1-6 In-Class Exercise

1. Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

$$\begin{aligned} 6x_1 - 4x_2 &= b_1 \\ 3x_1 - 2x_2 &= b_2 \end{aligned}$$

$$\left[\begin{array}{cc|c} 6 & -4 & b_1 \\ 3 & -2 & b_2 \end{array}\right] \longleftarrow \text{The augmented matrix for the system.}$$

$$\left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{1}{6}b_1 \\ 3 & -2 & b_2 \end{array}\right] \longleftarrow \text{The first row was multiplied by } \tfrac{1}{6}.$$

$$\left[\begin{array}{cc|c} 1 & -\frac{2}{3} & \frac{1}{6}b_1 \\ 0 & 0 & -\frac{1}{2}b_1 + b_2 \end{array}\right] \longleftarrow -3 \text{ times the first row was added to the second row.}$$

The system is consistent if and only if $-\frac{1}{2}b_1 + b_2 = 0$, i.e. $b_1 = 2b_2$.

1-6 Suggested Exercise

1. Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

$$\begin{aligned}
x_1 - x_2 + 3x_3 + 2x_4 &= b_1 \\
-2x_1 + x_2 + 5x_3 + x_4 &= b_2 \\
-3x_1 + 2x_2 + 2x_3 - x_4 &= b_3 \\
4x_1 - 3x_2 + x_3 + 3x_4 &= b_4
\end{aligned}$$

$$\left[\begin{array}{rrrr|l} 1 & -1 & 3 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right] \longleftarrow \text{The augmented matrix for the system.}$$

$$\left[\begin{array}{rrrr|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & -1 & 11 & 5 & 2b_1 + b_2 \\ 0 & -1 & 11 & 5 & 3b_1 + b_3 \\ 0 & 1 & -11 & -5 & -4b_1 + b_4 \end{array}\right]$$

$\leftarrow$ 2 times the first row was added to the second row, 3 times the first row was added to the third row, and -4 times the first row was added to the fourth row.

$$\left[\begin{array}{rrrr|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1 - b_2 \\ 0 & -1 & 11 & 5 & 3b_1 + b_3 \\ 0 & 1 & -11 & -5 & -4b_1 + b_4 \end{array}\right]$$

$\leftarrow$ The second row was multiplied by -1.

$$\left[\begin{array}{rrrr|c} 1 & -1 & 3 & 2 & b_1 \\ 0 & 1 & -11 & -5 & -2b_1 - b_2 \\ 0 & 0 & 0 & 0 & b_1 - b_2 + b_3 \\ 0 & 0 & 0 & 0 & -2b_1 + b_2 + b_4 \end{array}\right]$$

$\leftarrow$ The second row was added to the third row and -1 times the second row was added to the fourth row.

The system is consistent for all values of b_1, b_2, b_3, and b_4 that satisfy the equations

$b_1 - b_2 + b_3 = 0$ and $-2b_1 + b_2 + b_4 = 0$.

2. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

a) Show that the equation $A\mathbf{x} = \mathbf{x}$ can be rewritten as $(A - I)\mathbf{x} = \mathbf{0}$ and use this result to solve $A\mathbf{x} = \mathbf{x}$ for $\mathbf{x}$.

b) Solve $A\mathbf{x} = 4\mathbf{x}$.

a)

The equation $A\text{x} = \text{x}$ can be rewritten as $A\text{x} = I\text{x}$, which yields $A\text{x} - I\text{x} = 0$ and $(A - I)\text{x} = 0$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 2 & 1 & -2 & 0 \\ 3 & 1 & 0 & 0 \end{array}\right] \longleftarrow$$

The augmented matrix for the homogeneous system $(A - I)\text{x} = 0$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & -1 & -6 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right] \longleftarrow$$

-2 times the first row was added to the second row and -3 times the first row was added to the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & -2 & -6 & 0 \end{array}\right] \longleftarrow$$

The second row was multiplied by -1.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \longleftarrow$$

2 times the second row was added to the third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longleftarrow$$

The third row was multiplied by $\frac{1}{6}$.

Using back-substitution, we obtain the unique solution: $x_1 = x_2 = x_3 = 0$.

b)

As was done in part (a), the equation $A\mathrm{x} = 4\mathrm{x}$ can be rewritten as $(A - 4I)\mathrm{x} = 0$.

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$

The augmented matrix for the homogeneous system $(A - 4I)\mathrm{x} = 0$.

$$\left[\begin{array}{ccc|c} 2 & -2 & -2 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$

The first and second rows were interchanged.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ -2 & 1 & 2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right]$$

The first row was multiplied by $\frac{1}{2}$.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

2 times the first row was added to the second row and -3 times the first row was added to the third row.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{array}\right]$$

The second row was multiplied by -1.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

-4 times the second row was added to the third row and the second row was added to the first row.

If we assign x_3 an arbitrary value t, the general solution is given by the formulas

$x_1 = t$, $x_2 = 0$, and $x_3 = t$.

3. Solve the matrix equation for X.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix} X = \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix}. \text{ Let us find } \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1}:$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \longleftarrow$$

The identity matrix was adjoined to the matrix.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$\longleftarrow$ -2 times the first row was added to the second row.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$\longleftarrow$ -2 times the third row was added to the second row.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & -1 & 4 & -2 & 5 \end{array}\right]$$

$\longleftarrow$ -2 times the second row was added to the third row.

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right]$$

$\longleftarrow$ The third row was multiplied by -1 .

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 5 & -2 & 5 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \quad -1 \text{ times the third row was added to the first row.}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & -1 & 3 \\ 0 & 1 & 0 & -2 & 1 & -2 \\ 0 & 0 & 1 & -4 & 2 & -5 \end{array}\right] \longleftarrow \quad \text{The second row was added to the first row.}$$

Using $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & 2 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix}$ we obtain

$$X = \begin{bmatrix} 3 & -1 & 3 \\ -2 & 1 & -2 \\ -4 & 2 & -5 \end{bmatrix} \begin{bmatrix} 2 & -1 & 5 & 7 & 8 \\ 4 & 0 & -3 & 0 & 1 \\ 3 & 5 & -7 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 12 & -3 & 27 & 26 \\ -6 & -8 & 1 & -18 & -17 \\ -15 & -21 & 9 & -38 & -35 \end{bmatrix}$$