2-2 In-Class Exercise

1. Evaluate the determinant, given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$$

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = ?$$

$$\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g+3a & h+3b & i+3c \end{vmatrix} = \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix}$$

-3 times the first row was added to the last row.

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

A common factor of 2 from the second row was taken through the determinant sign.

$$= (2)(-6) = -12$$

2-2 Suggested Exercise

1. Evaluate the determinant of the matrix. $\begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$\begin{vmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} = (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \quad \longleftarrow \text{The first and second rows were interchanged.}$$

$$= (-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{vmatrix} \quad \longleftarrow -2 \text{ times the first row was added to the second row.}$$

$$=(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 1 & 2 & 3 \end{vmatrix}$$

-2 times the second row was added to the third row.

$$=(-1)\begin{vmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{vmatrix}$$

-1 times the second row was added to the fourth row.

$$=(-1)(1)\begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 1 & 4 \end{vmatrix}=(-1)(1)(1)\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix}$$

cofactor expansions along the first columns

$$=(-1)(1)(1)(-6)=6\,.$$

2. Evaluate the determinant of the matrix. $\begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = (-1) * \begin{vmatrix} 1 & -2 & 3 & 1 \\ -1 & 2 & -6 & -2 \\ 5 & -9 & 6 & 3 \\ 2 & 8 & 6 & 1 \end{vmatrix} \quad row2 \Leftrightarrow row3$$

$$= (-1)*(-1)* \begin{vmatrix} 1 & -2 & 3 & 1 \\ 1 & -2 & 6 & 2 \\ 5 & -9 & 6 & 3 \\ 2 & 8 & 6 & 1 \end{vmatrix} \quad (-1) * row2$$

$$= \begin{vmatrix} 1 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 \\ 5 & -9 & -3 & 3 \\ 2 & 8 & 3 & 1 \end{vmatrix} \quad col3 + (-1) * col4$$

$$= \begin{vmatrix} 1 & -2 & 0 & 1 \\ 1 & -2 & 0 & 2 \\ 7 & -1 & 0 & 4 \\ 2 & 8 & 3 & 1 \end{vmatrix}$$ row 3 + (1) * row 4

$$=(-3) \begin{vmatrix} 1 & -2 & 1 \\ 1 & -2 & 2 \\ 7 & -1 & 4 \end{vmatrix}$$ cofactor expansion on col 3.

$$= (-3) \begin{vmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 7 & -1 & 4 \end{vmatrix}$$ row 2 + (-1) * row 1

$$=(3) \begin{vmatrix} 1 & -2 \\ 7 & -1 \end{vmatrix}$$ cofactor expansion on row 2

$$= 3 \times 13 = 39$$

3. Evaluate the determinant, given that $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = -6$

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix} = ?$$

$$\begin{vmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{vmatrix}$$

A common factor of -3 from the first row was taken through the determinant sign.

$$= -3\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

4 times the second row was added to the last row.

$$= (-3)(-6) = 18$$

4. Show that $\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ a_1t+b_1 & a_2t+b_2 & a_3t+b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ (1-t^2)b_1 & (1-t^2)b_2 & (1-t^2)b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$-t$ times the first row was added to the second row.

$$= (1-t^2)\begin{vmatrix} a_1+b_1t & a_2+b_2t & a_3+b_3t \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

A common factor of $1-t^2$ from the second row was taken through the determinant sign.

$$= (1-t^2)\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$-t$ times the second row was added to the first row.

5. Show that $\det(A) = 0$ without directly evaluating the determinant.

$$A = \begin{bmatrix} -2 & 8 & 1 & 4 \\ 3 & 2 & 5 & 1 \\ 1 & 10 & 6 & 5 \\ 4 & -6 & 4 & -3 \end{bmatrix}$$

The second column vector is a scalar multiple of the fourth. Therefore the determinant is 0.

6. Let A be an $n \times n$ matrix, and let B be the matrix that results when the rows of A are written in reverse order. State a theorem that describes how $\det(A)$ and $\det(B)$ are related.

$$A = \begin{bmatrix} \text{row } 1 \\ \text{row } 2 \\ \vdots \\ \text{row } n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \text{row } n \\ \text{row } n-1 \\ \vdots \\ \text{row } 1 \end{bmatrix}$$

matrix B is obtained by interchanging

row 1 with row n

row 2 with row $n-1$

$\vdots$

row $\lfloor \frac{n}{2} \rfloor$ with row $\lfloor \frac{n+3}{2} \rfloor$ of A

Therefore, $\det(B) = (-1)^{\lfloor \frac{n}{2} \rfloor} * \det(A)$