1-2 In-Class Exercise

1.

(a) If A is a 3 x 5 matrix, then what is the maximum possible number of leading 1's in its reduced row echelon form?

(b) If B is a 3 x 6 matrix, and B is not a zero matrix, then what is the maximum possible number of parameters in the general solution of the linear system with augmented matrix B?

(c) If C is a 5 x 3 matrix, then what is the minimum possible number of rows of zeros in any row echelon form of C?

(a) 3 (this will be the number of leading 1's if the matrix has no rows of zeros).
(b) 4
(c) 2 (this will be the number of rows of zeros if each column contains a leading 1)

1-2 Suggested Exercise

1. Discuss the existence and uniqueness of solutions to the corresponding linear systems.

$$(a)\begin{bmatrix}1 & -3 & 7 & 2 & 5\\0 & 1 & 2 & -4 & 1\\0 & 0 & 1 & 6 & 9\\0 & 0 & 0 & 0 & 1\end{bmatrix} \qquad (b)\begin{bmatrix}1 & -3 & 7 & 2 & 5\\0 & 1 & 2 & -4 & 1\\0 & 0 & 1 & 6 & 9\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$(c)\begin{bmatrix}1 & -3 & 7 & 2 & 5\\0 & 1 & 2 & -4 & 1\\0 & 0 & 1 & 6 & 9\\0 & 0 & 0 & 1 & 0\end{bmatrix}$$

Solution (*a*) The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

from which it is evident that the system is inconsistent.

Solution (*b*) The last row corresponds to the equation

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$$

which has no effect on the solution set. In the remaining three equations the variables x_1, x_2, and x_3 correspond to leading 1's and hence are leading variables. The variable x_4 is a free variable. With a little algebra, the leading variables can be expressed in terms of the free variable, and the free variable can be assigned an arbitrary value. Thus, the system must have infinitely many solutions.

Solution (*c*) The last row corresponds to the equation

$$x_4 = 0$$

which gives us a numerical value for x_4. If we substitute this value into the third equation, namely,

$$x_3 + 6x_4 = 9$$

we obtain $x_3 = 9$. You should now be able to see that if we continue this process and substitute the known values of x_3 and x_4 into the equation corresponding to the second row, we will obtain a unique numerical value for x_2; and if, finally, we substitute the known values of x_4, x_3, and x_2 into the equation corresponding to the first row, we will produce a unique numerical value for x_1. Thus, the system has a unique solution.

2. Solve the linear system.

$$\begin{aligned} && Z_3 + Z_4 + Z_5 &= 0 \\ -Z_1 - Z_2 &+ 2Z_3 - 3Z_4 + Z_5 &= 0 \\ Z_1 + Z_2 &- 2Z_3 \qquad\quad - Z_5 &= 0 \\ 2Z_1 + 2Z_2 &- Z_3 \qquad\quad + Z_5 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 2 & 2 & -1 & 0 & 1 & 0 \end{bmatrix} \longleftarrow$$

The first and third rows were interchanged.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix} \longleftarrow$$

The first row was added to the second row and -2 times the first row was added to the last row.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 \end{bmatrix} \longleftarrow$$

The second and third rows were interchanged.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \longleftarrow$$

-3 times the second row was added to the fourth row.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \end{bmatrix} \longleftarrow$$

The third row was multiplied by $-\frac{1}{3}$.

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{3 times the third row was added to the fourth row.}$$

$$\begin{bmatrix} 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow -1 \text{ times the third row was added to the second row.}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longleftarrow \text{2 times the second row was added to the first row.}$$

If we assign Z_2 and Z_5 the arbitrary values s and t, respectively, the general solution is given by the formulas

$$Z_1 = -s - t, \quad Z_2 = s, \quad Z_3 = -t, \quad Z_4 = 0, \quad Z_5 = t\,.$$

3. What condition, if any, must a, b, and c satisfy for the linear system to be consistent?

$$\begin{aligned} x + 3y + z &= a \\ -x - 2y + z &= b \\ 3x + 7y - z &= c \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & 1 & a \\ -1 & -2 & 1 & b \\ 3 & 7 & -1 & c \end{bmatrix}$$

The augmented matrix for the system.

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & -2 & -4 & -3a+c \end{bmatrix}$$

The first row was added to the second row and -3 times the first row was added to the third row.

$$\begin{bmatrix} 1 & 3 & 1 & a \\ 0 & 1 & 2 & a+b \\ 0 & 0 & 0 & -a+2b+c \end{bmatrix}$$

2 times the second row was added to the third row.

If $-a+2b+c=0$ then the linear system is consistent.
Otherwise (if $-a+2b+c\neq 0$) it is inconsistent.

4. Solve the following systems, where a, b, and c are constants.

$$\begin{aligned} x_1 + x_2 + x_3 &= a \\ 2x_1 \qquad + 2x_3 &= b \\ 3x_2 + 3x_3 &= c \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 2 & 0 & 2 & b \\ 0 & 3 & 3 & c \end{bmatrix} \longleftarrow$$ The augmented matrix for the system.

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & 0 & -2a+b \\ 0 & 3 & 3 & c \end{bmatrix} \longleftarrow$$ -2 times the first row was added to the second row.

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 3 & 3 & c \end{bmatrix} \longleftarrow$$ The second row was multiplied by $-\frac{1}{2}$.

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 3 & -3a+\frac{3}{2}b+c \end{bmatrix} \longleftarrow$$ -3 times the second row was added to the third row.

$$\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix} \longleftarrow \text{The third row was multiplied by } \tfrac{1}{3}.$$

$$\begin{bmatrix} 1 & 1 & 0 & 2a-\frac{b}{2}-\frac{c}{3} \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix} \longleftarrow -1 \text{ times the third row was added to the first row.}$$

$$\begin{bmatrix} 1 & 0 & 0 & a-\frac{c}{3} \\ 0 & 1 & 0 & a-\frac{b}{2} \\ 0 & 0 & 1 & -a+\frac{b}{2}+\frac{c}{3} \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the first row.}$$

The system has exactly one solution: $x_1 = a - \frac{c}{3}$, $x_2 = a - \frac{b}{2}$, and $x_3 = -a + \frac{b}{2} + \frac{c}{3}$.