Counting
In-class Exercises

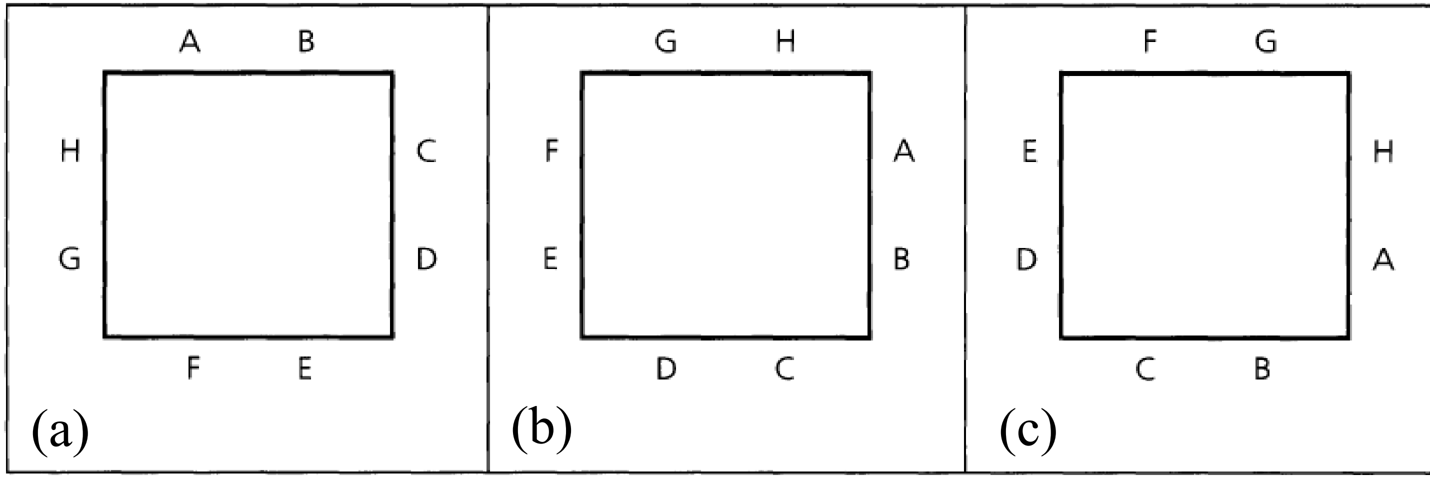
Over the Internet, data are transmitted in structured blocks of bits called *datagrams*.

a) In how many ways can the letters in DATAGRAM be arranged?

b) For the arrangements of part (a), how many have all three A's together?

a) $\frac{8!}{3!} = 6720$ arrangements

b) $6! = 720$ ways



1. **a)** In how many ways can eight people, denoted A, B, . . . , H be seated about the square table, where situations (a) and (b) are considered the same, but are distinct from (c).

b) If two of the eight people, say A and B, do not get along well, how many different seatings are possible with A and B not sitting next to each other? (Note: all (a), (b), (c) are not allowed)

a) $2 \times 7! = 10080$ ways

b) $10080 - 4 \times 6! = 7200$ ways

2. Find the coefficient of $w^2x^2y^2z^2$ in the expansion of $(2w - x + 3y + z - 2)^{12}$

$$\binom{12}{2,2,2,2,4} \cdot (2)^2 \cdot (-1)^2 \cdot (3)^2 \cdot (1)^2 \cdot (-2)^4 = 718{,}502{,}400$$

3. In how many ways can a teacher distribute eight chocolate donuts and seven jelly donuts among three student helpers if each helper wants at least one donut of each kind?

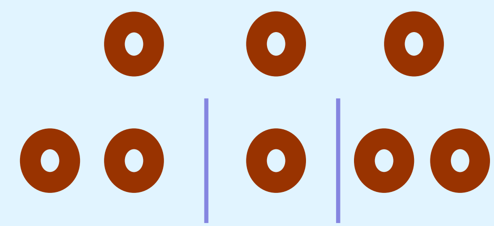
註： 3 位小朋友視為 A, B, C

Chocolate donuts $\binom{5+3-1}{5} = \binom{7}{5} = 21$ distributions

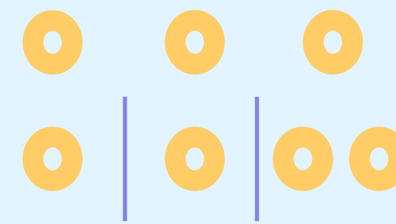
Jelly donuts $\binom{4+3-1}{4} = \binom{6}{4} = 15$ ways

By the rule of product, there are $\binom{7}{5} \cdot \binom{6}{4} = 315$ ways to distribute the donuts.

Chocolate donuts $\binom{5+3-1}{5} = \binom{7}{5}$



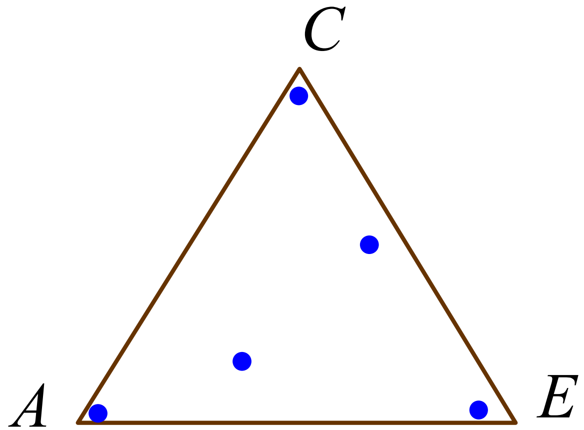
Jelly donuts $\binom{4+3-1}{4} = \binom{6}{4}$



By the rule of product, there are $\binom{7}{5} * \binom{6}{4}$ ways to distribute the donuts.

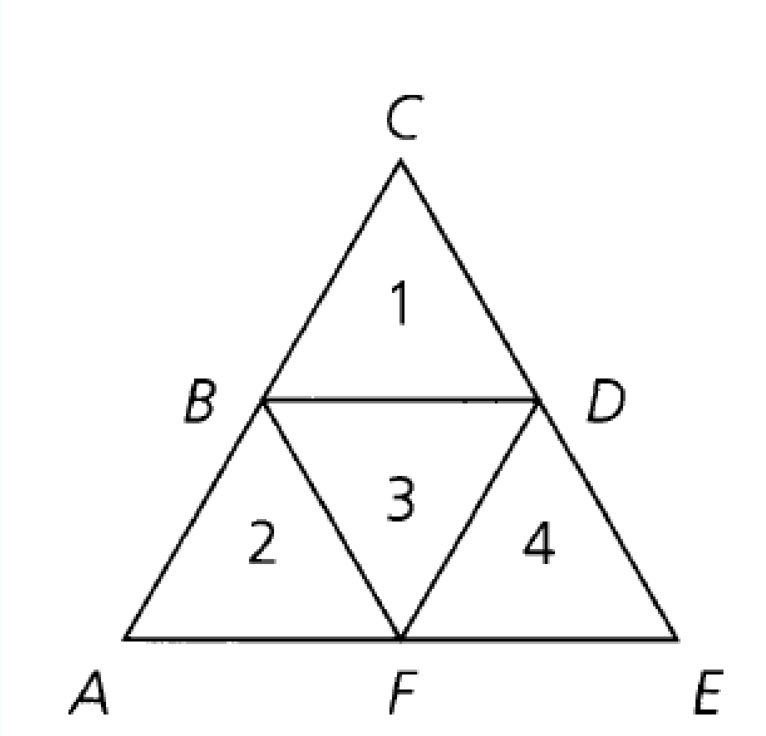
4. Triangle ACE is equilateral with $AC = 1$.

If five points are selected from the interior of the triangle, there are at least two whose distance apart is less than $1/2$.



鴿子 : 5個點

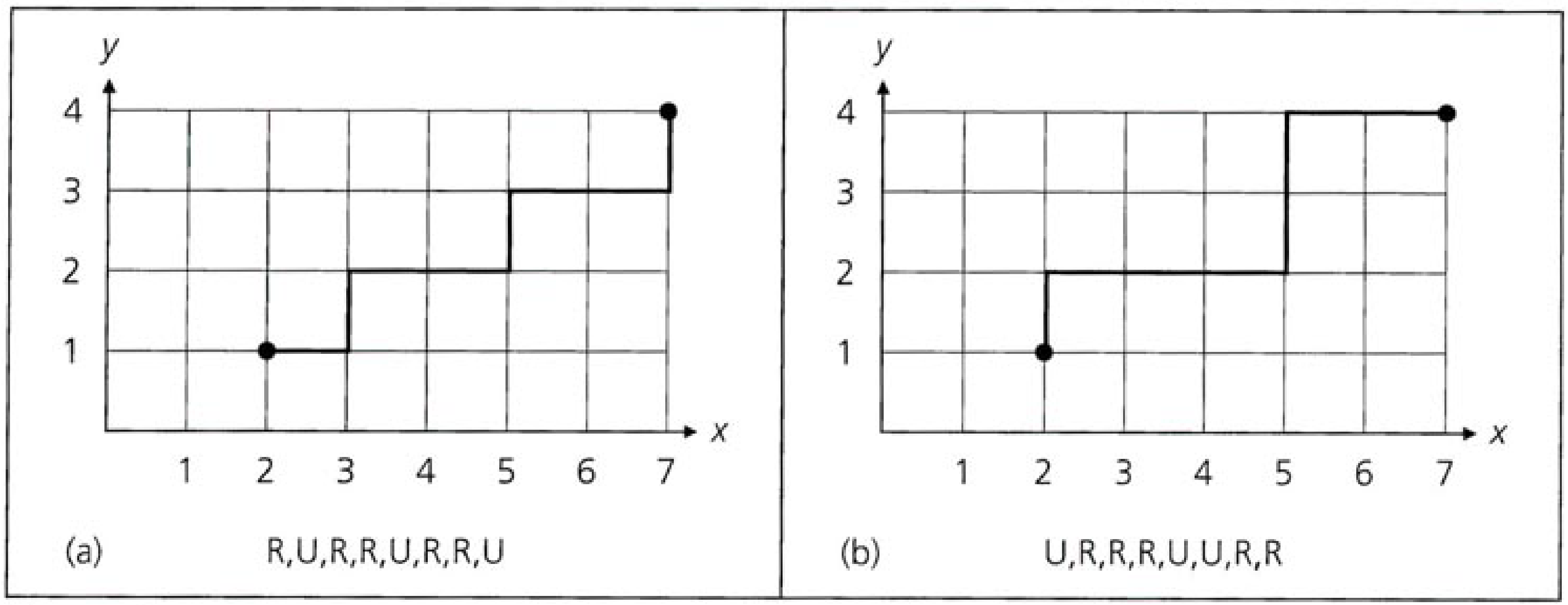
鴿洞 : 4個三角形如右圖

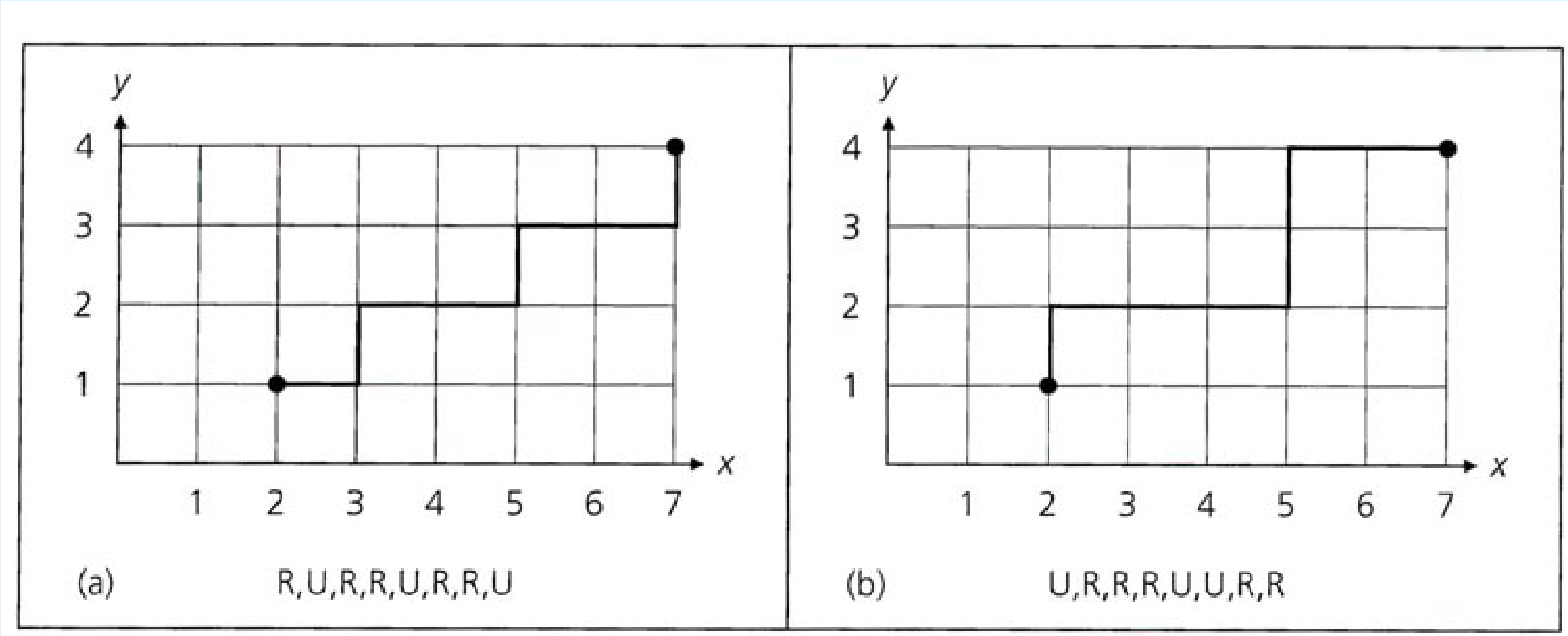


因為 5>4，基於鴿洞原理，
至少有兩個點落於其中一個三角形內，
因此至少有兩個點之間距離小於 ½。

Counting
Suggested Exercises

1. Determine the number of (staircase) paths in the xy-plane from (2, 1) to (7, 4), where each such path is made up of individual steps going one unit to the right (R) or one unit upward (U).





It needs 5 R and 3 U,

so $8!/(5!3!) = 56$

2. How many positive integers n can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want n to exceed 5,000,000?

Case 1: Leading digit 5 _ _ _ _ _ _ $\qquad \dfrac{6!}{2!}$

Case 2: Leading digit 6 _ _ _ _ _ _ $\qquad \dfrac{6!}{2!2!}$

Case 3: Leading digit 7 _ _ _ _ _ _ $\qquad \dfrac{6!}{2!2!}$

$$\frac{6!}{2!} + 2 \times \frac{6!}{2!2!}$$

$= 720$ positive integers n.

3. Determine the sum of all the coefficients in the expansions of

$$(x + y + z)^{10}$$

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n} \binom{n!}{k_1, k_2, \cdots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$$

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n!}{k_1, k_2, \cdots, k_m} = m^n$$

3^{10}

4. In how many ways can we distribute eight identical white balls into four distinct containers so that (a) no container is left empty? (b) the fourth container has an odd number of balls in it?

a) $\binom{4+4-1}{4} = \binom{7}{4}$

b)

One marble : $\binom{3+7-1}{7}$

Three marble : $\binom{3+5-1}{5}$

Five marble : $\binom{3+3-1}{3}$

Seven marble : $\binom{3+1-1}{1}$

$$\binom{3+7-1}{7} + \binom{3+5-1}{5} + \binom{3+3-1}{3} + \binom{3+1-1}{1} = \sum_{i=0}^{3} \binom{9-2i}{7-2i}$$

5.

In how many ways can the letters in WONDERING be arranged with exactly two consecutive vowels?

the seven symbols OE,W,N,N,D,R,G : $\left(\frac{7!}{2!}\right)$

there are 6 locations for that the I so that it is not adjacent to a vowel : $(6)\left(\frac{7!}{2!}\right)$

the three vowels can be divided up into a pair and a single vowel in six ways : $(6^2)\left(\frac{7!}{2!}\right)$

6.

How many times must we roll a single die in order to get the same score (a) at least twice? (b) at least three times? (c) at least n times, for $n \geq 4$?

a) 7 b) 13 c) 6(n-1)+1