1-8 In-Class Exercise

1. Find the standard matrix for the transformation and use it to compute $T(\mathbf{x})$.

$$T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2); \ \mathbf{x} = (1, 0, 5)$$

2. Find the standard matrix A for the linear transformation $T : R^2 \rightarrow R^2$ for which

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \ T\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

1. $T(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 - x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} (1)(1)+(0)(0)+(0)(5) \\ (0)(1)+(1)(0)-(1)(5) \\ (0)(1)+(1)(0)+(0)(5) \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$ matches $T(1,0,5) = (1,-5,0)$.

2. The standard matrix for T is $A = \begin{bmatrix} T(\mathbf{e}_1) & | & T(\mathbf{e}_2) \end{bmatrix}$. Observe that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Because

T_A is a transformation, $T_A(\mathbf{e}_1) = T_A\left(3\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 3T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ -11 \end{bmatrix}$.

Likewise, $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so we obtain

$T_A(\mathbf{e}_2) = T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = T_A\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) - 2T_A\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \end{bmatrix} - 2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -4 \\ 9 \end{bmatrix}$.

Therefore, the matrix for T_A is $A = \begin{bmatrix} 5 & -4 \\ -11 & 9 \end{bmatrix}$.

1-8 Suggested Exercise

1. Find the domain and codomain of the transformation defined by the equations.

$$
\begin{aligned}
w_1 &= 5x_1 - 7x_2 \\
w_2 &= 6x_1 + x_2 \\
w_3 &= 2x_1 + 3x_2
\end{aligned}
$$

The transformation maps any vector $\mathbf{x}$ in R^2 into a vector $\mathbf{w}$ in R^3.
Its domain is R^2; the codomain is R^3.

2. Find the domain and codomain of the transformation T defined by the formula.

a) $T(x_1, x_2, x_3) = (x_1, x_2 - x_3, x_2)$

b) $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_1 - x_3 \\ 0 \end{bmatrix}$$

a) The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^3.
Its domain is R^3; the codomain is R^3.

b)
The transformation maps any vector $\mathbf{x}$ in R^3 into a vector in R^4.
Its domain is R^3; the codomain is R^4.

3. Find the standard matrix for the transformation defined by the equation or formula.

a)
$$
\begin{aligned}
w_1 &= 2x_1 - 3x_2 + x_3 \\
w_2 &= 3x_1 + 5x_2 - x_3
\end{aligned}
$$

b) $T(x_1, x_2, x_3) = (0, 0, 0, 0, 0)$

(a) The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix} 2 & -3 & 1 \\ 3 & 5 & -1 \end{bmatrix}$

(b) $T(x_1, x_2, x_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; the standard matrix is $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

4. Find the standard matrix for the transformation $T : R^4 \rightarrow R^2$ defined by.

$$\begin{aligned} w_1 &= 2x_1 + 3x_2 - 5x_3 - x_4 \\ w_2 &= x_1 - 5x_2 + 2x_3 - 3x_4 \end{aligned}$$

and then compute $T(1, -1, 2, 4)$ by directly substituting in the equations and then by matrix multiplication.

The given equations can be expressed in matrix form as $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

therefore the standard matrix for this transformation is $\begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix}$.

By directly substituting $(1, -1, 2, 4)$ for (x_1, x_2, x_3, x_4) into the given equation we obtain

$$w_1 = (2)(1) - (3)(1) - (5)(2) - (1)(4) = -15$$

$$w_2 = (1)(1) + (5)(1) + (2)(2) - (3)(4) = -2$$

By matrix multiplication,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -5 & -1 \\ 1 & -5 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} (2)(1) - (3)(1) - (5)(2) - (1)(4) \\ (1)(1) + (5)(1) + (2)(2) - (3)(4) \end{bmatrix} = \begin{bmatrix} -15 \\ -2 \end{bmatrix}.$$

5. Find $T_A(\mathbf{x})$, and express your answer in matrix form.

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix};\ \mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

$$T_A(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$$

6. Use Theorem 1.8.2 to show that T is a matrix transformation.

$$T(x, y, z) = (x + y, y + z, x)$$

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ then

$$T(\mathbf{u}+\mathbf{v}) = T(u_1 + v_1, u_2 + v_2, u_3 + v_3)$$

$$= (u_1 + v_1 + u_2 + v_2, u_2 + v_2 + u_3 + v_3, u_1 + v_1)$$

$$= (u_1 + u_2, u_2 + u_3, u_1) + (v_1 + v_2, v_2 + v_3, v_1)$$

$$= T(\mathbf{u}) + T(\mathbf{v})$$

and $T(k\mathbf{u}) = T(ku_1, ku_2, ku_3) = (ku_1 + ku_2, ku_2 + ku_3, ku_1) = k(u_1 + u_2, u_2 + u_3, u_1) = kT(\mathbf{u})$.

7. Use Theorem 1.8.2 to show that T is <u>not</u> a matrix transformation.

$$T(x, y) = (x, y + 1)$$

The homogeneity property fails to hold since $T(kx, ky) = (kx, ky + 1)$

does not generally equal $kT(x, y) = k(x, y+1) = (kx, ky + k)$.

(It can be shown that the additivity property fails to hold as well.)

8. The images of the standard basis vectors for R^3 are given for a linear transformation $T: R^3 \to R^3$. Find the standard matrix for the transformation, and find $T(\mathbf{x})$.

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad T(\mathbf{e}_3) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}; \quad \mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

the standard matrix for T is $A = \left[\; T(\mathbf{e}_1) \;\middle|\; T(\mathbf{e}_2) \;\middle|\; T(\mathbf{e}_3) \;\right]$.

. Therefore

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 0 & -3 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} (1)(2)+(0)(1)+(4)(0) \\ (3)(2)+(0)(1)-(3)(0) \\ (0)(2)+(1)(1)-(1)(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 1 \end{bmatrix}.$$

9. Let $T_A : R^3 \to R^3$ be multiplication by

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 2 \\ 4 & 5 & -3 \end{bmatrix}$$

and let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the standard basis vectors for R^3. Find the following vectors by inspection.

$$T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$$

Since T_A is a matrix transformation,

$$T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = T_A(\mathbf{e}_1) + T_A(\mathbf{e}_2) + T_A(\mathbf{e}_3) = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}.$$

10. Find the standard matrix A for the linear transformation $T: R^3 \to R^3$ for which

$$T\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right)=\begin{bmatrix}2\\-3\\10\end{bmatrix},\ T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\3\\8\end{bmatrix},\ T\left(\begin{bmatrix}-3\\-1\\2\end{bmatrix}\right)=\begin{bmatrix}-5\\-11\\7\end{bmatrix}$$

The standard matrix for T is $A=\left[\ T(\mathbf{e}_1)\ \middle|\ T(\mathbf{e}_2)\ \middle|\ T(\mathbf{e}_3)\ \right]$, so we need to express the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$, and $\mathbf{e}_3$ as linear combinations of the vectors $\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$, and $\begin{bmatrix}-3\\-1\\2\end{bmatrix}$.

To do this, we compute the inverse of $\begin{bmatrix}1 & 1 & -3\\0 & 1 & -1\\2 & 1 & 2\end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

The identity matrix was adjoined to the original matrix.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 8 & -2 & 0 & 1 \end{array}\right]$$

-2 times the first row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 7 & -2 & 1 & 1 \end{array}\right]$$

The second row was added to the third row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \end{array}\right]$$

The third row was multiplied by $\frac{1}{7}$.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{7} & \frac{8}{7} & \frac{1}{7} \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \end{array}\right]$$

The third row was added to the second row.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{7} & \frac{3}{7} & \frac{3}{7} \\ 0 & 1 & 0 & -\frac{2}{7} & \frac{8}{7} & \frac{1}{7} \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \end{array}\right]$$

3 times the third row was added to the first row.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{7} & -\frac{5}{7} & \frac{2}{7} \\ 0 & 1 & 0 & -\frac{2}{7} & \frac{8}{7} & \frac{1}{7} \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \end{array}\right] \longleftarrow \text{-1 times the second row was added to the first row.}$$

We obtain $\begin{bmatrix} \frac{3}{7} & -\frac{5}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{8}{7} & \frac{1}{7} \\ -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ -\frac{2}{7} \\ -\frac{2}{7} \end{bmatrix}$, $\begin{bmatrix} \frac{3}{7} & -\frac{5}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{8}{7} & \frac{1}{7} \\ -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix} -\frac{5}{7} \\ \frac{8}{7} \\ \frac{1}{7} \end{bmatrix}$, and $\begin{bmatrix} \frac{3}{7} & -\frac{5}{7} & \frac{2}{7} \\ -\frac{2}{7} & \frac{8}{7} & \frac{1}{7} \\ -\frac{2}{7} & \frac{1}{7} & \frac{1}{7} \end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{1}{7} \\ \frac{1}{7} \end{bmatrix}$

so that

$$T(\mathbf{e}_1) = T\left(\frac{3}{7}\begin{bmatrix}1\\0\\2\end{bmatrix} - \frac{2}{7}\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{2}{7}\begin{bmatrix}-3\\-1\\2\end{bmatrix}\right) = \frac{3}{7}T\left(\begin{bmatrix}1\\0\\2\end{bmatrix}\right) - \frac{2}{7}T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - \frac{2}{7}T\left(\begin{bmatrix}-3\\-1\\2\end{bmatrix}\right) = \frac{3}{7}\begin{bmatrix}2\\-3\\10\end{bmatrix} - \frac{2}{7}\begin{bmatrix}1\\3\\8\end{bmatrix} - \frac{2}{7}\begin{bmatrix}-5\\-11\\7\end{bmatrix} = \begin{bmatrix}2\\1\\0\end{bmatrix}.$$

$T(\mathbf{e}_2) = -\frac{5}{7}\begin{bmatrix}2\\-3\\10\end{bmatrix} + \frac{8}{7}\begin{bmatrix}1\\3\\8\end{bmatrix} + \frac{1}{7}\begin{bmatrix}-5\\-11\\7\end{bmatrix} = \begin{bmatrix}-1\\4\\3\end{bmatrix}$ and $T(\mathbf{e}_3) = \frac{2}{7}\begin{bmatrix}2\\-3\\10\end{bmatrix} + \frac{1}{7}\begin{bmatrix}1\\3\\8\end{bmatrix} + \frac{1}{7}\begin{bmatrix}-5\\-11\\7\end{bmatrix} = \begin{bmatrix}0\\-2\\5\end{bmatrix}$.

Therefore, the standard matrix for T is $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 4 & -2 \\ 0 & 3 & 5 \end{bmatrix}$.