Logic
In-class Exercises

1. Determine whether each of the following sentences is a statement.

a) In 2003 George W. Bush was the president of the United States.

b) $x + 3$ is a positive integer.

c) Fifteen is an even number.

d) If Jennifer is late for the party, then her cousin Zachary will be quite angry.

e) What time is it?

f) As of June 30, 2003, Christine Marie Evert had won the French Open a record seven times.

2. Identify the primitive statements in Exercise 1.

1. a, c, d and f are statements.
2. a, c and f are primitive statements.

3. Let p, q, r, s denote the following statements:

p: I finish writing my computer program before lunch.
q: I shall play tennis in the afternoon.
r: The sun is shining.
s: The humidity is low.

Write the following in symbolic form.

a) If the sun is shining, I shall play tennis this afternoon.

b) Finishing the writing of my computer program before lunch is necessary for my playing tennis this afternoon.

c) Low humidity and sunshine are sufficient for me to play tennis this afternoon.

a) $r \to q$
b) $q \to p$
c) $(s \land r) \to q$

4. Construct a truth table for each of the following compound statements, where p, q, r denote primitive statements.

a) $[p \wedge (p \rightarrow q)] \rightarrow q$

b) $q \leftrightarrow (\neg p \vee \neg q)$

5. Which of the compound statements in Exercise 4 are tautologies?

4. a)

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

b)

p	q	$\neg p$	$\neg q$	$(\neg p \vee \neg q)$	$q \leftrightarrow (\neg p \vee \neg q)$
0	0	1	1	1	0
0	1	1	0	1	1
1	0	0	1	1	0
1	1	0	0	0	0

5. (a) is tautology.

6. Use the substitution rules to verify that each of the following is a tautology. (Here p, q, and r are primitive statements.)

$$[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$$

For primitive statements s, t, we have shown that $(s \rightarrow t) \Leftrightarrow (\neg t \rightarrow \neg s)$ (NOTE: contrapositive) thus, $(s \rightarrow t) \leftrightarrow (\neg t \rightarrow \neg s)$ is tautology.

Replace each S by $(p \rightarrow q)$,and each t by r.

By First Substitution Rule, we may conclude that $[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$ is also a tautology.

7. Give the reasons for each step in the following simplifications of compound statements.

	Reasons
$(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$	
$\Leftrightarrow (p \rightarrow q) \wedge \neg q$	(Commutation Law and) Absorption Law
$\Leftrightarrow (\neg p \vee q) \wedge \neg q$	$(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$
$\Leftrightarrow \neg q \wedge (\neg p \vee q)$	Commutation Law
$\Leftrightarrow (\neg q \wedge \neg p) \vee (\neg q \wedge q)$	Distributive Law
$\Leftrightarrow (\neg q \wedge \neg p) \vee F_0$	Inverse Law
$\Leftrightarrow \neg q \wedge \neg p$	Identity Law
$\Leftrightarrow \neg(q \vee p)$	DeMorgan's Laws

8. For primitive statements p, q, r, and s, simplify the compound statement

$$[[[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r]] \vee \neg q] \to s.$$

$[[[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r]] \vee \neg q] \to s$	Reasons
$\Leftrightarrow [[(p \wedge q) \wedge (r \vee \neg r)] \vee \neg q] \to s$	Distributive Law
$\Leftrightarrow [[(p \wedge q) \wedge T_0] \vee \neg q] \to s$	Inverse Law
$\Leftrightarrow [(p \wedge q) \vee \neg q] \to s$	Identity Law
$\Leftrightarrow [\neg q \vee (p \wedge q)] \to s$	Commutation Law
$\Leftrightarrow [(\neg q \vee p) \wedge (\neg q \vee q)] \to s$	Distributive Law
$\Leftrightarrow (\neg q \vee p) \to s$	Inverse Law and Identity Law
$\Leftrightarrow$Or $(q \to p) \to s$	$(q \to p) \Leftrightarrow (\neg q \vee p)$
$\Leftrightarrow$Or $(\neg p \wedge q) \vee s$	

9. Show that each of the following arguments is invalid by providing a counterexample

$$
\begin{array}{l}
p \leftrightarrow q \\
q \rightarrow r \\
r \vee \neg s \\
\neg s \rightarrow q \\
\hline
\therefore s
\end{array}
$$

A counterexample

p : 1 $\quad$ q : 1

r : 1 $\quad$ s : 0

10. Establish the validity of the following arguments.

$$
\begin{array}{l}
p \rightarrow (q \rightarrow r) \\
p \vee s \\
t \rightarrow q \\
\neg s \\
\hline
\therefore \neg r \rightarrow \neg t
\end{array}
$$

Steps	Reasons
1. $\neg s,\ p \vee s$	Premises
2. p	1, Rule of Disjunctive Syllogism
3. $p \to (q \to r)$	Premises
4. $q \to r$	2, 3, Rule of Detachment
5. $t \to q$	Premises
6. $t \to r$	4, 5 Law of the Syllogism
7. $\therefore \neg r \to \neg t$	6, $(t \to r) \Leftrightarrow (\neg r \to \neg t)$

OR

1. $t \to q$	Premises
2. $\neg q \to \neg t$	1, contrapositive ($t \to q \Leftrightarrow \neg q \to \neg t$)
3. $p \to (q \to r)$	premises
4. $\neg s$	Premises
5. $p \vee s$	Premises
6. p	4, 5, Rule of Disjunctive Syllogism
7. $q \to r$	3, 6, Rule of Detachment (Modus Ponens)
8. $\neg q \vee r$	7, $q \to r \Leftrightarrow \neg q \vee r$
9. $\neg r$	Premise
10. $\neg q$	8, 9, Rule of Disjunctive Syllogism
11. $\therefore \neg t$	2, 10, Rule of Detachment (Modus Ponens)

11. Let $p(x)$, $q(x)$, and $r(x)$ denote the following open statements.

$$p(x): \quad x^2 - 8x + 15 = 0$$

$$q(x): \quad x \text{ is odd}$$

$$r(x): \quad x > 0$$

For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counterexample.

a) $\forall x\, [p(x) \rightarrow q(x)]$

b) $\forall x\, [q(x) \rightarrow p(x)]$

c) $\exists x\, [p(x) \rightarrow q(x)]$

d) $\forall x\, [(p(x) \vee q(x)) \rightarrow r(x)]$

e) $\exists x\, [r(x) \rightarrow p(x)]$

a) True
b) False, e.g. $x = 1$
c) True
d) False, e.g. $x = -1$
e) True

13. Negate and simplify each of the following.

$$\exists x\ [(p(x) \vee q(x)) \rightarrow r(x)]$$

$$\neg\exists x\left[\left(p(x) \vee q(x)\right) \longrightarrow r(x)\right]$$

$$\Leftrightarrow \forall x \neg\left[\left(p(x) \vee q(x)\right) \longrightarrow r(x)\right]$$

$$\Leftrightarrow \forall x \neg\left[\neg\left(p(x) \vee q(x)\right) \vee r(x)\right]$$

$$\Leftrightarrow \forall x \left(\neg\neg\left(p(x) \vee q(x)\right) \wedge \neg r(x)\right)$$

$$\Leftrightarrow \forall x \left(\left(p(x) \vee q(x)\right) \wedge \neg r(x)\right)$$

Logic
Suggested Exercises

1. Let p, q, r denote the following statements about a particular triangle ABC.

p: Triangle ABC is isosceles.
q: Triangle ABC is equilateral.
r: Triangle ABC is equiangular.

Translate each of the following into an English sentence.

a) $q \rightarrow p$
b) $\neg p \rightarrow \neg q$
c) $q \leftrightarrow r$
d) $p \wedge \neg q$
e) $r \rightarrow p$

a) If triangle ABC is equilateral, then it is isosceles.
b) If triangle ABC is not isosceles, then it is not equilateral.
c) Triangle ABC is equilateral if and only if it is equiangular.
d) Triangle ABC is isosceles but it is not equilateral.
e) If triangle ABC is equiangular, then it is isosceles.

2. The integer variables m and n are assigned the values 3 and 8, respectively, during the execution of a program (written in pseudocode). Each of the following *successive* statements is then encountered during program execution. [Here the values of m, n following the execution of the statement in part (a) become the values of m, n for the statement in part (b), and so on, through the statement in part (e).] What are the values of m, n after each of these statements is encountered?

a) **if** $n - m = 5$ **then** $n := n - 2$

b) **if** $((2 * m = n)$ **and** $(\lfloor n/4 \rfloor = 1))$ **then**
$n := 4 * m - 3$

c) **if** $((n < 8)$ **or** $(\lfloor m/2 \rfloor = 2))$ **then** $n := 2 * m$
else $m := 2 * n$

d) **if** $((m < 20)$ **and** $(\lfloor n/6 \rfloor = 1))$ **then**
$m := m - n - 5$

e) **if** $((n = 2 * m)$ **or** $(\lfloor n/2 \rfloor = 5))$ **then**
$m := m + 2$

a) $m = 3, n = 6$
b) $m = 3, n = 9$
c) $m = 18, n = 9$
d) $m = 4, n = 9$
e) $m = 4, n = 9$

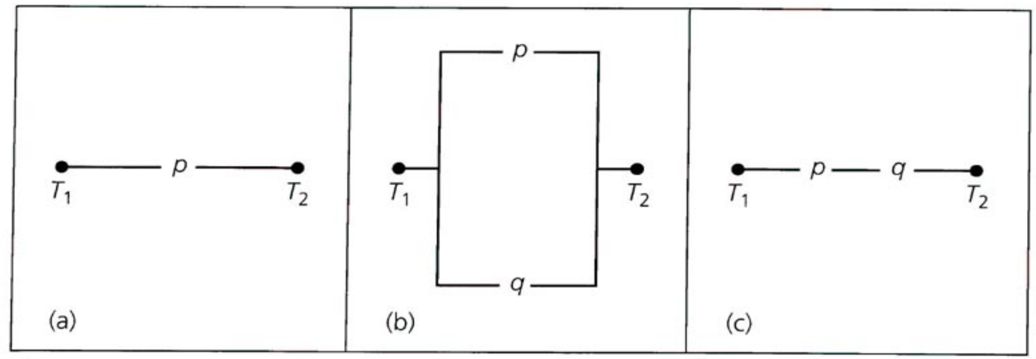


Figure 2.1

$p \vee q$ $p \wedge q$

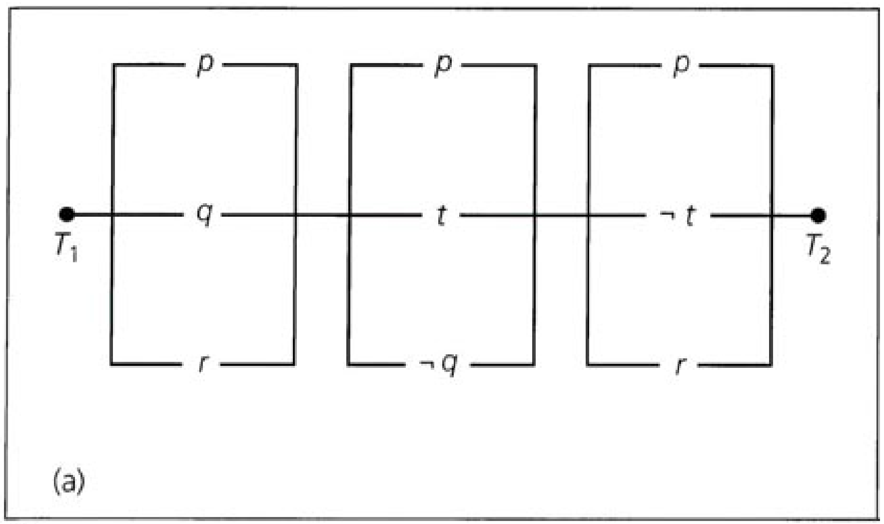
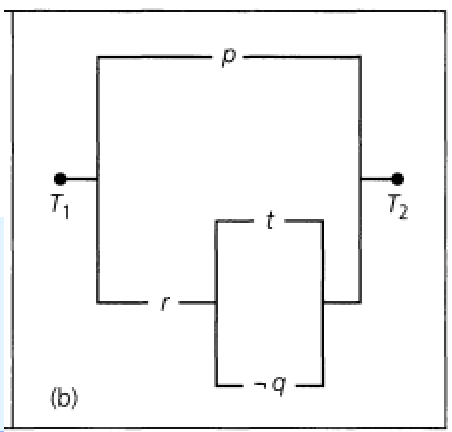


Figure 2.2

3. Simplify $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$

	Reasons
$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$	
$\Leftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$	Distributive Law of $\vee$ over $\wedge$
$\Leftrightarrow p \vee [(q \vee r) \wedge (\neg t \vee r) \wedge (t \vee \neg q)]$	Commutative Law of $\wedge$
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)]$	Distributive Law of $\vee$ over $\wedge$
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (\neg\neg t \vee \neg q)]$	Law of Double Negation
$\Leftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)]$	DeMorgan's Law
$\Leftrightarrow p \vee [\neg(\neg t \wedge q) \wedge ((\neg t \wedge q) \vee r)]$	Commutative Law of $\wedge$ (twice)
$\Leftrightarrow p \vee [(\neg(\neg t \wedge q) \wedge (\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)]$	Distributive Law of $\wedge$ over $\vee$
$\Leftrightarrow p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)]$	$\neg s \wedge s \Leftrightarrow F_0$, for any statement s
$\Leftrightarrow p \vee [(\neg(\neg t \wedge q)) \wedge r]$	F_0 is the identity for $\vee$
$\Leftrightarrow p \vee [r \wedge \neg(\neg t \wedge q)]$	Commutative Law of $\wedge$
$\Leftrightarrow p \vee [r \wedge (t \vee \neg q)]$	DeMorgan's Law and the Law of Double Negation



(b)

4. For primitive statements p, q,

a) verify that $p \rightarrow [q \rightarrow (p \wedge q)]$ is a tautology.

b) verify that $(p \vee q) \rightarrow [q \rightarrow q]$ is a tautology by using the result from part (a) along with the substitution rules and the laws of logic.

c) is $(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$ a tautology?

a)

p	q	$p \wedge q$	$q \rightarrow (p \wedge q)$	$p \rightarrow [q \rightarrow (p \wedge q)]$
0	0	0	1	1
0	1	0	0	1
1	0	0	1	1
1	1	1	1	1

$\Big\}$ tautology

b) By first substitution rule, since $p \rightarrow [q \rightarrow (p \wedge q)]$ is tautology, replace each occurrence of p by $p \vee q$,
$(p \vee q) \rightarrow [q \rightarrow [(p \vee q) \wedge q]]$ will also be a tautology.

$(p \vee q) \rightarrow [q \rightarrow [(p \vee q) \wedge q]] \Leftrightarrow (p \vee q) \rightarrow [q \rightarrow q]$
because $(p \vee q) \wedge q \Leftrightarrow q$.

c)

p	q	$p \vee q$	$p \wedge q$	$q \rightarrow (p \wedge q)$	$(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$
0	0	0	0	1	1
0	1	1	0	0	0
1	0	1	0	1	1
1	1	1	1	1	1

Not a tautology

$\because$ p is not logically equivalent to $p \vee q$.

5. Establish the validity of the argument

$$
\begin{array}{l}
p \rightarrow q \\
q \rightarrow (r \wedge s) \\
\neg r \vee (\neg t \vee u) \\
p \wedge t \\
\hline
\therefore u
\end{array}
$$

Steps	**Reasons**
1) $p \rightarrow q$	Premise
2) $q \rightarrow (r \wedge s)$	Premise
3) $p \rightarrow (r \wedge s)$	Steps (1) and (2) and the Law of the Syllogism
4) $p \wedge t$	Premise
5) p	Step (4) and the Rule of Conjunctive Simplification
6) $r \wedge s$	Steps (5) and (3) and the Rule of Detachment
7) r	Step (6) and the Rule of Conjunctive Simplification
8) $\neg r \vee (\neg t \vee u)$	Premise
9) $\neg(r \wedge t) \vee u$	Step (8), the Associative Law of $\vee$, and DeMorgan's Laws
10) t	Step (4) and the Rule of Conjunctive Simplification
11) $r \wedge t$	Steps (7) and (10) and the Rule of Conjunction
12) $\therefore u$	Steps (9) and (11), the Law of Double Negation, and the Rule of Disjunctive Syllogism

6. Give the reasons for the steps verifying the following argument.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \rightarrow \neg t \\ \hline \therefore p \end{array}$$

6.

Steps	*Reasons*
1)$\neg s \wedge \neg u$	Premise
2) $\neg u$	Step (1) and the Rule of Conjunctive Simplification
3) $\neg u \rightarrow \neg t$	Premise
4) $\neg t$	Step (1), (3) and the Rule of Detachment
5) $\neg s$	Step (1) and the Rule of Conjunctive Simplification
6) $\neg s \wedge \neg t$	Step (4), (5) and the Rule of Conjunctive
7) $r \rightarrow (s \vee t)$	Premise
8) $\neg(s \vee t) \rightarrow \neg r$	Step (7) and $[r \rightarrow (s \vee t)] \Leftrightarrow [\neg(s \vee t) \rightarrow \neg r]$
9)$(\neg s \wedge \neg t) \rightarrow \neg r$	Step (8) and DeMorgan's Laws
10) $\neg r$	Step (6), (9) and the Rule of Detachment
11)$(\neg p \vee q) \rightarrow r$	Premise
12) $\neg r \rightarrow \neg(\neg p \vee q)$	Step (11) and $[(\neg p \vee q) \rightarrow r] \Leftrightarrow [\neg r \rightarrow \neg(\neg p \vee q)]$
13) $\neg r \rightarrow (p \wedge \neg q)$	Step (12) and DeMorgan's Laws and Double Negation
14) $p \wedge \neg q$	Step (10), (13) and the Rule of Detachment
15)$\therefore p$	Step (14) and the Rule of Conjunctive Simplification

7. For the universe of all integers, let $p(x)$, $q(x)$, $r(x)$, $s(x)$, and $t(x)$ be the following open statements.

$p(x)$: $x > 0$

$q(x)$: x is even

$r(x)$: x is a perfect square

$s(x)$: x is (exactly) divisible by 4

$t(x)$: x is (exactly) divisible by 5

a) Write the following statements in symbolic form.

- **i)** At least one integer is even.
- **ii)** There exists a positive integer that is even.
- **iii)** If x is even, then x is not divisible by 5.
- **iv)** No even integer is divisible by 5.
- **v)** There exists an even integer divisible by 5.
- **vi)** If x is even and x is a perfect square, then x is divisible by 4.

b) Determine whether each of the six statements in part (a) is true or false. For each false statement, provide a counterexample.

c) Express each of the following symbolic representations in words.

i) $\forall x\,[r(x) \rightarrow p(x)]$ **ii)** $\forall x\,[s(x) \rightarrow q(x)]$

iii) $\forall x\,[s(x) \rightarrow \neg t(x)]$ **iv)** $\exists x\,[s(x) \wedge \neg r(x)]$

d) Provide a counterexample for each false statement in part (c).

a)

i $\exists x\, q(x)$

ii $\exists x\, [p(x) \wedge q(x)]$

iii $\forall x\, [q(x) \rightarrow \neg t(x)]$

iv $\forall x\, [q(x) \rightarrow \neg t(x)]$

v $\exists x\, [p(x) \wedge t(x)]$

vi $\forall x\, [(q(x) \wedge t(x)) \rightarrow s(x)]$

b) Statements (i), (ii), (v), and (vi) are true. Statements (iii) and (iv) are false: $x = 10$ provides a counterexample for either statement.

c)

i. If x is a perfect square, then $x > 0$.

ii. If x is divisible by 4, then x is even.

iii. If x is divisible by 4, then x is not divisible by 5.

iv. There exists an integer that is divisible by 4 but it is not a perfect square.

d)(i) Let $x = 0$. (iii)Let $x = 20$