Chapter 6

圖論導引

An Introduction to Graph Theory

6.1　基本定義

6.2 連通性、子圖及圖同構

6.3 圖的種類、頂點次數及平面圖

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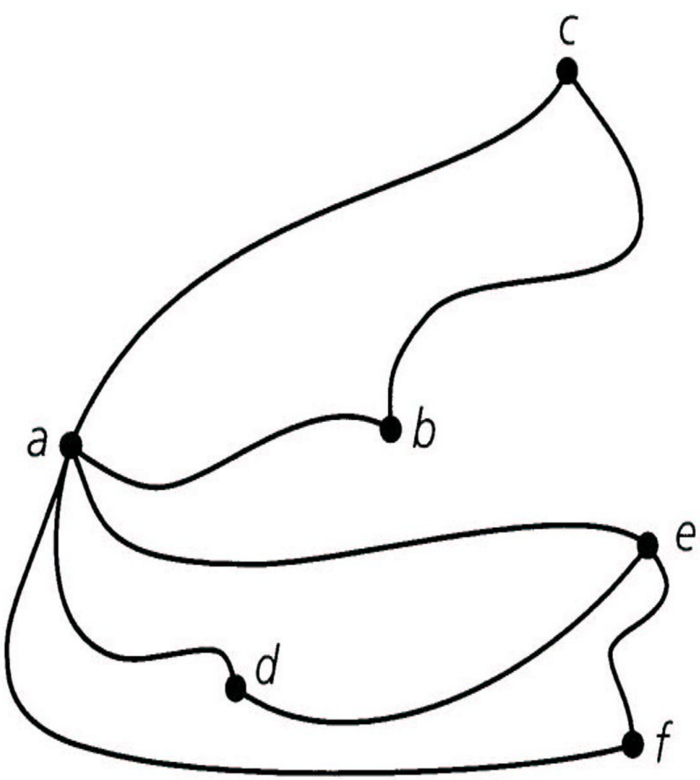
6.1 基本定義



Set of *vertices* or *nodes*

$$V = \{a, b, c, d, e, f\}$$

Set of *edges*

E = a set of edges

a—b, a—c,
a—d, a—e,
a—f, b—c,
d—e, e—f

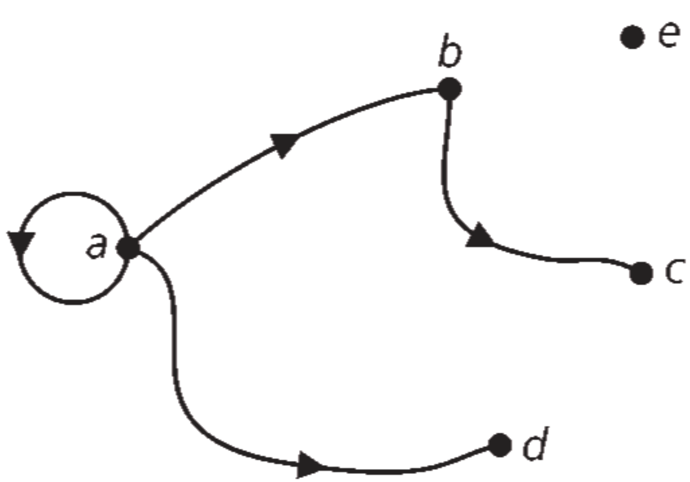
Directed or undirected?

Definition 6.1

Let V be a finite nonempty set, and let $E \subseteq V \times V$.

The pair (V, E) is then called a *directed graph* or *digraph* (on V).

We write $\boxed{G = (V, E)}$ to denote such a graph.



$V = \{a, b, c, d, e\}$

$E = \{(a, a), (a, b), (a, d), (b, c)\}.$

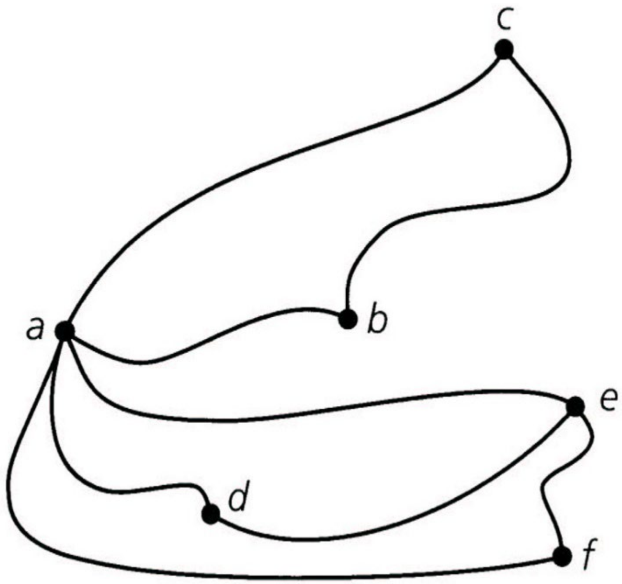
Definition 6.1 cont.

When there is no concern about the direction of any edge, we still write $\boxed{G = (V, E)}$.

But now E is a set of unordered pairs of elements taken from V, and G is called an *undirected graph*.

In general, if a graph G is not specified as directed or undirected, it is assumed to be undirected.

Whether $G = (V, E)$ is directed or undirected, we often call V the *vertex set* of G and E the *edge set* of G.



$V = \{a, b, c, d, e, f\}$

$E = \{ \{a, b\}, \{a, c\}, \{a, d\}, \{a, e\}, \{a, f\}, \{b, c\}, \{d, e\}, \{e, f\} \}$

or

$E = \{ a \leftrightarrow b, a \leftrightarrow c, a \leftrightarrow d, a \leftrightarrow e, a \leftrightarrow f, b \leftrightarrow c, d \leftrightarrow e, e \leftrightarrow f \}$

Definition 6.2

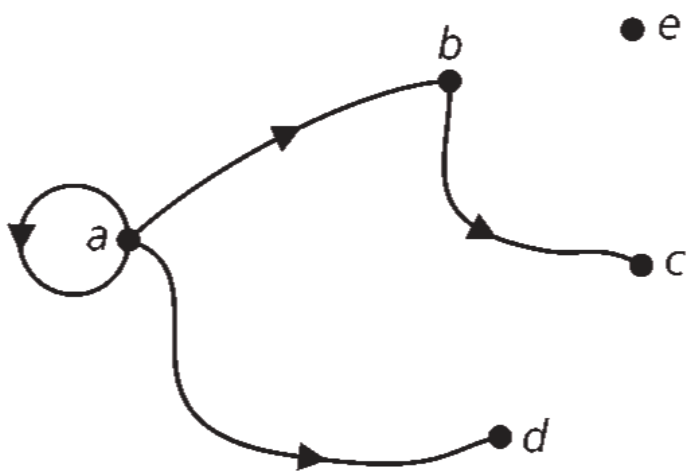
For any edge, such as (b, c) or $\{b, c\}$, we say that the edge is *incident* with the vertices b, c, and we say that vertices b, c are *adjacent*.

Note:

edge --- *incident* --- vertex

edge --- edge

vertex --- *adjacent* --- vertex



$$V = \{a, b, c, d, e\}$$

$$E = \{(a, a), (a, b), (a, d), (b, c)\}.$$

Definition 6.3

The edge (a, a) is an example of a *loop*, and the vertex e is called an *isolated* vertex.

When it contains no loops it is called *loop-free*.

A *trivial graph* is a graph with only one vertex and no edges.

Definition 6.4

Let V be a finite nonempty set.

We say that the pair (V, E) determines a *multigraph* G with vertex set V and edge set E if, for some $x, y \in V$, there are two or more edges in E of the form

(a) (x, y) (for a directed multigraph),

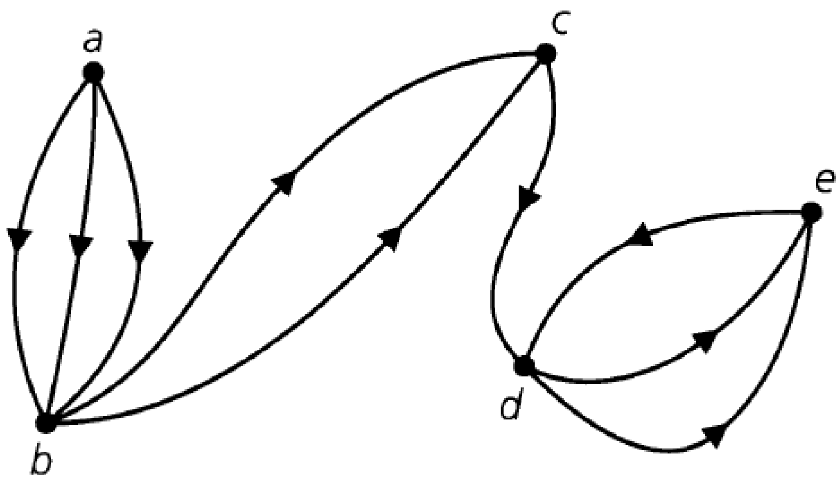
or (b) $\{x, y\}$ (for an undirected multigraph).

Definition 6.5

A graph $G = (V, E)$ is called a *simple graph* if it is loop-free and has no multi-edges.

EXAMPLE 6.1

an example of a directed multigraph



we say that the edge (a, b) has *multiplicity* 3.

The edges (b, c) and (d, e) both have multiplicity

Definition 6.6

Let x, y be (not necessarily distinct) vertices in an undirected graph $G = (V, E)$.

An *x-y walk* in G is a (loop-free) finite alternating sequence

$$\boxed{x = x_0, e_1, x_1, e_2, x_2, e_3, \ldots, e_{n-1}, x_{n-1}, e_n, x_n = y}$$

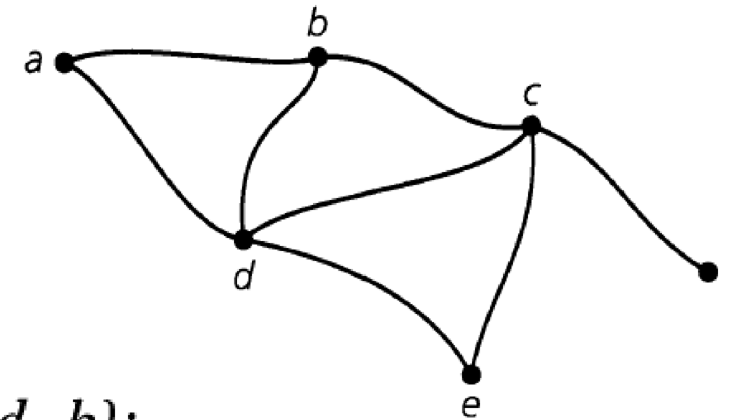
of vertices and edges from G, starting at vertex x and ending at vertex y and involving the n edges $e_i = \{x_{i-1}, x_i\}$, where $1 \le i \le n$.

The *length* of this walk is n, the number of edges in the walk.

(When $n = 0$ the walk is called *trivial*.)

Any x-y walk where $x = y$ (and $n > 1$) is called a *closed walk*. Otherwise the walk is called an *open walk*.

EXAMPLE 6.2



Followings are 3 open walks.

1) $\{a, b\}, \{b, d\}, \{d, c\}, \{c, e\}, \{e, d\}, \{d, b\}$:

This is an a-b walk of length 6 in which we find the vertices d and b repeated, as well as the edge $\{b, d\}$ ($= \{d, b\}$).

2) $b \rightarrow c \rightarrow d \rightarrow e \rightarrow c \rightarrow f$:

a b-f walk where the length is 5 and the vertex c is repeated,

3) $\{f, c\}, \{c, e\}, \{e, d\}, \{d, a\}$:

the given f-a walk has length 4 with no repetition of either vertices or edges.

Definition 6.7

Consider any x-y walk in an undirected graph $G = (V, E)$.

a) If no edge in the x-y walk is repeated, then the walk is called an *x-y trail*. （小徑）

A closed x-x trail is called a *circuit*. （環道）

b) If no vertex of the x-y walk occurs more than once, then the walk is called an *x-y path*. （路徑）

When $x = y$, the term *cycle* is used to describe such a closed path.

（循環）

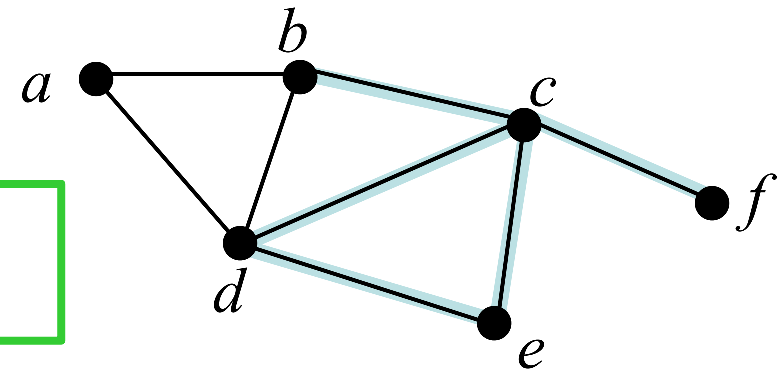
EXAMPLE 6.3

a)

a b-f walk

$b \rightarrow c \rightarrow d \rightarrow$ $\boxed{}$

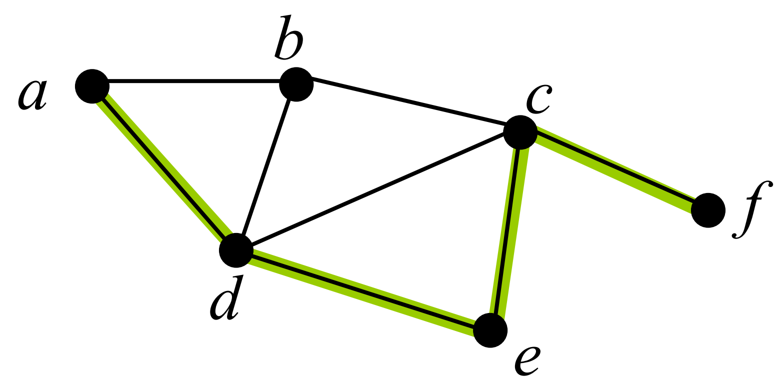
is a b-f trail, but it is not a b-f path



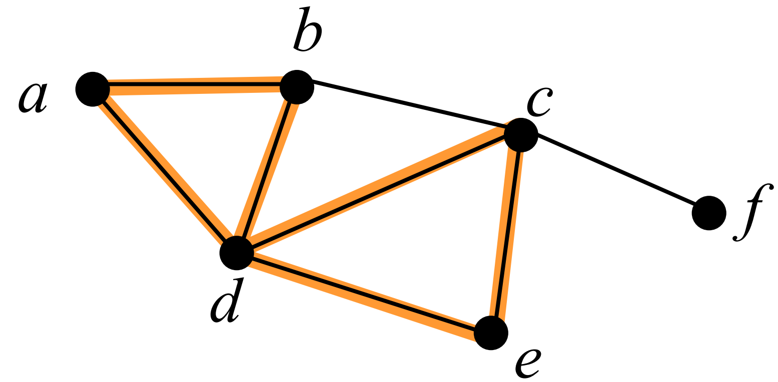
the f-a walk

$\{f, c\}, \{c, e\}, \{e, d\}, \{d, a\}$

is both an f-a trail (of length 4) and an f-a path (of length 4).



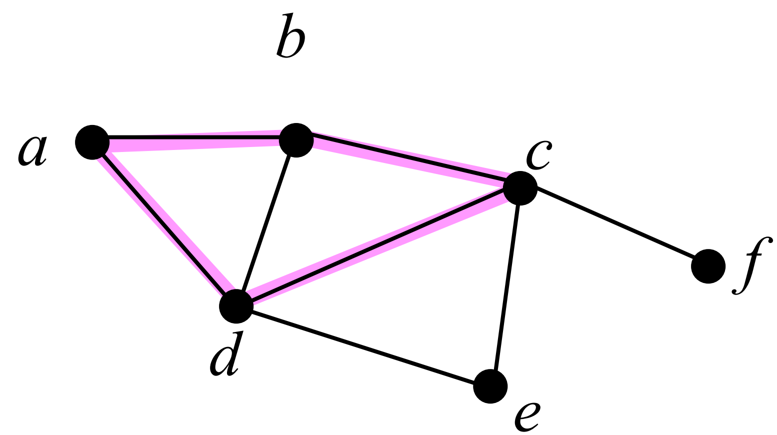
EXAMPLE 6.3 Cont.



b) the edges $\{a, b\}$, $\{b, d\}$, $\{d, c\}$, $\{c, e\}$, $\{e, d\}$, and $\{d, a\}$ provide an a-a circuit.

The vertex d is repeated, so the edges do *not* give us an a-a cycle.

c) The edges $\{a, b\}$, $\{b, c\}$, $\{c, d\}$, and $\{d, a\}$ provide an a-a cycle (of length 4), which is also a circuit.



Convention:

* In dealing with circuits, we shall always understand the presence of at least one edge.

  When there is only one edge, then the circuit is a loop (and the graph is no longer loop-free).

  Circuits with two edges arise in multigraphs (Later!)

* The term *cycle* will always imply the presence of at least three distinct edges (from the graph).

Note:

For a directed graph we shall use

directed walks, *directed paths*, and *directed cycles*.

Summary:

Repeated Vertex (Vertices)	Repeated Edge(s)	Open	Closed	Name
Yes		Yes		Walk (open)
Yes			Yes	Walk (closed)
Yes		Yes		Trail
Yes			Yes	Circuit
No		Yes		Path
No			Yes	Cycle

THEOREM 6.1

Let $G = (V, E)$ be an undirected graph, with $a, b \in V, a \neq b$. If there exists a trail (in G) from a to b, then there is a path (in G) from a to b.

Proof: Since there is a trail from a to b, we select one of shortest length, say $\{a, x_1\}, \{x_1, x_2\}, \ldots, \{x_n, b\}$.

THEOREM 6.1

Proof: (cont.)

Shortest length trail $\{a, x_1\}, \{x_1, x_2\}, \ldots, \{x_n, b\}$.

If this trail is not a path, we have the situation

$$\{a, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_{k+1}\}, \{x_{k+1}, x_{k+2}\},$$
$$\ldots, \{x_{m-1}, x_m\}, \{x_m, x_{m+1}\}, \ldots, \{x_n, b\},$$

where $k < m$ and $x_k = x_m$,
possibly with $k = 0$ and $a\ (= x_0) = x_m$, or $m = n + 1$
and $x_k = b\ (= x_{n+1})$.

But then we have a contradiction because
$\{a, x_1\}, \{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_m, x_{m+1}\}, \ldots, \{x_n, b\}$
is a shorter trail from a to b.

6.2 連通性、子圖及圖同構

Graph Connectivity, Subgraph, and Isomorphism

Definition 6.8

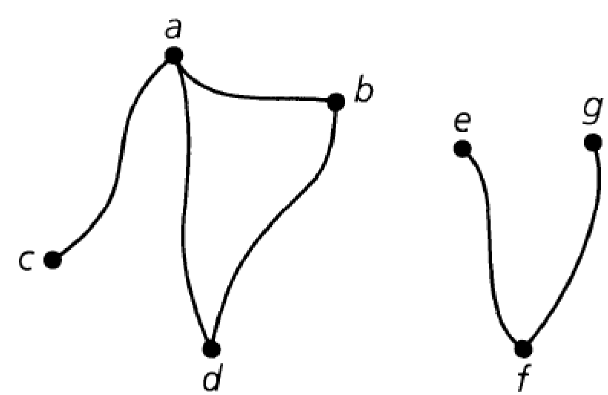
Let $G = (V, E)$ be an undirected graph.

We call G *connected* if there is a path between any two distinct vertices of G.

A graph that is not connected is called *disconnected*.

EXAMPLE 6.4

This graph is disconnected.
However, it is composed of two part that are themselves connected, and they are called the *connected components* of the graph.

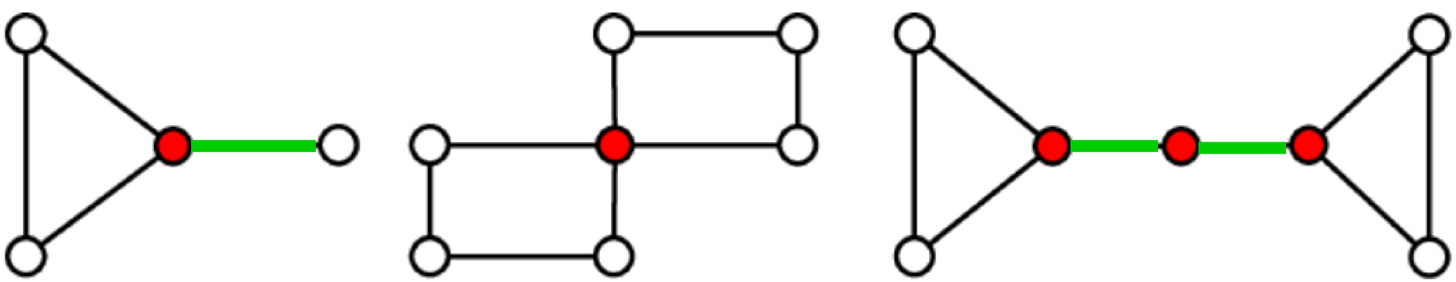


Definition 6.9

Let $G = (V, E)$ be a connected graph, and $v \in V$ be a vertex. If removing v and the edges incident to it results in a disconnected graph, then v is a *cut vertex*.

Likewise, if $e \in E$ is an edge such that removing it from G results in a disconnected graph, then e is a *cut edge*.

EXAMPLE 6.5



Definition 6.10

Let $G = (V, E)$ be a connected graph.
The minimum number of vertices whose removal makes G either a disconnected or reduces G in to a trivial graph is called its *vertex connectivity*, dented as $\kappa(G)$.

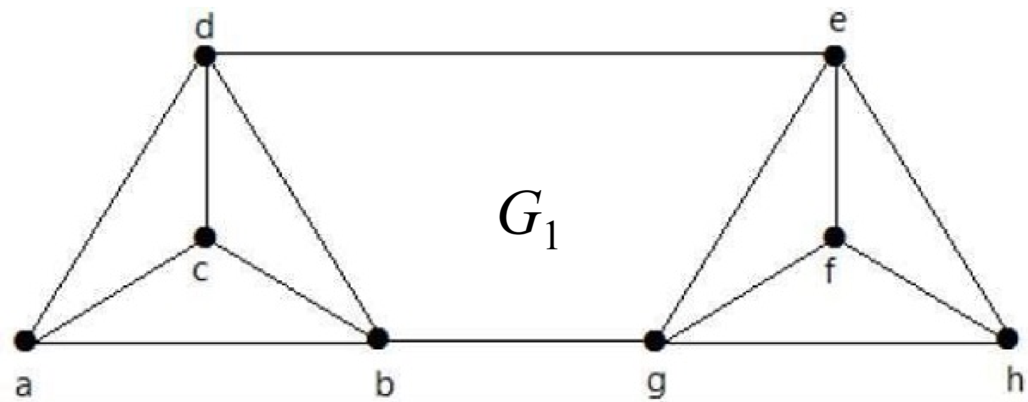
A graph G is said to be *k-connected* if $k \leq \kappa(G)$.

Likewise, the minimum number of edges whose removal makes G either a disconnected or reduces G in to a trivial graph is called *edge connectivity* of G dented as $\lambda(G)$.

A graph G is said to be *k-edge-connected* if $k \leq \lambda(G)$.

EXAMPLE 6.6

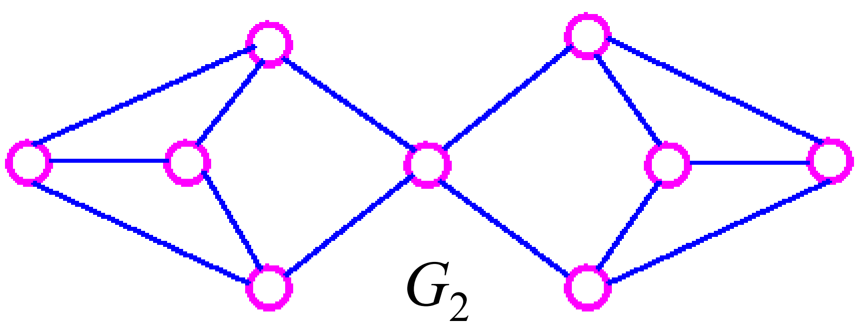
Calculate $\kappa(G_1)$ and $\lambda(G_1)$ for the following graphs.



Deleting the edges $\{d, e\}$ and $\{b, g\}$, we can disconnect G_1.

$\lambda(G_1) = 2$

$\kappa(G_1) = 2$ E.g. Deleting the vertices d and b.



$\kappa(G_2) =$

$\lambda(G_2) =$

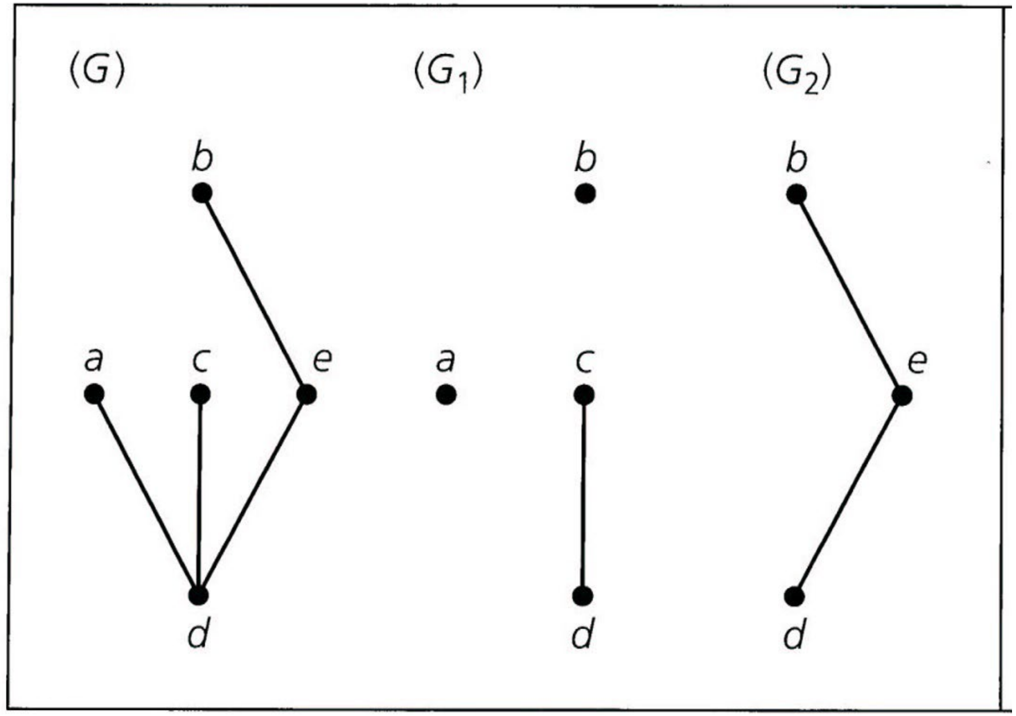
Definition 6.11

If $G = (V, E)$ is a graph (directed or undirected),
then $G_1 = (V_1, E_1)$ is called a *subgraph* of G
if $\emptyset \neq V_1 \subseteq V$ and $E_1 \subseteq E$,
where each edge in E_1 is incident with vertices in V_1.

Definition 6.12

Given a (directed or undirected) graph $G = (V, E)$,
let $G_1 = (V_1, E_1)$ be a subgraph of G.
If $V_1 = V$, then G_1 is called a *spanning subgraph* of G

EXAMPLE 6. 7

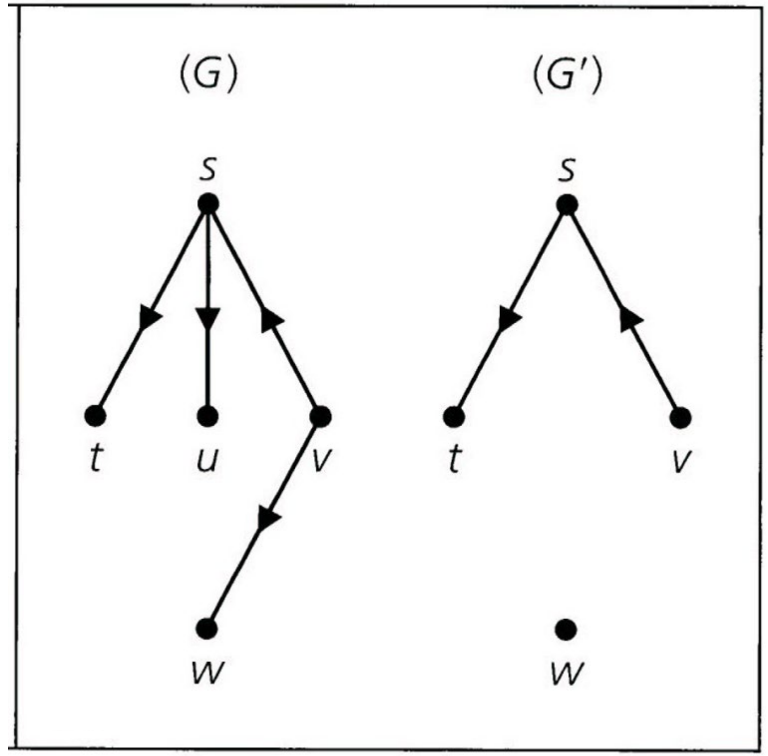


an undirected graph G and two of its subgraphs, G_1 and G_2.

The vertices a, b are isolated in subgraph G_1

neither G_1 nor G_2 is a spanning subgraph of G.

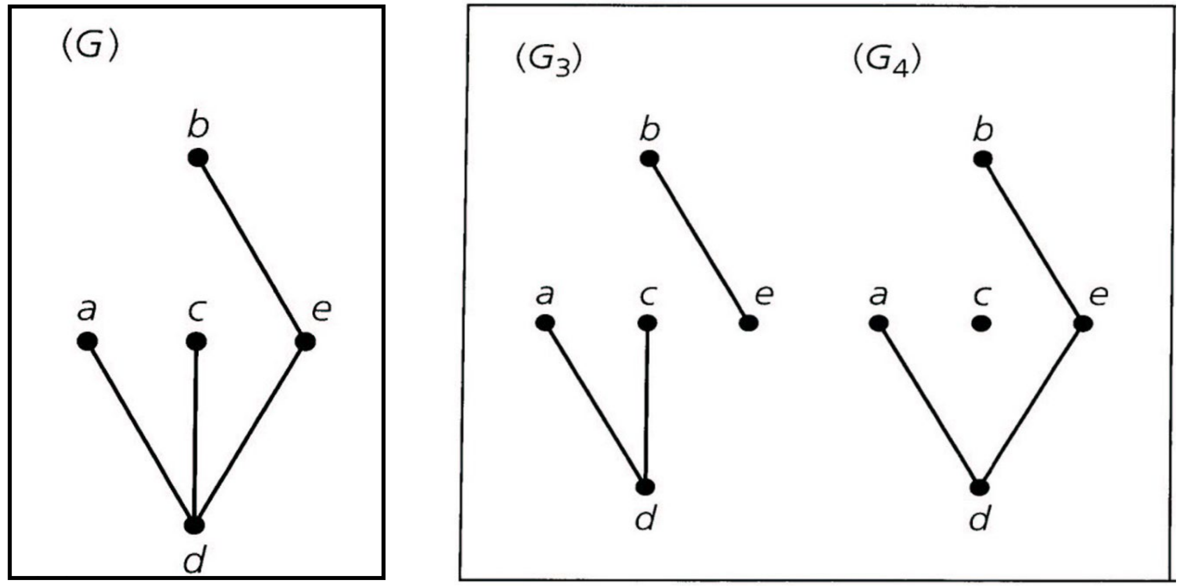
EXAMPLE 6.7 cont.



a directed G and the subgraph G'. Here vertex w is isolated in G'.

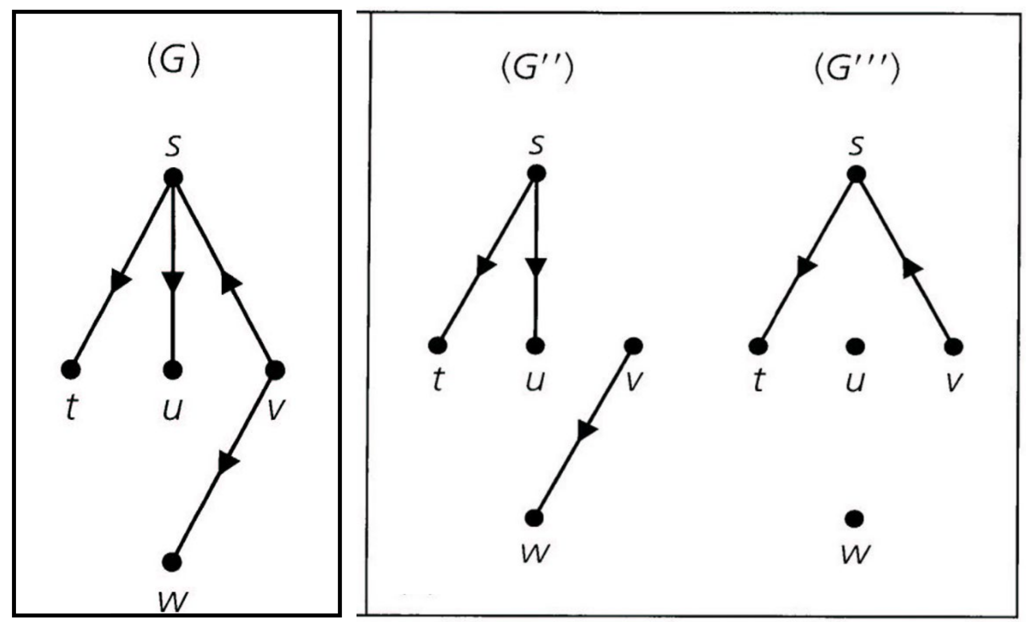
directed graph G' is a subgraph, but *not* a spanning subgraph of G.

EXAMPLE 6.7 cont.



G_3 and G_4 are both spanning subgraphs of G.

directed graphs G'' and G''' are two of the $2^4 = 16$ possible spanning subgraphs of G.



Definition 6.13

Let $G = (V, E)$ be a graph (directed or undirected).

If $\emptyset \neq U \subseteq V$, the *subgraph of G induced by U* is the subgraph whose vertex set is U and which contains all edges (from G) of either the form

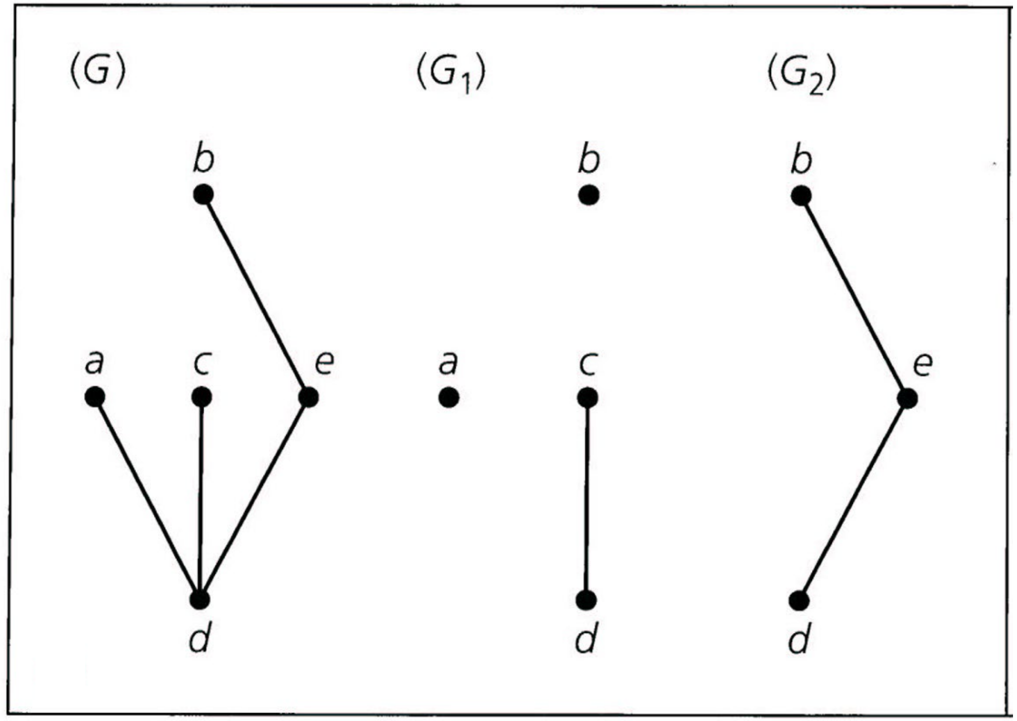
(a) (x, y), for $x, y \in U$ (when G is directed),

or (b) $\{x, y\}$, for $x, y \in U$ (when G is undirected).

We denote this subgraph by $\langle U \rangle$.

A subgraph G' of a graph $G = (V, E)$ is called an *induced subgraph* if there exists $\emptyset \neq U \subseteq V$, where $G' = \langle U \rangle$.

EXAMPLE 6.8

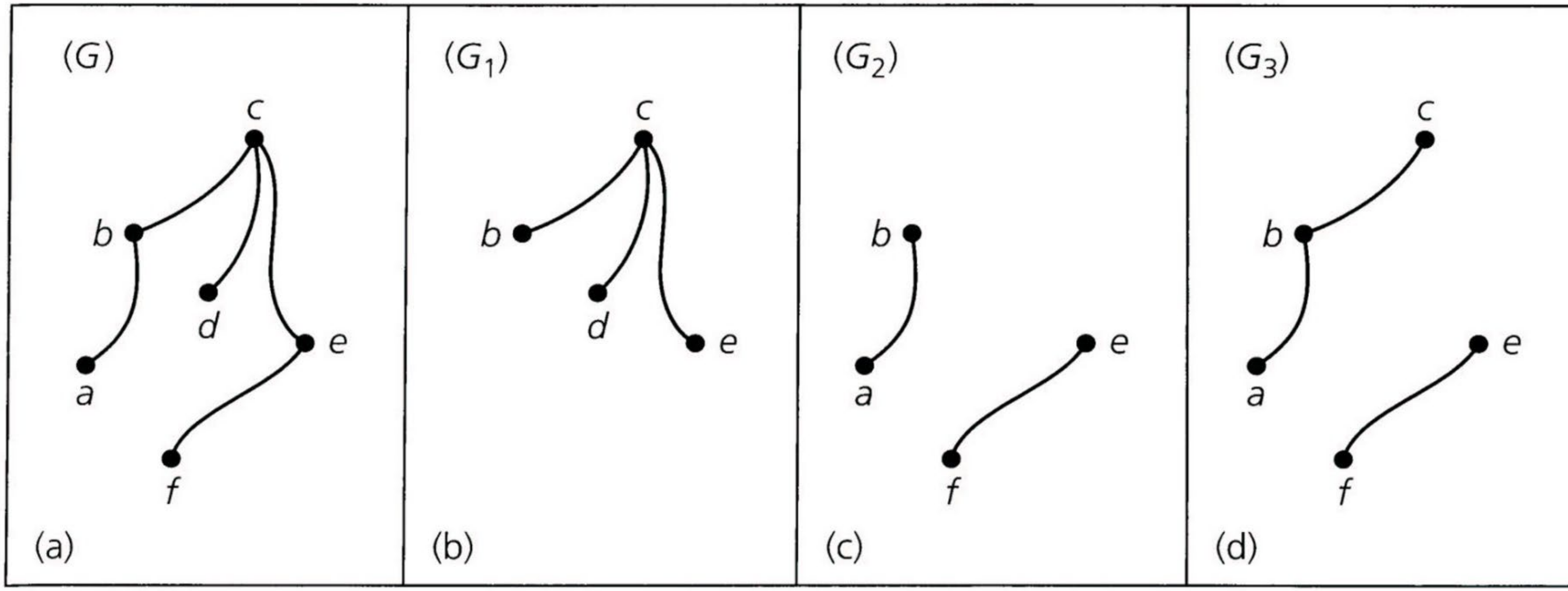


G_2 is an induced subgraph of G

but the subgraph G_1 is not an induced subgraph

because edge $\{a, d\}$ is missing.

EXAMPLE 6.9



The subgraphs in parts (b) and (c) are induced subgraphs of G.

$G_1 = \langle U_1 \rangle$ for $U_1 = \{b, c, d, e\}$.

$G_2 = \langle U_2 \rangle$ for $U_2 =$

G_3 is not an induced subgraph (edge $\{c, e\}$ is not present.)

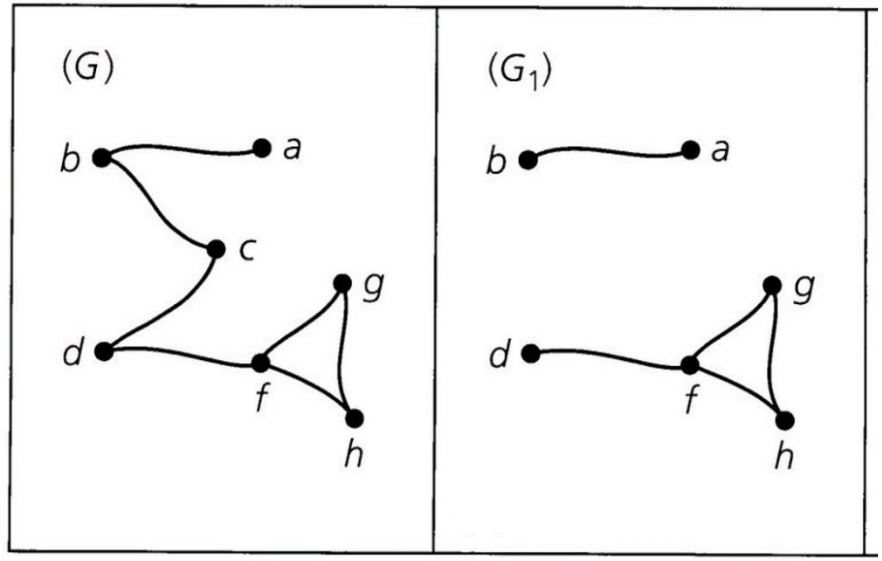
Definition 6.14

Let v be a vertex in a directed or an undirected graph $G = (V, E)$. The subgraph of G denoted by $G - v$ has the vertex set $V_1 = V - \{v\}$ and the edge set $E_1 \subseteq E$, where E_1 contains all the edges in E except for those that are incident with the vertex v.

(Hence $G - v$ is the subgraph of G induced by V_1.)

In a similar way, if e is an edge of a directed or an undirected $G = (V, E)$, we obtain the subgraph $G - e = (V_1, E_1)$ of G, where the set of edges $E_1 = E - \{e\}$, and the vertex set is unchanged (that is, $V_1 = V$).

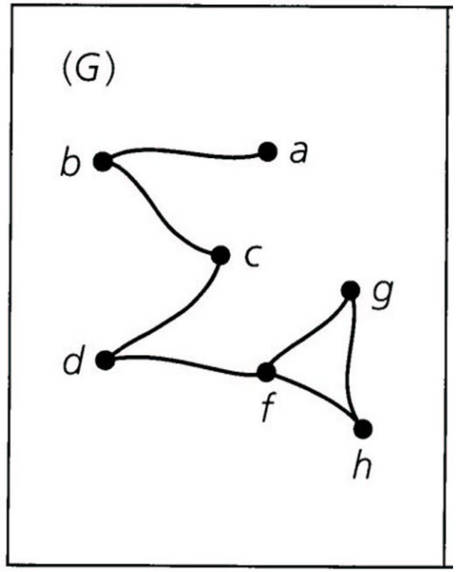
EXAMPLE 6.10

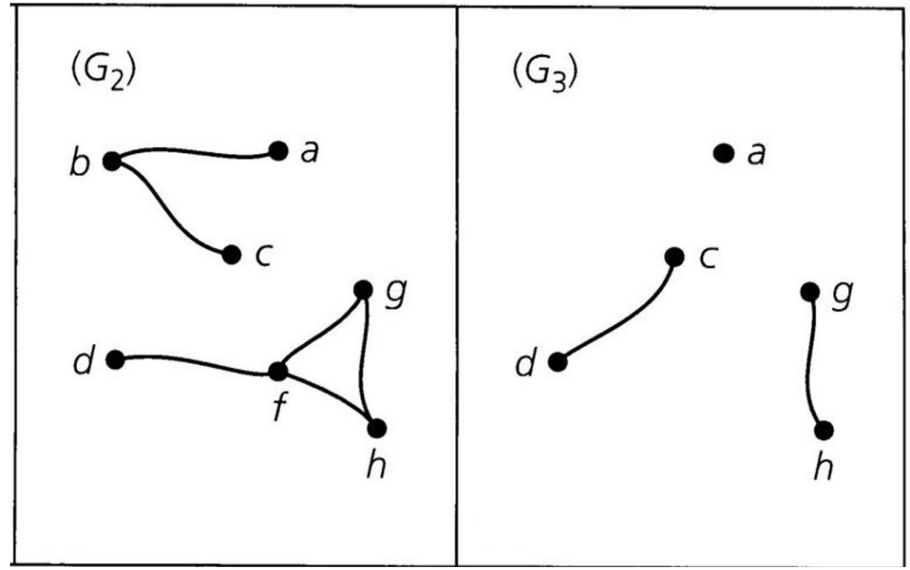


subgraph G_1 (of G), where $G_1 = G - c$.

It is also the subgraph of G induced by the set of vertices $U_1 = \{a, b, d, f, g, h\}$, so $G_1 = \langle V - \{c\} \rangle = \langle U_1 \rangle$.

EXAMPLE 6.10 Cont.





subgraph G_2 of G, where $G_2 = G - e$ for e the edge $\{c, d\}$.

$$G_3 = (G - b) - f = (G - f) - b$$
$$= G - \{b, f\} = \langle U_3 \rangle, \text{ for } U_3 = \{a, c, d, g, h\}.$$

Definition 6.15

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two undirected graphs.

A function $f : V_1 \to V_2$ is called a *graph isomorphism* if

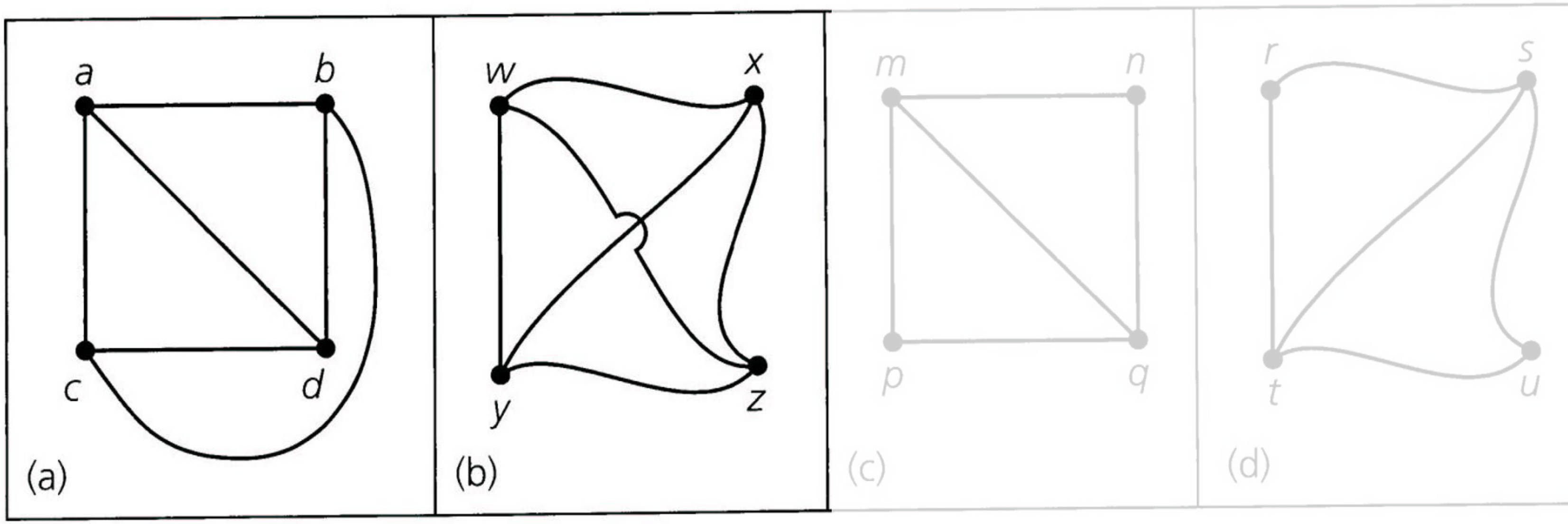
(a) f is one-to-one and onto, and

(b) for all $a, b \in V_1$, $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$.

When such a function exists, G_1 and G_2 are called *isomorphic graph*.

同構圖形

EXAMPLE 6.11

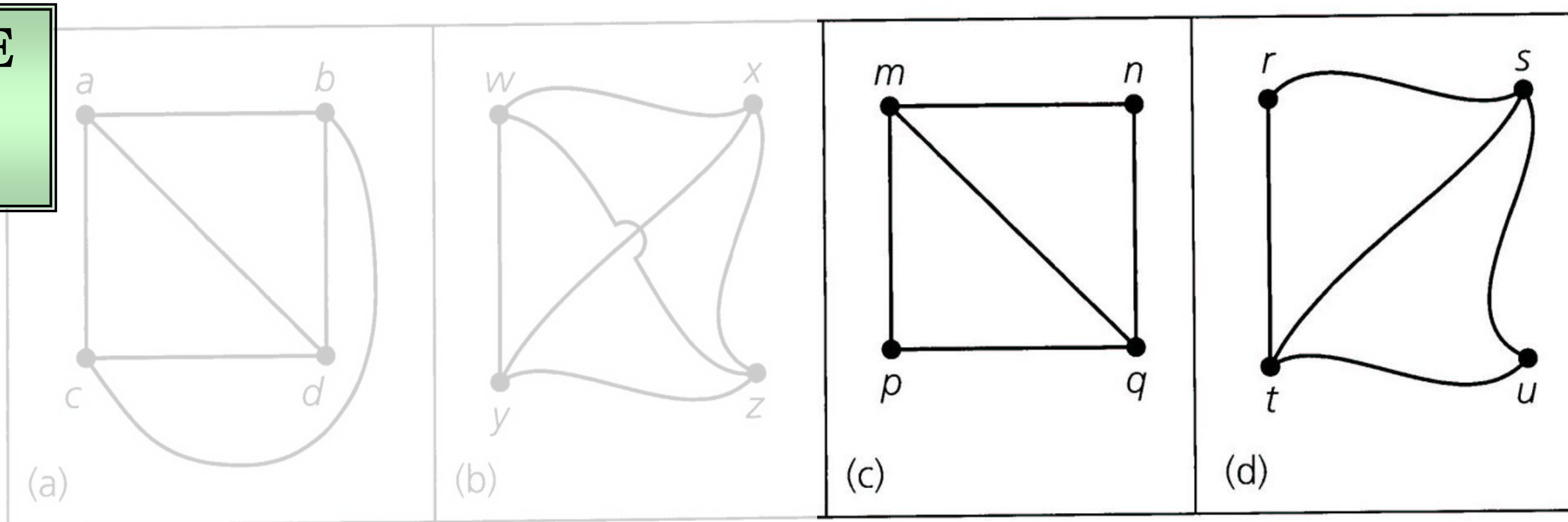


For the graphs in parts (a) and (b), the function f defined by

$$f(a) = w, \qquad f(b) = x, \qquad f(c) = y, \qquad f(d) = z$$

provides an isomorphism.

EXAMPLE 6.11 Cont.



For the graphs in parts (c) and (d), function g defined by

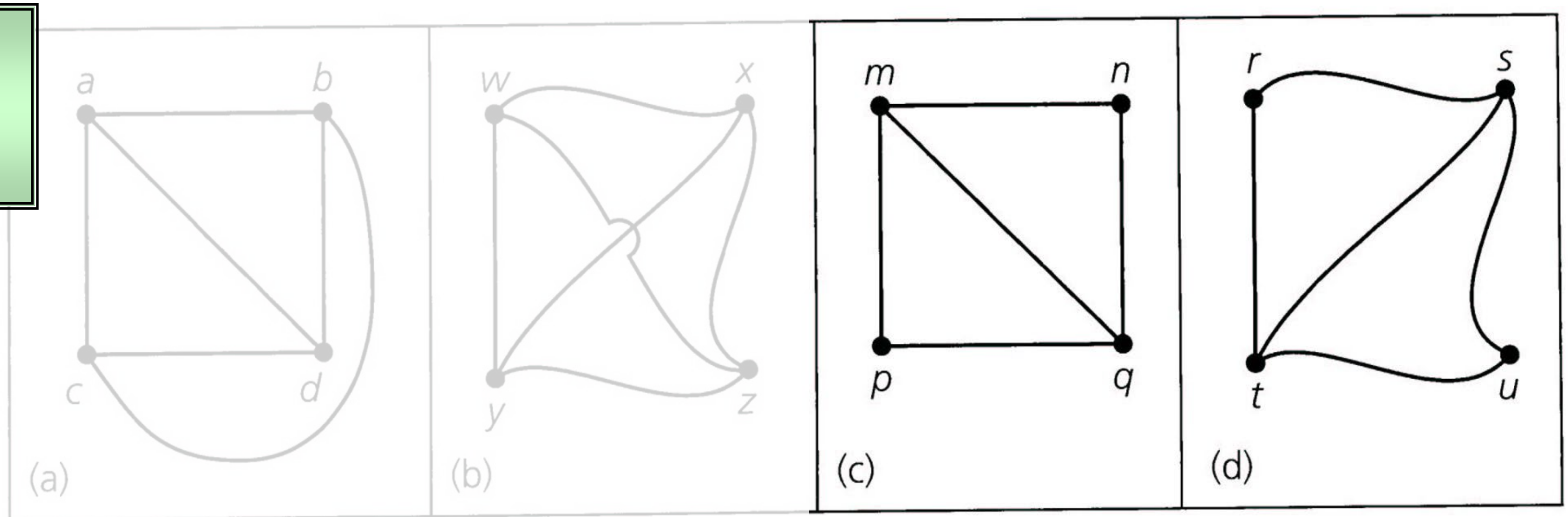
$$g(m) = r, \qquad g(n) = s, \qquad g(p) = t, \qquad g(q) = u$$

is one-to-one and onto

However, although $\{m, q\}$ is an edge in the graph of part (c), $\{g(m), g(q)\} = \{r, u\}$ is not an edge in the graph of part (d).

Thus, function g does *not* define a graph isomorphism.

EXAMPLE 6.11 Cont.



consider the one-to-one onto function h where

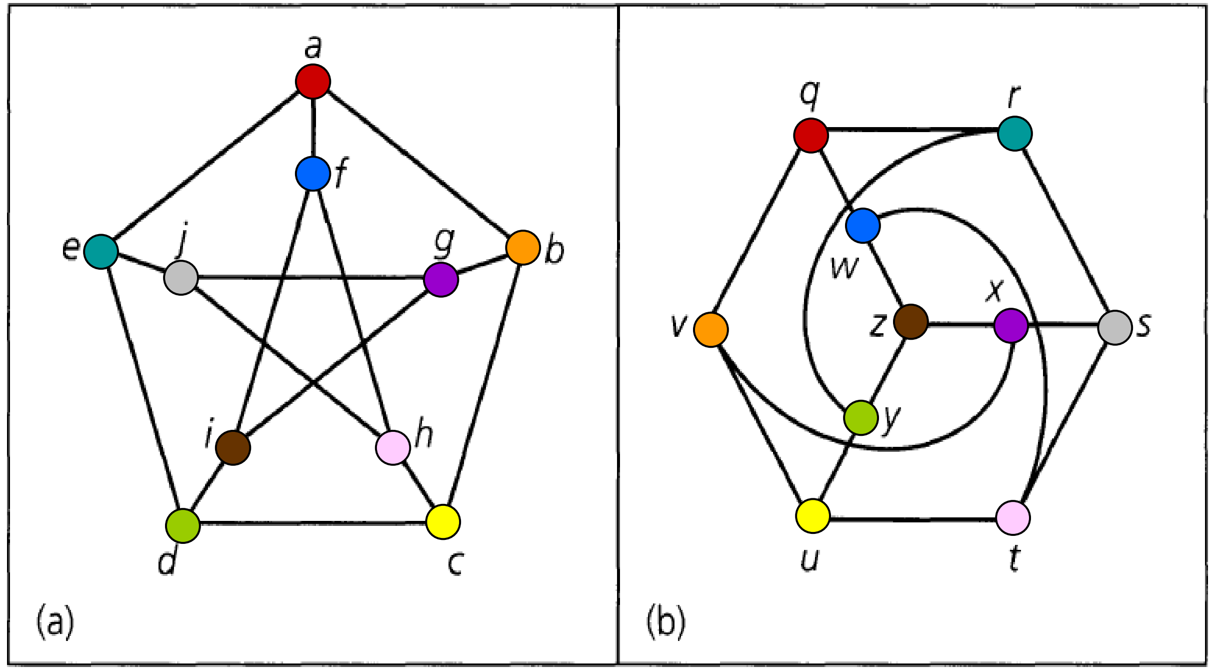
$$h(m) = \boxed{\quad} \qquad h(n) = \boxed{\quad} \qquad h(p) = \boxed{\quad} \qquad h(q) = \boxed{\quad}$$

In this case we have the edge correspondences

so h is a graph isomorphism.

EXAMPLE 6.12

Two graphs have identical numbers of edges and vertices.



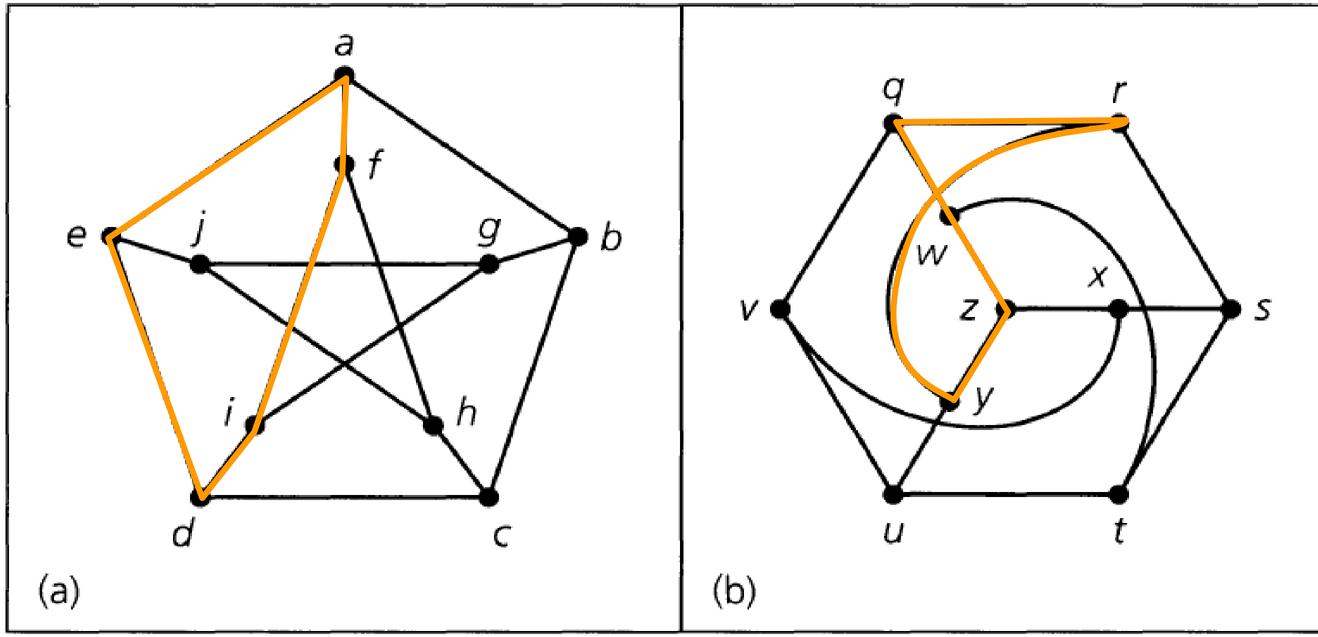
But, it is not immediately apparent whether or not these graphs are isomorphic.

One finds that the correspondence given by

$$\begin{array}{lllll} a \to q & c \to u & e \to r & g \to x & i \to z \\ b \to v & d \to y & f \to w & h \to t & j \to s \end{array}$$

preserves all adjacencies.

EXAMPLE 6.12 Cont.



We note that because an isomorphism preserves adjacencies, it preserves graph substructures such as paths and cycles.

In graph (a) the edges $\{a, f\}$, $\{f, i\}$, $\{i, d\}$, $\{d, e\}$, and $\{e, a\}$ constitute a cycle of length 5.

One possibility for the corresponding edges in graph (b) is $\{q, w\}$, $\{w, z\}$, $\{z, y\}$, $\{y, r\}$, and $\{r, q\}$, which also provides a cycle of length 5.

Note:

These are some of the ideas we can use to try to develop an isomorphism and determine whether two graphs are isomorphic.

However, there is no simple, foolproof method.

6.3 圖的種類、頂點次數及平面圖

Types of Graphs, Vertex Degree, and Planar Graph

Definition 6.16

Let G be an undirected graph or multigraph.

For each vertex v of G, the *degree of v*, written $\deg(v)$, is the number of edges in G that are incident with v.

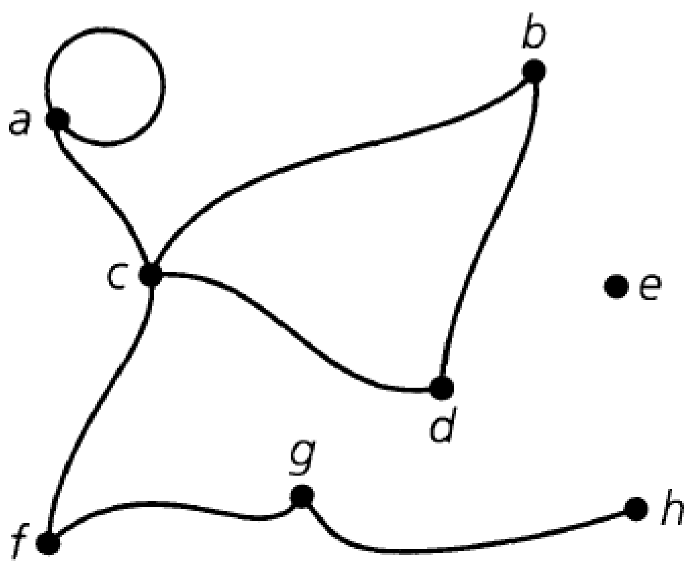
Here a loop at a vertex v is considered as two incident edges for v.

EXAMPLE 6.13

$$\deg(b) = \deg(d) = \deg(f) = \deg(g) = 2,$$

$$\deg(c) = 4, \deg(e) = 0, \text{ and } \deg(h) = 1.$$

For vertex a we have $\deg(a) = 3$ because we count a loop twice.



Since h has degree 1,

it is called a *pendant* vertex.

THEOREM 6.2

If $G = (V, E)$ is an undirected graph or multigraph, then $\sum_{v \in V} \deg(v) = 2|E|$.

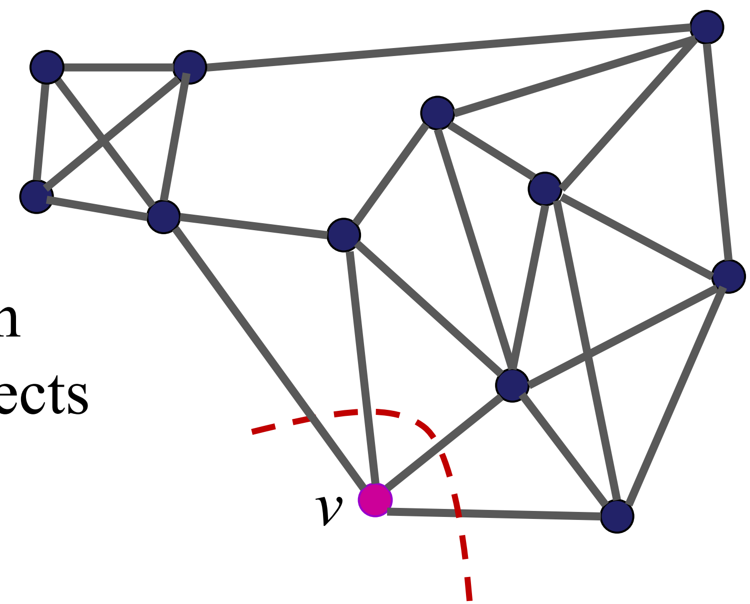
Proof:

consider each edge $\{a, b\}$ in graph G, we find that the edge contributes a count of 1 to each of $\deg(a)$, $\deg(b)$, and consequently a count of 2 to $\sum_{v \in V} \deg(v)$.

Thus $2|E|$ accounts for $\deg(v)$, for all $v \in V$, and $\sum_{v \in V} \deg(v) = 2|E|$.

THEOREM 6.3

Let G be an simple graph, then $\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$



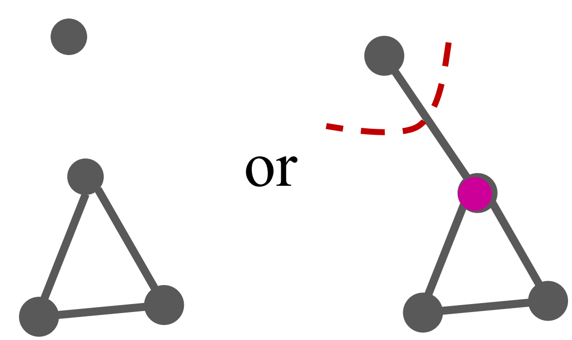
Proof:

Let $\delta(G) = \min_{v \in V} \deg(v)$

Let v be a vertex with $\deg(v) = \delta(G)$, then removing all edges incident to v disconnects v from the other vertices of G.

Therefore, $\lambda(G) \leq \delta(G)$.

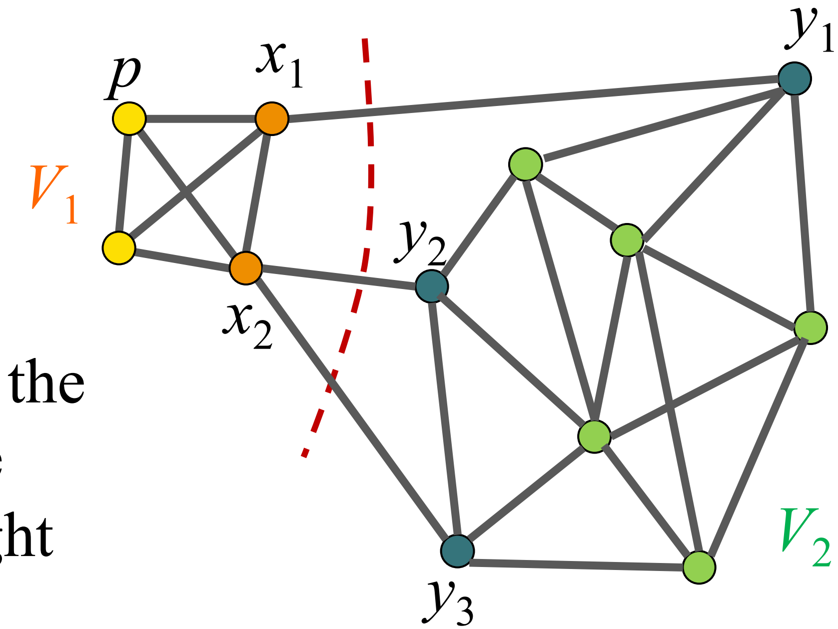
If $\lambda(G) = 0$ or 1, then $\kappa(G) = \lambda(G)$.



THEOREM 6.3 cont.

Let G be an arbitrary graph, then $\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$

Proof cont.:



If $\lambda(G) = k \geq 2$,

let $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_k, y_k\}$ be the edges whose removal causes G to be disconnected, where some x_i (y_i) might be identical.

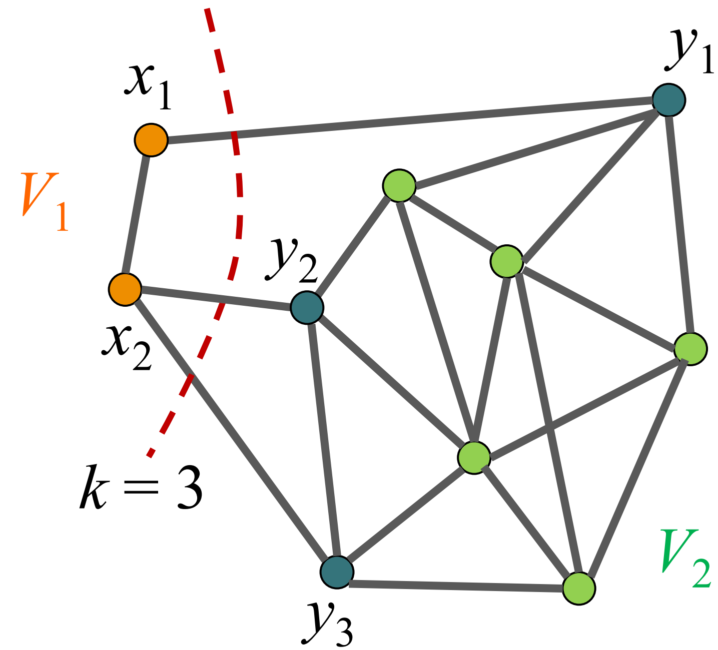
Denote V_1 and V_2 as the components of this disconnected graph.

Then, either V_1 contains a vertex p different from $x_1, x_2, \ldots, x_k$, meaning that removing $x_1, x_2, \ldots, x_k$, causes v to be disconnected from V_2. In this case, $\kappa(G) \leq k$. (some x_i might be identical).

THEOREM 6.3 cont.

Let G be an arbitrary graph, then $\kappa(G) \le \lambda(G) \le \min_{v \in V} \deg(v)$

Proof cont.:



Or, $V_1 = \{x_1, x_2, \ldots, x_k\}$, where $|V_1| \le k$ (some x_i might be identical).

Now, in this case, $\deg(x_i) \le k$, that is x_i has at most k neighbors.

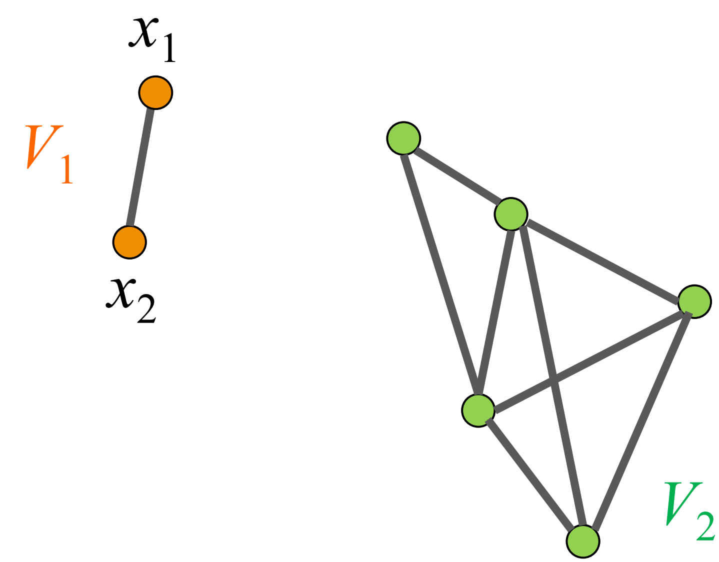
Since, $\lambda(G) = k$, and removal of the neighbors of x_i cause G to be disconnected. Thus, $\kappa(G) \le \lambda(G) = k$.

Therefore, we may conclude $\kappa(G) \le \lambda(G) \le \min_{v \in V} \deg(v)$

THEOREM 6.3 cont.

Let G be an arbitrary graph, then $\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$

Proof cont.:



Or, $V_1 = \{x_1, x_2, \ldots, x_k\}$, where $|V_1| \leq k$ (some x_i might be identical).

Now, in this case, $\deg(x_i) \leq k$, that is x_i has at most k neighbors.

Since, $\lambda(G) = k$, and removal of the neighbors of x_i cause G to be disconnected. Thus, $\kappa(G) \leq \lambda(G) = k$.

Therefore, we may conclude $\kappa(G) \leq \lambda(G) \leq \min_{v \in V} \deg(v)$

Definition 6.17

An undirected graph (or multigraph) where each vertex has the same degree is called a *regular* graph.

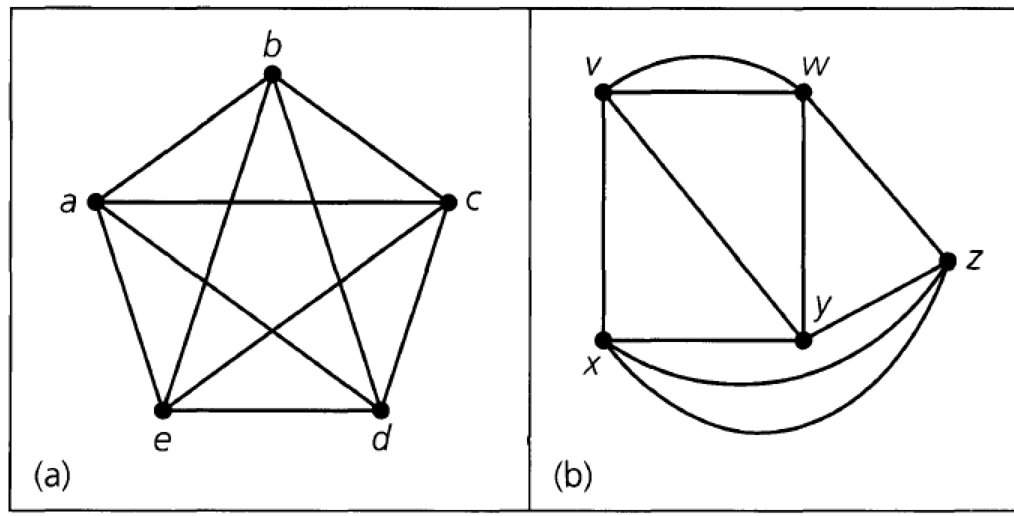
If $\deg(v) = k$ for all vertices v, then the graph is called *k-regular*.

EXAMPLE 6. 14

Is it possible to have a 4-regular graph with 10 edges?

$$2|E| = 20 = 4|V|,$$

so we have five vertices of degree 4.



(a) (b)

How about 15 edges?

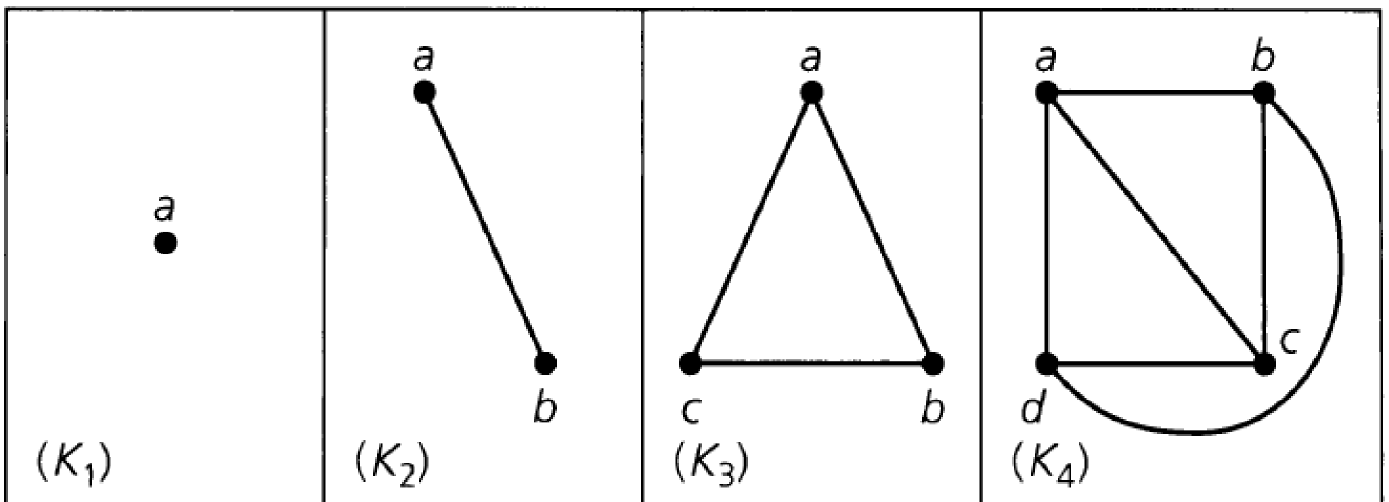
$$2|E| = 30 = 4|V|,$$

from which it follows that no such graph is possible.

Definition 6.18

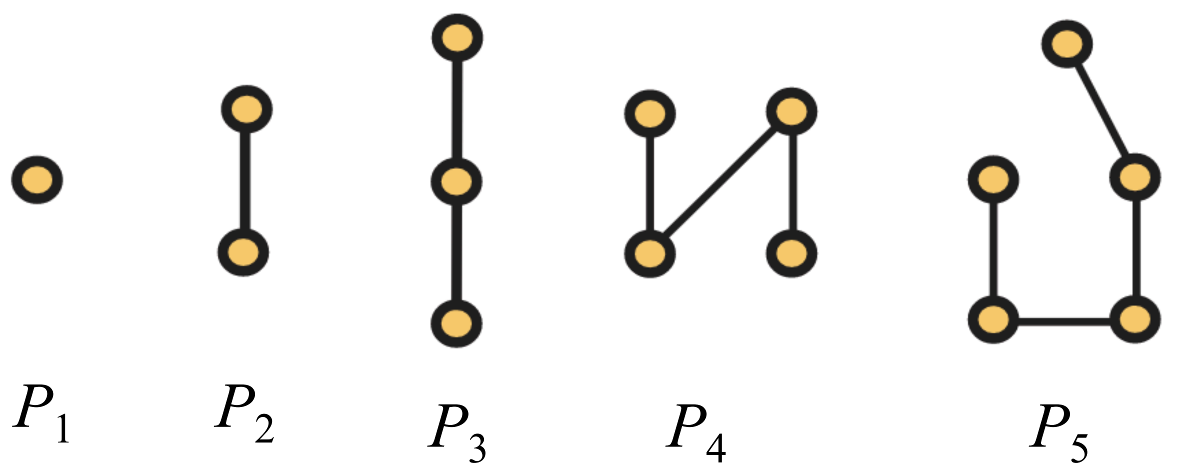
Let V be a set of n vertices.

The *complete graph* on V, denoted K_n is a loop-free undirected graph, where for all $a, b \in V$, $a \neq b$, there is an edge $\{a, b\}$.



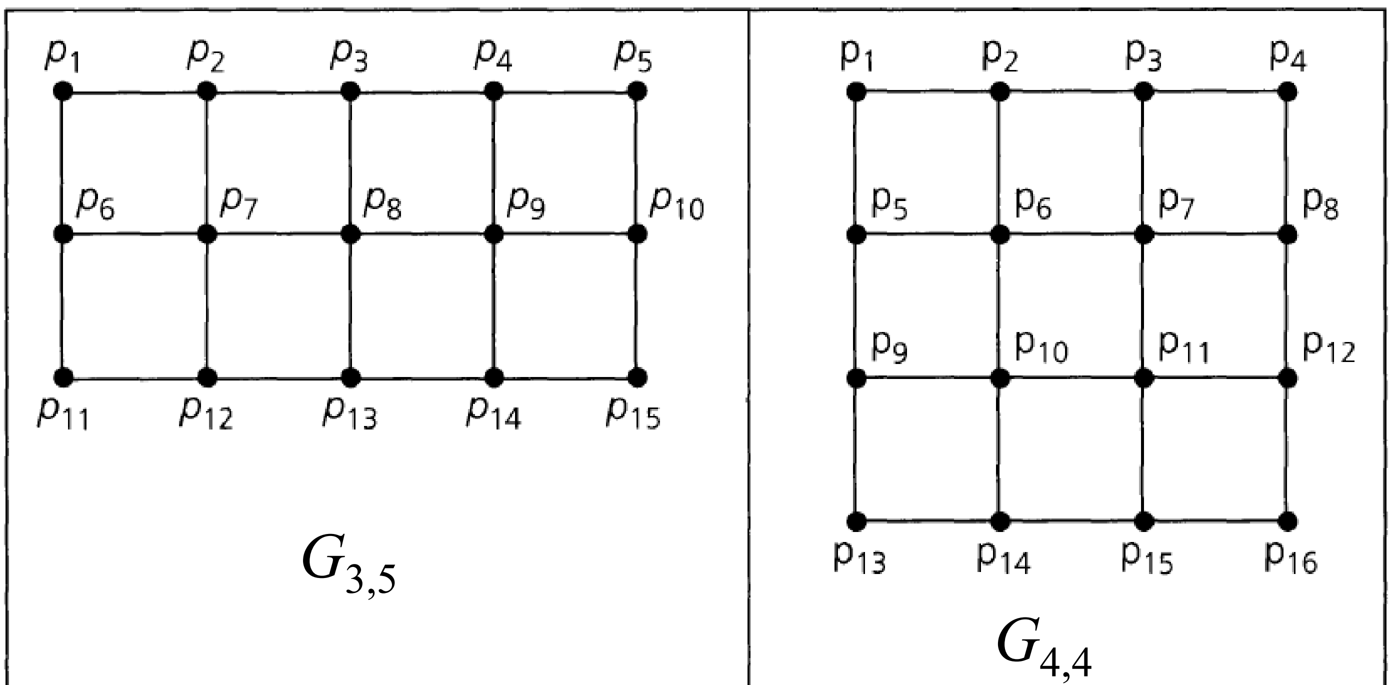
Definition 6.19

A *path* (or *linear*) graph, denoted as P_n, is a graph whose vertices can be listed in the order $v_1, v_2, \ldots, v_n$ such that the edges are $\{v_i, v_{i+1}\}$ where $i = 1, 2, \ldots, n-1$.



Definition 6.20

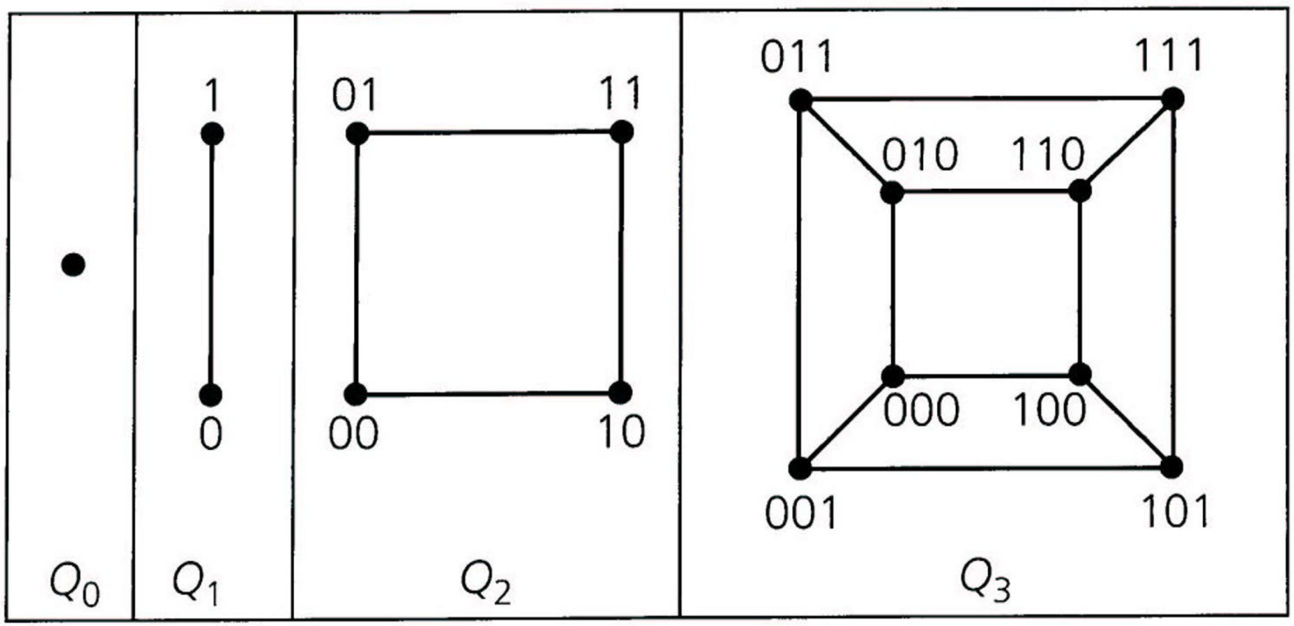
A two-dimensional *grid* (or *mesh* or *lattice*) graph, denoted as $G_{m,n}$, is a Cartesian product of two path graphs with m and n vertices.



Definition 6.21

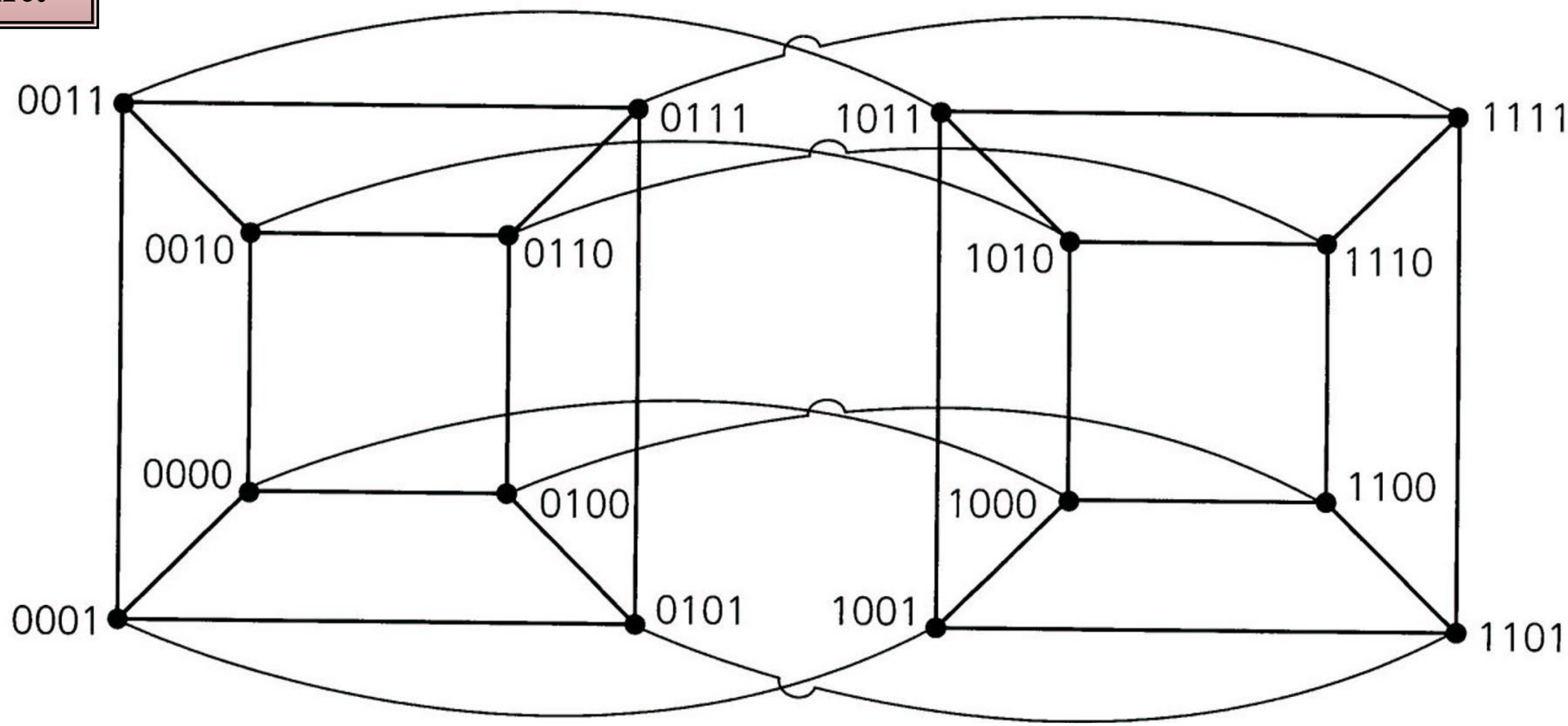
For $n \in \mathbf{N}$, the n-dimensional *hypercube* or *n-cube*, denoted by Q_n.

It is a loop-free connected undirected graph with 2^n vertices.



Two vertices are adjacent iff the symbols differ in exactly one coordinate.

Definition 6.21 cont.

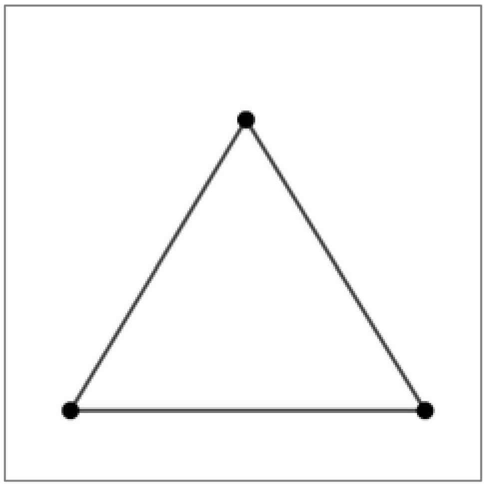


Q_4

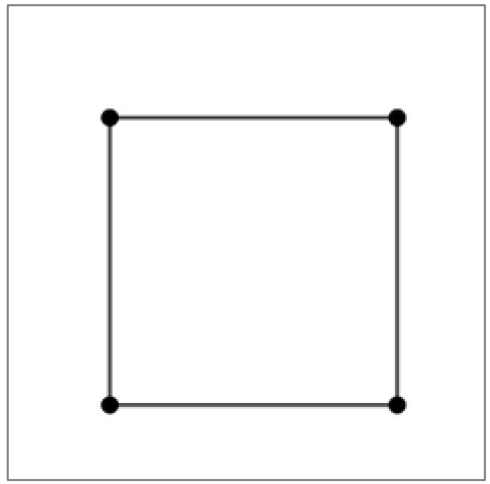
The hypercube Q_n is an n-regular loop-free undirected graph with $\boxed{}$ vertices.

Definition 6.22

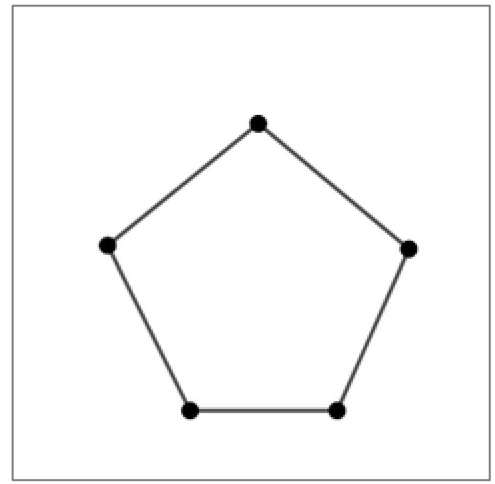
A *cycle* or *circular* graph, denoted as C_n, is a graph that consists of a single cycle with n vertices, where $n \geq 3$.



C_3



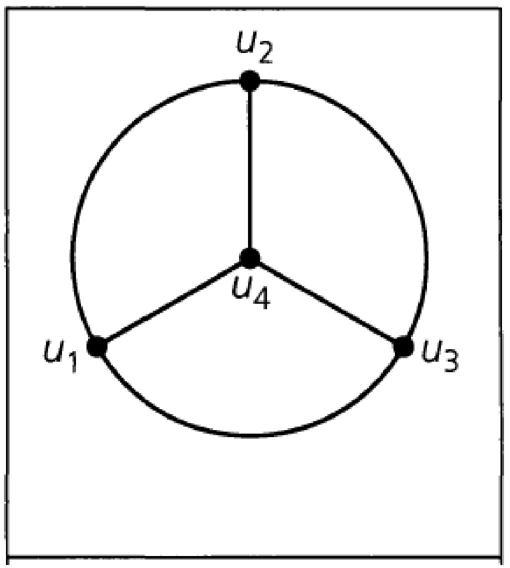
C_4



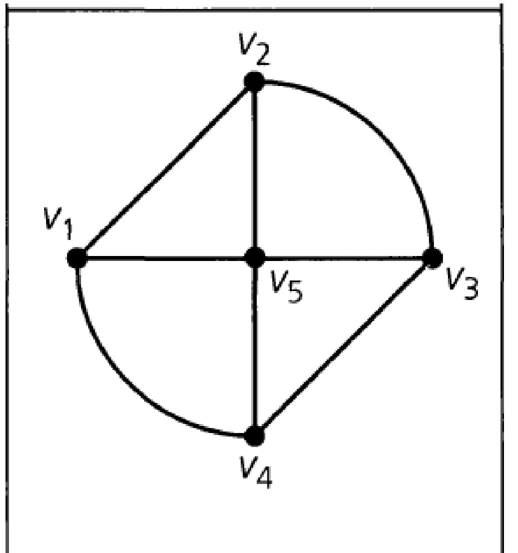
C_5

Definition 6.23

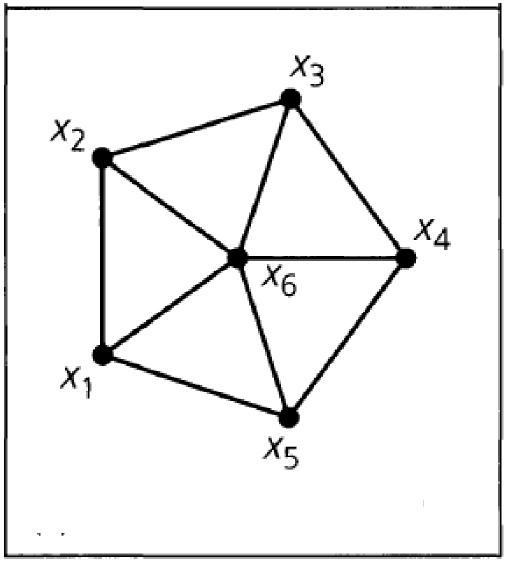
A *wheel* graph with n vertices, denoted as W_n, is a graph formed by connecting a single universal vertex to all vertices of a cycle.



W_4



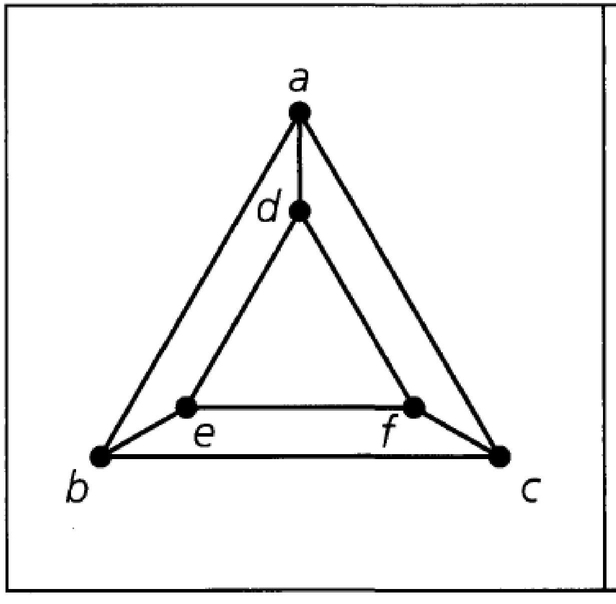
W_5



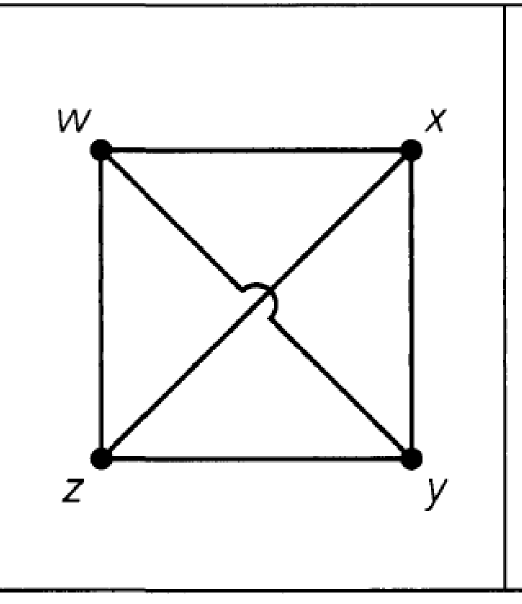
W_6

Definition 6.24

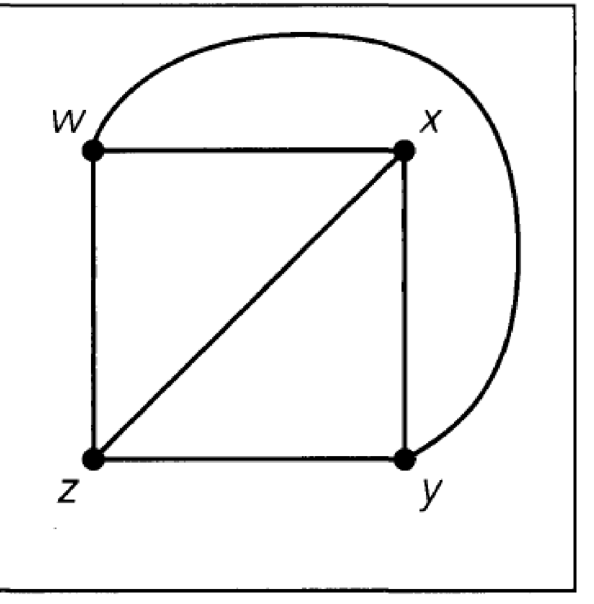
A graph (or multigraph) G is called *planar* if G can be drawn in the plane with its edges intersecting only at vertices of G. Such a drawing of G is called an *embedding* of G in the plane.



3-regular graph



K_4

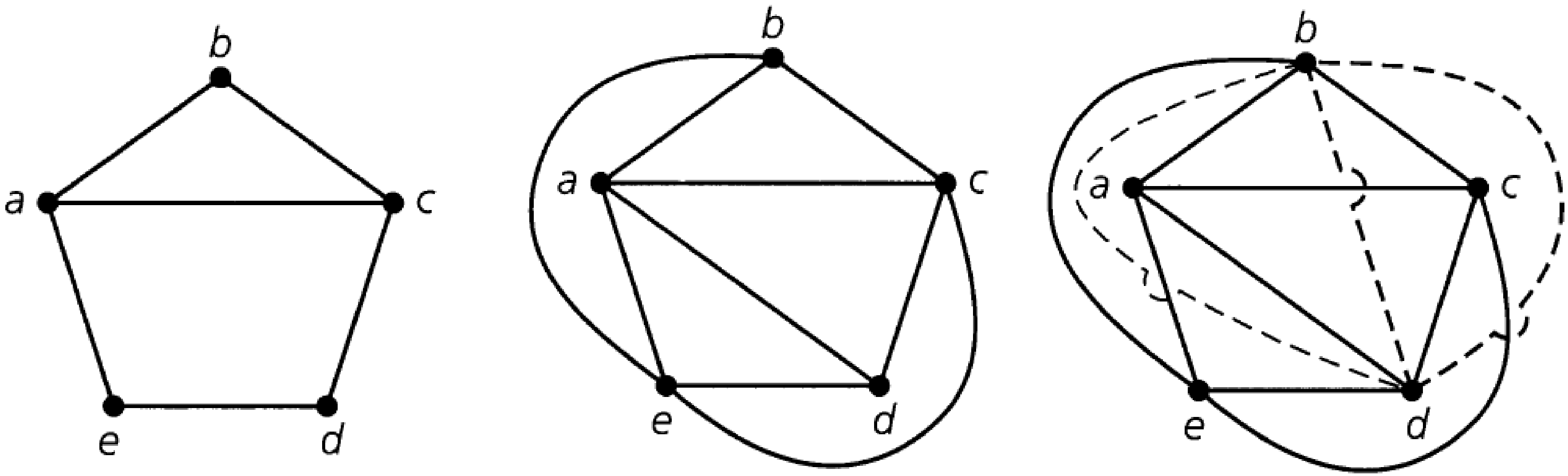


Embedding of K_4

EXAMPLE 6.15

Just as K_4 is planar, so are the graphs K_1, K_2, and K_3.

attempt to embed K_5 in the plane

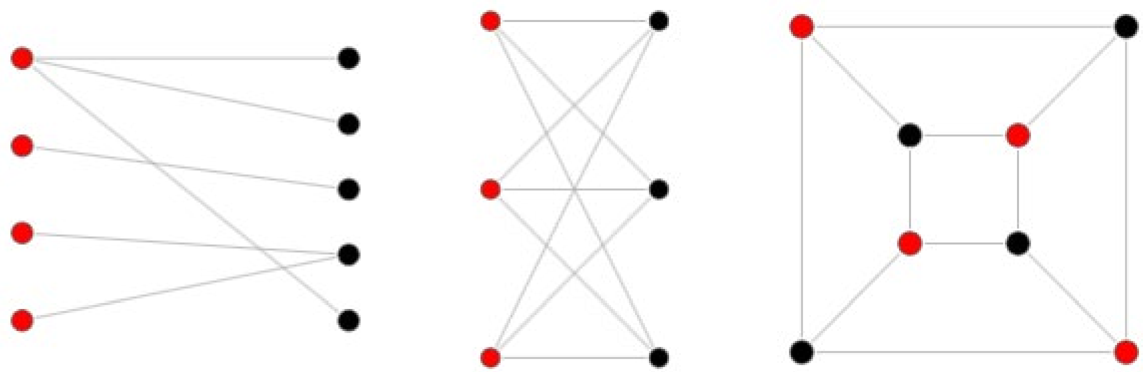


K_5 is nonplanar.

Definition 6.25

A graph $G = (V, E)$ is called *bipartite* if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, and every edge of G is of the form $\{a, b\}$ with $a \in V_1$ and $b \in V_2$.

If each vertex in V_1 is joined with every vertex in V_2, we have a *complete bipartite* graph. In this case, if $|V_1| = m$, $|V_2| = n$, the graph is denoted by $K_{m,n}$

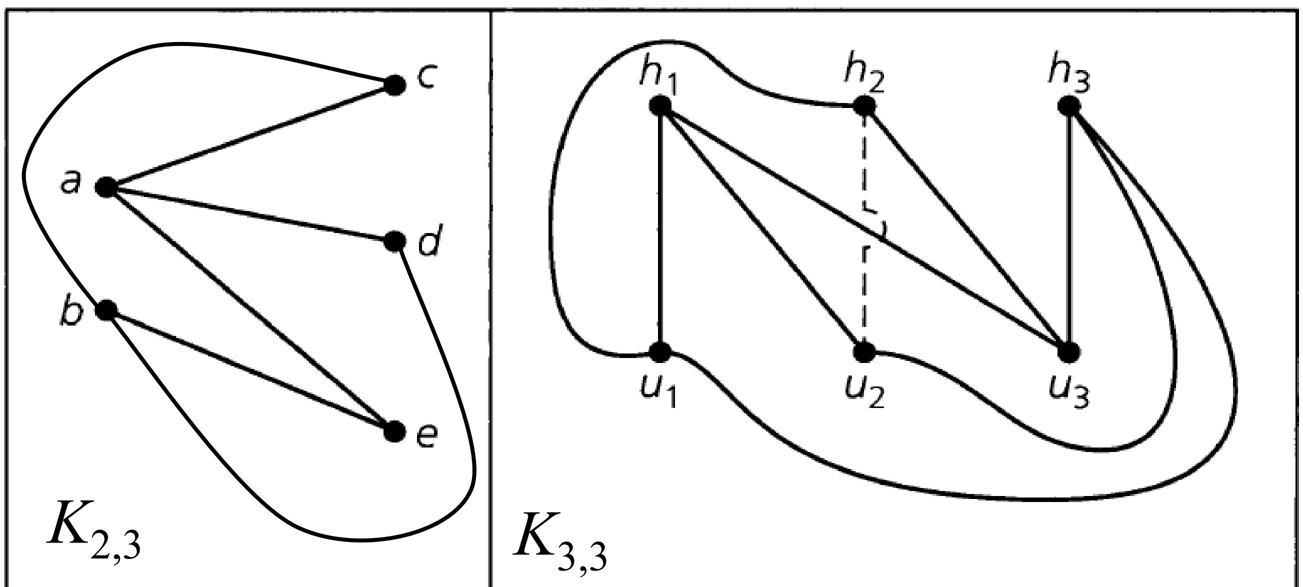


EXAMPLE 6.16

(a) hypercubes Q_n is bipartite for all $n \geq 1$.

(b) complete bipartite graph $K_{2,3}$ is planar.

(c) $K_{3,3}$ is



6.4 Eulerian & Hamiltonian 迴路

Definition 6.26

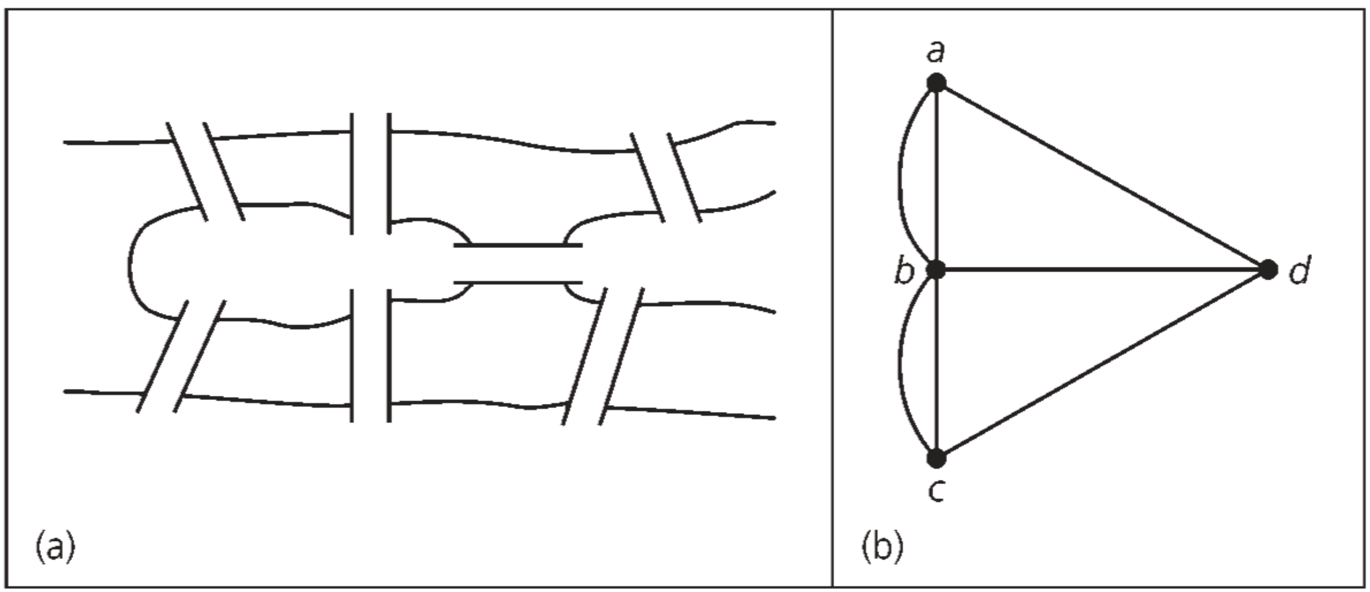
Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices. Then G is said to have an *Eulerian circuit* if there is a circuit in G that traverses every edge of the graph exactly once.

If there is an open trail from a to b in G and this trail traverses each edge in G exactly once, the trail is called an *Eulerian trail*.

EXAMPLE 6.17

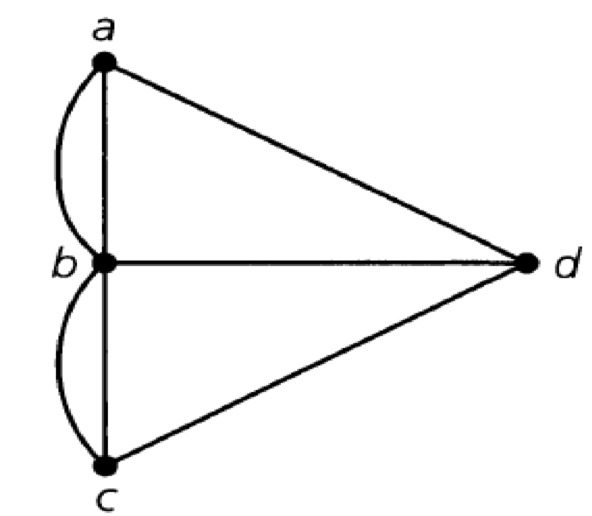
The Seven Bridges of Königsberg.

Königsberg 七橋問題。在 18 世紀間，Königsberg 這個城市 (在東普魯士) 被 Pregel 河分成四個區域 (包括 Kneiphof 島)。有七座橋連接這些區域，如 下圖所示。 據說居民花費他們的週日時間試著找一個走法使得每座橋恰經過一次且回到出發點。



EXAMPLE 6.17 Cont.

In order to determine whether or not such a circuit existed, Euler represented it by the multigraph.



(b)

he found four vertices with

$\deg(a) = \deg(c) = \deg(d) = 3$

and $\deg(b) = 5$.

the existence of Euler circuit depended on the number of vertices of odd degree in the graph.

THEOREM 6. 6

Let $G = (V, E)$ be an undirected graph or multigraph with no isolated vertices.

Then G has an Euler circuit if and only if G is connected and every vertex in G has even degree.

Proof:

($\Rightarrow$) Suppose that $G = (V, E)$ is a graph with a Eulerian circuit. Write down that Eulerian circuit here, as $\{v_0, v_1\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$, . . . , $\{v_{n-1}, v_n\}$, $\{v_n, v_0\}$. Here, v_i can be repeated. Every vertex v_i shows up an even number of times in the circuit, that is, deg(v) is even for every vertex $v \in V$.

(⇒) For example:

Please draw a graph without lifting your pen, and make sure the endpoint returns to the starting point.
（一筆畫完成任何圖，並返回起點）

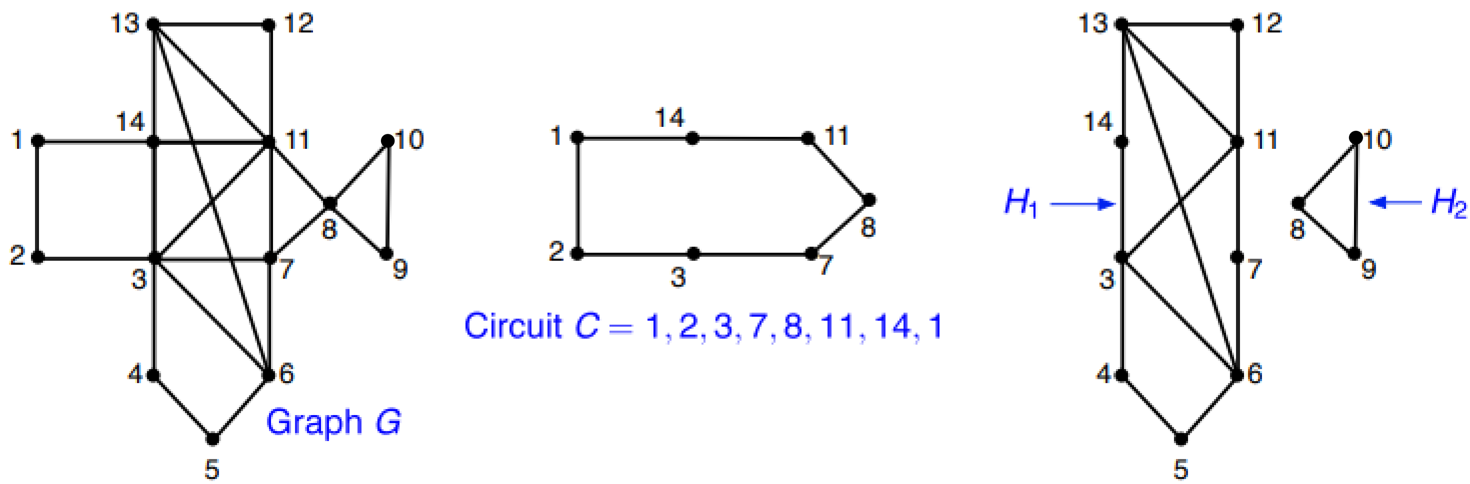
The circuit :

Every vertex v_i shows up an even number of times in the circuit, that is, deg(v) is even for every vertex $v \in V$.

THEOREM 6.6 Cont.

Proof cont.:

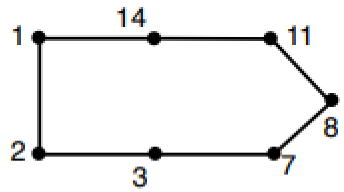
(⇐) Consider the following process for generating a cycle in G:



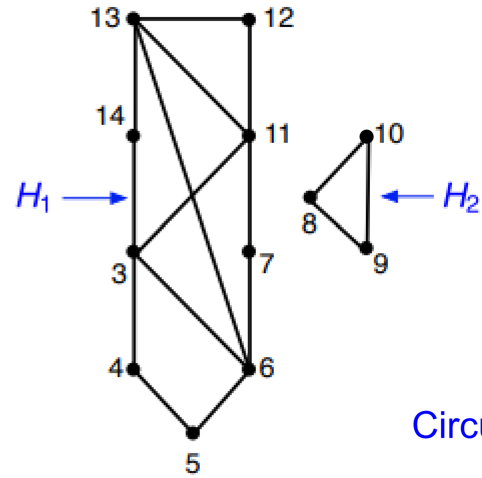
1. Pick a vertex v at random from V. Randomly find a circuit C in G. (it is always possible because every vertex has even degree)
2. Form a graph G' by removing from G all edges in the circuit C. In this case, G' is not connected, and H_1 and H_2 are two connected components of G. And all vertices in H_1 and H_2 have even degree.

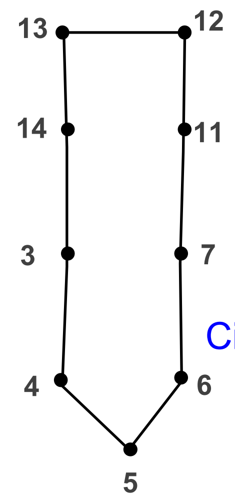
THEOREM 6.6 Cont.

Proof cont.: (⇐)

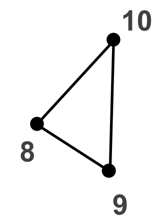


Circuit $C = 1, 2, 3, 7, 8, 11, 14, 1$

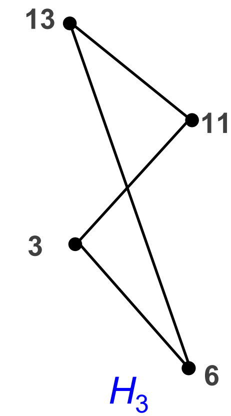




Circuit $C_1 = 3, 4, 5, 6, 7, 11, 12, 13, 14, 3$



Circuit $C_2 = 8, 9, 10, 8$

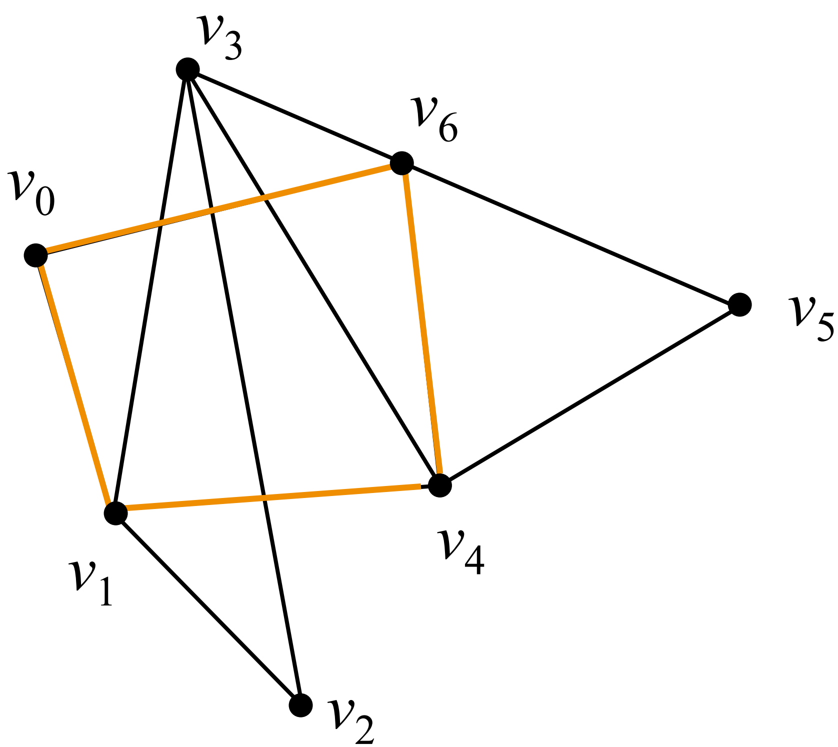


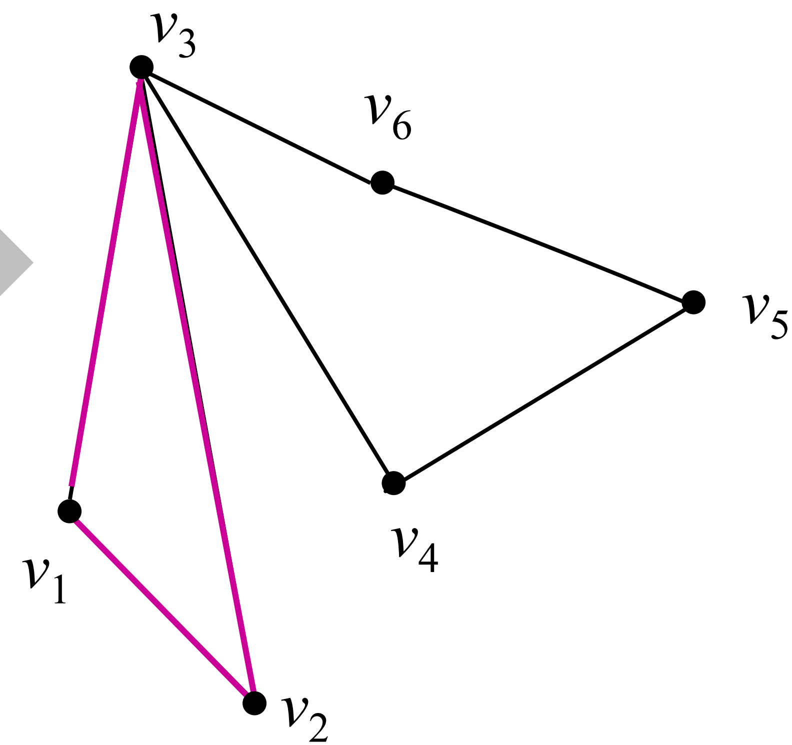
Circuit $C_3 = 3, 6, 13, 11, 3$

3. Repeat steps 1 and 2 for all the connected components of G’ until there is no edges left.
   In this case, pick C_1 and C_2, and results H_3, and then again pick C_3.
4. Now, paste circuits C_n to C_m one stage before, by traveling along C_n until we get to v_i , then taking the circuit C_m which starts and ends back at v_i, and then resuming the original circuit C_n.
5. We should finally obtain an Eulerian circuit of G.

Eulerian Circuit $C = 1, 2, 3, 6, 13, 11, 3, 4, 5, 6, 7, 11, 12, 13, 14, 3, 7, 8, 9, 10, 8, 11, 14, 1$

(⇐) Another example:





The circuit : $\{v_0, v_1\}$, $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_1\}$, $\{v_1, v_4\}$, $\{v_4, v_5\}$, $\{v_5, v_6\}$, $\{v_6, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_6\}$, $\{v_6, v_0\}$.

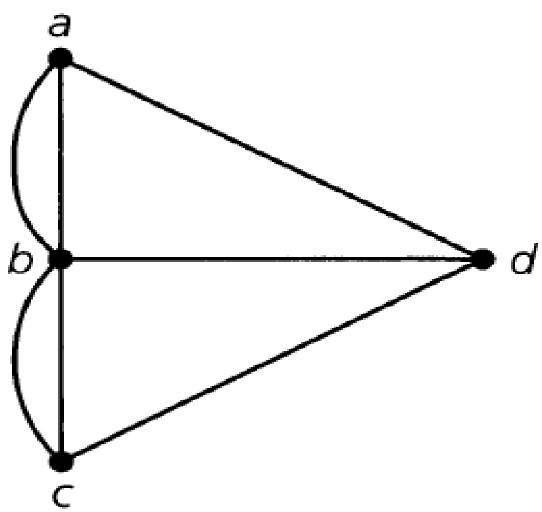
COROLLARY 6. 7

If G is an undirected graph or multigraph with no isolated vertices, then we can construct an Euler trail in G if and only if G is connected and has exactly two vertices of odd degree.

Proof: If G is connected and a and b are the vertices of G that have odd degree, add an additional edge $\{a, b\}$ to G.
We now have a graph G_1 that is connected and has every vertex of even degree. Hence G_1 has an Euler circuit C, and when the edge $\{a, b\}$ is removed from C, we obtain an Euler trail for G.

(Thus the Euler trail starts at one of the vertices of odd degree and terminates at the other odd vertex.)

EXAMPLE 6.17 Cont.



(b)

$\deg(a) = \deg(c) = \deg(d) = 3$

and $\deg(b) = 5$.

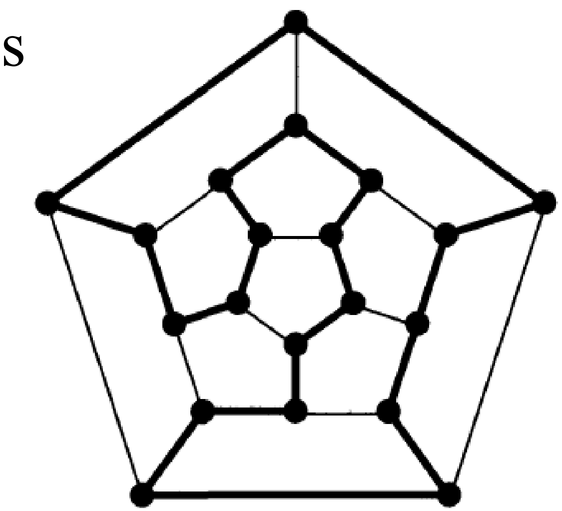
Returning now to the seven bridges of Königsberg,

it has four vertices of odd degree.

it has no Euler trail or Euler circuit.

Definition 6.27

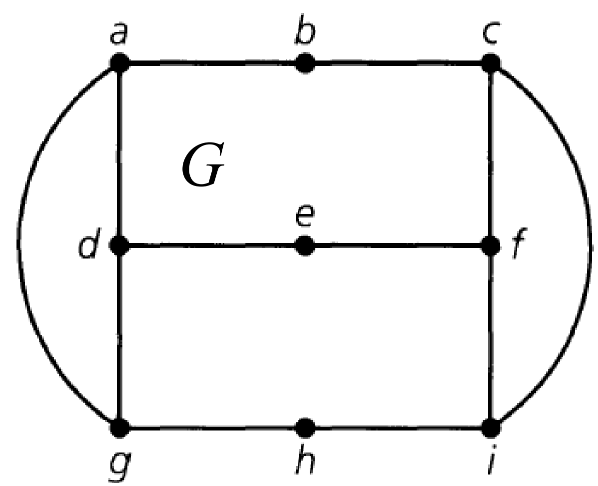
If $G = (V, E)$ is a graph or multigraph with $|V| \geq 3$, we say that G has a *Hamilton cycle* if there is a cycle in G that contains every vertex in V. A *Hamilton path* is a path (and not a cycle) in G that contains each vertex.



EXAMPLE 6.18

the edges $\{a, b\}$, $\{b, c\}$, $\{c, f\}$, $\{f, e\}$, $\{e, d\}$, $\{d, g\}$, $\{g, h\}$, $\{h, i\}$ yield a Hamilton path for G.

But does G have a Hamilton cycle?



Observations

1) If G has a Hamilton cycle, then for all $v \in V$, $\deg(v) \geq 2$.

2) If $a \in V$ and $\deg(a) = 2$, then the two edges incident with vertex a must appear in every Hamilton cycle for G.

3) If $a \in V$ and $\deg(a) > 2$, then as we try to build a Hamilton cycle, once we pass through vertex a, any unused edges incident with a are deleted from further consideration.

4) In building a Hamilton cycle for G, we cannot obtain a cycle for a subgraph of G unless it contains all the vertices of G.