1-4 In-Class Exercise

1. $$(5A^T)^{-1} = \begin{bmatrix} -3 & -1 \\ 5 & 2 \end{bmatrix}$$

Find A

inverse of $\left(5A^T\right)^{-1}$ is $5A^T$.

Thus $5A^T = \frac{1}{-1}\begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 5 & 3 \end{bmatrix}$. Consequently, $A = \begin{bmatrix} -\frac{2}{5} & 1 \\ -\frac{1}{5} & \frac{3}{5} \end{bmatrix}$.

Suggested Exercise

$$A = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}$$

1. Compute $A^2 - 2A + I$

$$A^2 - 2A + I = \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 \\ 12 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$$

2. Compute $p(A)$ for the polynomial $p(x) = 2x^2 - x + 1$

$$2A^2 - A + I = \begin{bmatrix} 7 & 0 \\ 20 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

3. Find all values of a, b, c, and d (if any) for which the matrices A and B commute.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}; \; BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}.$$

The matrices A and B commute if $\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$, i.e. $0 = c$ and $a = d$

Therefore, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ commute if $c = 0$ and $a = d$.

If we assign b and d the arbitrary values s and t, respectively,

the general solution is given by the formulas $\quad a = t, \quad b = s, \quad c = 0, \quad d = t$

4. Simplify the expression assuming that A, B, C, and D are invertible.

$$(AB)^{-1}(AC^{-1})(D^{-1}C^{-1})^{-1}D^{-1}$$

$$(AB)^{-1}\left(AC^{-1}\right)\left(D^{-1}C^{-1}\right)^{-1}D^{-1}$$

$$=(B^{-1}A^{-1})\left(AC^{-1}\right)\left(\left(C^{-1}\right)^{-1}\left(D^{-1}\right)^{-1}\right)D^{-1}$$

$$=(B^{-1}A^{-1})\left(AC^{-1}\right)(CD)D^{-1}$$

$$=B^{-1}\left(A^{-1}A\right)\left(C^{-1}C\right)\left(DD^{-1}\right)$$

$$=B^{-1}III$$

$$=B^{-1}$$

5. Show that if A, B, and $A + B$ are invertible matrices with the same size, then

$$A(A^{-1} + B^{-1})B(A + B)^{-1} = I$$

$$\begin{aligned}
&A\left(A^{-1}+B^{-1}\right)B\left(A+B\right)^{-1} \\
&=\left(AA^{-1}B+AB^{-1}B\right)\left(A+B\right)^{-1} \\
&=\left(IB+AI\right)\left(A+B\right)^{-1} \\
&=\left(B+A\right)\left(A+B\right)^{-1} \\
&=\left(A+B\right)\left(A+B\right)^{-1} \\
&=I
\end{aligned}$$

6. A square matrix A is said to be ***idempotent*** if $A^2 = A$.

a. Show that if A is idempotent, then so is $I - A$.

b. Show that if A is idempotent, then $2A - I$ is invertible and is its own inverse.

(a)

$$(I-A)^2$$
$$=(I-A)(I-A)$$
$$=II-IA-AI+AA$$
$$=I-A-A+A^2$$
$$=I-A-A+A$$
$$=I-A$$

(b)

$$(2A-I)(2A-I)$$
$$=(2A)(2A)-2AI-I(2A)+II$$
$$=4A^2-2A-2A+I$$
$$=4A-4A+I$$
$$=I$$

7. Assuming that all matrices are $n \times n$ and invertible, solve for D.

$$ABC^TDBA^TC = AB^T$$

$$ABC^TDBA^TC = AB^T$$

$$BC^TDBA^T = B^TC^{-1}$$

$$C^TDB = B^{-1}B^TC^{-1}(A^T)^{-1}$$

$$D = (C^T)^{-1}B^{-1}B^TC^{-1}(A^T)^{-1}B^{-1}$$

$$= (BC^T)^{-1}B^T(BA^TC)^{-1}$$