Chapter 1

Systems of Linear Equations and Matrices

Chapter 1.4

Inverses; Algebraic Properties of Matrices

Arithmetic Properties of Matrix Operations

THEOREM 1.4.1

(a) $A + B = B + A$

(b) $A + (B + C) = (A + B) + C$

(c) $A(BC) = (AB)C$

(d) $A(B + C) = AB + AC$

(e) $(B + C)A = BA + CA$

(f) $A(B - C) = AB - AC$

(g) $(B - C)A = BA - CA$

(h) $a(B + C) = aB + aC$

(i) $a(B - C) = aB - aC$

(j) $(a + b)C = aC + bC$

(k) $(a - b)C = aC - bC$

(l) $a(bC) = (ab)C$

(m) $a(BC) = (aB)C = B(aC)$

In general $AB \neq BA$

Properties of Matrix Multiplication

1. *AB* may be defined and *BA* may not
(for example, if *A* is 2 x 3 and *B* is 3 x 4).

2. *AB* and *BA* may both be defined, but they may have different sizes
(for example, if *A* is 2 x 3 and *B* is 3 x 2).

3. *AB* and *BA* may both be defined and have the same size, but the two matrices may be different (see example 1)

4. If it happens that $AB = BA$, then we say that *AB* and *BA* *commute*.

Properties of Matrix Multiplication

EXAMPLE 1

Consider the matrices

$$A = \begin{bmatrix} -1 & 0 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}$$

Multiplying gives

$$AB = \begin{bmatrix} -1 & -2 \\ 11 & 4 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 3 & 6 \\ -3 & 0 \end{bmatrix}$$

Thus, $AB \neq BA$.

Zero Matrices

A matrix whose entries are all zero is called a *zero matrix.*

Some examples are

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [0]$$

We will denote a zero matrix by *0*

Properties of Zero Matrices

THEOREM 1.4.2

If c is a scalar, and if the sizes of the matrices are such that the operations can be performed, then:

(a) $A + 0 = 0 + A = A$

(b) $A - 0 = A$

(c) $A - A = A + (-A) = 0$

(d) $0A = 0$

(e) If $cA = 0$, then $c = 0$ or $A = 0$.

When a, b, c are real numbers, consider the followings:

If $ab = ac$ and $a \neq 0$, then $b = c$.

[The cancellation law]

If $ab = 0$, then at least one of the factors on the left is 0.

How about matrices?

Properties of Zero Matrices

EXAMPLE 2 Failure of the Cancellation Law

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix}$$

$$AB = AC = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}$$

Although $A \neq 0$, canceling A from both sides of the equation $AB = AC$ would lead to the incorrect conclusion that $B = C$.

Properties of Zero Matrices

EXAMPLE 3 A Zero Product with Nonzero Factors

Here are two matrices for which $AB = 0$, but $A \neq 0$ and $B \neq 0$:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 7 \\ 0 & 0 \end{bmatrix}$$

Identity Matrices

A square matrix with 1's on the main diagonal and zeros elsewhere is called an *identity matrix*.

Some examples are

$$[1], \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

An identity matrix is denoted by the letter I *or* I_n

Properties of Identity Matrices

if A is any $m \times n$ matrix, then

$$\boxed{AI_n = A \quad \text{and} \quad I_m A = A}$$

Reduced Row Echelon Form

THEOREM 1.4.3

If R is the reduced row echelon form of an $n \times n$ matrix A, then either R has a row of zeros or R is the identity matrix I_n.

Proof Suppose that the reduced row echelon form of A is

$$R = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{bmatrix}$$

Either R has n leading 1's $\longrightarrow$ I_n

or less than n leading 1's $\longrightarrow$ rows of zeros

Inverse of a Matrix

DEFINITION 1

If A is a square matrix, and if a matrix B of the same size can be found such that $AB = BA = I$, then A is said to be *invertible* (or *nonsingular*) and B is called an *inverse* of A.

A and B are *inverses of one another*.

If no such matrix B can be found, then A is said to be *singular*.

Inverse of a Matrix

EXAMPLE 4 An Invertible Matrix

$$A = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$BA = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Thus, A and B are invertible and each is an inverse of the other.

Singular Matrix

EXAMPLE 5 Class of Singular Matrices

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

To prove that A is singular we must show that there is no 3×3 matrix B such that

$$AB = BA = I$$

For this purpose let $\mathbf{c}_1$, $\mathbf{c}_2$, $\mathbf{0}$ be the column vectors of A.

$$BA = B[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}] = [B\mathbf{c}_1 \quad B\mathbf{c}_2 \quad \mathbf{0}]$$

The column of zeros shows that $BA \neq I$ and hence that A is singular.

Properties of Inverses

THEOREM 1.4.4

If B and C are both inverses of the matrix A, then $B = C$.

Proof

Since B is an inverse of A, we have $BA = I$.

Multiplying both sides on the right by C gives
$(BA)C = IC = C$.

But it is also true that $(BA)C = B(AC) = BI = B$, so $C = B$.

Notation of Inverses

If A is invertible, then its inverse will be denoted by the symbol A^{-1}.

Thus,

$$\boxed{AA^{-1} = I \quad \text{and} \quad A^{-1}A = I}$$

Inverse of a 2x2 Matrix

THEOREM 1.4.5

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Properties of the Matrix Inverse

THEOREM 1.4.6

If A and B are invertible matrices with the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Proof

by showing that

$$(AB)(B^{-1}A^{-1}) = (B^{-1}A^{-1})(AB) = I$$

But

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly, $(B^{-1}A^{-1})(AB) = I$.

Properties of the Matrix Inverse

A product of any number of invertible matrices is invertible, and the inverse of the product is the product of the inverses in the reverse order.

EXAMPLE 6 The Inverse of a Product

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix} \qquad AB = \begin{bmatrix} 7 & 6 \\ 9 & 8 \end{bmatrix}, \quad (AB)^{-1} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix},$$

$$B^{-1}A^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -\frac{9}{2} & \frac{7}{2} \end{bmatrix}$$

Thus, $(AB)^{-1} = B^{-1}A^{-1}$

Powers of a Matrix

If A is a square matrix, then we define the nonnegative integer powers of A to be

$$A^0 = I \quad \text{and} \quad A^n = AA\cdots A \quad [n \text{ factors}]$$

and if A is invertible, then we define the negative integer powers of A to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1} \quad [n \text{ factors}]$$

laws of nonnegative exponents hold;

$$A^rA^s = A^{r+s} \quad \text{and} \quad (A^r)^s = A^{rs}$$

Properties of the Matrix Inverse

THEOREM 1.4.7

If A is invertible and n is a nonnegative integer, then:

(a) A^{-1} is invertible and $(A^{-1})^{-1} = A$.
(b) A^n is invertible and $(A^n)^{-1} = A^{-n} = (A^{-1})^n$.
(c) kA is invertible for any nonzero scalar k, and $(kA)^{-1} = k^{-1}A^{-1}$.

Proof

[Note]

Properties of the Matrix Inverse

EXAMPLE 7 Properties of Exponents

exe

EXAMPLE 8 The Square of a Matrix Sum

$$(A+B)^2 = A^2 + AB + BA + B^2$$

It is only in the special case where A and B commute (i.e., $AB = BA$) that we can write

$$(A+B)^2 = A^2 + 2AB + B^2$$

EXAMPLE 7 Properties of Exponents

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

$$A^{-3} = (A^{-1})^3 = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 30 \\ 15 & 41 \end{bmatrix}$$

$$(A^3)^{-1} = \frac{1}{(11)(41) - (30)(15)}\begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = \begin{bmatrix} 41 & -30 \\ -15 & 11 \end{bmatrix} = (A^{-1})^3$$

Matrix Polynomials

If A is a square matrix, say $n \times n$, and if

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

is any polynomial, then we define the $n \times n$ matrix $p(A)$ to be

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_m A^m$$

$p(A)$ is called a *matrix polynomial in A*.

Matrix Polynomials

EXAMPLE 9 A Matrix Polynomial

Find $p(A)$ for $\quad p(x) = x^2 - 2x - 5 \quad$ and $\quad A = \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}$

$$\begin{aligned} p(A) &= A^2 - 2A - 5I \\ &= \begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix}^2 - 2\begin{bmatrix} -1 & 2 \\ 1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 2 & 11 \end{bmatrix} - \begin{bmatrix} -2 & 4 \\ 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

or more briefly, $p(A) = 0$.

Properties of the Matrix Transpose

THEOREM 1.4.8

$$(a)\quad (A^T)^T = A$$
$$(b)\quad (A+B)^T = A^T + B^T$$
$$(c)\quad (A-B)^T = A^T - B^T$$
$$(d)\quad (kA)^T = kA^T$$
$$(e)\quad (AB)^T = B^T A^T$$

The transpose of a product of any number of matrices is the product of the transposes in the reverse order.

Properties of the Matrix Transpose

THEOREM 1.4.9

If A is an invertible matrix, then A^T is also invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

Proof

[Note]

Properties of the Matrix Transpose

EXAMPLE 10 Inverse of a Transpose

Consider a general 2×2 invertible matrix and its transpose:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Verify $\quad (A^T)^{-1} = (A^{-1})^T$

exe

Chapter 1-4 Objectives

- Know the arithmetic properties of matrix operations.
- Be able to prove arithmetic properties of matrices.
- Know the properties of zero matrices.
- Know the properties of identity matrices.
- Be able to recognize when two square matrices are inverses of each other.
- Be able to determine whether a 2 x 2 matrix is invertible.
- Be able to solve a linear system of two equations in two unknowns whose coefficient matrix is invertible.

Chapter 1-4 Objectives

- Be able to prove basic properties involving invertible matrices.
- Know the properties of the matrix transpose and its relationship with invertible matrices.