2-1 True-False Exercises

a. The determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad + bc$. ✗

b. Two square matrices that have the same determinant must have the same size. ✗

c. The minor M_{ij} is the same as the cofactor C_{ij} if $i + j$ is even. ○

d. If A is a 3×3 symmetric matrix, then $C_{ij} = C_{ji}$ for all i and j. ○

e. The number obtained by a cofactor expansion of a matrix A is independent of the row or column chosen for the expansion. ○

f. If A is a square matrix whose minors are all zero, then $\det(A) = 0$. ○

g. The determinant of a lower triangular matrix is the sum of the entries along the main diagonal. ×

h. For every square matrix A and every scalar c, it is true that $\det(cA) = c\det(A)$. ×

i. For all square matrices A and B, it is true that ×

$$\det(A + B) = \det(A) + \det(B)$$

j. For every 2×2 matrix A it is true that ○

$$\det(A^2) = (\det(A))^2$$

(a) False. The determinant is $ad-bc$.

(b) False. E.g., $\det(I_2)=\det(I_3)=1$.

(c) True. If $i+j$ is even then $(-1)^{i+j}=1$ therefore $C_{ij}=(-1)^{i+j}M_{ij}=M_{ij}$.

(d) True. Let $A=\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$.

Then $C_{12}=(-1)^{1+2}\begin{vmatrix} b & e \\ c & f \end{vmatrix}=-(bf-ec)$ and $C_{21}=(-1)^{2+1}\begin{vmatrix} b & c \\ e & f \end{vmatrix}=-(bf-ce)$ therefore $C_{12}=C_{21}$. In the same way, one can show $C_{13}=C_{31}$ and $C_{23}=C_{32}$.

(e) True. This follows from Theorem 2.1.1.

(f) True. In formulas (7) and (8), each cofactor C_{ij} is zero.

(g) False. The determinant of a lower triangular matrix is the *product* of the entries along the main diagonal.

(h) False. E.g. $\det(2I_2)=4\neq 2=2\det(I_2)$.

(i) False. E.g., $\det(I_2+I_2)=4\neq 2=\det(I_2)+\det(I_2)$.

(j) True. $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right)=\begin{vmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{vmatrix}=(a^2+bc)(bc+d^2)-(ab+bd)(ac+cd)$

$=a^2bc+a^2d^2+b^2c^2+bcd^2-a^2bc-abcd-abcd-bcd^2=a^2d^2+b^2c^2-2abcd$.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}^2=(ad-bc)^2=a^2d^2-2adbc+b^2c^2$ therefore $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2\right)=\left(\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)^2$.

2-2 True-False Exercises

a. If A is a 4×4 matrix and B is obtained from A by interchanging the first two rows and then interchanging the last two rows, then $\det(B) = \det(A)$. ○

b. If A is a 3×3 matrix and B is obtained from A by multiplying the first column by 4 and multiplying the third column by $\frac{3}{4}$, then $\det(B) = 3\det(A)$. ○

c. If A is a 3×3 matrix and B is obtained from A by adding 5 times the first row to each of the second and third rows, then $\det(B) = 25\det(A)$. ×

d. If A is an $n \times n$ matrix and B is obtained from A by multiplying each row of A by its row number, then

$$\det(B) = \frac{n(n+1)}{2}\det(A)$$

e. If A is a square matrix with two identical columns, then $\det(A) = 0$.

f. If the sum of the second and fourth row vectors of a 6×6 matrix A is equal to the last row vector, then $\det(A) = 0$.

(a) True. $\det(B)=(-1)(-1)\det(A)=\det(A)$.

(b) True. $\det(B)=(4)(\tfrac{3}{4})\det(A)=3\det(A)$.

(c) False. $\det(B)=\det(A)$.

(d) False. $\det(B)=n(n-1)\cdots 3\cdot 2\cdot 1\cdot\det(A)=(n!)\det(A)$.

(e) True. This follows from Theorem 2.2.5.

(f) True. Let B be obtained from A by adding the second row to the fourth row, so $\det(A)=\det(B)$. Since the fourth row and the sixth row of B are identical, by Theorem 2.2.5 $\det(B)=0$.

2-3 True-False Exercises

a. If A is a 3×3 matrix, then $\det(2A) = 2\det(A)$. ✗

b. If A and B are square matrices of the same size such that $\det(A) = \det(B)$, then $\det(A + B) = 2\det(A)$. ✗

c. If A and B are square matrices of the same size and A is invertible, then

$$\det(A^{-1}BA) = \det(B)$$

○

d. A square matrix A is invertible if and only if $\det(A) = 0$. ✗

e. The matrix of cofactors of A is precisely $[\operatorname{adj}(A)]^T$. ○

f. For every $n \times n$ matrix A, we have

$$A \cdot \operatorname{adj}(A) = (\det(A))I_n$$

g. If A is a square matrix and the linear system $A\mathbf{x} = \mathbf{0}$ has multiple solutions for $\mathbf{x}$, then $\det(A) = 0$.

h. If A is an $n \times n$ matrix and there exists an $n \times 1$ matrix $\mathbf{b}$ such that the linear system $A\mathbf{x} = \mathbf{b}$ has no solutions, then the reduced row echelon form of A cannot be I_n.

i. If E is an elementary matrix, then $E\mathbf{x} = \mathbf{0}$ has only the trivial solution.

j. If A is an invertible matrix, then the linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution if and only if the linear system $A^{-1}\mathbf{x} = \mathbf{0}$ has only the trivial solution.

k. If A is invertible, then $\text{adj}(A)$ must also be invertible. ○

l. If A has a row of zeros, then so does $\text{adj}(A)$. ×

(a) False. By Formula (1), $\det(2A) = 2^3 \det(A) = 8\det(A)$.

(b) False. E.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have $\det(A) = \det(B) = 0$ but $\det(A+B) = 1 \neq 2\det(A)$.

(c) True. By Theorems 2.3.4 and 2.3.5,
$\det(A^{-1}BA) = \det(A^{-1})\det(B)\det(A) = \frac{1}{\det(A)}\det(B)\det(A) = \det(B)$.

(d) False. A square matrix A is invertible if and only if $\det(A) \neq 0$.

(e) True. This follows from Definition 1.

(f) True. This is Formula (8).

(g) True. If $\det(A) \neq 0$ then by Theorem 2.3.8 $A\mathbf{x} = 0$ must have only the trivial solution, which contradicts our assumption. Consequently, $\det(A) = 0$.

(h) True. If the reduced row echelon form of A is I_n then by Theorem 2.3.8 $A\mathbf{x}=\mathbf{b}$ is consistent for every $\mathbf{b}$, which contradicts our assumption. Consequently, the reduced row echelon form of A cannot be I_n.

(i) True. Since the reduced row echelon form of E is I then by Theorem 2.3.8 $E\mathbf{x}=0$ must have only the trivial solution.

(j) True. If A is invertible, so is A^{-1}. By Theorem 2.3.8, each system has only the trivial solution.

(k) True. From Theorem 2.3.6, $A^{-1}=\frac{1}{\det(A)}\operatorname{adj}(A)$ therefore $\operatorname{adj}(A)=\det(A)A^{-1}$. Consequently,

$\left(\frac{1}{\det(A)}A\right)\operatorname{adj}(A)=\left(\frac{1}{\det(A)}A\right)\left(\det(A)\,A^{-1}\right)=\frac{\det(A)}{\det(A)}\left(AA^{-1}\right)=I_n$ so $\left(\operatorname{adj}(A)\right)^{-1}=\frac{1}{\det(A)}A$.

(l) False. If the kth row of A contains only zeros then all cofactors C_{jk} where $j\neq i$ are zero (since each of them involves a determinant of a matrix with a zero row). This means the matrix of cofactors contains at least one zero row, therefore $\operatorname{adj}(A)$ has a *column* of zeros.