4-6 In-Class Exercise

1. In each part, find a basis for the given subspace of R^4, and state its dimension.

a. All vectors of the form $(a, b, c, 0)$.

b. All vectors of the form (a, b, c, d), where $d = a + b$ and $c = a - b$.

c. All vectors of the form (a, b, c, d), where $a = b = c = d$.

1.

(a) The given subspace can be expressed as $\text{span}(S)$ where $S=\{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$ is a set of linearly independent vectors. Therefore S forms a basis for the subspace, so its dimension is 3.

(b) The subspace contains all vectors $(a,b,a+b,a-b)=a(1,0,1,1)+b(0,1,1,-1)$ thus we can express it as $\text{span}(S)$ where $S=\{(1,0,1,1),(0,1,1,-1)\}$. By Theorem 4.4.2(c), S is linearly independent since neither vector in the set is a scalar multiple of the other. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 2.

(c) The subspace contains all vectors $(a,a,a,a)=a(1,1,1,1)$ thus we can express it as as $\text{span}(S)$ where $S=\{(1,1,1,1)\}$. By Theorem 4.4.2(b), S is linearly independent since it contains a single nonzero vector. Consequently, S forms a basis for the given subspace. The dimension of the subspace is 1.

4-6 Suggested Exercises

1. Find a basis for the solution space of the homogeneous linear system, and find the dimension of that space.

(a)
$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

(b)
$$\begin{aligned} x + y + z &= 0 \\ 3x + 2y - 2z &= 0 \\ 4x + 3y - z &= 0 \\ 6x + 5y + z &= 0 \end{aligned}$$

1.

(a) The augmented matrix of the linear system $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 2 & -8 & 6 & -2 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -4 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = 4r - 3s + t$, $x_2 = r$, $x_3 = s$, $x_4 = t$. In vector form

$$(x_1, x_2, x_3, x_4) = (4r - 3s + t, r, s, t) = r(4,1,0,0) + s(-3,0,1,0) + t(1,0,0,1)$$

therefore the solution space is spanned by vectors $\mathbf{v}_1 = (4,1,0,0)$, $\mathbf{v}_2 = (-3,0,1,0)$, and $\mathbf{v}_3 = (1,0,0,1)$. By inspection, these vectors are linearly independent since $r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3 = \mathbf{0}$ implies $r = s = t = 0$. We conclude that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for the solution space and that the dimension of the solution space is 3.

(b) The augmented matrix $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & 2 & -2 & 0 \\ 4 & 3 & -1 & 0 \\ 6 & 5 & 1 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

The general solution is $x = 4t$, $y = -5t$, $z = t$. In vector form $(x, y, z) = (4t, -5t, t) = t(4, -5, 1)$

therefore the solution space is spanned by vector $\mathbf{v}_1 = (4,-5,1)$. By Theorem 4.4.2(b), this vector forms a linearly independent set since it is not the zero vector. We conclude that $\mathbf{v}_1$ forms a basis for the solution space and that the dimension of the solution space is 1.

2. Find the dimension of each of the following vector spaces.

a. The vector space of all diagonal $n \times n$ matrices.

b. The vector space of all symmetric $n \times n$ matrices.

c. The vector space of all upper triangular $n \times n$ matrices.

(a) Let W be the space of all diagonal $n \times n$ matrices. We can write

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1 \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_1} + d_2 \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}}_{A_2} + \cdots + d_n \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{A_n}$$

The matrices $A_1, \ldots, A_n$ are linearly independent and they span W; hence, $A_1, \ldots, A_n$ form a basis for W. Consequently, the dimension of W is n.

(b) A basis for this space can be constructed by including the n matrices $A_1, \ldots, A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1=\frac{n(n-1)}{2}$ matrices B_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) and (j,i) entries, which are both 1.

For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{A_1},\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{A_2},\underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{A_3},\underbrace{\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{B_{12}},\underbrace{\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}}_{B_{13}},\underbrace{\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{B_{23}}$$

The dimension is $n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$.

(c) A basis for this space can be constructed by including the n matrices $A_1, \ldots, A_n$ from part (a), as well as $(n-1)+(n-2)+\cdots+3+2+1=\frac{n(n-1)}{2}$ matrices C_{ij} (for all $i<j$) where all entries are 0 except for the (i,j) entry, which is 1.

For instance, for $n=3$, such a basis would be:

$$\underbrace{\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}}_{A_1},\underbrace{\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}}_{A_2},\underbrace{\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}}_{A_3},\underbrace{\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}}_{C_{12}},\underbrace{\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}}_{C_{13}},\underbrace{\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix}}_{C_{23}}$$

The dimension is $n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$.

3. Show that the set W of all polynomials in P_2 such that $p(1) = 0$ is a subspace of P_2.

W is the set of all polynomials $a_0 + a_1 x + a_2 x^2$ for which $a_0 + a_1 + a_2 = 0$, i.e. all polynomials that can be expressed in the form $-a_1 - a_2 + a_1 x + a_2 x^2$.

Adding two polynomials in W results in another polynomial in W

$$\left(-a_1 - a_2 + a_1 x + a_2 x^2\right) + \left(-b_1 - b_2 + b_1 x + b_2 x^2\right)$$

$$= \left(-a_1 - a_2 - b_1 - b_2\right) + \left(a_1 + b_1\right) x + \left(a_2 + b_2\right) x^2$$

since we have $\left(-a_1 - a_2 - b_1 - b_2\right) + \left(a_1 + b_1\right) + \left(a_2 + b_2\right) = 0$.

Likewise, a scalar multiple of a polynomial in W is also in W

$$k\left(-a_1 - a_2 + a_1 x + a_2 x^2\right) = -ka_1 - ka_2 + ka_1 x + ka_2 x^2$$

since it meets the condition $\left(-ka_1 - ka_2\right) + \left(ka_1\right) + \left(ka_2\right) = 0$.

According to Theorem 4.2.1, W is a subspace of P_2.

4. Find a standard basis vector for R^3 that can be added to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ to produce a basis for R^3.

$$\mathbf{v}_1 = (-1, 2, 3),\ \mathbf{v}_2 = (1, -2, -2)$$

Either $(1,0,0)$ or $(0,1,0)$ can be used since neither is in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$

(e.g., with $(1,0,0)$, linear independence can be easily shown calculating $\begin{vmatrix} -1 & 1 & 1 \\ 2 & -2 & 0 \\ 3 & -2 & 0 \end{vmatrix} = 2 \neq 0$ then using parts (b) and (g) of Theorem 2.3.8; the set forms a basis by Theorem 4.6.4)

5. Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for a vector space V. Show that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is also a basis, where $\mathbf{u}_1 = \mathbf{v}_1$, $\mathbf{u}_2 = \mathbf{v}_1 + \mathbf{v}_2$, and $\mathbf{u}_3 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

The equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = 0$ implies $c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 + \mathbf{v}_2) + c_3(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = 0$, i.e., $(c_1 + c_2 + c_3)\mathbf{v}_1 + (c_2 + c_3)\mathbf{v}_2 + c_3\mathbf{v}_3 = 0$, which by linear independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ requires that

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_3 &= 0 \\ c_3 &= 0 \end{aligned}$$

Solving this system by back-substitution yields $c_1 = c_2 = c_3 = 0$ therefore $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is linearly independent. Since the dimension of V is 3 (as its basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains three vectors), by Theorem 4.6.4 $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ must also be a basis for V.

6. The vectors $\mathbf{v}_1 = (1, -2, 3)$ and $\mathbf{v}_2 = (0, 5, -3)$ are linearly independent. Enlarge $\{\mathbf{v}_1, \mathbf{v}_2\}$ to a basis for R^3.

The equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 + k_4\mathbf{e}_2 + k_5\mathbf{e}_3 = \mathbf{0}$ can be rewritten as a linear system

$$\begin{array}{rcrcrcrcrcl} k_1 & & & + & k_3 & & & & & = & 0 \\ -2k_1 & + & 5k_2 & & & + & k_4 & & & = & 0 \\ 3k_1 & - & 3k_2 & & & & & + & k_5 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{5}{9} & 0 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{2}{9} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & -\frac{5}{9} & 0 \end{bmatrix}$.

Based on the leading entries in the first three columns, the vector equation $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{e}_1 = \mathbf{0}$ has only the trivial solution (the corresponding augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$).

Therefore the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ are linearly independent. Since $\dim\left(R^3\right) = 3$, it follows by Theorem 4.6.4 that the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{e}_1$ form a basis for R^3. (The answer is not unique.)