4-9 In-Class Exercise

1. In each part, use the information in the table to:

i. find the dimensions of the row space of A, column space of A, null space of A, and null space of A^T;

ii. determine whether the linear system $A\mathbf{x} = \mathbf{b}$ is consistent;

iii. find the number of parameters in the general solution of each system in (ii) that is consistent.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)
Size of A	3×3	3×3	3×3	5×9	5×9	4×4	6×2
Rank(A)	3	2	1	2	2	0	2
Rank$[A \mid \mathbf{b}]$	3	3	1	2	3	0	2

		(a)	(b)	(c)	(d)	(e)	(f)	(g)
	Size of A:	3×3	3×3	3×3	5×9	5×9	4×4	6×2
	rank(A)	3	2	1	2	2	0	2
	rank($A \mid \mathbf{b}$)	3	3	1	2	3	0	2
(i)	dimension of the row space of A							
	dimension of the column space of A							
	dimension of the null space of A							
	dimension of the null space of A^T							
(ii)	is the system $A\mathbf{x}=\mathbf{b}$ consistent?							
(iii)	number of parameters in the general solution of $A\mathbf{x}=\mathbf{b}$							

1.

			(a)	(b)	(c)	(d)	(e)	(f)	(g)
	Size of A:	$m \times n$	3×3	3×3	3×3	5×9	5×9	4×4	6×2
	$\text{rank}(A)$	$=r$	3	2	1	2	2	0	2
	$\text{rank}(A\,\vert\,\mathbf{b})$	$=s$	3	3	1	2	3	0	2
(i)	dimension of the row space of A	$=r$	3	2	1	2	2	0	2
	dimension of the column space of A	$=r$	3	2	1	2	2	0	2
	dimension of the null space of A	$=n-r$	0	1	2	7	7	4	0
	dimension of the null space of A^T	$=m-r$	0	1	2	3	3	4	4
(ii)	is the system $A\mathbf{x}=\mathbf{b}$ consistent?	Is $r=s$?	Yes	No	Yes	Yes	No	Yes	Yes
(iii)	number of parameters in the general solution of $A\mathbf{x}=\mathbf{b}$	$=n-r$ if consistent	0	-	2	7	-	4	0

2. Find the dimensions and bases for the four fundamental spaces of the matrix.

$$A = \begin{bmatrix} 0 & -1 & -4 \\ -1 & 0 & -4 \\ -2 & 3 & 4 \end{bmatrix}$$

2.

$$[\; A \;|\; I \;]$$

$$\left[\begin{array}{ccc|ccc} 0 & -1 & -4 & 1 & 0 & 0 \\ -1 & 0 & -4 & 0 & 1 & 0 \\ -2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} +1 & 0 & +4 & 0 & -1 & 0 \\ -2 & 3 & 4 & 0 & 0 & 1 \\ 0 & -1 & -4 & 1 & 0 & 0 \end{array}\right]$$

exchanging rows

row1 * (-1)

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & -1 & 0 \\ 0 & 3 & 12 & 0 & -2 & 1 \\ 0 & -1 & -4 & 1 & 0 & 0 \end{array}\right]$$

row2 + 2*row 1

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & -1 & 0 \\ 0 & 1 & 4 & -1 & 0 & 0 \\ 0 & 3 & 12 & 0 & -2 & 1 \end{array}\right]$$

exchanging rows.

row2 * (-1)

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & -1 & 0 \\ 0 & 1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & 3 & -2 & 1 \end{array}\right]$$

row3 + (-3) * row2

$$\text{row}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 0 \\ -1 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix} \right\}$$

$$\text{col}(A) = \left\{ \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\}$$

$$\text{null}(A) = \left\{ \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix} \right\}$$

$$\text{left null}(A) \text{ or } \text{null}(A^T) = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$$

4-9 Suggested Exercises

1. Find bases for the four fundamental spaces of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 2 & 8 & 0 & 1 & 2 \\ 0 & 4 & -6 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Discuss how the rank of A varies with t.

$$A = \begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$$

1.

Following Example 5, we find that the reduced row echelon form of the

Augmented matrix $\begin{bmatrix} 1 & 2 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 8 & 0 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 4 & -6 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{8} & 0 & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{12} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$

Hence, a basis for

the column space basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, the row space basis is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -6 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$,the null space basis is $\left\{ \begin{bmatrix} 0 \\ -\frac{1}{4} \\ 0 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\}$, and the left null space contains only the zero vector.

2. The determinant of A is

$$\begin{vmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ t-1 & 0 & 1-t \end{vmatrix}$$

-1 times the first row was added to the second row and to the third row.

$$= \begin{vmatrix} 1 & 1+t & t \\ 0 & 0 & 1-t \\ t-1 & 1-t & 1-t \end{vmatrix}$$

Last column was added to the second column.

$$= -(1-t)\begin{vmatrix} 1 & 1+t \\ t-1 & 1-t \end{vmatrix}$$

Cofactor expansion along the second row.

$$= -(1-t)\big((1-t)-(1+t)(t-1)\big) \qquad = -(1-t)^2(2+t)$$

From parts (g) and (n) of Theorem 4.9.8, $\operatorname{rank}(A)=3$ when $\det(A)\neq 0$, i.e. for all t values other than 1 or -2.

If $t=1$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 1.

If $t=-2$, the matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ so that its rank is 2.

3.

a. If A is a 3×5 matrix, then the rank of A is at most ________. Why?

b. If A is a 3×5 matrix, then the nullity of A is at most ________. Why?

c. If A is a 3×5 matrix, then the rank of A^T is at most ________. Why?

d. If A is a 3×5 matrix, then the nullity of A^T is at most ________. Why?

4. Let A be a 5×7 matrix with rank 4.

a. What is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$?

b. Is $A\mathbf{x} = \mathbf{b}$ consistent for all vectors $\mathbf{b}$ in R^5? Explain.

3.

(a) 3; reduced row echelon form of A can contain at most 3 leading 1's when each of its rows is nonzero;

(b) 5; if A is the zero matrix, then the general solution of $A\mathbf{x} = \mathbf{0}$ has five parameters;

(c) 3; reduced row echelon form of A^T can contain at most 3 leading 1's when each of its columns has a leading 1;

(d) 3; if A is the zero matrix, then the general solution of $A^T\mathbf{x} = \mathbf{0}$ has three parameters;

4.

(a) By Formula (4), $\text{nullity}(A) = 7 - 4 = 3$ thus the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$ is 3.

(b) No, the column space of A is a subspace of R^5 of dimension 4, therefore there exist vectors $\mathbf{b}$ in R^5 that are outside this column space. For any such vector, the system $A\mathbf{x} = \mathbf{b}$ is inconsistent.