Chapter 2

邏輯基礎

Fundamentals of Logics

2.1　基本聯結及真假值表

2.2　邏輯等價：邏輯定律

2.3　邏輯蘊涵：推論規則

2.4　量詞的使用

2.1 基本聯結及真假值表

$\neg$
$\wedge, \vee, \oplus, \veebar$
$\rightarrow, \leftrightarrow$

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

Statement or Proposition

任何數學理論的發展，語句的斷定是需做的。此類斷定，被稱為**敘述** (statements) (或**命題** (propositions))，為陳述的語句，其不是為真就是為假——不可為真假不分。例如，下面各個為敘述，且以小寫英文字母 (如 p，q 及 r) 來表示這些敘述。

p：組合學是大二生必修的課程。

q：Margaret Mitchell 撰寫飄。

r：$2+3=5$

反之，我們不認為感嘆語句

"多美的傍晚啊！" (What a beautiful evening!)

或命令句

"起床，做您的功課！" (Get up and do your exercise.)

為敘述，因為它們沒有真假值 (真或假)。

A **statement** is a sentence that is either true or false, but not both.

The following sentences are statements.

(a) $2 + 2 = 4$.

(b) $2 + 2 \neq 4$.

(c) $\sqrt{4} = 2$ and $\sqrt{5} > 2$.

(d) The sine function is periodic and 2π is an integer.

(e) $10^2 > 2^{10}$ or $2^{10} > 10^2$.

(f) If $e > 2$, then $e^2 > 4$.

(g) $\sqrt{-1}$ is a real number.

The following sentences are not statements.

(a) What is the sum 2 + 2?

(b) Evaluate the sum 2 + 2.

(c) This sentence is false.

(d) The number x is an integer.

Primitive and Compound Statement

The preceding statements represented by the letters p, q, and r are considered to be *primitive* statements, for there is really no way to break them down into anything simpler.

New statements can be obtained from existing ones in two ways.

p：組合學是大二生必修的課程。

q：Margaret Mitchell 撰寫飄。

r：$2+3=5$

前述由字母 p，q 及 r 所代表的敘述為**原本敘述** (primitive statement)，因為無法將它們打破成任何較簡單的形式。新的敘述可由已存在的敘述以兩種方法獲得。

Primitive and Compound Statement

1) Transform a given statement p into the statement $\neg p$, which denotes its *negation* and is read "Not p."

否定

> p：組合學是大二生必修的課程。
>
> 敘述 $\neg p$ 表"組合學不是大二生的必修課程。"

We do not consider the negation of a primitive statement to be a primitive statement.

2) Combine two or more statements into a *compound* statement, using the following *logical connectives.*

複合敘述

邏輯聯結

Logical Connectives

Symbol	Name	Read	中文
$\wedge$	conjunction	and	合取
$\vee$	disjunction	or	析取
$\veebar$	exclusive disjunction	exclusive or	異或
$\rightarrow$	implication	implies	蘊涵
$\leftrightarrow$	biconditional	if and only if	雙條件

Form	**Translation**
$\neg p$	not p ; it is not the case that p
$p \wedge q$	p and q ; p but q ; p however q
$p \vee q$	p or q ; p and/or q
$p \veebar q$	p or q ; p or q, but not both
$p \rightarrow q$	if p, then q ; p is sufficient for q ; q is necessary for p ; p implies q ; p is a sufficient condition for q ; p only if q ; q is a necessary condition for p ; q whenever p ; q follows from p ; q when p ; q if p.
$p \leftrightarrow q$	p if and only if q ; p iff q ; p just in case q

Truth Table

Truth Table for the Negation of a Proposition.

p	$\neg p$
0	**1**
1	**0**

Write "**0**" or "**F**" for false and "**1**" or "**T**" for true.

The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
0	0	
0	1	
1	0	
1	1	

The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
0	0	
0	1	
1	0	
1	1	

The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \veebar q$
0	0	
0	1	
1	0	
1	1	

The Truth Table for the Implication $p \rightarrow q$.

p	q	$p \rightarrow q$
0	0	
0	1	
1	0	
1	1	

EXAMPLE 2.1

Implication

Penny says: "If I weigh more than 54 kg, then I shall enroll in an exercise class."

p: On Dec. 26, Penny weighs more than 54 kg.

q: Penny enrolls in an exercise class.

p	q	$p \rightarrow q$
0	0	
0	1	
1	0	
1	1	

$p \rightarrow q$	if p, then q ; p is sufficient for q ; q is necessary for p ; p implies q ; p is a sufficient condition for q ; p only if q ; q is a necessary condition for p ; q whenever p ; q follows from p ; q when p ; q if p.

p is sufficient for q

p 是 q 的充分條件

若 p 成立，則 q 必定成立

q is necessary for p

q 是 p 的必要條件

只有 q 成立時，p 才能成立

The Truth Table for Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
0	0	
0	1	
1	0	
1	1	

The order of precedence of the basic operations listed from highest to lowest is

$\neg$
$\wedge$, $\vee$, $\oplus$, $\veebar$
$\rightarrow$, $\leftrightarrow$

EXAMPLE 2.2

Statement to English/(Chinese) Sentence

Let s, t, and u denote the following primitive statements:

s: Phyllis goes out for a walk.

t: The moon is out.

u: It is snowing.

The following sentences provide possible translations for the given (symbolic) compound statements.

a) $(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk.

b) $t \to (\neg u \to s)$: If the moon is out, then if it is not snowing Phyllis goes out for a walk.

c) $\neg(s \leftrightarrow (u \vee t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

English/(Chinese) Sentence to Statement

Now we will work in reverse order and examine the logical (or symbolic) notation for three given sentences:

d) "Phyllis will go out walking if and only if the moon is out."

e) “If it is snowing and the moon is not out, then Phyllis will not go out for a walk.”

$$(u \wedge \neg t) \rightarrow \neg s$$

f) “It is snowing but Phyllis will still go out for a walk.”

Note: the connectives *but* and *and* convey the same meaning.

EXAMPLE 2.3

Floor and Ceiling

floor(x) = $\lfloor x \rfloor$ is the largest integer not greater than x and

ceiling(x) = $\lceil x \rceil$ is the smallest integer not less than x.

Examples

Sample value x	Floor(x) $\lfloor x \rfloor$	Ceiling(x) $\lceil x \rceil$
12/5 = 2.4	2	3
2.9	2	3
−2.7	−3	−2
−2	−2	−2

EXAMPLE 2.4

Make a **truth table for the compound statement** $q \wedge (\neg r \rightarrow p)$.

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0			
0	0	1			
0	1	0			
0	1	1			
1	0	0			
1	0	1			
1	1	0			
1	1	1			

EXAMPLE 2.5

Below we develop the truth tables for the compound statements

$$p \vee (q \wedge r) \text{ and } (p \vee q) \wedge r$$

結果不同

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

EXAMPLE 2.5 Cont.

$$p \vee (q \wedge r) \text{ and } (p \vee q) \wedge r$$

結果不同

Therefore,

we must avoid writing a compound statement such as $p \vee q \wedge r$.

EXAMPLE 2.6

Table reveal that the statement $p \rightarrow (p \vee q)$ is true and that the statement $p \wedge (\neg p \wedge q)$ is false for all truth value assignments for the component statements p, q.

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0		1	0	
0	1	1		1	1	
1	0	1		0	0	
1	1	1		0	0	

Definition 2.1

重言

A compound statement is a ***tautology*** if it is true for all truth value assignments for its component statements. We denote a tautology by T_0.

If a compound statement is false for all such assignment, then it is a ***contradiction***. 矛盾
We denote a contradiction by F_0.

2.2 邏輯等價：邏輯定律

EXAMPLE 2.7

Table provides the truth tables for the compound statements $\neg p \vee q$ and $p \rightarrow q$. Here we see that the corresponding truth tables for the two statements $\neg p \vee q$ and $p \rightarrow q$ are exactly the same.

結果相同

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Definition 2.2

邏輯等價

Two statements s_1, s_2 are said to be *logically equivalent*, and we write $s_1 \Longleftrightarrow s_2$, when the statement s_1 is true (respectively, false) if and only if the statement s_2 is true (respectively, false).

記為 $\boxed{s_1 \Leftrightarrow s_2}$

由 Example 2.7 得知 $(p \rightarrow q) \Longleftrightarrow \neg p \vee q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

p	q	$p \veebar q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	0

Examining these two tables, we may conclude the followings,

(a) for the **exclusive or** operation $\veebar$:

$$p \veebar q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q).$$

(b) for the **biconditional** operation $\leftrightarrow$:

$$p \leftrightarrow q \Leftrightarrow (p \rightarrow \mathrm{q}) \wedge (q \rightarrow p).$$

$$p \leftrightarrow q \Leftrightarrow (\neg p \vee \mathrm{q}) \wedge (\neg q \vee p).$$

$\veebar$ $\leftrightarrow$ and $\rightarrow$ can be replaced by $\neg$, $\wedge$, and $\vee$.

$$p \veebar q \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q).$$

$$p \leftrightarrow q \Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p).$$

$$(p \rightarrow q) \Leftrightarrow \neg p \vee q$$

One of $\wedge$ or $\vee$ can also be replaced by $\neg$ and the other.

Two statement forms p and q are **logically equivalent**, written $p \Leftrightarrow q$, if and only if the statement $p \leftrightarrow q$ is a tautology.

E.g. Verify that $\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$.

Solution.

p	q	$\neg q$	$p \rightarrow q$	$\neg(p \rightarrow q)$	$p \wedge \neg q$	$(\neg(p \rightarrow q)) \leftrightarrow (p \wedge \neg q)$
0	0	1	1	0	0	1
0	1	0	1	0	0	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1

Since $(\neg(p \rightarrow q)) \Leftrightarrow (p \wedge \neg q)$ is a tautology, we conclude that $\neg(p \rightarrow q) \Leftrightarrow p \wedge \neg q$.

$$-(a+b) = (-a)+(-b),$$

$$\neg(p \wedge q) \overset{?}{\Leftrightarrow} \neg p \wedge \neg q$$

$$\neg(p \vee q) \overset{?}{\Leftrightarrow} \neg p \vee \neg q$$

EXAMPLE 2.8

Below we have constructed the truth tables for the statements $\neg(p \wedge q)$, $\neg p \vee \neg q$, $\neg(p \vee q)$, and $\neg p \wedge \neg q$, where p, q are primitive statements.

$$\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q; \qquad \neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q.$$

These results are known as *DeMorgan's Laws.*

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	1	0	0	0	0	1	0	0

EXAMPLE 2.9

$$a \times (b + c) = (a \times b) + (a \times c)$$

$$p \wedge (q \vee r) \overset{?}{\Leftrightarrow} (p \wedge q) \vee (p \wedge r)$$

$$p \vee (q \wedge r) \overset{?}{\Leftrightarrow} (p \vee q) \wedge (p \vee r)$$

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

By the truth table, we may conclude the followings:

$$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$$

The Distributive Law of ∧ over ∨

$$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$$

The Distributive Law of ∨ over ∧

The Laws of Logic

For any primitive statements p, q, r, any tautology T_0, and any contradiction F_0.

1)	$\neg\neg p \Leftrightarrow p$	Law of Double Negation
2)	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	DeMorgan's Law
3)	$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	Commutative Law
4)	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	Associative Law

5)	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	Distributive Law
6)	$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	Idempotent Law
7)	$p \vee F_0 \Leftrightarrow p$ $p \wedge T_0 \Leftrightarrow p$	Identity Law
8)	$p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$	Inverse Law
9)	$p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$	Domination Laws
10)	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	Absorption Law

邏輯定律

對任意原本敘述 p，q，r，任意重言 T_0 及任意矛盾 F_0。

1)	$\neg\neg p \Leftrightarrow p$	雙否定定律
2)	$\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$	DeMorgan 定律
3)	$p \vee q \Leftrightarrow q \vee p$ $p \wedge q \Leftrightarrow q \wedge p$	交換律
4)	$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee r^{\dagger}$ $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge r$	結合律
5)	$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$	分配律
6)	$p \vee p \Leftrightarrow p$ $p \wedge p \Leftrightarrow p$	冪等定律
7)	$p \vee F_0 \Leftrightarrow p$ $p \wedge T_0 \Leftrightarrow p$	恒等定律
8)	$p \vee \neg p \Leftrightarrow T_0$ $p \wedge \neg p \Leftrightarrow F_0$	逆定律
9)	$p \vee T_0 \Leftrightarrow T_0$ $p \wedge F_0 \Leftrightarrow F_0$	優控定律
10)	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$	吸收定律

Definition 2.3

Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the *dual* of s, denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

E.g.

$$s: (p \wedge \neg q) \vee (r \wedge T_0)$$

$$s^d: (p \vee \neg q) \wedge (r \vee F_0)$$

Note: $\neg q$ is unchanged.

THEOREM 2.1

對偶原理 *The Principle of Duality.* Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$.

If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

E.g., 邏輯定律 (2)-(10)

We also find that it is possible to derive many other logical equivalences.

E.g.

$$(r \wedge s) \rightarrow q \Leftrightarrow \neg(r \wedge s) \vee q$$

Instead of always constructing more (and, unfortunately, larger) truth tables we introduce the following two *substitution rules*

First Substitution Rule

Suppose that the compound statement P is a tautology.
If p is a primitive statement that appears in P and
we replace each occurrence of p by the same statement q,
then the resulting compound statement P_1 is also a tautology.

From Example 2.7 we have known that

$$[p \rightarrow q] \leftrightarrow [\neg p \vee q] \text{ is a tautology.}$$

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Replace each p by $(r \wedge s)$, then

$$[(r \wedge s) \rightarrow q] \leftrightarrow [\neg(r \wedge s) \vee q] \text{ is also a tautology.}$$

We may verify this from Table in the next slide.

q	r	s	$r \wedge s$	$(r \wedge s) \rightarrow q$	$\neg(r \wedge s)$	$\neg(r \wedge s) \vee q$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	1	1
0	1	1	1	0	0	0
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

EXAMPLE 2.10

$$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q) \quad \text{is a tautology.}$$

Now, replace each occurrence of p by $(r \wedge s)$ and each q by $(t \rightarrow u)$

$$\neg[(r \wedge s) \vee (t \rightarrow u)] \leftrightarrow [\neg(r \wedge s) \wedge \neg (t \rightarrow u)]$$

is also a tautology.

Thus, $\neg[(r \wedge s) \vee (t \rightarrow u)] \Leftrightarrow [\neg(r \wedge s) \wedge \neg (t \rightarrow u)]$.

Second Substitution Rule

Let P be a compound statement where p is an arbitrary statement that appears in P, and let q be a statement such that $q \Leftrightarrow p$. Suppose that in P we replace one or more occurrences of p by q. Then this replacement yields the compound statement P_1. Under these circumstances $P_1 \Leftrightarrow P$.

EXAMPLE 2.11

(a) $P: (p \rightarrow q) \rightarrow r$

$P_1: (\neg p \vee q) \rightarrow r$

Then, $P \Leftrightarrow P_1$

because $(p \rightarrow q) \Leftrightarrow (\neg p \vee q)$

(b) $P: p \rightarrow (p \vee q)$, since $\neg\neg p \Leftrightarrow p$.

$P_1: p \rightarrow (\neg\neg p \vee q)$

$P_2: \neg\neg p \rightarrow (\neg\neg p \vee q)$

We have $P \Leftrightarrow P_1$ and also $P \Leftrightarrow P_2$.

EXAMPLE 2.12

Given a statement $p \to q$,

(a) its **converse** is $q \to p$;

(b) its **contrapositive** is $\neg q \to \neg p$; and

(c) its **inverse** is $\neg p \to \neg q$.

p	q	$p \to q$	$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1
			(b)	(a)	(c)

$p \to q \not\Leftrightarrow q \to p$ (converse)

$p \to q \not\Leftrightarrow \neg p \to \neg q$ (inverse)

(contrapositive)

EXAMPLE 2.12 Cont.

Consider the statement "If a solution exists, then the program terminates."

(a) Its converse is "If the program terminates, then a solution exists."

(b) Its contrapositive is "If the program does not terminate, then a solution does not exist."

(c) Its inverse is "If a solution does not exist, then the program does not terminate."

EXAMPLE 2.13

What is the negation of "If Joan passes the test, then Mary will buy him a cake."?

Because $\neg(p \to q) \Leftrightarrow \neg(\neg p \vee q) \Leftrightarrow p \wedge \neg q$

$$\neg(p \to q) \Leftrightarrow (p \wedge \neg q)$$

The negation is " Joan passes the test, but (or and) Mary does not buy him a cake."

First Substitution Rule

P (compound statement) is a tautology

p is a primitive statement that appears in P

replace **each occurrence** of p by q (any statement)

the resulting P_1 is also a tautology

Second Substitution Rule

P (compound statement)

p is an arbitrary statement that appears in P, and we know $q \Leftrightarrow p$

replace **one or more occurrences** of p by q.

the resulting P_1, we have $P_1 \Leftrightarrow P$

Simplification of Compound Statements

* We shall list the major laws of logic being used.

* Shall not mention any applications of our two substitution rules.

EXAMPLE 2.14

For primitive statements p, q, is there any simpler way to express the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$ that is, can we find a simpler statement that is logically equivalent to the one given?

	Reasons
$(p \vee q) \wedge \neg(\neg p \wedge q)$	
$\Leftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	DeMorgan's Law
$\Leftrightarrow (p \vee q) \wedge (p \vee \neg q)$	Law of Double Negation
$\Leftrightarrow p \vee (q \wedge \neg q)$	Distributive Law of $\vee$ over $\wedge$
$\Leftrightarrow p \vee F_0$	Inverse Law
	Identity Law

$$(p \vee q) \wedge \neg(\neg p \wedge q) \Leftrightarrow p,$$

EXAMPLE 2.15

Consider the compound statement

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q],$$

where p, q, r are primitive statements.

From the laws of logic it follows that

	Reasons
$\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$	
$\Leftrightarrow \neg\neg[(p \vee q) \wedge r] \wedge \neg\neg q$	DeMorgan's Law
$\Leftrightarrow [(p \vee q) \wedge r] \wedge q$	Law of Double Negation
$\Leftrightarrow (p \vee q) \wedge (r \wedge q)$	Associative Law of $\wedge$
$\Leftrightarrow (p \vee q) \wedge (q \wedge r)$	Commutative Law of $\wedge$
$\Leftrightarrow [(p \vee q) \wedge q] \wedge r$	Associative Law of $\wedge$
	Absorption Law (as well as the Commutative Laws for $\wedge$ and $\vee$)

The original statement is logically equivalent to the much simpler statement $q \wedge r$.

2.3　邏輯蘊涵：推論規則

We start by considering the general form of an argument, one we wish to show is valid. So let us consider the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q.$$

Here n is a positive integer, the statements $p_1, p_2, p_3, \ldots, p_n$ are called the *premises* 前提 of the argument, and the statement q is the *conclusion* 結論 for the argument.

To establish an argument $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$

is a valid argument is to show that the statement

is a tautology

EXAMPLE 2.16

Let p, q, r denote the primitive statements given as

p: Roger studies.

q: Roger plays racketball.

r: Roger passes discrete mathematics.

用功讀書。
打籃球。
通過離散數學。

Now let p_1, p_2, p_3 denote the premises

若 Roger 用功讀書，則他將通過離散數學。
若 Roger 不打籃球，則他將用功讀書。
Roger 沒有通過離散數學。

p_1: If Roger studies, then he will pass discrete mathematics.

p_2: If Roger doesn't play racketball, then he'll study.

p_3: Roger failed discrete mathematics.

We want to determine whether the argument

$$(p_1 \wedge p_2 \wedge p_3) \to q$$ is valid.

EXAMPLE 2.16 Cont.

To do so, we rewrite p_1, p_2, p_3 as

$$p_1: \quad p \to r \qquad p_2: \quad \neg q \to p \qquad p_3: \quad \neg r$$

and examine the truth table for the implication

$$[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$$

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \to q$
p	q	r	$p \to r$	$\neg q \to p$	$\neg r$	$[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

tautology

Hence we can say that $(p_1 \wedge p_2 \wedge p_3) \to q$ is a valid argument.

EXAMPLE 2.16 Cont.

			p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3) \to q$
p	q	r	$p \to r$	$\neg q \to p$	$\neg r$	$[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

什麼情況這裡的值會是 0 ？

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

什麼情況 p 的值會是 1 ？

p_1	p_2	p_3	$(p_1 \wedge p_2 \wedge p_3)$
1	1	1	1
1	1	0	0
1	0	1	0
⋮	⋮	⋮	0

Definition 2.4

If p, q are arbitrary statements such that $p \to q$ is a tautology, then we say that p *logically implies* q and

邏輯蘊涵

we write $p \Rightarrow q$ to denote this situation.

$$p \Leftrightarrow q \quad \textit{means} \quad p \leftrightarrow q \quad \textit{is a tautology.}$$

$$p \Rightarrow q \quad \textit{means} \quad p \to q \quad \textit{is a tautology.}$$

E.g., recall from Example 2.6

$p \to (p \vee q)$ is a tautology,

thus,

p	q	$p \vee q$	$p \to (p \vee q)$
0	0	0	1
0	1	1	1
1	0	1	1
1	1	1	1

EXAMPLE 2.17

$$[p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$$

p_1				p_2	q	$(p_1 \wedge p_2) \rightarrow q$
p	r	s	$p \wedge r$	$(p \wedge r) \rightarrow s$	$r \rightarrow s$	$[(p \wedge ((p \wedge r) \rightarrow s)] \rightarrow (r \rightarrow s)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

The truth of the conclusion q is **deduced** or **inferred** from the truth of the two premises p_1 and p_2.

EXAMPLE 2.18

我們考慮的推論規則，被稱為**斷言法** (Modus Ponens) 或**分離規則** (Rule of Detachment)。 (Modus Ponens 來自拉丁文且可被譯為 "斷言法")

Consider argument $[p \wedge (p \rightarrow q)] \rightarrow q$,

p	q	$p \rightarrow q$	$p \wedge (p \rightarrow q)$	$[p \wedge (p \rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

EXAMPLE 2.18 Cont.

The actual rule will be written in the tabular form

$$\begin{array}{l} \quad p \\ \quad p \rightarrow q \\ \hline \therefore q \end{array}$$

it follows from the first substitution rule that (all occurrences of)

p or q may be replaced by compound statements

— and the resulting implication will also be a tautology.

Consequently,

$$\begin{array}{l} \quad r \vee s \\ \quad (r \vee s) \rightarrow (\neg t \wedge u) \\ \hline \therefore \neg t \wedge u \end{array}$$

is a valid argument, by the Rule of Detachment

EXAMPLE 2.19

A second rule of inference is given by the logical implication

$$[(p \to q) \land (q \to r)] \to (p \to r),$$

where p, q, and r are any statements.

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

三段論法定律

This rule, which is referred to as the *Law of the Syllogism*,

1) If the integer 35244 is divisible by 396, then the integer 35244 is divisible by 66. $p \to q$
2) If the integer 35244 is divisible by 66, then the integer 35244 is divisible by 3. $q \to r$
3) Therefore, if the integer 35244 is divisible by 396, then the integer 35244 is divisible by 3. $\therefore p \to r$

EXAMPLE 2.20

Modus Tollens (method of denying)

$$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$$

$$\begin{array}{l} p \rightarrow q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

Note: using contrapositive

p	q	$\neg q$	$p \rightarrow q$	$\neg p$	.....	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$
0	0	1	1	1		1
0	1	0	1	1		1
1	0	1	0	0		1
1	1	0	1	0		1

Rule of Conjunction

EXAMPLE 2. 21

$$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

EXAMPLE 2. 22

Rule of Disjunctive Syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore\ q \end{array}$$

Bart's wallet is in his back pocket or it is on his desk.
Bart's wallet is not in his back pocket.

Therefore, Bart's wallet is on his desk.

EXAMPLE 2.23

Rule of Contradiction

$$\frac{\neg p \rightarrow F_0}{\therefore p}$$

p	$\neg p$	F_0	$\neg p \rightarrow F_0$	$(\neg p \rightarrow F_0) \rightarrow p$
1	0	0	1	1
0	1	0	0	1

Rules of Inference

Rule of Inference	中文名稱	Name of Rule
1) p $p \to q$ $\therefore q$	分離規則 (斷言法)	Rule of Detachment (Modus Ponens)
2) $p \to q$ $q \to r$ $\therefore p \to r$	三段論法定律	Law of the Syllogism
3) $p \to q$ $\neg q$ $\therefore \neg p$	否定法	Modus Tollens
4) p q $\therefore p \wedge q$	合取規則	Rule of Conjunction
5) $p \vee q$ $\neg p$ $\therefore q$	析取三段論法規則	Rule of Disjunctive Syllogism

6)	$\dfrac{\neg p \rightarrow F_0}{\therefore p}$	矛盾規則	Rule of Contradiction
7)	$\dfrac{p \wedge q}{\therefore p}$	合取簡化規則	Rule of Conjunctive Simplification
8)	$\dfrac{p}{\therefore p \vee q}$	析取放大規則	Rule of Disjunctive Amplification
9)	$\dfrac{\begin{array}{l} p \wedge q \\ p \rightarrow (q \rightarrow r) \end{array}}{\therefore r}$	條件證明規則	Rule of Conditional Proof
10)	$\dfrac{\begin{array}{l} p \rightarrow r \\ q \rightarrow r \end{array}}{\therefore (p \vee q) \rightarrow r}$	訴訟證明規則	Rule for Proof by Cases
11)	$\dfrac{\begin{array}{l} p \rightarrow q \\ r \rightarrow s \\ p \vee r \end{array}}{\therefore q \vee s}$	構造性兩難法規則	Rule of the Constructive Dilemma
12)	$\dfrac{\begin{array}{l} p \rightarrow q \\ r \rightarrow s \\ \neg q \vee \neg s \end{array}}{\therefore \neg p \vee \neg r}$	破壞性兩難法規則	Rule of the Destructive Dilemma

利用 Rules of Inference 來驗證
當 p_1、p_2 和 p_3 都為 1 時 q 值也為 1 而非 0。

$$
\begin{array}{ll}
p_1: & p \to r \\
p_2: & \neg q \to p \\
p_3: & \neg r \\
\hline
 & \therefore q
\end{array}
$$

Previously EXAMPLE 2.16

Steps	**Reasons**
1) $\neg q \to p$	Premise p_2
2) $p \to r$	Premise p_1
3) $\neg q \to r$	Steps (1)&(2) by Law of the Syllogism
4) $\neg r$	Premise p_3
5)	Steps (3)&(4) by Modus Tollens
6)	Step (5) by Law of Double Negation

EXAMPLE 2.24

Our first example demonstrates the validity of the argument

$$\begin{array}{l} p \to r \\ \neg p \to q \\ q \to s \\ \hline \therefore \neg r \to s \end{array}$$

Steps	**Reasons**
1) $p \to r$	Premise
2) $\neg r \to \neg p$	Step (1) and $p \to r \Leftrightarrow \neg r \to \neg p$
3) $\neg p \to q$	Premise
4) $\neg r \to q$	Steps (2) and (3) and the Law of the Syllogism
5) $q \to s$	Premise
6) $\therefore \neg r \to s$	Steps (4) and (5) and the Law of the Syllogism

EXAMPLE 2.24 Cont.

$$\begin{array}{l} p \to r \\ \neg p \to q \\ q \to s \\ \hline \therefore \neg r \to s \end{array}$$

A second way to validate the given argument

Steps	Reasons
1) $p \to r$	Premise
2) $q \to s$	Premise
3) $\neg p \to q$	Premise
4) $p \vee q$	Step (3) and $(\neg p \to q) \Leftrightarrow (\neg\neg p \vee q) \Leftrightarrow (p \vee q)$, where the second logical equivalence follows by the Law of Double Negation
5) $r \vee s$	Steps (1), (2), and (4) and the Rule of the Constructive Dilemma
6) $\therefore \neg r \to s$	Step (5) and $(r \vee s) \Leftrightarrow (\neg\neg r \vee s) \Leftrightarrow (\neg r \to s)$, where the Law of Double Negation is used in the first logical equivalence

EXAMPLE 2.25

This example will provide a way to show that the following argument is valid.

If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry.

If the party were canceled, then refunds would have had to be made.

No refunds were made.

Therefore the band could play rock music.

EXAMPLE 2.25

This example will provide a way to show that the following argument is valid.

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If the party were canceled, then refunds would have had to be made.
No refunds were made.
Therefore the band could play rock music.

p q r s t

$\neg p$ $\vee$ $\neg q$

p_1: If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry.

$\rightarrow$ r $\wedge$ s

r $\rightarrow$ t

p_2: If the party were canceled, then refunds would have had to be made.

$\neg t$

p_3: No refunds were made.

$\therefore$ p

Therefore the band could play rock music.

p: The band could play rock music.
q: The refreshments were delivered on time.
r: The New Year's party was canceled.
s: Alicia was angry.
t: Refunds had to be made.

$$
\begin{array}{l}
(\neg p \vee \neg q) \rightarrow (r \wedge s) \\
r \rightarrow t \\
\neg t \\
\hline
\therefore p
\end{array}
$$

Steps	**Reasons**
1) $r \rightarrow t$	Premise
2) $\neg t$	Premise
3) $\neg r$	Steps (1) and (2) and Modus Tollens
4) $\neg r \vee \neg s$	Step (3) and the Rule of Disjunctive Amplification
5) $\neg(r \wedge s)$	Step (4) and DeMorgan's Laws
6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Premise
7) $\neg(\neg p \vee \neg q)$	Steps (6) and (5) and Modus Tollens
8) $p \wedge q$	Step (7), DeMorgan's Laws, and the Law of Double Negation
9) $\therefore p$	Step (8) and the Rule of Conjunctive Simplification

For EXAMPLE 2.26

$$\frac{\neg p \rightarrow F_0}{\therefore p}$$ ***Rule of Contradiction***

Proof by Contradiction

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$$

To prove $\neg((p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q) \rightarrow F_0 \Leftrightarrow$

we prove $(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q) \rightarrow F_0$

From Example 2.13, we know

EXAMPLE 2.26

Proof by Contradiction. Consider the argument

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \neg r \\ \hline \therefore p \end{array}$$

we assume the negation $\neg p$ of the conclusion p as another premise.

The objective now is to use these four premises to derive a contradiction F_0.

Steps	Reasons
1) $\neg p \leftrightarrow q$	Premise
2) $(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	Step (1) and $(\neg p \leftrightarrow q) \Leftrightarrow [(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)]$
3) $\neg p \rightarrow q$	Step (2) and the Rule of Conjunctive Simplification
4) $q \rightarrow r$	Premise
5) $\neg p \rightarrow r$	Steps (3) and (4) and the Law of the Syllogism
6) $\neg p$	Premise (the one assumed)
7) r	Steps (5) and (6) and the Rule of Detachment
8) $\neg r$	Premise
9) $r \wedge \neg r (\Leftrightarrow F_0)$	Steps (7) and (8) and the Rule of Conjunction
10) $\therefore p$	Steps (6) and (9) and the method of Proof by Contradiction

For EXAMPLE 2.27

$$\begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_n \\ \hline \therefore q \rightarrow r \end{array} \longrightarrow \begin{array}{l} p_1 \\ p_2 \\ \vdots \\ \vdots \\ p_n \\ q \\ \hline \therefore r \end{array}$$

reasoning

$$\begin{aligned} & p \rightarrow (q \rightarrow r) \\ \Leftrightarrow\ & \neg p \vee (q \rightarrow r) \\ \Leftrightarrow\ & \neg p \vee (\neg q \vee r) \\ \Leftrightarrow\ & (\neg p \vee \neg q) \vee r \\ \Leftrightarrow\ & \neg(p \wedge q) \vee r \\ \Leftrightarrow\ & (p \wedge q) \rightarrow r \end{aligned}$$

This can also be verified by truth table.

EXAMPLE 2.27

In order to establish the validity of the argument

(*)

$$\begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ \hline \therefore q \rightarrow p \end{array}$$

we consider the corresponding argument

(**)

$$\begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ q \\ \hline \therefore p \end{array}$$

EXAMPLE 2.27 Cont.

$$\begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ q \\ \hline \therefore p \end{array}$$

Steps		Reasons
1)	q	Premise
2)	$q \rightarrow (u \wedge s)$	Premise
3)	$u \wedge s$	Steps (1) and (2) and the Rule of Detachment
4)	u	Step (3) and the Rule of Conjunctive Simplification
5)	$u \rightarrow r$	Premise
6)	r	Steps (4) and (5) and the Rule of Detachment
7)	s	Step (3) and the Rule of Conjunctive Simplification
8)	$r \wedge s$	Steps (6) and (7) and the Rule of Conjunction
9)	$(r \wedge s) \rightarrow (p \vee t)$	Premise
10)	$p \vee t$	Steps (8) and (9) and the Rule of Detachment
11)	$\neg t$	Premise
12)	$\therefore p$	Steps (10) and (11) and the Rule of Disjunctive Syllogism

EXAMPLE 2.28

Consider the primitive statements p, q, r, s, and t and the argument

$$\begin{array}{l} p \\ p \vee q \\ q \rightarrow (r \rightarrow s) \\ t \rightarrow r \\ \hline \therefore \neg s \rightarrow \neg t \end{array}$$

show that this is an invalid argument.

Find truth values for each of the statements p, q, r, s, and t such that the four premises are all true, but the conclusion $\neg s \rightarrow \neg t$ is false.

EXAMPLE 2.28 Cont.

$$p:\ 1 \qquad q:\ 0 \qquad r:\ 1 \qquad s:\ 0 \qquad t:\ 1,$$

the four premises

$$p \qquad p \vee q \qquad q \rightarrow (r \rightarrow s) \qquad t \rightarrow r$$

all have the truth value 1, while the conclusion

$$\neg s \rightarrow \neg t$$

has the truth value 0.

2.4 量詞的使用

Definition 2.5

A declarative sentence is an *open statement* if
(1) it contains one or more **variables**, and
(2) it is not a statement, but
(3) it becomes a statement when the variables in it are replaced by certain **allowable choices**.

universe

定義 2.5　一個陳述語句是一個**開放敘述** (open statement) 若

1) 它含有一個或更多的變數，且
2) 它不是一個敘述，但
3) 當變數以某種允許的選擇來取代時，它變為一個敘述。

Examples of open statements:

(1) The number $x + 2$ is an even integer.

(2) The numbers $y + 2$, $x - y$, and $x + 3y$ are even integers.

(3) $x^2 - 3 > 0$

Notations

(1) $p(x)$: The number $x + 2$ is an even integer.

(2) $q(x, y)$: The numbers $y + 2$, $x - y$, and $x + 3y$ are even integers.

Lets consider only positive integers for these two examples.

$p(5)$: FALSE, $\neg p(7)$: $q(4, 2)$:

$p(6)$: TRUE, $\neg p(8)$: $q(3, 4)$:

Therefore,

For some x, $p(x)$ is true. For some x, $\neg p(x)$ is true.

For some x, y, $q(x, y)$ is true. For some x, y, $\neg q(x, y)$ is true.

Quantifiers

(1) For some x, $p(x)$ is true. (2) For some x, y, $q(x, y)$ is true.

are said to quantify (量化) the open statements $p(x)$ and $q(x, y)$.

這些結果係由兩種型態的**量詞** (quantifiers) 而得，其一被稱為**存在量詞** (existential quantifiers)，而另一被稱為**全稱量詞** (universal quantifiers)。

Statement (1) uses the *existential quantifier* "For some x," which can also be expressed as "For at least one x" or "There exists an x such that." Written in symbolic form as $\boxed{\exists x.}$

Statement (2) becomes $\boxed{\exists x\ \exists y\ q(x, y)}$ in symbolic form.

Also $\boxed{\exists x, y\ q(x, y).}$

Quantifiers

這些結果係由兩種型態的**量詞** (quantifiers) 而得，其一被稱為**存在量詞** (existential quantifiers)，而另一被稱為**全稱量詞** (universal quantifiers)。

The *universal quantifier* is denoted by $\boxed{\forall x}$ and is read "For all x," "For any x," "For each x," or "For every x." "For all x, y," "For any x, y," "For every x, y," or "For all x and y" is denoted by $\boxed{\forall x\ \forall y,}$ which can be abbreviated to $\boxed{\forall x, y.}$

EXAMPLE 2.29

Universe: Real numbers

$p(x):\ x \geq 0$

$q(x):\ x^2 \geq 0$

$r(x):\ x^2 - 3x - 4 = 0$

$s(x):\ x^2 - 3 > 0$

$\exists x[p(x) \wedge r(x)]: TRUE$ $\quad x = 4$

$\forall x[p(x) \rightarrow q(x)]: TRUE$

$\exists x[p(x) \rightarrow q(x)]: TRUE$

$\forall x[q(x) \rightarrow s(x)]: FALSE$

$\forall x[r(x) \vee s(x)]: FALSE$

$\forall x[r(x) \rightarrow p(x)]: FALSE$

翻譯 e.g. For every real number x, if x is a solution of the equation $x^2 - 3x - 4 = 0$, then $x \geq 0$.

小結論

if $\forall x\ p(x)$ is true, so is $\exists x\ p(x)$

$$\forall x\ p(x) \Rightarrow \exists x\ p(x)$$

$\exists x\ p(x)$ is true whenever $\forall x\ p(x)$ is true.

EXAMPLE 2.30

Implicit Quantification

Here, in (1) and (2), the universe comprises all real numbers, and in (3), the universe is all positive integers.

(1) "If a number is rational, the it is a real number."

$$\forall x\ [p(x) \rightarrow q(x)]$$ explicit

(2) $\sin^2 x + \cos^2 x = 1$

$$\forall x(\sin^2 x + \cos^2 x = 1)$$ explicit

(3) "The integer 41 is equal to the sum of two perfect squares."

explicit

EXAMPLE 2.31

Consider $q(x): x^2 \geq 1$

If the universe consists of all positive integers, then the quantified statement $\forall x \, q(x)$ is true.

For the universe of all positive real numbers, however, the same quantified statement $\forall x \, q(x)$ is false. E.g. $x = 0.5$

EXAMPLE 2.32

A is an array

```
for n := 1 to 20 do
  A[n] := n * n - n
```

1) Every entry in the array is nonnegative:

2) There exist two consecutive entries in A where the larger entry is twice the smaller:

3) The entries in the array are sorted in (strictly) ascending order:

4) The entries in the array are distinct:

EXAMPLE 2.32

A is an array

```
for n := 1 to 20 do
    A[n] := n * n - n
```

1) Every entry in the array is nonnegative: $\forall n\,(A[n] \geq 0)$.

2) There exist two consecutive entries in A where the larger entry is twice the smaller: $\exists n\,(A[n+1] = 2A[n])$.

3) The entries in the array are sorted in (strictly) ascending order: $\forall n\,[(1 \leq n \leq 19) \rightarrow (A[n] < A[n+1])]$.

4) The entries in the array are distinct:

$$\forall m\,\forall n\,[(m \neq n) \rightarrow (A[m] \neq A[n])], \quad \text{or}$$

$$\forall m, n\,[(m < n) \rightarrow (A[m] \neq A[n])].$$

Definition 2.6

Let $p(x), q(x)$ be open statements defined for a given universe.

(邏輯) 等價

The open statements $p(x)$ and $q(x)$ are called (*logically*) *equivalent*, and we write $\forall x[p(x) \Leftrightarrow q(x)]$ when the biconditional $p(a) \leftrightarrow q(a)$ is true for each replacement a from the universe.

(that is, $p(a) \Leftrightarrow q(a)$ for each a in the universe)

If the implication $p(a) \rightarrow q(a)$ is true for each a in the universe,

(that is, $p(a) \Rightarrow q(a)$ for each a in the universe)

then we write $\forall x\,[p(x) \Rightarrow q(x)]$ and say that $p(x)$ *logically implies* $q(x)$.

邏輯蘊涵

EXAMPLE 2.33

For the universe of all triangles in the plane,
let $p(x)$, $q(x)$ denote the open statements:

$p(x)$: x is equiangular (等角),

$q(x)$: x is equilateral (等邊).

$\because p(a) \leftrightarrow q(a)$ is true for every triangle a in the plane

$\therefore \forall x\, [p(x) \Leftrightarrow q(x)]$

($p(x)$ and $q(x)$ are logically equivalent)

Summary Table

Statement	When Is It True?	When Is It False?
$\exists x\ p(x)$	For some *a* in the universe, *p(a)* is true.	For every *a* in the universe, *p(a)* is false.
$\forall x\ p(x)$	For every *a* in the universe, *p(a)* is true.	For some *a* in the universe, *p(a)* is false.
$\exists x\ \neg p(x)$	For some *a* in the universe, *p(a)* is false.	For every *a* in the universe, *p(a)* is true.
$\forall x\ \neg p(x)$	For every *a* in the universe, *p(a)* is false.	For some *a* in the universe, *p(a)* is true.

Negating Statement with One Quantifier

$$\neg[\forall x\, p(x)] \Rightarrow \exists x\, \neg p(x)$$

- Suppose $\neg[\forall x\, p(x)]$ is true.
- $\forall x\, p(x)$ is false.
- $p(a)$ is false for some a.
- $\neg p(a)$ is true for some a.
- $\exists x\, \neg p(x)$ is true.

否定含一個量詞的敘述之規則

$$\neg[\forall x\ p(x)] \Leftrightarrow \exists x\ \neg p(x)$$

$$\neg[\exists x\ p(x)] \Leftrightarrow \forall x\ \neg p(x)$$

$$\neg[\forall x\ \neg p(x)] \Leftrightarrow \exists x\ \neg\neg p(x) \Leftrightarrow \exists x\ p(x)$$

$$\neg[\exists x\ \neg p(x)] \Leftrightarrow \forall x\ \neg\neg p(x) \Leftrightarrow \forall x\ p(x)$$

EXAMPLE 2.34

Universe: all integers

$r(x)$: $2x+1=5$

$s(x)$: $x^2=9$

then $\exists x\,[r(x)\wedge s(x)]$ is false

but $\exists x\, r(x)\wedge \exists x\, s(x)$ is true

Therefore, $\exists x\,[r(x)\wedge s(x)]\ \boxed{}\ \exists x\, r(x)\wedge \exists x\, s(x)$

But $\exists x\,[p(x)\wedge q(x)]\Rightarrow[\exists x\, p(x)\wedge \exists x\, q(x)]$
for any $p(x)$, $q(x)$ and universe

單變數的量化敘述之邏輯等價及邏輯蘊涵

對一個預設的宇集及含變數 x 的任意開放敘述 $p(x)$，$q(x)$：

$$\exists x\, [p(x) \wedge q(x)] \Rightarrow [\exists x\; p(x) \wedge \exists x\; q(x)]$$

$$\exists x\, [p(x) \vee q(x)] \Leftrightarrow [\exists x\; p(x) \vee \exists x\; q(x)]$$

$$\forall x\, [p(x) \wedge q(x)] \Leftrightarrow [\forall x\; p(x) \wedge \forall x\; q(x)]$$

$$[\forall x\; p(x) \vee \forall x\; q(x)] \Rightarrow \forall x\, [p(x) \vee q(x)]$$

對一個預設的宇集及含變數 x 的任意開放敘述 $p(x)$，$q(x)$：

$$\exists x\,[p(x) \wedge q(x)] \Rightarrow [\exists x\ p(x) \wedge \exists x\ q(x)]$$

$$\exists x\,[p(x) \vee q(x)] \Leftrightarrow [\exists x\ p(x) \vee \exists x\ q(x)]$$

$$\forall x\,[p(x) \wedge q(x)] \Leftrightarrow [\forall x\ p(x) \wedge \forall x\ q(x)]$$

$$[\forall x\ p(x) \vee \forall x\ q(x)] \Rightarrow \forall x\,[p(x) \vee q(x)]$$

Example of $\not\Leftarrow$

$p(x):$

$q(x):$

Universe: integers

EXAMPLE 2.35

$$\forall x\ [p(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow \forall x\ [(p(x) \wedge q(x)) \wedge r(x)].$$

$$\forall x\ \neg\neg p(x) \Leftrightarrow \forall x\ p(x)$$
$$\forall x\ \neg[p(x) \wedge q(x)] \Leftrightarrow \forall x\ [\neg p(x) \vee \neg q(x)]$$
$$\forall x\ \neg[p(x) \vee q(x)] \Leftrightarrow \forall x\ [\neg p(x) \wedge \neg q(x)]$$

$$\exists x\ [p(x) \rightarrow q(x)] \Leftrightarrow \exists x\ [\neg p(x) \vee q(x)].$$

EXAMPLE 2.36 find the negation of two statements, where the universe comprises all of the integers.

1) Let $p(x)$ and $q(x)$ be given by

$$p(x): \quad x \text{ is odd} \qquad\qquad q(x): \quad x^2 - 1 \text{ is even.}$$

$\forall x\, [p(x) \to q(x)]$. (This is a true statement.)

$$\neg[\forall x\, (p(x) \to q(x))] \Leftrightarrow \exists x\, [\neg(p(x) \to q(x))]$$
$$\Leftrightarrow \exists x\, [\neg(\neg p(x) \vee q(x))] \Leftrightarrow \exists x\, [\neg\neg p(x) \wedge \neg q(x)]$$
$$\Leftrightarrow \exists x\, [p(x) \wedge \neg q(x)]$$

"There exists an integer x such that x is odd and $x^2 - 1$ is odd ." (This statement is false.)

EXAMPLE 2.36 Cont.

2) As in Example 2.36, let $r(x)$ and $s(x)$ be the open statements

$$r(x):\quad 2x+1=5 \qquad\qquad s(x):\quad x^2=9.$$

$\exists x\,[r(x) \wedge s(x)]$ is false

$$\neg[\exists x\,(r(x) \wedge s(x))] \Leftrightarrow \forall x\,[\neg(r(x) \wedge s(x))] \Leftrightarrow \forall x\,[\neg r(x) \vee \neg s(x)]$$

This negation "For every integer x, $2x+1 \neq 5$ or $x^2 \neq 9$." is true.

EXAMPLE 2.37

$$\forall x \forall y\ p(x, y) \Leftrightarrow \forall y \forall x\ p(x, y)$$

$$\exists x \exists y\ p(x, y) \Leftrightarrow \exists y \exists x\ p(x, y)$$

BUT

$p(x, y)$: $x + y = 17$.

$\forall x \exists y\ p(x, y)$: For every integer x, there exists an integer y such that $x + y = 17$. (TRUE)

$\exists y \forall x\ p(x, y)$: There exists an integer y so that for all integer x, $x + y = 17$. (FALSE)

Therefore, $\forall x \exists y\ p(x, y)\ \boxed{}\ \exists y \forall x\ p(x, y)$

EXAMPLE 2.38

Let $p(x, y)$, $q(x, y)$, and $r(x, y)$ represent three open statements. What is the negation of the following statement?

$$\forall x\, \exists y\, [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$

$$\neg[\forall x\, \exists y\, [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]]$$
$$\Longleftrightarrow \exists x\, [\neg \exists y\, [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]]$$
$$\Longleftrightarrow \exists x\, \forall y\, \neg[(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$
$$\Longleftrightarrow \exists x\, \forall y\, \neg[\neg[p(x, y) \wedge q(x, y)] \vee r(x, y)]$$
$$\Longleftrightarrow \exists x\, \forall y\, [\neg\neg[p(x, y) \wedge q(x, y)] \wedge \neg r(x, y)]$$
$$\Longleftrightarrow \exists x\, \forall y\, [(p(x, y) \wedge q(x, y)) \wedge \neg r(x, y)].$$