1-5 In-Class Exercise

1. Find an elementary matrix E that satisfies the stated equation.

$$EB = F \qquad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix} \qquad F = \begin{bmatrix} 8 & 1 & 5 \\ 8 & 1 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (F was obtained from B by adding 2 times the third row to the second row)

2. Find the inverse of the 4×4 matrix, where k_1 , k_2 , k_3 , k_4 , are all nonzero.

$$\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$$

2. Find the inverse of the 4×4 matrix, where k_1 , k_2 , k_3 , k_4 , are all nonzero.

$$\begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$

The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{cccc|cccc} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{array}\right] \longleftarrow$$

The first and fourth rows were interchanged;
the second and third rows were interchanged.

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{k_1} & 0 & 0 & 0 \end{array}\right] \longleftarrow$$

The first row was multiplied by $1/k_4$,
the second row was multiplied by $1/k_3$,
the third row was multiplied by $1/k_2$, and
the fourth row was multiplied by $1/k_1$.

The inverse is $\begin{bmatrix} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 & 0 \end{bmatrix}$.

1-5 Suggested Exercise

1. Find a row operation and the corresponding elementary matrix that will restore the given elementary matrix to the identity matrix.

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Add 5 times the first row to the third row: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$

Interchange the first and fourth rows: $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & -7 & -1 \\ 8 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 8 & 1 & 5 \\ 2 & -7 & -1 \\ 3 & 4 & 1 \end{bmatrix}$$

2. Find an elementary matrix E that satisfies the stated equation $EB = A$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$ (A was obtained from B by interchanging the first row and the third row)

3. Use the inversion algorithm to find the inverse of the matrix (if the inverse exists).

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]$$

← The identity matrix was adjoined to the given matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right]$$

← -2 times the first row was added to the second row and -1 times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right]$$

← 2 times the second row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

← The third row was multiplied by -1.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -14 & 6 & 3 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

3 times the third row was added to the second row and -3 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right]$$

-2 times the second row was added to the first row.

The inverse is $\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$.

4. Find all values of c, if any, for which the given matrix is invertible.

$$\begin{bmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{bmatrix}$$

$$\begin{bmatrix} 1 & c & 1 \\ c & 1 & 0 \\ 0 & 1 & c \end{bmatrix} \longleftarrow \text{The first and second rows were interchanged.}$$

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ c & 1 & 0 \end{bmatrix} \longleftarrow \text{The second and third rows were interchanged.}$$

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 1-c^2 & -c \end{bmatrix} \longleftarrow -c \text{ times the first row was added to the third row.}$$

$$\begin{bmatrix} 1 & c & 1 \\ 0 & 1 & c \\ 0 & 0 & c^3-2c \end{bmatrix} \longleftarrow c^2-1 \text{ times the second row was added to the third.}$$

If $c^3 - 2c = c(c^2 - 2) = 0$, i.e. if $c = 0$, $c = \sqrt{2}$ or $c = -\sqrt{2}$ the last matrix contains a row of zeros, therefore it cannot be reduced to I by elementary row operations.

Otherwise (if $c^3 - 2c \neq 0$), multiplying the last row by $\frac{1}{c^3 - 2c}$ would result in a row echelon form with 1's on the main diagonal. Subsequent elementary row operations would then lead to the identity matrix.

We conclude that for any value of c other than 0, $\sqrt{2}$ and $-\sqrt{2}$ the matrix is invertible.

5. Express the matrix and its inverse as products of elementary matrices.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

We perform a sequence of elementary row operations to reduce the given matrix to the identity matrix. As we do so, we keep track of each corresponding elementary matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longleftarrow -1 \text{ times the first row was added to the second row.} \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow \text{The second and third rows were interchanged} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow -1 \text{ times the third row was added to the second.} \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longleftarrow -1 \text{ times the second row was added to the first row.} \qquad E_4 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $E_4 E_3 E_2 E_1 A = I$, we have

$$A = (E_4 E_3 E_2 E_1)^{-1} I = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$A^{-1} = E_4 E_3 E_2 E_1 = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that this answer is not unique since a different sequence of elementary row operations (and the corresponding elementary matrices) could be used instead.

6. Show that

$$A = \begin{bmatrix} 0 & a & 0 & 0 & 0 \\ b & 0 & c & 0 & 0 \\ 0 & d & 0 & e & 0 \\ 0 & 0 & f & 0 & g \\ 0 & 0 & 0 & h & 0 \end{bmatrix}$$

is not invertible for any values of the entries.

Consider three cases:

- If $a=0$ then A has a row of zeros (first row).
- If $a\neq 0$ and $h=0$ then A has a row of zeros (fifth row).
- If $a\neq 0$ and $h\neq 0$ then adding $-\frac{d}{a}$ times the first row to the third, and adding $-\frac{e}{h}$ times the fifth row to the third results in the third row becoming a row of zeros.

In all three cases, the reduced row echelon form of A is not I_5. By Theorem 1.5.3, A is not invertible.