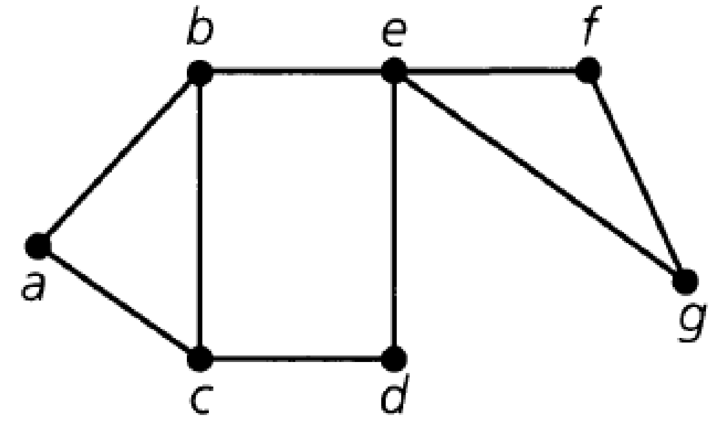
Graph Theory
In-class Exercises

1.



① b → e → f → g → e → b

② e → b → a → c → d → e

③ b → e → f → g → e → d

④ f → g → e → b → c → a

⑤ d → c → b → e → g → f → e → b

⑥ b → e → f → g → e → d → c → b

(A) a path

(B) a circuit that is not a cycle

(C) an open walk that is not a trail

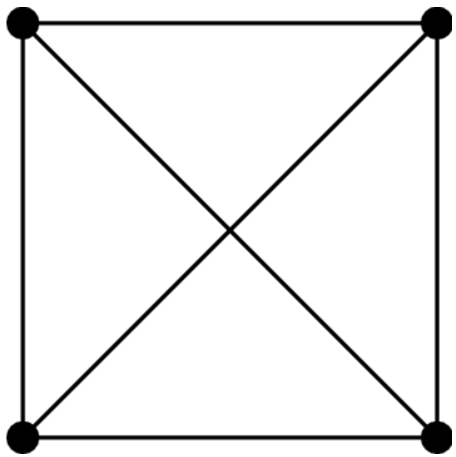
(D) a cycle

(E) a trail that is not a path

(F) a closed walk that is not a circuit

2. Calculate $\kappa(G)$ and $\lambda(G)$ for the following graphs.

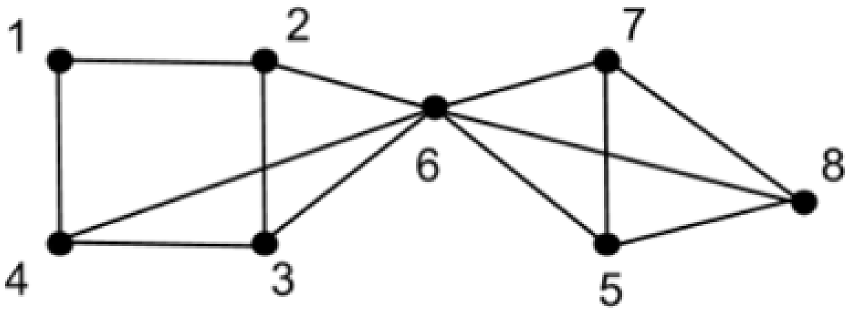
(a)



$\kappa(G) = 3$

$\lambda(G) = 3$

(b)



$\kappa(G) = 1$

$\lambda(G) = 2$

3. Let G be the undirected graph

a) How many connected subgraphs of G have 4 vertices and include a cycle?

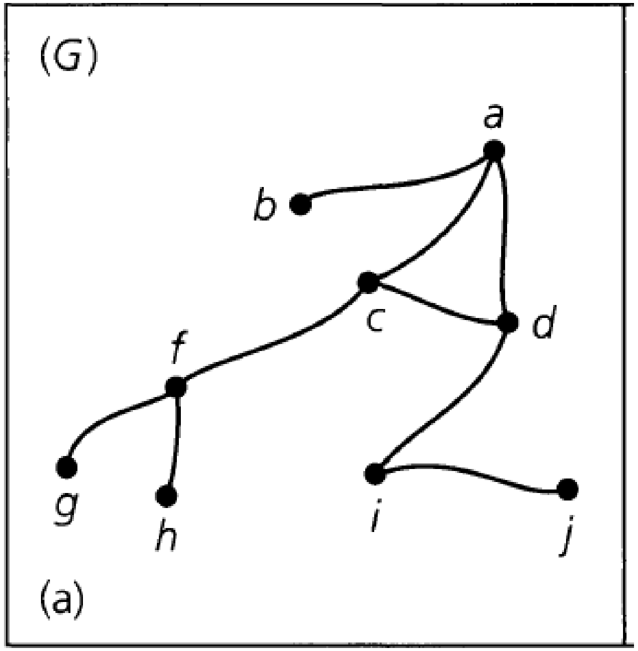
b) How many connected spanning subgraphs are there in G?

c) Draw the subgraph of G induced by the set of vertices

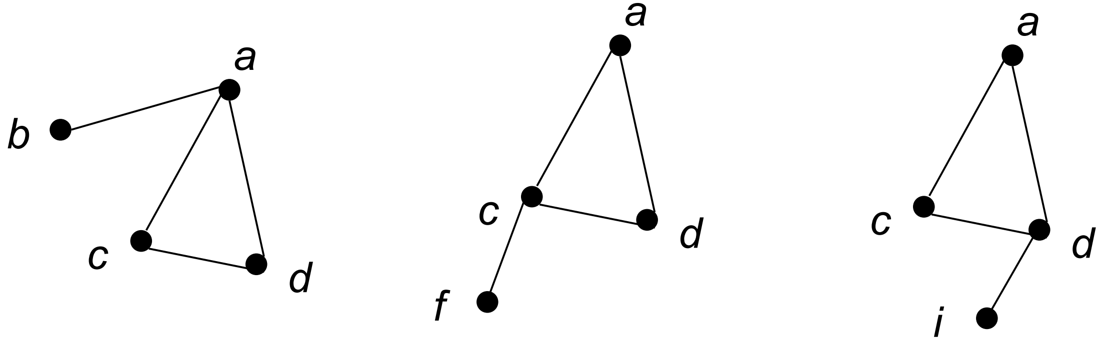
vertices $U = \{b, c, d, f, i, j\}$.

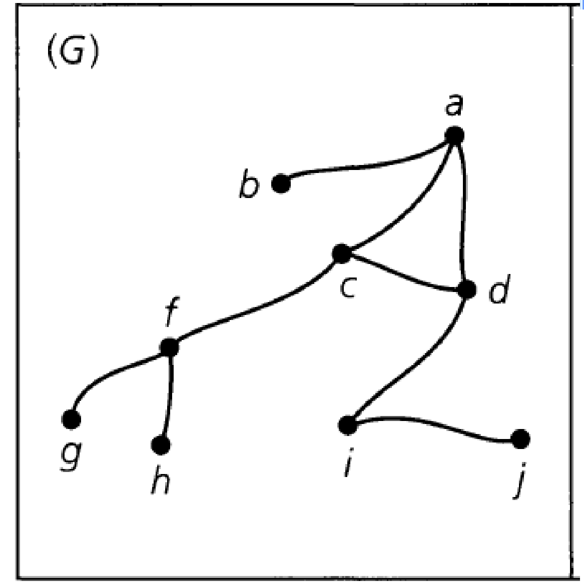
d) For the graph G, let the edge $e = \{c, f\}$.

Draw the subgraph $G - e$.

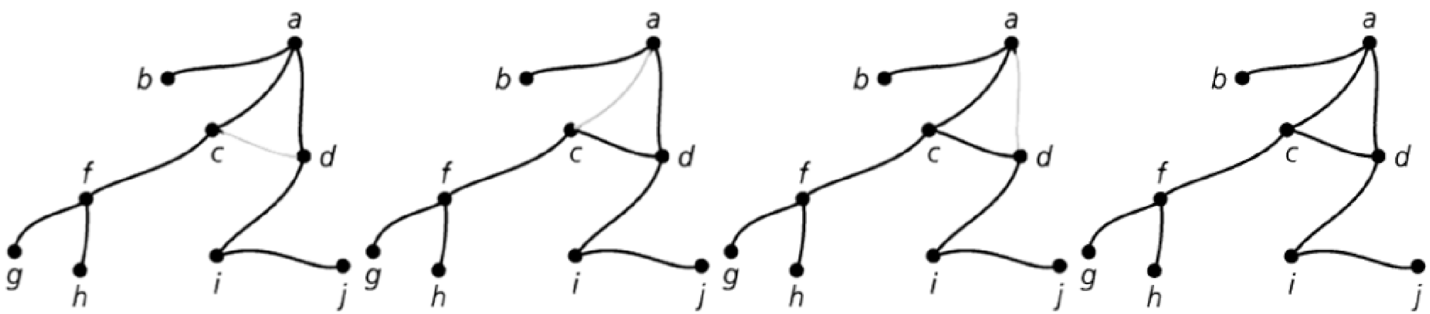


a) 3

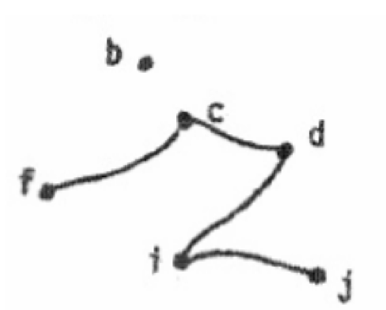




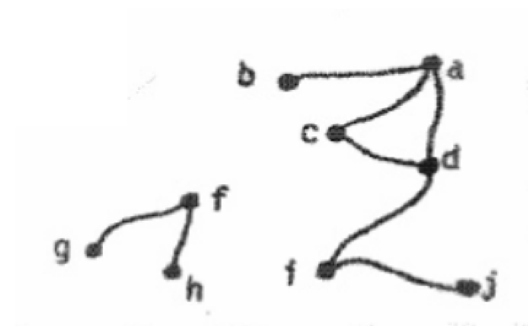
b) 4



c)



d)



4.

(a) K_n has $\underline{\quad \frac{n(n-1)}{2} \quad}$ edges and $\underline{\quad n \quad}$ vertices.

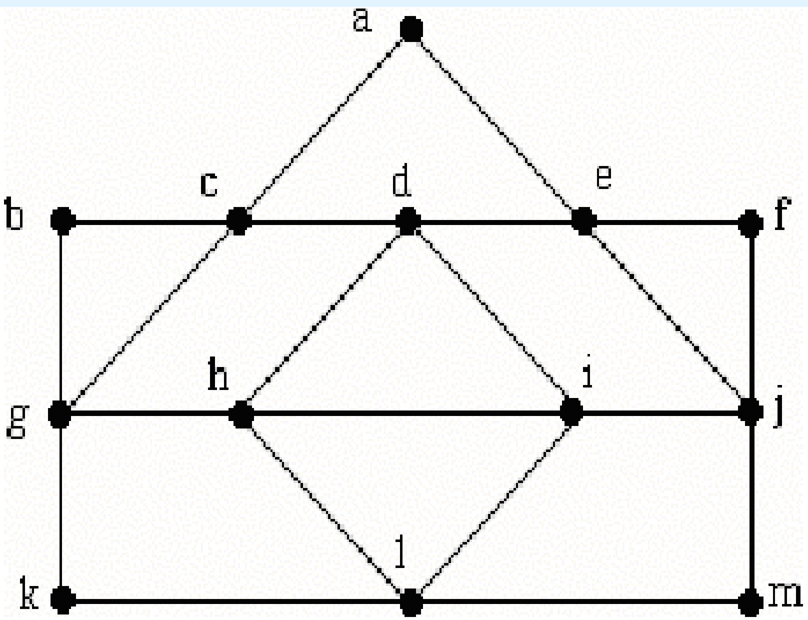
(b) $K_{m,n}$ has $\underline{\quad mn \quad}$ edges and $\underline{\quad m+n \quad}$ vertices.

(c) W_n has $\underline{\quad 2(n-1) \quad}$ edges and $\underline{\quad n \quad}$ vertices.

(d) Q_n has $\underline{\quad n \cdot 2^{n-1} \quad}$ edges and $\underline{\quad 2^n \quad}$ vertices.

(e) $G_{m,n}$ has $\underline{\quad 2mn-(m+n) \quad}$ edges and $\underline{\quad mn \quad}$ vertices.

5. Consider the graph here.



(a) Does it have an Euler circuit? Yes

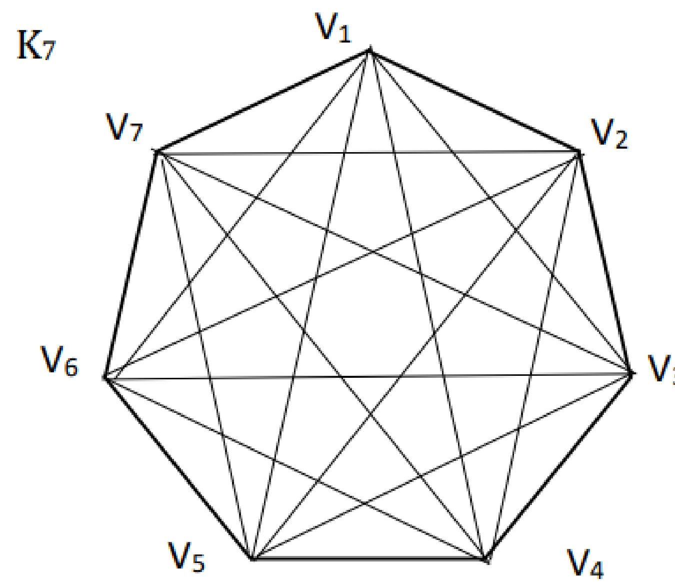
(b) Does it have an Euler trail? Yes

(c) Does it have a Hamilton cycle? No

(d) Does it have a Hamilton path? Yes

Graph Theory
Suggested Exercises

1. How many paths of length 4 are there in the complete graph K_7? (Remember that a path such as $v_1 \to v_2 \to v_3 \to v_4 \to v_5$ is considered to be the same as the path $v_5 \to v_4 \to v_3 \to v_2 \to v_1$.)



$$\binom{7}{5} \times \frac{5!}{2}$$

$$= \frac{7 \times 6 \times 5 \times 3 \times 2}{2 \times 2}$$

$$= 1260$$

e.g. $V_1 \to V_2 \to V_3 \to V_4 \to V_5$

Collect 5 vertices from 7 vertices. $\binom{7}{5}$ permutations of 5 vertices (i.e. 5!) .

Reversing path is considered to be the same, thus divides 2.

2. Determine $|V|$ for the following graphs or multigraphs G.

a) G has nine edges and all vertices have degree 3.

b) G is regular with 15 edges.

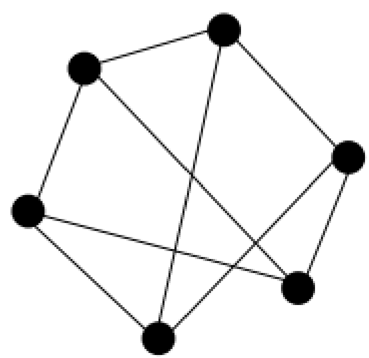
c) G has 10 edges with two vertices of degree 4 and all others of degree 3.

a) $|V| = n$

$3n = 2|E|$

$\quad = 2 \times 9$

$n = 6$



c) $|V| = n$

$2 \times 10 = 2 \times 4 + (n-2) \times 3$

$20 = 8 + 3n - 6$

$n = 6$

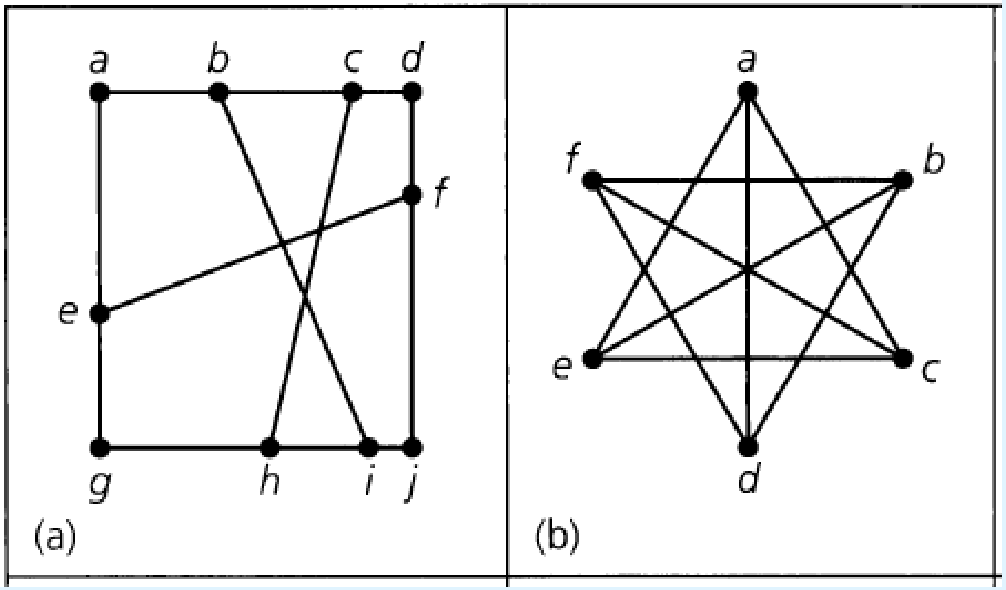
$|V| = 6$

b) $|E| = 15, |V| = n, m$ - regular.

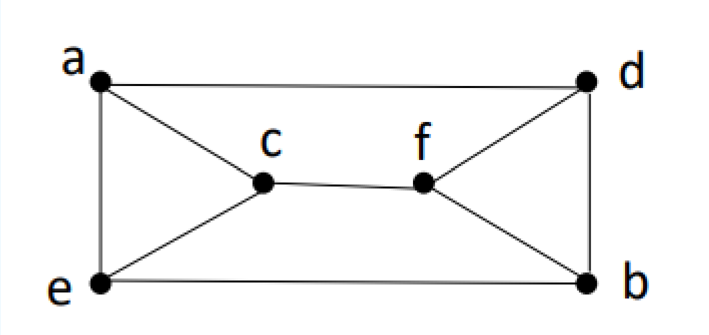
$2 \times 15 = m \cdot n = 1 \times 30 \; or \; 2 \times 15 \; or \; 3 \times 10, 5 \times 6, 6 \times 5, 10 \times 3$

$or \; 30 \times 1$

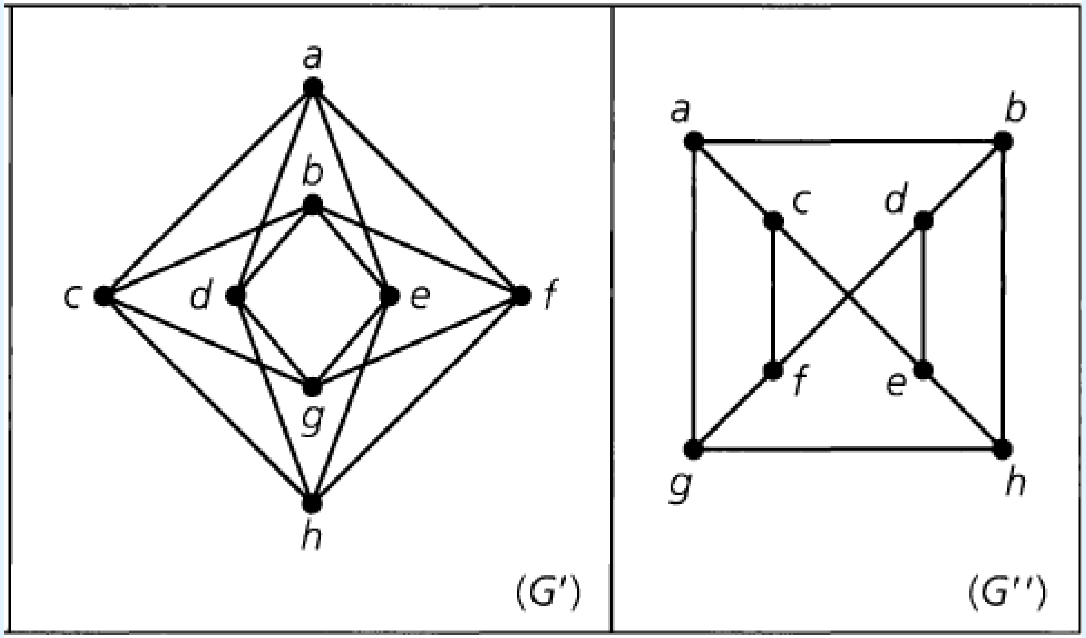
3. Determine which of the graphs are planar. It a graph is planar, redraw it with no edges overlapping.



a) Not planar, b) Planar



4. Determine whether or not the graph is bipartite.



G' bipartite, *G''* Not *bipartite*

G ′ : $V_1 = \{a, b, g, h\}$
$V_2 = \{c, d, e, f\}$

5. Let $G = (V, E)$ be a loop-free connected undirected graph with $|V| \geq 2$. Prove that G contains two vertices v, w, where $\deg(v) = \deg(w)$.

Assume that a finite graph G has n vertices. Then each vertex has a degree between $n - 1$ and 0. But if any vertex has degree 0, then no vertex can have degree $n - 1$, so it's not possible for the degrees of the graph's vertices to include both 0 and $n - 1$. Thus, the n vertices of the graph can only have $n - 1$ different degrees, so by the pigeonhole principle at least two vertices must have the same degree.