Chapter 1

Systems of Linear Equations and Matrices

Chapter 1.8

Introduction to Linear Transformations

Standard Basis Vectors

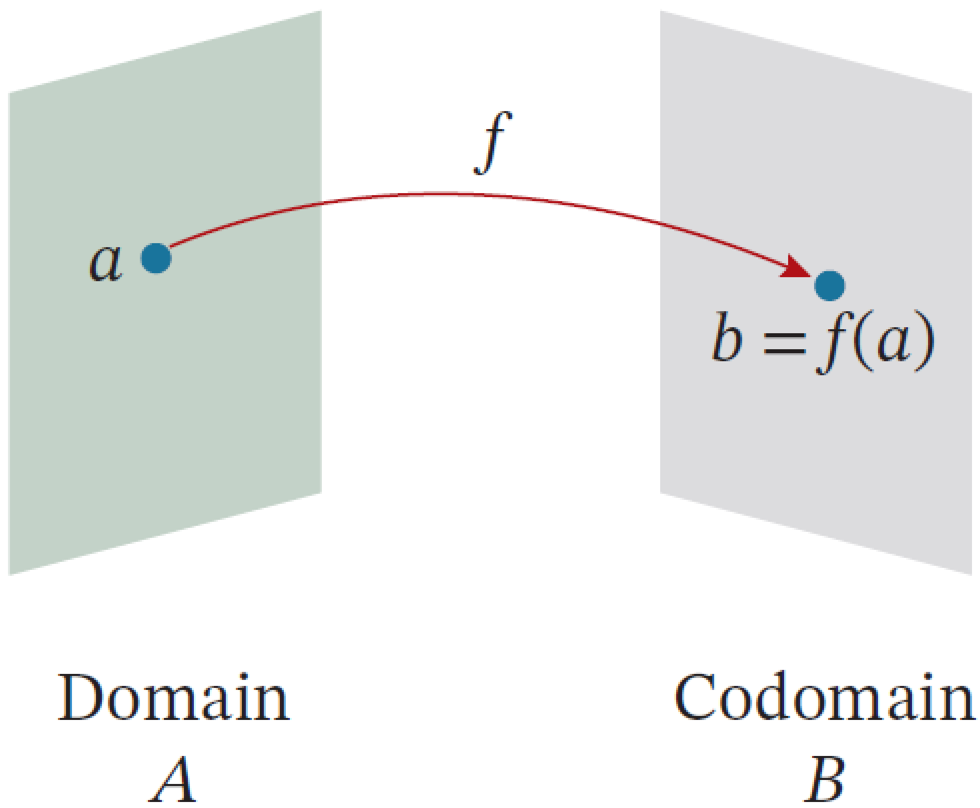
$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \qquad \text{for } R^n$$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \text{for } R^3$$

$$\forall\, \mathbf{x} \in R^n \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$$

Functions and Transformations

Recall



Functions and Transformations

DEFINITION 1

If T is a function with domain R^n and codomain R^m, then we say that T is a *transformation* from R^n to R^m or that T *maps* from R^n to R^m , which we denote by writing

$$T : R^n \rightarrow R^m$$

In the special case when $m = n$, a transformation is sometimes called an *operator* on R^n.

Matrix Transformations

$$
\begin{aligned}
w_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\
w_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\
\vdots \quad & \quad\ \vdots \qquad\quad\ \vdots \qquad\qquad\quad \vdots \\
w_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n
\end{aligned}
$$

$$
\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
$$

$$\mathbf{w} = A\mathbf{x}$$

We call this a *matrix transformation* (or *matrix operator* in the special case where $m = n$).

We denote it by $T_A : R^n \to R^m$

Matrix Transformations

$$\mathbf{w} = A\mathbf{x} \qquad \text{where} \quad \begin{array}{l} \mathbf{x} \in R^n \\ \mathbf{w} \in R^m \end{array}$$

$$T_A : R^n \to R^m$$

$$\mathbf{w} = T_A(\mathbf{x})$$

We also call the transformation T_A *multiplication by A*.

$$\mathbf{x} \xrightarrow{T_A} \mathbf{w}$$

T_A maps $\mathbf{x}$ into $\mathbf{w}$.

Matrix Transformations

EXAMPLE 1

The transformation from R^4 to R^3 defined by the equations

$$
\begin{aligned}
w_1 &= 2x_1 - 3x_2 + x_3 - 5x_4 \\
w_2 &= 4x_1 + x_2 - 2x_3 + x_4 \\
w_3 &= 5x_1 - x_2 + 4x_3
\end{aligned}
$$

$$
\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}
\qquad
A = \begin{bmatrix} 2 & -3 & 1 & -5 \\ 4 & 1 & -2 & 1 \\ 5 & -1 & 4 & 0 \end{bmatrix}
$$

$$\mathbf{w} = A\mathbf{x}$$

Zero Transformations

EXAMPLE 2

If 0 is the $m \times n$ zero matrix, then

$$T_0(\mathbf{x}) = 0\mathbf{x} = \mathbf{0}$$

multiplication by zero maps every vector in R^n into the zero vector in R^m.

We call T_0 the *zero transformation* from R^n to R^m.

Identity Operators

EXAMPLE 3

If I is the $n \times n$ identity matrix, then

$$T_I(\mathbf{x}) = I\mathbf{x} = \mathbf{x}$$

multiplication by I maps every vector in R^n to itself.

We call T_I the *identity operator* on R^n.

Properties of Matrix Transformations

THEOREM 1.8.1

For every matrix A the matrix transformation $T_A : R^n \to R^m$ has the following properties for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for every scalar k :

$$(a)\quad T_A(\mathbf{0}) = \mathbf{0}$$
$$(b)\quad T_A(k\mathbf{u}) = kT_A(\mathbf{u})$$
$$(c)\quad T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$$
$$(d)\quad T_A(\mathbf{u}-\mathbf{v}) = T_A(\mathbf{u}) - T_A(\mathbf{v})$$

Properties of Matrix Transformations

It follows from

$$(b)\quad T_A(k\mathbf{u}) = kT_A(\mathbf{u})$$

$$(c)\quad T_A(\mathbf{u}+\mathbf{v}) = T_A(\mathbf{u}) + T_A(\mathbf{v})$$

that a matrix transformation maps linear combinations of vectors in R^n into the corresponding linear combinations in R^m

$$\boxed{T_A(k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_r\mathbf{u}_r) = k_1T_A(\mathbf{u}_1) + k_2T_A(\mathbf{u}_2) + \cdots + k_rT_A(\mathbf{u}_r)}$$

Matrix Transformations

Q 1: How do we know that a transformation $T : R^n \to R^m$ is a matrix transformation?

Q 2: If we know that a transformation $T : R^n \to R^m$ is a matrix transformation, how can we find a matrix A for which $T = T_A$?

Linearity Conditions

THEOREM 1.8.2

$T : R^n \rightarrow R^m$ is a matrix transformation if and only if the following relationships hold for all vectors $\mathbf{u}$ and $\mathbf{v}$ in R^n and for every scalar k :

$$\text{(i)}\quad T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

$$\text{(ii)}\quad T(k\mathbf{u}) = kT(\mathbf{u})$$

Linear Transformation

The additivity and homogeneity properties in Theorem 1.8.2 are called *linearity conditions*, and a transformation that satisfies these conditions is called a *linear transformation*.

YouTube: Linear transformations and matrices

https://www.youtube.com/watch?v=kYB8IZa5AuE

Linear Transformation

THEOREM 1.8.3

Every linear transformation from R^n to R^m is a matrix transformation and conversely every matrix transformation from R^n to R^m is a linear transformation.

Properties of Matrix Transformations

THEOREM 1.8.4

If $T_A : R^n \rightarrow R^m$ and $T_B : R^n \rightarrow R^m$ are matrix transformations, and if $T_A(\mathbf{x}) = T_B(\mathbf{x})$ for every $\mathbf{x}$ in R^n, then $A = B$.

Standard Matrix for a Matrix Transformation

If $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors for R^n (in column form), then the *standard matrix* for a linear transformation $T : R^n \rightarrow R^m$ is given by

$$A = [\,(T(\mathbf{e}_1) \mid (T(\mathbf{e}_2) \mid \cdots \mid (T(\mathbf{e}_n)\,]$$

Finding a Standard Matrix

EXAMPLE 4 Find the standard matrix A for the linear transformation $T: R^2 \to R^3$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - 3x_2 \\ -x_1 + x_2 \end{bmatrix}$$

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

standard matrix is $$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

transformation $T: R^2 \rightarrow R^3$

$$T(\mathbf{e}_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

$$A = [T(\mathbf{e}_1) \mid T(\mathbf{e}_2)] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}$$

$$A\mathbf{e}_1 = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \qquad A\mathbf{e}_2 = \begin{bmatrix} 2 & 1 \\ 1 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

Standard Matrix

EXAMPLE 5 Computing with Standard Matrices exe

EXAMPLE 6 $T(x_1, x_2) = (3x_1 + x_2, 2x_1 - 4x_2)$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ 2x_1 - 4x_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Thus, the standard matrix is $\begin{bmatrix} 3 & 1 \\ 2 & -4 \end{bmatrix}$

EXAMPLE 5 Computing with Standard Matrices

For the linear transformation in Example 4, use the standard matrix A obtained in that example to find

$$T\left(\begin{bmatrix}1\\4\end{bmatrix}\right)$$

The transformation is multiplication by A, so

$$T\left(\begin{bmatrix}1\\4\end{bmatrix}\right)=\begin{bmatrix}2&1\\1&-3\\-1&1\end{bmatrix}\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}6\\-11\\3\end{bmatrix}$$

Standard Matrix

EXAMPLE 7

Find the standard matrix A for the linear transformation $T : R^2 \to R^2$

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -5 \\ 5 \end{bmatrix}, \quad T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

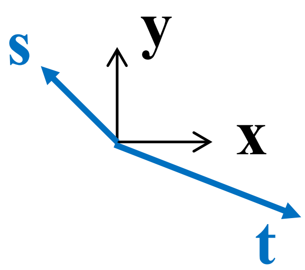
$$\Rightarrow \begin{cases} -a + b = -5 \\ -c + d = 5 \\ 2a - b = 7 \\ 2c - d = -6 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -3 \\ c = -1 \\ d = 4 \end{cases}$$

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

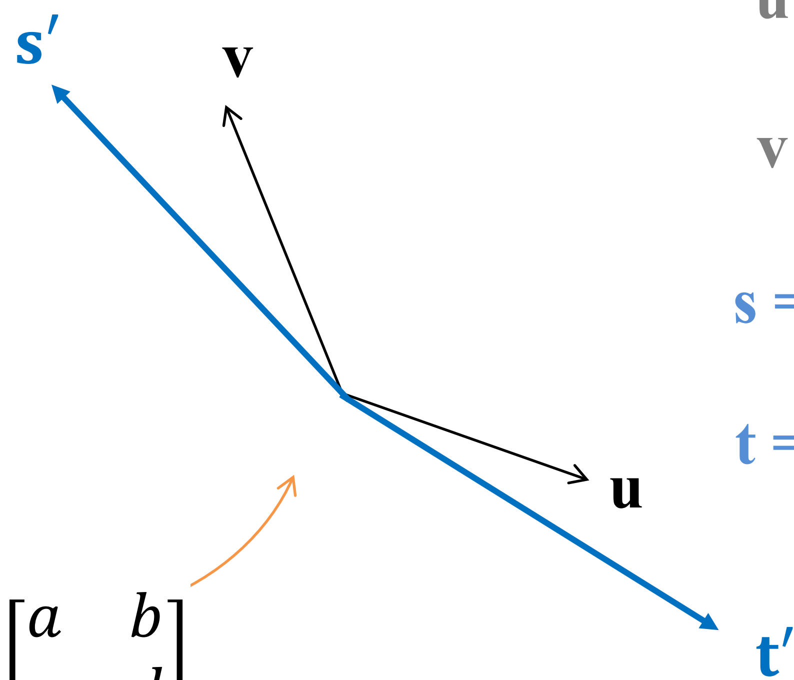
EXAMPLE 7

$$\mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$\mathbf{s} = \begin{bmatrix}-1\\1\end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix}2\\-1\end{bmatrix}$$



$$A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$$



$$\mathbf{u} = \begin{bmatrix}a\\c\end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix}b\\d\end{bmatrix}$$

$$\mathbf{s} = \begin{bmatrix}-5\\5\end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix}7\\-6\end{bmatrix}$$

Given

$$\underbrace{T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)}_{\mathbf{s}} = \underbrace{\begin{bmatrix}-5\\5\end{bmatrix}}_{\mathbf{s}'}, \quad \underbrace{T\left(\begin{bmatrix}2\\-1\end{bmatrix}\right)}_{\mathbf{t}} = \underbrace{\begin{bmatrix}7\\-6\end{bmatrix}}_{\mathbf{t}'}$$

Find A

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad T\left(\underset{\mathbf{s}}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}\right) = \underset{\mathbf{s}'}{\begin{bmatrix} -5 \\ 5 \end{bmatrix}}, \; T\left(\underset{\mathbf{t}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}\right) = \underset{\mathbf{t}'}{\begin{bmatrix} 7 \\ -6 \end{bmatrix}}$$

$$\underset{\mathbf{s}'}{\begin{bmatrix} -5 \\ 5 \end{bmatrix}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{\mathbf{s}}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \quad \text{and} \quad \underset{\mathbf{t}'}{\begin{bmatrix} 7 \\ -6 \end{bmatrix}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underset{\mathbf{t}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

$$\begin{array}{c} \begin{bmatrix} -5 & 7 \\ 5 & -6 \end{bmatrix} \\ \mathbf{s}' \quad \mathbf{t}' \end{array} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{array}{c} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \\ \mathbf{s} \quad \mathbf{t} \end{array}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -5 & 7 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

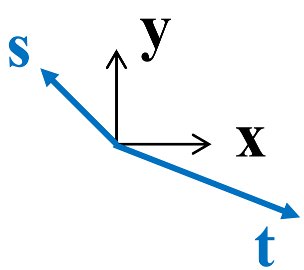
$$A = [\mathbf{s}' \quad \mathbf{t}'][\mathbf{s} \quad \mathbf{t}]^{-1}$$

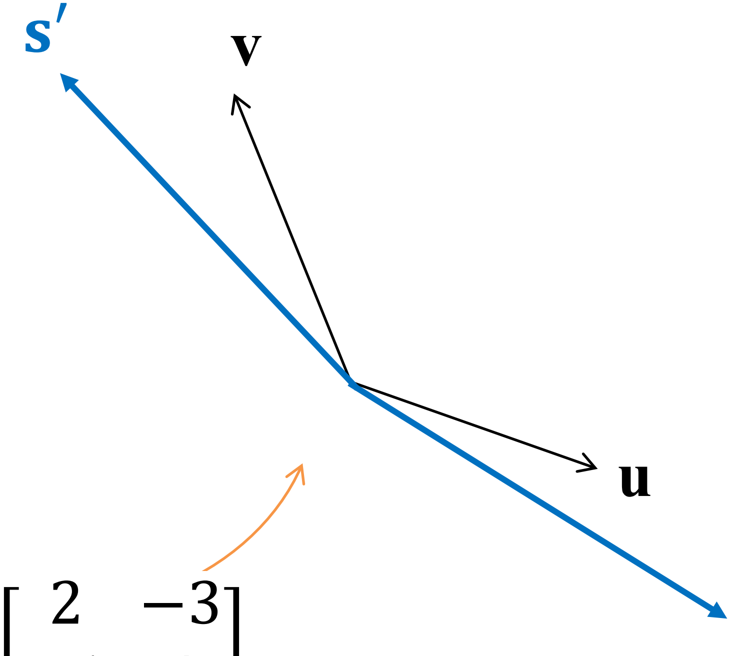
$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\mathbf{s} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\mathbf{t} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$





$$A = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix}$$

$\mathbf{u} = \begin{bmatrix} a \\ c \end{bmatrix}$

$\mathbf{v} = \begin{bmatrix} b \\ d \end{bmatrix}$

$\mathbf{s} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$

$\mathbf{t} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$

Verify

$$\underset{\mathbf{u}}{\begin{bmatrix} 2 \\ -1 \end{bmatrix}} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \underset{\mathbf{x}}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \quad \text{and} \quad \underset{\mathbf{v}}{\begin{bmatrix} -3 \\ 4 \end{bmatrix}} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \underset{\mathbf{y}}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

So $\quad A = [\mathbf{u} \quad \mathbf{v}] = [A\mathbf{x} \quad A\mathbf{y}] = [T(\mathbf{x}) \quad T(\mathbf{y})]$

Chapter 1-8 Objectives

- Determine whether a function is a linear transformation.
- Find the standard matrix for a matrix transformation.