Relation & Function In-class Exercises

1. Prove Theorem 5.1 (d)

對任意集合 A，B，$C \subseteq \mathcal{U}$：

d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

2. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{w, x, y, z\}$, how many elements are there in $\mathcal{P}(A \times B)$?

3. Consider the relation $\mathcal{R}$ on the set $\mathbf{Z}$ where we define $a \mathcal{R} b$ when $ab \geq 0$.

Whether this binary relation $\mathcal{R}$ is reflexive, symmetric, or transitive?

4. For $x, y \in$ R define $x \mathcal{R} y$ to mean that $x - y \in$ Z. Prove that $\mathcal{R}$ is an equivalence relation on R. Please show all workings.

5. For each of the following functions, determine whether it is one-to-one and determine its range.

a) $f: \mathbf{Z} \rightarrow \mathbf{Z}$, $f(x) = 2x + 1$

b) $f: \mathbf{R} \rightarrow \mathbf{R}$, $f(x) = e^x$

c) $f: [0, \pi] \rightarrow \mathbf{R}$, $f(x) = \sin x$

6. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$.

(a) How many functions are there from A to B?

How many of these are one-to-one?

How many are onto?

(b) How many functions are there from B to A?

How many of these are onto?

How many are one-to-one?

7.

Let $g: \mathbf{N} \rightarrow \mathbf{N}$ be defined by $g(n) = 2n$. If $A = \{1, 2, 3, 4\}$ and $f: A \rightarrow \mathbf{N}$ is given by

$$f = \{(1, 2), (2, 3), (3, 5), (4, 7)\},$$

find $g \circ f$.

8. Let $f, g: \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ where for all $x \in \mathbf{Z}^+$, $f(x) = x + 1$ and $g(x) = \max\{1, x - 1\}$, the maximum of 1 and $x - 1$.

a) Is g an onto function?

b) Is the function g one-to-one?

c) Show that $g \circ f = 1_{\mathbf{Z}^+}$.

Relation & Function
Suggested Exercises

1. Prove Theorem 5.1 (c)

對任意集合 A，B，$C \subseteq \mathscr{U}$：

c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

2. If $A = \{1, 2, 3, 4\}$, give an example of a relation $\mathscr{R}$ on A that is

a) reflexive and symmetric, but not transitive

b) reflexive and transitive, but not symmetric

c) symmetric and transitive, but not reflexive

3. **a)** Rephrase the definitions for the reflexive, symmetric, transitive, and antisymmetric properties of a relation $\mathcal{R}$ (on a set A), using quantifiers.

b) Use the results of part (a) to specify when a relation $\mathcal{R}$ (on a set A) is (i) *not* reflexive; (ii) *not* symmetric; (iii) *not* transitive; and (iv) *not* antisymmetric.

4. If $A = \{w, x, y, z\}$, determine the number of relations on A that are (a) reflexive; (b) symmetric; (c) reflexive and symmetric; (d) reflexive and contain (x, y); (e) symmetric and contain (x, y); (f) antisymmetric; (g) antisymmetric and contain (x, y); (h) symmetric and antisymmetric; and (i) reflexive, symmetric, and antisymmetric.

5. Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$.

(a) List a possible function from A to B.

(b) How many functions $f: A \to B$ are there?

(c) How many functions $f: A \to B$ are one-to-one? (d) How many functions $g: B \to A$ are there? (e) How many functions $g: B \to A$ are one-to-one? (f) How many functions $f: A \to B$ satisfy $f(1) = x$? (g) How many functions $f: A \to B$ satisfy $f(1) = f(2) = x$? (h) How many functions $f: A \to B$ satisfy $f(1) = x$ and $f(2) = y$?

6. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9, 10, 11, 12\}$. How many functions $f: A \to B$ are such that $f^{-1}(\{6, 7, 8\}) = \{1, 2\}$?

7. Let $f: A \to B$, with $A_1, A_2 \subseteq A$. Then prove that

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2);$$