of images, reduce it to a much smaller set of eigenfaces, and then use it to recognize a face or create an approximation, even with some disguising.

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