Chapter 4 (Part 2)

數學歸納法

Mathematical Induction

THEOREM 4.1

數學歸納法原理。令 $S(n)$ 表一個開放的數學敘述 (或此類開放敘述的集合)，其含有一個或多個變數 n 發生，n 為一正整數。

a) 若 $S(1)$ 為真；且
b) 若當 $S(k)$ 為真時 (對某個特別，但任意選的，$k \in \mathbf{Z}^+$)，$S(k+1)$ 為真；

則 $S(n)$ 為真對所有 $n \in \mathbf{Z}^+$。

Recall

A declarative sentence is an *open statement* if
(1) it contains one or more **variables**, and
(2) it is not a statement, but
(3) it becomes a statement when the variables in it are replaced by certain **allowable choices**.

universe

Definition 2.5

Examples of open statements:

$$p(x):\ x \geq 0$$

$$q(x):\ x^2 \geq 0$$

$$r(x):\ x^2 - 3x - 4 = 0$$

$$s(x):\ x^2 - 3 > 0$$

Universe: Real numbers

THEOREM 4.1 Cont.

Principle of Mathematical Induction

Let $S(n)$ denote an open mathematical statement that involves one or more occurrences of the variable n, where n is a positive integer.

(a) If $S(1)$ is true; and **** basis step, not necessarily from 1 ****

(b) If whenever $S(k)$ is true (for some particular, but arbitrarily chosen, $k \in \mathbf{Z}^+$), then $S(k+1)$ is true; (**inductive step**)

then $S(n)$ is true for all $n \in \mathbf{Z}^+$

THEOREM 4.1 Cont.

Principle of Mathematical Induction

Proof:

Let $S(n)$ be such an open statement satisfying conditions (a) and (b), and let $F = \{t \in \mathbf{Z}^+ | S(t) \text{ is false}\}$. We wish to prove that $F = \emptyset$

Then by the Well-Ordering Principle, F has a least element m.
Since $S(1)$ is true, it follows that $m \neq 1$, so $m > 1$,
and consequently $m - 1 \in \mathbf{Z}^+$.
With $m - 1 \notin F$, we have $S(m - 1)$ true.
So by condition (b) it follows that $S((m - 1) + 1) = S(m)$ is true,
contradicting $m \in F$. Consequently, $F = \emptyset$.

EXAMPLE 4.1

For all $n \in \mathbf{Z}^+$, $\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

$$\text{Proof}: S(n): \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Induction basis: show $S(1)$ is true

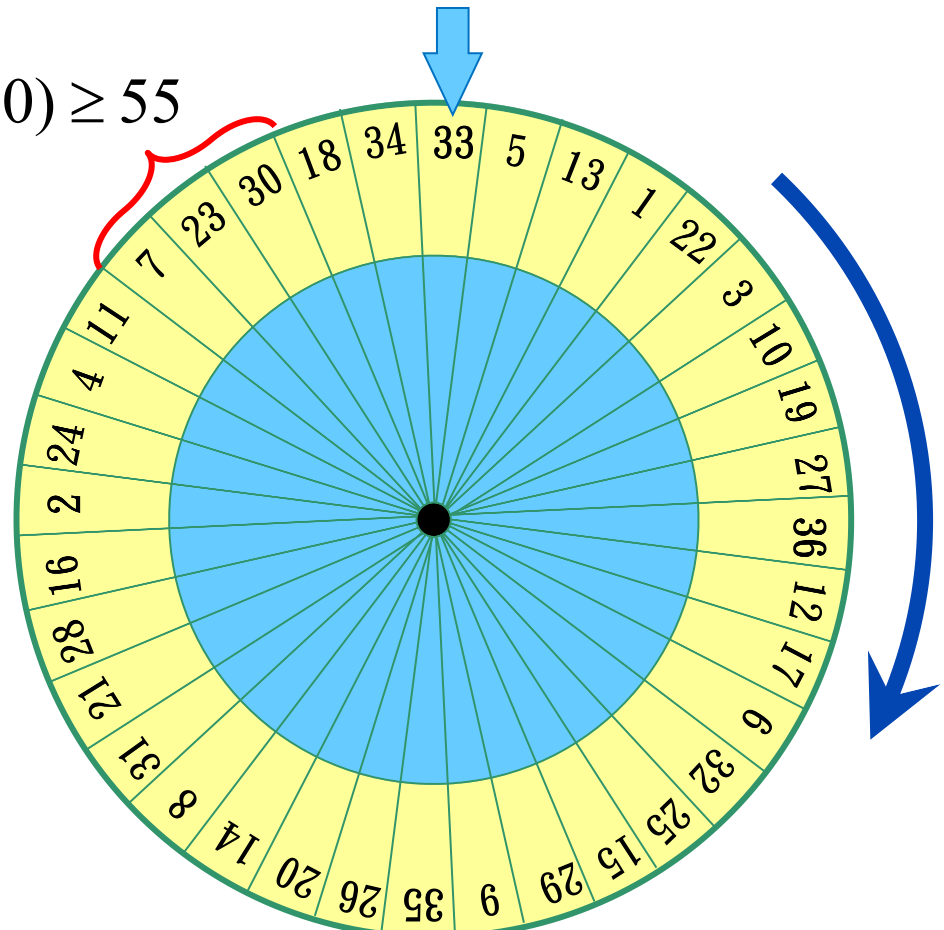
Induction hypothesis: assume $S(k): \sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ is true

Establish the truth of $S(k+1)$

EXAMPLE 4.2

A wheel of fortune has the numbers from 1 to 36 painted on it in a random manner. Show that regardless of how the numbers are situated, there are three consecutive (on the wheel) numbers whose total is 55 or more.

$$\underbrace{(7+23+30)}\geq 55$$



For all $n \in \mathbf{Z}^+$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

EXAMPLE 4.3

Prove that for each $n \in \mathbf{Z}^+$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$

EXAMPLE 4. 4

Prove that $2^n < n!$, where $n \geq 4$ and $n \in \mathbf{Z}^+$.

- Let $P(n)$ be the proposition $2^n < n!$.
- Basis step: $p(4)$ is true, because $2^4 < 4!$.
- Inductive step:
- **Firstly, assume that $P(k)$ is true, where $k \geq 4$ and $k \in \mathbb{Z}^+$.**
- **Second, if $P(k)$ is true, $P(k+1)$ is also true.**
- $2^{k+1} = 2 \times 2^k =< 2 \times k! < (k+1)k! = (k+1)!$.
- Thus, $P(k+1)$ is true when $P(k)$ is true.
- Hence, $2^n < n!$ is true, where $k \geq 4$ and $k \in \mathbb{Z}^+$, through the completion of basis and inductive steps.

EXAMPLE 4.5

If $n \in \mathbf{Z}^+$, establish the validity of the open statement

$$S(n): \quad \sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n^2 + n + 2}{2}.$$

EXAMPLE 4.6

E.g., $14 = 3 + 3 + 8$

We can express 14 using only 3's and 8's as summands.

$S(n)$: n can be written as a sum of 3's and/or 8's (with no regard to order).

Prove $S(n)$ is true for all $n \geq 14$.

THEOREM 4. 2 *The Principle of Mathematical Induction — Alternative Form.*

Let $S(n)$ denote an open mathematical statement (or set of such open statements) that involves one or more occurrences of the variable n, which represents a positive integer. Also let $n_0, n_1 \in \mathbf{Z}^+$ with $n_0 \leq n_1$.

a) If $S(n_0)$, $S(n_0 + 1)$, $S(n_0 + 2)$, . . . , $S(n_1 - 1)$, and $S(n_1)$ are true;

b) If whenever $S(n_0)$, $S(n_0 + 1)$, . . . , $S(k - 1)$, and $S(k)$ are true for some (particular but arbitrarily chosen) $k \in \mathbf{Z}^+$, where $k \geq n_1$, then the statement $S(k + 1)$ is also true;

then $S(n)$ is true for all $n \geq n_0$.

EXAMPLE 4.7

For every $n \in \mathbf{Z}^+$ where $n \geq 14$,

$S(n)$: n can be written as a sum of 3's and/or 8's.