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Inner Product of Vectors (Dot Product / Scalar Product)

Inner product of 2 vectors $u = (u_1, u_2, \ldots u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ from the vector space R^n is defined as $u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$.

It's also defined as $u \cdot v = u^T v$.

Note: Inner product is a scalar. ~~Compute u·v and~~

1. Compute $u \cdot v$ and $v \cdot u$ for $u = \begin{bmatrix} 2 \\ -5 \\ -1 \end{bmatrix}$ & $v = \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix}$

$$\begin{aligned} u \cdot v &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= 2(3) + (-5)(2) + (-1)(-3) \\ &= 6 - 10 + 3 \\ &= -1 \end{aligned}$$

$$\begin{aligned} v \cdot u &= v_1u_1 + v_2u_2 + v_3u_3 \\ &= 3(2) + (2)(-5) + (-3)(-1) \\ &= 6 - 10 + 3 \\ &= -1 \end{aligned}$$

Teacher's Sign/Remarks

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Theorem: Let u, v, w be vectors in R^n and c be a scalar then

i) $u \cdot v = v \cdot u$

ii) $(u+v) \cdot w = (u \cdot w) + (v \cdot w)$

iii) $(cu \cdot v) = c(u \cdot v) = (u \cdot cv)$.

iv) $u \cdot u \geq 0$

$u \cdot u = 0$ if and only if $u = 0$

Defn: Length (norm) of a vector $v = (v_1, v_2, \ldots, v_n)$ is a non-negative scalar denoted by $\|v\|$ is defined as $\sqrt{v \cdot v} = \sqrt{v_1^2 + v_2^2 + \ldots v_n^2}$

norm

For any scalar c $\|cv\| = |c| \, \|v\|$

$c = -1$ $\|cv\| = \|-v\| = |-1| \, \|v\|$
$= \|v\|$

Unit Vector: A vector with unit length is called unit vector.

Given a non zero vector v we obtain the unit vector by dividing v by its length.

i.e. $\hat{v} = \dfrac{\vec{v}}{\|v\|}$

This process of creating a unit vector from a given non zero vector is called normalization.

Distance b/w 2 vectors:

If $u = (u_1, u_2, \ldots, u_n)$ & $v = (v_1, v_2, \ldots, v_n)$ are 2 vectors, then

$$\|u - v\| = \|v - u\| = \sqrt{(u_1-v_1)^2 + (u_2-v_2)^2 + \cdots + (u_n-v_n)^2}$$

Angle b/w 2 vectors: u and v is given by the formula

$u \cdot v = \|u\| \, \|v\| \cdot \cos\theta$ (or)

$$\theta = \cos^{-1}\left(\frac{u \cdot v}{\|u\| \, \|v\|}\right)$$

If $\theta = 90^\circ$; $\cos 90 = 0$; $\therefore u \cdot v = 0$ and we say that u & v are orthogonal.

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Orthogonal Vectors :

Two vectors ~~in~~ u & v in R^n are orthogonal to each other if $u \cdot v = 0$ (zero)

Note: 0 vector is orthognal to every vector in R^n.

Since $0 \cdot u = 0$ for all u.

Orthognal Set:

A set of vectors says $\{v_1, v_2, \dots, v_p\}$ in R^n is said to be an orthognal set if $v_i \cdot v_j = 0$ (or) each pair of distinct vector from the set is orthognal

for $i \neq j$

Theorem: If $S = \{v_1, v_2, \dots, v_p\}$ is an orthognal set of non zero vectors in R^n then S is linearly indipendent and hence a basis for the subspace spanned by S.

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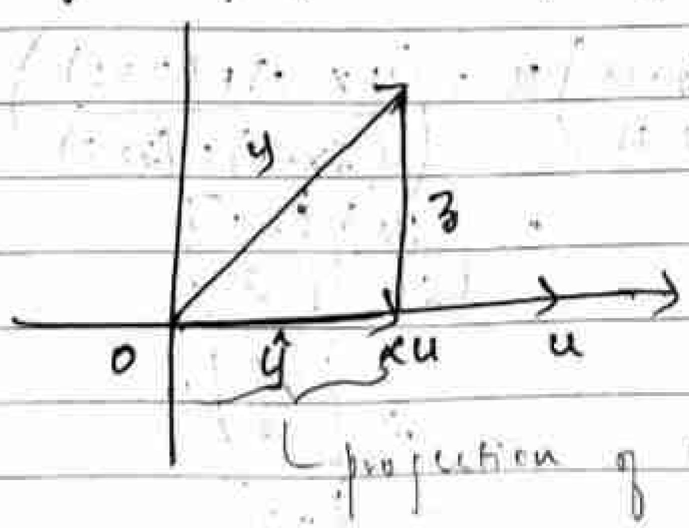
Orthognal Basis for a subspace W (R^n) is a basis for W that is also a orthogonal set.

Orthognal Projection: given a non zero vector u in R^n consider the problem of decomposing a vector y in R^n into the sum of 2 vectors, one a multiple of u and the other orthogonal to u i.e. we write $y = \hat{y} + z$.

Here $\hat{y}$ is multiple of u and z is orthogonal to u.

The vector $\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) u$ is called the orthognal projection of y onto u.



Ex 1. Let $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ find the orthogonal projection of y onto u.

Solⁿ: $\hat{y}$ is orthogonal projection of y onto u

$$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) u = \left(\frac{(7\times4)+(6\times2)}{(4\times4)+(2\times2)}\right)\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= \left(\frac{40}{20}\right)\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$= 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

2. Compute the orthogonal projection of $\begin{bmatrix} 1 \\ 7 \end{bmatrix}$ onto the line through $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and the origin

Solⁿ:

$$\hat{y} = \left(\frac{y \cdot u}{u \cdot u}\right) u = \left(\frac{(1\times-4)+(7\times2)}{(-4\times-4)+(2\times2)}\right)\begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$= \left(\frac{10}{20}\right)\begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Orthonormal set: A set $\{u_1, u_2, \ldots, u_p\}$ is an orthonormal set if it is an orthogonal set of unit vectors.

If W is the subspace spanned by such a set then this set is called an orthonormal basis for W. Since ~~orthonor~~ the set is automatically linear independent.

Example:

The standard basis $\{e_1, e_2, \ldots, e_n\}$ of R^n is an orthonormal basis for R^n.

1. Show that the set $\{v_1, v_2, v_3\}$ is an orthonormal basis for R^3 where

$$v_1 = \begin{bmatrix} 3/\sqrt{11} \\ 1/\sqrt{11} \\ 1/\sqrt{11} \end{bmatrix} \quad v_2 = \begin{bmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} \quad v_3 = \begin{bmatrix} -1/\sqrt{66} \\ -4/\sqrt{66} \\ 7/\sqrt{66} \end{bmatrix}$$

Solⁿ:

$$v_1 \cdot v_2 = \left(-3/\sqrt{66} + 2/\sqrt{66} + 1/\sqrt{66}\right)$$
$$= 0$$

$$v_2 \cdot v_3 = \left(1/\sqrt{66\times6} + -8/\sqrt{66\times6} + 7/\sqrt{66\times6}\right)$$
$$= 0$$

$$v_1 \cdot v_3 = \left(-3/\sqrt{11\times66} - 4/\sqrt{11\times66} + 7/\sqrt{11\times66}\right)$$
$$= 0$$

$\therefore \{v_1, v_2, v_3\}$ is orthognal

$$\|v_1\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$= \sqrt{(3/\sqrt{11})^2 + (1/\sqrt{11})^2 + (1/\sqrt{11})^2}$$

$$= \sqrt{\frac{9}{11} + \frac{1}{11} + \frac{1}{11}}$$

$$= \sqrt{\frac{11}{11}} = 1$$

$$\|v_2\| = \sqrt{\left(\frac{-1}{\sqrt{6}}\right)^2 + \left(\frac{2}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2}$$

$$= \sqrt{\frac{1}{6} + \frac{4}{6} + \frac{1}{6}}$$

$$= \sqrt{\frac{6}{6}} = 1$$

$$\|v_3\| = \sqrt{\frac{1}{66} + \frac{16}{66} + \frac{49}{66}}$$

$$= \sqrt{\frac{66}{66}} = 1$$

$\therefore$ The given is orthonormal basis for $\mathbb{R}^3$.

04/08/23

Gram - Schmidt's Orthogonalization process :

Given a basis $\{x_1, x_2, \cdots, x_p\}$ for a subspace $W (\mathbb{R}^n)$, The following algorithm namely gram-schmidt algorithm forms a orthogonal basis for W namely $\{v_1, v_2, \cdots, v_p\}$ where $v_1 = x_1$, $v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$v_4 = x_4 - \left(\frac{x_4 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_4 \cdot v_2}{v_2 \cdot v_2}\right) v_2 - \left(\frac{x_4 \cdot v_3}{v_3 \cdot v_3}\right) v_3$$

$$v_p = x_p - \left(\frac{x_p \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \cdots - \left(\frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}}\right) v_{p-1}$$

Note: Normalizing each of the vectors $v_1, v_2, \ldots v_p$, we get an orthogonal basis $\{u_1, u_2, \cdots, u_p\}$ where $u_i = \frac{v_i}{\|v_i\|}$, $1 \le i \le p$.

Imp 1. Given a basis $\{x_1, x_2, x_3\}$ where $x_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ $x_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $x_3 = \begin{bmatrix}2\\1\\0\end{bmatrix}$ of a subspace W, construct an orthonormal basis.

Sol^n Using Gram-Schimdt's orthogonal process,

$$v_1 = x_1 = \begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$v_2 = x_2 - \left(\frac{x_2 \cdot v_1}{v_1 \cdot v_1}\right) v_1$$

$$v_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \left(\frac{1}{2}\right)\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}$$

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= \begin{bmatrix}2\\1\\0\end{bmatrix} - \left(\frac{2}{2}\right)\begin{bmatrix}1\\0\\1\end{bmatrix} - \left(\frac{1}{1/2}\right)\begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}$$

$1+\frac{1}{2}$

$\frac{3/2}{2}$

$$v_3 = \begin{bmatrix}2\\1\\0\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} - 2\begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}$$

$$v_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Thus $\{v_1, v_2, v_3\}$ is an orthogonal basis

i.e. $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1/2\\0\\-1/2\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$ is an orthogonal basis

To obtain orthonormal basis we have to normalize each of the v_1, v_2, v_3

Now $u_1 = \frac{v_1}{\|v_1\|} = \frac{(1, 0, 1)}{\sqrt{1^2+0^2+1^2}} = \left(\frac{1}{\sqrt2}, \frac{0}{\sqrt2}, \frac{1}{\sqrt2}\right)$

$u_2 = \frac{v_2}{\|v_2\|} = \frac{(1/2, 0, -1/2)}{\sqrt{(1/2)^2+0^2+(-1/2)^2}} = \frac{(1/2, 0, -1/2)}{\sqrt{2/4}}$

$= \frac{(1/2, 0, -1/2)}{\sqrt2/2} = \frac{(1/2, 0, -1/2)}{1/\sqrt2}$

$\because \frac{\sqrt2/2}{\sqrt2 \times \sqrt2} = 1/\sqrt2$

$= \left(\frac{\sqrt2}{2}, 0, \frac{-\sqrt2}{2}\right) = \left(\frac{1}{\sqrt2}, 0, \frac{-1}{\sqrt2}\right)$

$\frac{\sqrt2}{2} = \frac{\sqrt2}{\sqrt2 \times \sqrt2}$

$= \frac{1}{\sqrt2}$

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$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(0,1,0)}{\sqrt{0^2+1^2+0^2}} = \frac{(0,1,0)}{1}$$
$$= (0,1,0)$$

$\therefore$ Orthonormal basis for W is $\{u_1, u_2, u_3\}$

is $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

v.v.imp

2. Construct an orthonormal basis for a subspace W of R^4 where $\{x_1, x_2, x_3\}$ with $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ is a basis for W.

Sol^n: $v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$v_2 = x_2 - \left(\frac{x_2 v_1}{v_1 v_1}\right) \cdot v_1$$
$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{3}{4}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

On scaling v_2, we get. $(v_2 \times 4)$

$v_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ (Scaling is applied only to simplify the calculation (multiply vector by suitable no. so all components becoming whole no.)

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2'}{v_2' \cdot v_2'}\right) v_2'$$
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{4}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{12} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

$\therefore \{v_1, v_2', v_3\}$ are orthogonal basis for W

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1,1)}{\sqrt{1^2+1^2+1^2+1^2}}$$

$$= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{(-3,1,1,1)}{\sqrt{(-3)^2+(1^2)+1^2+1^2}}$$

$$= \left(\frac{-3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)$$

$$= \cancel{(2\sqrt{3}}$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(0, -2/3, 1/3, 1/3)}{\sqrt{0^2+(-2/3)^2+(1/3)^2+(1/3)^2}}$$

$$= 0, \frac{\frac{-2}{3}}{\frac{\sqrt{6}}{3}}, \frac{\frac{1}{3}}{\sqrt{6}/3}, \frac{\frac{1}{3}}{\sqrt{6}/3}, \cancel{\frac{\frac{1}{3}}{\sqrt{6}/3}}$$

$$= \left(0, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \cancel{\frac{1}{\sqrt{6}}}\right)$$

Then $\{u_1, u_2, u_3\}$ forms an orthonormal basis.

Unit - III

CO - V

Quadratic forms: A Quadratic form on R^n is a function $Q(x)$ defined as $Q(x) = x^T A x$, where A is an $n \times n$ symmetric matrix and x is a vector in R^n.

In other words quadratic form is a homogeneous expression of 2nd degree in any number of variables.

Examples:
1. $Q(x) = x^2$
2. $Q(x,y) = x^2 + y^2$
3. $Q(x,y) = x^2 + y^2 + 2xy$
4. $Q(x,y,z) = 2x^2 + 3y^2 + 5z^2 + \underline{2xy} + \underline{6yz}$

cross product

In example 3 the term $2xy$ is called cross product term.

1. Let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, compute $x^T A x$ for the following matrices.

i) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ ii) $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

Sol^n i) $A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$

$x^T A x$:

$\Rightarrow Ax = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} 4x_1 & 0 \\ 0 & 3x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}$

$= \begin{bmatrix} 4x_1^2 & 0 \\ 0 & 3x_2^2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} 4x_1^2 \\ 3x_2^2 \end{bmatrix}$

$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 4x_1 \\ 3x_2 \end{bmatrix}$

$= 4x_1^2 + 3x_2^2$

$\therefore$ A is called matrix of quadratic form

ii) $A = \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}$

$\Rightarrow x^T A x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3x_1 - 2x_2 \\ -2x_1 + 7x_2 \end{bmatrix}$

$= x_1(3x_1 - 2x_2) + x_2(-2x_1 + 7x_2)$

$= 3x_1^2 - 2x_1x_2 - 2x_1x_2 + 7x_2^2$

$= 3x_1^2 + 7x_2^2 - 4x_1x_2$

The cross product term $-4x_1x_2$ appears because of the off diagonal element of A

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2. For x in $\mathbb{R}^3$, let $Q(x) = 5x_1^2 + 3x_2^2 + 2x_3^2 - x_1x_2 + 8x_2x_3$, write this quadratic form as x^TAx

Diagonal elements

Sol^n: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Matrix of $Q(x)$ is

$A = \begin{bmatrix} 5 & -1/2 & 0 \\ -1/2 & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix}$

$x_1x_2 = -1$
$\div 2$
cos need to write in 2 places
12×21

x_2x_3
$= 8/2$
$= 4$
$a_{23} = 4$
$a_{32} = 4$

$$Q(x) = x^T A x = [x_1 \ \ x_2 \ \ x_3] \begin{bmatrix} 5 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 3 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= [x_1 \ x_2 \ x_3] \left[5x_1 - \right.$$

<u>Classifications</u> (<u>Nature of Quadratic Form</u>):

A Quadratic form $Q(x)$ is

i) Positive definite if $Q(x) > 0$ for all $x \neq 0$

ii) Negative definite if $Q(x) < 0$ for all $x \neq 0$

iii) Indefinite if $Q(x) \neq \emptyset$ assumes both +ve and -ve values

iv) Positive semidefinite if $Q(x) \geq 0$ for all x

v) Negative semidefinite if $Q(x) \leq 0$ for all x

<u>Classification in terms of Eigen Values</u>:

A Quadratic form $Q(x)$ is

i) Positive definite if all its eigen values are +ve i.e. > 0

ii) Negative definite if all its eigen values are -ve i.e. < 0

iii) Indefinite if some eigen values are positive and some are negative. (no zero)

iv) Positive semidefinite if its one eigen values ~~are~~ is zero and others may be +ve (if all eigen values are ≥ 0 & one is 0)

v) Negative semidefinite if its eigen values are ≤ 0 with atleast one is 0)

1. Find the nature of following quadratic forms

i) $Q(x) = x^2 + 5y^2 + z^2 + 2xy + 2yz + 6xz$

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

$$\lambda^3 - (tr(A))\lambda^2 + \left(\text{sum of cofactors of diagonal ele}\right)\lambda - det(A) = 0$$

$$\lambda^3 - (7)\lambda^2 + (4-8+4)\lambda - (-36) = 0.$$

$$\lambda^3 - 7\lambda^2 + 36 = 0$$

$$\cancel{\lambda_1 = -2} \quad \lambda = -2, 6, 3$$

$\therefore$ Eigen values are $-2, 6, 3$

$Q(x)$ is indefinite since A has -ve & +ve eigen values.

ii) $Q(x) = 3x^2 + 5y^2 + 3z^2 - \underset{23}{2yz} + \underset{31}{2zx} - \underset{12}{2xy}$

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$$\lambda^3 - (tr(A))\lambda^2 + \left(\text{sum of cofactors of Diagonal ele}\right)\lambda - det(A) = 0.$$

$$\lambda^3 - 11\lambda^2 + (14 + 8 + 14)\lambda - 36 = 0.$$

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

$$\lambda = 6, 3, 2$$

$\therefore$ Eigen values are $6, 3, 2$

$Q(x)$ is positive definite, since A has +ve eigen values.

iii) Is $Q(x) = 3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 + 4x_2x_3$ positive definite.

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\lambda^3 - (tr(A))\lambda^2 + \left(\text{sum of cofactors of Diagonal ele}\right)\lambda - det(A) = 0$$

$$\lambda^3 - 6\lambda^2 + (-2 + 3 + 2)\lambda - (-10) = 0$$

$$\lambda^3 - 6\lambda^2 + 3\lambda + 10 = 0$$

$$\lambda = -1, 5, 2$$

$\therefore$ Eigen values are $-1, 5, 2$

$Q(x)$ is indefinite since A has +ve & -ve eigen values.

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V.Imp Problems

Reduction of Quadratic form to Cannonical form by change of variables:

5M

If the quadratic form $Q(x)$ where x is a variable vector in R^n to write it in the cannonical form (only the sum of square terms) we make change of variable in the equation

i.e. $x = Py$ (or) $y = P^{-1}x$ —— (1)

Here P is an invertible matrix whose columns are eigen vectors of the matrix of the quadratic form.

So the change of variable (1) made in quadratic form $Q(x)$ we get

$Q(x) = x^T A x = (Py)^T A Py = y^T P^T A Py$

$= y^T (P^T A P) y$

and the new matrix of the quadratic form is $P^T A P$

If P orthogonally diagonalizes A then $P^T = P^{-1}$

∴ Matrix of the new quadratic form $= P^T A P = P^{-1} A P = D$

Thus the Matrix of the new quadratic form is Diagonal of A

Here eigen values are the diagonal elements of D

Problems

1. Make the change of variable in the quadratic form $Q(x) = x_1^2 - 8x_1x_2 - 5x_2^2$ so that the new quadratic form is cannonical form. (no cross product terms only square terms)

Soln: The matrix of the given quadratic form is

$$A = \begin{bmatrix} +1 & -4 \\ -4 & -5 \end{bmatrix}$$

$\lambda^2 - tr(A)\lambda + det(A) = 0$

$\lambda^2 - (-4)\lambda + (-21) = 0$

$\lambda^2 + 4\lambda - 21 = 0$

$\lambda = 3, -7$

∴ eigen values are 3 & -7.

Let $x = Py$ or $y = P^{-1}x$

P contains the eigen vectors as its columns

$\therefore$ new quadratic form is written using the matrix $D = \begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$

the new quadratic form is

$$Q(y) = 3y_1^2 - 7y_2^2$$

2. Reduce the quadratic form $3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$ to the cannonical form by changing variables.

Sol^n: Here $Q(x) = 3x^2 + 5y^2 + 3z^2 - 2yz + 2zx - 2xy$

where $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ Let $x = Py$ where P contains columns of the eigen vector of the matrix of quadratic form $Q(x)$ and y is new vector in R^3

$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

Let $y = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$

$\lambda^3 - (\text{tr } A)\lambda^2 + \left(\begin{matrix}\text{sum of diagonal} \\ \text{elements cofactors of } A\end{matrix}\right)\lambda - \det(A) = 0$

$\lambda^3 - 11\lambda^2 + (14 + 8 + 14)\lambda - 36 = 0$

$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$

$\lambda = 6, 3, 2$

eigen values are $6, 3, 2$. D should be in ↑ order

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$\therefore$ New quadratic form $Q(y) = 2p^2 + 3q^2 + 6r^2$.

3. Find the maximum and minimum values of the quadratic form $Q(x) = 9x_1^2 + 4x_2^2 + 3x_3^2$ subject to the contraint $x^T x = 1$

Sol^n: $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Given $x^T x = 1$

Then $x^T x = [x_1 \ x_2 \ x_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Given

i.e. $x_1^2 + x_2^2 + x_3^2 = 1$

look for max coefficient.

observe that $4x_2^2 \le 9x_2^2$, $3x_3^2 \le 9x_3^2$

$$Q(x) = 9x_1^2 + 4x_2^2 + 3x_3^2$$
$$\le 9x_1^2 + 9x_2^2 + 9x_3^2$$
$$= 9(x_1^2 + x_2^2 + x_3^2)$$
$$= 9(1)$$
$$= 9$$

$\therefore Q(x) \le 9$

$\therefore$ 9 is the maximum value of $Q(x)$

observe that $4x_2^2 \ge 3x_2^2$, $9x_1^2 \ge 3x_1^2$

$$Q(x) = 9x_1^2 + 4x_2^2 + 3x_3^2$$
$$\ge 3x_1^2 + 3x_2^2 + 3x_3^2$$
$$= 3(x_1^2 + x_2^2 + x_3^2)$$
$$= \cancel{3(x)+}\ 3 \times 1$$
$$= 3$$

$Q(x) \ge 3$

$\therefore$ 3 is the minimum value of $Q(x)$

Date 18/07/2023

QR factorization of matrix :

If A is $m \times n$ matrix with linearly independent columns, then by applying gramschith's process with normalization A can be factorised has

$A = QR$ where Q is an $m \times n$ matrix whose columns form an orthonormal basis for column space of A and R is an $n \times n$ matrix with positive entries on its diagonal.

V.V.Imp

Ex 1. Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{4\times3}$

observe that all the columns of A are linearly independent

$\therefore$ These columns form a basis for a subspace W of $\mathbb{R}^4$.

By applying Gram-schimdt's process to get an orthonormal basis

Let $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = x_2 - \left(\frac{x_2 v_1}{v_1 v_1}\right) v_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{3}{4}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \quad \text{Multiply by 4 (on scaling)} = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = v_2'$$

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2'}{v_2' \cdot v_2'}\right) v_2'$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{4}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{2}{12}\right) \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

On scaling v_3 ie multiply by 3

$$v_3' = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Normalizing;

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1,1)}{\sqrt{1^2+1^2+1^2+1^2}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$= \sqrt{12}$

$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{(-3,1,1,1)}{\sqrt{(-3)^2+1^2+1^2+1^2}} = \left(\frac{-3}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}\right)$$

$$u_3 = \frac{v_3'}{\|v_3'\|} = \frac{(0,-2,1,1)}{\sqrt{(0)^2+(-2)^2+1^2+1^2}} = \left(\frac{0}{\sqrt{6}} = 0, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\therefore Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ 1/2 & 1/(2\sqrt{3}) & -2/\sqrt{6} \\ 1/2 & 1/(2\sqrt{3}) & 1/\sqrt{6} \\ 1/2 & 1/(2\sqrt{3}) & 1/\sqrt{6} \end{bmatrix} = [u_1 \quad u_2 \quad u_3]$$

Let $A = QR$ for some R

Pre multiply by Q^T

$$Q^T A = Q^T Q R = (Q^{-1} Q) R = IR = R$$

$\because Q$ = It is orthogonal matrix $\therefore Q^T = Q^{-1}$

$$R = Q^T A = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -\sqrt{3}/2 & 1/2\sqrt{3} & 1/2\sqrt{3} & 1/2\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & \sqrt{3}/2 & 1/\sqrt{3} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$ Thus $A = QR$ where Q & R are as above

Verification: $$QR = \begin{bmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ 1/2 & 1/2\sqrt{3} & -2/\sqrt{6} \\ 1/2 & 1/2\sqrt{3} & 1/\sqrt{6} \\ 1/2 & 1/2\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & \sqrt{3}/2 & 1/\sqrt{3} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

2. Obtain QR factorization of $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}$

Mostly they give l.i col only

l.i

Observe that all the columns are l.i $\therefore$ columns form a basis for subspace W of R^4.

By applying Gram schmidths process.

$x_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ $x_3 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \end{bmatrix}$

$x_1 = v_1$

$$v_2 = x_2 - \left(\frac{x_2 v_1}{v_1 v_1}\right) v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \left(\frac{4}{7}\right) \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2 = \begin{pmatrix} 3/7 \\ -1/7 \\ 3/7 \\ -4/7 \end{pmatrix}$$ on scaling $v_2' = \begin{bmatrix} 3 \\ -1 \\ 3 \\ -4 \end{bmatrix}$

$$v_3 = x_3 - \left(\frac{x_3 v_1}{v_1 v_1}\right) v_1 - \left(\frac{x_3 v_2'}{v_2' v_2'}\right) v_2'$$

$$= \begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \end{bmatrix} - \left(\frac{9}{7}\right) \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{6}{35}\right) \begin{bmatrix} 3 \\ -1 \\ 3 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} -9/5 \\ -2/5 \\ 11/5 \\ 2/5 \end{bmatrix} \cancel{\begin{bmatrix} -15/7 \\ -2/7 \\ 13/7 \\ -8/35 \end{bmatrix}}$$ on scaling $v_3' = \cancel{\begin{bmatrix} -15 \\ -2 \\ 13 \\ 6 \end{bmatrix}} \begin{bmatrix} -9 \\ -2 \\ 11 \\ 2 \end{bmatrix}$

$$u_1 = \frac{\cancel{\text{that}}\ v_1}{\|v_1\|} = \frac{(1,2,1,1)}{\sqrt{1^2+2^2+1^2+1^2}} = \left(\frac{1}{\sqrt7}, \frac{1}{\sqrt7}, \frac{1}{\sqrt7}, \frac{1}{\sqrt7}\right)$$

$$u_2 = \frac{v_2'}{\|v_2'\|} = \frac{3,-1,3,4}{\sqrt{3^2+(-1)^2+3^2+4^2}} = \left(\frac{3}{\sqrt{35}}, \frac{-1}{\sqrt{35}}, \frac{3}{\sqrt{35}}, \frac{4}{\sqrt{35}}\right)$$

$$u_3 = \frac{v_3'}{\|v_3'\|} = \frac{-9,-2,11,2}{\sqrt{-9^2+-2^2+11^2+2^2}} = \left(\frac{-9}{\sqrt{210}}, \frac{-2}{\sqrt{210}}, \frac{11}{\sqrt{210}}, \frac{2}{\sqrt{210}}\right)$$

$$Q = \begin{bmatrix} 1/\sqrt7 & 3/\sqrt{35} & -9/\sqrt{210} \\ 1/\sqrt7 & -1/\sqrt{35} & -2/\sqrt{210} \\ 1/\sqrt7 & 3/\sqrt{35} & 11/\sqrt{210} \\ 1/\sqrt7 & 4/\sqrt{35} & 2/\sqrt{210} \end{bmatrix}$$

Let $A = Q^T R$ for some R

Premultiply by Q^T

$$Q^T A = Q Q^T R \qquad Q^{-1} Q R = IR = R$$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt7 & 1/\sqrt7 & 1/\sqrt7 & 1/\sqrt7 \\ 3/\sqrt{35} & -1/\sqrt{35} & 3/\sqrt{35} & 4/\sqrt{35} \\ -9/\sqrt{210} & -2/\sqrt{210} & 11/\sqrt{210} & 2/\sqrt{210} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}$$

Date: 19.07.2023

<u>Least Square Solution</u>:

Inconsistent systems $AX = B$ arise often in applications.

When a solⁿ is demanded but none exists, the best one can do is to find a solution x such that the difference between AX & B is very very small.

The general least square problem is to find a solution x that makes $\|B - AX\|$ as small as possible.

The name 'least square' arises from the fact that this $\|B-AX\|$ is square root of sum of squares of the component.

If A is an $m \times n$ matrix and B is in R^m, a least square solution of $AX = B$, is a solution denoted by $\hat{x}$ in R^n such that this $\|B - A\hat{x}\| \le \|B - AX\|$

The matrix equation $A^TAX = A^TB$ is called normal equations for $AX = B$

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Solution of normal equations is the least square solution $\hat{x}$.

1. Find a least square solution of $AX = B$ where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}_{3\times 2}$ and $B = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}_{3\times 1}$

Solⁿ: The normal equations of $AX = B$ is
$A^TAX = A^TB$.

$$\begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$17x_1 + x_2 = 19$

$x_1 + 5x_2 = 11$

$\cancel{x_1 = -5x_2 + 11}$

$x_1 = 1, \; x_2 = 2$

$\therefore$ Least square solⁿ $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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2. Find a least square solⁿ of $AX = B$ where $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{6\times 4}$ $B = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$

The normal equations of $AX = B$ is
$A^TAX = A^TB$

wrong

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

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$$\begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

$$\begin{aligned} 6x_1 + 2x_2 + 2x_3 + 2x_4 &= 4 \\ 2x_1 + 2x_2 &= -4 \\ 2x_1 + 2x_3 &= 2 \\ 2x_1 + 2x_4 &= 6 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{array}\right] \quad R_3 \leftrightarrow R_1$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 6 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 6 & 2 & 2 & 2 & 4 \end{array}\right] \quad \begin{aligned} &R_2 \to R_2 - R_1 \\ &R_3 \to R_3 - R_1 \\ &R_4 \to R_4 - 3R_1 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 6 \\ 0 & 2 & 0 & -2 & -10 \\ 0 & 0 & 2 & -2 & -4 \\ 0 & 2 & 2 & -4 & -14 \end{array}\right] \quad R_4 \to R_4 - R_2$$

$$\left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 6 \\ 0 & 2 & 0 & -2 & -10 \\ 0 & 0 & 2 & -2 & -4 \\ 0 & 0 & 2 & -2 & -4 \end{array}\right] \quad R_4 \to R_4 - R_3$$

$$\sim \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 6 \\ 0 & 2 & 0 & -2 & -10 \\ 0 & 0 & 2 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2, & x_3 are basic variables

x_4 is free variable.

$\therefore$ General sol^n is

$$\begin{aligned} 2x_3 + 2x_4 &= -4 &\Rightarrow\ & x_3 = -2 + x_4 \\ +2x_2 - 2x_4 &= -10 &\Rightarrow\ & x_2 = -5 + x_4 \\ +2x_1 + 2x_4 &= 6 &\Rightarrow\ & x_1 = 3 - x_4 \end{aligned}$$

Thus the least square solution is

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 - x_4 \\ -5 + x_4 \\ -2 + x_4 \\ x_4 \end{bmatrix}$$

where x_4 is free variable

If $x_4 = 0$, the l.s. sol^n is $\hat{x} = \begin{bmatrix} 3 \\ -5 \\ -2 \\ 1 \end{bmatrix}$

3. Find a least square solution of $AX = B$ when $A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}$ & $B = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$.

Sol^n. $A^T A X = A^T B$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -6 & -2 & 1 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 \\ 0 & 90 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 33 \end{bmatrix}$$

$4x_1 = 8 \qquad x_1 = 2$

$90x_2 = 33 \qquad x_2 = 11/30$

The least square sol^n. $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 11/30 \end{bmatrix}$

Till here unit III and part B

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4. Let $A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$

Find least square solⁿ of $AX = B$

$A^TAX = A^TB$

$$\begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 9 & 0 \\ 3 & 41 & 16 \\ 6 & 58 & 20 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -35 \\ -46 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 3 & 41 & 16 & -35 \\ 6 & 58 & 20 & -46 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 0 & 32 & 16 & -32 \\ 0 & 40 & 20 & -40 \end{array}\right] \quad R_3 \to R_3 - \frac{40 R_2}{32}$$

$$\left[\begin{array}{ccc|c} 3 & 9 & 0 & -3 \\ 0 & 32 & 16 & -32 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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x_1 & x_2 are basic variable

x_3 is free variable

General solution is

$32x_2 + 16x_3 = -32$

$3x_1 + 9x_2 = -3$

$32x_2 = -32 - 16x_3$

$x_2 = -1 - \frac{1}{2}x_3$

$3x_1 + 9\left(-1 - \frac{1}{2}x_3\right) = -3$

$3x_1 - 9 - \frac{9}{2}x_3 = -3$

$3x_1 = 6 + \frac{9}{2}x_3$

$x_1 = 2 + \frac{3}{2}x_3$

Thus the least square solution is

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 + \frac{3}{2}x_3 \\ -1 - \frac{1}{2}x_3 \\ x_3 \end{bmatrix}$$

$x_3 = 0$ the L.S. solⁿ is $\hat{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

Date 20 07 2023

Alternative method to find the least square sol^n of $AX=B$

If A is $m \times n$ matrix with l.i column we can factorize A as $A = QR$

$\therefore$ each B in R^m the system $AX = B$ has unique least square sol^n.

$$\hat{X} = R^{-1}Q^TB$$

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Alternative method:

1. $A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$

Apply Gram Schmidt's process.

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = x_2 - \frac{(x_2 \cdot v_1)}{(v_1 \cdot v_1)} v_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{8}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$v_3 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1}\right) v_1 - \left(\frac{x_3 \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \left(\frac{10}{4}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{6}{4}\right)\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Normalizing $v_1\ v_2\ v_3$ &

$$u_1 = \frac{v_1}{\|v_1\|} = \frac{(1,1,1,1)}{\sqrt{1^2+1^2+1^2+1^2}} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{(1,-1,-1,1)}{\sqrt{1^2+(-1)^2+(-1)^2+(1^2)}} = \left(\frac{1}{2}, \frac{-1}{2}, \frac{-1}{2}, \frac{1}{2}\right)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{(1,-1,1,-1)}{\sqrt{1^2+-1^2+1^2+-1^2}} = \left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}\right)$$

$$Q = [u_1\ u_2\ u_3]$$

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}$$

Since $A = QR$ we have $R = Q^{-1}A$

ie $R = Q^T A$ ($\because Q$ is orthogonal matrix $Q^{-1} = Q^T$)

$$QR = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Thus $A = QR$ where Q and R are as above.

Least square solⁿ of $AX = B$ is ie $QRX = B$ is obtained by

$$R\hat{X} = Q^{-1}B$$

ie $R\hat{X} = Q^T B$ —— (*)

$$Q^T B = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}$$

(*) now becomes

$$\begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \\ 4 \end{bmatrix}$$

$2\hat{x}_3 = 4$

$\hat{x}_3 = 2$

$2\hat{x}_2 + 3\hat{x}_3 = -6$

$2\hat{x}_2 + 6 = -6$

$2\hat{x}_2 = -12$

$\hat{x}_2 = -6$

$2\hat{x}_1 + 4\hat{x}_2 + 5\hat{x}_3 = 6$

$2\hat{x}_1 - 24 + 10 = 6$

$\hat{x}_1 = 10$

∴ The least square solution

$$\hat{x} = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$$

Date 25/07/2023

Singular Value Decomposition: (SVD)

Unfortunately Not all matrices can be diagonalizable or factorized as $A = PDP^{-1}$

However a factorization $A = QDP^{-1}$ is possible for any $m \times n$ matrix A. This special factorization is called singular value decomposition of A and this factorization is most useful in applied linear algebra. SVD is widely used in signal processing, noise deduction and image compressing are sum of the applications of SVD. In Data Science it helps to reduce the dimension of the data.

Singular Values of a Matrix :

Let A be an $m \times n$ matrix then A^TA is symmetric and its eigen values are non negative, also it can be orthogonally diagonalized, the singular values of A are the square roots of eigen values of A^TA.

These singular values are denoted by $\sigma_1, \sigma_2, \ldots, \sigma_n$ and they are arranged in decreasing order.

$$\sigma_i = \sqrt{\lambda_i}, \quad 1 \le i \le n$$

1. Obtain the singular values of a matrix $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

Solⁿ: $A^TA = \begin{bmatrix} 4 & 8 \\ 11 & 7 \\ 14 & -2 \end{bmatrix} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

Characteristic equation of A^TA is $\lambda^3 - tr(A)\lambda^2 + \begin{pmatrix}\text{sum of cofactors} \\ \text{of diagonal ele of } A^TA\end{pmatrix}\lambda - det(A^TA) = 0$

$$\lambda^3 - 450\lambda^2 + (14400 + 14400 + 3600)\lambda - 0 = 0$$

$$= \lambda^3 - 450\lambda^2 + 32400\lambda = 0$$

$$\lambda = \underset{\lambda_1}{360}, \underset{\lambda_2}{90}, \underset{\lambda_3}{0}$$

$\therefore$ Singular values of A are

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{360} = 6\sqrt{10}$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{90} = 3\sqrt{10}$$
$$\sigma_3 = \sqrt{\lambda_3} = \sqrt{0} = 0.$$

SVD Theorem:

Let A be an $m \times n$ matrix with a rank r then there exists a $m \times n$ matrix $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ with +ve diagonal entries in D and which are singular values of matrix A.

i.e. $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$.

there exists an $m \times m$ orthonormal matrix U, and an $n \times n$ orthonormal matrix V, such that $A = U \Sigma V^T$
Here the matrices U and V are not uniquely determined by A.

1. Find the SVD of $A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}$

Sol^n: $A^TA = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

characteristic eq^n of A^TA =

$$\lambda^2 - tr(A^TA)\lambda + det(A^TA) = 0$$

always write in ↓ order

$$\lambda^2 - 50\lambda + 225 = 0$$

$\lambda_1 = 45$, $\lambda_2 = 5$ are eigen values

$\therefore$ Singular values of A are $\sigma_1 = \sqrt{\lambda_1} = \sqrt{45}$
$\sigma_2 = \sqrt{\lambda_2} = \sqrt{5}$

$$\therefore \Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

$\therefore \Sigma = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$ as size of A is 2×2

To find matrix V, we find the eigen vectors of A^TA corresponding to eigen values $\lambda_1 = 45$ i.e.

(i.e. soln of $(A^TA)x = \lambda_1 x$))

$(A - \lambda I)X = 0$

i.e. sol^n of $(A^TA - \lambda_1 I)x_1 = 0$

i.e. $\begin{bmatrix} -20 & 20 \\ 20 & -20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Echelon form

$$\begin{bmatrix} -20 & 20 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-20x_1 + 20x_2 = 0$$

$$x_1 = x_2$$

$\therefore$ eigen vector $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\frac{X_1}{\|X_1\|} = \left(\frac{1, 1}{\sqrt{2}}\right)$ $\therefore$ unit eigen vector is $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$(A^TA - \lambda_2 I)X_2 = 0.$$

$$\left(\begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{bmatrix} 20 & 20 \\ 20 & 20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$20x_1 + 20x_2 = 0$$
$$x_1 = -x_2$$

$\therefore$ eigen vectors $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ +x_2 \end{bmatrix} =$

Say $x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with $x_2 = 1$

$\therefore$ unit eigen vector $V_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$$\therefore V = [v_1 \ \ v_2]$$
$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

To find U:

$$u_1 = \frac{Av_1}{\|Av_1\|} \qquad Av_1 = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ 9/\sqrt{2} \end{bmatrix}$$

$$= \frac{(3/\sqrt{2},\ 9/\sqrt{2})}{\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{9}{\sqrt{2}}\right)^2}} = \frac{3/\sqrt{2},\ 9/\sqrt{2}}{3\sqrt{5}}$$

$$= \left(\frac{\frac{3}{\sqrt{2}}}{3\sqrt{5}},\ \frac{9/\sqrt{2}}{3/\sqrt{5}} \right)$$

$$= \left(\frac{1}{\sqrt{10}},\ \frac{3}{\sqrt{10}} \right)$$

$$u_2 = \frac{Av_2}{\|Av_2\|} \Rightarrow Av_2 = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -3/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \frac{-3/\sqrt{2},\ 1/\sqrt{2}}{\sqrt{\left(\frac{-3}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}} = \left(\frac{\frac{-3}{\sqrt{2}}}{\sqrt{5}},\ \frac{\frac{1}{\sqrt{2}}}{\sqrt{5}} \right)$$

$$= \left(\frac{-3}{\sqrt{10}},\ \frac{1}{\sqrt{10}} \right)$$

$$\therefore U = [u_1 \ \ u_2]$$
$$= \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

$$\therefore \text{SVD of } A = U \Sigma V^T$$
$$= \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix} \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

Note: To find U, alternate method is find the unit eigen vectors of AA^T

Principal Component Analysis (PCA):

PCA is a technique to analyse multi variate data, it can be applied to any data that consists list's of measurements (attributes) made on a collection of objects or individuals.

Ex: consider a two dimensional data that represents weight and height of "N" students.

Let x_j denote the weight and height of j^{th} student.

This data can be represented by a $2 \times N$ matrix where the columns are $x_1, x_2, \dots x_N$

$$\begin{bmatrix} w_1 & w_2 & \cdots & w_N \\ h_1 & h_2 & \cdots & h_N \end{bmatrix}$$

This data can be visualized as a 2D scatter plot.

In a similar way we can think of higher dimensional data which is difficult to visualise.

Mean & Covariance:

Let $x_1, x_2, \dots x_N$ represent p-Dimensional data represented by a $p \times N$ matrix. The sample mean M of these observations $x_1, x_2, \dots, x_N$ is given by

$M = \frac{1}{N}(x_1 + x_2 + \dots + x_N)$

for the above figure sample mean is the point in the centre of the scatter plot.

Let $\hat{X}_k = X_k - M$ for, $k = 1, 2, \dots, N$

The matrix $B = [\hat{x}_1 \; \hat{x}_2 \; \dots \; \hat{x}_N]$ is known as mean deviation form of the data. This has zero sample mean.

The sample covariance matrix is the $p \times p$ defined by $S = \frac{BB^T}{N-1}$

or $S = \frac{x_1^2 + x_2^2 + \dots x_N^2}{N-1}$

1. Compute the sample mean and the covariance matrix of 3 measurements/attributes made on each of 4 individuals in a random sample given by $X_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 4 \\ 2 \\ 13 \end{bmatrix}$,

$X_3 = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}$, $X_4 = \begin{bmatrix} 8 \\ 4 \\ 5 \end{bmatrix}$

Sample Mean $M = \frac{X_1 + X_2 + X_3 + X_4}{4}$

$$M = \frac{1}{4}\begin{bmatrix} 1+4+7+8 \\ 2+2+8+4 \\ 1+13+1+5 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix}$$

Subtract the sample mean from X_1, X_2, X_3 & X_4

$$\hat{X}_1 = X_1 - M = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -2 \\ -4 \end{bmatrix}$$

$$\hat{X}_2 = \begin{bmatrix} -1 \\ -2 \\ 8 \end{bmatrix} \quad \hat{X}_3 = \begin{bmatrix} 2 \\ 4 \\ -4 \end{bmatrix} \quad \hat{X}_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

Matrix of mean deviation $B = [\hat{X}_1\ \hat{X}_2\ \hat{X}_3\ \hat{X}_4]$

$$B = \begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix}$$

Covariance Matrix, $S = \frac{BB^T}{N-1}$

$$= \frac{\begin{bmatrix} -4 & -1 & 2 & 3 \\ -2 & -2 & 4 & 0 \\ -4 & 8 & -4 & 0 \end{bmatrix}\begin{bmatrix} -4 & -2 & -4 \\ -1 & -2 & 8 \\ 2 & 4 & -4 \\ 3 & 0 & 0 \end{bmatrix}}{4-1}$$

$$= \frac{1}{3}\begin{bmatrix} 30 & 18 & 0 \\ 18 & 24 & -24 \\ 0 & -24 & 96 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 6 & 0 \\ 6 & 8 & -8 \\ 0 & -8 & 32 \end{bmatrix}$$

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The diagonal entry s_{jj} in the covariance matrix S is called the variance of the attribute x_j.

Total variance $= s_{11} + s_{22} + \cdots + s_{pp} = \text{trace}(S)$

The entry s_{ij} for $i \neq j$ in S is called covariance of x_i and x_j.

In the above problem covariance b/w x_1 & x_3 is 0 since the entry $s_{13} = 0$.
i.e. x_1 & x_3 are uncorrelated.