Chapter 4
General Vector Spaces

Chapter 4.4

Linear Independence

Linear Independence

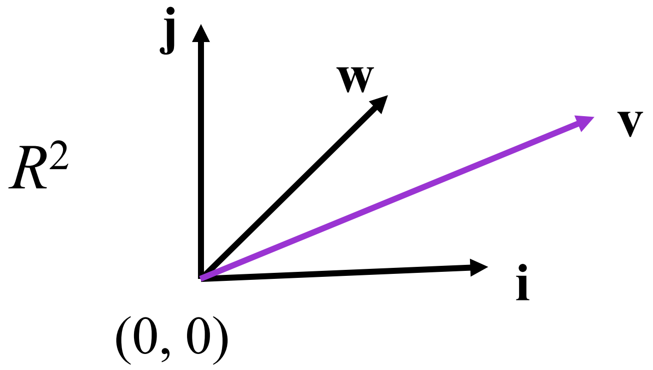
EXAMPLE 1

Consider R^2: $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ are linearly independent.

Each vector in R^2 can be expressed in exactly one way as a linearly combination of $\mathbf{i}$ and $\mathbf{j}$.

Now, let $S = \{\mathbf{i}\ , \mathbf{j}\ , \mathbf{w}\}$ where

$$\mathbf{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$



EXAMPLE 1 Cont.

There are infinitely many ways to express vector (3, 2) as a linear combination of **i**, **j**, and **w**. Thus S is a linearly dependent set.

Three possibilities are

$$(3,2) = 3(1,0) + 2(0,1) + 0\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 3\mathbf{i} + 2\mathbf{j} + 0\mathbf{w}$$

$$(3,2) = 2(1,0) + (0,1) + \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2\mathbf{i} + \mathbf{j} + \sqrt{2}\mathbf{w}$$

$$(3,2) = 4(1,0) + 3(0,1) - \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 4\mathbf{i} + 3\mathbf{j} - \sqrt{2}\mathbf{w}$$

In fact, $\mathbf{w} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}$

Linear Independence and Dependence

DEFINITION 1

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V, then S is said to be a *linearly independent set* if no vector in S can be expressed as a linear combination of the others.

A set that is not linearly independent is said to be *linearly dependent*.

If S has only one vector, we will agree that it is linearly independent <u>if and only if</u> that vector is nonzero.

Linear Independence and Dependence

THEOREM 4.4.1

A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

Linear Independence and Dependence

EXAMPLE 2

$$\left.\begin{aligned}\mathbf{v}_1 &= (1,-2,3),\\ \mathbf{v}_2 &= (5,6,-1),\\ \mathbf{v}_3 &= (3,2,1)\end{aligned}\right\}$$

Linearly independent or dependent in R^3 ?

Solution

$$k_1(1,-2,3) + k_2(5,6,-1) + k_3(3,2,1) = (0,0,0)$$

$$\begin{aligned} k_1 + 5k_2 + 3k_3 &= 0\\ -2k_1 + 6k_2 + 2k_3 &= 0\\ 3k_1 - k_2 + k_3 &= 0 \end{aligned} \Rightarrow \begin{aligned} k_1 &= -\tfrac{1}{2}t,\\ k_2 &= -\tfrac{1}{2}t,\\ k_3 &= t \end{aligned}$$

Thus, linearly dependent.

Linear Independence and Dependence

EXAMPLE 3

$$\left.\begin{aligned}\mathbf{v}_1 &= (1, 2, 2, -1),\\ \mathbf{v}_2 &= (4, 9, 9, -4),\\ \mathbf{v}_3 &= (5, 8, 9, -5)\end{aligned}\right\}$$

Linearly independent or dependent in R^4?

Solution

$$k_1(1, 2, 2, -1) + k_2(4, 9, 9, -4) + k_3(5, 8, 9, -5) = (0, 0, 0, 0)$$

$$\begin{aligned} k_1 + 4k_2 + 5k_3 &= 0\\ 2k_1 + 9k_2 + 8k_3 &= 0\\ 2k_1 + 9k_2 + 9k_3 &= 0\\ -k_1 - 4k_2 - 5k_3 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} k_1 &= 0,\\ k_2 &= 0,\\ k_3 &= 0 \end{aligned}$$

Thus, $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Linear Independence and Dependence

EXAMPLE 4

$$\left.\begin{aligned} \mathbf{p}_1 &= 1 - x, \\ \mathbf{p}_2 &= 5 + 3x - 2x^2, \\ \mathbf{p}_3 &= 1 + 3x - x^2 \end{aligned}\right\}$$

Linearly independent or dependent in P_2 ?

Solution

$$k_1(1 - x) + k_2(5 + 3x - 2x^2) + k_3(1 + 3x - x^2) = 0$$

$$(k_1 + 5k_2 + k_3) + (-k_1 + 3k_2 + 3k_3)x + (-2k_2 - k_3)x^2 = 0$$

$$\begin{aligned} k_1 + 5k_2 + k_3 &= 0 \\ -k_1 + 3k_2 + 3k_3 &= 0 \\ -2k_2 - k_3 &= 0 \end{aligned} \Rightarrow$$

exe

has a nontrivial solutions

Thus, linearly dependent.

Linear Independence and Dependence

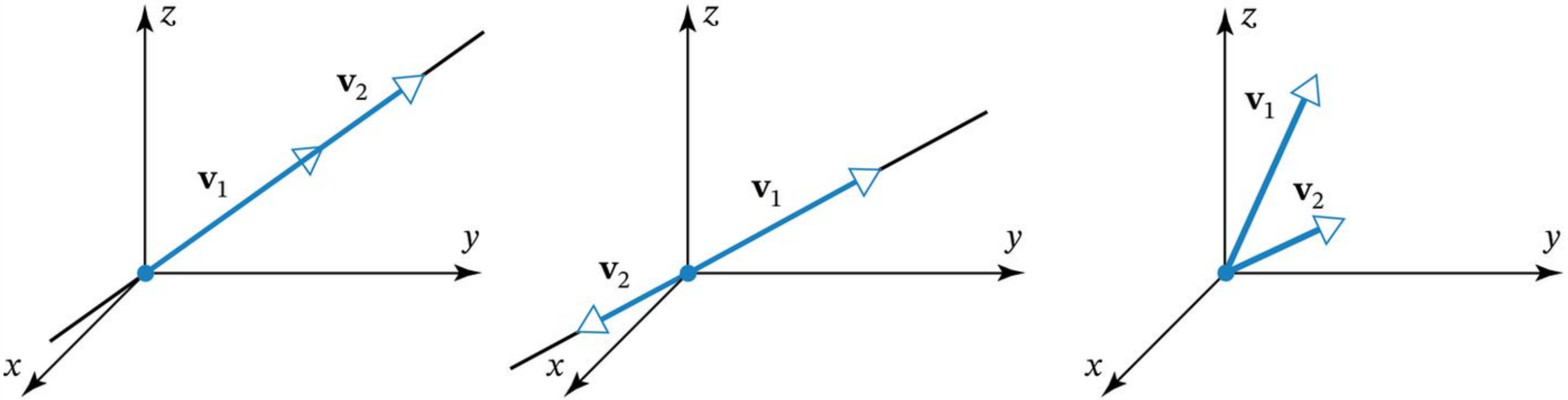
THEOREM 4.4.2

(a) A set with finitely many vectors that contains $\mathbf{0}$ is linearly dependent.

(b) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

A Geometric Interpretation of Linear Independence

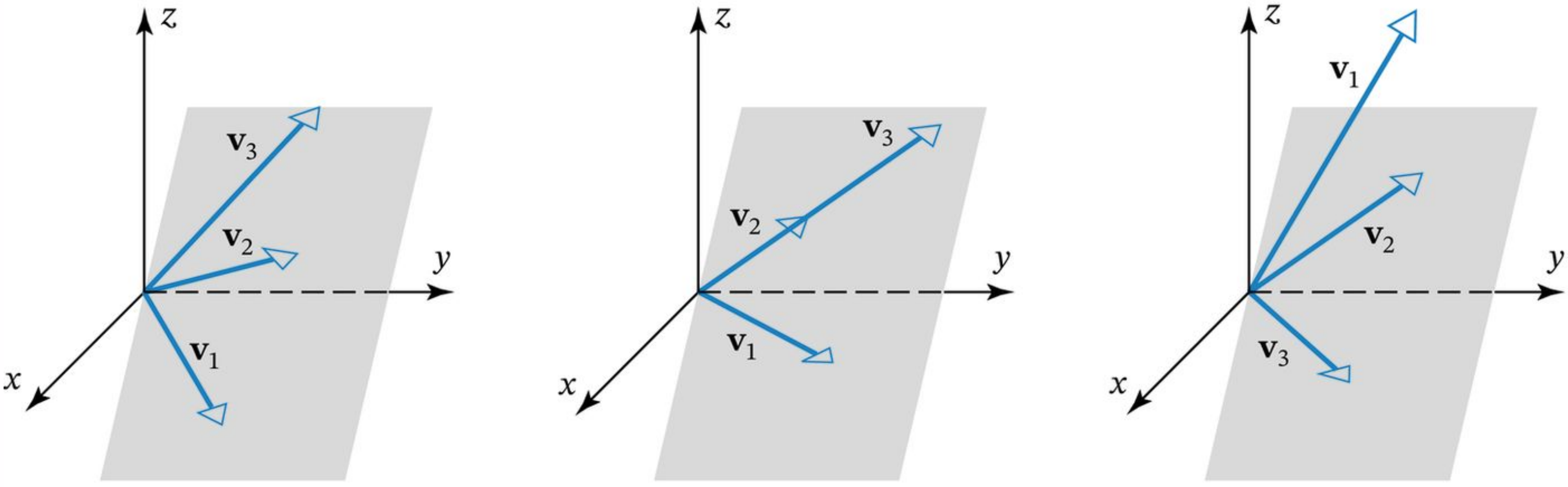
Two vectors in R^3



(a) Linearly dependent   **(b)** Linearly dependent   **(c)** Linearly independent

A Geometric Interpretation of Linear Independence

Three vectors in R^3



(*a*) Linearly dependent (*b*) Linearly dependent (*c*) Linearly independent

Linearly Dependent Set of Vectors

THEOREM 4.4.3

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n.

If $r > n$, then S is linearly dependent.

Proof

consider the equation $\quad k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = \mathbf{0}$

This is a homogeneous system of n equations in the r unknowns $k_1, \ldots, k_r$. Since $r > n$, Theorem 1.2.2 implies that the system has nontrivial solutions, so $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a linearly dependent set.

Recall

THEOREM 1.2.2

A homogeneous linear system with more unknowns than equations has infinitely many solutions.

$$
n \left\{ \underbrace{\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots\cdots & \mathbf{v}_k \\ & & & \\ & & & \end{pmatrix}}_{r} \right. \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ \vdots \\ k_r \end{bmatrix} = \mathbf{0}
$$

Linear Independence of Row Vectors

EXAMPLE 5

It is an important fact that the nonzero row vectors of a matrix in (reduced) row echelon form are linearly independent.

e.g.

$$R = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denoting the row vectors by $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3,$ we must show that the only solution of the vector equation $c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3 = \mathbf{0}$ is the trivial solution $c_1 = c_2 = c_3 = 0.$

EXAMPLE 5 Cont.

Rewrite the equation in row-vector form

$$[c_1 \quad c_1a_{12} + c_2 \quad c_1a_{13} + c_2a_{23} \quad c_1a_{14} + c_2a_{24} + c_3] = [0 \quad 0 \quad 0 \quad 0]$$

and comparing corresponding components.

The solution is $c_1 = c_2 = c_3 = 0$.

Chapter 4-4 Objectives

- Determine whether a set of vectors is linearly independent or linearly dependent.
- Express one vector in a linearly dependent set as a linear combination of the other vectors in the set.