~~5x4~~ (5x3)

Theorem

Unit - 2

If & only if

Union

Date 03/02/25

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PART A :- Vector Space

(problm / Theorem) 4 mcq

1) Vector space & subspace.

5 3

problm Theorem

2) Linearly independent & linearly dependent vectors

3) Basis & dimensions

4) Row space, column space & NULL space.

PART B :- Linear Transformation

1) Transformation :- Algerbra of Transformation, Represtation of transformation by Matrix.

2) Range space & NULL space for linear Transformation

V.V.Imp 3) Rank - Nullity theorem

4) Innerproduct space, Orthogonal set of projections

V.V.Imp 5) Gram - Schmidt's Orthogonalization Process.

PART A :- Vector Space

Introduction :-

A vector is one that has both magnitude & direction

~~Ag~~ Any vector, $\vec{a} = 2\hat{i} + 3\hat{j}$ can be written as a matrix as $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ in the region R^2 where IR^2 is 2 dimentional

~~eklq~~ Euclidian space

A vector space is collection of all these elements i.e R^2....

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Group:- Let $*$ be a binary operator defined over non empty set G then G is said to be a group under the operator $*$ if it satisfied following condition

i) Closure law :- $\forall a, b \in G$; $a * b \in G$.

ii) Associative law :- $\forall a, b, c \in G$: $a*(b*c) = (a*b)*c$

iii) Inverse law :- $\forall a \in G$: $a * a^{-1} = a^{-1} a = e$.

iv) Identity law :- $\exists a \in G$, such that $a * e = e * a = a$

There exists — where e is a identity element

Abelian Group:- A group is said to be abelian group with respect to the operator $*$ if it satisfied following condition above 4 condition +

v) Commutative law :- $\forall a, b \in G$; $a * b = b * a$.

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Field:- Let 'F' be any non empty set along with the 2 binary operator '+' & '·' is said to be a field if its satisfies

(i) $(F, +)$ Should be a Abelian group.

(ii) $(F, \cdot)$ Should be a Abelian group.

(iii) Multiplication is distributed over addition i.e

$a(b+c) = ab + ac$.

defination — Imp

Vector Space :- Let F be a field then the non empty set 'V' along with the 2 binary operator say vector addition '+' & scalar multiplication '·' is said to be a vector space over the field F if it satisfies the following Conditions

C, A, I, I, C.

(i) $(V, +)$ Should be a Abelian group (where write all abelian group)

(ii) For any vector $u \in V$ (vector space) & for any scalar $c \in F$ (Field), then $c \cdot u \in V$ must satisfy

(a) scalar multiplication is distributive over addition

i.e $c(u+v) = cu + cv$, where $u, v \in V$, $c \in F$

(b) vector multiplication is distributive over Scalar addition

i.e $u(c_1 + c_2) = uc_1 + uc_2$, where $u \in V$, $c_1, c_2 \in F$

<u>Subspace</u> :- Let 'V' be a vector space over field 'F' then a <u>non empty subset</u> 'W' of V is said to be a subspace if its satisfies the following condition

(i) $0 \in W$

(ii) W is closed under vector addition i.e $w_1 + w_2 \in W$ where $w_1, w_2 \in W$

(iii) W is closed under Scalar multiplication i.e $\alpha w \in W$ where $w \in W$, $\alpha \in F$ (any constant)

Prblm 1) Check whether H is subspace of a $\mathbb{R}^3$, where

$$H = \left\{ \begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix} ; c_1, c_2 \in \mathbb{R} \right\}$$

Constant → scalar

⇒ H is non empty

(i) Let $c_1 = 0$ & $c_2 = 0 \in \mathbb{R}$

Then, $H = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$

(ii) Let $H_1 = \begin{bmatrix} a_1 \\ 0 \\ b_1 \end{bmatrix}$ & $H_2 = \begin{bmatrix} a_2 \\ 0 \\ b_2 \end{bmatrix} \in H$ where $a_1, b_1 \in \mathbb{R}$, $a_2, b_2 \in \mathbb{R}$.

then vector addition $H_1 + H_2 = \begin{bmatrix} a_1 + a_2 \\ 0 \\ b_1 + b_2 \end{bmatrix} \in H$.

$a(b+c) = ab + ac$

$\therefore H_1 + H_2 \in H$.

(iii) Let $\alpha \in F$ Then $\alpha H = \begin{bmatrix} \alpha c_1 \\ 0 \\ \alpha c_2 \end{bmatrix} \in H$

Since all 3 condition are satisfied $\therefore$ H is a subspace.

2) $H = \left\{ \begin{bmatrix} c_1 \\ 4 \\ c_2 \end{bmatrix} ; c_1, c_2 \in \mathbb{R} \right\}$

$\Rightarrow$ H is non Empty

(i) Let $c_1 = 0$ & $c_2 = 0 \in \mathbb{R}$, Then $H = \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \notin H$

$0 \notin H$

$\therefore$ Not subspace.

Some prblm
is there.
didn't write

Linearly dependent & Independent Vector.

Let V be a vector space over a field F then the set $\{v_1, v_2, v_3 \dots v_n\} \in V$ are said to be linearly dependent if there exists a sclar $C_1, C_2, C_3 \dots C_n \in F$ such that not all C_i's are zero. but it's linear combination can be zero.

(i.e $C_1v_1 + C_2v_2 + \dots C_nv_n = 0$)

lly, if all the scalars are zero (i.e $C_1, C_2, C_3, \dots C_n = 0$) & the linear combination also zero (i.e $C_1v_1 + C_2v_2 \dots C_nv_n = 0$) then the given set of vectors are said to be linearly independent.

NOTE :- * If the no. of unknows is equal to the numb of pivotal column then it is said to be linearly independent.

1) check whether the given set of vector are linearly dependent or independent $\{(1,4,5),(4,4,8),(3,-3,0)\}$

=> Let LC be $C_1V_1 + C_2V_2 + C_3V_3 = 0$

$$c_1(1,4,5) + c_2(4,4,8) + c_3(3,-3,0) = (0,0,0)$$

$$1c_1 + 4c_2 + 3c_3 = 0$$

$$4c_1 + 4c_2 - 3c_3 = 0$$

$$5c_1 + 8c_2 + 0c_3 = 0$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 4 & 3 & 0 \\ 4 & 4 & -3 & 0 \\ 5 & 8 & 0 & 0 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 5R_1$

$$\left[\begin{array}{ccc:c} 1 & 4 & 3 & 0 \\ 0 & -12 & -15 & 0 \\ 0 & -12 & -15 & 0 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$\left[\begin{array}{ccc:c} 1 & 4 & 3 & 0 \\ 0 & -12 & -15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivotal Column = 2 no. of unknown = 3.

$\therefore$ The given set of vectors are linearly dependent

2) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ -2 \\ 3 \end{bmatrix} \right\}$ is LD | LI

$\Rightarrow$ Let LC be $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$.

$$c_1(1, 0, -1, 2) + c_2(4, 2, 0, -1) + c_3(6, 4, -2, 3) = (0, 0, 0, 0)$$

$$1c_1 + 4c_2 + 6c_3 = 0$$

$$0 + 2c_2 + 4c_3 = 0$$

$$-1c_1 + 0 + (-2)c_3 = 0$$

$$2c_1 - c_2 + 3c_3 = 0.$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 4 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ -1 & 0 & -2 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 2R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 4 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 4 & 4 & 0 \\ 0 & -9 & -9 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - 2R_2 \\ R_4 \to 2R_4 + 9R_2 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 4 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & -0 & 18 & 0 \end{array}\right] \quad \cancel{R_4 \to 4R_4} \quad R_4 \to 4R_4 + 18R_3$$

$$\left[\begin{array}{ccc:c} 1 & 4 & 6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivotal column $= 3$ no. of unknown $= 3$

$\therefore$ Linearly independent

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Theorem :-

A set of vectors $\{u_1, u_2, u_3 \dots u_n\}$ is linearly dependet if and only if one of them is linear combination of other since $\{u_1, u_2, u_3, \dots u_m\}$ is linearly dependent then by de fin ation of linearly dependent we can write $c_1u_1 + c_2u_2 + \dots + c_mu_m = 0$. Such that not all c_i's equal to zero.

$\therefore$ Let us consider $c_i \neq 0$.

ie $c_1u_1 + c_2u_2 \dots c_iu_i \dots + c_nu_n = 0$.

OR

$c_iu_i = -c_1u_1 - c_2u_2 \dots \dots - c_nu_n = 0$.

OR

$$u_i = -\frac{c_1}{c_i}u_1 - \frac{c_2}{c_i}u_2 \dots -\frac{c_n}{c_i}u_n$$

OR

$$u_i = -\alpha_1 u_1 + \alpha_2 u_2 \dots - \alpha_n u_n \quad ; \alpha_2 = \frac{-c_2}{c_i} \dots$$

∴ One can be written α C of others.

Conversely:-

Given $u_i = \alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_n u_n$ ⊛ we need to PT $\{u_1, u_2 \dots u_n\}$ is linearly dependent

From ⊛ $\alpha_1 u_1 + \alpha_2 u_2 + \dots + (-1) u_i \dots + \alpha_n u_n$

clearly $-1 = \alpha_i \neq 0$.

∴ $\{u_1, u_2 \dots u_n\}$ is linearly dependent

Basis & dimensions:-

Let V be a vector space over a field F & let S be a non empty subset (ie. $S = \{u_1, u_2 \dots u_n\}$), then S is said to be the basis of V if its satisfy

1) Vector S should be linearly independent

2) S spans V

The no of elements in the bases set S is called dimensions

any one for me question 2a → see

1) PT $\{(1,3,1), (0,2,0), (0,0,7)\} \in \mathbb{R}^3$ forms a Basis of $\mathbb{R}^3$ also find dimensions.

(i) let ~~S~~ u of T set is Linearly independent

Let $c_1u_1 + c_2u_2 + c_3u_3 = (0,0,0)$

$c_1(1,3,1) + c_2(0,2,4) + c_3(0,0,7) = (0,0,0)$

$1c_1 + 0c_2 + 0c_3 + \cancel{3c_1} + = 0.$

$3c_1 + 2c_2 + 0c_3 = 0.$

$1c_1 + 4c_2 + 7c_3 = 0.$

OR

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 1 & 4 & 7 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 - R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 4 & 7 & 0 \end{array}\right] \quad R_3 \to R_3 - 2R_1$$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right]$$

Since No of pivotal column = No of unknown given set of vector is LS

(ii) S spaces V

Let linear combination be $c_1u_1 + c_2u_2 + c_3u_3 = (x, y, z)$

$c_1(1,3,1) + c_2(0,2,4) + c_3(0,0,7) = (x, y, z)$

$1c_1 + 0c_2 + 0c_3 = x$ — ①

$3c_1 + 2c_2 + 0c_3 = y$ — ②

$1c_1 + 4c_2 + 7c_3 = z$ — ③

from ① $\Rightarrow$ $x = c_1$

Substitute in ② $3x + 2c_2 = y$

$c_2 = \frac{y-3x}{2}$

Sub in ③

$c_3 = \frac{z-x}{7}$

$\boxed{\text{Dimension} = 3}$

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2) ~~Ck~~ check $S = \{(2,3), (0,4)\}$ forms a basis of $\mathbb{R}^2$

3) $S = \{(3,1,0), (2,1,0), (1,1,3)\}$ P.T S is a basis of $\mathbb{R}^3$ & Find its dimension

=> Let PT ~~it~~ S is LI

let $c_1u_1 + c_2u_2 + c_3u_3 = 0$.

$c_1(3,1,0) + c_2(2,1,0) + c_3(1,1,3) = 0$

$3c_1 + 2c_2 + 1c_3 = 0$

$1c_1 + 1c_2 + 1c_3 = 0$

$0 + 0 + 3c_3 = 0.$

$$[A:B] = \left[\begin{array}{ccc:c} 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \quad R_2 \to 3R_2 - R_1$$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right] \quad R$$

Dimension is 3

(ii) we PT S span V

$\boxed{c_3 = z/3}$

let $c_1u_1 + c_2u_2 + c_3u_3 = 0.$

$3c_1 + 2c_2 + 1c_3 = x$ $\quad$ $3c_1 + 2c_2 = x - z/3$ $\quad$ mult by 3

$1c_1 + 1c_2 + 1c_3 = y$ $\quad$ $1c_1 + 1c_2 = y - z/3$

$0 + 0 + 3c_3 = z$

$\cancel{3}c_1 + 2c_2 = x - z/3$

$\cancel{3}c_1 + 3c_2 = (y - z/3)3$

$-\cancel{1}c_2 = -3(y - z/3) + (x - z/3)$

$c_2 = x + y/3 - (z/3)\,3$

$c_1 = x - 2y + z/3$

$\therefore$ S Space N

$\therefore$ S basis $\mathbb{R}^3$

Imp

4) Let w be a subspace of $\mathbb{R}^4$ ~~spand~~ spanned by vector $u_1 = (1, -2, 5, -3)$, $u_2 = (3, 8, -3, -5)$ $u_3 = (2, 3, 1, -4)$

i) Find basis & dimension of W

ii) Extend the basis of w to basis of $\mathbb{R}^4$.

=> Let LC be $c_1u_1 + c_2u_2 + c_3u_3 = 0$.

$c_1(1, -2, 5, -3) + c_2(3, 8, -3, -5) + c_3(2, 3, 1, -4) = 0$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 3 & 2 & 0 \\ -2 & 8 & 3 & 0 \\ 5 & -3 & 1 & 0 \\ -3 & -5 & -4 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 5R_1 \\ R_4 \to R_4 + 3R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & 3 & 2 & 0 \\ 0 & 14 & 7 & 0 \\ 0 & -18 & -9 & 0 \\ 0 & 4 & +2 & 0 \end{array}\right] \quad \begin{array}{l} R_3 \to 14R_3 + 18R_2 \\ R_4 \to 14R_4 - 4R_2 \end{array}$$

$$= \begin{bmatrix} \boxed{1} & 3 & 2 & : & 0 \\ 0 & \boxed{14} & 7 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

The num of pivotal column ≠ nu of unknown.

∴ The given set of vector is LD

∴ The dimension = 2

∴ Basis set is $\{(1,-2,5,-3), (3,8-3,-5)\}$

(ii) To extend basis of w to $\mathbb{R}^4$

$\Rightarrow \{(1,-2,5,-3), (3,8,-3,-5), (0,0,1,0), (0,0,0,1)\}$

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Coordinate :-

Suppose S is the basis set of V then we know that any element can be linear combination of S.

$u = \alpha_1 u_1 + \alpha_2 u_2 + \dots \alpha_n u_n$ then $\alpha_1, \alpha_2 \dots \alpha_m$ are called as coordinates

Ex:- Find the coordinates of $u = (3,1,-4)$ related to the basis set $u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$ find the coordinates.

LC is $u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$

$(3,1,-4) = \alpha_1(1,1,1) + \alpha_2(0,1,1) + \alpha_3(0,0,1)$

$3 = 1\alpha_1 + 0\alpha_2 + 1\alpha_3$

$1 = 1\alpha_1 + 1\alpha_2 + 0\alpha_3$

$-4 = 1\alpha_1 + 1\alpha_2 + 1\alpha_3$

$\therefore$ Coordinates $\alpha_1 =$ $\alpha_2 =$ $\alpha_3 =$

Column space of a Matrix :-

Column space of a $m \times n$ matrix is denoted by $col(A)$ and it's defined as set of all linear combination of columns of A

NOTE :- $col(A) = \{ B \mid AX = B ;$ Should be consistent $\}$

Row space of a Matrix :-

Let A be a $m \times n$ matrix the subset of $\mathbb{R}^n$ consisting of all vectors that are linear combination of row of A is called Row space of matrix A.
It is denoted as $ROW(A)$

Null space of Matrix :-

The null space of $m \times n$ matrix is denoted by $null(A)$ and defined as set of all soln of homogenious eqn. [i.e eqn should be equated to zero].

1) Check whether $B = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$ is a column space of

$$A = \begin{bmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{bmatrix}$$

Let LC be $\alpha_1 C_1 + \alpha_2 C_2 + \alpha_3 C_3 = B$

$$\alpha_1 \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -3 \\ 6 \\ 7 \end{bmatrix} + \alpha_3 \begin{bmatrix} -4 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix}$$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & -3 & -4 & 3 \\ -4 & 6 & -2 & 3 \\ -3 & 7 & 6 & -4 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 4R_1 \\ R_3 \to R_3 + 3R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & -2 & -6 & 5 \end{array}\right] \quad R_3 \to 3R_3 + R_2$$

$$\left[\begin{array}{ccc:c} 1 & -3 & -4 & 3 \\ 0 & -6 & -18 & 15 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = \rho(A:B) < n$

$\therefore$ consistent and infinite soln

2) Find the basis of a column space for the given matrix

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \begin{array}{l} R_2 \to 2R_2 + R_1 \\ R_3 \to 2R_3 + 3R_1 \\ R_4 \to 2R_4 + R_1 \end{array}$$

$$\left[\begin{array}{cccc:c} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 7 & -14 & 14 & -49 \\ 0 & 9 & -18 & 10 & -23 \end{array}\right]$$

$R_3 \to R_3 - 7R_2$
$R_4 \to R_4 - 9R_2$

$$\left[\begin{array}{ccccc} \boxed{-2} & -5 & 8 & 0 & -17 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{-08} & 40 \end{array}\right]$$

$\therefore$ Pivotal colum C_1, C_2, C_4

$\therefore$ Basis set of $Col(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

3) check $B = \begin{bmatrix} -2 \\ 3 \\ 2 \end{bmatrix}$ is a col space of A. $\begin{bmatrix} 1 & -1 & 5 \\ 2 & 0 & 7 \\ -3 & 5 & -3 \end{bmatrix}$

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4) check whether $(1, 3, 4) \in Row(A)$ where $A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 0 & 1 \\ 2 & 4 & -3 \end{bmatrix}$

$\to$ LT is $\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_3 = B$

$\alpha_1 [3 \quad 1 \quad -2] + \alpha_2 [4 \quad 0 \quad 1] + \alpha_3 [2 \quad 4 \quad -3] =$

$[1, \quad 3 \quad 4]$

$3\alpha_1 + 4\alpha_2 -$

$3\alpha_1 + 4\alpha_2 + 2\alpha_3 = 1$

$1\alpha_1 + 0\alpha_2 + 4\alpha_3 = 3$

$-2\alpha_1 + 1\alpha_2 - 3\alpha_3 = 4$

$$[A:B] = \left[\begin{array}{ccc:c} 3 & 4 & 2 & 1 \\ 1 & 0 & 4 & 3 \\ -2 & 1 & -3 & 4 \end{array}\right] \quad \begin{array}{l} R_2 \to 3R_2 - R_1 \\ R_3 \to 3R_3 + 2R_1 \end{array}$$

$$\left[\begin{array}{ccc:c} 3 & 4 & 2 & 1 \\ 0 & -4 & 10 & 8 \\ 0 & 11 & -5 & 14 \end{array}\right] \quad R_3 \to 4R_3 + 11R_2$$

$$\left[\begin{array}{ccc:c} 3 & 4 & 2 & 1 \\ 0 & -4 & 10 & 8 \\ 0 & 0 & 90 & 144 \end{array}\right]$$

$\rho(A) = 3 \quad \rho(A:B) = 3 \quad n = 3$

$\therefore$ consistent & finite soln.

$\therefore$ B is row space of A

****5) Find the basis of Row space $A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ -2 & 6 & -6 & -1 & -10 \\ -3 & 9 & -6 & -6 & -3 \\ 3 & -9 & 4 & 9 & 0 \end{bmatrix}$

$R_2 \to R_2 + 2R_1 \qquad R_4 \to R_4 - 3R_1$

$R_3 \to R_3 + 3R_1$

$$A = \begin{bmatrix} 1 & -3 & 4 & -1 & 9 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & +6 & -9 & +24 \\ 0 & 0 & -8 & 12 & -27 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 + 4R_2 \end{matrix}$$

$$\begin{bmatrix} \boxed{1} & -3 & 4 & -1 & 9 \\ 0 & 0 & \boxed{2} & -3 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{5} \end{bmatrix}$$

*** $\therefore$ Basis set of Row(A) $= \left\{ \begin{matrix} [1 & -3 & 4 & -1 & 9] \\ [0 & 0 & 2 & -3 & 8] \\ [0 & 0 & 0 & 0 & 5] \end{matrix} \right\}$

6) check $(5, 17, 20) \in$ Row(A) ; $A = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 1 & 1 \\ -2 & 4 & -3 \end{bmatrix}$

7) when $A = \begin{bmatrix} 1 & 3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ Determine

whether $U \in$ NULL (A)

$$AU = \begin{bmatrix} 1 & 3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 18 \\ 0 \\ \end{bmatrix}$$

//

Date: 10/03/25

PART-B :-

Linear Transformation:-

Linear Transformation:- If V & W are 2 vector space over the field F, then the linear transformation from V to W ($T: V \to W$) is defined as $T(\alpha u + v) = \alpha T(u) + T(v)$

NOTE :- * If $T(0) \neq 0$ for any tranformation from V to W then T is not a linear transformation

Ex:- $T: V_3 \to V_3$ defined by $T(x_1, x_2, x_3) \to (2x_1 - 3x_2, x_1 + 4, 5x_3)$ check whether its linear transformation

$\Rightarrow T(0,0,0) = (0,4,0) \Rightarrow T(0) \neq 0$

$\therefore$ not linear transformation.

(Sure 1 Q will come)

1) $T: V_3(R) \to V_2(R)$ defined by $T(x,y,z) = (x+y, y+z)$ PT T is a LT / linear map.

Ans:- we need to PT $T(\alpha u + v) = \alpha T(u) + T(v)$

Let $u = x_1, y_1, z_1$

$v = x_2, y_2, z_2$

Then $T(\alpha u + v) = T[\alpha(x_1, y_1, z_1) + (x_2, y_2, z_2)]$

$= T[\alpha x_1 + x_2, \alpha y_1 + y_2 + \alpha z_1 + z_2]$

Given $T(x,y,z) = (x+y, y+z)$

$= (\alpha x_1 + x_2 + \alpha y_1 + y_2, \alpha y_1 + y_2 + \alpha z_1 + z_2)$

$= (\alpha(x_1 + y_1) + (x_2 + y_2), \alpha(y_1 + z_1) + (y_2 + z_2))$

$= (\alpha(x_1 + y_1), (y_1 + z_1)) + (x_2 + y_2), (y_2 + z_2)$

$= \alpha T(x_1, y_1, z_1) + (x_2 + y_2), (y_2 + z_2)$

$= \alpha T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$= \alpha T(u) + T(v)$

2) $T: V_3(R) \to V_2(R)$ defined by $T(x_1, x_2, x_3) =$
$(x_1 - x_2, x_1 + x_3)$

Ans:- we need to PT $T(\alpha u + v) = \alpha T(u) + T(v)$

Let $T(x_1, x_2, x_3) = T(x_1 - x_2, x_1 + x_3) \Rightarrow$

$T(x, y, z) = (x - y, x + z)$

$u = x_1, y_1, z_1$

$v = x_2, y_2, z_2$

Then $T(\alpha u + v) = T[\alpha(x_1, y_1 + z_1) + (x_2 + y_2 + z_2)]$

$= T[\alpha x_1 + x_2, \alpha y_1 + y_2, \alpha z_1 + z_2]$

Given $T(x, y, z) = (x - y, x + z)$

$= (\alpha x_1 + x_2 - \alpha y_1 + y_2, \alpha x_1 + x_2 + \alpha z_1 + z_2)$

$= (\alpha(x_1 - y_1) + (x_2 - y_2), \alpha(x_1 + z_1) + x_2 + z_2)$

$= (\alpha(x_1 - y_1), (x_1 + z_1)) + (x_2 - y_2), (x_2 + z_2)$

$= \alpha T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$

$= \alpha T(u) + T(v)$

** 3) $T: R^2 \to R^2$; $T(x, y) = (x + y, x)$ PT T is LT

we need to PT $T(\alpha u + v) = \alpha T(u) + T(v)$

let $u = x_1, y_1$

$v = x_2, y_2$

then $T(\alpha u + v) = T[\alpha(x_1, y_1) + (x_2, y_2)]$

$= T[\alpha x_1 + x_2, \alpha y_1 + y_2]$

Given $T(x, y) = (x + y, x)$

$= (\alpha x_1 + x_2 + \alpha y_1 + y_2, \alpha x_1 + x_2)$

$= (\alpha(x_1 + y_1) + x_2 + y_2, \alpha x_1 + x_2)$

$= \alpha[(x_1 + y_1), (x_1)] + (x_2 + y_2, x_2)$

$= \alpha T(x_1, y_1) + T(x_2, y_2)$

$= \alpha T(u) + T(v)$

*** **Algebra of Linear Transformation**

Theorem:- Let V & W be a 2 vector space over a field F, Let T & U be a linear-tranformation from V into W then

a) $(T+U)$ is defined by $(T+U)v = T(v) + U(v)$ is a LT

b) If $c \in F$ the $(cT)v = cT(v)$ in a LT

Proof:- we know that :. $T(\alpha u + v) = \alpha T(u) + T(v)$

a) $(T+U)$ is a LT

i.e $(T+U)(\alpha u + v) = \alpha(T+U)(u) + (T+U)(v)$

LHS $= (T+U)(\alpha u + v) = T(\alpha u + v) + U(\alpha u + v)$

$= \alpha T(u) + T(v) + \alpha U(u) + U(v)$

$= \alpha(T+U)(u) + (T+U)(v)$

$=$ RHS

b) we need to PT cT is a LT

i.e $cT(\alpha u + v) = \alpha cT(u) + cT(v)$

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LHS $= cT(\alpha u + v) = c[\alpha T(u) + T(v)]$

$= \alpha cT(u) + cT(v)$

$=$ RHS

$\therefore$ cT is a LT

Prove (TU) is a LT

Theorem 2:- Let V, W & Z be a 3 vector space over the field F let T be a LT $(T: V \to W)$ and $U: W \to Z$ then the composite function $U.T$ which is defined as

$U.T(v) = U(T(v))$ is a LT

Range space & NULL space of LT

Let V, W 2 vector space over field F & let $T: V \to W$ be LT then the range space of T / image of T is denoted as $R(T)$ & defined as

$R(T) = \{ w \in W \mid T(v) = w \ \forall \ v \in V \}$

&

the null space of T / kernel of T is denoted as $N(T)$ and defined as $N(T) = \{ v \in V \mid T(v) = 0 \}$

Rank - Nullity theorem.

If U & V are 2 vector space over the field F $T: U \to V$ be LT then $\dim(R(T)) + \dim(N(T)) = \dim U$

OR Rank (T) + Nullity $(T) = \dim(U)$

1) For LT $T: V \to W$ defined as $T(x,y,z) = (y-x, y-z)$

Find $R(T)$, $N(T)$, $\dim(R(T))$, $\dim(N(T))$.

Ans:- WKT $R(T) = \{ w \in W \mid T(v) = w \ \forall \ v \in V \}$

$= \{ T(x,y,z) \in W \mid x,y,z \in \mathbb{R}^3 \}$

$= \{ (y-x, y-z) \in W \mid x,y,z \in \mathbb{R}^3 \}$

$N(T) = \{ v \in V \mid T(v) = 0 \}$

$\{ x,y,z \in \mathbb{R}^3 \mid T(x,y,z) = 0 \}$

$\{ x,y,z \in \mathbb{R}^3 \mid (y-x, y-z) = 0 \}$

$y - x = 0 \Rightarrow y = x$

$y - z = 0 \Rightarrow y = z$

$\therefore x = y = z = t$

$\therefore N(T) = \{ x,y,z \in \mathbb{R}^3 \mid T(k,k,k) = 0 \}$

$\therefore \dim(N(T)) = 1$

By Rank - nullity theorem

$\dim(R(T)) + \dim(N(T)) = \dim U$

$\dim(R(T)) + 1 = 3$

$\therefore \dim(R(T)) = 2$

2) LT $T: V \to W$ defined as $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$. Find $R(T)$, $N(T)$, $\dim(R(T))$, $\dim(N(T))$.

Ans:- WKT $R(T) = \{ w \in W \mid T(v) = w \ \forall \ v \in V \}$

$R(T) = \{ T(x,y,z) \in W \mid x,y,z \in \mathbb{R}^3 \}$

$= \{ (x+2y-z, y+z, x+y-2z) \in W \mid x,y,z \in \mathbb{R}^3 \}$

$N(T) = \{ v \in U \mid T(v) = 0 \}$

$= \{ x, y, z \in \mathbb{R}^3 \mid T(x, y, z) = 0 \}$

$= \{ x, y, z \in \mathbb{R}^3 \mid (x + 2y - z,\ y + z,\ x + y - 2z) = 0 \}$

$x + 2y - z = 0$

$y + z = 0$

$x + y - 2z = 0.$

$$[A : B] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \quad R_3 \to R_3 + R_2$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \rho(A) = 2 \quad \rho(A:B) = 2 \quad n = 3$$

$\therefore$ Infinite soln.

let $z = k$

$y + z = 0 \rightarrow \underline{y = -k.}$

$x + 2y - z = 0. \quad \underline{x = 3k}$

$\therefore R(T) = \left\{ x, y, z \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} x = 3k \\ y = -k \\ z = k \end{array} \right\}$

$\therefore \underline{\dim(N(T)) = 1}$

By Rank nullity Theorem

$\dim(R(T)) + \dim(N(T)) = \dim U \Rightarrow \dim(R(T)) + 1 = 3$

$\dim(R(T)) = 2$

$\therefore \dim = 1$

$0, 0, k$

*** 3) Let $T: V_3 \to V_2$ defined by $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

Find $R(T)$, $N(T)$, $\dim(R(T))$, $\dim(N(T))$.

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*** 4) $T: \mathbb{R}^3 \to \mathbb{R}^4$ such that $T(x, y, z) = (y+z, x+z, x+y, x+y+z)$

Find $R(T)$, $N(T)$, $\dim(R(T))$, $\dim(N(T))$.

Ans:- WKT $R(T) = \{ w \in W \mid T(v) = w \ \forall v \in V \}$

$R(T) = \{ T(x, y, z) \in W \mid x, y, z \in \mathbb{R}^3 \}$

$R(T) = \{ (y+z, x+z, x+y, x+y+z) \in W \mid x, y, z \in \mathbb{R}^3 \}$

$N(T) = \{ v \in V \mid T(v) = 0 \}$

$= \{ x, y, z \in \mathbb{R}^3 \mid T(x, y, z) = 0 \}$

$= \{ x, y, z \in \mathbb{R}^3 \mid (y+z, x+z, x+y, x+y+z) = 0 \}$

$y + z = 0$

$x + z = 0$

$x + y = 0$

$x + y + z = 0$

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 - R_1, \quad R_4 \to R_4 - R_1$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \quad R_3 \to R_3 + R_2 \Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$R_4 \to R_4 + R_3 \Rightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\rho(A) = 3$, $\rho(A:B) = 3$, $n = 3$

$\therefore$ unique solution

$x = y = z = 0$

$\therefore \dim(N(T)) = 0$.

By back substitution method

$x + 0 + z = 0 \quad \therefore z = 0.$

$0 + y + z = 0 \quad \therefore y = 0$

$x + y + z = 0 \quad \therefore x = 0.$

$\therefore x = y = z = 0. \Rightarrow N(T) = \{x, y, z \in \mathbb{R}^3 \mid (0,0,0)\}$

$\therefore \dim(N(T)) = 0.$

By Rank nullity Theorem

$\dim(R(T)) + \dim(N(T)) = \dim(U)$

$\dim(R(T)) + 0 = 3$

5) Prove that $N(T)$ is subspace of V

Ans:- ~~$N(T) = \{v \in V \mid x, y, z$~~ $N(T) = \{v \in V \mid T(v) = 0\}$

proof :- we need to prove that $N(T)$ is subspace of V

(i) Clearly $T(v) = 0 \quad \therefore 0 \in V \quad \therefore$ It's non empty

(ii) To prove vector addition

Let $v_1, v_2 \in V$ Then $T(v_1) = 0$

$T(v_1 + v_2) = T(v_1) + T(v_2)$

$= 0 + 0$

$= 0$

$\therefore v_1 + v_2 \in V$

(iii) To prove scalar multiplication

let $v \in V$ & $c \in F$

Then $c\,T(v) = T(cv) = 0.$

$\therefore$ scalar multiplication is True

$\therefore N(T)$ is subspace of V.

6) Prove that $R(T)$ is subspace of W

Matrix of Linear Transformation.

Let $T: U \to V$ & $B_1 = \{u_1, u_2, u_3 \ldots\ldots u_n\}$

$B_2 = \{v_1, v_2, v_3 \ldots\ldots v_n\}$ be the basis of U & V respectively such that

$T(u_1) = a_{11}v_1 + a_{21}v_2 + a_{31}v_3 + \ldots\ldots a_{n1}v_n$

$T(u_2) = a_{12}v_1 + a_{22}v_2 + a_{32}v_3 + \ldots\ldots a_{n2}v_n$

$\vdots$

$\therefore$ The matrix of (T) is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots \\ \vdots & & & \\ a_{n1} & a_{n2} & a_{n3} & \end{bmatrix}$$

1) Find the matrix of the linear transformation $T: R^2 \to R^2$ defined by $T(x, y) = (x, -y)$ wrt basis $B_1 = \{\underbrace{(1,1)}_{u_1}, \underbrace{(1,0)}_{u_2}\}$

$B_2 = \{\underbrace{(2,3)}_{v_1}, \underbrace{(4,5)}_{v_2}\}$

Ans:- $T(u_1) = a_{11}v_1 + a_{21}v_2$

$T(1, 1) \Rightarrow (1, -1) = a_{11}(2, 3) + a_{21}(4, 5)$

$1 = 2a_{11} + 4a_{21}$ $\qquad \therefore a_{11} = \cancel{8}\ -4.5$

$-1 = 3a_{11} + 5a_{21}$ $\qquad a_{21} = 2.5$

$T(u_2) = a_{12}v_1 + a_{22}v_2$

$T(1, 0) \Rightarrow (1, 0) = a_{12}(2, 3) + a_{22}(4, 5)$

$1 = 2a_{12} + 4a_{22}$ $\qquad \therefore a_{12} = -2.5$

$0 = 3a_{12} + 5a_{22}$ $\qquad \therefore a_{22} = 1.5$

$\therefore$ Matrix of LT is $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -4.5 & -2.5 \\ 2.5 & 1.5 \end{bmatrix}$

*** 2) $T: V_2 \to V_3$ defined by $T(x, y) = (x+y, x, 3x-y)$

3X2 wrt basis $B_1 = \{(1, 1), (3, 1)\}$ $B_2 = \{(1,1,1)\ (1,1,0)\ (1,0,0)\}$

Ans:- $T(u_1) = a_{11} v_1 + a_{21} v_2 + a_{31} v_3$

$T(1,1) \Rightarrow (2, 1, 2) = a_{11}(1,1,1) + a_{21}(1,1,0) + a_{31}(1,0,0)$

$2 = 1a_{11} + 1a_{21} + 1a_{31}$ $\therefore a_{11} = 2$

$1 = 1a_{11} + 1a_{21} + 0a_{31}$ $a_{21} = -1$

$2 = 1a_{11} + 0 + 0.$ $a_{31} = 1$

$T(u_2) = a_{12} v_1 + a_{22} v_2 + a_{32} v_3$

$T(3,1) \Rightarrow (4, 3, 8) = a_{12}(1,1,1) + a_{22}(1,1,0) + a_{32}(1,0,0)$

$4 = 1a_{12} + 1a_{22} + 1a_{32}$ $\therefore a_{12} = 8$

$3 = 1a_{12} + 1a_{22} + 0a_{32}$ $a_{22} = -5$

$8 = 1a_{12} + 0 + 0.$ $a_{32} = 1$

~~$\therefore$ Matrix of LT is~~ $\cancel{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -1 & -5 \end{bmatrix}}$

$\therefore$ Matrix of LT is $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 2 & 8 \\ -1 & -5 \\ 1 & 1 \end{bmatrix}$

a_{11} a_{12} a_{13}
a_{21} a_{22} a_{23}
a_{31} a_{32} a_{33}

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$V \times U$ 3) Find the matrix of LT $T: V_3 \to V_2$; $T(x,y,z) = x-y+2z,$
2×3 $3x+y)$ Basis $B_1 = \{(1,1,1), (1,2,3), (1,0,0)\}$ $B_2 = \{(1,1), (1,-1)\}$

$\to$ $T(u_1) = a_{11}v_1 + a_{21}v_2$

$T(1,1,1) = (2,4) = a_{11}(1,1) + a_{21}(1,-1)$

$2 = 1a_{11} + 1a_{21}$ $\therefore a_{11} = 3$

$4 = 1a_{11} - 1a_{21}$ $a_{21} = -1$

$T(1,2,3) = a_{12}v_1 + a_{22}v_2$

$(5,5) = a_{12}(1,1) + a_{22}(1,-1)$

$5 = 1a_{12} + 1a_{22}$ $\therefore a_{12} = 5$

$5 = 1a_{12} - 1a_{22}$ $a_{22} = 0$

$T(1,0,0) = a_{13}v_1 + a_{23}v_2$

$1, 3 = a_{13}(1,1) + a_{23}(1,-1)$

$1 = 1a_{13} + 1a_{23}$ $\therefore a_{13} = 2$

$3 = 1a_{13} - 1a_{23}$ $a_{23} = -1$

$$\begin{bmatrix} 3 & 5 & 2 \\ -1 & 0 & -1 \end{bmatrix}$$

can skip for now

4) Given $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$ Find LT for $T: V_3 \to V_2$ related to basis $B_1 = \{(1,2,0)\ (0,-1,0)\ (1,-1,1)\}$

$B_2 = \{(1,0)\ (2,-1)\}$

$\to$ $T(x,y,z) = (\ldots, \ldots)$

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$T(u_1) = a_{11}v_1 + a_{21}v_2$

$T(1,2,0) = -1(1,0) + 1(2,-1)$

$T(1,2,0) = (-1,1)$ —— ①

$T(u_2) = a_{12}v_1 + a_{22}v_2$

$T(0,-1,0) = (2,0)$ —— ②

$T(u_3) = a_{13}u_1 + a_{23}v_2$

$T(1,-1,1) = (1,3)$ —— ③

Let $(x,y,z) \in V_3$ or $\mathbb{R}^3$

Then LC : $(x,y,z) = c_1u_1 + c_2u_2 + c_3u_3$

$(x,y,z) = c_1(1,2,0) + c_2(0,-1,0) + c_3(1,-1,1)$

$x \quad = 1c_1 + 0c_2 + 1c_3 \qquad c_1 = x - z$

$y \quad = 2c_1 + (-1)c_2 + (-1)(c_3) \qquad c_2 = 2x - y - 3z.$

$z \quad = 0c_1 + 0c_2 + 1c_3 \quad \Rightarrow c_3 = z$

$\therefore$ LT is $T: V_3 \to V_2$

$T(x,y,z) = c_1T(u_1) + c_2(T(u_2)) + c_3T(u_3)$

$= (x-z)((-1,1)) + 2x-y-3z(2,0) + z(1,3)$

$= (-x+z, x-z) + (4x-2y-6z, 0) + (1z, 3z)$

$= (3x-2y-4z, x+2z)$

Till here

Task Monday (MCQ → QB)