Chapter 4
General Vector Spaces

Chapter 4.1

Real Vector Spaces

Vector Space Axioms

DEFINITION 1

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By *addition* we mean a rule for associating with each pair of objects $\mathbf{u}$ and $\mathbf{v}$ in V an object $\mathbf{u} + \mathbf{v}$, called the *sum* of $\mathbf{u}$ and $\mathbf{v}$; by *scalar multiplication* we mean a rule for associating with each scalar k and each object $\mathbf{u}$ in V an object $k\mathbf{u}$, called the *scalar multiple* of $\mathbf{u}$ by k.

If the following axioms are satisfied by all objects $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V and all scalars k and m, then we call V a *vector space* and we call the objects in V *vectors*.

Vector Space Axioms

DEFINITION 1

1. If $\mathbf{u}$ and $\mathbf{v}$ are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V.

2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

4. There exists an object in V, called the *zero vector*, that is denoted by $\mathbf{0}$ and has the property that $\mathbf{0} + \mathbf{u} = \mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V.

5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called a *negative* of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.

Vector Space Axioms

DEFINITION 1

6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V.

7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

9. $k(m\mathbf{u}) = (km)(\mathbf{u})$

10. $1\mathbf{u} = \mathbf{u}$

Group Definition (explained)
Abstract Algebra

https://www.youtube.com/watch?v=g7L_r6zw4-c

Examples of groups
https://planetmath.org/examplesofgroups

Show that a Set with Two Operations is a Vector Space

Step 1 Identify the set V of objects that will become vectors.

Step 2 Identify the addition and scalar multiplication operations on V.

Step 3 Verify Axioms 1 and 6; that is, adding two vectors in V produces a vector in V, and multiplying a vector in V by a scalar also produces a vector in V.
Axiom 1 is called *closure under addition*, and
Axiom 6 is called *closure under scalar multiplication*.

Step 4 Confirm that Axioms 2, 3, 4, 5, 7, 8, 9, and 10 hold.

The Zero Vector Space

EXAMPLE 1

Let V consist of a single object, which we denote by $\mathbf{0}$, and define

$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0} \qquad \text{for all scalars } k.$$

It is easy to check that all the vector space axioms are satisfied.

We call this the *zero vector space*.

Vector Space R^n

EXAMPLE 2

Let $V = R^n$, and define the vector space operations on V to be the usual operations of addition and scalar multiplication of n-tuples;

$$\mathbf{u} + \mathbf{v} = (u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)$$
$$k\mathbf{u} = (ku_1, ku_2, \ldots, ku_n)$$

The set $V = R^n$ is closed under addition and scalar multiplication because the foregoing operations produce n-tuples as their end result, and these operations satisfy Axioms 2, 3, 4, 5, 7, 8, 9, and 10 by virtue of Theorem 3.1.1.

A Vector Space of 2 × 2 Matrices

EXAMPLE 3

Let V be the set of 2×2 matrices with real entries,

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{bmatrix}$$

$$k\mathbf{u} = k\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix}$$

Axioms 1, 2, 3, 6, 7, 8, 9 holds.
This leaves Axioms 4, 5, and 10 that remain to be verified.

EXAMPLE 3 Continued

To confirm that Axiom 4 is satisfied, we must find a 2×2 matrix $\mathbf{0}$ in V for which $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u}$ for all 2×2 matrices in V. We can do this by taking

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{0} + \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

and similarly $\mathbf{u} + \mathbf{0} = \mathbf{u}$.

EXAMPLE 3 Continued

To verify that Axiom 5 holds we must show that each object $\mathbf{u}$ in V has a negative $-\mathbf{u}$ in V such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ and $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$. This can be done by defining the negative of $\mathbf{u}$ to be

$$-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$$

$$\mathbf{u} + (-\mathbf{u}) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$$

and similarly $(-\mathbf{u}) + \mathbf{u} = \mathbf{0}$.

Finally, Axiom 10 holds because

$$1\mathbf{u} = 1\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \mathbf{u}$$

A Set That Is Not a Vector Space

EXAMPLE 4

Let $V = R^2$ and define addition and scalar multiplication operations as follows:

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$, then define

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2)$$

and if k is any real number, then define

$$k\mathbf{u} = (ku_1, 0)$$

[Note]

An Unusual Vector Space

EXAMPLE 5

Let V be the set of positive real numbers, let $\mathbf{u} = u$ and $\mathbf{v} = v$ be any vectors (i.e., positive real numbers) in V, and let k be any scalar. Define the operations on V to be

$u + v = uv$ **[Vector addition is numerical multiplication.]**

$ku = u^k$ **[Scalar multiplication is numerical exponentiation.]**

[Note]

Some Properties of Vectors

THEOREM 4.1.1

Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:

(*a*) $0\mathbf{u} = \mathbf{0}$

(*b*) $k\mathbf{0} = \mathbf{0}$

(*c*) $(-1)\mathbf{u} = -\mathbf{u}$

(*d*) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$.

Proof

[same as in Chapter 3]

Chapter 4-1 Objectives

- Determine whether a given set with two operations is a vector space.
- Show that a set with two operations is not a vector space by demonstrating that at least one of the vector space axioms fails.