Chapter 1

Systems of Linear Equations and Matrices

Chapter 1.5

Elementary Matrices and

a Method for Finding Inverse

Row Equivalent

DEFINITION 1

Matrices A and B are said to be *row equivalent* if either (hence each) can be obtained from the other by a sequence of elementary row operations.

$$\begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 1 & 4 & 1 \\ 1 & 0 & 8 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 4 & 5 \\ -2 & 1 & -11 & -8 \\ -1 & 2 & 2 & 2 \end{bmatrix}$$

$$\searrow \qquad \swarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & -18 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Elementary Matrix

DEFINITION 2

An n x n matrix is called an *elementary matrix* if it can be obtained from the n x n identity matrix I_n by performing a single elementary row operation.

EXAMPLE 1 Elementary Matrices and Row Operations

$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply the second row of I_2 by -3.

Interchange the second and fourth rows of I_4.

Add 3 times the third row of I_3 to the first row.

Multiply the first row of I_3 by 1.

Row Operations by Matrix Multiplication

THEOREM 1.5.1

If the elementary matrix E results from performing a certain row operation on I_m and if A is an m x n matrix, then the product EA is the matrix that results when this same row operation is performed on A.

If $\begin{array}{c} I_m \\ A_{mxn} \end{array} \xrightarrow{\text{row operation}} \begin{array}{c} E \\ B \end{array}$ then $B = EA$

Row Operations by Matrix Multiplication

EXAMPLE 2 Using Elementary Matrices

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 1 & 4 & 4 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

adding 3 times the first row of I_3 to the third row.

$$EA = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 2 & -1 & 3 & 6 \\ 4 & 4 & 10 & 9 \end{bmatrix}$$

precisely the matrix that results when we add 3 times the first row of A to the third row.

Inverse Operations

Row Operation on I That Produces E	Row Operation on E That Reproduces I
Multiply row i by $c \neq 0$	Multiply row i by $1/c$
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add $-c$ times row i to row j

inverse operations

Inverse Operations

EXAMPLE 3 Row Operations and Inverse Row Operations

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Example [Note]

EXAMPLE 3 Row Operations and Inverse Row Operations

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Multiply the second row by 7.

Multiply the second row by $\frac{1}{7}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Interchange the first and second rows.

Interchange the first and second rows.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Add 5 times the second row to the first.

Add -5 times the second row to the first.

Inverse of Elementary Matrices

THEOREM 1.5.2

Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Proof

The inverse of an elementary matrix is its inverse row operation matrix.

Row Operation on I That Produces E	Row Operation on E That Reproduces I
Multiply row i by $c \neq 0$	Multiply row i by $1/c$
Interchange rows i and j	Interchange rows i and j
Add c times row i to row j	Add $-c$ times row i to row j

Equivalence Theorem

THEOREM 1.5.3

If A is an $n \times n$ matrix, then the following statements are equivalent, that is, all true or all false.

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(c) The reduced row echelon form of A is I_n.

(d) A is expressible as a product of elementary matrices.

(Later) Equivalence Theorem

Theorem 5.1.5

Equivalent Statements

If A is an $n \times n$ matrix in which there are no duplicate rows and no duplicate columns, then the following statements are equivalent.

(*a*) A is invertible.

(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*c*) The reduced row echelon form of A is I_n.

(*d*) A is expressible as a product of elementary matrices.

(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(*g*) $\det(A) \neq 0$.

(*h*) The column vectors of A are linearly independent.

(*i*) The row vectors of A are linearly independent.

(*j*) The column vectors of A span R^n.

(*k*) The row vectors of A span R^n.

(*l*) The column vectors of A form a basis for R^n.

(*m*) The row vectors of A form a basis for R^n.

(*n*) A has rank n.

(*o*) A has nullity 0.

(*p*) The orthogonal complement of the null space of A is R^n.

(*q*) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

(*r*) $\lambda = 0$ is not an eigenvalue of A.

Equivalence Theorem

THEOREM 1.5.3

Proof $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (a)$.

$\mathbf{(a) \Rightarrow (b)}$ Assume A is invertible and let $\mathbf{x}_0$ be any solution of $A\mathbf{x} = \mathbf{0}$.

$$(A^{-1}A)\mathbf{x}_0 = A^{-1}\mathbf{0}$$

$\mathbf{x}_0 = \mathbf{0}$, so $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$\mathbf{(b) \Rightarrow (c)}$ Let the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 \end{bmatrix} \xrightarrow{\text{E.R.O.s}} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

the reduced row echelon form of A is I_n.

Equivalence Theorem

THEOREM 1.5.3

Proof

$(c) \Rightarrow (d)$ Assume that the reduced row echelon form of A is I_n, so that A can be reduced to I_n by a finite sequence of elementary row operations.

$$E_k \cdots E_2 E_1 A = I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

$(d) \Rightarrow (a)$ If A is a product of elementary matrices, then the matrix A is a product of invertible matrices and hence is invertible.

A Method for Inverting Matrices

$$E_k \cdots E_2 E_1 A = I_n$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$A^{-1} = E_k \cdots E_2 E_1 I_n$$

the same sequence of row operations that reduces A to I_n will transform I_n to A^{-1}.

Inversion Algorithm

To find the inverse of an invertible matrix A, find a sequence of elementary row operations that reduces A to the identity and then perform that same sequence of operations on I_n to obtain A^{-1}.

A Method for Inverting Matrices

$$\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right. \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} = [A \mid \mathbf{b}]$$

For the case when $m = n$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \xrightarrow{E_1} \begin{bmatrix} \quad \end{bmatrix} \xrightarrow{E_2} \cdots\cdots \xrightarrow{E_k} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = I_n$$

$$E_k \cdots E_2 E_1 A = I_n$$

$$\boxed{E_k \cdots E_2 E_1 A = I_n}$$

$$E_k^{-1} E_k \cdots E_2 E_1 A = E_k^{-1} I_n$$

$$\|$$

$$I_n$$

$$\vdots$$

$$\vdots$$

$$A = E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n$$

$$A^{-1} = (E_1^{-1} E_2^{-1} \cdots E_k^{-1} I_n)^{-1}$$

$$= (I_n)^{-1} (E_k^{-1})^{-1} \cdots (E_1^{-1})^{-1}$$

$$\boxed{A^{-1} = E_k \cdots E_2 E_1 I_n}$$

$$\left[\begin{array}{c|c} A & I_n \end{array}\right] = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right] \xrightarrow{E_1} \left[\begin{array}{c|c} \quad & \quad \end{array}\right]$$

$$\xrightarrow{E_2} \cdots\cdots \xrightarrow{E_k} \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & A^{-1} \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 & \end{array}\right]$$

$$I_n$$

$$\boxed{E_k \cdots E_2 E_1 A = I_n}$$

$$\boxed{E_k \cdots E_2 E_1 I_n = A^{-1}}$$

Inversion Algorithm

EXAMPLE 4 Using Row Operations to Find A^{-1}

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

exe

EXAMPLE 5 Showing That a Matrix Is Not Invertible

exe

EXAMPLE 6 Analyzing Homogeneous Systems

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EXAMPLE 4 Using Row Operations to Find A^{-1}

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

Solution We want to reduce A to the identity matrix by row operations and simultaneously apply these operations to I to produce A^{-1}. To accomplish this we will adjoin the identity matrix to the right side of A, thereby producing a partitioned matrix of the form

$$[A \mid I]$$

Then we will apply row operations to this matrix until the left side is reduced to I; these operations will convert the right side to A^{-1}, so the final matrix will have the form

$$[I \mid A^{-1}]$$

The computations are as follows:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

← We added -2 times the first row to the second and -1 times the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

← We added 2 times the second row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We multiplied the third row by -1.

$$\left[\begin{array}{rrr|rrr} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We added 3 times the third row to the second and -3 times the third row to the first.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← We added -2 times the second row to the first.

Thus,

$$A^{-1} = \begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

EXAMPLE 5 Showing That a Matrix Is Not Invertible

Consider the matrix

$$A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$$

Applying the procedure of Example 4 yields

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right]$$

We added -2 times the first row to the second and added the first row to the third.

$$\left[\begin{array}{rrr|rrr} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right]$$

We added the second row to the third.

Since we have obtained a row of zeros on the left side, A is not invertible.

EXAMPLE 6 Analyzing Homogeneous Systems

Use Theorem 1.5.3 to determine whether the given homogeneous system has nontrivial solutions.

(a)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 3x_3 &= 0 \\ x_1 \qquad\quad + 8x_3 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 6x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 - x_3 &= 0 \\ -x_1 + 2x_2 + 5x_3 &= 0 \end{aligned}$$

Solution From parts (*a*) and (*b*) of Theorem 1.5.3 a homogeneous linear system has only the trivial solution if and only if its coefficient matrix is invertible. From Examples 4 and 5 the coefficient matrix of system (a) is invertible and that of system (b) is not. Thus, system (a) has only the trivial solution while system (b) has nontrivial solutions.

Chapter 1-5 Objectives

- Determine whether a given square matrix is an elementary.
- Determine whether two square matrices are row equivalent.
- Apply the inverse of a given elementary row operation to a matrix.
- Apply elementary row operations to reduce a given square matrix to the identity matrix.
- Understand the relationships between statements that are equivalent to the invertibility of a square matrix (Theorem 1.5.3).
- Use the inversion algorithm to find the inverse of an invertible matrix.
- Express an invertible matrix as a product of elementary matrices.