Chapter 4
General Vector Spaces

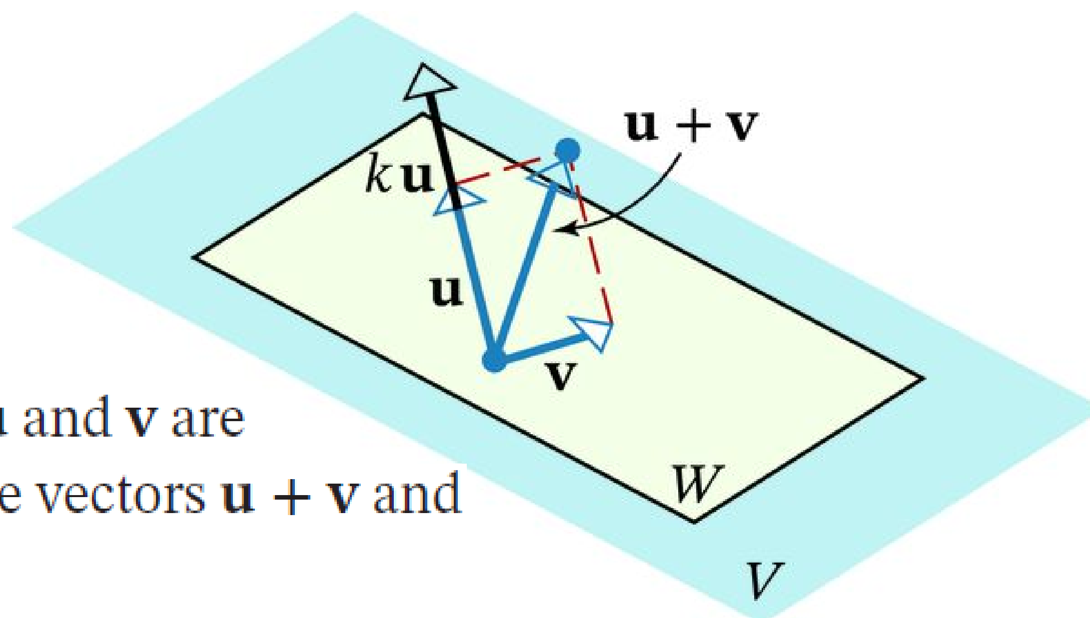
Chapter 4.2

Subspaces

Subspaces

DEFINITION 1

A subset W of a vector space V is called a *subspace* of V if W is itself a vector space under the addition and scalar multiplication defined on V.



E.g. The vectors $\mathbf{u}$ and $\mathbf{v}$ are in W, but the vectors $\mathbf{u}+\mathbf{v}$ and $k\mathbf{u}$ are not.

Show that a Nonempty Set is a Subspace

Those axioms that are not inherited by W are

Axiom 1 — Closure of W under addition
Axiom 4 — Existence of a zero vector in W
Axiom 5 — Existence of a negative in W for every vector in W
Axiom 6 — Closure of W under scalar multiplication

so these must be verified to prove that it is a subspace of V.

However, the following theorem shows that if Axiom 1 and Axiom 6 hold in W, then Axioms 4 and 5 hold in W as a consequence and hence need not be verified.

Subspace Test

THEOREM 4.2.1

If W is a nonempty set of vectors in a vector space V, then W is a subspace of V if and only if the following conditions are satisfied.

(a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.

(b) If k is a scalar and $\mathbf{u}$ is a vector in W, then $k\mathbf{u}$ is in W.

Proof

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THEOREM 4.2.1 *Proof*

If W is a subspace of V, then all the vector space axioms hold in W, including Axioms 1 and 6, which are precisely conditions (a) and (b).

Conversely, assume that conditions (a) and (b) hold. Since these are Axioms 1 and 6, and since Axioms 2, 3, 7, 8, 9, and 10 are inherited from V, we only need to show that Axioms 4 and 5 hold in W.

For this purpose, let $\mathbf{u}$ be any vector in W.
It follows from condition (b) that the product $k\mathbf{u}$ is also a vector in W for every scalar k. In particular, $0\mathbf{u} = \mathbf{0}$ and $(-1)\mathbf{u} = -\mathbf{u}$ are in W, which shows that Axioms 4 and 5 hold in W.

The Zero Subspace

EXAMPLE 1

If V is any vector space, and if $W = \{\mathbf{0}\}$ is the subset of V that consists of the zero vector only, then W is closed under addition and scalar multiplication since

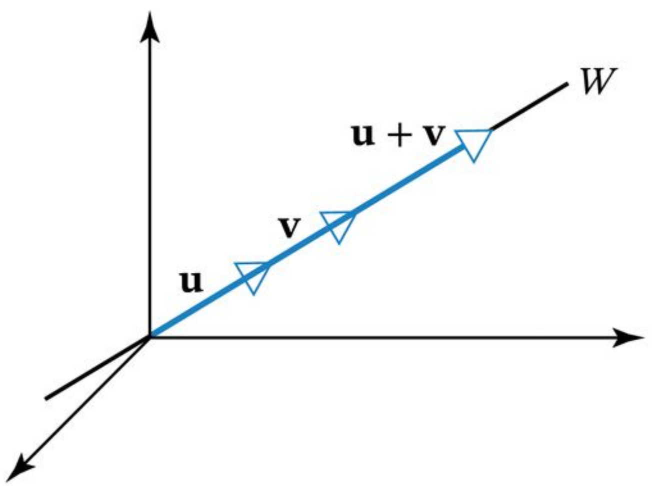
$$\mathbf{0} + \mathbf{0} = \mathbf{0} \quad \text{and} \quad k\mathbf{0} = \mathbf{0} \qquad \text{for any scalar } k.$$

We call W the *zero subspace* of V.

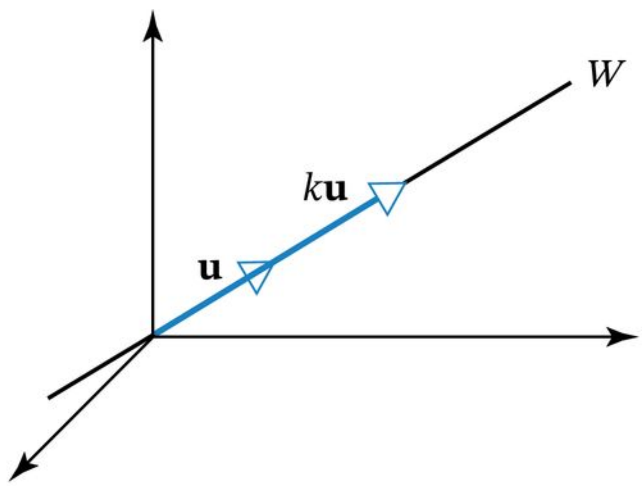
Subspaces of R^2 and of R^3

EXAMPLE 2

Lines through the origin are subspaces of R^2 and of R^3



(a) W is closed under addition.

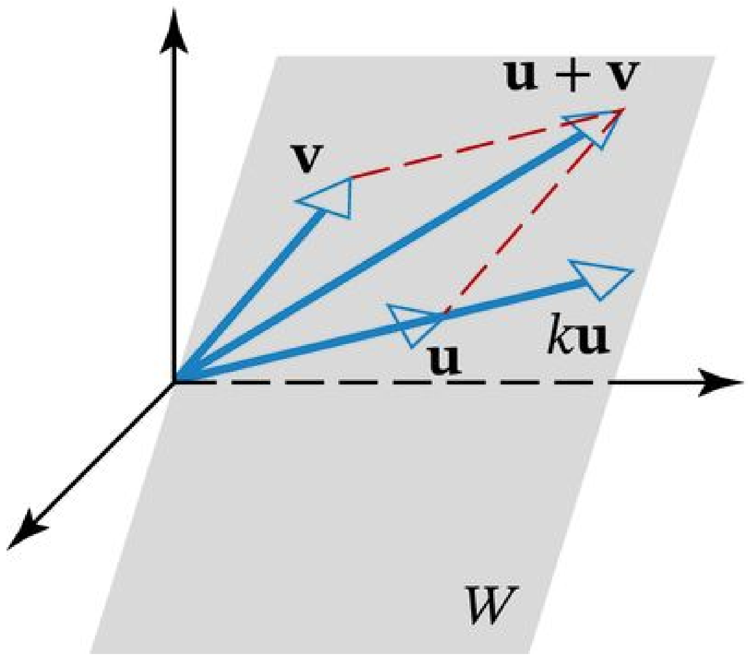


(b) W is closed under scalar multiplication.

Subspaces of R^2 and of R^3

EXAMPLE 3

Planes through the origin are subspaces of R^3



The vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ both lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$.

Subspaces of R^2 and of R^3

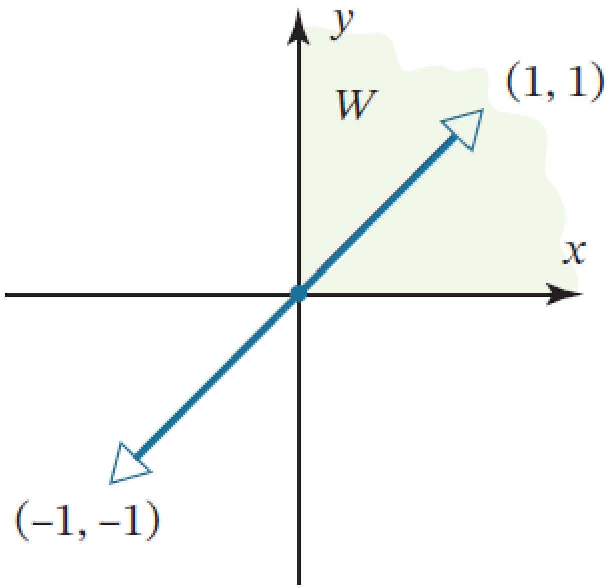
Subspaces of R^2	Subspaces of R^3
• $\{\mathbf{0}\}$	• $\{\mathbf{0}\}$
• Lines through the origin	• Lines through the origin
• R^2	• Planes through the origin
	• R^3

A Subset of R^2 That Is Not a Subspace

EXAMPLE 4

Let W be the set of all points (x, y) in R^2 for which $x \geq 0$ and $y \geq 0$

This set is not a subspace of R^2 because it is not closed under scalar multiplication.



$\mathbf{v} = (1, 1)$ is a vector in W,

but $(-1)\mathbf{v} = (-1, -1)$ is not.

Subspaces of M_{nn}

EXAMPLE 5

the set of symmetric $n \times n$ matrices is closed under addition and scalar multiplication and hence is a subspace of M_{nn}.

Similarly, the sets of upper triangular matrices, lower triangular matrices, and diagonal matrices are subspaces of M_{nn}.

A Subset of M_{nn} That Is Not a Subspace

EXAMPLE 6

The set W of invertible $n \times n$ matrices is not a subspace of M_{nn}.

$$U = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -1 & 2 \\ -2 & 5 \end{bmatrix}$$

The matrix $0U$ is the 2×2 zero matrix and hence is not invertible, and the matrix $U + V$ has a column of zeros so it also is not invertible.

The Subspace of All Polynomials

EXAMPLE 7

Recall that a *polynomial* is a function that can be expressed in the form

$$p(x) = a_0 + a_1x + \cdots + a_nx^n$$

where $a_0, a_1, \ldots, a_n$ are constants. Denote this space by P_n.

It is evident that the sum of two polynomials is a polynomial and that a constant times a polynomial is a polynomial.

Thus, the set W of all polynomials is a subspace of $F(-\infty, \infty)$.

(i.e. denote this space by P_∞.) (i.e. a function space)

Subspaces Test

EXAMPLE 8

Determine whether the indicated set of matrices is a subspace of M_{22}.

(a) The set U consisting of all matrices of the form

$$\begin{bmatrix} x & 0 \\ 2x & y \end{bmatrix}$$

(b) The set W consisting of all 2×2 matrices A such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

EXAMPLE 8 cont.

(a) If A and B are matrices in U, then they can be expressed in the form

$$A = \begin{bmatrix} a & 0 \\ 2a & b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & 0 \\ 2c & d \end{bmatrix}$$

for some real numbers a, b, c, and d.

$$A + B = \begin{bmatrix} a + c & 0 \\ 2(a + c) & b + d \end{bmatrix}$$ is a matrix in U

Thus, U is closed under addition.

$$kA = \begin{bmatrix} ka & 0 \\ 2ka & kb \end{bmatrix}$$ is also a matrix in U

Moreover, U is closed under scalar multiplication.

Hence, U is a subspace of M_{22}.

EXAMPLE 8 cont.

(b)

Let $$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

This is a vector in W since

$$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

However, $2A$ does not satisfy Equation (3) since

$$(2A)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

This alone establishes that W is not a subspace of M_{22}.

Subspaces Test

EXAMPLE 9

Determine whether the indicated set of polynomials is a subspace of P_2.

(a) The set U consisting of all polynomials of the form $\mathbf{p} = 1 + ax - ax^2$, where a is a real number.

(b) The set W consisting of all polynomials $\mathbf{p}$ in P_2 such that $\mathbf{p}(2) = 0$.

EXAMPLE 9 cont.

(a)

The set U is not a subspace of P_2 because it is not closed under addition.

polynomials $\mathbf{p} = 1 + x - x^2$ and $\mathbf{q} = 1 + 2x - 2x^2$ are in U, but

$\mathbf{p} + \mathbf{q} = 2 + 3x - 3x^2$ is not.

(b)

If $\mathbf{p}$ and $\mathbf{q}$ are polynomials in W, and k is any real number, then

$$(\mathbf{p} + \mathbf{q})(2) = \mathbf{p}(2) + \mathbf{q}(2) = 0 + 0 = 0$$

$$(k\mathbf{p})(2) = k \cdot \mathbf{p}(2) = k \cdot 0 = 0.$$

Since $\mathbf{p} + \mathbf{q}$ and $k\mathbf{p}$ are in W, it follows that W is a subspace of P_2.

Building Subspaces

THEOREM 4.2.2

If $W_1, W_2, \ldots, W_r$ are subspaces of a vector space V, then the intersection of these subspaces is also a subspace of V.

Proof

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THEOREM 4.2.2

Proof Let W be the intersection of the subspaces $W_1, W_2, \ldots, W_r$. This set is not empty because each of these subspaces contains the zero vector of V, and hence so does their intersection. Thus, it remains to show that W is closed under addition and scalar multiplication.

To prove closure under addition, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in W. Since W is the intersection of $W_1, W_2, \ldots, W_r$, it follows that $\mathbf{u}$ and $\mathbf{v}$ also lie in each of these subspaces. Moreover, since these subspaces are closed under addition and scalar multiplication, they also all contain the vectors $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for every scalar k, and hence so does their intersection W. This proves that W is closed under addition and scalar multiplication. ■

Solution Spaces of Homogeneous Systems

THEOREM 4.2.3

The solution set of a homogeneous system $A\mathbf{x} = \mathbf{0}$ of m equations in n unknowns is a subspace of R^n.

Proof

[Note]

Because the solution set of a homogeneous system in n unknowns is actually a subspace of R^n, we will generally refer to it as the *solution space* of the system.

Solution Spaces of Homogeneous Systems

EXAMPLE 10 Give a geometric description of the solution set.

(a) $\begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ 3 & -6 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ -2 & 4 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 \\ -3 & 7 & -8 \\ 4 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

EXAMPLE 10 cont.

(a) The solutions are $x = 2s - 3t, \quad y = s, \quad z = t$

it follows that $x = 2y - 3z$ or $x - 2y + 3z = 0$

This is a plane through the origin that has $\mathbf{n} = (1, -2, 3)$ as a normal.

(b) The solutions are $x = -5t, \quad y = -t, \quad z = t$

A parametric equations for the line through the origin that is parallel to the vector $\mathbf{v} = (-5, -1, 1)$.

(c) The only solution is $x = 0, y = 0, z = 0$,

so the solution space consists of the single point $\{\mathbf{0}\}$.

(d) This linear system is satisfied by all real values of x, y, and z, so the solution space is all of R^3.

The Linear Transformation Viewpoint

The solution space of $A\mathbf{x} = \mathbf{0}$ is the set of vectors in R^n, that T_A maps into the zero vector in R^m. This set is sometimes called the *kernel* of the transformation.

THEOREM 4.2.4.

If A is an $m \times n$ matrix, then the kernel of the matrix transformation $T_A : R^n \to R^m$ is a subspace of R^n.

Chapter 4-2 Objectives

- Determine whether a subset of a vector space is a subspace.
- Show that a subset of a vector space is a subspace.
- Show that a nonempty subset of a vector space is not a subspace by demonstrating that the set is either not closed under addition or not closed under scalar multiplication.