Chapter 2
Determinants

2.1. Determinants by Cofactor Expansion

2.2. Evaluating Determinants by Row Reduction

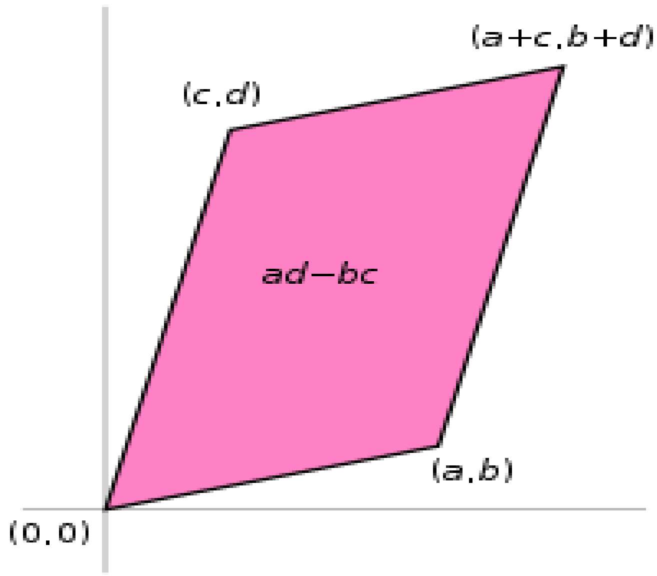
2.3. Properties of Determinants; Cramer's Rule

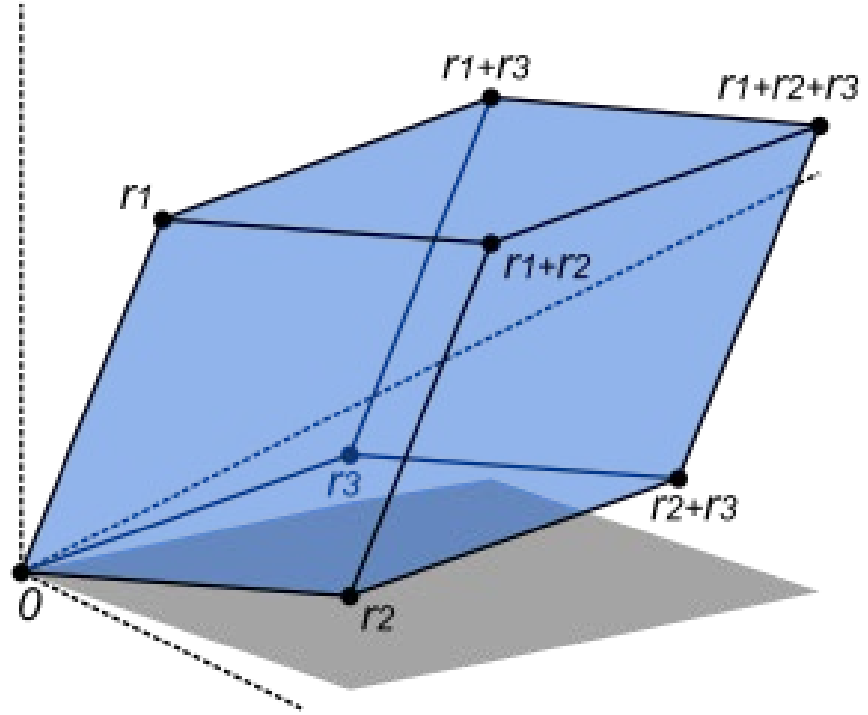
Chapter 2.1

Determinants by Cofactor Expansion

Things to Know in Advance

Determinants in 2D is the area defined by two row/column vectors and in 3D is the volume defined by three row/column vectors.





Determinants

For a 2 x 2 matrix A, $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

the *determinant* is a real number defined by

$$\det(A) = ad - bc \quad \text{or} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinants are commonly used to test if a matrix is invertible and to find the area of certain geometric figures.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Minors and Cofactors

In order to find the determinants of larger square matrices, we need to understand the concept of minors and cofactors.

DEFINITION 1

If A is a square matrix, then the *minor of entry a_{ij}* is denoted by M_{ij} and is defined to be the determinant of the submatrix that remains after deleting the i^{th} row and j^{th} column from A.

The *cofactor for entry a_{ij}* is denoted by C_{ij} and is defined to be the number $(-1)^{i+j} M_{ij}$.

Minors and Cofactors

EXAMPLE 1 Finding Minors and Cofactors

Let

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

Note:

$$\begin{bmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$M_{32} = 26$$

$$C_{32} = -26$$

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EXAMPLE 1

$$A = \begin{bmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{bmatrix}$$

The minor of entry a_{11} is

$$M_{11} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 16$$

The cofactor of a_{11} is

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 16$$

Similarly, the minor of entry a_{32} is

$$M_{32} = \begin{vmatrix} 3 & 1 & -4 \\ 2 & 5 & 6 \\ 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 26$$

The cofactor of a_{32} is

$$C_{32} = (-1)^{3+2} M_{32} = -M_{32} = -26$$

Determinant of any Square Matrix

DEFINITION 2

If A is an $n \times n$ matrix, then the number obtained by multiplying the entries in any row or column of A by the corresponding cofactors and adding the resulting products is called the *determinant of A*, denoted by $\det(A)$. That is,

$$\det(A) = \sum_{i=1}^{n} a_{ij}\, C_{ij} \quad \text{or} \quad \sum_{j=1}^{n} a_{ij}\, C_{ij}$$

for any column j for any row i

These are called *cofactor expansion of A.*

Determinant of any Square Matrix

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

[cofactor expansion along the *j*th column]

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

[cofactor expansion along the *i*th row]

THEOREM 2.1.1

If A is an n x n matrix, then regardless of which row or column of A is chosen, the number obtained by multiplying the entries in that row or column by the corresponding cofactors and adding the resulting products is always the same.

Cofactor Expansions

We can find the determinant of any square matrix by cofactors. You may pick any row or column, but the calculation is easier if some elements in the selected row or column equal 0.

EXAMPLE 2 Cofactor Expansion Along the First Row

Find the determinant of the matrix
$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

by cofactor expansion along the first row.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} -2 & 3 \\ 5 & -2 \end{vmatrix} + 0\begin{vmatrix} -2 & -4 \\ 5 & 4 \end{vmatrix}$$

$$= 3(-4) - (1)(-11) + 0 = -1$$

EXAMPLE 3 Cofactor Expansion Along the First Column

Find the determinant of the matrix $A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$

by cofactor expansion along the first column.

$$\det(A) = \begin{vmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{vmatrix} = 3\begin{vmatrix} -4 & 3 \\ 4 & -2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 1 & 0 \\ -4 & 3 \end{vmatrix}$$

$$= 3(-4) - (-2)(-2) + 5(3) = -1$$

Cofactor Expansions

EXAMPLE 4 Smart Choice of Row or Column

If A is the 4×4 matrix

$$A = \begin{bmatrix} 1 & \boxed{0} & 0 & -1 \\ 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 0 & -1 \\ 1 & -2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$\det(A) = (1)(-2)\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$$

$$= -2(1+2)$$

$$= -6$$

Determinant of Triangular Matrices

THEOREM 2.1.2

If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then $\det(A)$ is the product of the entries on the main diagonal of the matrix. That is,

$$\det(A) = a_{11}a_{22}\cdots a_{nn}.$$

EXAMPLE 5

$$\begin{vmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & 0 & 0 \\ a_{32} & a_{33} & 0 \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}\begin{vmatrix} a_{33} & 0 \\ a_{43} & a_{44} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33}|a_{44}| = a_{11}a_{22}a_{33}a_{44}$$

A Technique for Determinants of 2x2 and 3x3 Matrices <u>Only</u>

Arrow Technique

EXAMPLE 6

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \left|\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & -8 & 9 \end{matrix}\right|\begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

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Chapter 2-1 Objectives

- Find the minors and cofactors of a square matrix.
- Use cofactor expansion to evaluate the determinant of a square matrix.
- Use the arrow technique to evaluate the determinant of a 2 x 2 or 3 x 3 matrix.
- Use the determinant of a 2 x 2 invertible matrix to find the inverse of that matrix.
- Find the determinant of an upper triangular, lower triangular, or diagonal matrix by inspection.