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Elimantary Row Operations:

- Interchange of any 2 rows
- Addition of any row to the other non-zero row
-

Echelon form of a matrix:

Every row must have a leading non-zero element and all the entries below that should be zero.

ex:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

V.IMP **Rank of a matrix:**

It is defined as the no. of non-zero rows in the echelon form of a matrix.

It is denoted by $\rho(A)$ or Rank (A)

For the above mentioned ex: the rank of the matrix is $\underline{3}$

Pivotal positions:

The leading non-zero entries in the echelon form of a matrix is known as pivotal elements and the corresponding coloums is called as pivotal coloumns.

eg:

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivotal elements

pivotal coloumns.

IMP Define Basic and free variable

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Problems:

1. Reduce the following matrices to echelon form and find rank of the identify the pivotal coloumns matrix.

(i)
$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & 1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

:
$$A = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$$

$R_2 \to 1R_2 - 3R_1$

$R_3 \to R_3 + R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 2 & -3 & 10 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -2 & 3 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ coloumns 1 & 2 are pivotal coloumns

and $\rho(A) = 2$.

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2. $$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$$A \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

$R_3 \to 1R_3 - 2R_1$

$R_4 \to R_4 + R_1$

$$= A = \begin{bmatrix} -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 1 & 2 & -3 & -3 \\ 0 & 2 & 4 & -6 & \cancel{8}-6 \end{bmatrix}$$

$R_3 \to 3R_3 + R_2$

$R_4 \to 3R_4 + 2R_2$

$$A = \begin{bmatrix} -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -10 & 0 \end{bmatrix}$$

$R_4 \to 5R_4 + 10R_3$

OR

$R_4 \to R_4 + 2R_3$

$$A = \begin{bmatrix} -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivotal elements $\to -1, -3, -5$

" coloumns $\to 1, 3, 4$

$\rho(A) = 1, 2, 3 \Rightarrow \underline{3}$

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(ii) Find the rank of $A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

: $A = \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$R_2 \to 2R_2 - R_1$, $R_3 \to 2R_3 - R_1$

$A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 7 & 9 & -1 \\ 0 & 1 & 5 & 3 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$R_3 \to 7R_3 - R_2$ $\qquad$ $R_4 \to 7R_4 - R_2$

$A = \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 7 & 9 & -1 \\ 0 & 0 & 26 & 22 \\ 0 & 0 & -2 & -6 \end{bmatrix}$

$R_4 \to 26R_4 + 2R_3$ or $13R_4 + R_3$

$A = \begin{bmatrix} 2 & -1 & -3 & -1 \\ 0 & 7 & 9 & -1 \\ 0 & 0 & 26 & 22 \\ 0 & 0 & 0 & -112 \end{bmatrix}$ or -56

$\therefore$ $\rho(A) = 4$

(iii)

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$$\therefore \quad A = \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - R_1$

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\underline{\rho(A) = 2}}$, Pivotal coloumns $\rightarrow$ 2.

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2G for SEE

Solution of system of linear Equations:

If a linear equation has a solution then the given system of equations is said to be consistent else it is inconsistent.

Conditions for consistency of equations [Gauss Elimination method]

(i) If rank of matrix A $\rho(A) = \rho(A:B) = n$ [no. of unknowns] then the system is consistent and has unique solution

(ii) if $\rho(A) = \rho(A:B) < n$ then system is consistent and has infinitely many solution

(iii) if $\rho(A) \neq \rho(B)$, system is inconsistent and has no solutions

1) Test the consistency of the eq^n and solve the eq^n using G. elimination Method:

(i)
$$5x + y + 3z = 20$$
$$2x + 5y + 2z = 18$$
$$3x + 2y + z = 14$$

$x = 3$
$y = 2$
$z = 1$

$$[A:B] = \left[\begin{array}{ccc:c} 5 & 1 & 3 & 20 \\ 2 & 5 & 2 & 18 \\ 3 & 2 & 1 & 14 \end{array}\right]$$

Reduce to echelon form.

$$[A:B] = \left[\begin{array}{ccc:c} \boxed{5} & 1 & 3 & 20 \\ 2 & 5 & 2 & 18 \\ 3 & 2 & 1 & 14 \end{array}\right]$$

$R_2 \to 5R_2 - 2R_1$
$R_3 \to 5R_3 - 3R_1$

$$= \begin{bmatrix} 5 & 1 & 3 & 20 \\ 0 & \boxed{23} & 4 & 50 \\ 0 & \cancel{12}\ 7 & -4 & 10 \end{bmatrix}$$

$R_3 \to 23R_3 - 7R_2$

$$= \begin{bmatrix} 5 & 1 & 3 & : & 20 \\ 0 & 23 & 4 & : & 50 \\ 0 & 0 & -120 & : & -120 \end{bmatrix}$$

$\rho(A:B) = 3$

$\rho(A) = 3.$

$n = 3$

$\rho(A:B) = \rho(A) = n$

$\therefore$ system of eq^n are consistent ~~for~~ and has unique solution.

To find the solution:

By back substitution method,

$5x + y + 3z = 20$ — ①

$0x + 23y + 4z = 50$ — ②

$0x + 0y + -120z = -120.$ — ③

from ③

$\to -120z = -120$

$z = 1$

from ②

$23y + 4z = 50$

$23y + 4 = 50$

$y = 2.$

from ①

$5x + 1(2) + 3(1) = 20$

$\underline{\underline{x = 3}}$

2) $5x + 3y + 7z = 5$

$3x + 26y + 2z = 9$

$7x + 2y + 10z = 5$

$$[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

math error
so it is either no sol^n
or infinite sol^n

Reduce to echelon form.

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$[A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$

$R_2 \to 5R_2 - 3R_1 \qquad R_3 \to 5R_3 - 7R_1$

$[A:B] \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & -11 & 1 & : & -10 \end{bmatrix}$

$R_3 \to \cancel{121}R_3 + 11R_2 \Rightarrow R_3 \to 11R_3 + R_2$

$[A:B] \begin{bmatrix} 5 & 3 & 7 & : & 5 \\ 0 & 121 & -11 & : & 30 \\ 0 & 0 & 0 & : & -880 \end{bmatrix}$

$\rho(A:B) = 3$

$\rho(A) = 2$

$\rho(A:B) \neq \rho(A)$

$\therefore$ It has no solution, and inconsistent.

3. $5x_1 - x_2 + 2x_3 = 7$

$-2x_1 + 6x_2 + 9x_3 = 0$

$-7x_1 + 5x_2 - 3x_3 = -7.$

unique sol^n.

$\therefore [A:B] = \begin{bmatrix} 5 & -1 & 2 & : & 7 \\ -2 & 6 & 9 & : & 0 \\ -7 & 5 & -3 & : & -7 \end{bmatrix}$

echelon form $\therefore R_2 \to 5R_2 + 2R_1 \qquad R_3 \to 5R_3 + 7R_1$

$[A:B] \begin{bmatrix} 5 & -1 & 2 & 7 \\ 0 & 28 & 49 & 14 \\ 0 & 18 & -1 & 14 \end{bmatrix}$

$R_3 \to \cancel{28}R_3 - 18R_2 \quad , \quad R_3 \to 14R_3 - 9R_2$

$[A:B] = \begin{bmatrix} 5 & -1 & 2 & : & 7 \\ 0 & 28 & 49 & : & 14 \\ 0 & 0 & -445 & : & 70 \end{bmatrix}$

$\rho(A:B) = 3 \qquad \rho(A) = 3 \qquad n = 3$ It has unique solution.

4. $x_2 - 4x_3 = 8$

$2x_1 - 3x_2 + 2x_3 = 1$

$4x_1 - 8x_2 + 12x_3 = 1$

:

$$\left[\begin{array}{ccc:c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array}\right]$$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array}\right]$$

$R_3 \to 2R_3 - 4R_1$ OR $R_3 \to R_3 - 2R_1$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -4 & 16 & -2 \end{array}\right]$$

$R_3 \to 1R_3 \overset{(-)\times(-)}{+} 4R_2$

$$[A:B] = \left[\begin{array}{ccc:c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 30 \end{array}\right]$$

$\rho(A:B) = 3$

$\rho(A) = 2$

$\rho(A:B) \neq \rho(A)$

$\therefore$ The system is inconsistent, i.e it has no solution

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Find the general solution of the equation also define basic and free variables

$x + 3y + 4z = 7$

$2x + 9y + 7z = 6$

:

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 3 & 4 & 7 \\ 2 & 9 & 7 & 6 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$

$$\begin{array}{c} \\ [A:B] = \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc:c} 1 & 3 & 4 & 7 \\ 0 & 0 & -5 & -15 \end{array}\right] \end{array}$$

Here basic variables are x and z

and free variable is y

$\rho(A:B) = 2$

$\rho(A) = 2$

$n = 3$ $\quad x, y, z$

Since $\rho(A:B) = \rho(A) < n$

so eq^n are consistent and has infinitely many sol^n.

By back substitution.

$-5z = -15$

$z = 3$ and $y = y$.

$1x + 3y + 4z = 7$

$x + 3y + 12 = 7$

$x = 7 - 12 - 3y$

$x = -5 - 3y$

IMP 2. Identify the basic and free variable and also find the general sol^n whose augmented matrix is given by

$$[A:B] = \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \left[\begin{array}{ccccc:c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right] \end{array}$$

∴ The given matrix is in echelon form

∴ Basic (dependent) variable = Pivotal elements are a, c, e

and free (independent) variables are b and d.

$\rho(A:B) = 3$

$\rho(A) = 3.$

$n = 5$

$\rho(A:B) = \rho(A) < n$

system is consistent and has infinitely many sol^n

By back sol^n

$e = 7.$ or $x_5 = 7$

$d = \cancel{d_5}$ or $x_4 = x_4$

$$0x_1 + 0x_2 + 2x_3 - 8x_4 - x_5 = 3$$

$$2x_3 - 8x_4 - 7 = 3$$

$$2x_3 = 10 + 8x_4$$

$$x_3 = 5 + 4x_4 \qquad x_2 = x_2$$

$$x_1 + 6x_2 + 2x_3 - 5x_4 - 2x_5 = -4$$

$$x_1 + 6x_2 + 2(5 + 4x_4) - 5x_4 - 2(7) = -4$$

$$x_1 + 6x_2 + 3x_4 - 4 = -4$$

$$x_1 = -6x_2 - 3x_4$$

3. $2x_1 + 3x_2 + 5x_3 - x_4 + x_5 = 0$

$3x_1 - 2x_2 + x_3 + 2x_4 = 0$

$x_2 + 2x_3 + 4x_4 + x_5 = 0$

$$\therefore [A:B] = \begin{bmatrix} 2 & 3 & 5 & -1 & 1 & 0 \\ 3 & -2 & 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 4 & 1 & 0 \end{bmatrix}$$

$R_2 \to 2R_2 - 3R_1$

$$\therefore [A:B] \begin{bmatrix} 2 & 3 & 5 & -1 & 1 & 0 \\ 0 & -13 & -13 & 7 & -3 & 0 \\ 0 & 8 & 13 & 59 & 10 & 0 \end{bmatrix}$$

~~$R_3 \to 13R_3 + 8R_2$~~ $R_3 \to 13R_3 + 8R_2$

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$$[A:B] = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\ \left[\begin{array}{cccccc} 2 & 3 & 5 & -1 & 1 & 0 \\ 0 & -13 & -13 & 7 & -3 & 0 \\ 0 & 0 & 13 & 59 & 10 & 0 \end{array}\right] \end{array}$$

Here $\rho(A:B) = 3$ $\quad \rho(A) = 3$; $n = 5$
since $\rho(A:B) = \rho(A) < n$ then given system of eq^n is consistent and has infinitely many sol^n.
Here x_1, x_2 and x_3 are basic variables and x_4 and x_5 are free variable

by back substitution. $\quad x_5 = x_5 \quad x_4 = x_4$ free

(1)
$$13x_3 + 59x_4 + 10x_5 = 0$$
$$13x_3 = -59x_4 - 10x_5$$
$$x_3 = \frac{-59}{13}x_4 - \frac{10}{13}x_5$$

(2)
$$-13x_2 - 13x_3 + 7x_4 - 3x_5 = 0$$
$$\cancel{-13x_2 + 13x_3 - 7x_4 + 3x_5}$$
$$-13x_2 = 13\left(-\frac{59}{13}x_4 - \frac{10}{13}x_5\right) + 7x_4 - 3x_5 = 0$$
$$-13x_2 + 59x_4 + 10x_5 + 7x_4 - 3x_5 = 0$$
$$-13x_2 + 66x_4 + 7x_5 = 0$$
$$\cancel{-13}\ \frac{66x_4}{13} + \frac{7}{13}x_5 = x_2$$

x_3 (3) $\rightarrow$ $2x_1 + 3x_2 + 5x_3 - x_4 + x_5 = 0$
$$2x_1 + 3\left[\frac{66x_4}{13} + \frac{7}{13}x_5\right] + 5\left[\frac{-59}{13}x_4 - \frac{10}{13}x_5\right] - x_4 + x_5 = 0$$
$$2x_1 + \frac{19}{13}x_4 + \frac{21}{13}x_5 - \frac{295}{13}x_4 - \frac{50}{13}x_5 - x_4 + x_5 = 0$$
$$2x_1 - \frac{97}{13}x_4 - \frac{29}{13}x_5 - x_4 + x_5 = 0$$
$$2x_1 - \frac{110}{13}x_4 - \frac{16}{13}x_5 = 0$$
$$x_1 = \frac{55}{13}x_4 - \frac{8x_5}{13} \quad //$$

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LU factorization

A matrix can be represented as a product of 2 or more matrices and this conversion is called as matrix factorization.

Given a matrix A can be reduced to 2 factors L and U where L represents lower Triangular matrix with the diagonal entries as 1 and U represents upper Triangular matrix or echelon form of the matrix.

Procedure:

- For a system of matrix $AX = B$, take $A = LU$ $LUX = B$.
- Take $LUX = B$ Take $UX = Y$ (where Y is dummy variable)

$\Rightarrow LY = B$

Problems:

1. For a given matrix find LU factorization.

IMP (i) $A = \begin{bmatrix} \boxed{4} & 1 & 1 \\ 1 & 4 & -2 \\ -3 & 2 & 4 \end{bmatrix}$

S:

$R_2 \to 4R_2 - R_1$, $R_3 \to 4R_3 + 3R_1$

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & \boxed{15} & -9 \\ 0 & 11 & 19 \end{bmatrix}$$

$R_3 \to 15R_3 - 11R_2$

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 15 & -9 \\ 0 & 0 & 384 \end{bmatrix} = U$$

identify first pivotal element and below ~~that~~ ~~then~~ right side should be made '0'

$$L = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 15 & 0 \\ -3 & 11 & 384 \end{bmatrix}$$

$c_1/4$ $c_2/15$ $c_3/384$

diagonal elements must be made 1 so divide by respective number.

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$$= \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -3/4 & 11/15 & 1 \end{bmatrix} = L$$

2.

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 + 3R_1$

$$\begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$
$R_4 \to R_4 - 4R_2$

$$= \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix}$$

$R_4 \to 2R_4 - 4R_3 = R_4 \to R_4 - 2R_3$

$$L = \begin{bmatrix} \boxed{2} & 4 & -1 & 5 & -2 \\ 0 & \boxed{3} & 1 & 2 & -3 \\ 0 & 0 & 0 & \boxed{2} & 1 \\ 0 & 0 & 0 & 0 & \boxed{5} \end{bmatrix} = U$$

no. of coloums of should be equal to

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$$L = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 2 & -9 & 2 & 0 \\ 6 & 12 & 4 & 5 \end{bmatrix}$$

OR

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}$$

VIMP 2

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ -1 & -5 & 8 & 4 \\ 4 & 2 & -5 & -7 \\ -2 & -4 & 7 & 5 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to 1R_3 - 4R_1$, $R_4 \to 1R_4 + 2R_1$

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & -10 & 15 & 5 \\ 0 & 2 & -3 & -1 \end{bmatrix}$$

$-5 - (-5 \times 2)$
$-5 - (-20)$
$-5 + 20$
$-7 - (4 \times -3)$
$-7 - (-12)$
$-7 + 12$
$-4 + (2 \times 3)$
$-4 + 6$
$7 + 2(-5)$
$7 - 10$
-3

$R_3 \to 2R_3 - 10R_2$ (5) $R_4 \to 2R_4 + 2R_2$

$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & -0 & \\ 0 & 0 & 0 & \end{bmatrix}$$

$15 + 5(-2)$
$15 + (-10)$
-5
$-3 + 3$
5

$5 + (2 \times -3)$
$5 + (-6)$
-1

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$$A = \begin{bmatrix} 1 & 3 & -5 & -3 \\ 0 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 4 & -10 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix}$$

or.

$$\cancel{L} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

3. Using LU factorization method solve the system of equation

$x + y + z = 1$

$4x + 3y - z = 6$

$3x + 5y + 3z = 4$

consider $AX = B$

wher $A = \begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

$R_2 \to R_2 - 4R_1$ $R_3 = R_3 - 3R_1$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 2 & 0 \end{bmatrix}$$

$R_3 = R_3 + 2R_2$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} = U$$

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$R_2/-1 \quad R_3/-10$

Now $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & -1 & 0 \\ 3 & 2 & -10 \end{bmatrix}$ OR $L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ +3 & -2 & 1 \end{bmatrix}$

$AX = B$

$LUX = B$

let $UX = Y$ —(*)

$LY = B$

Then $LY = B$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \times \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$

By solving

$y_1 = 1$

$y_2 = 2$

$y_3 = 5$

sub Y in (*).

i.e $UX = Y$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

on solving

$z = \frac{-1}{2} \Rightarrow z = -\frac{1}{2}$

$-y - 5z = 2$

$y = \frac{1}{2}$

$x + y + z = 1$

$\underline{x = 1}$

2. $5x + y + 2z = 20$

$2x + 5y + 2z = 18$

HW $3x + 2y + z = 14.$

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3. $4x_1 + x_2 + x_3 = 4$

$x + 4x_2 - 2x_3 = 4$

$3x_1 + 2x_2 - 4x_3 = 6$

consider $AX = B$

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & -2 \\ 3 & 2 & -4 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$R_2 \to 4R_2 - R_1 \qquad R_3 \to 4R_3 - 3R_1$

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 15 & -9 \\ 0 & 5 & -19 \end{bmatrix}$$

$R_3 \to 15R_3 - 5R_2$

$R_3 \to 3R_3 - R_2$

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 0 & 15 & -9 \\ 0 & 0 & -48 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 15 & 0 \\ 3 & 5 & -48 \end{bmatrix} \quad \text{OR} \quad \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & 1/3 & 1 \end{bmatrix}$$

$AX = B$

$LUX = B$

$UX = Y$

$LY = B$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$L \begin{bmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ 3/4 & 1/3 & 1 \end{bmatrix} Y \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 6 \end{bmatrix}$$

$y_1 = 4$

$y_2 = 3$

$y_3 = 2$

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sub y in (x)

$UX = Y$

$$\begin{bmatrix} 4 & 1 & 1 \\ 0 & 15 & -9 \\ 0 & 0 & -48 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$x_1 = \frac{29}{30}$ $\quad x_2 = \frac{7}{40}$ $\quad x_3 = \frac{-1}{24}$

Gauss – Seidel Method

it is a iterative method where to solve these equation we need to following steps

- check whether the eq^n are diagonally dominant
- solve for x, y and z from equations 1, 2 and 3 respectively
- Take the initial approximations as $x^{(0)} = 0$ $y^{(0)} = 0$ $z^{(0)} = 0$
- Go on incrementing the values of x, y and z respectively and stop the procedure when the values are same in any 2 iterations.

Using gauss - seidal iterative methods solve the system of equation.

1) $54x + y + z = 110$

$2x + 15y + 6z = 72$.

$-x + 6y + 27z = 85$

Given $x^{(0)} = y^{(0)} = z^{(0)} = 0$;

carry-out 3 iteration.

∴ Given system of eq^n are diagonally dominant

From (1)

$x = \frac{1}{54}[110 - y - z]$

From (2)

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$y = \frac{1}{15}[72 - 2x - 6z]$

From (3), $z = \frac{1}{27}[85 + x - 6y]$

Given $x^{(0)} = 0;\ y^{(0)} = 0;\ z^{(0)} = 0.$

1st iteration:

$x^{(1)} = \frac{1}{54}[110 - y^{(0)} - z^{(0)}]$

$= \frac{1}{54}[110 - 0 - 0]$

$x^{(1)} = 2.0370$

$y^{(1)} = \frac{1}{15}(72 - 2x^{(1)} - 6z^{(0)}]$

$= \frac{1}{15}(72 - 2\times 2.0370 - 6(0))]$

$y^{(1)} = 4.5284$

$z^{(1)} = \frac{1}{27}[85 + x^{(1)} - 6y^{(1)}]$

$= \frac{1}{27}[85 + 2.0370 - 6(4.5284))]$

$z^{(1)} = 2.2173$

$\cancel{2^{nd}}$

$\cancel{x^{(2)} = \frac{1}{15}[72 - 2x^{(2)} - 6z^{(1)}]}$

$\cancel{= \frac{1}{15}[72 - 2(1.9121) - 6(2.2173)]}$

$\cancel{= 3.6581}$

2nd

$x^{(2)} = \frac{1}{54}(110 - y^{(1)} - z^{(1)}]$

$= \frac{1}{54}[110 - 4.5284 - 2.2173$

$= 1.9121$

$y^{(2)} = \frac{1}{15}[72 - 2x^{(2)} - 6z^{(1)}]$

$= \frac{1}{15}[72 - 2(1.9121) - 6(2.2173)]$

$= 3.6581$

$z^{(2)} = \frac{1}{27}[85 + 1.9121 - 6(3.6581)]$

$= 2.4061$

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3^{rd}:

$$x^{(3)} = \frac{1}{54}\left[110 - y^{(2)} - z^{(2)}\right]$$

$$= \frac{1}{54}(110 - 3.6581 - 2.4061)$$

$$= \underline{1.9247}$$

$$y^{(3)} = \frac{1}{15}\left[72 - 2x^{(3)} - 6z^{(2)}\right]$$

$$= \frac{1}{15}\left[72 - 2(1.9247) - 6(2.4061)\right]$$

$$= \underline{3.5809}$$

$$z^{(3)} = \frac{1}{27}\left(85 + x^{(3)} - 6y^{(3)}\right]$$

$$= \frac{1}{27}\left[85 + 1.9247 - 6(3.5809)\right]$$

$$= \underline{2.4237}$$

(ii) $3x + 8y + 29z = 71$

$83x + 11y - 4z = 95$

$7x + 52y + 13z = 104$

last page. ∴

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Part - B :

Trace of a Matrix :

If A is a square matrix then Trace of A is equ to the sum of diagonal elements of matrix A

or $\sum_{ij}$ $\sum_{i,j=1}^{n} a_{i,j}$

Properties of Trace :

1. if A and B are

- $Trace(cA) = c\,trace(A)$
- $Trace(A \pm B) = Trace(A) \pm Trace(B)$
- $Trace(AB) = Trace(BA)$
- $Trace(B^{-1}AB) = Trace(A)$
- $Trace(AA^{-1}) = \sum_{i,j=1}^{n} a_{i,j}^2$

1] $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}$ Prove that

$Trace(B^{-1}AB) = Trace(A)$

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LHS:

$$B^{-1} = \frac{1}{|B|} \text{adj } B.$$

$$= \frac{1}{17} \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} \frac{-13}{17} & \frac{-12}{17} \\ \frac{58}{17} & \frac{98}{17} \end{bmatrix}$$

LHS:

$$\text{Trace}(B^{-1}AB) = \frac{-13}{17} + \frac{98}{17} = 5$$

RHS

$$\text{Trace}(A) = 1+4 = 5$$

LHS = RHS

V. IMP

Eigen value and Eigen vectors:

If A is a square matrix of order $n \times n$ then, ~~AX=λX~~ $AX = \lambda X$ where, X is a non zero ~~on~~ and λ is eigen values (real and complex) and X indicates corresponding eigen vectors.

Note:

- Trace of A = sum of eigen values of A.
- the eigen values are calculated using the characteristic equations $|A - \lambda I| = 0$
- IMP For a 2x2 matrix the characteristics equations can be considered as $\lambda^2 - \Sigma D \lambda + |A| = 0$.
- For a 3x3 matrix, $\lambda^3 - \Sigma D \lambda^2 + \Sigma m D \lambda - |A| = 0$
- IMP For a upper Triangular matrix, eigen values are the diagonal values.

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1. Check whether sum of eigen values = trace of the following matrix.

(i)
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

: eq^n is $|A - \lambda I| = 0$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{vmatrix} = 0$$

OR

$$\lambda^3 - \Sigma D \lambda^2 + \Sigma m D \lambda - |A| = 0$$

where $\Sigma D = 1 + 3 + 3 = 7$

$$\Sigma m D = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}$$

$8 + 3 + 3 = 14$

$|A|$

$= 8$

$\therefore$ Eq^n is $\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

$\lambda = 4, 2, 1$

$4 + 2 + 1 = 7$

Trace $(A) = 1 + 3 + 3 = 7$

(ii)
$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

: ~~eq^n is~~

3, 1, 0 bcz it is upper triangular matrix

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$\Rightarrow \lambda^3 - \Sigma D\lambda^2 + \Sigma mD\lambda - |A| = 0$

where $\Sigma D = 3+1+0 = 4$

$$\Sigma mD = \begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix}$$

$\Sigma mD = 0 + 0 + 3 = \underline{3}$

$|A| = 0$

Eqn is $\lambda^3 - 4\lambda^2 + 3\lambda - 0 = 0$

$\lambda = 3, 1, 0$

$3+1 = 4$

Trace $(A) = 3+1 = 4$

~~Trace~~ Find the eigen values and eigen vectors for the matrix

1. $A = \begin{bmatrix} 13 & -4 & 2 \\ -4 & 13 & -2 \\ 2 & -2 & 10 \end{bmatrix}$

char eqn, $\lambda^3 - \Sigma D\lambda^2 + \Sigma mD\lambda - |A| = 0$

$\Sigma D = 13 + 13 + 10 = 36$

$$\Sigma mD = \begin{vmatrix} 13 & -2 \\ -2 & 10 \end{vmatrix} + \begin{vmatrix} 13 & 2 \\ 2 & 10 \end{vmatrix} + \begin{vmatrix} 13 & -4 \\ -4 & 13 \end{vmatrix}$$

$= 126 + 126 + 153$

$= \underline{405}$

$|A| = \underline{1458}$

Eqn is $\lambda^3 - 36\lambda^2 + 405 - 1458 = 0$

$\lambda = 18, 9$

$\lambda + \lambda_2 + \lambda_3 = 36 \rightarrow$ Trace(A)

$18 + 9 + \lambda_3 = 36$

$\lambda_3 = 9$

$\lambda = \underline{18, 9, 9}$

To find eigen vectors:

$AX = \lambda X$

$[A - \lambda I]X = 0$

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$$\begin{bmatrix} 13-\lambda & -4 & 2 \\ -4 & 13-\lambda & -2 \\ 2 & -2 & 10-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

case 1:

when $\lambda = 18$

$-5x - 4y + 2z = 0$ — ①

$-4x - 5y - 2z = 0$ — ②

$2x - 2y - 8z = 0$

$$\frac{x}{\begin{vmatrix} -4 & 2 \\ -5 & -2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & -5 \\ -2 & -4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -5 & -4 \\ -4 & -5 \end{vmatrix}}$$

$$\frac{x}{18} = \frac{y}{-18} = \frac{z}{9} = k$$

$$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 18k \\ -18k \\ 9k \end{pmatrix}$$

These are eigen vectors where $\lambda = 18$

case 2:

when $\lambda = 9$

sub in (*)

$4x - 4y + 2z = 0$ — ①

$-4x + 4y - 2z = 0$

$2x - 2y + z = 0$

let $z = k_1$, $y = k_2$ (since $\lambda = 9$ is repeated)

Then sub in ①

$4x - 4k_2 + 2k_1 = 0$

$4x = 4k_2 - 2k_1$

$x = k_2 - 1/2\, k_1$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_2 - 1/2\, k_1 \\ k_2 \\ k_3 \end{pmatrix}$$ where $\lambda = 9$

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2. $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$

3 Find eigen values and eigen vectors

$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$

char. eq^n, $\lambda^3 - \Sigma D \lambda^2 + \Sigma m D \lambda - |A| = 0$

$\Sigma D = 3+6+3 = 12$

$\Sigma mD = \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -2 \\ -2 & 6 \end{vmatrix}$

$= 14 \cancel{4} - 7 + \cancel{5}\, 14$

$= \cancel{14} \quad 21$

$|A| = 98$

$\cancel{3(14) + 2(-7) + 4(}$

Eq^n is $\lambda^3 - 12\lambda^2 + 21\lambda + 98 = 0$

$\lambda = -2, 7, 7$ → If u add this we should get trace of the matrix or else it is wrong

To find eigen vectors

$[A - \lambda I] X = 0$

$$\begin{bmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1:

when $\lambda = 2$

$5x - 2y + 4z = 0$ —(1)

$-2x + 8y + 2z = 0$ —(2)

$4x + 2y + 5z = 0$

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$$\frac{x}{\begin{vmatrix} -2 & 4 \\ 8 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 4 & 5 \\ 2 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 5 & -2 \\ -2 & 8 \end{vmatrix}} = k$$

$$\frac{x}{-36} = \frac{y}{-18} = \frac{z}{36} = k$$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -36k \\ -18k \\ 36k \end{pmatrix}$$

case 2: when $\lambda = 7$

$-4x - 2y + 4z = 0$ ①

$-2x - y + 2z = 0$

$4x + 2y - 4z = 0$

let $y = k_1$, & $z = k_2$

sub in ①

$-4x - 2k_1 + 4k_2 = 0$

$x = \frac{-2}{4}k_1 + \frac{4}{4}k_2$

$$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}k_1 + k_2 \\ k_1 \\ k_2 \end{pmatrix}$$

4

$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$$

char eqⁿ $\lambda^3 - \Sigma D \lambda^2 + \Sigma m D \lambda - |A| = 0$

$\cancel{\lambda^3 - 4\lambda^2 F}$

$\Sigma D = -2 + 1 + 5 = 4$

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$$\Sigma MD = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} -2 & -4 \\ -2 & 1 \end{vmatrix}$$

$= 1\cancel{5} - 18 - 10$

$= \underline{\underline{-27}}$

$|A| = 90$

eq^n is $\lambda^3 - \cancel{4}\lambda^2 - 27\lambda + 90 = 0$

$= \lambda = \underline{\underline{-5, 6, 3}}$

To find the eigen vectors,

$[A - \lambda I]X = 0$

$$\begin{bmatrix} -2-\lambda & -4 & 2 \\ -2 & 1-\lambda & 2 \\ 4 & 2\cancel{+\lambda} & 5-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Case 1, when $\lambda = -5$

$3x - 4y + 2z = 0$

$-2x + 6y + 2z = 0$

$$\frac{x}{\begin{vmatrix} -4 & 2 \\ 6 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 2 & 3 \\ 2 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & -4 \\ -2 & 6 \end{vmatrix}} = k$$

$$\frac{x}{-20} = \frac{y}{-10} = \frac{z}{10} = k$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -20k \\ -10k \\ 10k \end{pmatrix}$$

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Rayleigh's Power method

To find the "largest eigen value and the corresponding eigen vector

Given a matrix A and the initial eigen vector $X(0)$, then multiply $AX(0)$

- Divide the obtained matrix by the numerically largest value and call it as $\lambda_1^{(1)}$ and $X^{(1)}$
- For the second iteration, multiply $AX^{(1)}$ and follow the same procedure as previous one.
- ~~Using power metho~~

Using power method find the largest eigen value and eigen vector for the given matrix. carry out 6 iteration

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \qquad X^{(0)} = [1 \quad 0.8 \quad 0.8]$$

1st

$$AX = \underset{3\times3}{\begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 1 \\ 0.8 \\ 0.8 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 4 \\ 3.6 \\ 2.8 \end{bmatrix}} = 4 \begin{bmatrix} 1 \\ 0.9 \\ 0.7 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

2nd

$$AX^{(1)} \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.9 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 4.2 \\ 4 \\ 2.4 \end{bmatrix} = 4.2 \begin{bmatrix} 1 \\ 0.95 \\ 0.57 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

3rd

$$AX^{(2)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.95 \\ 0.57 \end{bmatrix} = \begin{bmatrix} 4.38 \\ 4.29 \\ 1.81 \end{bmatrix} = 4.38 \begin{bmatrix} 1 \\ 0.95 \\ 0.41 \end{bmatrix}$$

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4th.

$$Ax^{(3)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \\ 0.41 \end{bmatrix} = \begin{bmatrix} 4.57 \\ 4.52 \\ 1.02 \end{bmatrix} = 4.57 \begin{bmatrix} 1 \\ 0.99 \\ 0.23 \end{bmatrix}$$

5th

$$Ax^{(4)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0.99 \\ 0.23 \end{bmatrix} = \begin{bmatrix} 4.76 \\ 4.74 \\ 0.13 \end{bmatrix} = 4.76 \begin{bmatrix} 1 \\ 1 \\ 0.03 \end{bmatrix}$$

6th

$$Ax^{(5)} = \begin{bmatrix} 4 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0.03 \end{bmatrix} = \begin{bmatrix} 4.97 \\ 4.96 \\ -0.86 \end{bmatrix} = 4.97 \begin{bmatrix} 1 \\ 1 \\ -0.17 \end{bmatrix}$$

$\lambda = 4.97 \rightarrow$ largest eigen value

$$X = \begin{pmatrix} 1 \\ 1 \\ -0.17 \end{pmatrix}$$

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Q. Find the largest eigen value & corresponding eigen vector using power method for a given matrix: Carry out 6 iteration:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{Take } x^{(0)} = [1\ 0\ 0]$$

1st:

$$AX = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \lambda X^{(1)}$$

2nd

$$AX^1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

3rd

$$AX^{(2)} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.49 \end{bmatrix}$$

4th.

$$AX^{(3)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0.49 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.428 \\ 1.857 \end{bmatrix} = 3.428\begin{bmatrix} 0.875 \\ -1 \\ 0.542 \end{bmatrix} = X^{(4)}(x^4)$$

5th

$$AX^{(4)} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 0.875 \\ -1 \\ 0.542 \end{bmatrix} = \begin{bmatrix} 2.75 \\ -3.41 \\ 2.08 \end{bmatrix} = 3.41\begin{bmatrix} 0.805 \\ -1 \\ 0.6099 \end{bmatrix}$$

6th

$$A(X)^5 \begin{bmatrix} \\ \\ \end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 2.610 \\ -3.415 \\ 2.22 \end{bmatrix} = 3.415\begin{bmatrix} 0.830 \\ -1.086 \\ 0.705 \end{bmatrix}$$

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largest eigen value = 3.415

$$X = \begin{pmatrix} 0.830 \\ -1.086 \\ 0.705 \end{pmatrix}$$

2.

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$

Take $X^{(0)} = [1 \ 0 \ 0]$

1st

$$AX = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 0.333 \\ 1 \\ -0.33 \end{bmatrix}$$

2nd

$$= \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.333 \\ 1 \\ -0.333 \end{bmatrix} = \begin{bmatrix} 3.667 \\ 1.667 \\ 3.667 \end{bmatrix} = 3.667 \begin{bmatrix} 1 \\ 0.455 \\ 1 \end{bmatrix}$$

3rd

$$= \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0.455 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.364 \\ 7.908 \\ 0.818 \end{bmatrix} = 7.908 \begin{bmatrix} 0.172 \\ 1 \\ 0.103 \end{bmatrix}$$

4th

$$\begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.172 \\ 1 \\ 0.103 \end{bmatrix} = \begin{bmatrix} 3.069 \\ 2.931 \\ 3.828 \end{bmatrix}$$

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3. $A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$

Take $x^{(0)} = [1\ 0\ 0]$

: 1st

$$AX = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 3\begin{bmatrix} 0.333 \\ 1.000 \\ -0.333 \end{bmatrix} = \begin{bmatrix} 0.333 \\ 1.000 \\ -0.333 \end{bmatrix} = \lambda^{(1)} (X^{(1)})$$

2nd :

$$AX^{(1)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.333 \\ 1.000 \\ -0.333 \end{bmatrix} = \begin{bmatrix} 3.666 \\ 1.667 \\ 0.337 \end{bmatrix} = 3.666\begin{bmatrix} 1.000 \\ 0.455 \\ 0.092 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

3rd

$$AX^{(2)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 1.000 \\ 0.455 \\ 0.092 \end{bmatrix} = \begin{bmatrix} 2.273 \\ 4.278 \\ 1.740 \end{bmatrix} = 4.278\begin{bmatrix} 0.531 \\ 1 \\ 0.407 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

4th.

$$AX^{(3)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.531 \\ 1 \\ 0.407 \end{bmatrix} = \begin{bmatrix} 3.124 \\ 5.221 \\ 7.539 \end{bmatrix} = 7.539\begin{bmatrix} 0.414 \\ 0.6925 \\ 1 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

5th

$$AX^{(4)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.414 \\ 0.6925 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.492 \\ 6.627 \\ 12.356 \end{bmatrix} = 12.356\begin{bmatrix} 0.121 \\ 0.536 \\ 1 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

6th.

$$AX^{(5)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}\begin{bmatrix} 0.121 \\ 0.536 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.729 \\ 5.435 \\ 12.023 \end{bmatrix} = 12.023\begin{bmatrix} 0.061 \\ 0.452 \\ 1 \end{bmatrix} \quad \lambda^{(6)} . X^{(6)}$$

largest eigen value is $\lambda = 12.023$ and

$$X = \begin{pmatrix} 0.061 \\ 0.452 \\ 1 \end{pmatrix}$$

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4\. Given: $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ check whether $U = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ is a eigen vector of A

: $AU = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -24 \begin{bmatrix} 6 \\ 5 \end{bmatrix}$, $AU = \lambda U$

$\lambda U = \lambda U$ $\quad \lambda = -24$ & $U = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ $\quad \therefore U$ is eigen vector.

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Diagonalization of Matrix:

Let A be a square matrix of order n, $\lambda, \lambda_2, \lambda_3 \dots \lambda_n$ are the eigen value of $X_1, X_2 \dots X_n$ are corresponding vectors then the matrix p is called as Diagonalising of ~~eqn~~ model matrix and is represented by the corresponding eigen vectors. The diagonalization of a matrix is calculated as

$D = P^{-1}AP$ (∵ the diagonalization matrix will always be the eigen values in the diagonal)

or it can also be true when

$$PD = AP$$

Note!

Powers of a square matrix can be calculated using $A^k = PD^kP^{-1}$

Problems:

1 Diagonalize the matrix if possible.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

$\Sigma mD = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix}$

: $\lambda^3 - \Sigma D\lambda^2 + \Sigma mD\lambda - |A| = 0$

$\lambda^3 - 7\lambda^2 + 14\lambda - 8 = 0$

$\lambda = 4, 2, 1$

To find eigen vectors

$[A - \lambda I]X = 0$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 3-\lambda & -1 \\ 0 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case 1: when $\lambda = 4$

$-3x + 0y + 0z = 0$ —(1)

$0x - y - z = 0$ —(2)

$$\frac{x}{\begin{vmatrix} 0 & 0 \\ -1 & -1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 0 & -2 \\ -1 & 0 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix}} = k$$

$$\frac{x}{0} = \frac{y}{-2} = \frac{z}{2} = k$$

$$x_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2k \\ 2k \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$k = \frac{1}{2}.$$

Case 2

when $\lambda = 2$.

$-2x + 0y + 0z = 0$ —(1)

$0x + y - z = 0$ —(2)

$$\frac{x}{0} = \frac{y}{-1} = \frac{z}{-1} = k$$

$$x_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1k \\ -1k \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

Case 3:

when $\lambda = 1$

$0x + 2y - 2z = 0$ —(2)

$0x \cancel{1} - y + 2z = 0$ —(3)

$$\frac{x}{2} = \frac{y}{0} = \frac{z}{0} = k$$

$$X_3 = \begin{pmatrix} 2k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

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let $P = [x_1, x_2, x_3]$

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$D = P^{-1}AP$

$$D = \begin{bmatrix} 1 & -0.5 & 0.5 \\ 0 & -0.5 & -0.5 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} P \end{bmatrix}$$

: mat A × mat B, mat B^{-1}

: mat A × mat B × mat Ans.

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2)

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$\Sigma MD = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix}$

$= 2 + 2 + 4$

$= 8$

$\lambda^3 - \Sigma D\lambda^2 + \Sigma MD\lambda - |A| = 0$

$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$

$\lambda = 1, 2, 2$

eigen vector.

$[A - \lambda I]\, x = 0$

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

case 1:

when $\lambda = 1$

$$x_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

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case 2: when $\lambda = 2$ (repeated)

$x + 0y + z = 0$ — (2)

$-x + 0y - z = 0$ — (3).

let $y = k_2$ and $z = k_1$

sub in (2)

$x = -k_1$

$$x_j \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -k_1 \\ k_2 \\ k_1 \end{pmatrix},$$

when $k_1 = 0$, $k_2 = 1$

$$x_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

when $k_1 = 1$ and $k_2 = 0$

$$x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$P = [x_1, x_2, x_3]$

$$\begin{bmatrix} 0 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$D = P^{-1} A P$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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3. $\begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

: $\lambda^3 - \Sigma D\lambda^2 + \Sigma mD\lambda - |A| = 0$

$\cancel{\lambda^3 - 5\lambda^2}$ –

$\lambda^3 + 3\lambda^2 +$

$\Sigma mD = \begin{vmatrix} 6 & -3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ -4 & 6 \end{vmatrix}$

$= 6-(-9) \quad 2-9 \quad 12-(-16)$

$= 6+9 \quad 12+16$

$= 15 + -7 \quad 28$

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$\Sigma mD\lambda = \begin{vmatrix} -6 & -3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ -4 & -6 \end{vmatrix}$

$: -6+9 + 2-9 \quad -12+16$

$: 3 \cancel{4} - 7 + 4$

$: \underline{0}$

$\lambda^3 + 3\lambda^2 + 0\lambda - 4 = 0$

$\lambda = 1, -2, -2$

<u>eigen vectors :</u>

$[A - \lambda I]\, X = 0$

$$\begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

when $\lambda = -2$ [repeated]

$4x + 4y + 3z = 0$ — 1

$-4x - 4y - 3z = 0$ — 1

$3x + 3y + 3z = 0$ — 2

$$\frac{x}{\begin{vmatrix} 4 & 3 \\ 3 & 3 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & 4 \\ 3 & 3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 4 & 4 \\ 3 & 3 \end{vmatrix}} = k$$

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$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3k \\ -3k \\ k \end{pmatrix}$$

Since $z = 0$;
and only one independent variable is available for a repeated matrix we cannot diagonalize the given matrix

4. $\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ diagonalize and hence find A^6

: $\lambda^2 - \Sigma D \lambda + |A| = 0$.

$\lambda^2 - (-2\lambda) + (-3) = 0$

$\lambda^2 + 2\lambda - 3 = 0$

$\lambda = 1, -3$

eigen vectors

$[A - \lambda I]\, X = 0$

$$\begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

when $\lambda = 1$

$-2x + 2y = 0$ — (1)

$2x - 2y = 0$

let $y = k$

$-2x + 2k = 0$

$-2x = -2k$

$x = k$

$$x_1 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix}$$

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when $\lambda = -3$

$$x_2 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$P = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$D = P^{-1} A P$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

To find A^6

$$A^6 = P D^6 P^{-1}$$

$$= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^6 & 0 \\ 0 & (-3)^6 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

$=$

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For what value of a and b the system of equations has
(i) no solution
(ii) unique solution
(iii) infinite solution

$$x + 2y + 3z = 6$$
$$x + 3y + 5z = 9$$
$$2x + 5y + az = b.$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 2 & 5 & a \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 9 \\ b \end{bmatrix}$$

$$A:B = \left[\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 1 & 3 & 5 & 9 \\ 2 & 5 & a & b \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$A:B = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & a-6 & b-12 \end{bmatrix}$$

$R_3 \to \cancel{2}R_3 - R_2$

$$\left[\begin{array}{ccc:c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & a-8 & b-15 \end{array}\right]$$

(i) If $a \neq 8$ and $b \neq 15$ then we get unique solⁿ.
(ii) If $a = 8$ and $b = 15$ then $\rho(A:B) = \rho(A) = 2 < n = 3$
∴ we get infinitely many solⁿ
(iii) If $a = 8$ and $b \neq 15$ then $\rho(A:B) = 3$ $\rho(A) = 2$.
∴ we get no solution.