3-1 In-Class Exercise

1. Let $\mathbf{u} = (1, 2, -3, 5, 0)$, $\mathbf{v} = (0, 4, -1, 1, 2)$, and $\mathbf{w} = (7, 1, -4, -2, 3)$. Find the components of

$$\tfrac{1}{2}(\mathbf{w} - 5\mathbf{v} + 2\mathbf{u}) + \mathbf{v}$$

$$\tfrac{1}{2}(\mathbf{w} - 5\mathbf{v} + 2\mathbf{u}) + \mathbf{v} = \tfrac{1}{2}\left[(7,1,-4,-2,3) - (0,20,-5,5,10) + (2,4,-6,10,0)\right] + (0,4,-1,1,2)$$

$$= \tfrac{1}{2}(9,-15,-5,3,-7) + (0,4,-1,1,2) = \left(\tfrac{9}{2}, -\tfrac{7}{2}, -\tfrac{7}{2}, \tfrac{5}{2}, -\tfrac{3}{2}\right)$$

3-1 Suggested Exercise

1. Which of the following vectors in R^6, if any, are parallel to $\mathbf{u} = (-2, 1, 0, 3, 5, 1)$?

a. $(4, 2, 0, 6, 10, 2)$

b. $(4, -2, 0, -6, -10, -2)$

c. $(0, 0, 0, 0, 0, 0)$

Vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel (collinear) if one of them is a scalar multiple of the other one, i.e. either $\mathbf{u} = a\mathbf{v}$ for some scalar a or $\mathbf{v} = b\mathbf{u}$ for some scalar b or both (the two conditions are not equivalent if one of the vectors is a zero vector, but the other one is not.)

(a) $\mathbf{v} = (4,2,0,6,10,2)$ does not equal $k\mathbf{u} = (-2k, k, 0, 3k, 5k, k)$ for any scalar k; $\mathbf{v}$ is not parallel to $\mathbf{u}$

(b) $\mathbf{v} = (4,-2,0,-6,-10,-2) = -2\mathbf{u}$; $\mathbf{v}$ is parallel to $\mathbf{u}$

(c) $\mathbf{v} = (0,0,0,0,0,0) = 0\mathbf{u}$; $\mathbf{v}$ is parallel to $\mathbf{u}$

2. Show that there do not exist scalars c_1, c_2, and c_3 such that

$$c_1(-2,9,6)+c_2(-3,2,1)+c_3(1,7,5)=(0,5,4)$$

The vector equation $c_1(-2,9,6)+c_2(-3,2,1)+c_3(1,7,5)=(0,5,4)$ is equivalent to the linear system

$$\begin{aligned} -2c_1 - 3c_2 + 1c_3 &= 0 \\ 9c_1 + 2c_2 + 7c_3 &= 5 \\ 6c_1 + 1c_2 + 5c_3 &= 4 \end{aligned}$$

whose augmented matrix $\left[\begin{array}{ccc|c} -2 & -3 & 1 & 0 \\ 9 & 2 & 7 & 5 \\ 6 & 1 & 5 & 4 \end{array}\right]$ has the reduced row echelon form $\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$.

The system has no solution.

3. Let P be the point $(2, 3, -2)$ and Q the point $(7, -4, 1)$. Find the midpoint of the line segment connecting the points P and Q.

The midpoint of the segment is the terminal point of the vector

$$\overrightarrow{OM} = \overrightarrow{OP} + \tfrac{1}{2}\overrightarrow{PQ} = (2,3,-2) + \tfrac{1}{2}\big(7-2,-4-3,1-(-2)\big) = \left(\tfrac{9}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right)$$

therefore the midpoint has coordinates $\left(\tfrac{9}{2},-\tfrac{1}{2},-\tfrac{1}{2}\right)$.

3.2 In-Class Exercise

1. Determine whether the expression makes sense mathematically.

a. $\|\mathbf{u}\| \cdot \|\mathbf{v}\|$

b. $(\mathbf{u} \cdot \mathbf{v}) - \mathbf{w}$

c. $(\mathbf{u} \cdot \mathbf{v}) - k$

d. $k \cdot \mathbf{u}$

(a) $\|\mathbf{u}\| \cdot \|\mathbf{v}\|$ does not make sense: $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are scalars, whereas the dot product is only defined for vectors

(b) $(\mathbf{u} \cdot \mathbf{v}) - \mathbf{w}$ does not make sense: $\mathbf{u} \cdot \mathbf{v}$ is a scalar so the vector $\mathbf{w}$ cannot be subtracted from it

(c) $(\mathbf{u} \cdot \mathbf{v}) - k$ makes sense (the result is a scalar)

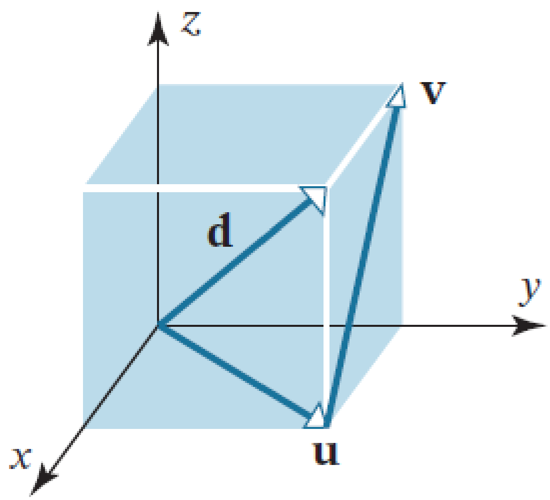
(d) $k \cdot \mathbf{u}$ does not make sense: k is a scalar, whereas the dot product is only defined for vectors

3.2 Suggested Exercise

1. Let $\mathbf{v} = (1, 1, 2, -3, 1)$. Find all scalars k such that $\|k\mathbf{v}\| = 4$.

$\|k\mathbf{v}\| = \sqrt{k^2 + k^2 + (2k)^2 + (-3k)^2 + k^2} = \sqrt{16k^2} = 4\sqrt{k^2}$; this quantity equals 4 if $k = 1$ or $k = -1$

2. Figure shows a cube. Find the angle between the vectors **d** and **u** to the nearest degree.



We have $\mathbf{d} = (1,1,1)$ and $\mathbf{u} = (1,1,0)$.

$\cos\theta = \frac{\mathbf{d}\cdot\mathbf{u}}{\|\mathbf{d}\|\|\mathbf{u}\|} = \frac{(1)(1)+(1)(1)+(1)(0)}{\sqrt{1^2+1^2+1^2}\sqrt{1^2+1^2+0^2}} = \frac{2}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}}$ therefore $\theta = \cos^{-1}\sqrt{\frac{2}{3}} \approx 35^\circ$.

3.3 In-Class Exercise

1. Find the vector component of **u** along **a** and the vector component of **u** orthogonal to **a**.

$$\mathbf{u} = (-1, -2),\ \mathbf{a} = (-2, 3)$$

$\mathbf{u}\cdot\mathbf{a} = (-1)(-2)+(-2)(3) = -4$, $\|\mathbf{a}\|^2 = (-2)^2 + 3^2 = 13$,

the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = -\frac{4}{13}(-2,3) = \left(\frac{8}{13}, -\frac{12}{13}\right)$,

the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is $\mathbf{u} - \operatorname{proj}_{\mathbf{a}}\mathbf{u} = (-1,-2) - \left(\frac{8}{13}, -\frac{12}{13}\right) = \left(-\frac{21}{13}, -\frac{14}{13}\right)$

3.3 Suggested Exercise

1. Show that if $\mathbf{v}$ is orthogonal to both $\mathbf{w}_1$ and $\mathbf{w}_2$, then $\mathbf{v}$ is orthogonal to $k_1\mathbf{w}_1 + k_2\mathbf{w}_2$ for all scalars k_1 and k_2.

Assuming $\mathbf{v}\cdot\mathbf{w}_1 = \mathbf{v}\cdot\mathbf{w}_2 = 0$ and using Theorem 3.2.2, we have

$$\mathbf{v}\cdot(k_1\mathbf{w}_1 + k_2\mathbf{w}_2) = \mathbf{v}\cdot(k_1\mathbf{w}_1) + \mathbf{v}\cdot(k_2\mathbf{w}_2) = k_1(\mathbf{v}\cdot\mathbf{w}_1) + k_2(\mathbf{v}\cdot\mathbf{w}_2) = (k_1)(0) + (k_2)(0) = 0.$$

2. Is it possible to have $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{u}}\mathbf{a}$? Explain.

Yes.

One possible scenario is when $\mathbf{u} = \mathbf{a}$ - in this case, $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{u}}\mathbf{a} = \text{proj}_{\mathbf{u}}\mathbf{u} = \mathbf{u}$.

Another possibility is to take $\mathbf{u}$ and $\mathbf{a}$ to be orthogonal vectors, so that $\text{proj}_{\mathbf{a}}\mathbf{u} = \text{proj}_{\mathbf{u}}\mathbf{a} = 0$.

3.5 In-Class Exercise

1. Calculate the scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ of the vectors

$$\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} - 5\mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 4\mathbf{j} - 4\mathbf{k}, \quad \mathbf{w} = 3\mathbf{j} + 2\mathbf{k}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 3 & -2 & -5 \\ 1 & 4 & -4 \\ 0 & 3 & 2 \end{vmatrix}$$

$$= 3\begin{vmatrix} 4 & -4 \\ 3 & 2 \end{vmatrix} - (-2)\begin{vmatrix} 1 & -4 \\ 0 & 2 \end{vmatrix} + (-5)\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix}$$

$$= 60 + 4 - 15 = 49$$

3.5 Suggested Exercise

1. Find the area of the triangle in 3-space that has the given vertices.

$$P_1(2,6,-1),\ \ P_2(1,1,1),\ \ P_3(4,6,2)$$

$$\overrightarrow{P_1P_2}=(-1,-5,2), \qquad \overrightarrow{P_1P_3}=(2,0,3)$$

$$\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}=\left(\begin{vmatrix}-5 & 2\\ 0 & 3\end{vmatrix},-\begin{vmatrix}-1 & 2\\ 2 & 3\end{vmatrix},\begin{vmatrix}-1 & -5\\ 2 & 0\end{vmatrix}\right)=(-15,7,10).$$

The area of the triangle is $\frac{1}{2}\left\|\overrightarrow{P_1P_2}\times\overrightarrow{P_1P_3}\right\|=\frac{1}{2}\sqrt{(-15)^2+7^2+10^2}=\frac{\sqrt{374}}{2}$.

2. Simplify $(\mathbf{u} + \mathbf{v}) \times (\mathbf{u} - \mathbf{v})$.

Using parts (a), (b), (c), and (f) of Theorem 3.5.2, we can write $(\mathbf{u}+\mathbf{v})\times(\mathbf{u}-\mathbf{v})$ $=(\mathbf{u}\times\mathbf{u})-(\mathbf{u}\times\mathbf{v})+(\mathbf{v}\times\mathbf{u})+(\mathbf{v}\times\mathbf{v})=\mathbf{0}-(-(\mathbf{v}\times\mathbf{u}))+(\mathbf{v}\times\mathbf{u})+\mathbf{0}=2(\mathbf{v}\times\mathbf{u})$.

3. Suppose that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 3$. Find

a) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$
b) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$
c) $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$

(a) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v})=\begin{vmatrix} u_1 & u_2 & u_3 \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$ can be obtained from $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

by interchanging the second row and the third row.

This reverses the sign of the determinant, therefore $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v})=-3$.

(b) By Theorem 3.2.2(a), $(\mathbf{v}\times\mathbf{w})\cdot\mathbf{u}=\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=3$.

(c) $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$ Row 1 $\leftrightarrow$ Row 3

Row 2 $\leftrightarrow$ Row 3 $\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=\begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

$\mathbf{w}\cdot(\mathbf{u}\times\mathbf{v})=(-1)(-1)3=3.$