4-1 In-Class Exercise

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad k\mathbf{u} = (ku_1, ku_2)$$

a. Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (0, 4), \mathbf{v} = (1, -3)$, and $k = 2$.

b. Show that $(0, 0) \neq \mathbf{0}$.

c. Show that $(-1, -1) = \mathbf{0}$.

d. Show that Axiom 5 holds by producing a vector $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ for $\mathbf{u} = (u_1, u_2)$.

e. Find two vector space axioms that fail to hold.

4-1 Suggested Exercises

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \quad k\mathbf{u} = (0, ku_2)$$

a. Compute $\mathbf{u} + \mathbf{v}$ and $k\mathbf{u}$ for $\mathbf{u} = (-1, 2), \mathbf{v} = (3, 4)$, and $k = 3$.

b. In words, explain why V is closed under addition and scalar multiplication.

c. Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?

d. Show that Axioms 7, 8, and 9 hold.

e. Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

Determine whether each of the following sets equipped with the given operations is a vector space.

2. The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on R^2.

3. The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = (k^2x, k^2y, k^2z)$$

4. The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2.

5. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.