Chapter 4
General Vector Spaces

4.1. Real Vector Spaces

4.2. Subspaces

4.3. Spanning Sets

4.4. Linear Independence

4.5. Coordinates and Basis

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Chapter 4.3

Spanning Sets

Linear Combination

DEFINITION 1

If $\mathbf{w}$ is a vector in a vector space V, then $\mathbf{w}$ is said to be a *linear combination* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in V if $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

These scalars are called the *coefficients* of the linear combination.

Linear Combination

THEOREM 4.3.1

If $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ is a nonempty set of vectors in a vector space V, then:

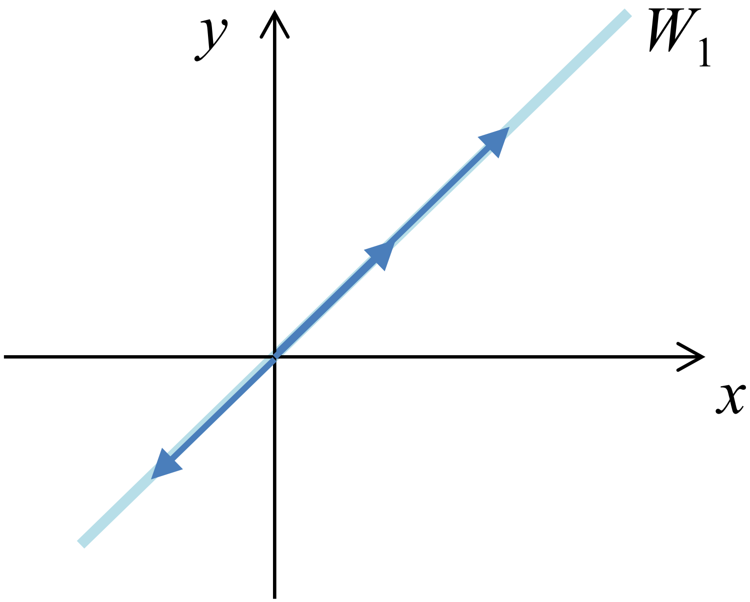
> The set W of all possible linear combinations of the vectors in S is a subspace of V.

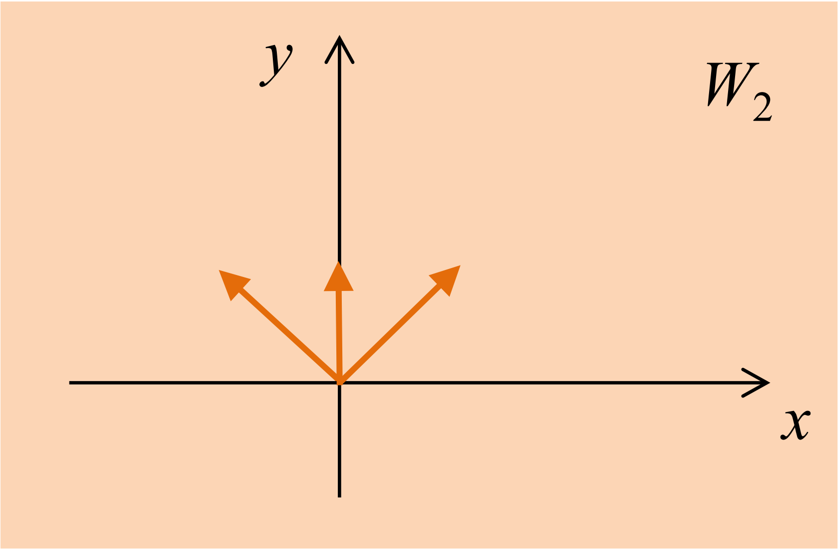
Proof exe

Let $V = R^2$

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$





THEOREM 4.3.1

Proof Let W be the set of all possible linear combinations of the vectors in S. We must show that W is closed under addition and scalar multiplication. To prove closure under addition, let

$$\mathbf{u} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_r\mathbf{w}_r \quad \text{and} \quad \mathbf{v} = k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r$$

be two vectors in W. It follows that their sum can be written as

$$\mathbf{u} + \mathbf{v} = (c_1 + k_1)\mathbf{w}_1 + (c_2 + k_2)\mathbf{w}_2 + \cdots + (c_r + k_r)\mathbf{w}_r$$

which is a linear combination of the vectors in S. Thus, W is closed under addition. We leave it for you to prove that W is also closed under scalar multiplication and hence is a subspace of V.

Span

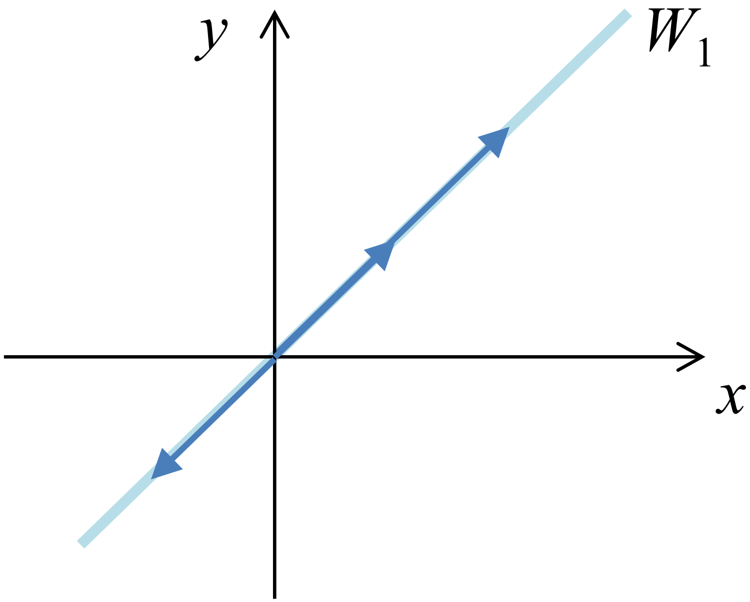
The subspace W in Theorem 4.3.1 is called the subspace of V *spanned* by S. The vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r$ in S are said to *span* W, and we write

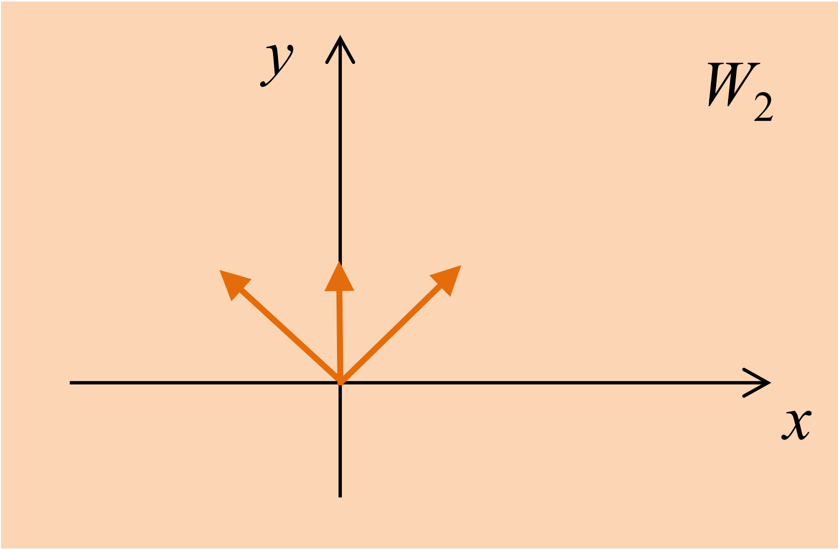
$$W = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\} \quad \text{or} \quad W = \text{span}(S)$$

Let $V = R^2$

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

$$S_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$





$$W_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

$$W_2 = \text{span}(S_2)$$

The Standard Unit Vectors Span R^n

EXAMPLE 1

Recall that the standard unit vectors in R^n are

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

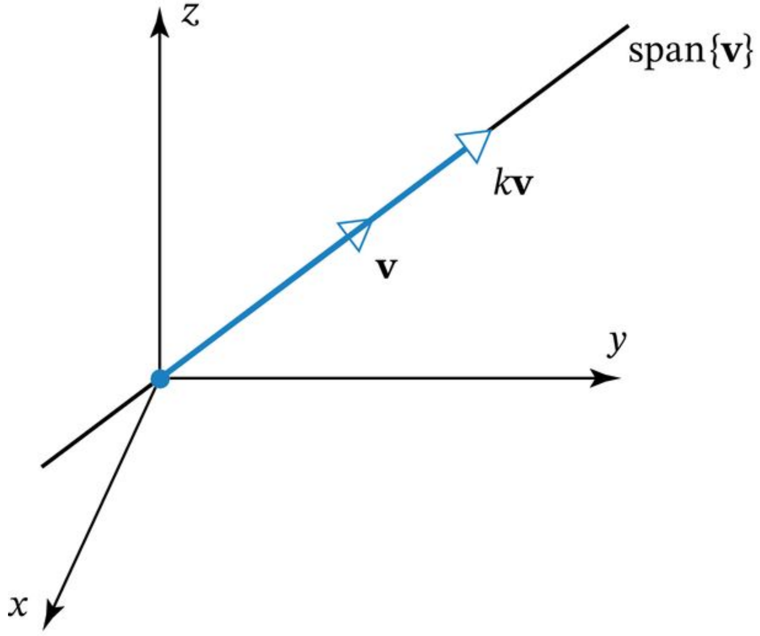
These vectors span R^n since every vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in R^n can be expressed as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$$

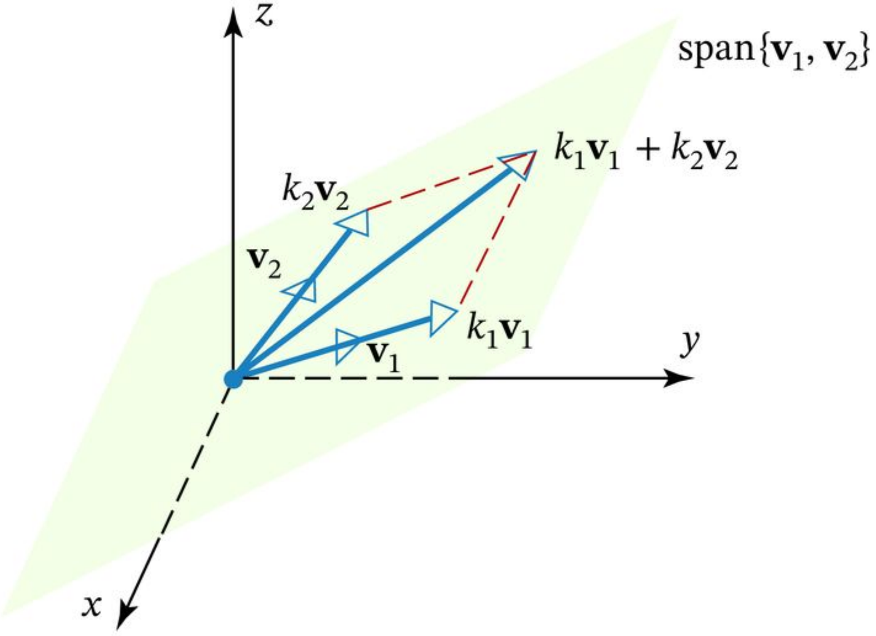
which is a linear combination of $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$.

A Geometric View of Spanning in R^2 and R^3

EXAMPLE 2



(a) span{**v**} is the line through the origin determined by **v**



(b) span{$\mathbf{v}_1$, $\mathbf{v}_2$} is the plane through the origin determined by $\mathbf{v}_1$ and $\mathbf{v}_2$

A Spanning Set for P_n

EXAMPLE 3

The polynomials $1, x, x^2, \ldots, x^n$ span the vector space P_n since each polynomial $\mathbf{p}$ in P_n can be written as

$$\mathbf{p} = a_0 + a_1 x + \cdots + a_n x^n$$

which is a linear combination of $1, x, x^2, \ldots, x^n$.

We can denote this by writing $P_n = \operatorname{span}\{1, x, x^2, \ldots, x^n\}$

Two Important Types of Problems

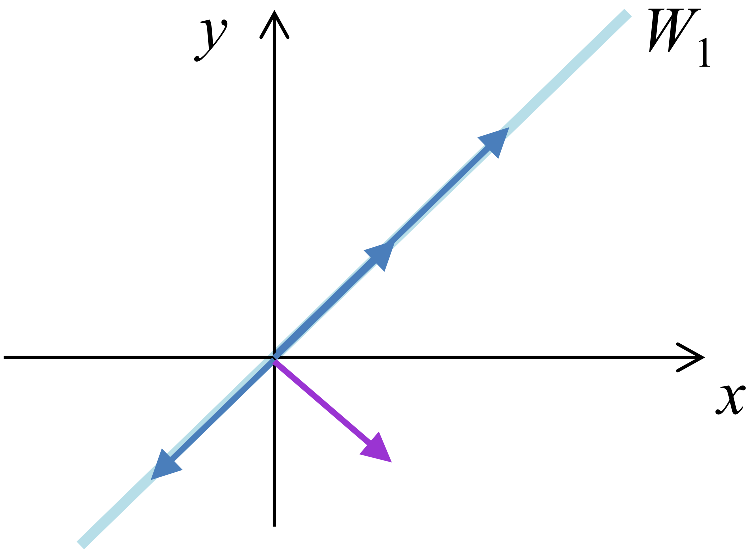
- Given a nonempty set S of vectors in R^n and a vector $\mathbf{v}$ in R^n, determine whether $\mathbf{v}$ is a linear combination of the vectors in S.

- Given a nonempty set S of vectors in R^n, determine whether the vectors span R^n.

Let $V = R^2$

$$S_1 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

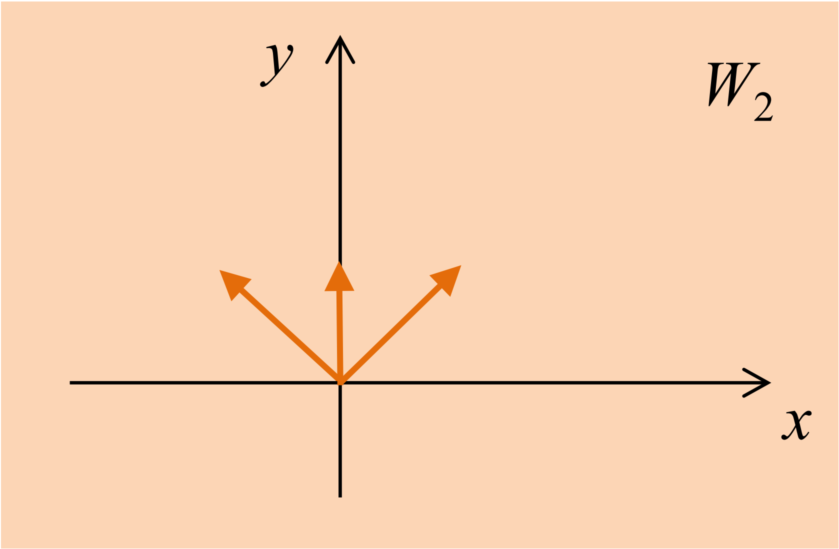
$$S_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$



If $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in R^2$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} \overset{?}{=} a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 2 \end{bmatrix} + c \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

for some real numbers a, b, c



If span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \overset{?}{=} R^2$

Linear Combination Problem

EXAMPLE 4

Consider the vectors $\mathbf{u} = (1, 2, -1)$ and $\mathbf{v} = (6, 4, 2)$ in R^3.
Show that $\mathbf{w} = (9, 2, 7)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{w}' = (4, -1, 8)$ is *not* a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Solution

$$\mathbf{w} = k_1\mathbf{u} + k_2\mathbf{v}$$

$$(9, 2, 7) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

EXAMPLE 4 Cont.

Equating corresponding components gives

$$\begin{aligned} k_1 + 6k_2 &= 9 \\ 2k_1 + 4k_2 &= 2 \\ -k_1 + 2k_2 &= 7 \end{aligned}$$

Solving this system

$$k_1 = -3, k_2 = 2, \text{ so} \quad \mathbf{w} = -3\mathbf{u} + 2\mathbf{v}$$

Similarly,

$$\mathbf{w}' = k_1\mathbf{u} + k_2\mathbf{v}$$

$$(4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2) = (k_1 + 6k_2, 2k_1 + 4k_2, -k_1 + 2k_2)$$

$$\begin{aligned} k_1 + 6k_2 &= 4 \\ 2k_1 + 4k_2 &= -1 \\ -k_1 + 2k_2 &= 8 \end{aligned}$$

This system of equations is inconsistent

$\mathbf{w}'$ is not a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Testing for Spanning

EXAMPLE 5

Determine whether the vectors $\mathbf{v}_1 = (1, 1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (2, 1, 3)$ span the vector space R^3.

Solution

determine whether an arbitrary vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as a linear combination

$$\mathbf{b} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$$

$$(b_1, b_2, b_3) = k_1(1, 1, 2) + k_2(1, 0, 1) + k_3(2, 1, 3)$$

$$(b_1, b_2, b_3) = (k_1 + k_2 + 2k_3, k_1 + k_3, 2k_1 + k_2 + 3k_3)$$

EXAMPLE 5 Cont.

$$\begin{aligned} k_1 + k_2 + 2k_3 &= b_1 \\ k_1 \qquad + \ k_3 &= b_2 \\ 2k_1 + k_2 + 3k_3 &= b_3 \end{aligned}$$

Theorem 2.3.8, the system is consistent if and only if its coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$ has a nonzero determinant.

But this is *not* the case here; $\det(A) = 0$ so $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not span R^3.

The Procedure for Identifying Spanning Sets

Step 1. Let $S = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r\}$ be a given set of vectors in V, and let $\mathbf{x}$ be an arbitrary vector in V.

Step 2. Set up the augmented matrix for the linear system that results by equating corresponding components on the two sides of the vector equation $k_1\mathbf{w}_1 + k_2\mathbf{w}_2 + \cdots + k_r\mathbf{w}_r = \mathbf{x}$

Step 3. Use the techniques developed in Chapters 1 and 2 to investigate the consistency or inconsistency of that system.

If it is consistent for *all* choices of $\mathbf{x}$, the vectors in S span V, and if it is inconsistent for *some* vector $\mathbf{x}$, they do not.

Testing for Spanning

EXAMPLE 6

Determine whether the set S spans P_2.

(a) $S = \{1 + x + x^2, -1 - x, 2 + 2x + x^2\}$

(b) $S = \{x + x^2, x - x^2, 1 + x, 1 - x\}$

Solution (*a*) An arbitrary vector in P_2 is of the form $\mathbf{p} = a + bx + cx^2$,

$$k_1(1 + x + x^2) + k_2(-1 - x) + k_3(2 + 2x + x^2) = a + bx + cx^2$$

$$(k_1 - k_2 + 2k_3) + (k_1 - k_2 + 2k_3)x + (k_1 + k_3)x^2 = a + bx + cx^2$$

EXAMPLE 6 Cont.

$$\begin{bmatrix} 1 & -1 & 2 & a \\ 1 & -1 & 2 & b \\ 1 & 0 & 1 & c \end{bmatrix}$$

and whose coefficient matrix is

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

S does **not** span P_2.

Solution (*b*) Using the same procedure the augmented matrix

$$\begin{bmatrix} 0 & 0 & 1 & -1 & a \\ 1 & 1 & 1 & -1 & b \\ 1 & -1 & 0 & 0 & c \end{bmatrix}$$

EXAMPLE 6 Cont.

The reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{-a+b+c}{2} \\ 0 & 1 & 0 & 0 & \frac{-a+b-c}{2} \\ 0 & 0 & 1 & -1 & a \end{bmatrix}$$

is consistent for every choice a, b, and c. Thus,

$$\text{span}(S) = P_2$$

Testing for Spanning

EXAMPLE 7

In each part, determine whether the set S spans M_{22}.

(a) $S = \left\{ \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

(b) $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$

exe

EXAMPLE 7

Solution (*a*) An arbitrary vector in M_{22} is of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so Equation (2) becomes

$$k_1\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + k_3\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} + k_4\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

which we can rewrite as

$$\begin{bmatrix} k_1 + k_2 + k_3 + k_4 & 2k_1 + 2k_3 + k_4 \\ k_3 + k_4 & k_1 + k_2 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Equating corresponding entries produces a linear system whose augmented matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & a \\ 2 & 0 & 2 & 1 & b \\ 0 & 0 & 1 & 1 & c \\ 1 & 1 & 0 & 1 & d \end{bmatrix} \text{ and whose coefficient matrix is } A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

As in part (a) of Example 6, the coefficient matrix is square, so we can apply parts (e) and (g) of Theorem 2.3.8. We leave it for you to verify that $\det(A) = -2 \neq 0$, so the system is consistent for *every* choice of a, b, c, and d, which implies that $\text{span}(S) = M_{22}$.

EXAMPLE 7

Solution (*b*) Using the same procedure as in part (a), the augmented matrix for the linear system corresponding to Equation (2) is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & a \\ 0 & 0 & 0 & 1 & b \\ 0 & 1 & 1 & -1 & c \\ 0 & 0 & 0 & 1 & d \end{bmatrix} \text{ and the coefficient matrix is } A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ which}$$

is square, so once again we can apply parts (e) and (g) of Theorem 2.3.8. Since the second and fourth rows of this matrix are identical, it follows that $\det(A) = 0$. Thus, the system is inconsistent for *some* choice of a, b, c, and d, which implies that S does not span M_{22}.

Spanning Sets are not Unique

THEOREM 4.3.2

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ and $S' = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ are nonempty sets of vectors in a vector space V, then

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} = \text{span}\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$$

if and only if each vector in S is a linear combination of those in S', and each vector in S' is a linear combination of those in S.

This Theorem states that two different sets of vectors may span the same space.

Chapter 4-3 Objectives

- Given a set S of vectors in R^n and a vector $\mathbf{v}$ in R^n, determine whether $\mathbf{v}$ is a linear combination of the vectors in S.
- Given a set S of vectors in R^n, determine whether the vectors in S span R^n.
- Determine whether two nonempty sets of vectors in a vector space V span the same subspace of V.