4-1 True-False Exercises

a. A vector is any element of a vector space. ○

b. A vector space must contain at least two vectors. ×

c. If $\mathbf{u}$ is a vector and k is a scalar such that $k\mathbf{u} = \mathbf{0}$, then it must be true that $k = 0$. ×

d. The set of positive real numbers is a vector space if vector addition and scalar multiplication are the usual operations of addition and multiplication of real numbers. ×

e. In every vector space the vectors $(-1)\mathbf{u}$ and $-\mathbf{u}$ are the same. ○

(a) True. This is a part of Definition 1.

(b) False. Example 1 discusses a vector space containing only one vector.

(c) False. By part (d) of Theorem 4.1.1, if $k\mathbf{u} = \mathbf{0}$ then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

(d) False. Axiom 6 fails to hold if $k < 0$. (Also, Axiom 4 fails to hold.)

(e) True. This follows from part (c) of Theorem 4.1.1.

(f) False. This function must have a value of zero at *every* point in $(-\infty, \infty)$.

4-2 True-False Exercises

a. Every subspace of a vector space is itself a vector space. ○

b. Every vector space is a subspace of itself. ○

c. Every subset of a vector space V that contains the zero vector in V is a subspace of V. ✕

d. The kernel of a matrix transformation $T_A : R^n \to R^m$ is a subspace of R^m. ✕

e. The solution set of a consistent linear system $A\mathbf{x} = \mathbf{b}$ of m equations in n unknowns is a subspace of R^n. ✕

f. The intersection of any two subspaces of a vector space V is a subspace of V.

g. The union of any two subspaces of a vector space V is a subspace of V.

h. The set of upper triangular $n \times n$ matrices is a subspace of the vector space of all $n \times n$ matrices.

(a) True. This follows from Definition 1.

(b) True.

(c) False. The set of all nonnegative real numbers is a subset of the vector space R containing 0, but it is not closed under scalar multiplication.

(d) False. By Theorem 4.2.4, the kernel of $T_A : R^n \to R^m$ is a subspace of R^n.

(e) False. The solution set of a nonhomogeneous system is not closed under addition: $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{y} = \mathbf{b}$ do not imply $A(\mathbf{x} + \mathbf{y}) = \mathbf{b}$.

(f) True. This follows from Theorem 4.2.2.

(g) False. Consider $W_1 = \text{span}\{(1,0)\}$ and $W_2 = \text{span}\{(0,1)\}$. The union of these sets is not closed under vector addition, e.g. $(1,0) + (0,1) = (1,1)$ is outside the union.

(h) True. This set contains at least one matrix (e.g., I_n). A sum of two upper triangular matrices is also upper triangular, therefore the set is closed under addition. A scalar multiple of an upper triangular matrix is also upper triangular, hence the set is closed under scalar multiplication.

4-3 True-False Exercises

a. An expression of the form $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots k_r\mathbf{v}_r$ is called a linear combination. ○

b. The span of a single vector in R^2 is a line. ✕

c. The span of two vectors in R^3 is a plane. ✕

d. The span of a nonempty set S of vectors in V is the smallest subspace of V that contains S. ○

e. The span of any finite set of vectors in a vector space is closed under addition and scalar multiplication. ○

f. Two subsets of a vector space V that span the same subspace of V must be equal. ✗

g. The polynomials $x - 1$, $(x - 1)^2$, and $(x - 1)^3$ span P_3. ✗

(a) True.

(b) False. The span of the zero vector is just the zero vector.

(c) False. For example the vectors $(1,1,1)$ and $(2,2,2)$ span a line.

(d) True.

(e) True. This follows from part (a) of Theorem 4.2.1.

(f) False. For any nonzero vector $\mathbf{v}$ in a vector space V, both $\{\mathbf{v}\}$ and $\{2\mathbf{v}\}$ span the same subspace of V.

(g) False. The constant polynomial $p(x)=1$ cannot be represented as a linear combination of these, since at $x=1$ all three are zero, whereas $p(1)=1$.

4-4 True-False Exercises

a. A set containing a single vector is linearly independent. ✗

b. No linearly independent set contains the zero vector. ○

c. Every linearly dependent set contains the zero vector. ✗

d. If the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, then $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ is also linearly independent for every nonzero scalar k. ○

e. If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent nonzero vectors, then at least one vector $\mathbf{v}_k$ is a unique linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$. ○

f. The set of 2×2 matrices that contain exactly two 1's and two 0's is a linearly independent set in M_{22}.

g. The three polynomials $(x-1)(x+2)$, $x(x+2)$, and $x(x-1)$ are linearly independent.

h. The functions f_1 and f_2 are linearly dependent if there is a real number x such that $k_1 f_1(x) + k_2 f_2(x) = 0$ for some scalars k_1 and k_2.

(a) False. By part (b) of Theorem 4.4.2, a set containing a single *nonzero* vector is linearly independent.

(b) True. This follows directly from Definition 1.

(c) False. For instance $\{(1,1),(2,2)\}$ is a linearly dependent set that does not contain $(0,0)$.

(d) True. If $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$ has only one solution $a = b = c = 0$ then $a(k\mathbf{v}_1) + b(k\mathbf{v}_2) + c(k\mathbf{v}_3) = k(a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3)$ can only equal $\mathbf{0}$ when $a = b = c = 0$ as well.

(e) True. Since the vectors must be nonzero, $\{\mathbf{v}_1\}$ must be linearly independent.

Let us begin adding vectors to the set until the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ becomes linearly dependent, therefore, by construction, $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$ is linearly independent. The equation $c_1\mathbf{v}_1 + \cdots + c_{k-1}\mathbf{v}_{k-1} + c_k\mathbf{v}_k = \mathbf{0}$ must have a solution with $c_k \neq 0$, therefore $\mathbf{v}_k = -\frac{c_1}{c_k}\mathbf{v}_1 - \cdots - \frac{c_{k-1}}{c_k}\mathbf{v}_{k-1}$. Let us assume there exists another representation $\mathbf{v}_k = d_1\mathbf{v}_1 + \cdots + d_{k-1}\mathbf{v}_{k-1}$. Subtracting both sides yields $0 = \left(d_1 + \frac{c_1}{c_k}\right)\mathbf{v}_1 + \cdots + \left(d_{k-1} + \frac{c_{k-1}}{c_k}\right)\mathbf{v}_{k-1}$. By linear independence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}\}$, we must have $d_1 = -\frac{c_1}{c_k}$, …, $d_{k-1} = -\frac{c_{k-1}}{c_k}$, which shows that $\mathbf{v}_k$ is a *unique* linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$.

(f) False. The set $\left\{\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right\}$ is linearly dependent since

$$\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} = (-1)\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix} + \begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}.$$

(g) True. Requiring that for all x values $a(x-1)(x+2)+bx(x+2)+cx(x-1)=0$ holds true implies that the equality must be true for any specific x value. Setting $x=0$ yields $a=0$. Likewise, $x=1$ implies $b=0$, and $x=-2$ implies $c=0$. Since $a=b=c=0$ is required, we conclude that the three given polynomials are linearly independent.

(h) False. The functions f_1 and f_2 are linearly dependent if there exist scalars k_1 and k_2, <u>not both equal 0</u>, such that $k_1 f_1(x)+k_2 f_2(x)=0$ for <u>all</u> real numbers x.

4-5 True-False Exercises

a. If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. ×

b. Every linearly independent subset of a vector space V is a basis for V. ×

c. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. ○

d. The coordinate vector of a vector $\mathbf{x}$ in R^n relative to the standard basis for R^n is $\mathbf{x}$. ○

e. Every basis of P_4 contains at least one polynomial of degree 3 or less. ×

(a) False. The set must also be linearly independent.

(b) False. The subset must also span V.

(c) True. This follows from Theorem 4.5.1.

(d) True. For any vector $\mathbf{v} = (a_1, \ldots, a_n)$ in R^n, we have $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n$ therefore the coordinate vector of $\mathbf{v}$ with respect to the standard basis $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is $(\mathbf{v})_S = (a_1, \ldots, a_n) = \mathbf{v}$.

(e) False. For instance, $\{1+t^4, t+t^4, t^2+t^4, t^3+t^4, t^4\}$ is a basis for P_4.

4-6 True-False Exercises

a. The zero vector space has dimension zero. O

b. There is a set of 17 linearly independent vectors in R^{17}. O

c. There is a set of 11 vectors that span R^{17}. ×

d. Every linearly independent set of five vectors in R^5 is a basis for R^5. O

e. Every set of five vectors that spans R^5 is a basis for R^5. O

f. Every set of vectors that spans R^n contains a basis for R^n. O

g. Every linearly independent set of vectors in R^n is contained in some basis for R^n.

h. There is a basis for M_{22} consisting of invertible matrices.

i. If A has size $n \times n$ and $I_n, A, A^2, \dots, A^{n^2}$ are distinct matrices, then $\{I_n, A, A^2, \dots, A^{n^2}\}$ is a linearly dependent set.

j. There are at least two distinct three-dimensional subspaces of P_2.

k. There are only three distinct two-dimensional subspaces of P_2.

(a) True.

(b) True. For instance, $\mathbf{e}_1, \ldots, \mathbf{e}_{17}$.

(c) False. This follows from Theorem 4.6.2(b).

(d) True. This follows from Theorem 4.6.4.

(e) True. This follows from Theorem 4.6.4.

(f) True. This follows from Theorem 4.6.5(a).

(g) True. This follows from Theorem 4.6.5(b).

(h) True. For instance, invertible matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ form a basis for M_{22}.

(i) True. The set has n^2+1 matrices, which exceeds $\dim(M_{nn}) = n^2$.

(j) False. This follows from Theorem 4.6.6(c).

(k) False. For instance, for any constant c, $\operatorname{span}\{x-c, x^2-c^2\}$ is a two-dimensional subspace of P_2 consisting of all polynomials in P_2 for which $p(c)=0$. Clearly, there are infinitely many different subspaces of this type.

4-8 True-False Exercises

a. The span of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the column space of the matrix whose column vectors are $\mathbf{v}_1, \ldots, \mathbf{v}_n$. ○

b. The column space of a matrix A is the set of solutions of $A\mathbf{x} = \mathbf{b}$. ×

c. If R is the reduced row echelon form of A, then those column vectors of R that contain the leading 1's form a basis for the column space of A. ×

d. The set of nonzero row vectors of a matrix A is a basis for the row space of A. ×

e. If A and B are $n \times n$ matrices that have the same row space, then A and B have the same column space. ×

f. If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the null space of EA is the same as the null space of A. ○

g. If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the row space of EA is the same as the row space of A. ○

h. If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then the column space of EA is the same as the column space of A. ×

i. The system $A\mathbf{x} = \mathbf{b}$ is inconsistent if and only if $\mathbf{b}$ is not in the column space of A. ○

j. There is an invertible matrix A and a singular matrix B such that the row spaces of A and B are the same. ×

(a) True.

(b) False. The column space of A is the space spanned by all column vectors of A.

(c) False. Those column vectors form a basis for the column space of R.

(d) False. This would be true if A were in row echelon form.

(e) False. For instance $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ have the same row space, but different column spaces.

(f) True. This follows from Theorem 4.8.3.

(g) True. This follows from Theorem 4.8.3.

(h) False. Elementary row operations generally can change the column space of a matrix.

(i) True. This follows from Theorem 4.8.1.

(j) False. Let both A and B be $n \times n$ matrices. By Theorem 4.8.3, row operations do not change the row space of a matrix. An invertible matrix can be reduced to I thus its row space is always R^n. On the other hand, a singular matrix cannot be reduced to identity matrix - at least one row in its reduced row echelon form is made up of zeros. Consequently, its row space is spanned by fewer than n vectors, therefore the dimension of this space is less than n.

4-9 True-False Exercises

a. Either the row vectors or the column vectors of a square matrix are linearly independent. ✕

b. A matrix with linearly independent row vectors and linearly independent column vectors is square. ○

c. The nullity of a nonzero $m \times n$ matrix is at most m. ✕

d. Adding one additional column to a matrix increases its rank by one. ✕

e. The nullity of a square matrix with linearly dependent rows is at least one. ○

f. If A is square and $A\mathbf{x} = \mathbf{b}$ is inconsistent for some vector $\mathbf{b}$, then the nullity of A is zero.

g. If a matrix A has more rows than columns, then the dimension of the row space is greater than the dimension of the column space.

h. If $\operatorname{rank}(A^T) = \operatorname{rank}(A)$, then A is square.

i. There is no 3×3 matrix whose row space and null space are both lines in 3-space.

j. If V is a subspace of R^n and W is a subspace of V, then $W^\perp$ is a subspace of $V^\perp$.

(a) False. For instance, in, neither row vectors nor column $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ vectors are linearly independent.

(b) True. In an $m \times n$ matrix, if $m < n$ then by Theorem 4.6.2(a), the n columns in R^m must be linearly dependent. If $m > n$, then by the same theorem, the m rows in R^n must be linearly dependent. We conclude that $m = n$.

(c) False. The nullity in an $m \times n$ matrix is at most n.

(d) False. For instance, if the column contains all zeros, adding it to a matrix does not change the rank.

(e) True. In an $n \times n$ matrix A with linearly dependent rows, $\text{rank}(A) \le n-1$.

By Formula (4), $\text{nullity}(A) = n - \text{rank}(A) \ge 1$.

(f) False. By Theorem 4.9.7, the nullity must be nonzero.

(g) False. This follows from Theorem 4.9.1.

(h) False. By Theorem 4.9.4, $\text{rank}(A^T) = \text{rank}(A)$ for any matrix A.

(i) True. Since each of the two spaces has dimension 1, these dimensions would add up to 2 instead of 3 as required by Formula (4).

(j) False. For instance, if $n = 3$, $V = \text{span}\{\mathbf{i}, \mathbf{j}\}$ (the xy-plane), and $W = \text{span}\{\mathbf{i}\}$ (the x-axis) then $W^\perp = \text{span}\{\mathbf{j}, \mathbf{k}\}$ (the yz-plane) is not a subspace of $V^\perp = \text{span}\{\mathbf{k}\}$ (the z-axis).

(Note that it is true that $V^\perp$ is a subspace of $W^\perp$.)