4-1 In-Class Exercise

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad k\mathbf{u} = (ku_1, ku_2)$$

a. Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (0, 4), \mathbf{v} = (1, -3)$, and $k = 2$.

b. Show that $(0, 0) \neq \mathbf{0}$.

c. Show that $(-1, -1) = \mathbf{0}$.

d. Show that Axiom 5 holds by producing a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for $\mathbf{u} = (u_1, u_2)$.

e. Find two vector space axioms that fail to hold.

1.

(a) $\mathbf{u}+\mathbf{v}=(0+1+1,4-3+1)=(2,2)$; $k\mathbf{u}=(2\cdot 0,2\cdot 4)=(0,8)$

(b) $(0,0)+(u_1,u_2)=(0+u_1+1,0+u_2+1)=(u_1+1,\ u_2+1)\neq(u_1,u_2)$ therefore $(0,0)$ is not the zero vector $\mathbf{0}$ required by Axiom 4

(c) For all real numbers u_1 and u_2, we have

$(-1,-1)+(u_1,u_2)=(-1+u_1+1,\ -1+u_2+1)=(u_1,\ u_2)$ and

$(u_1,u_2)+(-1,-1)=(u_1-1+1,\ u_2-1+1)=(u_1,\ u_2)$ therefore Axiom 4 holds for

$\mathbf{0}=(-1,-1)$

d) For any pair of real numbers $\mathbf{u}=(u_1,u_2)$, letting $-\mathbf{u}=(-2-u_1,-2-u_2)$ yields

$$\mathbf{u}+(-\mathbf{u})=(u_1+(-2-u_1)+1,u_2+(-2-u_2)+1)=(-1,-1)=\mathbf{0};$$

Since $(-\mathbf{u})+\mathbf{u}=\mathbf{0}$ holds as well, Axiom 5 holds.

(e) Axiom 7 fails to hold:

$$k(\mathbf{u}+\mathbf{v}) = k(u_1 + v_1 + 1, u_2 + v_2 + 1) = (ku_1 + kv_1 + k, ku_2 + kv_2 + k)$$

$$k\mathbf{u} + k\mathbf{v} = (ku_1, ku_2) + (kv_1, kv_2) = (ku_1 + kv_1 + 1, ku_2 + kv_2 + 1)$$

therefore in general $k(\mathbf{u}+\mathbf{v}) \neq k\mathbf{u} + k\mathbf{v}$

Axiom 8 fails to hold:

$$(k+m)\mathbf{u} = ((k+m)u_1, (k+m)u_2) = (ku_1 + mu_1, ku_2 + mu_2)$$

$$k\mathbf{u} + m\mathbf{u} = (ku_1, ku_2) + (mu_1, mu_2) = (ku_1 + mu_1 + 1, ku_2 + mu_2 + 1)$$

therefore in general $(k+m)\mathbf{u} \neq k\mathbf{u} + m\mathbf{u}$

4-1 Suggested Exercises

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad k\mathbf{u} = (0, ku_2)$$

a. Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$, and $k = 3$.

b. In words, explain why V is closed under addition and scalar multiplication.

c. Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?

d. Show that Axioms 7, 8, and 9 hold.

e. Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

1.

(a) $\mathbf{u}+\mathbf{v}=(-1+3,2+4)=(2,6)$; $\quad k\mathbf{u}=(0,3\cdot 2)=(0,6)$

(b) For any $\mathbf{u}=(u_1,u_2)$ and $\mathbf{v}=(v_1,v_2)$ in V, $\mathbf{u}+\mathbf{v}=(u_1+v_1,u_2+v_2)$ is an ordered pair of real numbers, therefore $\mathbf{u}+\mathbf{v}$ is in V. Consequently, V is closed under addition.

For any $\mathbf{u}=(u_1,u_2)$ in V and for any scalar k, $k\mathbf{u}=(0,ku_2)$. is an ordered pair of real numbers, therefore $k\mathbf{u}$ is in V. Consequently, V is closed under scalar multiplication.

(c) Axioms 1-5 hold for V because they are known to hold for R^2.

(d) Axiom 7: $k((u_1,u_2)+(v_1,v_2))=k(u_1+v_1,u_2+v_2)=(0,k(u_2+v_2))=(0,ku_2)+(0,kv_2)$
$=k(u_1,u_2)+k(v_1,v_2)$ for all real k, u_1, u_2, v_1, and v_2;

Axiom 8: $(k+m)(u_1,u_2)=(0,(k+m)u_2)=(0,ku_2+mu_2)=(0,ku_2)+(0,mu_2)$
$=k(u_1,u_2)+m(u_1,u_2)$ for all real k, m, u_1, and u_2;

Axiom 9: $k(m(u_1,u_2))=k(0,mu_2)=(0,kmu_2)=(km)(u_1,u_2)$ for all real k, m, u_1, and u_2;

(e) Axiom 10 fails to hold: $1(u_1,u_2)=(0,u_2)$ does not generally equal (u_1,u_2).
Consequently, V is not a vector space.

Determine whether each of the following sets equipped with the given operations is a vector space.

2. The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on R^2.

3. The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2x, k^2y, k^2z)$$

4. The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2.

5. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.

2.

Axiom 5 fails whenever $x \neq 0$ since it is then impossible to find (x', y') satisfying $x' \geq 0$ for which $(x, y) + (x', y') = (0,0)$. (The zero vector from axiom 4 must be $\mathbf{0} = (0,0)$.)

Axiom 6 fails whenever $k < 0$ and $x \neq 0$.

This is not a vector space.

3. Axiom 8 fails to hold:

$$(k+m)\mathbf{u} = \left((k+m)^2 x, (k+m)^2 y, (k+m)^2 z\right)$$

$$k\mathbf{u} + m\mathbf{u} = \left(k^2 x, k^2 y, k^2 z\right) + \left(m^2 x, m^2 y, m^2 z\right) = \left(\left(k^2 + m^2\right) x, \left(k^2 + m^2\right) y, \left(k^2 + m^2\right) z\right)$$

therefore in general $(k+m)\mathbf{u} \neq k\mathbf{u} + m\mathbf{u}$.

This is not a vector space.

4. Let V denote the set of all pairs of real numbers of the form $(x,0)$.

Axiom 1: $(x,0)+(y,0)=(x+y,0)$ is in V for all real x and y;

Axiom 2: $(x,0)+(y,0)=(x+y,0)=(y+x,0)=(y,0)+(x,0)$ for all real x and y;

Axiom 3: $(x,0)+((y,0)+(z,0))=(x,0)+(y+z,0)=(x+y+z,0)=(x+y,0)+(z,0)$

$=((x,0)+(y,0))+(z,0)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0}=(0,0)$, we have $(0,0)+(x,0)=(x,0)$ and $(x,0)+(0,0)=(x,0)$ for all real x;

4.

Axiom 5: for each $\mathbf{u}=(x,0)$, let $-\mathbf{u}=(-x,0)$;

then $(x,0)+(-x,0)=(0,0)$ and $(-x,0)+(x,0)=(0,0)$;

Axiom 6: $k(x,0)=(kx,0)$ is in V for all real k and x;

Axiom 7: $k((x,0)+(y,0))=k(x+y,0)=(kx+ky,0)=k(x,0)+k(y,0)$

for all real k, x, and y;

Axiom 8: $(k+m)(x,0)=((k+m)x,0)=(kx+mx,0)=k(x,0)+m(x,0)$

for all real k, m, and x;

Axiom 9: $k(m(x,0))=k(mx,0)=(kmx,0)=(km)(x,0)$ for all real k, m, and x;

Axiom 10: $1(x,0)=(x,0)$ for all real x.

This is a vector space – all axioms hold.

5.

Let V be the set of all 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (i.e., all diagonal 2×2 matrices)

Axiom 1: the sum of two diagonal 2×2 matrices is also a diagonal 2×2 matrix.

Axiom 2: follows from part (a) of Theorem 1.4.1.

Axiom 3: follows from part (b) of Theorem 1.4.1.

Axiom 4: taking $0=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; follows from part (a) of Theorem 1.4.2.

Axiom 5: let the negative of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be $\begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix}$; follows from part (c) of Theorem 1.4.2 and Axiom 2.

Axiom 6: the scalar multiple of a diagonal 2×2 matrix is also a diagonal 2×2 matrix.

Axiom 7: follows from part (h) of Theorem 1.4.1.

Axiom 8: follows from part (j) of Theorem 1.4.1.

Axiom 9: follows from part (l) of Theorem 1.4.1.

Axiom 10: $1\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}=\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ for all real a and b.

This is a vector space – all axioms hold.