Relation & Function In-class Exercises

1. Prove Theorem 5.1 (d)

對任意集合 A，B，$C \subseteq \mathcal{U}$：

d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

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for $a, b \in \mathcal{U}$.

$(a,b) \in (A \cup B) \times C$

$\Leftrightarrow \quad a \in (A \cup B) \ \wedge \ b \in C$

$\Leftrightarrow \quad [(a \in A) \vee (a \in B)] \wedge (b \in C)$

$\Leftrightarrow \quad [(a \in A) \wedge (b \in C)] \vee [(a \in B) \wedge (b \in C)]$

$\Leftrightarrow \quad [(a,b) \in (A \times C)] \vee [(a,b) \in (B \times C)]$

$\Leftrightarrow \quad (a,b) \in (A \times C) \cup (B \times C)$

2. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{w, x, y, z\}$, how many elements are there in $\mathcal{P}(A \times B)$?

$|A \times B| = |A| * |B| = 5 * 4 = 20$

$|\mathcal{P}(A \times B)| = 2^{20}$

3. Consider the relation $\mathcal{R}$ on the set $\mathbf{Z}$ where we define $a \mathcal{R} b$ when $ab \geq 0$.

Whether this binary relation $\mathcal{R}$ is reflexive, symmetric, or transitive?

For all integers x we have $xx = x^2 \geq 0$, so $x \mathcal{R} x$ and $\mathcal{R}$ is reflexive.
Also, if $x, y \in \mathbf{Z}$ and $x \mathcal{R} y$, then $x \mathcal{R} y \Rightarrow xy \geq 0 \Rightarrow yx \geq 0 \Rightarrow y \mathcal{R} x$, so the relation $\mathcal{R}$ is symmetric as well.
However, here we find that $(3, 0), (0, -7) \in \mathcal{R}$ since $(3)(0) \geq 0$ and $(0)(-7) \geq 0$ but $(3, -7) \notin \mathcal{R}$ because $(3)(-7) < 0$. this relation is *not* transitive.

4. For $x, y \in \mathrm{R}$ define $x \mathcal{R} y$ to mean that $x - y \in \mathrm{Z}$. Prove that $\mathcal{R}$ is an equivalence relation on R. Please show all workings.

To see that $\mathscr{R}$ is **reflexive**, let $x \in \mathbf{R}$.

Then $x - x = 0$ and $0 \in \mathbf{Z}$, so $x \mathscr{R} x$ for all $x \in \mathbf{R}$.

To see that $\mathscr{R}$ is **symmetric**, let $a, b \in \mathbf{R}$.

Suppose $a \mathscr{R} b$. Then $a - b \in \mathbf{Z}$, say $a - b = m$ where $m \in \mathbf{Z}$.

Then $b - a = -(a - b) = -m$ and $-m \in \mathbf{Z}$ Thus, $b \mathscr{R} a$.

For any $a, b \in \mathbf{R}$, $a \mathscr{R} b \Rightarrow b \mathscr{R} a$

To see that $\mathscr{R}$ is **transitive**, let $a, b, c \in \mathbf{R}$.

Suppose that $a \mathscr{R} b$ and $b \mathscr{R} c$. Thus $a - b \in \mathbf{Z}$, and $b - c \in \mathbf{Z}$.

Suppose $a - b = m$ and $b - c = n$, where $m, n \in \mathbf{Z}$.

Then $a - c = (a-b) + (b-c) = m + n$.

Now $m + n \in \mathbf{Z}$, it means $a - c \in \mathbf{Z}$. Therefore $a \mathscr{R} c$.

For any $a, b, c \in \mathbf{R}$, $a \mathscr{R} b$ and $b \mathscr{R} c \Rightarrow a \mathscr{R} c$

It now follows that $\mathscr{R}$ is an equivalence relation on the set $\mathbf{R}$.

5. For each of the following functions, determine whether it is one-to-one and determine its range.

a) $f: \mathbf{Z} \to \mathbf{Z},\ f(x) = 2x + 1$

b) $f: \mathbf{R} \to \mathbf{R},\ f(x) = e^x$

c) $f: [0, \pi] \to \mathbf{R},\ f(x) = \sin x$

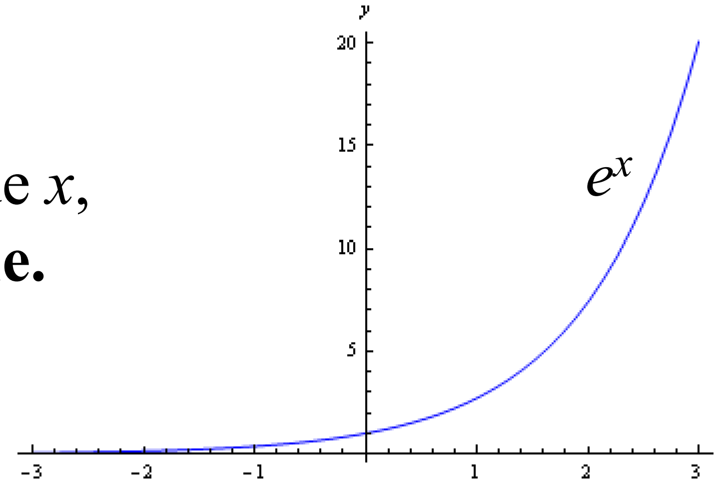
a) $f: \mathbf{Z} \to \mathbf{Z},\ f(x) = 2x + 1$

$$f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$$

One-to-one

Range is set of all odd integers.

b) $f: \mathbf{R} \to \mathbf{R},\ f(x) = e^x$



For each value of y, there is a unique x, such that $f(x) = y$. Thus, **One-to-one.**

Range is $\mathbf{R}^+$ or $(0, +)$

c) $f: [0, \pi] \to \mathbf{R},\ f(x) = \sin x$

Let $x_1 = 0$, $f(0) = \sin(0) = 0$,

$x_2 = \pi$, $f(\pi) = \sin(\pi) = 0$,

$f(x_1) = f(x_2)$ but $x_1 \neq x_2$. **Not One-to-one**

Range is range is [0, 1].

6. Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$.

(a) How many functions are there from A to B?

How many of these are one-to-one?

How many are onto?

(b) How many functions are there from B to A?

How many of these are onto?

How many are one-to-one?

(a) There are $6^4 (= |B|^{|A|})$ functions from A to B.

There are $P(|B|, |A|) = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times 2}{2} = 360$

one-to-one functions from A to B.

There is no/zero onto function from A to B, because $|B| \ngeq |A|$.

(b) There are $4^6 (= |A|^{|B|})$ functions from B to A.

There are 1560 onto functions from B to A.

$$\sum_{k=0}^{4} (-1)^k \binom{4}{4-k} (4-k)^6 = 1560$$

There is no/zero one-to-one function from B to A, because $|B| \nleq |A|$.

7.

Let $g: \mathbf{N} \to \mathbf{N}$ be defined by $g(n) = 2n$. If $A = \{1, 2, 3, 4\}$ and $f: A \to \mathbf{N}$ is given by

$$f = \{(1, 2), (2, 3), (3, 5), (4, 7)\},$$

find $g \circ f$.

$g \circ f = \{ (1, 4), (2, 6), (3, 10), (4, 14) \}$

8. Let $f, g: \mathbf{Z}^+ \to \mathbf{Z}^+$ where for all $x \in \mathbf{Z}^+$, $f(x) = x + 1$ and $g(x) = \max\{1, x - 1\}$, the maximum of 1 and $x - 1$.

a) Is g an onto function?

b) Is the function g one-to-one?

c) Show that $g \circ f = 1_{\mathbf{Z}^+}$.

a) For each value y in Z^+, there is a corresponding value x in Z^+ such that $g(x) = y$. E.g. if $y = 7$, then $x = 8$. Thus, g is an onto function.

b) We have $g(1) = 1 = g(2)$, but $1 \neq 2$, so g is not one-to-one.

c) For all $x \in \mathbf{Z}^+$,

$$
\begin{aligned}
(g \circ f)(x) &= g\big(f(x)\big) \\
&= g(x+1) \\
&= \max\{1, (x+1) - 1\}\} \\
&= \max\{1, x\} = x
\end{aligned}
$$

Here $x \in \mathbf{Z}^+$, thus $(g \circ f) = 1_{\mathbf{Z}^+}$

Relation & Function Suggested Exercises

1. Prove Theorem 5.1 (c)

對任意集合 A，B，$C \subseteq \mathcal{U}$：

c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$

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for $a, b \in \mathcal{U}$.

$(a, b) \in (A \cap B) \times C$

$\Leftrightarrow \quad a \in (A \cap B) \quad \wedge \quad b \in C$

$\Leftrightarrow \quad (a \in A) \wedge (a \in B) \wedge (b \in C)$

$\Leftrightarrow \quad (a \in A \wedge b \in C) \wedge (a \in B \wedge b \in C)$

$\Leftrightarrow \quad (a, b) \in (A \times C) \quad \wedge \quad (a, b) \in (B \times C)$

$\Leftrightarrow \quad (a, b) \in (A \times C) \cap (B \times C)$

2. If $A = \{1, 2, 3, 4\}$, give an example of a relation $\mathscr{R}$ on A that is

a) reflexive and symmetric, but not transitive

b) reflexive and transitive, but not symmetric

c) symmetric and transitive, but not reflexive

a) Reflexive and symmetric, but not transitive
examples are

$$\mathcal{R} = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (2,3), (3,2)\}$$
$$\mathcal{R} = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,1), (1,4), (4,1)\}$$

b) Reflexive and transitive, but not symmetric
examples are

$$\mathcal{R} = \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,3), (1,3)\}$$
$$\mathcal{R} = \{(1,1), (2,2), (3,3), (4,4), (1,2)\}$$

c) Symmetric and transitive, but not reflexive
examples are

$$\mathcal{R} = \{(1,2), (2,1), (1,1)\}$$
$$\mathcal{R} = \{(1,1), (2,2), (1,2), (2,1)\}$$

3. **a)** Rephrase the definitions for the reflexive, symmetric, transitive, and antisymmetric properties of a relation $\mathscr{R}$ (on a set A), using quantifiers.

b) Use the results of part (a) to specify when a relation $\mathscr{R}$ (on a set A) is (i) *not* reflexive; (ii) *not* symmetric; (iii) *not* transitive; and (iv) *not* antisymmetric.

a)

i. reflexive if $\forall x \in A (x,x) \in \mathcal{R}$

ii. symmetric if $\forall x, y \in A[(x,y) \in \mathcal{R} \Rightarrow (y,x) \in \mathcal{R}]$

iii. transitive if $\forall x, y, z \in A[(x,y),(y,z) \in \mathcal{R} \Rightarrow (x,z) \in \mathcal{R}]$

iv. antisymmetric if $\forall x, y \in A \quad [(x,y),(y,x) \in \mathcal{R} \Rightarrow x = y]$

b)

i. not reflexive if $\exists x \in A (x,x) \notin \mathcal{R}$

ii. not symmetric if $\exists x, y \in A[(x,y) \in \mathcal{R} \land (y,x) \notin \mathcal{R}]$

iii. not transitive if $\exists x, y, z \in A[(x,y),(y,z) \in \mathcal{R} \land (x,z) \notin \mathcal{R}]$

iv. not antisymmetric if $\exists x, y \in A[(x,y),(y,x) \in \mathcal{R} \land x \neq y]$

4. If $A = \{w, x, y, z\}$, determine the number of relations on A that are (a) reflexive; (b) symmetric; (c) reflexive and symmetric; (d) reflexive and contain (x, y); (e) symmetric and contain (x, y); (f) antisymmetric; (g) antisymmetric and contain (x, y); (h) symmetric and antisymmetric; and (i) reflexive, symmetric, and antisymmetric.

(a) $\binom{4}{2} = \frac{4!}{2!2!} = 6$

$2^{6\times2} = 2^{12}$

(b) $2^4 \times 2^6 = 2^{10}$

(c) 2^6

(d) 2^{11}

(e) $2^4 \times 2^5 = 2^9$

(f) $2^4 \times 3^6$

(g) $2^4 \times 3^5$

(h) 2^4

(i) 1

5. Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$.

(a) List a possible function from A to B.

(b) How many functions $f: A \to B$ are there?

(c) How many functions $f: A \to B$ are one-to-one? (d) How many functions $g: B \to A$ are there? (e) How many functions $g: B \to A$ are one-to-one? (f) How many functions $f: A \to B$ satisfy $f(1) = x$? (g) How many functions $f: A \to B$ satisfy $f(1) = f(2) = x$? (h) How many functions $f: A \to B$ satisfy $f(1) = x$ and $f(2) = y$?

(a) $\mathcal{F} = \{(1,\mathrm{b}), (2,\mathrm{b}), (3,\mathrm{b}), (4,\mathrm{b})\}$, where $b \in \{x, y, z\}$

(b) 3^4

(c) 0

(d) 4^3

(e) $4 \times 3 \times 2 = 24$

(f) 3^3

(g) 3^2

(h) 3^2

6. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{6, 7, 8, 9, 10, 11, 12\}$. How many functions $f: A \to B$ are such that $f^{-1}(\{6, 7, 8\}) = \{1, 2\}$?

Since $f^{-1}(\{6,7,8\}) = \{1,2\}$

$f\{(1, b), (2, b), (3, c), (4, c), (5, c)\}$

$b \in \{6,7,8\}$, $c \in \{9,10,11,12\}$

$3^2 \times 4^3 = 576$

7. Let $f: A \to B$, with $A_1, A_2 \subseteq A$. Then prove that

$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2);$$

For $b \in B$, $b \in f(A_1 \cup A_2)$

$\Leftrightarrow$ $b = f(a)$ for some $a \in (A_1 \cup A_2)$

$\Leftrightarrow$ $b = f(a)$ for some $(a \in A_1) \vee (a \in A_2)$

$\Leftrightarrow$ $[b = f(a)$ for some $a \in A_1] \vee [b = f(a)$ for some $a \in A_2]$

$\Leftrightarrow$ $b \in f(A_1) \vee b \in f(A_2)$

$\Leftrightarrow$ $b \in f(A_1) \cup f(A_2)$

Thus, $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$