Chapter 1

計數的基本原理

Fundamental Principle of Counting

1.1 和與積的規則

1.2 排列

1.3 組合：二項式定理

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1.1 和與積的規則

和規則 (The Rule of Sum)：若第一個工作可以 m 種方法來完成，而第二個工作可以 n 種方法來完成，且這兩個工作不可同時被執行，則執行任一工作可以 $m+n$ 種方法中的任一種來完成。

第一件工作 第二件工作

m ways n ways

can not be done simultaneously

then performing **either** task can be accomplished in any one of

ways

EXAMPLE 1.1

A college library has 40 textbooks on sociology and 50 textbooks dealing with anthropology.

By the rule of sum, a student at this college can select among

$$40 + 50 = 90 \text{ textbooks}$$

in order to learn more about one or the other of these two subjects.

某學院圖書館有 40 本社會學方面的教科書及 50 本和人類學有關的教科書，由和規則，這個學院的學生若要多學習關於這兩個學科中的一個或另一個時，他可有 40＋50＝90 本教科書的選擇。

積規則 (The Rule of Product)：假若一個程序可被分成第一及第二階段，且若第一階段有 m 種可能結果，且第一階段的每一結果在第二階段有 n 種可能結果，則整個程序，依指派順序，共有 mn 種完成的方法。

第一階段工作	第二階段工作
m ways	n ways

then performing this task can be accomplished in any one of $\boxed{}$ ways

EXAMPLE 1.2

The drama club of Central University is holding tryouts for a spring play. With six men and eight women auditioning for the leading male and female roles, by the rule of product the director can cast his leading couple in $6 \times 8 = 48$ ways.

中央大學戲劇社正準備進行春季公演試演。有 6 男 8 女為爭取成男主角及女主角正在進行試鏡；由積規則，導演可有 $6 \times 8 = 48$ 種方法來選取男女主角。

EXAMPLE 1.3

The license plate: 4 digits - 2 letters

(a) no letter or digit can be repeated

$$10\times 9\times 8\times 7\times 26\times 25$$

(b) with repetitions allowed

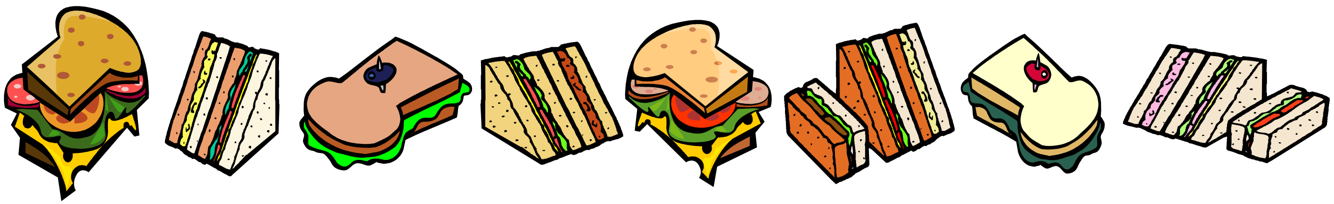
$$10\times 10\times 10\times 10\times 26\times 26$$

(c) same as (b), but only even digits and vowels

$$5^4\times 5^2$$

EXAMPLE 1.4

Quick Snack Coffee Shop



&

OR

&

有幾種早餐組合?

1.2 排列

Permutations

EXAMPLE 1.5

In a class of 10 students, five are to be chosen and seated in a row for a picture.

How many such linear arrangements are possible?

The key word here is *arrangement*, which designates the importance of *order*.

10	$\times$	9	$\times$	8	$\times$	7	$\times$	6
1st position		**2nd position**		**3rd position**		**4th position**		**5th position**

This yields a total of 30,240 possible arrangements

Definition 1.1

For an integer $n \geq 0$, n ***factorial*** (denoted $n!$) is defined by

$$0! = 1,$$

$$n! = (n)(n\text{-}1)(n\text{-}2) \cdots (3)(2)(1), \ \text{for } n \geq 1.$$

對一整數 $n \geq 0$，n **階乘** (factorial) (表為 $n!$) 被定義為

$$0! = 1,$$

$$n! = (n)(n-1)(n-2) \cdots (3)(2)(1), \ 對 \ n \geq 1.$$

Beware how fast $n!$ increases.

$$2^{10} = 1024$$

$10! = 3628800$ it just so happens that this is exactly the number of *seconds* in six *weeks*. Consequently,

11! exceeds the number of seconds in one *year*,

12! exceeds the number in 12 years, and

13! surpasses the number of seconds in a *century*.

If we make use of the factorial notation, the answer in Example 1.9 can be expressed in the following more compact form:

$$
\begin{aligned}
&10 \times 9 \times 8 \times 7 \times 6 \\
&= 10 \times 9 \times 8 \times 7 \times 6 \times \frac{5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} \\
&= \frac{10!}{5!}.
\end{aligned}
$$

Definition 1.2

Given a collection of n **distinct** objects, any (linear) arrangement of these objects is called a ***permutation*** of the collection.

給一個含 n 個不同物體的群體，則這些物體的任一 (線性) 安排被稱為這個群體的一個**排列** (permutation)。

If there are n distinct objects and r is an integer, with $1 \le r \le n$, then by the rule of product, the number of permutations of size r for the n objects is

若有 n 個不同物體且 r 為整數，滿足 $1 \le r \le n$，則由積規則，n 個物體中大小為 r 的排列數是

$$P(n, r) = \underset{\substack{\text{第一}\\\text{位置}}}{n} \times \underset{\substack{\text{第二}\\\text{位置}}}{(n-1)} \times \underset{\substack{\text{第三}\\\text{位置}}}{(n-2)} \times \cdots \times \underset{\substack{\text{第}r\\\text{位置}}}{(n-r+1)}$$

$$= (n)(n-1)(n-2)\cdots(n-r+1) \times \frac{(n-r)(n-r-1)\cdots(3)(2)(1)}{(n-r)(n-r-1)\cdots(3)(2)(1)}$$

$$= \frac{n!}{(n-r)!}.$$

EXAMPLE 1.6

The number of permutations of the letters in the word COMPUTER is $8!$.

If only five of the letters are used, the number of permutations (of size 5) is $P(8, 5) = 8!/(8-5)! = 8!/3! = 6720$.

If repetitions of letters are allowed, the number of possible 12-letter sequences is $8^{12} \doteq 6.872 \times 10^{10}$.

Note: The symbol "$\doteq$" is read "is approximately equal to."

EXAMPLE 1.7

The number of (linear) arrangements of the four letters in BALL is 12, not 4! (= 24).

Table 1.1

A	B	L	L	A	B	L_1	L_2	A	B	L_2	L_1
A	L	B	L	A	L_1	B	L_2	A	L_2	B	L_1
A	L	L	B	A	L_1	L_2	B	A	L_2	L_1	B
B	A	L	L	B	A	L_1	L_2	B	A	L_2	L_1
B	L	A	L	B	L_1	A	L_2	B	L_2	A	L_1
B	L	L	A	B	L_1	L_2	A	B	L_2	L_1	A
L	A	B	L	L_1	A	B	L_2	L_2	A	B	L_1
L	A	L	B	L_1	A	L_2	B	L_2	A	L_1	B
L	B	A	L	L_1	B	A	L_2	L_2	B	A	L_1
L	B	L	A	L_1	B	L_2	A	L_2	B	L_1	A
L	L	A	B	L_1	L_2	A	B	L_2	L_1	A	B
L	L	B	A	L_1	L_2	B	A	L_2	L_1	B	A

EXAMPLE 1.7 Cont.

$$2 \times (\text{Number of arrangements of the letters B, A, L, L})$$

$$= (\text{Number of permutations of the symbols B, A, } L_1, L_2),$$

So, the answer is $\boxed{}$

EXAMPLE 1.8

we now consider the arrangements of all nine letters in DATABASES.

DATABASES

$$\underbrace{(2!)}_{\text{S's}}\underbrace{(3!)}_{\text{A's}}(\text{Number of arrangements of the letters in DATABASES})$$
$$= (\text{Number of permutations of the symbols D, } A_1, \text{T}, A_2, \text{B}, A_3, S_1, \text{E}, S_2),$$

so the number of arrangements of the nine letters in DATABASES is $9!/(2!\,3!) = 30,240$.

If there are n objects with n_1 indistinguishable objects of a first type, n_2 indistinguishable objects of a second type, … , and n_r indistinguishable objects of an rth type, where, $n_1 + n_2 + \ldots + n_r = n$, then there are

$$\frac{n!}{n_1!\, n_2! \cdots n_r!}$$

(linear) arrangements of the given n objects.

Each arrangement of this type is called a permutation with repetitions.

若 n 個物體中，有 n_1 個第一類型相同物體，n_2 個第二類型相同物體，…，及 n_r 個第 r 類型相同物體，且 $n_1+n_2+\cdots+n_r=n$，則此 n 個物體共有 $\frac{n!}{n_1!\, n_2! \cdots n_r!}$ 個 (線性) 安排。

EXAMPLE 1.9

MASSASAUGA 為北美洲土產的褐白色毒蛇。

Arranging all of the letters in MASSASAUGA, we find that there are

$$\frac{10!}{4!\,3!\,1!\,1!\,1!} = 25{,}200$$

possible arrangements. Among these are

in which all four A's are together.

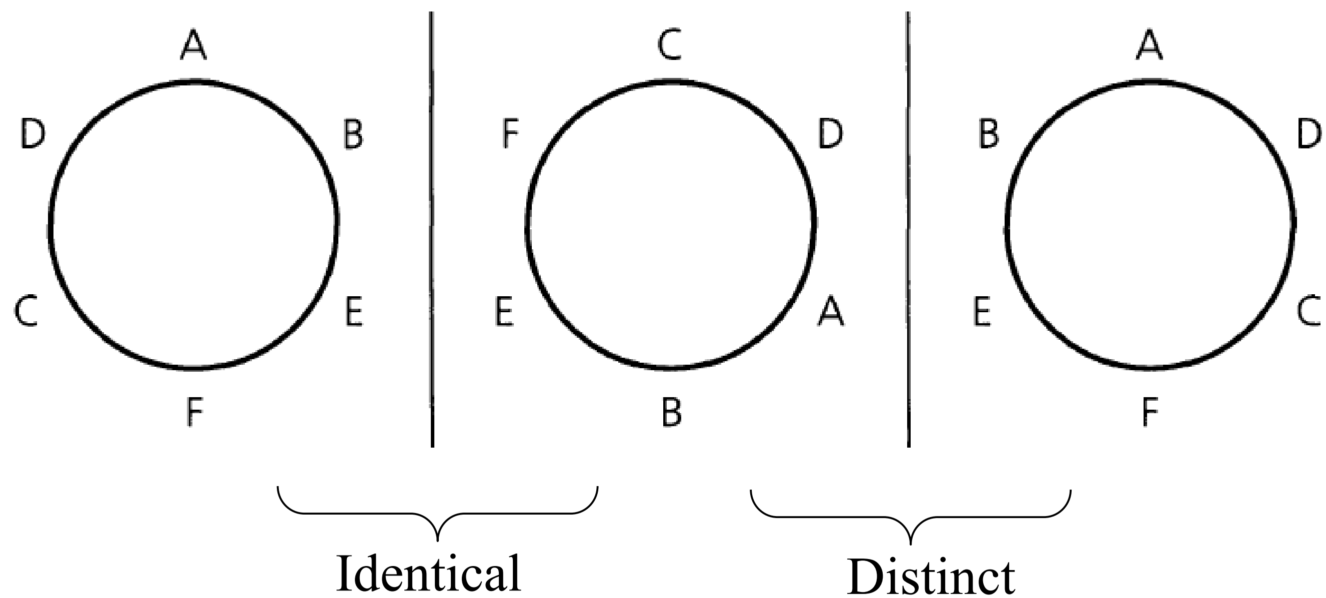
(Consider $\underbrace{A^4}_{AAAA}$, S, S, S, M, U, and G, seven symbols.)

EXAMPLE 1.10

Circular Permutation

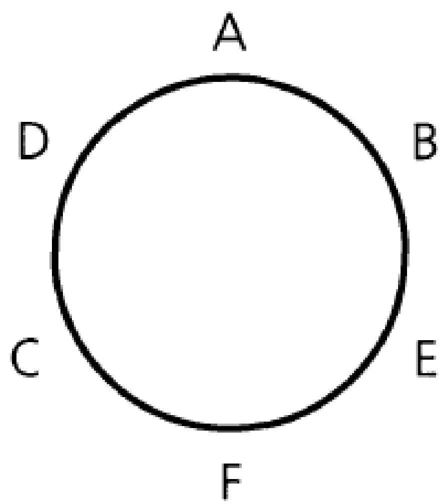
Six people are seated about a round table, how many different circular arrangements are possible, if arrangements are considered the same when one can be obtained from the other by rotations?

E.g.



EXAMPLE 1.10 Cont.

In particular,



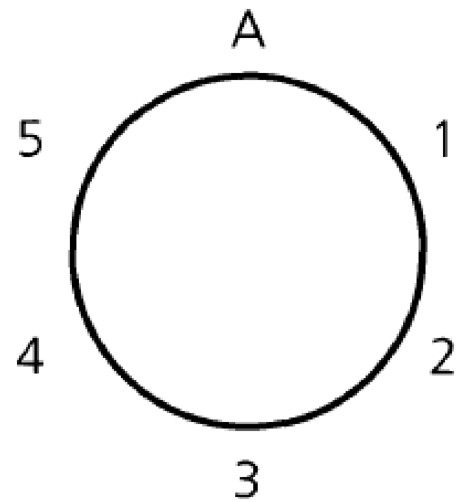
$$\left.\begin{array}{l}\text{ABEFCD}\\ \text{BEFCDA}\\ \text{EFCDAB}\\ \text{FCDABE}\\ \text{CDABEF}\\ \text{DABEFC}\end{array}\right\}$$ six of them are considered identical arrangements.

$$6 \times (\text{Number of circular arrangements of A, B, . . . , F}) = (\text{Number of linear arrangements of A, B, . . . , F}) = 6!.$$

Consequently, there are $6!/6 = 5! = 120$ arrangements of A, B, . . . , F around the circular table.

EXAMPLE 1.10 Cont.

Alternatively,

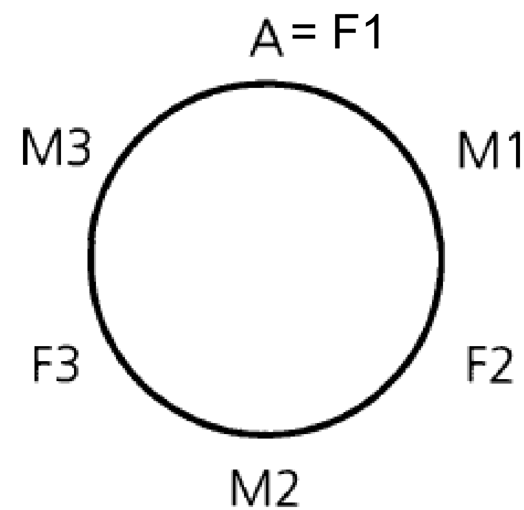


let us place A at the table as shown, then five locations (clockwise from A) remain to be filled.

These five positions is the problem of permuting B, C, . . . , F in a linear manner, and this can be done in $5! = 120$ ways.

EXAMPLE 1.11

We want to arrange three married couples around the table so that the sexes alternate.



M1 can be filled in three ways.

F2 can be filled in two ways.

Thus, there are $3 \times 2 \times 2 \times 1 \times 1 = 12$ ways in which these six people can be arranged with no two men or women seated next to each other.

1.3 組合：二項式定理

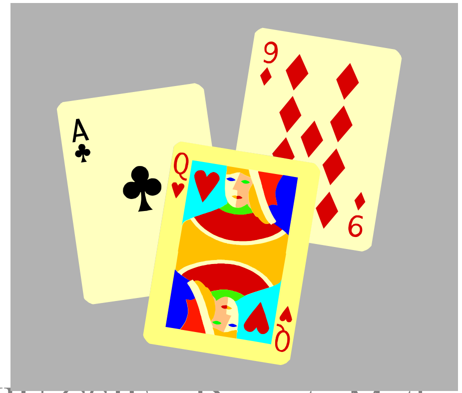
Combinations: The Binomial Theorem

Select 3 cards from a deck of playing cards (52 cards) without replacement:

$$P(n, r) = \frac{n!}{(n-r)!}.$$

order of selection is relevant: $P(52, 3) = 52 \times 51 \times 50$

order of selection is irrelevant: $P(52, 3)/3! = C(52, 3)$



A-Q-9	Q-A-9
A-9-Q	Q-9-A
9-A-Q	9-Q-A

3! identical collections.

If we start with n distinct objects, each *selection*, or *combination*, of r of these objects, with no reference to order, corresponds to $r!$ permutations of size r from the n objects. Thus the number of combinations of size r from a collection of size n is

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}, \qquad 0 \leq r \leq n.$$

若有 n 個不同物體在不考慮順位的情況下，每一個一次取 r 個物體的**選擇** (selection) 或**組合** (combination) 對應到 $r!$ 個由這 n 個物體中取大小為 r 的排列。因此，由大小為 n 的群體中取大小為 r 的組合數為

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}, \qquad 0 \leq r \leq n.$$

In addition to $C(n, r)$ the symbol $\binom{n}{r}$ is also frequently used.

Both $C(n, r)$ and $\binom{n}{r}$ are sometimes read "n choose r."

Note that for all $n \geq 0$, $C(n, 0) = C(n, n) = 1$.

for all $n \geq 1$, $C(n, 1) = C(n, n-1) = n$.

When $0 \leq n < r$, then $C(n, r) = \binom{n}{r} = 0$.

When dealing with any counting problem, we should **ask ourselves about the importance of order in the problem**.

When order is relevant, we think in terms of permutations and arrangements and the rule of product.

When order is not relevant, combinations could play a key role in solving the problem.

EXAMPLE 1.12

(a)考試時, 可回答十題中任七題
(b)十題中, 前五題答三題, 後五題答四題
(c)十題中, 前五題至少答三題, 但共需回答七題

分別可能有幾種答題方法?

(a) $$\binom{10}{7} = \frac{10!}{7!\,3!} = \frac{10\times 9\times 8}{3\times 2\times 1} = 120 \text{ ways.}$$

(b)

(c) $$\left.\begin{array}{l} \binom{5}{3}\binom{5}{4} = 10\times 5 = 50 \text{ ways} \\ \binom{5}{4}\binom{5}{3} = 5\times 10 = 50 \text{ ways} \\ \binom{5}{5}\binom{5}{2} = 1\times 10 = 10 \text{ ways} \end{array}\right\} 50+50+10 = 110 \text{ selections}$$

EXAMPLE 1.13

36個學生組成四隻球隊，編號A, B, C, D，每隊9人的方法

Method 1.

$$C(36,9) \times C(27,9) \times C(18,9) \times C(9,9) \doteq 2.145 \times 10^{19} \text{ ways.}$$

Method 2.

Students 1 2 3 4 ... 36

Teams ABCD ... B (9 A's, 9 B's, 9 C's, and 9'D's)

$$\frac{36!}{9!9!9!9!} \doteq 2.145 \times 10^{19} \text{ ways.}$$

EXAMPLE 1.14

The number of arrangements of the letters in TALLAHASSEE

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 831{,}600.$$

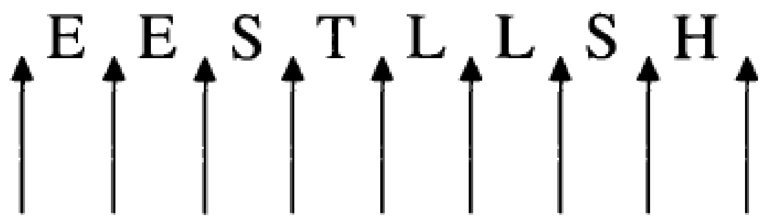
How many of these arrangements have no adjacent A's?

(i.e., 字母A不相鄰的排列)

Disregard A's first

$$\frac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$$

EXAMPLE 1.14 Cont.



arrows indicate nine possible locations for the three A's.

there are

$5040 \times 84 = 423{,}360$ arrangements of the letters in TALLAHASSEE with no consecutive A's.

Be careful of ***overcounting***!!!

EXAMPLE 1.15 From a standard deck of 52 cards select 5 cards which have at least 1 club. ♣

(a) In how many ways can her selection result in a hand with no clubs?

There are 39 cards in the deck that are not clubs.

Thus, $\binom{39}{5}$ ways.

(b)

And since there are $\binom{52}{5}$ possible five-card hands in total, we find that

$$\binom{52}{5} - \binom{39}{5} = 2{,}598{,}960 - 575{,}757 = 2{,}023{,}203$$

of all five-card hands contain at least one club.

EXAMPLE 1.15 Cont.

(c) Can we obtain the result in part (b) in another way?

Select 1 club first, $\binom{13}{1}$ ways

then other 4 cards. $\binom{51}{4}$ ways

$$\binom{13}{1}\binom{51}{4} = 13 \times 249{,}900 = 3{,}248{,}700.$$

Something here is definitely *wrong*!

$$\left.\begin{array}{l}\text{select C3 then C5, CK, H7, SJ}\\ \text{select C5 then C3, CK, H7, SJ}\\ \text{select CK then C5, C3, H7, SJ}\end{array}\right\}\ \textbf{All are the same selections.}$$

EXAMPLE 1.15 Cont.

(d) is there any other way to arrive at the answer in part (b)?

$$\sum_{i=1}^{5}\binom{13}{i}\binom{39}{5-i} = 2{,}023{,}203$$

Table 1.3

Number of Clubs	Number of Ways to Select This Number of Clubs	Number of Cards That Are Not Clubs	Number of Ways to Select This Number of Nonclubs
1	$\binom{13}{1}$	4	$\binom{39}{4}$
2	$\binom{13}{2}$	3	$\binom{39}{3}$
3	$\binom{13}{3}$	2	$\binom{39}{2}$
4	$\binom{13}{4}$	1	$\binom{39}{1}$
5	$\binom{13}{5}$	0	$\binom{39}{0}$

THEOREM 1.1

Note: $\binom{n}{k} = \binom{n}{n-k}$

二項式定理 (The Binomial Theorem)。

If x and y are variables and n is a positive integer, then

$$(x+y)^n = \binom{n}{0}x^0y^n + \binom{n}{1}x^1y^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n-1}x^{n-1}y^1 + \binom{n}{n}x^ny^0 = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$$

binomial coefficient

Select k x's from $(x+y)^n$

$$(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$$

$$= \ldots\ldots + \underline{\qquad\quad} x^2 y^2 + \ldots\ldots$$

$$
\begin{aligned}
&(x+y)^4 \\
&= (x+y)^2 \cdot (x+y)^2 \\
&= \left(x^2+2xy+y^2\right) \cdot \left(x^2+2xy+y^2\right) \\
&= x^4+2x^3y+x^2y^2+2x^3y+4x^2y^2+2xy^3+x^2y^2+2xy^3+y^4 \\
&= x^4+y^4+4x^3y+4xy^3+6x^2y^2
\end{aligned}
$$

EXAMPLE 1.16

a) From the binomial theorem it follows that the coefficient of x^5y^2 in the expansion of $(x+y)^7$ is $\binom{7}{5} = \binom{7}{2} = 21$.

b) To obtain the coefficient of a^5b^2 in the expansion of $(2a-3b)^7$, replace $2a$ by x and $-3b$ by y. From the binomial theorem the coefficient of x^5y^2 in $(x+y)^7$ is $\binom{7}{5}$, and

$$\binom{7}{5}x^5y^2 = \binom{7}{5}(2a)^5(-3b)^2 = \binom{7}{5}(2)^5(-3)^2a^5b^2 = 6048a^5b^2.$$

The Multinomial Theorem

THEOREM 1.2

For positive integer n, t, the coefficient of $x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_t^{n_t}$

in the expansion of $(x_1 + x_2 + \cdots + x_t)^n$ is

$$\frac{n!}{n_1! n_2! \cdots n_t!},$$

where each n_i is an integer with $0 \le n_i \le n$, for all $1 \le i \le t$,

and $n_1 + n_2 + \cdots + n_t = n$.

EXAMPLE 1.17

a) In the expansion of $(x+y+z)^7$ it follows from the multinomial theorem that the coefficient of $x^2y^2z^3$ is $\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = 210$, while the coefficient of xyz^5 is $\binom{7}{1,1,5} = 42$ and that of x^3z^4 is $\binom{7}{3,0,4} = \boxed{}$

b) The coefficient of $a^2b^3c^2d^5$ in $(a+2b-3c+2d+5)^{16}$ is $\dfrac{16!}{2!3!2!5!4!}(1)^2(2)^3(-3)^2(2)^5(5)^4$

determine the coefficient of $v^2w^3x^2y^5z^4$ in $(v+w+x+y+z)^{16}$

1.4 重複組合

Combinations with Repetition: Distributions

EXAMPLE 1.18

(c, h, t, f)

7個人買食物,有四種食物可選擇,有幾種買法?

(from the viewpoint of the restaurant)

Table 1.6

(a)		(b)	
1.	c, c, h, h, t, t, f	1.	x x | x x | x x | x
2.	c, c, c, c, h, t, f	2.	x x x x | x | x | x
3.	c, c, c, c, c, c, f	3.	x x x x x x | | | x
4.	h, t, t, f, f, f, f	4.	| x | x x | x x x x
5.	t, t, t, t, t, f, f	5.	| | x x x x x | x x
6.	t, t, t, t, t, t, t	6.	| | x x x x x x x |
7.	f, f, f, f, f, f, f	7.	| | | x x x x x x x

EXAMPLE 1.18 Cont.

In Table 1.6 we list some possible purchases in column (a) and another means of representing each purchase in column (b).

The seven x's (one for each freshman) correspond to the size of the selection and that the three bars are needed to separate the $3 + 1 = 4$ possible food items that can be chosen.

We are enumerating all arrangements of 10 symbols consisting of seven x's and three |'s, so the number of different purchases is

$$\frac{10!}{7!\,3!} = \binom{10}{7}.$$

When we wish to select, with repetition, r of n distinct objects, we find that we are considering all arrangements of r x's and n-1 |'s and that their number is

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

In Example 1.28, $n = 4$, $r = 7$

當我們想由 n 個不同物體中選 r 個時，允許重複，我們發現 (如表1.6) 我們正在考慮 r 個 x 及 $n-1$ 個 $|$ 的所有安排，且它們的安排數為

$$\frac{(n+r-1)!}{r!(n-1)!} = \binom{n+r-1}{r}$$

因此，n 個物體中一次取 r 個，允許重複，的組合數為 $C(n+r-1, r)$。

$xxxxx|xx||xxx$

EXAMPLE 1.19

Distribute \$1000 to 4 persons (in unit of \$100)

(a) no restriction $\binom{4+10-1}{10} = 286$ ways

(b) at least \$100 for anyone

$x \quad x \quad x \quad x$

$xx \mid xx \mid \mid xx$

(c) at least \$100 for anyone, one person has at least \$500

$$\binom{3+2-1}{2} + \binom{3+1-1}{1} + \binom{3+0-1}{0} = \binom{4+2-1}{2} = 10$$

That person gets \$500 \$600 \$700

EXAMPLE 1.20 Determine all integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 = 7, \quad \text{where } x_i \geq 0 \text{ for all } 1 \leq i \leq 4.$$

One solution of the equation is $x_1 = 3, x_2 = 3, x_3 = 0, x_4 = 1$.

○ ○ ○ | ○ ○ ○ | | ○

$C(4 + 7 - 1, 7) = 120$ solutions.

1.5　鴿洞原理

Pigeonhole Principle

鴿洞原理：若 m 隻鴿子佔據 n 個鴿洞且 $m > n$，則至少有一個鴿洞有兩隻或更多的鴿子棲息於內。

The Pigeonhole Principle　If pigeons are placed into pigeonholes and there are more pigeons than pigeonholes, then some pigeonhole must contain at least two pigeons. More generally, if the number of pigeons is more than k times the number of pigeonholes, then some pigeonhole must contain at least $k+1$ pigeons.

One situation for 6 (= m) pigeons and 4 (= n) pigeonholes

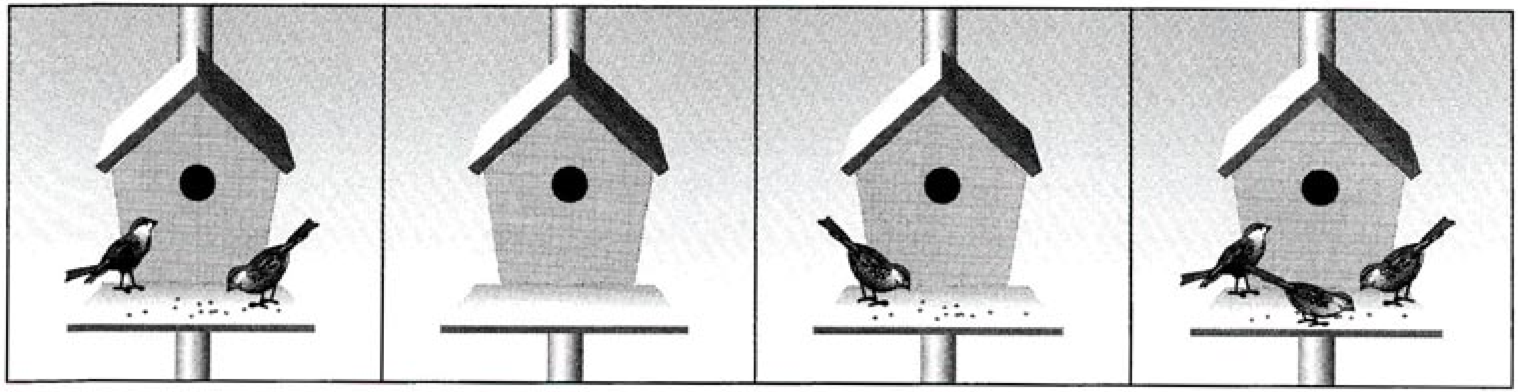
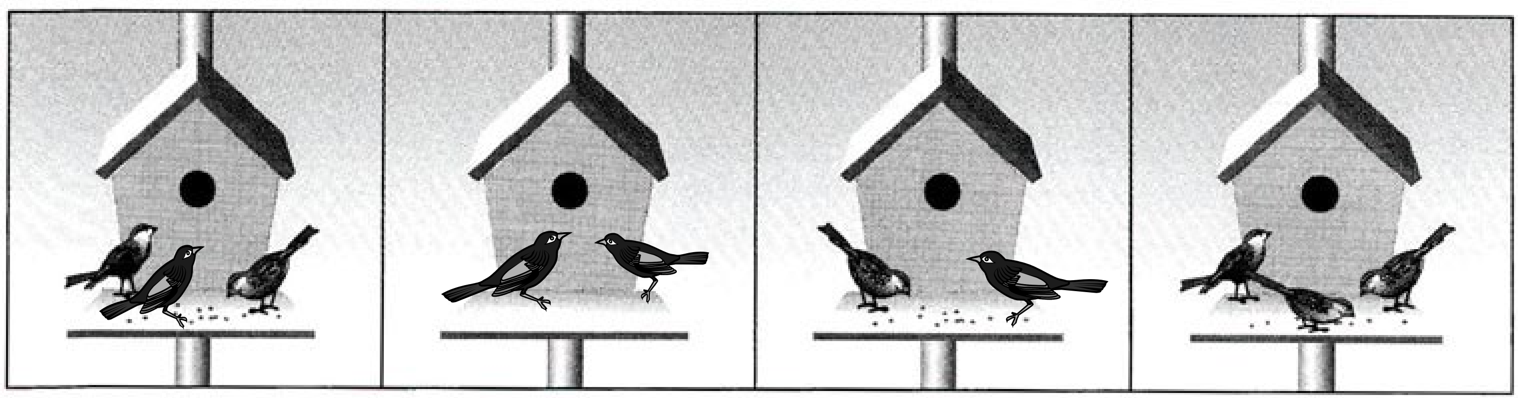


Figure 5.7

For situation of 10 pigeons and 4 pigeonholes. (Note: $k = 2$)



EXAMPLE 1.21

Among 13 people, at least two of them have birthdays during same month. Here we have 13 pigeons (people) and 12 pigeonholes (the months of the year).

EXAMPLE 1.22

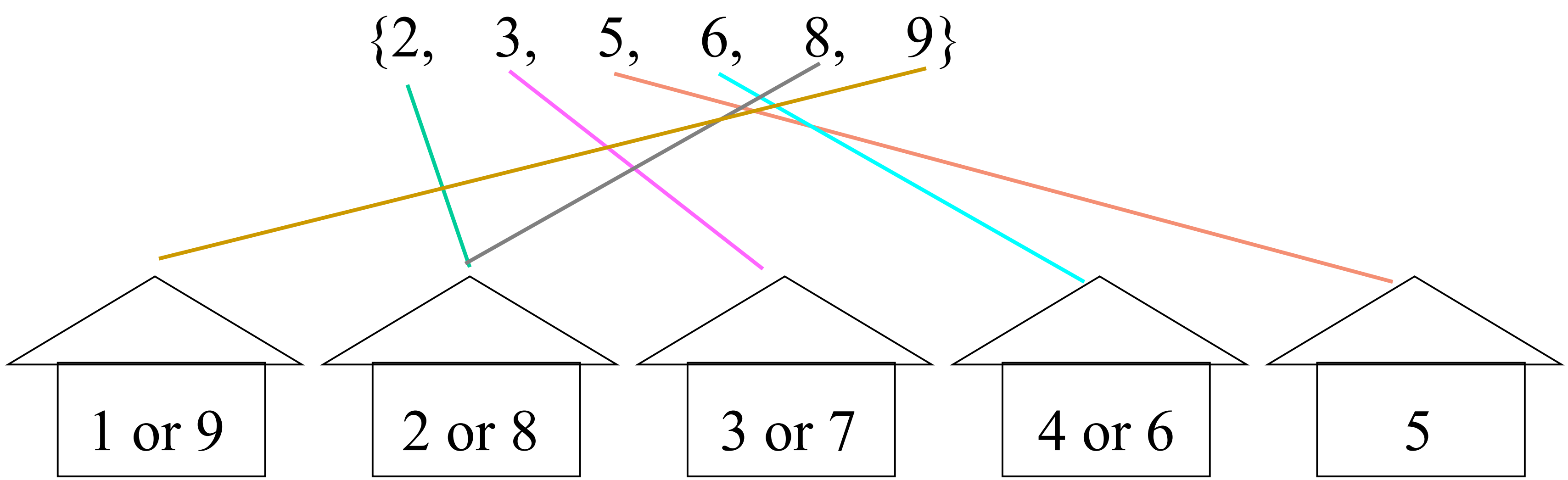
Larry returns from the laundromat with 12 pairs of socks (each pair a different color) in a laundry bag. Drawing the socks from the bag randomly, he'll have to draw 13 of them to ensure there is a matched pair.

EXAMPLE 1.23

$S = \{2,\ 3,\ 5,\ 6,\ 8,\ 9\}$

任何由集合 $S=\{1, 2, 3, \cdots, 9\}$ 中取出的大小為 6 的子集合必含兩個元素其和為 10。

這裡的鴿子組成一個 $\{1, 2, 3, \cdots, 9\}$ 的六個元素子集合，且鴿洞為子集合 $\{1, 9\}$，$\{2, 8\}$，$\{3, 7\}$，$\{4, 6\}$，$\{5\}$。當這六隻鴿子進入它們的個別鴿洞，它們必須至少進一個和為 10 的兩個元素子集合。



EXAMPLE 1.24

Given 8 Perl books, 17 Visual BASIC† books, 6 Java books, 12 SQL books, and 20 C++ books, how many of these books must we select to insure that we have 10 books dealing with the same computer language?

Consider the worse case, you have select the following books

8 本 Perl
9 本 VB
6 本 Java
9 本 SQL
9 本 C++

Totally 41 books. Then you need to select one more (any one) to ensure that 10 books of the same language.