4-4 In-Class Exercise

1. Determine whether the vectors are linearly independent or are linearly dependent in R^3.

$$(-3, 0, 4),\ (5, -1, 2),\ (1, 1, 3)$$

2. Determine whether the three vectors lie on the same line in R^3.

(a) $\mathbf{v}_1 = (-1, 2, 3),\ \mathbf{v}_2 = (2, -4, -6),\ \mathbf{v}_3 = (-3, 6, 0)$

(b) $\mathbf{v}_1 = (4, 6, 8),\ \mathbf{v}_2 = (2, 3, 4),\ \mathbf{v}_3 = (-2, -3, -4)$

1. The vector equation $a(-3,0,4)+b(5,-1,2)+c(1,1,3)=(0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcl} -3a & + & 5b & + & 1c & = & 0 \\ 0a & - & 1b & + & 1c & = & 0 \\ 4a & + & 2b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given set of vectors is linearly independent.

2.

(a) The set $\{\mathbf{v}_1,\mathbf{v}_3\}$ can be shown to be linearly independent since $a(-1,2,3)+b(-3,6,0)=(0,0,0)$ has only the trivial solution $a=b=0$. Therefore the three vectors do not lie on the same line (even though the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ *are* collinear).

(b) Each subset of two vectors chosen from these three vectors can be shown to be linearly dependent since $-1\mathbf{v}_1+2\mathbf{v}_2=\mathbf{0}$, $1\mathbf{v}_1+2\mathbf{v}_3=\mathbf{0}$, and $1\mathbf{v}_2+1\mathbf{v}_3=\mathbf{0}$. Therefore all three vectors lie on the same line.

4-4 Suggested Exercises

1. Determine whether the vectors are linearly independent or are linearly dependent in P_2.

$$2 - x + 4x^2,\ 3 + 6x + 2x^2,\ 2 + 10x - 4x^2$$

2. Determine whether the matrices are linearly independent or dependent.

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix},\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \text{ in } M_{22}$$

1. The terms in the equation

$$a\left(2-x+4x^2\right)+b\left(3+6x+2x^2\right)+c\left(2+10x-4x^2\right)=0$$

can be grouped according to the powers of x

$$(2a+3b+2c)+(-a+6b+10c)x+(4a+2b-4c)x^2=0+0x+0x^2$$

For this to hold for all real values of x, the coefficients corresponding to the same powers of x on both sides must match, which leads to the homogeneous linear system

$$\begin{array}{rcrcrcl} 2a & + & 3b & + & 2c & = & 0 \\ -a & + & 6b & + & 10c & = & 0 \\ 4a & + & 2b & - & 4c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given set of vectors in P_2 is linearly independent.

2.

The matrix equation $a\begin{bmatrix}1 & 0\\ 1 & 2\end{bmatrix}+b\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}+c\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ can be rewritten as a homogeneous linear system

$$\begin{array}{ccccccc} 1a & + & 1b & + & 0c & = & 0\\ 0a & + & 2b & + & 1c & = & 0\\ 1a & + & 2b & + & 2c & = & 0\\ 2a & + & 1b & + & 1c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$ therefore the system has only the trivial solution $a=b=c=0$. We conclude that the given matrices are linearly independent.

3. Show that the three vectors **v1** = (0, 3, 1, −1), **v2** = (6, 0, 5, 1), and **v3** = (4, −7, 1, 3) form a linearly dependent set in R^4.

The vector equation $a(0,3,1,-1)+b(6,0,5,1)+c(4,-7,1,3)=(0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcr} 0a & + & 6b & + & 4c & = & 0 \\ 3a & + & 0b & - & 7c & = & 0 \\ 1a & + & 5b & + & 1c & = & 0 \\ -1a & + & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -\frac{7}{3} & 0 \\ 0 & 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is $a=\frac{7}{3}t$, $b=-\frac{2}{3}t$, $c=t$.

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

4. In each part, let $T_A : R^3 \to R^3$ be multiplication by A, and let $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (2, -1, 1)$, and $\mathbf{u}_3 = (0, 1, 1)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly independent in R^3.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & 2 & 0 \end{bmatrix}$$

4.

We calculate $T_A(1,0,0)=(1,1,2)$, $T_A(2,-1,1)=(3,-1,2)$, and $T_A(0,1,1)=(3,-3,2)$. The vector equation

$$k_1(1,1,2)+k_2(3,-1,2)+k_3(3,-3,2)=(0,0,0)$$

can be rewritten as a homogeneous linear system

$$\begin{array}{rcrcrcr} 1k_1 & + & 3k_2 & + & 3k_3 & = & 0 \\ 1k_1 & - & 1k_2 & - & 3k_3 & = & 0 \\ 2k_1 & + & 2k_2 & + & 2k_3 & = & 0 \end{array}$$

The determinant of the coefficient matrix of this system is $\begin{vmatrix} 1 & 3 & 3 \\ 1 & -1 & -3 \\ 2 & 2 & 2 \end{vmatrix} = -8 \neq 0$, therefore by Theorem 2.3.8, the system has only the trivial solution. We conclude that the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ is linearly independent.