Chapter 3
Euclidean Vector Spaces

Chapter 3.1

Vectors in 2-Space, 3-Space, and n-Space

Properties of Vector Operations

THEOREM 3.1.1

If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, and if k and m are scalars, then:

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$

(f) $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$

(g) $k(m\mathbf{u}) = (km)\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

Proof

Trivial
(hints: using components)

Properties of Vector Operations

THEOREM 3.1.2

If $\mathbf{v}$ is a vector in R^n and k is a scalar, then:

$(a)\ \ 0\mathbf{v} = \mathbf{0}$

$(b)\ \ k\mathbf{0} = \mathbf{0}$

$(c)\ \ (-1)\mathbf{v} = -\mathbf{v}$

Proof

Trivial (hints: using components)

Linear Combinations

DEFINITION 1

If $\mathbf{w}$ is a vector in R^n, then $\mathbf{w}$ is said to be a *linear combinations* of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r$ in R^n if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r$$

where $k_1, k_2, \ldots, k_r$ are scalars.

These scalars are called the *coefficients* of the linear combination.

Alternative Notations for Vectors

Up to now we have been writing vectors in R^n using the notation

$$\mathbf{v} = (v_1, v_2, \ldots, v_n)$$

We call this the *comma-delimited* form.

can be written as $\mathbf{v} = [v_1 \quad v_2 \quad \cdots \quad v_n]$ *row-matrix* form,

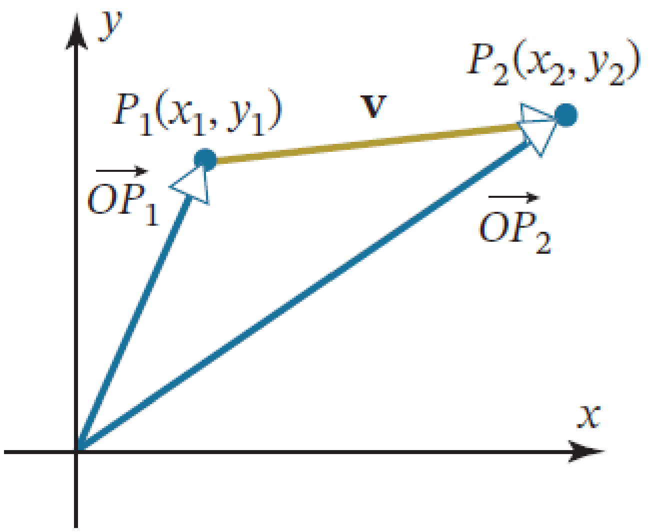
or as $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ *column-matrix* form.

Vectors Whose Initial Point Is Not at the Origin

It is sometimes necessary to consider vectors whose initial points are not at the origin. If $\overrightarrow{P_1P_2}$ denotes the vector with initial point $P_1(x_1, y_1)$ and terminal point $P_2(x_2, y_2)$, then the components of this vector are given by the formula

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1)$$

$$\mathbf{v} = \overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1}$$



Chapter 3.2

Norm, Dot Product, and Distance in R^n

Norm of a Vector

DEFINITION 1

If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is a vector in R^n, then the *norm* of $\mathbf{v}$ (also called the *length* of $\mathbf{v}$ or the *magnitude* of $\mathbf{v}$) is denoted by $\|\mathbf{v}\|$, and is defined by the formula

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Norm of a Vector

THEOREM 3.2.1

If $\mathbf{v}$ is a vector in R^n, and if k is any scalar, then:

(a) $||\mathbf{v}|| \geq 0$

(b) $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$

(c) $||k\mathbf{v}|| = |k|||\mathbf{v}||$

Proof If $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, then $k\mathbf{v} = (kv_1, kv_2, \ldots, kv_n)$, so

$$\begin{aligned} ||k\mathbf{v}|| &= \sqrt{(kv_1)^2 + (kv_2)^2 + \cdots + (kv_n)^2} \\ &= \sqrt{(k^2)(v_1^2 + v_2^2 + \cdots + v_n^2)} \\ &= |k|\sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \\ &= |k|||\mathbf{v}|| \end{aligned}$$

Unit Vectors

$$\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

The process is called *normalizing* **v**.

EXAMPLE 1

Find the unit vector **u** that has the same direction as **v** = (2, 2, −1).

$$\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \qquad \mathbf{u} = \tfrac{1}{3}(2, 2, -1) = \left(\tfrac{2}{3}, \tfrac{2}{3}, -\tfrac{1}{3}\right)$$

exe

Distance in R^n

DEFINITION 2

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are points in R^n,

then we denote the *distance* between $\mathbf{u}$ and $\mathbf{v}$ by

$d(\mathbf{u}, \mathbf{v})$ and define it to be

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$$

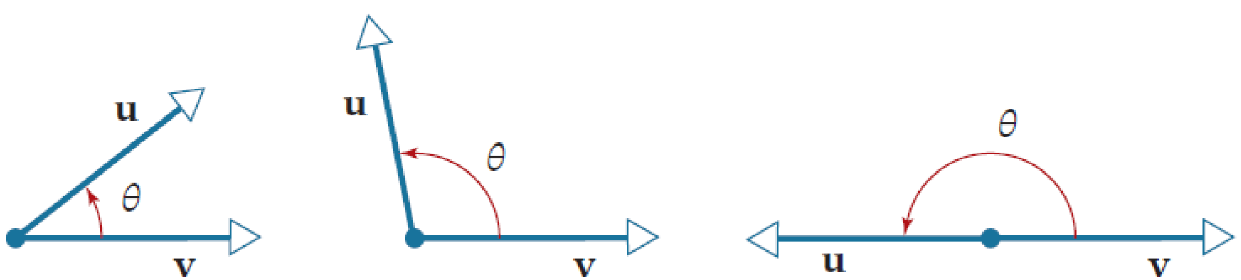
Dot Product

DEFINITION 3

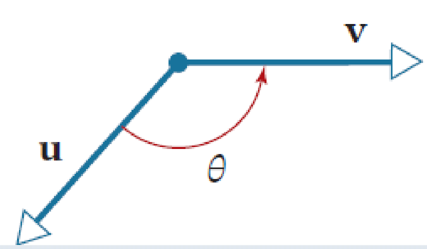
If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in R^2 or R^3, and if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then the *dot product* (also called the *Euclidean inner product*) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u}\cdot\mathbf{v}$ and is defined as

$$\boxed{\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta}$$

If $\mathbf{u}=\mathbf{0}$ or $\mathbf{v}=\mathbf{0}$, then we define $\mathbf{u}\cdot\mathbf{v}$ to be 0.



The angle θ between $\mathbf{u}$ and $\mathbf{v}$ satisfies $0 \le \theta \le \pi$.



$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \qquad 0 \le \theta \le \pi,$$

- θ is acute if $\mathbf{u}\cdot\mathbf{v} > 0$.
- θ is obtuse if $\mathbf{u}\cdot\mathbf{v} < 0$.
- $\theta = \pi/2$ if $\mathbf{u}\cdot\mathbf{v} = 0$.

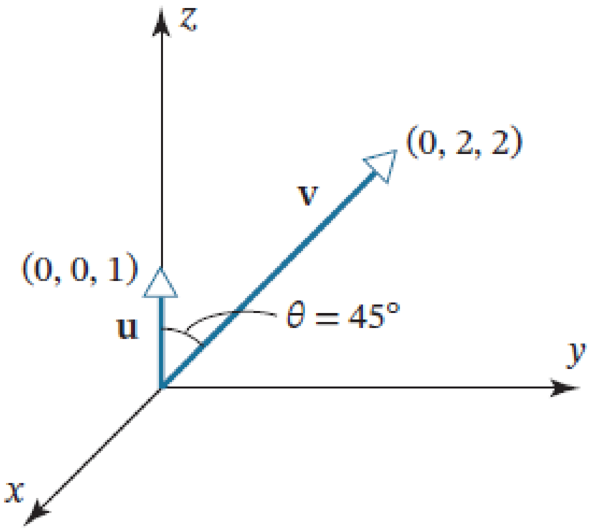
EXAMPLE 2

Find the dot product of the vectors

$\|\mathbf{u}\| = 1 \quad \text{and} \quad \|\mathbf{v}\| = \sqrt{8} = 2\sqrt{2}$

$\cos(45^\circ) = 1/\sqrt{2}$

$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = (1)(2\sqrt{2})(1/\sqrt{2}) = 2$



Component Form of the Dot Product

DEFINITION 4

If $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are vectors in R^n, then the *dot product* (also called the *Euclidean inner product*) of $\mathbf{u}$ and $\mathbf{v}$ is denoted by $\mathbf{u} \cdot \mathbf{v}$ and is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Algebraic Properties of the Dot Product

In the special case where **u** = **v**

$$\mathbf{v}\cdot\mathbf{v} = v_1^2 + v_2^2 + \cdots + v_n^2 = \|\mathbf{v}\|^2$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v}\cdot\mathbf{v}}$$

THEOREM 3.2.2

If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, and if k is a scalar, then:

(a) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$

(b) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}$

(c) $k(\mathbf{u}\cdot\mathbf{v}) = (k\mathbf{u})\cdot\mathbf{v}$

(d) $\mathbf{v}\cdot\mathbf{v} \geq 0$ and $\mathbf{v}\cdot\mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$

Proof [Note]

Algebraic Properties of the Dot Product

THEOREM 3.2.3

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in R^n, and if k is a scalar, then:

(a) $\mathbf{0} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{0} = 0$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(c) $\mathbf{u} \cdot (\mathbf{v} - \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{w}$

(d) $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} - \mathbf{v} \cdot \mathbf{w}$

(e) $k(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (k\mathbf{v})$

Proof [Note]

Algebraic Properties of the Dot Product

EXAMPLE 3

$$\begin{aligned}(\mathbf{u} - 2\mathbf{v}) \cdot (3\mathbf{u} + 4\mathbf{v}) &= \mathbf{u} \cdot (3\mathbf{u} + 4\mathbf{v}) - 2\mathbf{v} \cdot (3\mathbf{u} + 4\mathbf{v}) \\ &= 3(\mathbf{u} \cdot \mathbf{u}) + 4(\mathbf{u} \cdot \mathbf{v}) - 6(\mathbf{v} \cdot \mathbf{u}) - 8(\mathbf{v} \cdot \mathbf{v}) \\ &= 3\|\mathbf{u}\|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 8\|\mathbf{v}\|^2\end{aligned}$$

Dot Products as Matrix Multiplication

If A is an $n \times n$ matrix and $\mathbf{u}$ and $\mathbf{v}$ are $n \times 1$ matrices,

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T(A\mathbf{u}) = (\mathbf{v}^T A)\mathbf{u} = (A^T\mathbf{v})^T\mathbf{u} = \mathbf{u} \cdot A^T\mathbf{v}$$

$$\mathbf{u} \cdot A\mathbf{v} = (A\mathbf{v})^T\mathbf{u} = (\mathbf{v}^T A^T)\mathbf{u} = \mathbf{v}^T(A^T\mathbf{u}) = A^T\mathbf{u} \cdot \mathbf{v}$$

$$\boxed{A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}}$$

$$\boxed{\mathbf{u} \cdot A\mathbf{v} = A^T\mathbf{u} \cdot \mathbf{v}}$$

Form	Dot Product	Example	
u a column matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}^T\mathbf{v}=\mathbf{v}^T\mathbf{u}$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}^T\mathbf{v}=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
u a row matrix and **v** a column matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}=\mathbf{v}^T\mathbf{u}^T$	$\mathbf{u}=[1\quad -3\quad 5]$ $\mathbf{v}=\begin{bmatrix}5\\4\\0\end{bmatrix}$	$\mathbf{u}\mathbf{v}=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}^T\mathbf{u}^T=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$
u a column matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\mathbf{u}=\mathbf{u}^T\mathbf{v}^T$	$\mathbf{u}=\begin{bmatrix}1\\-3\\5\end{bmatrix}$ $\mathbf{v}=[5\quad 4\quad 0]$	$\mathbf{v}\mathbf{u}=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$ $\mathbf{u}^T\mathbf{v}^T=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$
u a row matrix and **v** a row matrix	$\mathbf{u}\cdot\mathbf{v}=\mathbf{u}\mathbf{v}^T=\mathbf{v}\mathbf{u}^T$	$\mathbf{u}=[1\quad -3\quad 5]$ $\mathbf{v}=[5\quad 4\quad 0]$	$\mathbf{u}\mathbf{v}^T=[1\quad -3\quad 5]\begin{bmatrix}5\\4\\0\end{bmatrix}=-7$ $\mathbf{v}\mathbf{u}^T=[5\quad 4\quad 0]\begin{bmatrix}1\\-3\\5\end{bmatrix}=-7$

Dot Products as Matrix Multiplication

EXAMPLE 4 Verifying That $A\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot A^T\mathbf{v}$

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$$

$$A\mathbf{u} = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 5 \end{bmatrix} \qquad A^T\mathbf{v} = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 0 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \\ -1 \end{bmatrix}$$

$$A\mathbf{u} \cdot \mathbf{v} = 7(-2) + 10(0) + 5(5) = 11$$
$$\mathbf{u} \cdot A^T\mathbf{v} = (-1)(-7) + 2(4) + 4(-1) = 11$$

A Dot Product View of Matrix Multiplication

$A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix,

the ijth entry of AB is

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ir}b_{rj}$$

which is the dot product of the ith row vector of A

$$[a_{i1} \quad a_{i2} \quad \cdots \quad a_{ir}]$$

and the jth column vector of B

$$\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{rj} \end{bmatrix}$$

A Dot Product View of Matrix Multiplication

Thus, if we denote the row vectors of A by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$ and the column vectors of the matrix B by $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$, then the matrix product AB can be expressed as

$$AB = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{c}_1 & \mathbf{r}_1 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_1 \cdot \mathbf{c}_n \\ \mathbf{r}_2 \cdot \mathbf{c}_1 & \mathbf{r}_2 \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_2 \cdot \mathbf{c}_n \\ \vdots & \vdots & & \vdots \\ \mathbf{r}_m \cdot \mathbf{c}_1 & \mathbf{r}_m \cdot \mathbf{c}_2 & \cdots & \mathbf{r}_m \cdot \mathbf{c}_n \end{bmatrix}$$

Chapter 3.3

Orthogonality

Orthogonal Vectors

$$\theta = \cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right)$$

It follows from this that $\theta = \pi/2$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

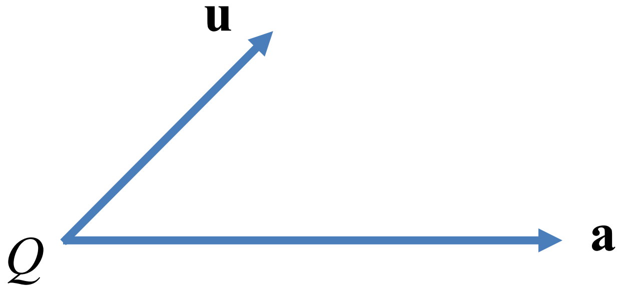
DEFINITION 1

Two nonzero vectors **u** and **v** in are said to be *orthogonal* (or *perpendicular*) if $\mathbf{u}\cdot\mathbf{v} = 0$.

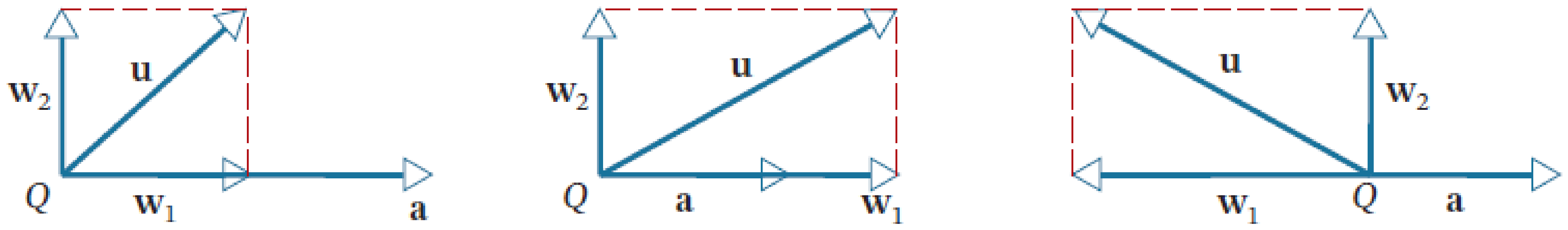
We will also agree that the zero vector in R^n is orthogonal to *every* vector in R^n.

Orthogonal Projections

A vector **u** can be decomposed into a sum of two terms, one term being a scalar multiple of a nonzero vector **a** and the other term being orthogonal to **a**.



$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$



Projection Theorem

THEOREM 3.3.2

If $\mathbf{u}$ and $\mathbf{a}$ are vectors in R^n, and if $\mathbf{a} \neq 0$, then $\mathbf{u}$ can be expressed in exactly one way in the form $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$,

where $\mathbf{w}_1$ is a scalar multiple of $\mathbf{a}$

Proof

and $\mathbf{w}_2$ is orthogonal to $\mathbf{a}$.

[Note]

Projection Notations

·the vector $\mathbf{w}_1$ is called the *orthogonal projection of* **u** *on* **a**

or sometimes *the vector component of* **u** *along* **a**

and the vector $\mathbf{w}_2$ is called the vector *component of* **u** *orthogonal to* **a**

vector $\mathbf{w}_1$ is commonly denoted by the symbol $\text{proj}_{\mathbf{a}}\mathbf{u}$,

$\mathbf{w}_2 = \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u}$.

$$\mathbf{w}_1 = \quad \text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}$$

$$\mathbf{w}_2 = \quad \mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{||\mathbf{a}||^2}\mathbf{a}$$

EXAMPLE 1

Let $\mathbf{u} = (2, -1, 3)$ and $\mathbf{a} = (4, -1, 2)$. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

$$\mathbf{u} \cdot \mathbf{a} = (2)(4) + (-1)(-1) + (3)(2) = 15$$
$$\|\mathbf{a}\|^2 = 4^2 + (-1)^2 + 2^2 = 21$$

$$\operatorname{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \tfrac{15}{21}(4, -1, 2) = \left(\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}\right)$$

$$\mathbf{u} - \operatorname{proj}_{\mathbf{a}}\mathbf{u} = (2, -1, 3) - \left(\tfrac{20}{7}, -\tfrac{5}{7}, \tfrac{10}{7}\right) = \left(-\tfrac{6}{7}, -\tfrac{2}{7}, \tfrac{11}{7}\right)$$

You may wish to verify that the vectors $\mathbf{u} - \operatorname{proj}_{\mathbf{a}}\mathbf{u}$ and $\mathbf{a}$ are perpendicular by showing that their dot product is zero.

Chapter 3.4

The Geometry of Linear Systems

Dot Product Form of a Linear System

Recall that a linear equation in the variables $x_1, x_2, \ldots, x_n$ has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (a_1, a_2, \ldots, a_n \text{ not all zero})$$

and that the corresponding homogenous equation is

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \quad (a_1, a_2, \ldots, a_n \text{ not all zero})$$

These equations can be rewritten in vector form by letting

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \quad \text{and} \quad \mathbf{x} = (x_1, x_2, \ldots, x_n)$$

$$\boxed{\mathbf{a} \cdot \mathbf{x} = b} \qquad \boxed{\mathbf{a} \cdot \mathbf{x} = 0}$$

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & 0 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & 0 \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & 0
\end{array}
$$

If we denote the successive row vectors of the coefficient matrix by $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m$, then

$$
\begin{array}{ccc}
\mathbf{r}_1 \cdot \mathbf{x} & = & 0 \\
\mathbf{r}_2 \cdot \mathbf{x} & = & 0 \\
\vdots & & \vdots \\
\mathbf{r}_m \cdot \mathbf{x} & = & 0
\end{array}
$$

every solution vector $\mathbf{x}$ is orthogonal to every row vector of the coefficient matrix.

Orthogonality of Row Vectors and Solution Vectors

THEOREM 3.4.1

If A is an $m \times n$ matrix, then the solution set of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ consists of all vectors in R^n that are orthogonal to every row vector of A.

EXAMPLE 1 Orthogonality of Row Vectors and Solution Vectors

The general solution of the homogeneous linear system

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is $x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = 0$

$$\mathbf{x} = (-3r - 4s - 2t, r, -2s, s, t, 0)$$

the vector **x** must be orthogonal to each of the row vectors

$$\begin{aligned} \mathbf{r}_1 &= (1, 3, -2, 0, 2, 0) \\ \mathbf{r}_2 &= (2, 6, -5, -2, 4, -3) \\ \mathbf{r}_3 &= (0, 0, 5, 10, 0, 15) \\ \mathbf{r}_4 &= (2, 6, 0, 8, 4, 18) \end{aligned}$$

$$\mathbf{r}_1 \cdot \mathbf{x} = 1(-3r - 4s - 2t) + 3(r) + (-2)(-2s) + 0(s) + 2(t) + 0(0) = 0$$

Chapter 3.5

Cross Product

Cross Product of Vectors

DEFINITION 1

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the *cross product* $\mathbf{u} \times \mathbf{v}$ is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$$

or, in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

EXAMPLE 1 Calculating a Cross Product

Find $\mathbf{u} \times \mathbf{v}$, where $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (3, 0, 1)$.

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix}, -\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \right)$$

$$= (2, -7, -6)$$

Relationships Involving Cross Product and Dot Product

THEOREM 3.5.1

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in 3-space, then

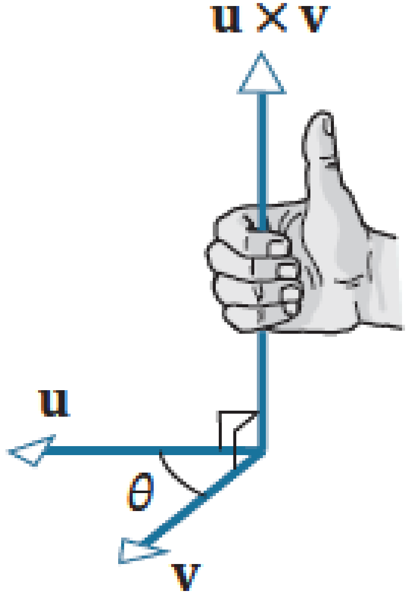
(a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

(b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

(c) $\|\mathbf{u} \times \mathbf{v}\|^2 = \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2$

(d) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

(e) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$



Proof (a) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$. Then

$$\begin{aligned}\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (u_1, u_2, u_3) \cdot (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1) \\ &= u_1(u_2 v_3 - u_3 v_2) + u_2(u_3 v_1 - u_1 v_3) + u_3(u_1 v_2 - u_2 v_1) = 0\end{aligned}$$

Proof (c)

Since

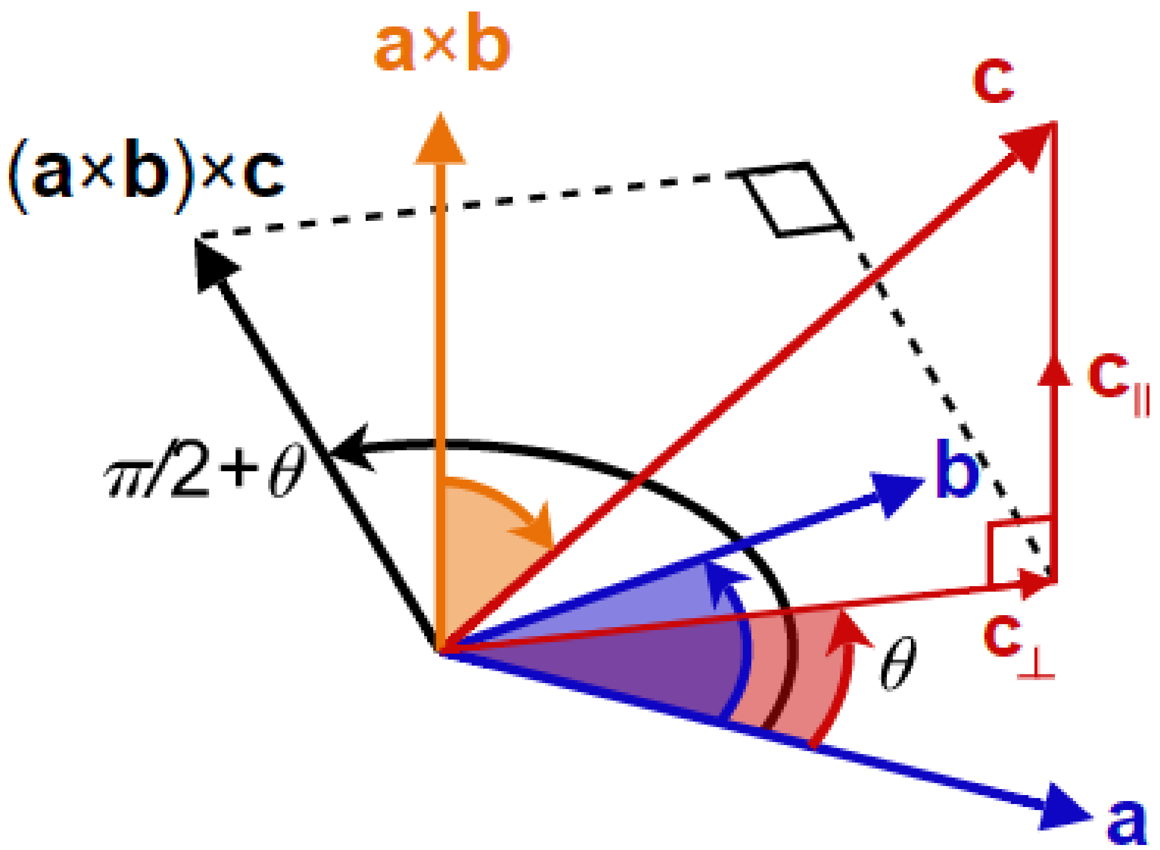
$$\|\mathbf{u} \times \mathbf{v}\|^2 = (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2$$

and

$$\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 = (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2$$

the proof can be completed by "multiplying out" the right sides of above equations and verifying their equality.

Visualization



Properties of Cross Product

THEOREM 3.5.2

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors in 3-space and k is any scalar, then:

(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

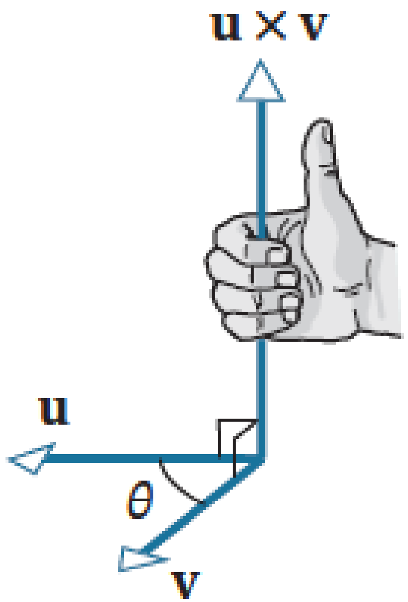
(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(d) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$



Proof

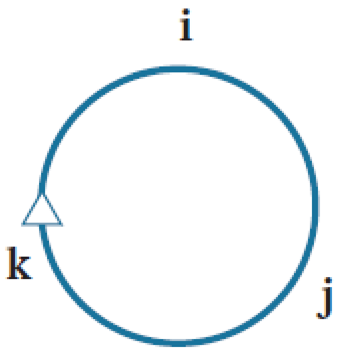
(Trivial)

EXAMPLE 2 Standard Unit Vectors

The standard unit vectors in 3-space are

$$\mathbf{i} = (1, 0, 0),\ \mathbf{j} = (0, 1, 0),\ \mathbf{k} = (0, 0, 1)$$

$$\begin{array}{lll} \mathbf{i}\times\mathbf{i}=\mathbf{0} & \mathbf{j}\times\mathbf{j}=\mathbf{0} & \mathbf{k}\times\mathbf{k}=\mathbf{0} \\ \mathbf{i}\times\mathbf{j}=\mathbf{k} & \mathbf{j}\times\mathbf{k}=\mathbf{i} & \mathbf{k}\times\mathbf{i}=\mathbf{j} \\ \mathbf{j}\times\mathbf{i}=-\mathbf{k} & \mathbf{k}\times\mathbf{j}=-\mathbf{i} & \mathbf{i}\times\mathbf{k}=-\mathbf{j} \end{array}$$



Every vector $\mathbf{v} = (v_1, v_2, v_3)$ in 3-space is expressible in terms of $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

Remark

It is not true in general that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

For example,

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) = \mathbf{i} \times \mathbf{0} = \mathbf{0}$$

$$(\mathbf{i} \times \mathbf{j}) \times \mathbf{j} = \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{i} \times (\mathbf{j} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$$

DEFINITION 2

If **u**, **v**, and **w** are vectors in 3-space, then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the *scalar triple product* of **u**, **v**, and **w**.

The scalar triple product of $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ can be calculated from the formula

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Remark

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$$