Chapter 2
Determinants

2.1. Determinants by Cofactor Expansion

2.2. Evaluating Determinants by Row Reduction

2.3. Properties of Determinants; Cramer's Rule

Chapter 2.3

Properties of Determinants; Cramer's Rule

Goals of This Chapter

- Computing the inverse of a nonsingular matrix using determinants.
- Solving $A\mathbf{x} = \mathbf{b}$ using determinants

Cramer's Rule

Basic Properties of Determinants

Given $\det(A)$ and $\det(B)$, aim to find out

$$\det(kA), \quad \det(A+B), \quad \text{and} \quad \det(AB)$$

A is an n x n matrix. It is easily understandable that

$$\det(kA) = k^n \det(A)$$

For example

$$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{vmatrix} = k^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Basic Properties of Determinants

But, in general

$$\det(A+B) \neq \det(A)+\det(B)$$

EXAMPLE 1 $\det(A+B) \neq \det(A)+\det(B)$

$$A=\begin{bmatrix}1 & 2\\ 2 & 5\end{bmatrix},\quad B=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix},\quad A+B=\begin{bmatrix}4 & 3\\ 3 & 8\end{bmatrix}$$

$\det(A)=1$, $\det(B)=8$, and $\det(A+B)=23$; thus

$$\det(A+B) \neq \det(A)+\det(B)$$

Basic Properties of Determinants

THEOREM 2.3.1

Let A, B, and C be $n \times n$ matrices that differ only in a single row, say the rth, and assume that the rth row of C can be obtained by adding corresponding entries in the rth rows of A and B.

Then

$$\det(C) = \det(A) + \det(B)$$

The same result holds for columns.

Basic Properties of Determinants

EXAMPLE 2 Sums of Determinants

$$\det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1+0 & 4+1 & 7+(-1) \end{bmatrix} = \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & 7 \end{bmatrix} + \det\begin{bmatrix} 1 & 7 & 5 \\ 2 & 0 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

Verify !

exe

Determinant of a Matrix Product

Next, we will show that if A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B)$$

Before that, lets first consider a special case

LEMMA 2.3.2

If B is an $n \times n$ matrix and E is an $n \times n$ elementary matrix, then

$$\det(EB) = \det(E)\det(B)$$

Proof [Note] Hint: use Theorem 2.2.3 & Theorem 2.2.4

Recalls:

$\begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(EB) = k\det(B)$	The first row of B is multiplied by k.
$\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = - \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(EB) = -\det(B)$	The first and second rows of B are interchanged.
$\begin{vmatrix} a_{11}+ka_{21} & a_{12}+ka_{22} & a_{13}+ka_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ $\det(EB) = \det(B)$	A multiple of the second row of B is added to the first row.

THEOREM 2.2.4

Let E be an $n \times n$ elementary matrix.

(a) If E results from multiplying a row of I_n by a nonzero number k, then $\det(E) = k$.

(b) If E results from interchanging two rows of I_n, then $\det(E) = -1$.

(c) If E results from adding a multiple of one row of I_n to another, then $\det(E) = 1$.

Determinant of a Matrix Product

It follows by repeated applications of Lemma 2.3.2 that if B is an $n \times n$ matrix and $E_1, E_2, \ldots, E_r$ are $n \times n$ elementary matrices, then

$$\det(E_1 E_2 \cdots E_r B) = \det(E_1)\det(E_2)\cdots\det(E_r)\det(B)$$

One more theorem before go into

$$\det(AB) = \det(A)\det(B)$$

Determinant Test for Invertibility

THEOREM 2.3.3

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof [Note]

$\Rightarrow$ Let R be the reduced row echelon form of A

$$R = E_r \cdots E_2 E_1 A$$

$$\det(R) = \det(E_r) \cdots \det(E_2)\det(E_1)\det(A)$$

If we assume first that A is invertible,
then it follows from Theorem 1.6.4 that $R = I$
and hence that $\det(R) = 1$ $(\neq 0)$.

This implies that $\det(A) \neq 0$.

Determinant Test for Invertibility

THEOREM 2.3.3

A square matrix A is invertible if and only if $\det(A) \neq 0$.

Proof [Note]

$\Leftarrow$

Assume that $\det(A) \neq 0$.
It follows from this that $\det(R) \neq 0$, which tells us that R cannot have a row of zeros.
Thus, it follows from Theorem 1.4.3 that $R = I$ and hence that A is invertible by Theorem 1.6.4.

Determinant of a Matrix Product

Now we are ready

THEOREM 2.3.4

If A and B are square matrices of the same size, then

$$\det(AB) = \det(A)\det(B)$$

Proof

[Note]

EXAMPLE 4 Verifying That det(*AB*) = det(*A*), det(*B*)

Consider the matrices

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 3 \\ 5 & 8 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 17 \\ 3 & 14 \end{bmatrix}$$

We leave it for you to verify that

$$\det(A) = 1, \quad \det(B) = -23, \quad \text{and} \quad \det(AB) = -23$$

Thus $\det(AB) = \det(A)\det(B)$, as guaranteed by Theorem 2.3.4.

exe

Determinant of a Matrix Inverse

THEOREM 2.3.5

If A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof

Since $A^{-1}A = I$, it follows that $\det(A^{-1}A) = \det(I)$.

Therefore, we must have $\det(A^{-1})\det(A) = 1$.

Since $\det(A) \neq 0$, the proof can be completed by dividing through by $\det(A)$.

Adjoint of a Matrix

DEFINITION 1

If A is any $n \times n$ matrix and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}$$

is called the *matrix of cofactors from A*. The *transpose* of this matrix is called the *adjoint of A* and is denoted by adj(A).

Adjoint of a Matrix

EXAMPLE 5 Adjoint of a 3 × 3 Matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

The cofactors of A are

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = 4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

matrix of cofactors is $\begin{bmatrix} 12 & 6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$

adjoint of A is

$$\operatorname{adj}(A) = \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$$

Inverse of a Matrix Using Its Adjoint

THEOREM 2.3.6

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

Before proving this theorem, lets take a look at

EXAMPLE 6

EXAMPLE 6 Entries and Cofactors from Different Rows

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

$$\begin{array}{lll} C_{11} = 12 & C_{12} = 6 & C_{13} = -16 \\ C_{21} = \;\;4 & C_{22} = 2 & C_{23} = 16 \\ C_{31} = 12 & C_{32} = -10 & C_{33} = 16 \end{array}$$

first row $\quad \det(A) = 3C_{11} + 2C_{12} + (-1)C_{13} = 36 + 12 + 16 = 64$

first column $\quad \det(A) = 3C_{11} + C_{21} + 2C_{31} = 36 + 4 + 24 = 64$

However, if we multiply the entries in the first row by the corresponding cofactors from the second row and add the resulting products.

The result is

$$3C_{21} + 2C_{22} + (-1)C_{23} = 12 + 4 - 16 = 0$$

Or we multiply the entries in the first column by the corresponding cofactors from the second column and add the resulting products. The result is again zero since

$$3C_{12} + 1C_{22} + 2C_{32} = 18 + 2 - 20 = 0$$

Inverse of a Matrix Using Its Adjoint

From the result of **EXAMPLE 6**

we may conclude that

if $i \neq j$

then $$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn} = 0$$

Inverse of a Matrix Using Its Adjoint

Back to proof of **THEOREM 2.3.6**

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$$

Proof We show first that $A\operatorname{adj}(A) = \det(A)I$

Consider the product

$$A\operatorname{adj}(A) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \dots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} & \dots & C_{j1} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{j2} & \dots & C_{n2} \\ \vdots & \vdots & & \vdots & & \vdots \\ C_{1n} & C_{2n} & \dots & C_{jn} & \dots & C_{nn} \end{bmatrix}$$

The entry in the ith row and jth column of the product $A\operatorname{adj}(A)$ is

$$a_{i1}C_{j1} + a_{i2}C_{j2} + \cdots + a_{in}C_{jn}$$

THEOREM 2.3.6 *Proof*

Because $a_{i1}C_{j1} + a_{i1}C_{j1} + \ldots + a_{i1}C_{j1} = \begin{cases} \det(A) & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

Therefore,

$$A\,\mathrm{adj}(A) = \begin{bmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \det(A) \end{bmatrix} = \det(A)I$$

Since A is invertible, $\det(A) \neq 0$.

$$\frac{1}{\det(A)}[A\,\mathrm{adj}(A)] = I \quad \text{or} \quad A\left[\frac{1}{\det(A)}\mathrm{adj}(A)\right] = I$$

Multiplying both sides on the left by A^{-1} yields

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$$

Inverse of a Matrix Using Its Adjoint

EXAMPLE 7 Using the Adjoint to Find an Inverse Matrix

find the inverse of the matrix A in Example 6.

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{bmatrix}$$

Solution We showed in Example 6 that $\det(A) = 64$. Thus,

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{64}\begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix} = \begin{bmatrix} \frac{12}{64} & \frac{4}{64} & \frac{12}{64} \\ \frac{6}{64} & \frac{2}{64} & -\frac{10}{64} \\ -\frac{16}{64} & \frac{16}{64} & \frac{16}{64} \end{bmatrix}$$

Cramer's Rule

THEOREM 2.3.7

If $A\mathbf{x} = \mathbf{b}$ is a system of n linear equations in n unknowns such that $\det(A) \neq 0$, then the system has a unique solution.
This solution is

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where A_j is the matrix obtained by replacing the entries in the jth column of A by the entries in the matrix

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Cramer's Rule

THEOREM 2.3.7

Proof

If $\det(A) \neq 0$, then A is invertible, and by Theorem 1.6.2, $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$.

Therefore, by Theorem 2.3.6 we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{\det(A)}\operatorname{adj}(A)\mathbf{b} = \frac{1}{\det(A)}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

THEOREM 2.3.7 *Proof*

Multiplying the matrices out gives

$$\mathbf{x} = \frac{1}{\det(A)}\begin{bmatrix} b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2} \\ \vdots \qquad \vdots \qquad \vdots \\ b_1C_{1n} + b_2C_{2n} + \cdots + b_nC_{nn} \end{bmatrix}$$

The entry in the *j*th row of **x** is therefore

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \cdots + b_nC_{nj}}{\det(A)}$$

Now let

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j-1} & b_1 & a_{1j+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j-1} & b_2 & a_{2j+1} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj-1} & b_n & a_{nj+1} & \cdots & a_{nn} \end{bmatrix}$$

THEOREM 2.3.7 *Proof*

Since A_j differs from A only in the jth column, it follows that the cofactors of entries $b_1, b_2, \ldots, b_n$ in A_j are the same as the cofactors of the corresponding entries in the jth column of A.

The cofactor expansion of $\det(A_j)$ along the jth column is therefore

$$\det(A_j) = b_1 C_{1j} + b_2 C_{2j} + \cdots + b_n C_{nj}$$

Thus

$$x_j = \frac{\det(A_j)}{\det(A)}$$

EXAMPLE 8 Using Cramer's Rule to Solve a Linear System

$$\begin{aligned} x_1 + + 2x_3 &= 6 \\ -3x_1 + 4x_2 + 6x_3 &= 30 \\ -x_1 - 2x_2 + 3x_3 &= 8 \end{aligned}$$

Solution

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix}$$

Therefore,

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-40}{44} = \frac{-10}{11}, \quad x_2 = \frac{\det(A_2)}{\det(A)} = \frac{72}{44} = \frac{18}{11},$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{152}{44} = \frac{38}{11}$$

Equivalence Theorem

THEOREM 2.3.8

If A is an $n \times n$ matrix, then the following statements are equivalent.

(*a*) A is invertible.

(*b*) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(*c*) The reduced row echelon form of A is I_n.

(*d*) A can be expressed as a product of elementary matrices.

(*e*) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

(*f*) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

(g) $\det(A) \neq 0$.

Chapter 2-3 Objectives

- Know how determinants behave with respect to basic arithmetic operations, as given in Equation 1,Theorem 2.3.1, Lemma 2.3.2, and Theorem 2.3.4.
- Use the determinant to test a matrix for invertibility.
- Know how det(A) and det(A^{-1}) are related.
- Compute the matrix of cofactors for a square matrix A.
- Compute det(A) for a square matrix A.
- Use the adjoint of an invertible matrix to find its inverse.
- Use Cramer's rule to solve linear systems of equations.
- Know the equivalent characterizations of an invertible matrix given in Theorem 2.3.8.