Chapter 5

關係和函數

Relations and Functions

5.1 笛卡兒積和關係

Definition 5.1

For sets $A, B \in \mathscr{U}$, the *Cartesian product* , or cross product, of A and B is denoted by $A \times B$ and equals $\{(a,b) | a \in A,\ b \in B\}$.

對集合 A，B，A 和 B 的**笛卡兒積** (Cartesian product) 或**叉積** (cross product) 被表為 $A \times B$，且等於 $\{ (a, b) \mid a \in A, b \in B \}$。

EXAMPLE 5.1

令 $A=\{2, 3, 4\}$，$B=\{4, 5\}$，則

a) $A \times B = \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}$.

b) $B \times A = \{(4, 2), (4, 3), (4, 4), (5, 2), (5, 3), (5, 4)\}$.

c) $B^2 = B \times B = \{(4, 4), (4, 5), (5, 4), (5, 5)\}$.

d) $B^3 = B \times B \times B = \{(a, b, c) | a, b, c \in B\}$; 例如 $(4, 5, 5) \in B^3$。

- the elements of $A\times B$ are ordered pairs

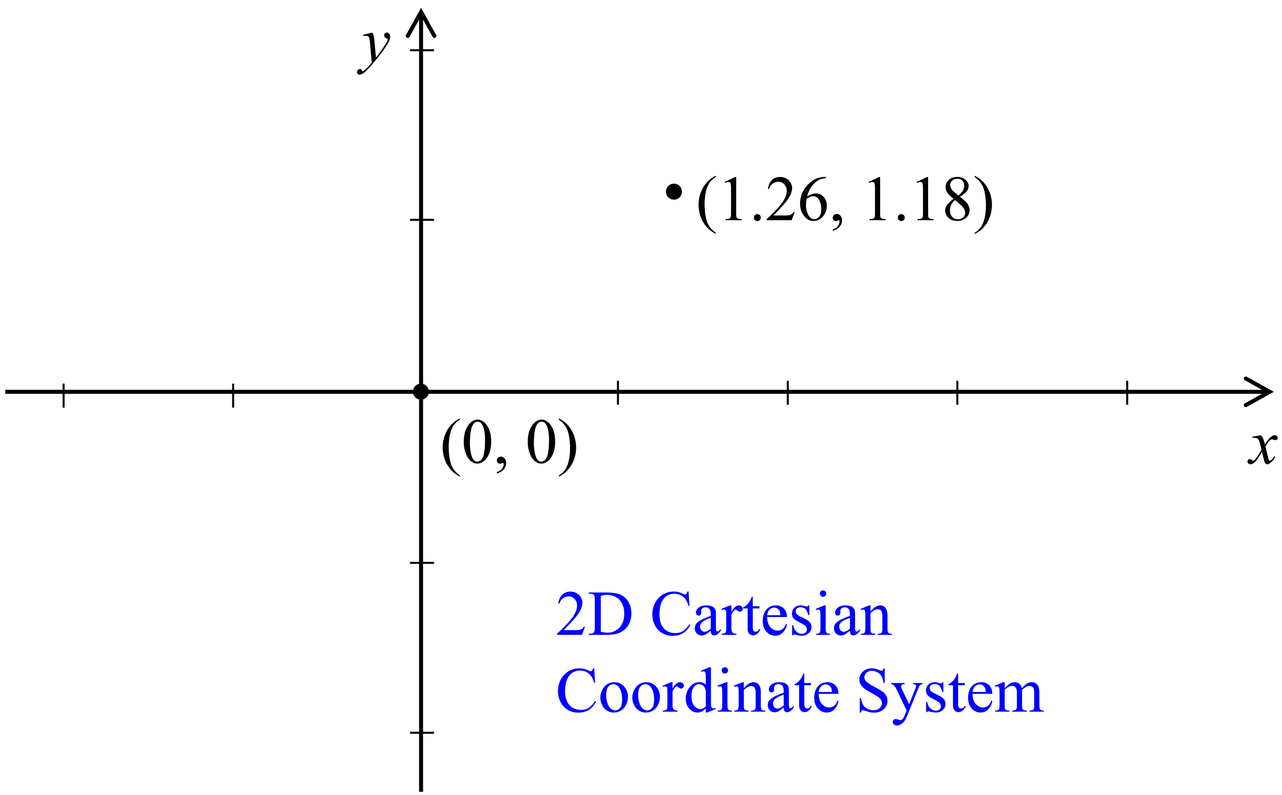
- $|A\times B|=|A|\times|B|=|B\times A|$

But, in general $A \times B \neq B \times A$. And

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) \mid a_i \in A_i, 1 \leq i \leq n\}.$$

EXAMPLE 5.2

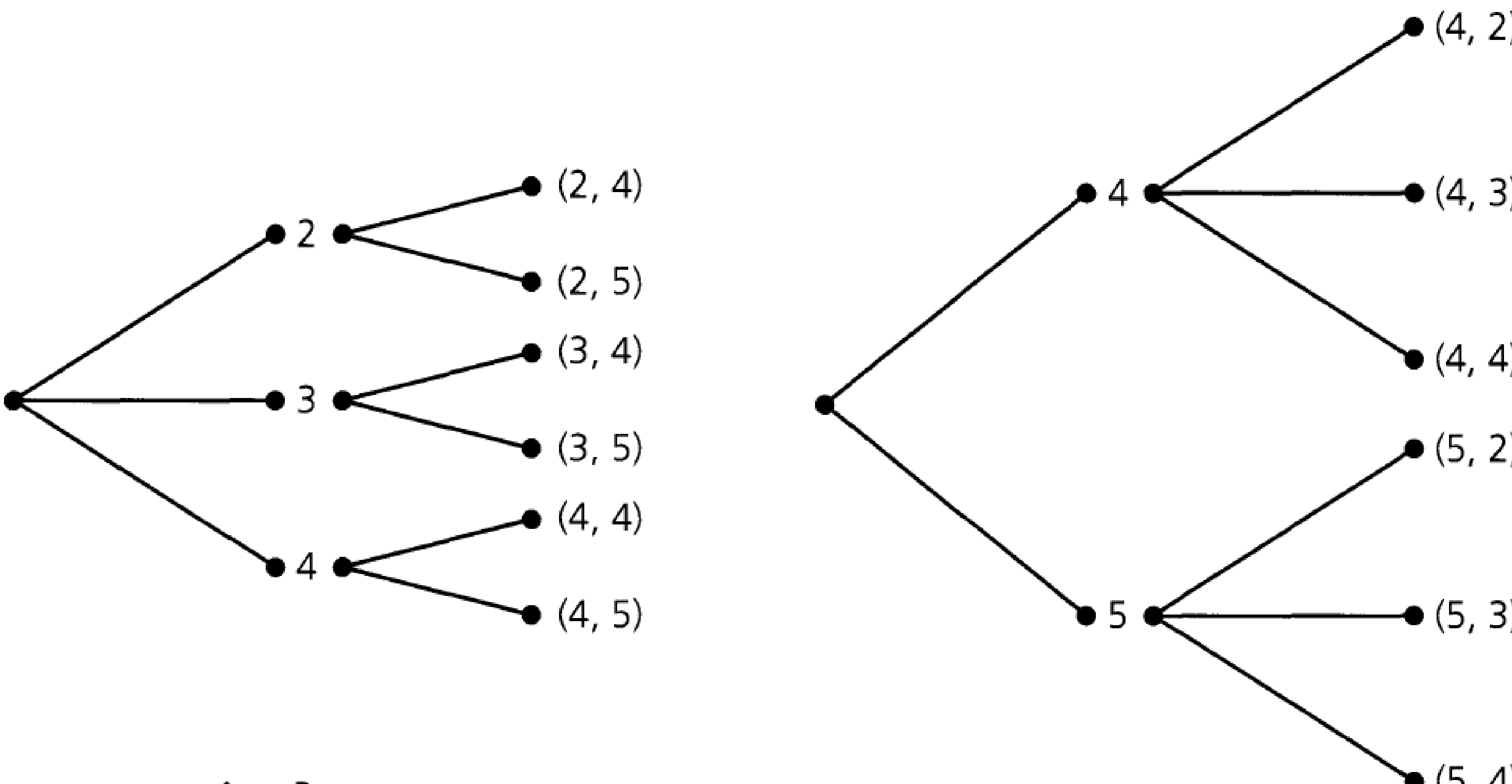
The set $\mathbf{R} \times \mathbf{R} = \{(x, y) | x, y \in \mathbf{R}\}$ is recognized as the real plane of coordinate geometry and two-dimensional calculus.



2D Cartesian Coordinate System

EXAMPLE 5.3

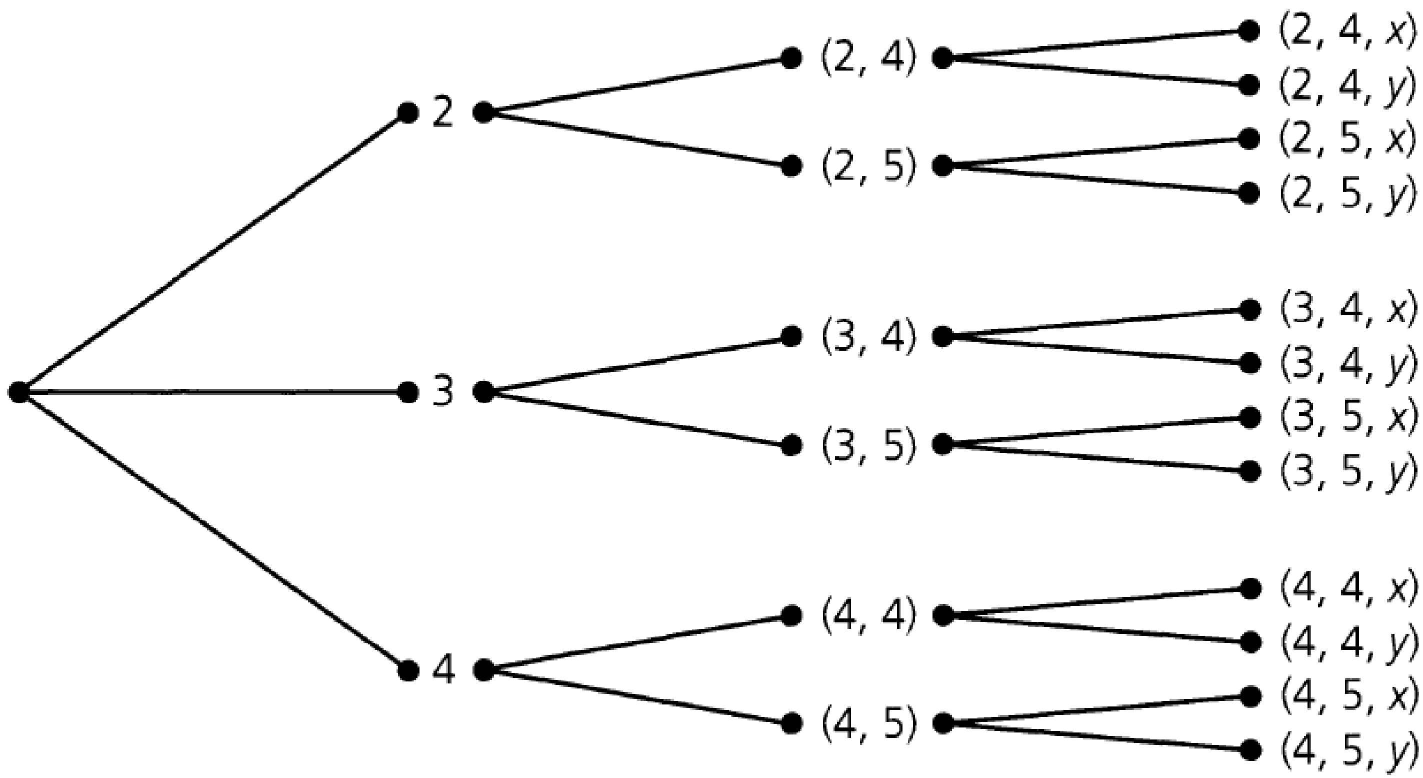
令 $A=\{2, 3, 4\}$，$B=\{4, 5\}$，則



$A \times B$ $B \times A$

EXAMPLE 5.3 Cont.

let $C = \{x, y\}$



$A \times B \times C$

$$|A \times B \times C| = 12 = 3 \times 2 \times 2 = \boxed{}$$

Definition 5.2

For sets A, B, any subset of $A \times B$ is called a (*binary*) *relation* from A to B. Any subset of $A \times A$ is called a (*binary*) *relation* on A.

對集合 A，B，$A \times B$ 的任一子集合被稱為一個由 A 到 B 的 (二元) **關係** [(binary) relation]。$A \times A$ 的任一子集合被稱為 A 上的 (二元) 關係。

EXAMPLE 5.4

$A=\{2, 3, 4\}$，$B=\{4, 5\}$

$$A \times B = \{(2, 4), (2, 5), (3, 4), (3, 5), (4, 4), (4, 5)\}.$$

Followings are some examples of relations from A to B.

a) $\emptyset$
b) $\{(2, 4)\}$
c) $\{(2, 4), (2, 5)\}$
d) $\{(2, 4), (3, 4), (4, 4)\}$
e) $\{(2, 4), (3, 4), (4, 5)\}$
f) $A \times B$

Since $|A \times B| = 6$, there are 2^6 possible relations from A to B (for there are 2^6 possible subsets of $A \times B$).

EXAMPLE 5.4 Cont.

對有限集合 A，B 具 $|A|=m$ 且 $|B|=n$，共有 2^{mn} 個由 A 到 B 的關係，包括空關係及關係 $A\times B$ 本身。

亦有 2^{nm} $(=2^{mn})$ 個由 B 到 A 的關係，其中亦含有 $\emptyset$ 及 $B\times A$。由 B 到 A 的關係個數和由 A 到 B 的關係個數相同的理由是由 B 到 A 的任一個關係 $\mathcal{R}_1$ 可由由 A 到 B 的一個唯一關係 $\mathcal{R}_2$ 得到，其方法僅是簡單的將 $\mathcal{R}_2$ 上的每個序對的分量對調即可 (且反過來亦可)。

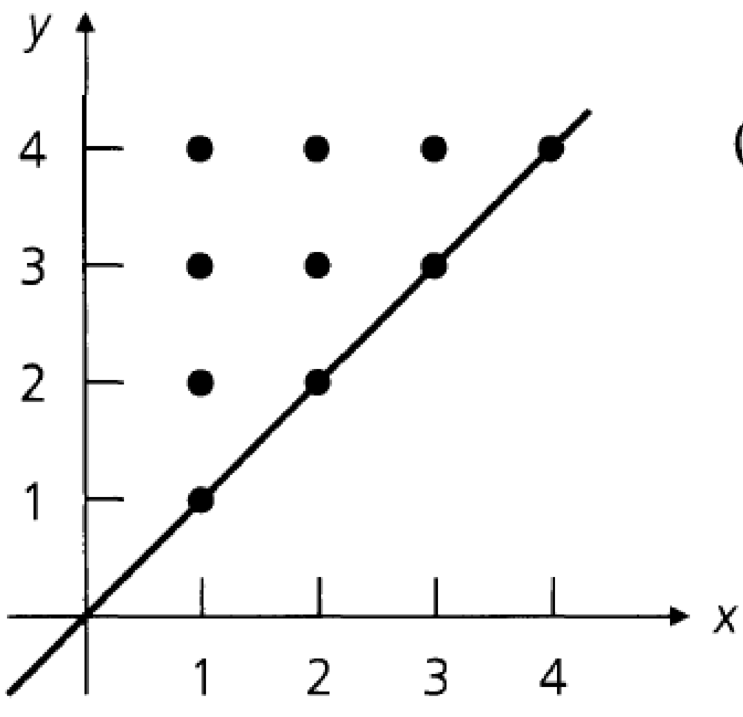
E.g. $A=\{2, 3, 4\}$，$B=\{4, 5\}$

Relations from A to B $\{(2, 4), (3, 4), (4, 4)\}$

Relations from B to A $\{(4, 2), (4, 3), (4, 4)\}$

EXAMPLE 5.5

With $A = \mathbf{Z}^+$, we may define a relation $\mathscr{R}$ on set A as $\{(x, y) | x \le y\}$.



$(7, 7), (7, 11) \in \mathscr{R}$, but $(8, 2) \notin \mathscr{R}$.

$(7, 11) \in \mathscr{R}$ can also be denoted by $7 \mathscr{R} 11$;

$(8, 2) \notin \mathscr{R}$ becomes $\boxed{\phantom{8 \not\mathscr{R} 2}}$

infix notation for a relation.

One observation

For any set A, $A \times \emptyset = \emptyset$.

(If $A \times \emptyset \neq \emptyset$, let $(a, b) \in A \times \emptyset$.

Then $a \in A$ and $b \in \emptyset$. Impossible!)

EXAMPLE 5.6

Real-Life Examples of Relations

Student and Grades: {(Alice, 80), (Bob, 75), (Charlie, 90)}.

Temperature and Time: {(8 am, 20°C), (12 pm, 25°C), (6 pm, 18°C)}.

A person and his/her FB friends in the class:
{(小明, 雅惠), (小明, 志豪), (雅婷, 怡君), (怡君, 雅婷), (志豪, 心怡) …..}.

What do I get for $120 in McDonalds?
{(安格斯牛肉堡, $114), (嫩煎雞腿堡, $114), (大麥克, $80), (凱撒辣脆雞沙拉, $104), …..}

THEOREM 5.1

對任意集合 A，B，$C \subseteq \mathscr{U}$：

a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Proof: (a)

For any $a, b \in \mathscr{U}$, $(a, b) \in A \times (B \cap C)$

$\Leftrightarrow a \in A \wedge b \in (B \cap C)$

$\Leftrightarrow a \in A, b \in B, b \in C$

$\Leftrightarrow (a, b) \in (A \times B) \wedge (a, b) \in (A \times C)$

$\Leftrightarrow (a, b) \in (A \times B) \cap (A \times C)$

THEOREM 5.1

對任意集合 A，B，$C \subseteq \mathcal{U}$：

a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$ 建議練習

d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$ 作業

Proof: (b)

5.2 關係的性質

EXAMPLE 5.7

1. Let $n \in \mathbf{Z}^+$. For $x, y \in \mathbf{Z}$, the *modulo n relation* $\mathscr{R}$ is defined by $x \mathscr{R} y$ if $x - y$ is a multiple of n.

Define $\mathscr{R}$ to be the binary relation on **Z**, such that $x \mathscr{R} y$ if

$$x \equiv y \pmod{n}$$

E.g. With $n = 7$, $9 \mathscr{R} 2$, $-3 \mathscr{R} 11$, $(14, 0) \in \mathscr{R}$,

but $3 \not\mathscr{R} 7$ (that is, 3 is *not* related to 7).

EXAMPLE 5.7 cont.

2. Define $\mathscr{R}$ to be the binary relation on $\mathscr{P}(\mathscr{U})$, such that $A \mathscr{R} B$ if $A \cap C = B \cap C$.

E.g. universe $\mathscr{U} = \{1, 2, 3, 4, 5, 6, 7\}$

$$C = \{1, 2, 3, 6\}$$

* Then the sets $\{1, 2, 4, 5\}$ and $\{1, 2, 5, 7\}$ are related since $\{1, 2, 4, 5\} \cap C = \{1, 2\} = \{1, 2, 5, 7\} \cap C$.

* $X = \{4, 5\}$ and $Y = \{7\}$ are related because $X \cap C = \emptyset = Y \cap C$.

* $S = \{1, 2, 3, 4, 5\}$ and $T = \{1, 2, 3, 6, 7\}$ are *not* related $S \not\mathscr{R} T$ — since $S \cap C = \{1, 2, 3\} \neq \{1, 2, 3, 6\} = T \cap C$.

EXAMPLE 5.8

(1) The relation $\mathcal{R}$ on $\{1, 2, 3, \ldots\}$ where $a\mathcal{R}b$ means $a \mid b$.

(2) The relation $\mathcal{R}$ on $\mathbf{Z}$ where $a\mathcal{R}b$ means $a \neq b$.

(3) The relation $\mathcal{R}$ on $\mathbf{Z}$ where $a\mathcal{R}b$ means $|a-b| \leq 1$.

Definition 5.3

A relation $\mathscr{R}$ on a set A is called *reflexive* if for all $x \in A$, $(x, x) \in \mathscr{R}$.

一個集合 A 上的關係被稱是**反身的** (reflexive)，若對所有 $x \in A$，$(x, x) \in \mathscr{R}$。

EXAMPLE 5.8 cont.

(1) The relation $\mathcal{R}$ on $\{1, 2, 3, \ldots\}$ where $a \mathcal{R} b$ means $a \mid b$.

(2) The relation $\mathcal{R}$ on $\mathbf{Z}$ where $a \mathcal{R} b$ means $a \neq b$.

(3) The relation $\mathcal{R}$ on $\mathbf{Z}$ where $a \mathcal{R} b$ means $|a - b| \leq 1$.

EXAMPLE 5.9

For $A = \{1, 2, 3, 4\}$, a relation $\mathscr{R} \subseteq A \times A$ will be reflexive if and only if $\{(1, 1), (2, 2), (3, 3), (4, 4) \subseteq \mathscr{R}$

Consequently, $\mathscr{R}_1 = \{(1, 1), (2, 2), (3, 3)\}$ is not a reflexive relation on A, whereas $\mathscr{R}_2 = \{(x, y) | x, y \in A, x \leq y\}$ is reflexive on A.

EXAMPLE 5.10

Given a finite set A with $|A| = n$, we have $|A \times A| = n^2$, so there are 2^{n^2} relations on A.

How many of these are reflexive?

If $A = \{a_1, a_2, \ldots, a_n\}$, a relation $\mathcal{R}$ on A is reflexive if and only if $\{(a_i, a_i) | 1 \leq i \leq n\} \subseteq \mathcal{R}$. Considering the other $n^2 - n$ ordered pairs in $A \times A$ [those of the form (a_i, a_j), where $i \neq j$ for $1 \leq i, j \leq n$] as we construct a reflexive relation $\mathcal{R}$ on A, we either include or exclude each of these ordered pairs, so by the rule of product there are $2^{(n^2-n)}$ reflexive relations on A.

Definition 5.4

Relation $\mathscr{R}$ on set A is called *symmetric* if $(x, y) \in \mathscr{R} \Rightarrow (y, x) \in \mathscr{R}$, for all $x, y \in A$.

集合 A 上的關係 $\mathscr{R}$ 被稱為是**對稱的** (symmetric)，若 $(x, y) \in \mathscr{R} \Rightarrow (y, x) \in \mathscr{R}$，對所有 x，$y \in A$。

EXAMPLE 5.11

以 $A=\{1, 2, 3\}$，我們有：

a) $\mathscr{R}_1=\{(1, 2), (2, 1), (1, 3), (3, 1)\}$，是 A 上一個對稱的但非反身的關係；
symmetric, but not reflexive

b) $\mathscr{R}_2=\{(1, 1), (2, 2), (3, 3), (2, 3)\}$，是 A 上一個反身的但非對稱的關係；
reflexive, but not symmetric

c) $\mathscr{R}_3=\{(1, 1), (2, 2), (3, 3)\}$ 及 $\mathscr{R}_4=\{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$，是 A 上兩個既反身且對稱的關係； both reflexive and symmetric

d) $\mathscr{R}_5=\{(1, 1), (2, 3), (3, 3)\}$ 是 A 上一個既不反身也不對稱的關係。
neither reflexive nor symmetric

EXAMPLE 5.12

To count the symmetric relations on $A = \{a_1, a_2, \ldots, a_n\}$, we write $A \times A$ as $A_1 \cup A_2$, where $A_1 = \{(a_i, a_i) | 1 \le i \le n\}$ and $A_2 = \{(a_i, a_j) | 1 \le i, j \le n, i \ne j\}$, so that every ordered pair in $A \times A$ is in exactly one of A_1, A_2. For A_2,

$$|A_2| = |A \times A| - |A_1| = n^2 - n = n(n-1), \text{ an even integer.}$$

The set A_2 contains $(1/2)(n^2 - n)$ subsets S_{ij} of the form $\{(a_i, a_j), (a_j, a_i)\}$ where $1 \le i < j \le n$.

EXAMPLE 5.12 Cont.

In constructing a symmetric relation $\mathcal{R}$ on A, for each ordered pair in A_1 we have our usual choice of exclusion or inclusion.

For each of the $(1/2)(n^2 - n)$ subsets S_{ij} $(1 \leq i < j \leq n)$ taken from A_2 we have the same two choices.

So by the rule of product there are

$$2^n \cdot 2^{(1/2)(n^2-n)} = 2^{(1/2)(n^2+n)} \text{ symmetric relations on } A.$$

EXAMPLE 5.12 Cont.

In counting those relations on A that are both reflexive and symmetric, we have only one choice for each ordered pair in A_1.

So we have $2^{(1/2)(n^2-n)}$ relations on A that are both reflexive and symmetric.

Definition 5.5

For a set A, a relation $\mathcal{R}$ on A is called *transitive* if, for all $x, y, z \in A$, $(x, y), (y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$.

(So if x "is related to" y, and y "is related to" z, we want x "related to" z, with y playing the role of "intermediary.")

對集合 A，A 上的關係 $\mathcal{R}$ 被稱是**遞移的** (transitive)，若對所有 x，y，$z \in A$ (x, y)，$(y, z) \in \mathcal{R} \Rightarrow (x, z) \in \mathcal{R}$。(所以若 x 和 y 有關係且 y 和 z 有關係，我們要 x 和 z 有關係，以 y 扮演中間媒介的角色。)

EXAMPLE 5.13

$x, y \in \mathbf{Z}$ $n \in \mathbf{Z}^+$

$\boxed{x \equiv y \pmod{n}}$

modulo n relation $\mathcal{R}$

defined by $x \mathcal{R} y$ if $x - y$ is a multiple of n.

---

relation $\mathcal{R}$ on the set $\mathbf{Z}$

$a \mathcal{R} b$ if $a \leq b$

Both relations are transitive.

EXAMPLE 5.14

Define the relation $\mathcal{R}$ on the set $\mathbf{Z}^+$ by

$$a \, \mathcal{R} \, b \text{ if } a \text{ divides } b$$

$a|b$

that is, $b = ca$ for some $c \in \mathbf{Z}^+$.

Now if $x \, \mathcal{R} \, y$ and $y \, \mathcal{R} \, z$, do we have $x \, \mathcal{R} \, z$?

$$x \, \mathcal{R} \, y \Rightarrow y = sx \quad \text{for some } s \in \mathbf{Z}^+$$

$$y \, \mathcal{R} \, z \Rightarrow z = ty \text{ where } t \in \mathbf{Z}^+$$

Consequently, $z = ty = t(sx) = (ts)x \quad$ for $ts \in \mathbf{Z}^+$

so $x \, \mathcal{R} \, z$ and $\mathcal{R}$ is transitive.

In addition, $\mathcal{R}$ is reflexive, but not symmetric.

EXAMPLE 5.15

If $A = \{1, 2, 3, 4\}$, then $\mathscr{R}_1 = \{(1, 1), (2, 3), (3, 4), (2, 4)\}$ is a transitive relation on A,

whereas $\mathscr{R}_2 = \{(1, 3), (3, 2)\}$ is not transitive

because $(1, 3), (3, 2) \in \mathscr{R}_2$ but $(1, 2) \notin \mathscr{R}_2$.

Note:

there is no known general formula for the total number of transitive relations on a finite set.

Definition 5.6

Given a relation $\mathscr{R}$ on a set A, $\mathscr{R}$ is called *antisymmetric* if for all $a, b \in A$, $(a \mathscr{R} b$ and $b \mathscr{R} a) \Rightarrow a = b$.

給集合 A 上的一個關係 $\mathscr{R}$，$\mathscr{R}$ 被稱為**反對稱的** (antisymmetric)，若對所有 a，$b \in A$，$(a \mathscr{R} b$ 且 $b \mathscr{R} a) \Rightarrow a=b$。(僅有一個方法我們可同時有 a 和 b 有關係及 b 和 a 有關係，此方法是 a 和 b 為 A 上的相同元素。)

EXAMPLE 5.16

For a given universal set $\mathscr{U}$, a relation $\mathscr{R}$ defined on $\mathscr{P}(\mathscr{U})$ is such that $(A, B) \in \mathscr{R}$ if and only if $A \subseteq$ B, where $A, B \subseteq \mathscr{U}$.

Therefore, $\mathscr{R}$ is the subset relation from Chapter 3.
If $A \mathscr{R} B$ and $B \mathscr{R} A$, then we have $A \subseteq B$ and $B \subseteq A$, which gives us $A = B$.

Hence, this relation is antisymmetric, reflexive, and transitive but not symmetric.

EXAMPLE 5.17

For $A = \{1, 2, 3\}$, the relation $\mathcal{R}$ on A given by

$\mathcal{R} = \{(1, 2), (2, 1), (2, 3)\}$ is not symmetric because $(3, 2) \notin \mathcal{R}$,

and it is not antisymmetric because $(1, 2), (2, 1) \in \mathcal{R}$ but $1 \neq 2$.

relation $\mathcal{R}_1 = \{(1, 1), (2, 2)\}$ is both symmetric and antisymmetric.

Definition 5.7

偏序

A relation $\mathcal{R}$ on a nonempty set A is called a *partial ordering* or a *partial-order relation* if $\mathcal{R}$ is reflexive, antisymmetric, and transitive.

We often use $\preceq$ to denote a partial ordering, and called $(A, \preceq)$ a *partially ordered set* or a *poset*.

Definition 5.8

全序

A relation $\mathcal{R}$ on a set A is called a *total order,* if $\mathcal{R}$ is partial order and for any a, b in A, either $a\mathcal{R}b$ or $b\mathcal{R}a$.

EXAMPLE 5.19

The relation $\mathscr{R}$ defined on the set of $\mathbf{Z}^+$, such that $(x, y) \in \mathscr{R}$, if $x \mid y$, is a partial order relation.

Reflexive: since for all $x \in \mathbf{Z}^+$, $x \mid x$. Thus, $(x, x) \in \mathscr{R}$.

Antisymmetric: since for all $x, y \in \mathbf{Z}^+$, if $x \mid y$ and $y \mid x$, then $x = y$. Thus, $\mathscr{R}$ is antisymmetric.

Transitive: since for all $x, y, z \in \mathbf{Z}^+$, if $x \mid y$ and $y \mid z$, then $x \mid z$. Thus, $\mathscr{R}$ is transitive.

But, the relation $\mathscr{R}$ is not a total order relation because for example we have neither $3 \nmid 7$ nor $7 \nmid 3$.

EXAMPLE 5.20

Define $A \mathscr{R} B$ to be "set A is a subset of or is equal to set B"
Then $\mathscr{R}$ is a partial order on $\{ \{\}, \{1\}, \{2\}, \{1,2\} \}$.

$\{\} \mathscr{R} \{\}$
$\{1\} \mathscr{R} \{1\}$
$\{2\} \mathscr{R} \{2\}$
$\{1,2\} \mathscr{R} \{1,2\}$

Reflexive

Subset is also antisymmetric.

$\{\} \mathscr{R} \{1\}$
$\{\} \mathscr{R} \{2\}$
$\{1\} \mathscr{R} \{1,2\}$
$\{2\} \mathscr{R} \{1,2\}$
$\{\} \mathscr{R} \{1,2\}$

Transitive

But neither
$\{1\} \mathscr{R} \{2\}$ nor $\{2\} \mathscr{R} \{1\}$, so $\mathscr{R}$ is not a total order on
$\{ \{\}, \{1\}, \{2\}, \{1,2\} \}$

Definition 5.9

等價

An *equivalence relation* $\mathcal{R}$ on a set A is a relation that is $\boxed{\text{reflexive}}$, $\boxed{\text{symmetric,}}$ and $\boxed{\text{transitive.}}$

EXAMPLE 5.21

The following are all equivalence relations:

- "equal to" on the set of real numbers.
- "similar to" on the set of all triangles.
- "congruence modulo *n*" on the integers.

EXAMPLE 5.22

If $A = \{1, 2, 3\}$, then

$\mathscr{R}_1 = \{(1, 1), (2, 2), (3, 3)\}$,

$\mathscr{R}_2 = \{(1, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$,

$\mathscr{R}_3 = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3)\}$, and

$\mathscr{R}_4 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\} = A \times A$

are all equivalence relations on A.

5.3 函數：容易的及一對一

Definition 5.10

For nonempty sets A, B, a *function*, or *mapping*, *f from A to B*, denoted $f: A \to B$, is a relation from A to B in which every element of A appears exactly once as the first component of an ordered pair in the relation.

對非空集合 A，B，一個**函數** (function)，或**映射** (mapping)，f 由 A 到 B，被表為 $f: A \to B$，是一個由 A 到 B 的關係，其中 A 的每個元素恰出現一次做為關係中序對的第一個分量。

EXAMPLE 5.23

For $A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$,

$f = \{(1, w), (2, x), (3, x)\}$ is a function,

and consequently a relation, from A to B.

$\mathscr{R}_1 = \{(1, w), (2, x)\}$ and $\mathscr{R}_2 = \{(1, w), (2, w), (2, x), (3, z)\}$
are relations, but not functions, from A to B. (Why?)

Notations

$f: A \to B$ We often write $f(a) = b$

$a \in A$ $b \in B$

(a, b) is an ordered pair in the function f

b is called *the image* of a under f, whereas a is a *preimage* of b.

$(a, b), (a, c) \in f$ implies $b = c$.

Definition 5.11

For the function $f: A \to B$,

A is called the *domain* of f and B the *codomain* of f.

The subset of B consisting of those elements that appear as second components in the ordered pairs of f is called the *range* of f and is also denoted by $f(A)$

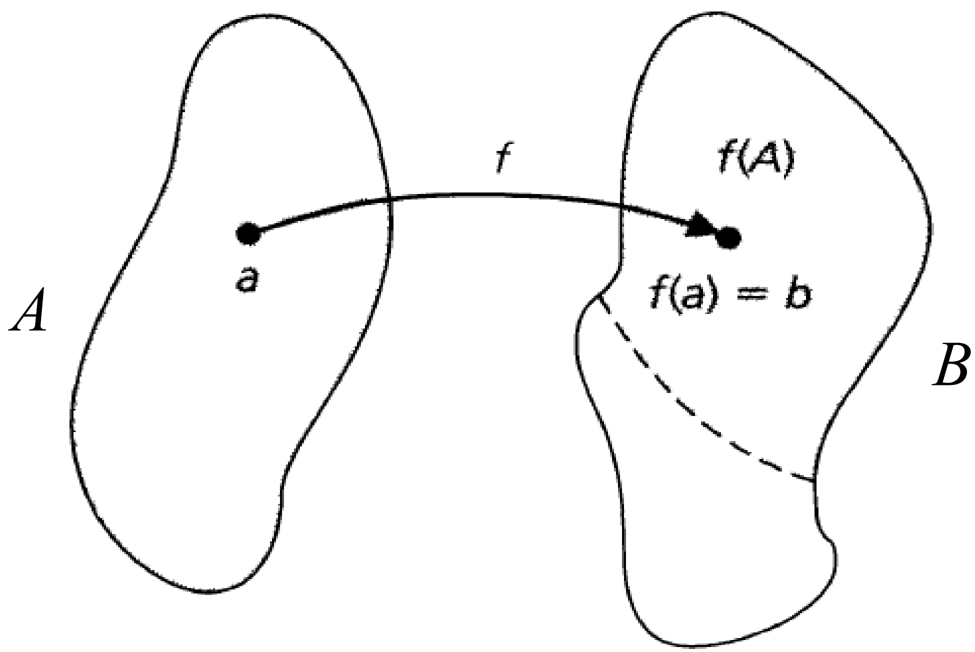
對函數 $f: A \to B$，A 被稱為 f 的**定義域** (domain) 且 B 被稱為 f 的**對應域** (codomain)。由 f 的所有序對中第二個分量所組成的 B 之子集合被稱為 f 的**值域** (range) 亦被表為 $f(A)$，因為它是 (A 的所有元素) 在 f 之下的像所成的集合。

EXAMPLE 5.23 Cont.

$A = \{1, 2, 3\}$ and $B = \{w, x, y, z\}$, $f = \{(1, w), (2, x), (3, x)\}$

the domain of $f = \{1, 2, 3\}$, the codomain of $f = \{w, x, y, z\}$,

and the range of $f = f(A) = \{w, x\}$.



EXAMPLE 5.24

$A=\{1, 2, 3\}$ 且 $B=\{w, x, y, z\}$

In Example 5.22, there are $2^{12} = 4096$ relations from A to B.

How many functions are there from A to B?

Let A, B be nonempty sets with $|A| = m$, $|B| = n$.

$A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$,

$f: A \rightarrow B$ can be described by

$$\{(a_1, x_1), (a_2, x_2), (a_3, x_3), \ldots, (a_m, x_m)\}.$$

We can select any of the n elements of B for each $\boldsymbol{x}_i$.

So, there are $n^m = |B|^{|A|}$ functions from A to B.

In Example 5.22, there are $\boxed{}$ functions from A to B.

EXAMPLE 5.25

Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$.

The function $f = \{(1, 1), (2, 3), (3, 4)\}$

is a one-to-one function from A to B;

$$g = \{(1, 1), (2, 3), (3, 3)\}$$

is a function from A to B,

but it fails to be one-to-one because $g(2) = g(3)$ but $2 \neq 3$.

EXAMPLE 5.26

Consider the function $f: \mathbf{R} \to \mathbf{R}$ where $f(x) = 3x + 7$ for all $x \in \mathbf{R}$.

Then for all $x_1, x_2 \in \mathbf{R}$, we find that

$$f(x_1) = f(x_2) \Rightarrow 3x_1 + 7 = 3x_2 + 7 \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2,$$

so the given function f is one-to-one.

On the other hand, suppose that $g: \mathbf{R} \to \mathbf{R}$ is the function defined by $g(x) = x^4 - x$ for each real number x. Then

$$g(0) = (0)^4 - 0 = 0 \quad \text{and} \quad g(1) = (1)^4 - (1) = 1 - 1 = 0.$$

Consequently, g is *not* one-to-one, since $g(0) = g(1)$ but $0 \neq 1$

EXAMPLE 5.27

$A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$.

there are 2^{15} relations from A to B and

there are 5^3 functions from A to B.

How many of these functions are one-to-one?

With $A = \{a_1, a_2, a_3, \ldots, a_m\}$ and $B = \{b_1, b_2, b_3, \ldots, b_n\}$, and $m \leq n$, a one-to-one function $f: A \rightarrow B$ has the form

$$\{(a_1, x_1), (a_2, x_2), (a_3, x_3), \ldots, (a_m, x_m)\}.$$

There are n choices for x_1 (that is, any element of B),

$n - 1$ choices for x_2 ,

$n - 2$ choices for x_3 , and so on,

EXAMPLE 5.27 Cont.

$n-(m-1) = n-m+1$ choices for x_m.

Thus,

$$n(n-1)(n-2)\cdots(n-m+1) = \frac{n!}{(n-m)!} = P(n, m)$$

$$= P(|B|, |A|).$$

$A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$.

Answer: there are $\boxed{}$ one-to-one functions $f: A \to B$.

5.4 映成函數

Definition 5.13

A function $f: A \to B$ is called *onto*, or *surjective*, if $f(A) = B$ — that is, if for all $b \in B$ there is at least one $a \in A$ with $f(a) = b$.

函數 f：$A \to B$ 被稱為**映成** (onto) 或**蓋射** (surjective)，若 $f(A)=B$，即若對所有 $b \in B$，至少存在一個 $a \in A$ 使得 $f(a)=b$。

EXAMPLE 5.29

If $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$, then

$f_1 = \{(1, z), (2, y), (3, x), (4, y)\}$ and

$f_2 = \{(1, x), (2, x), (3, y), (4, z)\}$

are both functions from A *onto* B.

However, the function $g = \{(1, x), (2, x), (3, y), (4, y)\}$ is not *onto*.

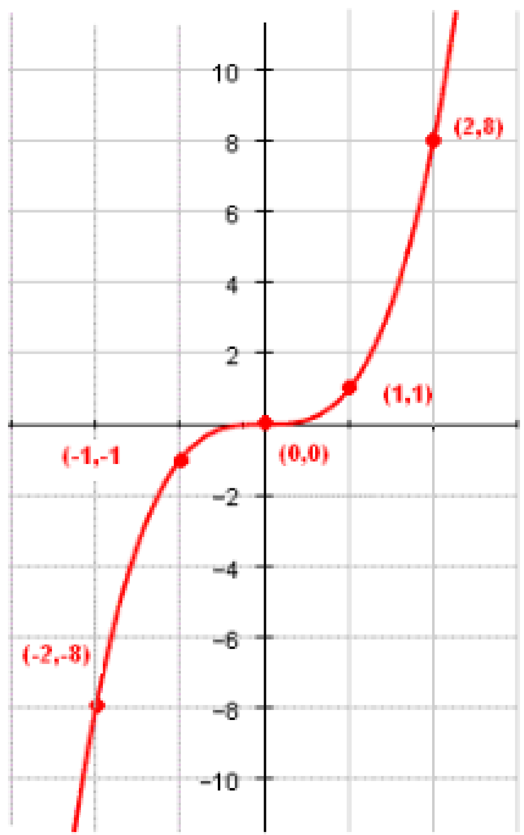
If A, B are finite sets, then for an onto function $f: A \to B$ to possibly exist we must have

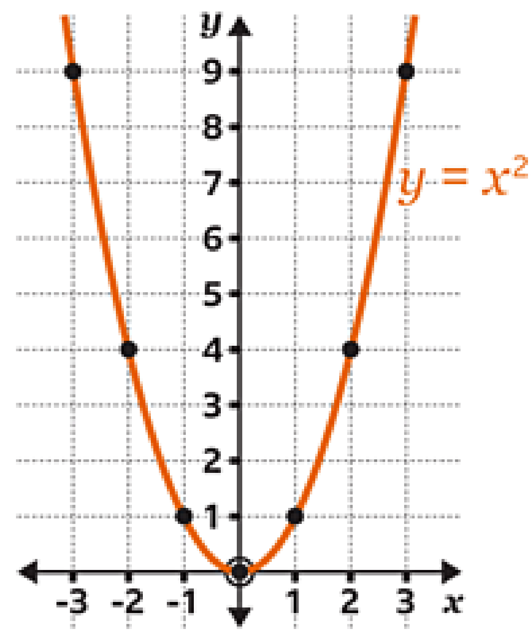
EXAMPLE 5.30

函數 f：$\mathbf{R} \to \mathbf{R}$ 被定義為 $f(x)=x^3$ 是一個映成函數。

函數 g：$\mathbf{R} \to \mathbf{R}$，其中 $g(x)=x^2$ 對每個實數 x，不是一個映成函數。

函數 h：$\mathbf{R} \to [0, +\infty)$ 定義為 $h(x)=x^2$ 是一個映成函數。





5.5 函數合成及反函數

Definition 5.14

If $f: A \to B$, then f is said to be *bijective*, or to be a *one-to-one correspondence*, if f is $\boxed{\text{both one-to-one and onto.}}$

若 $f: A \to B$，則 f 被稱為**單蓋射** (bijective)，或為**一對一對應** (one-to-one correspondence)，若 f 同時為一對一且映成。

EXAMPLE 5.33

If $A = \{1, 2, 3, 4\}$ and $B = \{w, x, y, z\}$,

then $f = \{(1, w), (2, x), (3, y), (4, z)\}$ is a one-to-one correspondence

from A (on)to B, and $g = \{(w, 1), (x, 2), (y, 3), (z, 4)\}$

is a one-to-one correspondence from B (on)to A.

Definition 5.15

The function $1_A: A \to A$, defined by $1_A(a) = a$ for all $a \in A$, is called the *identity function* for A.

函數 1_A：$A \to A$，定義為 $1_A(a) = a$ 對所有 $a \in A$，被稱為 A 的**恒等函數** (identity function)。

Definition 5.16

If $f, g: A \to B$, we say that f and g are *equal* and write $f = g$, if $f(a) = g(a)$ for all $a \in A$.

若 f，g：$A \to B$，我們稱 f 和 g 為**相等** (equal) 且記 $f = g$，若 $f(a) = g(a)$ 對所有 $a \in A$。

EXAMPLE 5.34

Let $f: \mathbf{Z} \to \mathbf{Z}$, $g: \mathbf{Z} \to \mathbf{Q}$ where $f(x) = x = g(x)$, for all $x \in \mathbf{Z}$.

Then f, g share the common domain $\mathbf{Z}$, have the same range $\mathbf{Z}$, and act the same on every element of $\mathbf{Z}$.

But, $f \neq g$! Here f is a one-to-one correspondence,

whereas g is one-to-one but not onto;

EXAMPLE 5.35

Consider the functions $f, g\colon \mathbf{R} \to \mathbf{Z}$ defined as follows:

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbf{Z} \\ \lfloor x \rfloor + 1, & \text{if } x \in \mathbf{R} - \mathbf{Z} \end{cases} \qquad g(x) = \lceil x \rceil, \text{ for all } x \in \mathbf{R}$$

If $x \in \mathbf{Z}$, then $f(x) = x = \lceil x \rceil = g(x)$.

For $x \in \mathbf{R} - \mathbf{Z}$, write $x = n + r$ where $n \in \mathbf{Z}$ and $0 < r < 1$.

Then $f(x) = \lfloor x \rfloor + 1 = n + 1 = \lceil x \rceil = g(x)$.

Consequently, even though f, g are defined by *different* formulas, they are the *same* function — because they have the same domain and codomain and $f(x) = g(x)$ for all x in the domain $\mathbf{R}$.

Definition 5.17

If $f: A \to B$ and $g: B \to C$, we define the *composite function*, which is denoted $g \circ f: A \to C$, by $(g \circ f)(a) = g(f(a))$, for each $a \in A$.

若 $f: A \to B$ 且 $g: B \to C$，我們定義**合成函數** (composite function)，其被表為 $g \circ f: A \to C$，為 $(g \circ f)(a) = g(f(a))$，對每個 $a \in A$。

EXAMPLE 5.36

Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, and $C = \{w, x, y, z\}$ with $f: A \rightarrow B$ and $g: B \rightarrow C$ given by

$f = \{(1, a), (2, a), (3, b), (4, c)\}$ and

$g = \{(a, x), (b, y), (c, z)\}$.

For each element of A we find:

$(g \circ f)(1) = g(f(1)) = g(a) = x$

$(g \circ f)(2) = g(f(2)) = g(a) = x$

$(g \circ f)(3) = g(f(3)) = g(b) = y$

$(g \circ f)(4) = g(f(4)) = g(c) = z$

So $g \circ f = \boxed{}$

Note: The composition $f \circ g$ is *not* defined.

EXAMPLE 5.37

Let $f: \mathbf{R} \to \mathbf{R}$, $g: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^2$, $g(x) = x + 5$.

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 5,$$

$$(f \circ g)(x) = f(g(x)) = f(x + 5) = (x + 5)^2 = x^2 + 10x + 25.$$

$$(g \circ f)(1) = 6 \neq 36 = (f \circ g)(1)$$

the composition of functions is not a commutative operation.

EXAMPLE 5. 38

Let $f, g, h: \mathbf{R} \to \mathbf{R}$,

where $f(x) = x^2$, $g(x) = x + 5$, and $h(x) = \sqrt{x^2 + 2}$.

Then $((h \circ g) \circ f)(x)$

$$= (h \circ g)(f(x))$$
$$= (h \circ g)(x^2)$$
$$= h(g(x^2))$$
$$= h(x^2 + 5)$$
$$= \sqrt{(x^2 + 5)^2 + 2}$$
$$= \sqrt{x^4 + 10x^2 + 27}.$$

$(h \circ (g \circ f))(x)$

$$= h((g \circ f)(x))$$
$$= h(g(f(x)))$$
$$= h(g(x^2))$$
$$= h(x^2 + 5)$$
$$= \sqrt{(x^2 + 5)^2 + 2}$$
$$= \sqrt{x^4 + 10x^2 + 27}$$

$(h \circ g) \circ f = h \circ (g \circ f)$ is true in general.

Definition 5.18

If $f: A \to B$, then f is said to be *invertible* if there is a function $g: B \to A$ such that

$$g \circ f = 1_A \text{ and } f \circ g = 1_B.$$

若 f：$A \to B$，則 f 被稱為**可逆** (invertible) 若存在一個函數 g：$B \to A$ 滿足 $g \circ f = 1_A$ 及 $f \circ g = 1_B$。

THEOREM 5.4

If a function $f: A \to B$ is invertible and a function $g: B \to A$ satisfies $g \circ f = 1_A$ and $f \circ g = 1_B$, then this function g is unique.

Proof:

If g is not unique, then there is another function $h: B \to A$ with $h \circ f = 1_A$ and $f \circ h = 1_B$.

Consequently,

$$h = h \circ 1_B = h \circ (f \circ g) = (h \circ f) \circ g = 1_A \circ g = g.$$

THEOREM 5.5

A function $f: A \to B$ is invertible if and only if it is one-to-one and onto.

Proof: Assuming that $f: A \to B$ is invertible, we have a unique function $g: B \to A$ with $g \circ f = 1_A$, $f \circ g = 1_B$.

If $a_1, a_2 \in A$ with $f(a_1) = f(a_2)$, then $g(f(a_1)) = g(f(a_2))$, or $(g \circ f)(a_1) = (g \circ f)(a_2)$.

With $g \circ f = 1_A$ it follows that $a_1 = a_2$, so f is one-to-one.

THEOREM 5.5 A function $f: A \to B$ is invertible if and only if it is one-to-one and onto.

Proof: (cont.)

For the onto property, let $b \in B$.

Then $g(b) \in A$, so we can talk about $f(g(b))$.

Since $f \circ g = 1_B$, we have $b = 1_B(b) = (f \circ g)(b) = f(g(b))$, so f is onto.

Conversely, suppose $f: A \to B$ is bijective.

Since f is onto, for each $b \in B$ there is an $a \in A$ with $f(a) = b$.

Consequently, we define the function $g: B \to A$ by $g(b) = a$, where $f(a) = b$.

THEOREM 5.5 A function $f: A \to B$ is invertible if and only if it is one-to-one and onto.

Proof: (cont.)

This definition yields a unique function.
The only problem that could arise is if

$$g(b) = a_1 \neq a_2 = g(b) \quad \text{because } f(a_1) = b = f(a_2).$$

However, this situation cannot arise because f is one-to-one.

Our definition of g is such that $g \circ f = 1_A$ and $f \circ g = 1_B$, so we find that f is invertible, with $g = f^{-1}$.