4-5 In-Class Exercise

1. Find the coordinate vector of $\mathbf{p}$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for P_2.

$$\mathbf{p} = 2 - x + x^2;\ \mathbf{p}_1 = 1 + x,\ \mathbf{p}_2 = 1 + x^2,\ \mathbf{p}_3 = x + x^2$$

1.

Expressing $\mathbf{p}$ as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ we obtain

$$2 - x + x^2 = c_1\left(1 + x\right) + c_2\left(1 + x^2\right) + c_3\left(x + x^2\right)$$

Grouping the terms on the right hand side according to powers of x yields

$$2 - x + x^2 = \left(c_1 + c_2\right) + \left(c_1 + c_3\right)x + \left(c_2 + c_3\right)x^2$$

For this equality to hold for all real x, the coefficients associated with the same power of x on both sides must match. This leads to the linear system

$$\begin{array}{ccccccc} c_1 & + & c_2 & & & = & 2 \\ c_1 & & & + & c_3 & = & -1 \\ & & c_2 & + & c_3 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is $c_1 = 0$, $c_2 = 2$, $c_3 = -1$, therefore the coordinate vector is $\left(\mathbf{p}\right)_S = \left(0, 2, -1\right)$.

4-5 Suggested Exercises

1. Show that the following vectors do not form a basis for P_2.

$$1 - 3x + 2x^2, \quad 1 + x + 4x^2, \quad 1 - 7x$$

2. Find the coordinate vector of $\mathbf{w}$ relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.

$$\mathbf{u}_1 = (1, 1), \ \mathbf{u}_2 = (0, 2); \ \mathbf{w} = (a, b)$$

1.

Vectors $\mathbf{p}_1 = 1 - 3x + 2x^2$, $\mathbf{p}_2 = 1 + x + 4x^2$, and $\mathbf{p}_3 = 1 - 7x$ are linearly independent if the vector equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ has only the trivial solution.

By grouping the terms on the left hand side as $c_1\left(1 - 3x + 2x^2\right) + c_2\left(1 + x + 4x^2\right) + c_3\left(1 - 7x\right) = \left(c_1 + c_2 + c_3\right) + \left(-3c_1 + c_2 - 7c_3\right)x + \left(2c_1 + 4c_2\right)x^2$ this equation can be rewritten as the linear system

$$\begin{array}{rcrcrcl} c_1 & + & c_2 & + & c_3 & = & 0 \\ -3c_1 & + & c_2 & - & 7c_3 & = & 0 \\ 2c_1 & + & 4c_2 & & & = & 0 \end{array}$$

The coefficient matrix of this system has determinant $\begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix} = 0$, thus it follows from parts (b) and (g) of Theorem 2.3.8 that the homogeneous system has nontrivial solutions. Since the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly dependent, we conclude that they do not form a basis for P_2.

2.

Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(a,\ b) = c_1(1,1) + c_2(0,2)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{ccccc} 1c_1 & + & 0c_2 & = & a \\ 1c_1 & + & 2c_2 & = & b \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & \frac{b-a}{2} \end{bmatrix}$. The solution of the linear system is $c_1 = a$, $c_2 = \frac{b-a}{2}$, therefore the coordinate vector is $(\mathbf{w})_S = \left(a, \frac{b-a}{2}\right)$.

3. First show that the set $S = \{A_1, A_2, A_3, A_4\}$ *is a* basis for M_{22}, then express A as a linear combination of the vectors in S, and then find the coordinate vector of A relative to S.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$$

3. Matrices (vectors in M_{22}) A_1, A_2, A_3, and A_4 are linearly independent if the equation

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = \mathbf{0}$$

has only the trivial solution. For these matrices to span M_{22}, it must be possible to express every matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as

$$k_1 A_1 + k_2 A_2 + k_3 A_3 + k_4 A_4 = B$$

The left hand side of each of these equations is the matrix $\begin{bmatrix} k_1 + k_2 + k_3 & k_2 \\ k_1 + k_4 & k_3 \end{bmatrix}$. Equating corresponding entries, these two equations can be rewritten as linear systems

$$\begin{array}{rcrcrcrcl} k_1 & + & k_2 & + & k_3 & & & = & 0 \\ & & k_2 & & & & & = & 0 \\ k_1 & & & & & + & k_4 & = & 0 \\ & & & & k_3 & & & = & 0 \end{array} \quad \text{and} \quad \begin{array}{rcrcrcrcl} k_1 & + & k_2 & + & k_3 & & & = & a \\ & & k_2 & & & & & = & b \\ k_1 & & & & & + & k_4 & = & c \\ & & & & k_3 & & & = & d \end{array}$$

3. cont.

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a, b, c and d. Therefore the matrices A_1, A_2, A_3, and A_4 are linearly independent and span M_{22} so that they form a basis for M_{22}.

To express $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ as a linear combination of the matrices A_1, A_2, A_3, and A_4, we form the nonhomogeneous system as above, with the appropriate right hand side values

$$\begin{array}{ccccccccc} k_1 & + & k_2 & + & k_3 & & & = & 6 \\ & & k_2 & & & & & = & 2 \\ k_1 & & & & & + & k_4 & = & 5 \\ & & & & k_3 & & & = & 3 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}$ therefore the solution is $k_1 = 1$, $k_2 = 2$, $k_3 = 3$, $k_4 = 4$.

This allows us to express $A = 1A_1 + 2A_2 + 3A_3 + 4A_4$. The coordinate vector is $(A)_S = (1,2,3,4)$.

4. In each part, let $T_A : R^3 \rightarrow R^3$ be multiplication by A, and let $\mathbf{u} = (1, -2, -1)$. Find the coordinate vector of $T_A(\mathbf{u})$ relative to the basis $S = \{(1, 1, 0), (0, 1, 1), (1, 1, 1)\}$ for R^3.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

4.

Expressing $T_A(\mathbf{u}) = (4,-2,0)$ as a linear combination of the vectors in S we obtain

$$(4,-2,0) = c_1(1,1,0) + c_2(0,1,1) + c_3(1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{ccccccc} 1c_1 & + & 0c_2 & + & 1c_3 & = & 4 \\ 1c_1 & + & 1c_2 & + & 1c_3 & = & -2 \\ 0c_1 & + & 1c_2 & + & 1c_3 & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 6 \end{bmatrix}$.

The solution of the linear system is $c_1 = -2$, $c_2 = -6$, and $c_3 = 6$.

The coordinate vector is $(T_A(\mathbf{u}))_S = (-2,-6,6)$.