4-8 In-Class Exercise

1. Find the vector form of the general solution of the linear system $A\mathbf{x} = \mathbf{b}$, and then use that result to find the vector form of the general solution of $A\mathbf{x} = \mathbf{0}$.

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 5 \\ x_1 \qquad + x_3 &= -2 \\ 2x_1 + x_2 + 3x_3 &= 3 \end{aligned}$$

2. Find a subset of the given vectors that forms a basis for the space spanned by those vectors, and then express each vector that is not in the basis as a linear combination of the basis vectors.

$$\mathbf{v}_1 = (1, 0, 1, 1),\ \mathbf{v}_2 = (-3, 3, 7, 1),$$
$$\mathbf{v}_3 = (-1, 3, 9, 3),\ \mathbf{v}_4 = (-5, 3, 5, -1)$$

1.

The reduced row echelon form of the augmented matrix of the system $A\mathbf{x}=\mathbf{b}$ is $\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution of this system is $x_1=-2-t$, $x_2=7-t$, $x_3=t$; in vector form,

$(x_1,x_2,x_3)=(-2-t,7-t,t)=(-2,7,0)+t(-1,-1,1)$.

The vector form of the general solution of $A\mathbf{x}=\mathbf{0}$ is $(x_1,x_2,x_3)=t(-1,-1,1)$.

2. Construct a matrix whose column vectors are the given vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$:

$A=\begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{bmatrix}$. Since its reduced row echelon form

$$\begin{array}{cccc} \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ \uparrow\ \ \uparrow\ \ \uparrow\ \ \uparrow \\ \mathbf{w}_1\ \mathbf{w}_2\ \mathbf{w}_3\ \mathbf{w}_4 \end{array}$$

contains leading 1's in the first two columns,

the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the column space of A,

and for $\text{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$. By inspection, $\mathbf{w}_3=2\mathbf{w}_1+\mathbf{w}_2$ and $\mathbf{w}_4=-2\mathbf{w}_1+\mathbf{w}_2$.

we conclude that $\mathbf{v}_3=2\mathbf{v}_1+\mathbf{v}_2$ and $\mathbf{v}_4=-2\mathbf{v}_1+\mathbf{v}_2$.

4-8 Suggested Exercises

1. Find bases for the null space and row space of A.

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

2. **a.** Find the bases for the row space and column space of A

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \\ -2 & 5 & -7 & 0 & -6 \\ -1 & 3 & -2 & 1 & -3 \\ -3 & 8 & -9 & 1 & -9 \end{bmatrix}$$

b. Find a basis for the row space of A that consists entirely of row vectors of A.

1.

The reduced row echelon form of A is $\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The reduced row echelon form of the augmented matrix of the homogeneous system $A\mathbf{x} = \mathbf{0}$ would have an additional column of zeros appended to this matrix. The general solution of the system $x_1 = -s + \frac{2}{7}t$, $x_2 = -s - \frac{4}{7}t$, $x_3 = s$, $x_4 = t$ can be written in the vector form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ therefore the vectors $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{7} \\ -\frac{4}{7} \\ 0 \\ 1 \end{bmatrix}$ form a basis for the null space of A.

A basis for the row space is formed by the nonzero rows of the reduced row echelon form of A: $\begin{bmatrix} 1 & 0 & 1 & -\frac{2}{7} \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & \frac{4}{7} \end{bmatrix}$.

2.

(a) The reduced row echelon form of A is $B=\begin{bmatrix}1&0&11&0&3\\0&1&3&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}$.

By Theorems 4.8.3 and 4.8.4, the nonzero rows of B form a basis for the row space of A:
$\mathbf{r}_1=\begin{bmatrix}1&0&11&0&3\end{bmatrix}$, $\mathbf{r}_2=\begin{bmatrix}0&1&3&0&0\end{bmatrix}$, and $\mathbf{r}_3=\begin{bmatrix}0&0&0&1&0\end{bmatrix}$.

By Theorem 4.8.4, columns of B containing leading 1's form a basis for the column space of B:
$\mathbf{c}_1'=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\mathbf{c}_2'=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$, and $\mathbf{c}_4'=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$. By Theorem 4.8.5(b), a basis for the column space of A is formed

by the corresponding columns of A: $\mathbf{c}_1=\begin{bmatrix}1\\-2\\-1\\-3\end{bmatrix}$, $\mathbf{c}_2=\begin{bmatrix}-2\\5\\3\\8\end{bmatrix}$, and $\mathbf{c}_4=\begin{bmatrix}0\\0\\1\\1\end{bmatrix}$.

2.

(b) We begin by transposing the matrix A.

We obtain $A^T = \begin{bmatrix} 1 & -2 & -1 & -3 \\ -2 & 5 & 3 & 8 \\ 5 & -7 & -2 & -9 \\ 0 & 0 & 1 & 1 \\ 3 & -6 & -3 & -9 \end{bmatrix}$, whose reduced row echelon form is $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. By Theorem 4.8.4, columns of C containing leading 1's form a basis for the column space of C:

$\mathbf{c}'_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{c}'_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{c}'_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. By Theorem 4.8.5(b), a basis for the column space of A^T is

formed by the corresponding columns of A^T: $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -2 \\ 5 \\ 0 \\ 3 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -2 \\ 5 \\ -7 \\ 0 \\ -6 \end{bmatrix}$, and $\mathbf{c}_3 = \begin{bmatrix} -1 \\ 3 \\ -2 \\ 1 \\ -3 \end{bmatrix}$.

Since columns of A^T are rows of A, a basis for the row space of A is formed by $\mathbf{r}_1 = \begin{bmatrix} 1 & -2 & 5 & 0 & 3 \end{bmatrix}$, $\mathbf{r}_2 = \begin{bmatrix} -2 & 5 & -7 & 0 & -6 \end{bmatrix}$, and $\mathbf{r}_3 = \begin{bmatrix} -1 & 3 & -2 & 1 & -3 \end{bmatrix}$.

3. Find a basis for the subspace of R^4 that is spanned by the given vectors.

$$(1,1,-4,-3),\ (2,0,2,-2),\ (2,-1,3,2)$$

We construct a matrix whose columns are the given vectors: $A=\begin{bmatrix} 1 & 2 & 2 \\ 1 & 0 & -1 \\ -4 & 2 & 3 \\ -3 & -2 & 2 \end{bmatrix}$. The reduced row echelon form of A is $B=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. By Theorem 4.8.4, the three columns of B form a basis for the column space of B. By Theorem 4.8.5(b), the three columns of A form a basis for the column space of A. We conclude that $\{(1,1,-4,-3),(2,0,2,-2),(2,-1,3,2)\}$ is a basis for the subspace of R^4 spanned by these vectors.