1-1 True-False Exercises

a) A linear system whose equations are all homogeneous must be consistent. ○

b) Multiplying a row of an augmented matrix through by zero is an acceptable elementary row operation. ✗

c) The linear system

$$\begin{aligned} x - y &= 3 \\ 2x - 2y &= k \end{aligned}$$

cannot have a unique solution, regardless of the value of k. ○

d) A single linear equation with two or more unknowns must have infinitely many solutions. ○

e) If the number of equations in a linear system exceeds the number of unknowns, then the system must be inconsistent. ✗

f) If each equation in a consistent linear system is multiplied through by a constant c, then all solutions to the new system can be obtained by multiplying solutions from the original system by c. ✗

g) Elementary row operations permit one row of an augmented matrix to be subtracted from another. ○

h) The linear system with corresponding augmented matrix

$$\begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

is consistent. ✗

(a) True. $(0,0,\ldots,0)$ is a solution.

(b) False. Only multiplication by a **non**zero constant is a valid elementary row operation.

(c) True. If $k=6$ then the system has infinitely many solutions; otherwise the system is inconsistent.

(d) True. According to the definition, $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is a linear equation if the a's are not all zero. Let us assume $a_j \neq 0$. The values of all x's except for x_j can be set to be arbitrary parameters, and the equation can be used to express x_j in terms of those parameters.

(e) False. E.g. if the equations are all homogeneous then the system must be consistent. (See True-False Exercise (a) above.)

(f) False. If $c \neq 0$ then the new system has the same solution set as the original one.

(g) True. Adding -1 times one row to another amounts to the same thing as subtracting one row from another.

(h) False. The second row corresponds to the equation $0 = -1$, which is contradictory.

1-2 True-False Exercises

a) If a matrix is in reduced row echelon form, then it is also in row echelon form. ○

b) If an elementary row operation is applied to a matrix that is in row echelon form, the resulting matrix will still be in row echelon form. ×

c) Every matrix has a unique row echelon form. ×

d) A homogeneous linear system in n unknowns whose corresponding augmented matrix has a reduced row echelon form with r leading 1's has $n - r$ free variables. ○

e) All leading 1's in a matrix in row echelon form must occur in different columns. ○

f) If every column of a matrix in row echelon form has a leading 1 then all entries that are not leading 1's are zero. ×

g) If a homogeneous linear system of *n* equations in *n* unknowns has a corresponding augmented matrix with a reduced row echelon form containing *n* leading 1's, then the linear system has only the trivial solution. ○

h) If the reduced row echelon form of the augmented matrix for a linear system has a row of zeros, then the system must have infinitely many solutions. ×

i) If a linear system has more unknowns than equations, then it must have infinitely many solutions. ×

(a) True. A matrix in reduced row echelon form has all properties required for the row echelon form.

(b) False. For instance, interchanging the rows of $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ yields a matrix that is not in row echelon form.

(c) False. See Exercise 31.

(d) True. In a reduced row echelon form, the number of nonzero rows equals to the number of leading 1's. The result follows from Theorem 1.2.1.

(e) True. This is implied by the third property of a row echelon form (see Section 1.2).

(f) False. Nonzero entries are permitted above the leading 1's in a row echelon form.

(g) True. In a reduced row echelon form, the number of nonzero rows equals to the number of leading 1's. From Theorem 1.2.1 we conclude that the system has $n-n=0$ free variables, i.e. it has only the trivial solution.

(h) False. The row of zeros imposes no restriction on the unknowns and can be omitted. Whether the system has infinitely many, one, or no solution(s) depends *solely* on the nonzero rows of the reduced row echelon form.

(i) False. For example, the following system is clearly inconsistent:

$$\begin{aligned} x+y+z&=1 \\ x+y+z&=2 \end{aligned}$$

1-3 True-False Exercises

a) The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ has no main diagonal. ○

b) An m x n matrix has m column vectors and n row vectors. ✕

c) If A and B are 2 x 2 matrices, then $AB = BA$ ✕

d) The ith row vector of a matrix product AB can be computed by multiplying A by the ith row vector of B. ✕

e) For every matrix A, it is true that $\left(A^T\right)^T = A$. ○

f) If A and B are square matrices of the same order, then ✕ $\operatorname{tr}(AB) = \operatorname{tr}(A)\operatorname{tr}(B)$.

g) If A and B are square matrices of the same order, then ✗ $(AB)^T = A^T B^T$.

h) For every square matrix A, it is true that $\mathrm{tr}\left(A^T\right) = \mathrm{tr}(A)$. ○

i) If A is a matrix 6 x 4 matrix and B is an m x n matrix such that $B^T A^T$ is a 2 x 6 matrix, then $m = 4$ and $n = 2$. ○

j) If A is an n x n matrix and c is a scalar, then $\mathrm{tr}(cA) = c\,\mathrm{tr}(A)$. ○

k) If A, B, and C are matrices of the same size such that $A - C = B - C$, then $A = B$. ○

l) If A, B, and C are square matrices of the same order such that $AC = BC$, then $A = B$. ✗

m) If $AB + BA$ is defined, then A and B are square matrices of the same size. ○

n) If B has a column of zeros, then so does AB if this product is defined. ○

o) If B has a column of zeros, then so does BA if this product is defined. ×

(a) True. The main diagonal is only defined for square matrices.

(b) False. An $m \times n$ matrix has m row vectors and n column vectors.

(c) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not equal $BA = B$.

(d) False. The i th row vector of AB can be computed by multiplying the i th row vector of A by B.

(e) True. Using Formula (14), $\left(\left(A^T\right)^T\right)_{ij} = \left(A^T\right)_{ji} = (A)_{ij}$.

(f) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ then the trace of $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is 0, which does not equal $\mathrm{tr}(A)\mathrm{tr}(B) = 1$.

(g) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $(AB)^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ does not equal $A^T B^T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

(h) True. The main diagonal entries in a square matrix A are the same as those in A^T.

(i) True. Since A^T is a 4×6 matrix, it follows from $B^T A^T$ being a 2×6 matrix that B^T must be a 2×4 matrix. Consequently, B is a 4×2 matrix.

(j) True.

$$\operatorname{tr}\left(c\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{n1} & \cdots & ca_{nn} \end{bmatrix}\right)$$

$$= ca_{11} + \cdots + ca_{nn} = c\left(a_{11} + \cdots + a_{nn}\right) = c\ \operatorname{tr}\left(\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\right)$$

(k) True. The equality of the matrices $A-C$ and $B-C$ implies that $a_{ij} - c_{ij} = b_{ij} - c_{ij}$ for all i and j. Adding c_{ij} to both sides yields $a_{ij} = b_{ij}$ for all i and j. Consequently, the matrices A and B are equal.

(l) False. E.g., if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ then $AC = BC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ even though $A \neq B$.

(m) True. If A is a $p \times q$ matrix and B is an $r \times s$ matrix then AB being defined requires $q = r$ and BA being defined requires $s = p$. For the $p \times p$ matrix AB to be possible to add to the $q \times q$ matrix BA, we must have $p = q$.

(n) True. If the j th column vector of B is $\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that

the j th column vector of $AB = A\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$.

(o) False. E.g., if $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ then $BA = A$ does not have a column of zeros even though B does.

1-4 True-False Exercises

a. Two $n \times n$ matrices, A and B, are inverses of one another if and only if $AB = BA = 0$.

b. For all square matrices A and B of the same size, it is true that $(A + B)^2 = A^2 + 2AB + B^2$.

c. For all square matrices A and B of the same size, it is true that $A^2 - B^2 = (A - B)(A + B)$.

d. If A and B are invertible matrices of the same size, then AB is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

e. If A and B are matrices such that AB is defined, then it is true that $(AB)^T = A^T B^T$.

f. The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$.

g. If A and B are matrices of the same size and k is a constant, then $(kA + B)^T = kA^T + B^T$.

h. If A is an invertible matrix, then so is A^T.

i. If $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_mx^m$ and I is an identity matrix, then $p(I) = a_0 + a_1 + a_2 + \cdots + a_m$.

j. A square matrix containing a row or column of zeros cannot be invertible.

k. The sum of two invertible matrices of the same size must be invertible.

(a) False. A and B are inverses of one another if and only if $AB = BA = I$.

(b) False. $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$ does not generally equal $A^2 + 2AB + B^2$ since AB may not equal BA.

(c) False. $(A-B)(A+B) = A^2 + AB - BA - B^2$ does not generally equal $A^2 - B^2$ since AB may not equal BA.

(d) False. $(AB)^{-1} = B^{-1}A^{-1}$ does not generally equal $A^{-1}B^{-1}$.

(e) False. $(AB)^T = B^T A^T$ does not generally equal $A^T B^T$.

(f) True. This follows from Theorem 1.4.5.

(g) True. This follows from Theorem 1.4.8.

(h) True. This follows from Theorem 1.4.9. (The inverse of A^T is the transpose of A^{-1}.)

(i) False. $p(I) = (a_0 + a_1 + a_2 + \cdots + a_m)I$.

(j) True.

If the i th row vector of A is $\begin{bmatrix}0 & \cdots & 0\end{bmatrix}$ then it follows from Formula (9) in Section 1.3 that

i th row vector of $AB = \begin{bmatrix}0 & \cdots & 0\end{bmatrix}B = \begin{bmatrix}0 & \cdots & 0\end{bmatrix}$.

Consequently no matrix B can be found to make the product $AB = I$ thus A does not have an inverse.

If the j th column vector of A is $\begin{bmatrix}0 \\ \vdots \\ 0\end{bmatrix}$ then it follows from Formula (8) in Section 1.3 that

the j th column vector of $BA = B\begin{bmatrix}0 \\ \vdots \\ 0\end{bmatrix} = \begin{bmatrix}0 \\ \vdots \\ 0\end{bmatrix}$.

Consequently no matrix B can be found to make the product $BA = I$ thus A does not have an inverse.

(k) False. E.g. I and $-I$ are both invertible but $I + (-I) = O$ is not.

1-5 True-False Exercises

a. The product of two elementary matrices of the same size must be an elementary matrix. ✕

b. Every elementary matrix is invertible. ○

c. If A and B are row equivalent, and if B and C are row equivalent, then A and C are row equivalent. ○

d. If A is an $n \times n$ matrix that is not invertible, then the linear system $A\mathbf{x} = 0$ has infinitely many solutions. ○

e. If A is an $n \times n$ matrix that is not invertible, then the matrix obtained by interchanging two rows of A cannot be invertible. ○

f. If A is invertible and a multiple of the first row of A is added to the second row, then the resulting matrix is invertible.

○

g. An expression of an invertible matrix A as a product of elementary matrices is unique. ×

(a) False. An elementary matrix results from performing a *single* elementary row operation on an identity matrix; a product of two elementary matrices would correspond to a sequence of two such operations instead, which generally is not equivalent to a single elementary operation.

(b) True. This follows from Theorem 1.5.2.

(c) True. If A and B are row equivalent then there exist elementary matrices $E_1,\ldots,E_p$ such that $B = E_p \cdots E_1 A$. Likewise, if B and C are row equivalent then there exist elementary matrices $E_1^*,\ldots,E_q^*$ such that $C = E_q^* \cdots E_1^* B$. Combining the two equalities yields $C = E_q^* \cdots E_1^* E_p \cdots E_1 A$ therefore A and C are row equivalent.

(d) True. A homogeneous system $A\mathbf{x} = 0$ has either one solution (the trivial solution) or infinitely many solutions. If A is not invertible, then by Theorem 1.5.3 the system cannot have just one solution. Consequently, it must have infinitely many solutions.

(e) True. If the matrix A is not invertible then by Theorem 1.5.3 its reduced row echelon form is not I_n. However, the matrix resulting from interchanging two rows of A (an elementary row operation) must have the same reduced row echelon form as A does, so by Theorem 1.5.3 that matrix is not invertible either.

(f) True. Adding a multiple of the first row of a matrix to its second row is an elementary row operation. Denoting by E be the corresponding elementary matrix we can write $(EA)^{-1} = A^{-1}E^{-1}$ so the resulting matrix EA is invertible if A is.

(g) False. For instance, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

1-6 True-False Exercises

a. It is impossible for a system of linear equations to have exactly two solutions.

b. If A is a square matrix, and if the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then the linear system $A\mathbf{x} = \mathbf{c}$ also must have a unique solution.

c. If A and B are $n \times n$ matrices such that $AB = I_n$, then $BA = I_n$.

d. If A and B are row equivalent matrices, then the linear systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set.

e. Let A be an $n \times n$ matrix and S is an $n \times n$ invertible matrix. If $\mathbf{x}$ is a solution to the system $(S^{-1}AS)\mathbf{x} = \mathbf{b}$, then $S\mathbf{x}$ is a solution to the system $A\mathbf{y} = S\mathbf{b}$. ○

f. Let A be an $n \times n$ matrix. The linear system $A\mathbf{x} = 4\mathbf{x}$ has a unique solution if and only if $A - 4I$ is an invertible matrix. ○

g. Let A and B be $n \times n$ matrices. If A or B (or both) are not invertible, then neither is AB. ○

(a) True. By Theorem 1.6.1, if a system of linear equation has more than one solution then it must have infinitely many.

(b) True. If A is a square matrix such that $A\mathbf{x} = \mathbf{b}$ has a unique solution then the reduced row echelon form of A must be I. Consequently, $A\mathbf{x} = \mathbf{c}$ must have a unique solution as well.

(c) True. Since B is a square matrix then by Theorem 1.6.3(b) $AB = I_n$ implies $B = A^{-1}$.

Therefore, $BA = A^{-1}A = I_n$.

(d) True. Since A and B are row equivalent matrices, it must be possible to perform a sequence of elementary row operations on A resulting in B. Let E be the product of the corresponding elementary matrices, i.e., $EA = B$. Note that E must be an invertible matrix thus $A = E^{-1}B$.

Any solution of $A\mathbf{x} = 0$ is also a solution of $B\mathbf{x} = 0$ since $B\mathbf{x} = EA\mathbf{x} = E0 = 0$.

Likewise, any solution of $B\mathbf{x} = 0$ is also a solution of $A\mathbf{x} = 0$ since $A\mathbf{x} = E^{-1}B\mathbf{x} = E^{-1}0 = 0$.

(e) True. If $(S^{-1}AS)\mathbf{x} = \mathbf{b}$ then $SS^{-1}AS\mathbf{x} = A(S\mathbf{x}) = S\mathbf{b}$. Consequently, $\mathbf{y} = S\mathbf{x}$ is a solution of $A\mathbf{y} = S\mathbf{b}$.

(f) True. $A\mathbf{x} = 4\mathbf{x}$ is equivalent to $A\mathbf{x} = 4I_n\mathbf{x}$, which can be rewritten as $(A - 4I_n)\mathbf{x} = 0$. By Theorem 1.6.4, this homogeneous system has a unique solution (the trivial solution) if and only if its coefficient matrix $A - 4I_n$ is invertible.

(g) True. If AB were invertible, then by Theorem 1.6.5 both A and B would be invertible.

1-7 True-False Exercises

a. The transpose of a diagonal matrix is a diagonal matrix. ○

b. The transpose of an upper triangular matrix is an upper triangular matrix. ×

c. The sum of an upper triangular matrix and a lower triangular matrix is a diagonal matrix. ×

d. All entries of a symmetric matrix are determined by the entries occurring on and above the main diagonal. ○

e. All entries of an upper triangular matrix are determined by the entries occurring on and above the main diagonal. ○

f. The inverse of an invertible lower triangular matrix is an upper triangular matrix. ✗

g. A diagonal matrix is invertible if and only if all of its diagonal entries are positive. ✗

h. The sum of a diagonal matrix and a lower triangular matrix is a lower triangular matrix. ○

i. A matrix that is both symmetric and upper triangular must be a diagonal matrix. ○

j. If A and B are $n \times n$ matrices such that $A + B$ is symmetric, then A and B are symmetric. ✗

k. If A and B are $n \times n$ matrices such that $A + B$ is upper triangular, then A and B are upper triangular. ✗

l. If A^2 is a symmetric matrix, then A is a symmetric matrix. ✗

m. If kA is a symmetric matrix for some $k \neq 0$, then A is a symmetric matrix. ○

(a) True. Every diagonal matrix is symmetric: its transpose equals to the original matrix.

(b) False. The transpose of an upper triangular matrix is a *lower* triangular matrix.

(c) False. E.g., $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ is not a diagonal matrix.

(d) True. Mirror images of entries across the main diagonal must be equal - see the margin note next to Example 4.

(e) True. All entries below the main diagonal must be zero.

(f) False. By Theorem 1.7.1(d), the inverse of an invertible lower triangular matrix is a lower triangular matrix.

(g) False. A diagonal matrix is invertible if and only if all or its diagonal entries are nonzero (positive or negative).

(h) True. The entries above the main diagonal are zero.

(i) True. If A is upper triangular then A^T is lower triangular. However, if A is also symmetric then it follows that $A^T = A$ must be both upper triangular and lower triangular. This requires A to be a diagonal matrix.

(j) False. For instance, neither $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ nor $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is symmetric even though $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is.

(k) False. For instance, neither $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ nor $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is upper triangular even though $A + B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is.

(l) False. For instance, $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is not symmetric even though $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is.

(m) True. By Theorem 1.4.8(d), $(kA)^T = kA^T$. Since kA is symmetric, we also have $(kA)^T = kA$. For nonzero k the equality of the right hand sides $kA^T = kA$ implies $A^T = A$.

1-8 True-False Exercises

a. If A is a 2×3 matrix, then the domain of the transformation T_A is R^2.

b. If A is an $m \times n$ matrix, then the codomain of the transformation T_A is R^n.

c. There is at least one linear transformation $T: R^n \to R^m$ for which $T(2\mathbf{x}) = 4T(\mathbf{x})$ for some vector $\mathbf{x}$ in R^n.

d. There are linear transformations from R^n to R^m that are not matrix transformations.

e. If $T_A : R^n \rightarrow R^n$ and if $T_A(\mathbf{x}) = \mathbf{0}$ for every vector $\mathbf{x}$ in R^n, then A is the $n \times n$ zero matrix. ○

f. There is only one matrix transformation $T : R^n \rightarrow R^m$ such that $T(-\mathbf{x}) = -T(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n. ×

g. If $\mathbf{b}$ is a nonzero vector in R^n, then $T(\mathbf{x}) = \mathbf{x} + \mathbf{b}$ is a matrix operator on R^n. ×

(a) False. The domain of T_A is R^3.

(b) False. The codomain of T_A is R^m.

(c) True. Since the statement requires the given equality to hold for <u>*some*</u> vector $\mathbf{x}$ in R^n, we can let $\mathbf{x} = 0$.

(d) False. (Refer to Theorem 1.8.3.)

(e) True. The columns of A are $T(\mathbf{e}_i) = 0$.

(f) False. The given equality must hold for every matrix transformation since it follows from the homogeneity property.

(g) False. The homogeneity property fails to hold since $T(k\mathbf{x}) = k\mathbf{x} + \mathbf{b}$ does not generally equal $kT(\mathbf{x}) = k(\mathbf{x} + \mathbf{b}) = k\mathbf{x} + k\mathbf{b}$.