2-1 In-Class Exercise

1. Evaluate det(A) using cofactor expansion method

$$A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$$

along the second row $\quad -0 + 5\begin{vmatrix} -3 & 7 \\ -1 & 5 \end{vmatrix} - 0 = 5(-8) = -40$

2-1 Suggested Exercise

1. Find all values of λ for which $\det(A) = 0$.

$$A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$$

Calculate the determinant by a cofactor expansion along the first row:

$$\det(A) = \begin{vmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{vmatrix} \quad = (\lambda - 4)\begin{vmatrix} \lambda & 2 \\ 3 & \lambda - 1 \end{vmatrix} - 0 + 0$$

$$= (\lambda - 4)\left[\lambda(\lambda - 1) - 6\right] = (\lambda - 4)\left[\lambda^2 - \lambda - 6\right] = (\lambda - 4)(\lambda - 3)(\lambda + 2)$$

The determinant is zero if $\lambda = -2$, $\lambda = 3$, or $\lambda = 4$.

2. Evaluate det(A) by a cofactor expansion along a row or column of your choice.

$$A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$$

along the first column:

$$1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} - 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} + 1\begin{vmatrix} k & k^2 \\ k & k^2 \end{vmatrix} = 1(0) - 1(0) + 1(0) = 0$$

3. Evaluate the determinant of the given matrix by inspection.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\det(A) = (1)(1)(2)(3) = 6.$$

4. Show that the matrices

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

$$AB = \begin{bmatrix} ad & ae+bf \\ 0 & cf \end{bmatrix}$$

$$BA = \begin{bmatrix} ad & bd+ce \\ 0 & cf \end{bmatrix}$$

$(\Rightarrow)$ if $AB = BA$ then $ae + bf = bd + ce$.

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = b(d-f) - e(a-c)$$
$$= bd - bf - ea + ec$$
$$= bd + ce - (ae + bf)$$
$$= 0$$

$(\Leftarrow)$ if $\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = bd + ce - (ae+bf)$
$$= 0$$

then $bd + ce = ae + bf$

thus $AB = BA$.

i.e. two matrices commute.

5. Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^2 = \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bc+d^2 \end{bmatrix}$

$tr(A) = a+d$, $tr(A^2) = a^2+bc+bc+d^2 = a^2+2bc+d^2$

$\det(A) = ad-bc$

$$\frac{1}{2}\begin{vmatrix} tr(A) & 1 \\ tr(A^2) & tr(A) \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a+d & 1 \\ a^2+2bc+d^2 & a+d \end{vmatrix}$$

$$= \frac{1}{2}\left[(a+d)^2 - (a^2+2bc+d^2)\right]$$

$$= \frac{1}{2}\left[a^2+2ad+d^2-a^2-2bc-d^2\right]$$

$$= \frac{1}{2}\left[2ad-2bc\right]$$

$$= ad-bc$$

$$= \det(A)$$

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6. What can you say about an nth-order determinant all of whose entries are 1? Explain.

If $n=1$ then the determinant is 1.

If $n=2$ then $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$.

If $n=3$ then a cofactor expansion will involves minors $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$. Therefore the determinant is 0.

By induction, we can show that the determinant will be 0 for all $n > 3$ as well.

7. What is the maximum number of zeros that a 3 × 3 matrix can have without having a zero determinant? Explain.

The answer is 6.
Because for a diagonal matrix, if any of the diagonal entry is 0, then the determinant = 0.

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$