4-2 In-Class Exercise

1. Use the Subspace Test to determine whether the set is a subspace of M_{nn}.

The set of all $n \times n$ matrices A such that $A^T = -A$.

Let W be the set of all $n \times n$ matrices such that $A^T = -A$.

This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.

Let us assume A and B are both in W, i.e. $A^T = -A$ and $B^T = -B$. By Theorem

$$(A+B)^T = A^T + B^T = -A - B = -(A+B)$$ therefore W is closed under addition.

we have $(kA)^T = kA^T = k(-A) = -kA$ which makes W closed under scalar multiplication.

W is a subspace of M_{nn}.

2. Use the Subspace Test to determine whether the set is a subspace of R^4.

All vectors of the form $(a, 0, b, 0)$.

Let W be the set of all vectors in R^4 of form $(a,0,b,0)$.

This set contains at least one vector, e.g. the zero vector.

Adding two vectors in W results in another vector in W:

$$(a,0,b,0)+(a',0,b',0)=(a+a',0,b+b',0).$$

a scalar multiple of a vector in W is also in W: $k(a,0,b,0)=(ka,0,kb,0)$.

W is a subspace of R^4.

4-2 Suggested Exercises

1. Use the Subspace Test to determine whether the set is a subspace of M_{nn}.

The set of all $n \times n$ matrices A such that $\operatorname{tr}(A) = 0$.

Let W be the set of all $n \times n$ matrices with zero trace.

This set contains at least one matrix, e.g., the zero $n \times n$ matrix is in W.

Let us assume $A = [a_{ij}]$ and $B = [b_{ij}]$ are both in W, i.e. $\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = 0$ and

$\operatorname{tr}(B) = b_{11} + b_{22} + \cdots + b_{nn} = 0$.

Since $\operatorname{tr}(A+B) = (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn})$

$= a_{11} + a_{22} + \cdots + a_{nn} + b_{11} + b_{22} + \cdots + b_{nn} = 0 + 0 = 0$, it follows that $A + B$ is in W.

A scalar multiple of the same matrix A with a scalar k has $\operatorname{tr}(kA) = ka_{11} + ka_{22} + \cdots +$

$ka_{nn} = k(a_{11} + a_{22} + \cdots + a_{nn}) = 0$ therefore kA is in W as well. W is a subspace of M_{nn}.

2. Use the Subspace Test to determine whether the set is a subspace of R^3.

All vectors of the form (a, b, c), where $b = a + c$.

Let W be the set of all vectors of the form (a, b, c), where $b = a + c$.

This set contains at least one vector, e.g. $(0,0,0)$. (The condition $b = a + c$ is satisfied when $a = b = c = 0$.)

Adding two vectors in W results in another vector in W

$(a, a+c, c) + (a', a'+c', c') = (a+a',\ a+c+a'+c',\ c+c')$ since in this result, the second component is the sum of the first and the third: $a+c+a'+c' = (a+a') + (c+c')$.

Likewise, a scalar multiple of a vector in W is also in W: $k(a, a+c, c) = (ka, k(a+c), kc)$ since in this result, the second component is once again the sum of the first and the third:

$k(a+c) = ka + kc$.

According to Theorem 4.2.1, W is a subspace of R^3.

3. Use the Subspace Test to determine whether the set is a subspace of P_3.

All polynomials of the form $a_0 + a_1x + a_2x^2 + a_3x^3$ in which $a_0, a_1, a_2,$ and a_3 are rational numbers.

Let W be the set of all polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ in which a_0, a_1, a_2, and a_3 are rational numbers. The set W is not closed under the operation of scalar multiplication, e.g., the scalar product of the polynomial x^3 in W by $k = \pi$ is πx^3, which is not in W.
According to Theorem 4.2.1, W is not a subspace of P_3.

4. Use the Subspace Test to determine whether the set is a subspace of M_{22}.

All 2×2 matrices A such that

$$A\begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix} A$$

Let W be the set of all 2×2 matrices A such that $A\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} A$. This set contains at least one matrix, e.g. the zero matrix. Adding two matrices in W results in another matrix in W: $(A+B)\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} = A\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} + B\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} A + \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} B = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}(A+B)$. Likewise, a scalar multiple of a matrix in W is also in W: $(kA)\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} = k\left(A\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}\right) = k\left(\begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix} A\right) = \begin{bmatrix} 0 & 2 \\ 2 & -1 \end{bmatrix}(kA)$. According to Theorem 4.2.1, W is a subspace of M_{22}.

5. Use the Subspace Test to determine whether the set is a subspace of R^4.

All vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where

$$A = \begin{bmatrix} 0 & -1 & 0 & 2 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Let W be the set of all vectors $\mathbf{x}$ in R^4 such that $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This set is not closed under scalar multiplication when the scalar is 0. Consequently, W is not a subspace of R^4.

6. If T_A is multiplication by a matrix A with three columns, then the kernel of T_A is one of four possible geometric objects. What are they? Explain how you reached your conclusion.

Since $T_A : R^3 \rightarrow R^m$, it follows from Theorem 4.2.5 that the kernel of T_A must be a subspace of R^3. Hence,

the kernel can be one of the following four geometric obects:

- the origin,
- a line through the origin,
- a plane through the origin,
- R^3.