Chapter 3

集合論

Set Theory

3.1 集合和子集合

3.2 集合運算及集合論定律

(3.3 計數及范恩圖)

3.1　集合和子集合

Set and Subset

A ***well-defined*** (unordered) collection of objects.

Should avoid: e.g. “the set of outstanding people”, where outstanding is very subjective

These objects are called ***elements*** and are said to be ***members*** of the set.

Notations

我們使用大寫字母，如 A，B，C，⋯ 來表示集合，且以小寫字母來表示元素。對集合 A，我們寫 $x \in A$ 若 x 為 A 的元素；$y \notin A$ 表示 y 不是 A 的一份子。

elements

EXAMPLE 3.1

一個集合可以把其所有元素列在集合括弧內之方式呈現。

For example, if A is the set consisting of the first five positive integers, then we write $A = \{1, 2, 3, 4, 5\}$.

Here $2 \in A$ but $6 \notin A$.

EXAMPLE 3.1 Cont.

Another standard notation

$$A = \{x \mid x \text{ is an integer and } x \leq 5\}.$$

在集合括弧內的垂直線 | 被讀做 “滿足”。符號 $\{x|\cdots\}$ 被讀做 “滿足…的所有 x 的集合”。| 之後的性質幫助我們決定被描述的集合之元素。

“ the set of all x such that …..”

Beware! The notation $\{x \mid 1 \leq x \leq 5\}$
is not an adequate description of the set A.

Universe of a Set

宇集

A *universe* is usually denoted by $\mathcal{U}$.

We select only elements from $\mathcal{U}$ to form our sets.

* if $\mathcal{U}$ denotes the set of all integers or the set of all positive integers, then $\{x \mid 1 \le x \le 5\}$ adequately describes A.
* If $\mathcal{U}$ is the set of all real numbers, then $\{x \mid 1 \le x \le 5\}$ would contain all of the real numbers between 1 and 5 inclusive
* if $\mathcal{U}$ consists of only even integers, then the only members of $\{x \mid 1 \le x \le 5\}$ would be 2 and 4.

EXAMPLE 3.2

For $\mathscr{U} = \{1, 2, 3, \ldots\}$, the set of positive integers, we consider the following sets. At the same time we introduce various notations one may use to describe such sets.

a) $A = \{1, 4, 9, \ldots, 64, 81\}$

$$= \{x^2 | x \in \mathscr{U}, x^2 < 100\} = \{x^2 | x \in \mathscr{U} \wedge x^2 < 100\}$$

b) $B = \{1, 4, 9, 16\}$

$$= \{y^2 | y \in \mathscr{U}, y^2 < 20\} = \{y^2 | y \in \mathscr{U}, y^2 < 23\}$$

$$= \{y^2 | y \in \mathscr{U} \wedge y^2 \leq 16\}.$$

c) $C = \{2, 4, 6, 8, \ldots\} = \{2k | k \in \mathscr{U}\}.$

EXAMPLE 3.2 Cont.

a) $A = \{1, 4, 9, \ldots, 64, 81\}$

b) $B = \{1, 4, 9, 16\}$

c) $C = \{2, 4, 6, 8, \ldots\}$

有限 無限

Sets A and B are examples of *finite* sets, whereas C is an *infinite* set.

For any finite set A, $|A|$ denotes the number of elements in A and is referred to as the *cardinality* or *size*

基數 大小

$|A| = 9$ $|B| = 4$

Definition 3.1

If C, D are sets from a universe $\mathscr{U}$, we say that C is a *subset* of D and write $C \subseteq D$, or $D \supseteq C$, if every element of C is an element of D. If, in addition, D contains an element that is not in C, then C is called a *proper subset* of D, and this is denoted by $C \subset D$ or $D \supset C$.

真子集

a) $A = \{1, 4, 9, \ldots, 64, 81\}$

b) $B = \{1, 4, 9, 16\}$

c) $C = \{2, 4, 6, 8, \ldots\}$

$B \subseteq A$ and also $B \subset A$

but $A \not\subseteq C$

Note that for all sets C, D from a universe $\mathcal{U}$, if $C \subseteq D$, then

$$\forall x\ [x \in C \Rightarrow x \in D],$$

and if $\forall x\ [x \in C \Rightarrow x \in D]$, then $C \subseteq D$.

Also, we find that for all subsets C, D of $\mathcal{U}$,

$$C \subset D \Rightarrow C \subseteq D,$$

and when C, D are finite,

$$C \subseteq D \Rightarrow |C| \leq |D|, \quad \text{and} \quad C \subset D \Rightarrow |C| < |D|.$$

However, for $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $C = \{1, 2\}$, and $D = \{1, 2\}$,

we see that C is a subset of D (that is, $C \subseteq D$),

but it is not a proper subset of D (or, $C \not\subset D$).

So, in general, we do *not* find that $C \subseteq D \Rightarrow C \subset D$.

EXAMPLE 3.3

For the universe $\mathcal{U} = \{1, 2, 3, 4, 5\}$, consider the set $A = \{1, 2\}$.

If $B = \{x | x^2 \in \mathcal{U}\}$, then the members of B are 1, 2.

Here A and B contain the same elements

sets A and B are *equal*.

it is also true here that $A \subseteq B$ and $B \subseteq A$

Definition 3.2

For a given universe $\mathscr{U}$, the sets C and D (taken from $\mathscr{U}$) are said to be *equal*, and we write $C = D$, when

相等的

$$C \subseteq D \text{ and } D \subseteq C.$$

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neither order nor repetition is relevant for a general set.

for example, that $\{1, 2, 3\} = \{3, 1, 2\} = \{2, 2, 1, 3\} = \{1, 2, 1, 3, 1\}$.

Negations of Subset and Set Equality

subsets $A \subseteq B \Leftrightarrow \forall x[x \in A \Rightarrow x \in B]$

$$A \not\subseteq B \Leftrightarrow \neg\forall x[x \in A \Rightarrow x \in B]$$

$$\Leftrightarrow \exists x \neg[\neg(x \in A) \vee x \in B)]$$

$$\Leftrightarrow$$

there is at least one element x in the universe

where x is a member of A but x is not a member of B.

Negations of Subset and Set Equality

set equality $C = D \Leftrightarrow (C \subseteq D) \wedge (D \subseteq C)$

$$C \neq D \Leftrightarrow \neg(C \subseteq D \wedge D \subseteq C)$$

$$\Leftrightarrow \boxed{}$$

(1) there exists at least one element x in $\mathcal{U}$ where $x \in C$ but $x \notin D$

or (2) there exists at least one element y in $\mathcal{U}$ where $y \in D$ and $y \notin C$

or perhaps both (1) and (2) occur.

EXAMPLE 3.4

Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, x, y, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ (where x, y are the 24th, 25th lowercase letters of the alphabet and do not represent anything else).

Then $|\mathcal{U}| = 11$.

If $A = \{1, 2, 3, 4\}$, then $|A| = 4$ and here we have

i) $A \subseteq \mathcal{U}$; **ii)** $A \subset \mathcal{U}$; **iii)** $A \in \mathcal{U}$;

iv) $\{A\} \subseteq \mathcal{U}$; **v)** $\{A\} \subset \mathcal{U}$; but **vi)** $\{A\} \notin \mathcal{U}$.

EXAMPLE 3.4 Cont.

Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, x, y, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ (where x, y are the 24th, 25th lowercase letters of the alphabet and do not represent anything else).

Then $|\mathcal{U}| = 11$.

$A = \{1, 2, 3, 4\}$

Now let $B = \{5, 6, x, y, A\} = \{5, 6, x, y, \{1, 2, 3, 4\}\}$.

Then $|B| = 5$, *not* 8.

we find that

i) $A \in B$; **ii)** $\{A\} \subseteq B$; and **iii)** $\{A\} \subset B$.

But

iv) $\{A\} \notin B$; **v)** $A \nsubseteq B$ **vi)** $A \not\subset B$

THEOREM 3.1

Let A, B, $C \subseteq \mathcal{U}$.

a) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

b) If $A \subset B$ and $B \subseteq C$, then $A \subset C$.

c) If $A \subseteq B$ and $B \subset C$, then $A \subset C$.

d) If $A \subset B$ and $B \subset C$, then $A \subset C$. 建議練習

Proof: **a)** To prove that $A \subseteq C$, we need to verify that for all $x \in \mathcal{U}$, if $x \in A$ then $x \in C$.

We start with an element x from A.

Since $A \subseteq B$, $x \in A$ implies $x \in B$.

Then with $B \subseteq C$, $x \in B$ implies $x \in C$.

So $x \in A$ implies $x \in C$, and $A \subseteq C$.

THEOREM 3.1 Cont.

Let $A, B, C \subseteq \mathscr{U}$.

b) If $A \subset B$ and $B \subseteq C$, then $A \subset C$.

Proof: **b)** Since $A \subset B$, if $x \in A$ then $x \in B$.

With $B \subseteq C$, it then follows that $x \in C$, so $A \subseteq C$.

However, $A \subset B \Rightarrow$ there exists an element $b \in B$ such that $b \notin A$.

Because $B \subseteq C$, $b \in B \Rightarrow b \in C$.

Thus $A \subseteq C$ and there exists an element $b \in C$ with $b \notin A$,

so $A \subset C$.

Let $A, B, C \subseteq \mathcal{U}$.

c) If $A \subseteq B$ and $B \subset C$, then $A \subset C$.

THEOREM 3.1 Cont.

Proof:

EXAMPLE 3.5

Let $\mathcal{U} = \{1, 2, 3, 4, 5\}$ with $A = \{1, 2, 3\}$, $B = \{3, 4\}$, and $C = \{1, 2, 3, 4\}$.

Then the following subset relations hold:

a) $A \subseteq C$

b) $A \subset C$

c) $B \subset C$

d) $A \subseteq A$

e) $B \nsubseteq A$

f) $A \not\subset A$ (that is, A is not a proper subset of A)

In this example, the sets A and B are both subsets of C.

How many subsets C has in total?

於 Example 3.6 回應

Definition 3.3

零集　　空集合

The *null set*, or *empty set*, is the (unique) set containing no elements.

It is denoted by $\emptyset$ or $\{\ \}$.

We note that $|\emptyset| = 0$ but $\{0\} \neq \emptyset$.

Also, $\emptyset \neq \{\emptyset\}$ because $\{\emptyset\}$ is a set with one element, namely, the null set.

THEOREM 3.2

For any universe $\mathcal{U}$, let $A \subseteq \mathcal{U}$. Then $\emptyset \subseteq A$, and if $A \neq \emptyset$, then $\emptyset \subset A$.

Proof: If the first result is not true, then $\emptyset \nsubseteq A$,

so there is an element x from the universe with $x \in \emptyset$ but $x \notin A$.

But $x \in \emptyset$ is impossible.

So we reject the assumption $\emptyset \nsubseteq A$ and find that $\emptyset \subseteq A$.

In addition, if $A \neq \emptyset$, then there is an element $a \in A$ (and $a \notin \emptyset$),

so $\emptyset \subset A$.

EXAMPLE 3.6

determine the number of subsets of the set $C = \{1, 2, 3, 4\}$.

In constructing a subset of C, we have, for each member x of C, two distinct choices: Either include it in the subset or exclude it. Consequently, there are $2 \times 2 \times 2 \times 2$ choices, resulting in $2^4 = 16$ subsets of C.

These include the empty set $\emptyset$ and the set C itself.

The total number of subsets of C, 2^4, is also the sum

$$\binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}$$

empty set

the four *singleton* subsets

the six subsets of size 2.

the four subsets of size 3

C

Definition 3.4

ㄇㄧˋ

冪集合

If A is a set from universe $\mathscr{U}$, the *power* set of A, denoted $\boxed{\mathscr{P}(A)}$, is the collection (or set) of all subsets of A.

$\mathscr{P}(A)$ is the set of all subsets of A.

EXAMPLE 3.7

For the set C of Example 3.6 $C = \{1, 2, 3, 4\}$

$$\mathscr{P}(C) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, C\}.$$

If $|A| = n$, then $|P(A)| = 2^n$.

For any finite set A with $|A| = n \geq 0$, and for any $0 \leq k \leq n$, there are $C(n, k)$ subsets of size k.

Counting the subsets of A according to the number, k, of elements in a subset, we have the combinatorial identity

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n, \text{ for } n \geq 0$$

Set of Numbers

(a) $\mathbf{Z}$ = the set of integers = {0, 1, -1 ,2, -1, 3, -3,...}

(b) $\mathbf{N}$ = the set of natural numbers or whole numbers = {0, 1, 2, 3,...}

(c) $\mathbf{Z}^+$ = the set of positive integers = {1, 2, 3,...}

(d) $\mathbf{Q}$ = the set of rational numbers = $\{a/b \mid a, b \in \mathbf{Z}, b \neq 0\}$

(e) $\mathbf{Q}^+$ = the set of positive rational numbers

(f) $\mathbf{Q}^*$ = the set of nonzero rational numbers

(g) $\mathbf{R}$ = the set of real numbers

(h) $\mathbf{R}^+$ = the set of positive real numbers

(i) $\mathbf{R}^*$ = the set of nonzero real numbers

(j) $\mathbf{C}$ = the set of complex numbers = $\{x + yi \mid x, y \in \mathbf{R}, i^2 = -1\}$

(k) $\mathbf{C}^*$ = the set of nonzero complex numbers

(l) For any $n \in \mathbf{Z}^+$, $\mathbf{Z}_n = \{0, 1, 2, 3, \ldots, n\text{-}1\}$

Set of Numbers

(m) For real numbers a, b with $a < b$,

$$[a,b] = \{x \in \mathbf{R} | a \le x \le b\}$$ closed interval

$$(a,b) = \{x \in \mathbf{R} | a < x < b\}$$ open interval

$$[a,b) = \{x \in \mathbf{R} | a \le x < b\}$$

$$(a,b] = \boxed{\phantom{\{x \in \mathbf{R} | a < x \le b\}}}$$ half-open interval

Other Forms

$\mathbb{Z}$: Integers
$\mathbb{Q}$: Rational numbers
$\mathbb{R}$: Real numbers
$\mathbb{C}$: Complex numbers
$\mathbb{N}$: Natural numbers

3.2 集合運算及集合論定律

Set Operations and the Laws of Set Theory

The addition and multiplication of positive integers are said to be *closed binary operations* on $\mathbf{Z}^+$.

封閉性的二元運算

For example, when we compute $a + b$, for $a, b \in \mathbf{Z}^+$, there are two *operands*, namely, a and b.

運算元

Hence the operation is called *binary*.

And since $a + b \in \mathbf{Z}^+$ when $a, b \in \mathbf{Z}^+$,
we say that the binary operation of addition (on $\mathbf{Z}^+$) is *closed*.

The binary operation of (nonzero) division, however, is *not* closed for $\mathbf{Z}^+$.

Yet this operation is closed when we consider the set $\mathbf{Q}^+$ instead of the set $\mathbf{Z}^+$.

Binary Operations for Sets

Definition 3.5

For A, B, $\subseteq \mathscr{U}$ we define the following:

a) $A \cup B$ (the *union* of A and B) $= \{x | x \in A \vee x \in B\}$.

b) $A \cap B$ (the *intersection* of A and B) $= \{x | x \in A \wedge x \in B\}$.

c) $A \triangle B$ (the *symmetric difference* of A and B)

$$= \{x | (x \in A \vee x \in B) \wedge x \notin A \cap B\} = \{x | x \in A \cup B \wedge x \notin A \cap B\}.$$

聯集、交集、對稱差集

Note that if A, $B \subseteq \mathscr{U}$, then $A \cup B$, $A \cap B$, $A \triangle B \subseteq \mathscr{U}$.

Consequently, $\cup$, $\cap$, and $\triangle$ are closed binary operations on $\mathscr{P}(\mathscr{U})$

EXAMPLE 3.8

With $\mathcal{U} = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, and $C = \{7, 8, 9\}$, we have:

a) $A \cap B = \{3, 4, 5\}$

b) $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

c) $B \cap C = \{7\}$

d) $A \cap C = \emptyset$

e) $A \triangle B = \{1, 2, 6, 7\}$

f) $A \cup C = \{1, 2, 3, 4, 5, 7, 8, 9\}$

g) $A \triangle C = \{1, 2, 3, 4, 5, 7, 8, 9\}$

Motivated by parts (d), (f), and (g) of Example 3.8 we introduce the following general ideas.

Definition 3.6

互斥的

Let $S, T \subseteq \mathcal{U}$. The sets S and T are called *disjoint*, or *mutually disjoint*, when $S \cap T = \emptyset$.

Such as example 3.8 (d), sets A and C are mutually disjoint.

THEOREM 3.3

$$\qquad\qquad\qquad\qquad\quad p \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q$$

If $S, T \subseteq \mathcal{U}$, then $\boxed{S \text{ and } T \text{ are disjoint}}$ if and only if $\boxed{S \cup T = S \,\triangle\, T.}$

$$\Longleftrightarrow$$

若 S，$T \subseteq \mathcal{U}$，則 S 和 T 為互斥的若且唯若 $S \cup T = S \triangle T$。

Proof:

$$p \Rightarrow q$$

$$p \Leftarrow q$$

THEOREM 3.3

If $S, T \subseteq \mathcal{U}$, then $\boxed{S \text{ and } T \text{ are disjoint}}$ if and only if $\boxed{S \cup T = S \bigtriangleup T.}$

$\Rightarrow$

If S and T are disjoint, then $S \cup T \subseteq S \Delta T$

If S and T are disjoint, then $S \Delta T \subseteq S \cup T$

Which implies $S \Delta T = S \cup T$

$\Leftarrow$

If $S \Delta T = S \cup T$, but we assume that S and T are NOT disjoint, then this leads to a ***contradiction***!!

It implies S and T are disjoint.

THEOREM 3.3 If $S, T \subseteq \mathcal{U}$, then

$$S \text{ and } T \text{ are disjoint if and only if } S \cup T = S \triangle T.$$

Proof: We start with S, T disjoint. (To prove that $S \cup T = S \triangle T$ we use Definition 3.2. In particular, we shall provide two element arguments, one for each inclusion.) Consider each x in $\mathcal{U}$. If $x \in S \cup T$, then $x \in S$ or $x \in T$ (or perhaps both). But with S and T disjoint, $x \notin S \cap T$ so $x \in S \triangle T$. Consequently, because $x \in S \cup T$ implies $x \in S \triangle T$, we have $S \cup T \subseteq S \triangle T$. For the opposite inclusion, if $y \in S \triangle T$, then $y \in S$ or $y \in T$. (But $y \notin S \cap T$; we don't actually use this here.) So $y \in S \cup T$. Therefore $S \triangle T \subseteq S \cup T$. And now that we have $S \cup T \subseteq S \triangle T$ and $S \triangle T \subseteq S \cup T$, it follows from Definition 3.2 that $S \triangle T = S \cup T$.

We prove the converse by the method of proof by contradiction. To do so we consider any $S, T \subseteq \mathcal{U}$ and keep the hypothesis (that is, that $S \cup T = S \triangle T$) as is, but we assume the negation of the conclusion (that is, we assume that S and T are *not* disjoint). So if $S \cap T \neq \emptyset$, let $x \in S \cap T$. Then $x \in S$ and $x \in T$, so $x \in S \cup T$ and

$$x \in S \triangle T (= S \cup T).$$

But when $x \in S \cup T$ and $x \in S \cap T$, then

$$x \notin S \triangle T.$$

From this contradiction — namely, $x \in S \triangle T \wedge x \notin S \triangle T$ — we realize that our original assumption was incorrect. Consequently, we have S and T disjoint.

Unary Operation for Sets

Definition 3.7

餘集

For a set $A \subseteq \mathcal{U}$, the *complement* of A, denoted $\mathcal{U} - A$, or $\overline{A}$, is given by $\{x | x \in \mathcal{U} \wedge x \notin A\}$.

EXAMPLE 3.9

With $\mathcal{U} = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, and $C = \{7, 8, 9\}$, we have:

$$\overline{A} = \{6, 7, 8, 9, 10\}, \overline{B} = \{1, 2, 8, 9, 10\},$$

$$\overline{C} =$$

Definition 3.8

相對餘集

For A, $B \subseteq \mathcal{U}$, the *(relative) complement* of A in B, denoted $B - A$, is given by $\{x | x \in B \wedge x \notin A\}$.

EXAMPLE 3.10

With $\mathcal{U} = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 2, 3, 4, 5\}$, $B = \{3, 4, 5, 6, 7\}$, and $C = \{7, 8, 9\}$, we have:

a) $B - A = \{6, 7\}$ **b)** $A - B = \{1, 2\}$ **c)** $A - C = A$

d) $C - A = C$ **e)** $A - A = \emptyset$ **f)** $\mathcal{U} - A = \overline{A}$

EXAMPLE 3.11

For $\mathcal{U} = \mathbf{R}$, let $A = [1, 2]$ and $B = [1, 3)$. Then we find that

a) $A = \{x | 1 \leq x \leq 2\} \subseteq \{x | 1 \leq x < 3\} = B$

b) $A \cup B = \{x | 1 \leq x < 3\} = B$

c) $A \cap B = \{x | 1 \leq x \leq 2\} = A$

d) $\overline{B} = (-\infty, 1) \cup [3, +\infty) \subseteq (-\infty, 1) \cup (2, +\infty) = \overline{A}$

THEOREM 3.4

For any universe $\mathcal{U}$ and any sets A, $B \subseteq \mathcal{U}$, the following statements are equivalent:

a) $A \subseteq B$

b) $A \cup B = B$

c) $A \cap B = A$

d) $\overline{B} \subseteq \overline{A}$

Proof:

reasoning process

(a) $\Rightarrow$ (b),

(b) $\Rightarrow$ (c), 建議練習

(c) $\Rightarrow$ (d), and

(d) $\Rightarrow$ (a) 建議練習

THEOREM 3.4

For any universe $\mathcal{U}$ and any sets A, $B \subseteq \mathcal{U}$,

$$A \subseteq B \Longrightarrow A \cup B = B$$

Proof:

THEOREM 3.4

For any universe $\mathcal{U}$ and any sets A, $B \subseteq \mathcal{U}$,

$$A \cap B = A \Rightarrow \overline{B} \subseteq \overline{A}$$

Proof:

The Laws of Set Theory

對取自宇集 $\mathcal{U}$ 的任意集合 A，B 及 C

1) $\overline{\overline{A}} = A$ 雙餘集定律 Law of Double Complement

2) $\overline{A \cup B} = \overline{A} \cap \overline{B}$ DeMorgan 定律
   $\overline{A \cap B} = \overline{A} \cup \overline{B}$

3) $A \cup B = B \cup A$ 交換律 Commutative Laws
   $A \cap B = B \cap A$

4) $A \cup (B \cup C) = (A \cup B) \cup C$ 結合律 Associative Laws
   $A \cap (B \cap C) = (A \cap B) \cap C$

5) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ 分配律 Distributive Laws
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6)	$A \cup A = A$ $A \cap A = A$	冪等定律	Idempotent Laws
7)	$A \cup \emptyset = A$ $A \cap \mathcal{U} = A$	恒等律	Identity Laws
8)	$A \cup \overline{A} = \mathcal{U}$ $A \cap \overline{A} = \emptyset$	逆定律	Inverse Laws
9)	$A \cup \mathcal{U} = \mathcal{U}$ $A \cap \emptyset = \emptyset$	優控律	Domination Laws
10)	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	吸收律	Absorption Laws

Definition 3.9

duals

令 s 為處理兩個集合表示式相等的 (一般) 敘述。每一個此類表示式可能含一個或更多個的集合因子 (諸如 A，$\overline{A}$，B，$\overline{B}$ 等等)，一個或多個 $\emptyset$ 及 $\mathscr{U}$，及僅有集合運算符號 $\cap$ 及 $\cup$。s 的對偶表為 s^d，由 s 以 (1) 將每一個 $\emptyset$ 及 $\mathscr{U}$ (在 s) 分別取代為 $\mathscr{U}$ 及 $\emptyset$；及 (2) 每一個 $\cap$ 及 $\cup$ (在 s) 分別取代為 $\cup$ 及 $\cap$ 之法獲得。

THEOREM 3.5

對偶原理 (The Principle of Duality)。

Let s denote a theorem dealing with the equality of two set expressions (involving only the set operations $\cap$ and $\cup$). Then s^d, the dual of s, is also a theorem.

Examples: The Laws of Set Theory (2) ~ (10).

EXAMPLE 3.12

Simplify the expression $\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$.

$\overline{\overline{(A \cup B) \cap C} \cup \overline{B}}$	**Reasons**
$= \overline{\overline{((A \cup B) \cap C)}} \cap \overline{\overline{B}}$	DeMorgan's Law
$= ((A \cup B) \cap C) \cap B$	Law of Double Complement
$= (A \cup B) \cap (C \cap B)$	Associative Law of Intersection
$= (A \cup B) \cap (B \cap C)$	Commutative Law of Intersection
$= [(A \cup B) \cap B] \cap C$	Associative Law of Intersection
$= B \cap C$	Absorption Law

EXAMPLE 3.13

Express $\overline{A - B}$ in terms of $\cup$ and $\overline{}$.

From the definition of relative complement,

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}.$$

Therefore,

	Reasons
$\overline{A - B} = \overline{A \cap \overline{B}}$	
$= \overline{A} \cup \overline{\overline{B}}$	DeMorgan's Law
$= \overline{A} \cup B$	Law of Double Complement

EXAMPLE 3.14

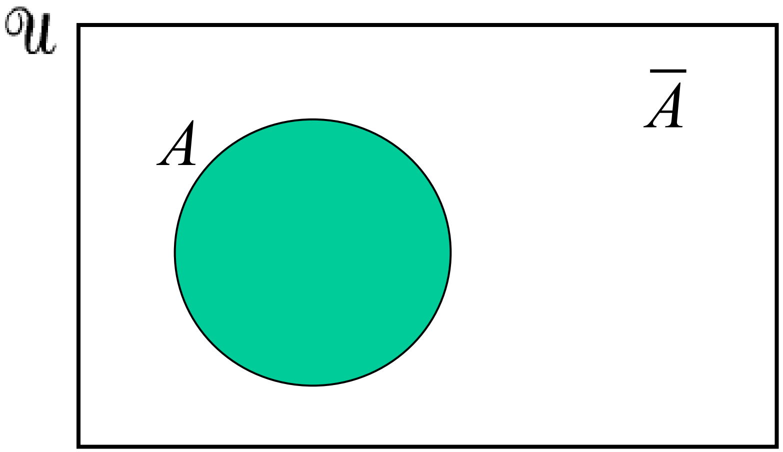
$$A \triangle B = \{x \mid x \in A \cup B \wedge x \notin A \cap B\}$$

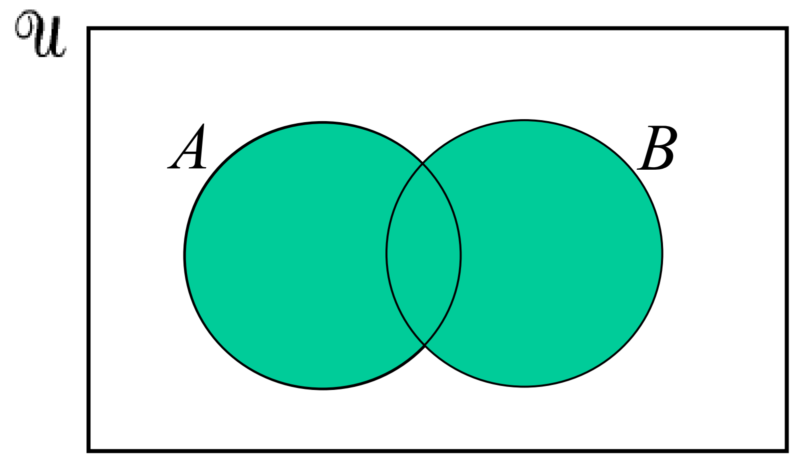
$$= (A \cup B) - (A \cap B) = (A \cup B) \cap \overline{(A \cap B)}, \text{ so}$$

	Reasons
$\overline{A \triangle B} = \overline{(A \cup B) \cap \overline{(A \cap B)}}$	
$= \overline{(A \cup B)} \cup \overline{\overline{(A \cap B)}}$	DeMorgan's Law
$= \overline{(A \cup B)} \cup (A \cap B)$	Law of Double Complement
$= (A \cap B) \cup \overline{(A \cup B)}$	Commutative Law of $\cup$
$= (A \cap B) \cup (\overline{A} \cap \overline{B})$	DeMorgan's Law
$= [(A \cap B) \cup \overline{A}] \cap [(A \cap B) \cup \overline{B}]$	Distributive Law of $\cup$ over $\cap$
$= [(A \cup \overline{A}) \cap (B \cup \overline{A})] \cap [(A \cup \overline{B}) \cap (B \cup \overline{B})]$	Distributive Law of $\cup$ over $\cap$
$= [\mathcal{U} \cap (B \cup \overline{A})] \cap [(A \cup \overline{B}) \cap \mathcal{U}]$	Inverse Law
$= (B \cup \overline{A}) \cap (A \cup \overline{B})$	Identity Law
$= (\overline{A} \cup B) \cap (A \cup \overline{B})$	Commutative Law of $\cup$
$= (\overline{A} \cup B) \cap \overline{(\overline{A} \cap B)}$	DeMorgan's Law
$= \overline{A} \triangle B$	
$= (A \cup \overline{B}) \cap (\overline{A} \cup B)$	Commutative Law of $\cap$
$= (A \cup \overline{B}) \cap \overline{(A \cap \overline{B})}$	DeMorgan's Law
$= A \triangle \overline{B}$	

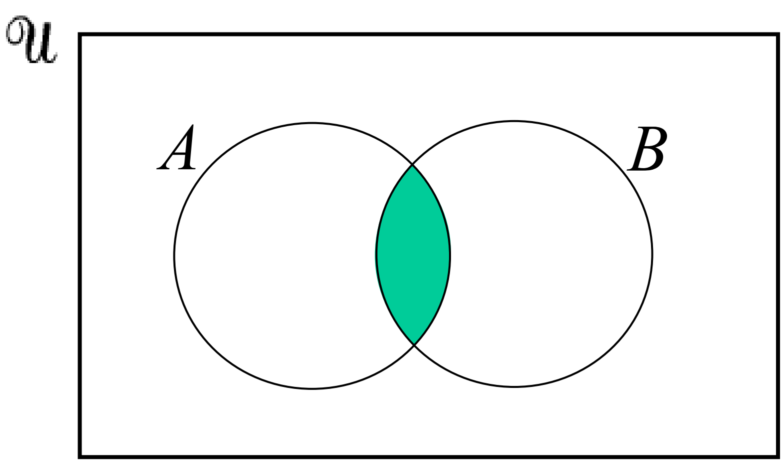
3.3 計數及范恩圖

Venn Diagram

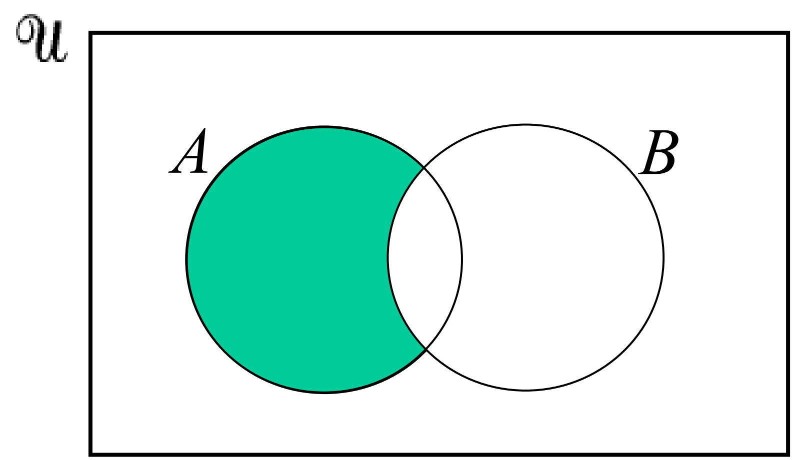




$A \cup B$

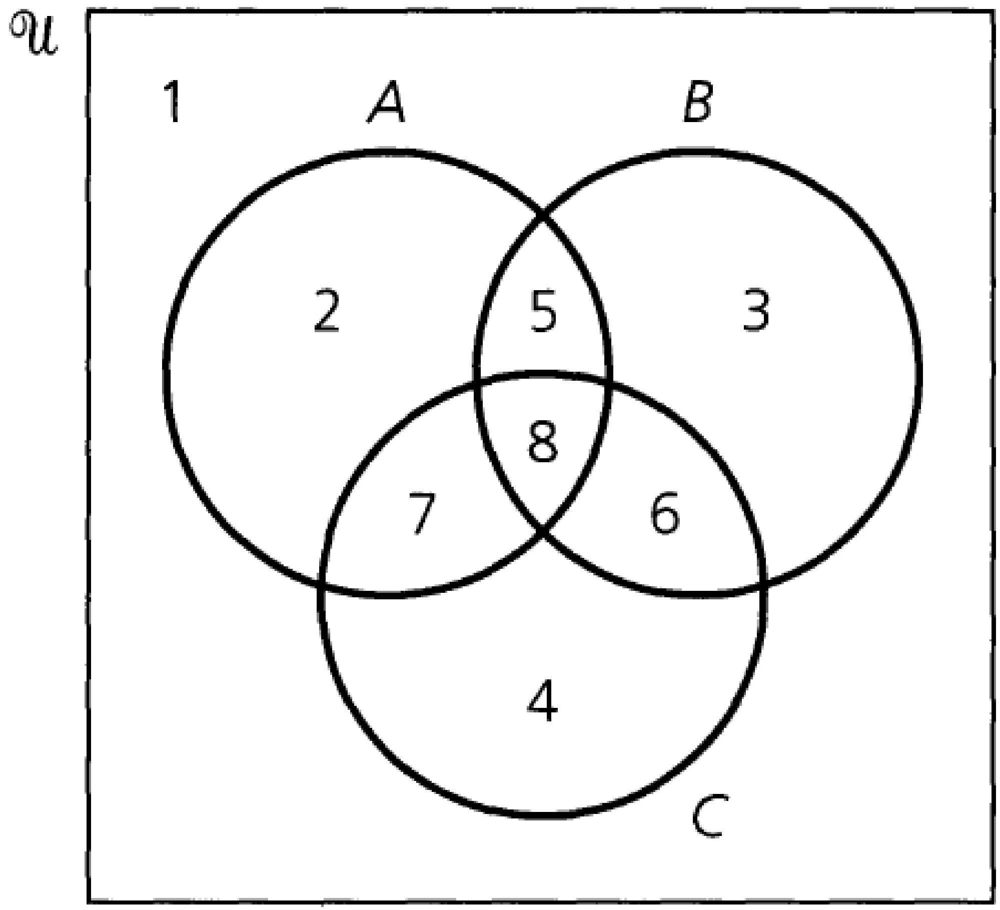


$A \cap B$

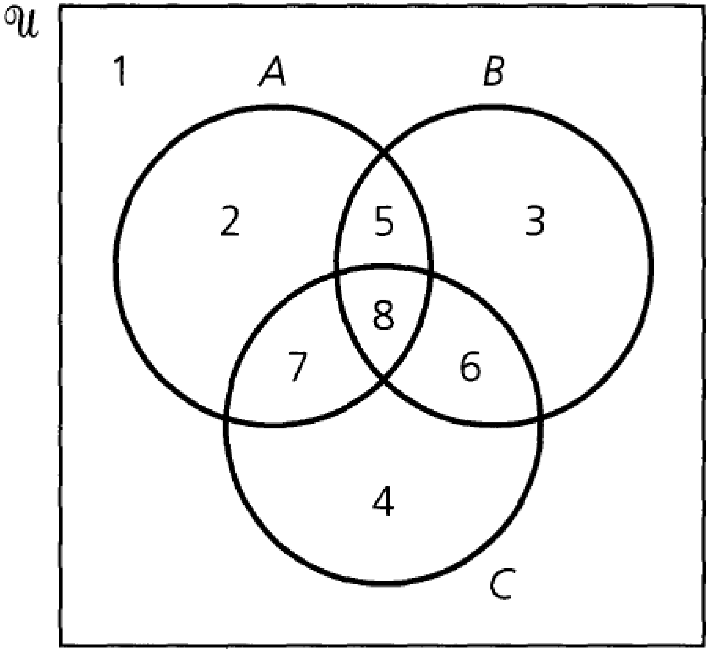


$A - B$

EXAMPLE 3.15



Show that $\overline{(A \cup B) \cap C} = (\overline{A} \cap \overline{B}) \cup \overline{C}$.



$$\overline{(A \cup B) \cap C} = (\overline{A} \cap \overline{B}) \cup \overline{C}$$

$A \cup B$ comprises regions $2, 3, 5, 6, 7, 8$

$(A \cup B) \cap C$ consists of regions $6, 7, 8$

$\overline{(A \cup B) \cap C}$ is made up of regions $1, 2, 3, 4, 5$

$\overline{A}$ comprises regions $1, 3, 4, 6$

$\overline{B}$ comprises regions $1, 2, 4, 7$

$\overline{A} \cap \overline{B}$ consists of regions $1, 4$

$\overline{C}$ comprises regions $1, 2, 3, 5$

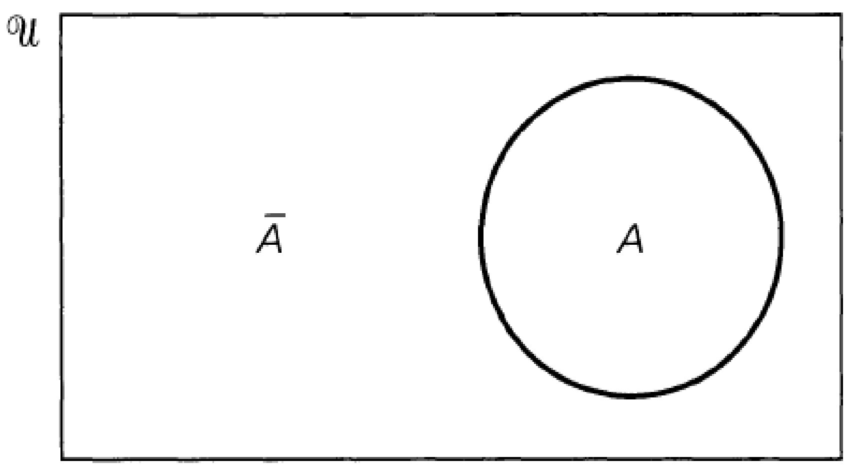
$(\overline{A} \cap \overline{B}) \cup \overline{C}$ is made up of regions $1, 2, 3, 4, 5$

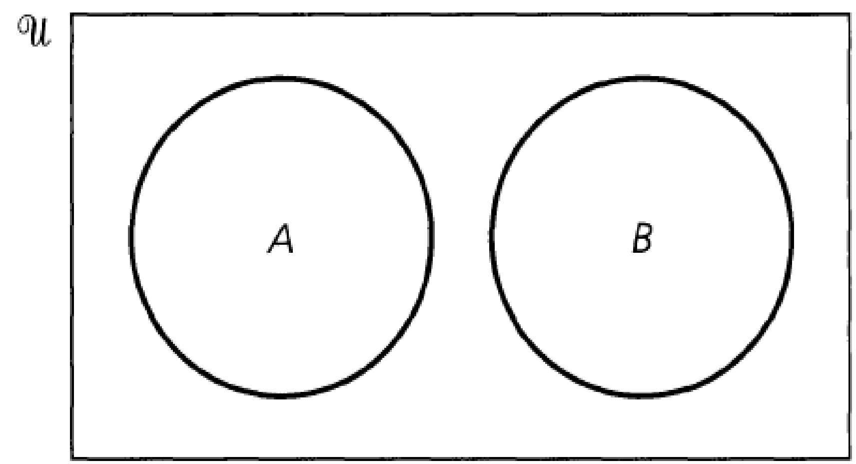
Counting

對有限宇集 $\mathcal{U}$ 上的集合 A，B，下面的范恩圖將助我們可以 $|\mathcal{U}|$，$|A|$，$|B|$ 及 $|A \cap B|$ 來得 $|\overline{A}|$ 及 $|A \cup B|$ 的計數公式。

$$A \cup \overline{A} = \mathcal{U} \text{ and } A \cap \overline{A} = \emptyset, \text{ so } |A| + |\overline{A}| = |\mathcal{U}|$$

$$A \cap B = \emptyset, \text{ so } |A \cup B| = |A| + |B|$$

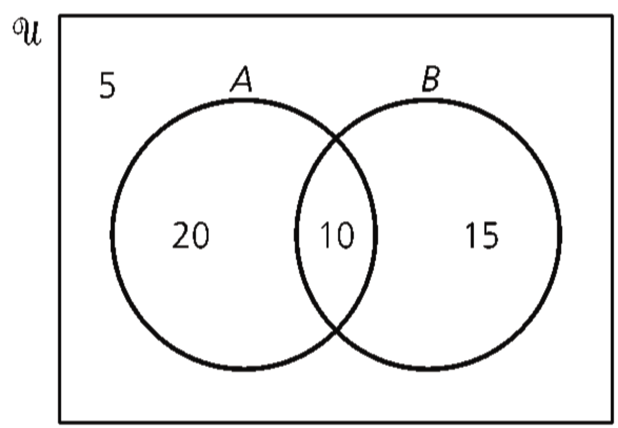




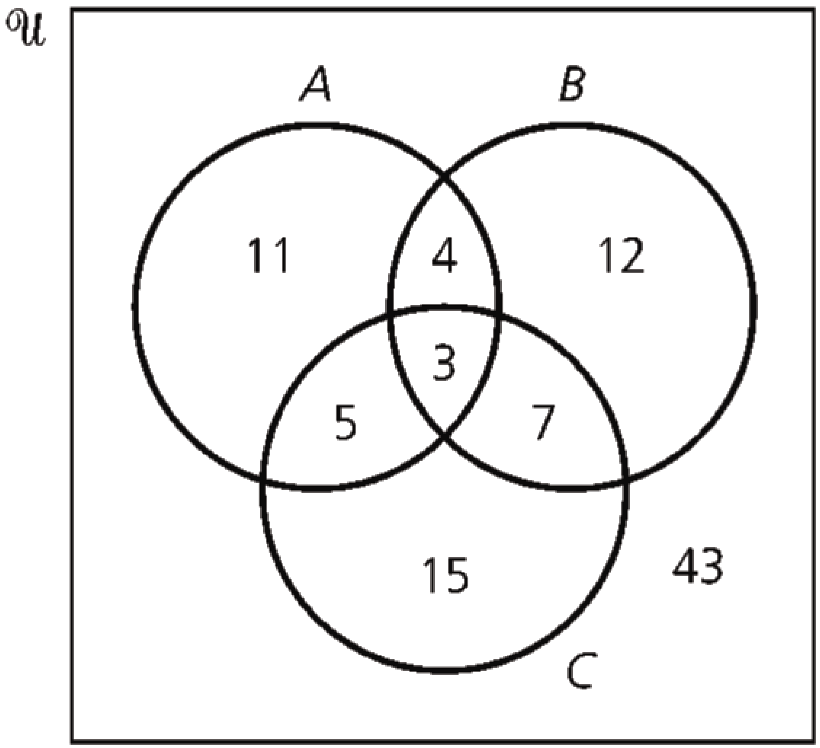
EXAMPLE 3.16

在一個 50 位大一新生的班級上，有 30 位學習 C++，25 位學習 Java，且有 10 位兩種語言均學習。試問有多少位大一新生學習 C++ 或 Java？

我們令 $\mathcal{U}$ 表 50 位大一新生的班級，A 為學習 C++ 的那些學生所成的子集合，且 B 為學習 Java 的那些學生所成的子集合。欲回答這個問題，我們需 $|A\cup B|$。圖 3.11 中，各區域的數目是由所給的資訊獲得：$|A|=30$，$|B|=25$，$|A\cap B|=10$。因此，$|A\cup B|=45\neq 55=30+25=|A|+|B|$，因為 $|A|+|B|$ 計數在 $A\cap B$ 的學生兩次。欲救濟這個重數，我們由 $|A|+|B|$ 減去 $|A\cap B|$ 而得正確公式：$|A\cup B|=|A|+|B|-|A\cap B|$。



EXAMPLE 3.17



$|A| = 23$, $|B| = 26$, $|C| = 30$,

$|A \cap B| = 7$, $|A \cap C| = 8$,

$|B \cap C| = 10$, and

$|A \cap B \cap C| = 3$

$|A \cup B \cup C| = ?$

$$
\begin{aligned}
|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\
&\quad + |A \cap B \cap C| \\
&= 23 + 26 + 30 - 7 - 8 - 10 + 3 = 57.
\end{aligned}
$$