

Chapter 2

Determinants

2.1. Determinants by Cofactor Expansion

2.2. Evaluating Determinants by Row Reduction

2.3. Properties of Determinants; Cramer's Rule

Chapter 2.2

Evaluating Determinants by Row Reduction

Things to Know in Advance

Determinants by cofactor expansion is time consuming especially when matrix is large.

Even with today's fastest computers it would take millions of years to calculate a 25×25 determinant by cofactor expansion, so methods based on row reduction are often used for large determinants.

A Basic Theorem

THEOREM 2.2.1

Let A be a square matrix. If A has a row of zeros or a column of zeros, then $\det(A) = 0$.

Proof

Since the determinant of A can be found by a cofactor expansion along any row or column, we can use the row or column of zeros.

$$\det(A) = 0 \cdot C_1 + 0 \cdot C_2 + \cdots + 0 \cdot C_n = 0$$

Determinant of the Transpose

THEOREM 2.2.2

Let A be a square matrix. Then $\det(A) = \det(A^T)$.

Proof

Trivial

Elementary Row Operations

THEOREM 2.2.3

Let A be an $n \times n$ matrix.

- (a) If B is the matrix that results when a single row or single column of A is multiplied by a scalar k , then $\det(B) = k \det(A)$.
- (b) If B is the matrix that results when two rows or two columns of A are interchanged, then $\det(B) = -\det(A)$.
- (c) If B is the matrix that results when a multiple of one row of A is added to another row or when a multiple of one column is added to another column, then $\det(B) = \det(A)$.

Elementary Row Operations

$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = k \det(A)$	<p>The first row of A is multiplied by k.</p>
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = -\det(A)$	<p>The first and second rows of A are interchanged.</p>
$\begin{vmatrix} a_{11} + ka_{21} & a_{12} + ka_{22} & a_{13} + ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(B) = \det(A)$	<p>A multiple of the second row of A is added to the first row.</p>

Special case of theorem 2.2.3

when $A = I_n$ and $B = E$ the elementary matrix

THEOREM 2.2.4

Let E be an $n \times n$ elementary matrix.

- (a) If E results from multiplying a row of I_n by a nonzero number k , then $\det(E) = k$.
- (b) If E results from interchanging two rows of I_n , then $\det(E) = -1$.
- (c) If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

EXAMPLE 1 Determinants of Elementary Matrices

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 3,$$

The second row of I_4 was multiplied by 3.

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix} = -1,$$

The first and last rows of I_4 were interchanged.

$$\begin{vmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1$$

7 times the last row of I_4 was added to the first row.

Observe that the determinant of an elementary matrix
cannot be zero.

Matrices with Proportional Rows or Columns

THEOREM 2.2.5

If A is a square matrix with two proportional rows or two proportional columns, then $\det(A) = 0$.

EXAMPLE 2 Introducing Zero Rows

$$\begin{vmatrix} 1 & 3 & -2 & 4 \\ 2 & 6 & -4 & 8 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 3 & 9 & 1 & 5 \\ 1 & 1 & 4 & 8 \end{vmatrix} = 0$$

Evaluating Determinants by Row Reduction

EXAMPLE 3

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{bmatrix}$$

Solution

$$\det(A) = \begin{vmatrix} 0 & 1 & 5 \\ 3 & -6 & 9 \\ 2 & 6 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & -6 & 9 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← The first and second rows of A were interchanged.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 2 & 6 & 1 \end{vmatrix}$$

← A common factor of 3 from the first row was taken through the determinant sign.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 10 & -5 \end{vmatrix}$$

← -2 times the first row was added to the third row.

$$= -3 \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -55 \end{vmatrix}$$

← -10 times the second row was added to the third row.

$$= (-3)(-55) \begin{vmatrix} 1 & -2 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{vmatrix}$$

← A common factor of -55 from the last row was taken through the determinant sign.

$$= (-3)(-55)(1) = 165$$

EXAMPLE 4 Using Column Operations to Evaluate Determinant

Compute the determinant of

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 2 & 7 & 0 & 6 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -5 \end{bmatrix}$$

Solution

we can put A in lower triangular form in one step by adding -3 times the first column to the fourth to obtain

$$\det(A) = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 7 & 0 & 0 \\ 0 & 6 & 3 & 0 \\ 7 & 3 & 1 & -26 \end{bmatrix} = (1)(7)(3)(-26) = -546$$

EXAMPLE 5 Row Operations and Cofactor Expansion

Evaluate $\det(A)$ where

$$A = \begin{bmatrix} 3 & 5 & -2 & 6 \\ 1 & 2 & -1 & 1 \\ 2 & 4 & 1 & 5 \\ 3 & 7 & 5 & 3 \end{bmatrix}$$

Solution

By adding suitable multiples of the second row to the remaining rows, we obtain

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$\det(A) = \begin{vmatrix} 0 & -1 & 1 & 3 \\ 1 & 2 & -1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & 1 & 8 & 0 \end{vmatrix}$$

$$= - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 1 & 8 & 0 \end{vmatrix}$$

← Cofactor expansion along the first column

$$= - \begin{vmatrix} -1 & 1 & 3 \\ 0 & 3 & 3 \\ 0 & 9 & 3 \end{vmatrix}$$

← We added the first row to the third row.

$$= -(-1) \begin{vmatrix} 3 & 3 \\ 9 & 3 \end{vmatrix} \\ = -18$$

← Cofactor expansion along the first column

Cofactor expansion and row or column operations can sometimes be used in combination to provide an effective method for evaluating determinants.

Chapter 2-2 Objectives

- ❑ Know the effect of elementary row operations on the value of a determinant.
- ❑ Know the determinants of the three types of elementary matrices.
- ❑ Know how to introduce zeros into the rows or columns of a matrix to facilitate the evaluation of its determinant.
- ❑ Use row reduction to evaluate the determinant of a matrix.
- ❑ Use column operations to evaluate the determinant of a matrix.
- ❑ Combine the use of row reduction and cofactor expansion to evaluate the determinant of a matrix.