Chapter 4
General Vector Spaces

4.1. Real Vector Spaces

4.2. Subspaces

4.3. Spanning Sets

4.4. Linear Independence

4.5. Coordinates and Basis

4.6. Dimension

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Chapter 4.6

Dimension

Number of Vectors in a Basis

THEOREM 4.6.1

All bases for a finite-dimensional vector space have the same number of vectors.

THEOREM 4.6.2

Let V be a finite-dimensional vector space, and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis for V.

(a) If a set in V has more than n vectors, then it is linearly dependent.
(b) If a set in V has fewer than n vectors, then it does not span V.

Dimension

DEFINITION 1

The *dimension* of a finite-dimensional vector space V is denoted by $\dim(V)$ and is defined to be the number of vectors in a basis for V.

In addition, the zero vector space is defined to have dimension zero.

Remarks

- Some writers regard the empty set to be a basis for the zero vector space. (The empty set has no vectors and the zero vector space has dimension zero.)

- Engineers often use the term *degrees of freedom* for dimension.

Dimensions of Some Familiar Vector Spaces

EXAMPLE 1

$\dim(R^n) = n$ **[The standard basis has n vectors.]**

$\dim(P_n) = n + 1$ **[The standard basis has $n + 1$ vectors.]**

$\dim(M_{mn}) = mn$ **[The standard basis has mn vectors.]**

Dimension of Span(S)

EXAMPLE 2

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly independent set in a vector space V, then vectors in S automatically form a basis for span(S).

$$\dim\,[\operatorname{span}(S)] = r \qquad \text{or} \qquad \dim\left[\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}\right] = r$$

The dimension of the space spanned by a linearly independent set of vectors is equal to the number of vectors in that set.

Dimension of a Solution Space

EXAMPLE 3

Find a basis for and the dimension of the solution space of the homogeneous system

$$\begin{aligned}
x_1 + 3x_2 - 2x_3 \qquad\qquad + 2x_5 \qquad\qquad &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= 0 \\
5x_3 + 10x_4 \qquad\qquad + 15x_6 &= 0 \\
2x_1 + 6x_2 \qquad\qquad + 8x_4 + 4x_5 + 18x_6 &= 0
\end{aligned}$$

Solution

The solution of this system

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$$

EXAMPLE 3 Cont.

which can be written in vector form as

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (-3r - 4s - 2t, r, -2s, s, t, 0)$$

$$(x_1, x_2, x_3, x_4, x_5, x_6) = r(-3, 1, 0, 0, 0, 0) + s(-4, 0, -2, 1, 0, 0) + t(-2, 0, 0, 0, 1, 0)$$

This shows that the vectors

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \quad \mathbf{v}_2 = (-4, 0, -2, 1, 0, 0), \quad \mathbf{v}_3 = (-2, 0, 0, 0, 1, 0)$$

span the solution space. So that these vectors are linearly independent.

Thus, the solution space has dimension 3.

Some Fundamental Theorems

A series of theorems that reveal the subtle interrelationships among the concepts of linear independence, basis, and dimension.

These theorems are essential to the understanding of vector spaces and the applications that build on them.

Plus/Minus Theorem

THEOREM 4.6.3

Let S be a nonempty set of vectors in a vector space V.

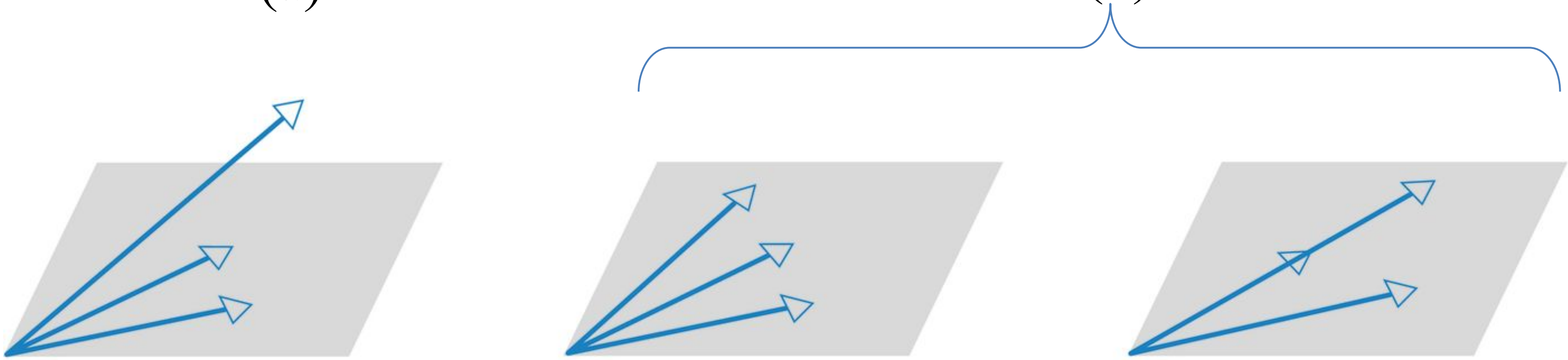
(a) If S is a linearly independent set, and if $\mathbf{v}$ is a vector in V that is outside of span(S), then the set $S \cup \{\mathbf{v}\}$ that results by inserting $\mathbf{v}$ into S is still linearly independent.

(b) If $\mathbf{v}$ is a vector in S that is expressible as a linear combination of other vectors in S, and if $S - \{\mathbf{v}\}$ denotes the set obtained by removing $\mathbf{v}$ from S, then S and $S - \{\mathbf{v}\}$ span the same space; that is,

$$\operatorname{span}(S) = \operatorname{span}(S - \{\mathbf{v}\})$$

Plus/Minus Theorem

(a) (b)



The vector outside the plane can be adjoined to the other two without affecting their linear independence.

Any of the vectors can be removed, and the remaining two will still span the plane.

Either of the collinear vectors can be removed, and the remaining two will still span the plane.

Plus/Minus Theorem

EXAMPLE 4

Show that $\mathbf{p}_1 = 1 - x^2$, $\mathbf{p}_2 = 2 - x^2$, and $\mathbf{p}_3 = x^3$ are linearly independent vectors.

Solution The set $S = \{\mathbf{p}_1, \mathbf{p}_2\}$ is linearly independent since neither vector in S is a scalar multiple of the other.

Since the vector $\mathbf{p}_3$ cannot be expressed as a linear combination of the vectors in S (why?), it can be adjoined to S to produce a linearly independent set $S \cup \{\mathbf{p}_3\} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$

Basis, Linearly Independence, Dimension

THEOREM 4.6.4

Let V be an n-dimensional vector space, and let S be a set in V with exactly n vectors. Then

S is a basis for V <u>if and only if</u> S spans V

or

S is linearly independent.

Bases by Inspection

EXAMPLE 5

(a) Explain why the vectors $\mathbf{v}_1 = (-3, 7)$ and $\mathbf{v}_2 = (5, 5)$ form a basis for R^2.
(b) Explain why the vectors $\mathbf{v}_1 = (2, 0, -1)$, $\mathbf{v}_2 = (4, 0, 7)$, and $\mathbf{v}_3 = (-1, 1, 4)$ form a basis for R^3.

Solution (*a*)

Since neither vector is a scalar multiple of the other, the two vectors form a linearly independent set in the two-dimensional space R^2, and hence they form a basis by Theorem 4.6.4.

EXAMPLE 5 cont.

Solution (*b*)

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a linearly independent set in the *xz*-plane

The vector $\mathbf{v}_3$ is outside of the *xz*-plane, so the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent. Since R^3 is three-dimensional, Theorem 4.6.4 implies that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the vector space R^3.

Basis, Linearly Independence, Dimension

THEOREM 4.6.5

Let S be a finite set of vectors in a finite-dimensional vector space V.

(a) If S spans V but is not a basis for V, then S can be reduced to a basis for V by removing appropriate vectors from S.

(b) If S is a linearly independent set that is not already a basis for V, then S can be enlarged to a basis for V by inserting appropriate vectors into S.

Basis, Linearly Independence, Dimension

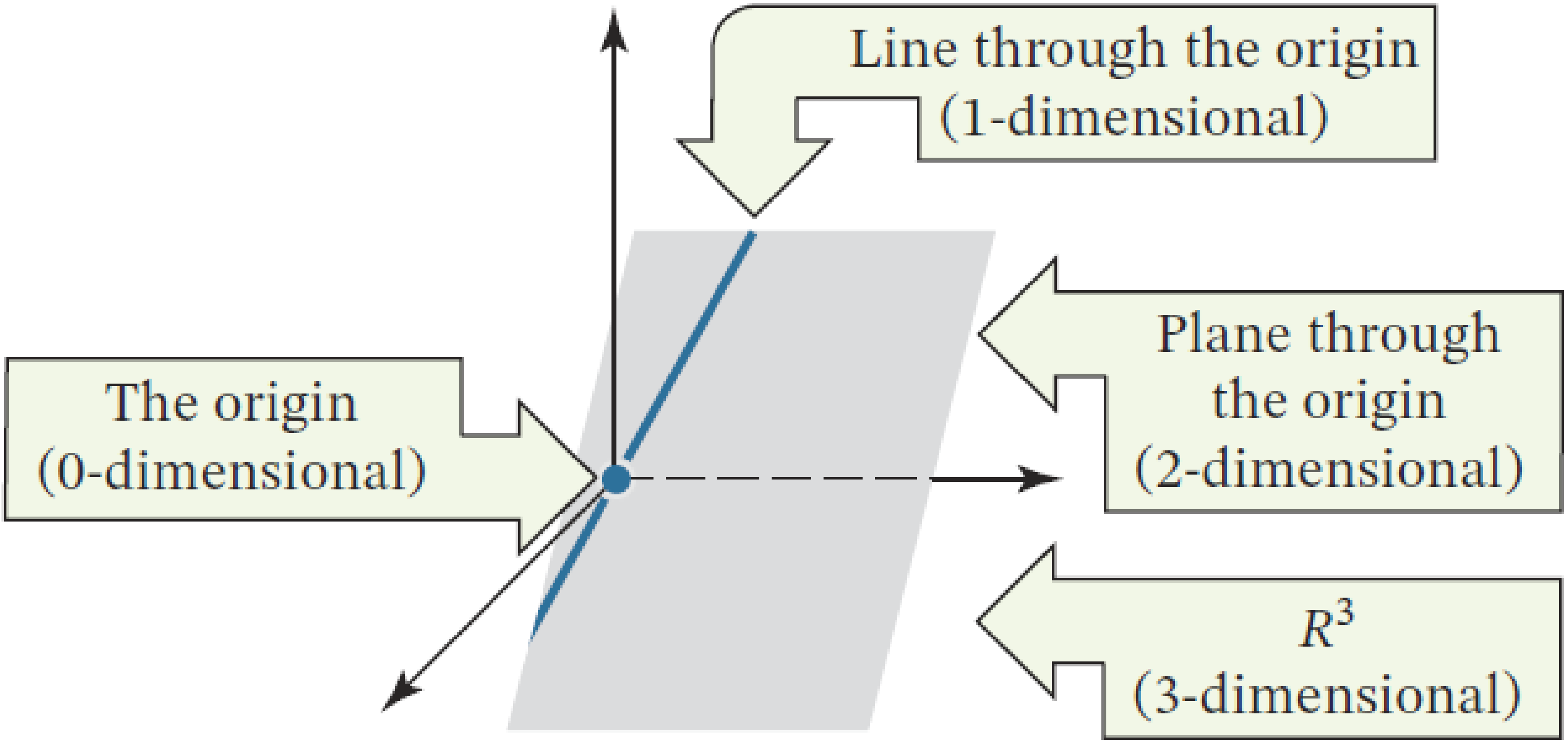
THEOREM 4.6.6

If W is a subspace of a finite-dimensional vector space V, then:

(a) W is finite-dimensional.

(b) $\dim(W) \leq \dim(V)$.

(c) $W = V$ if and only if $\dim(W) = \dim(V)$.



Line through the origin
(1-dimensional)
The origin
(0-dimensional)
Plane through
the origin
(2-dimensional)
R^3
(3-dimensional)

Chapter 4-6 Objectives

- Find a basis for and the dimension of the solution space of a homogeneous linear system.
- Use dimension to determine whether a set of vectors is a basis for a finite-dimensional vector space.
- Extend a linearly independent set to a basis.