4-3 In-Class Exercise

1. Determine whether the polynomial $1 + x + x^2$ is a linear combination of

$$\mathbf{p}_1 = 2 + x + x^2,\ \mathbf{p}_2 = 1 - x^2,\ \mathbf{p}_3 = 1 + 2x.$$

$$k_1\left(2+x+x^2\right)+k_2\left(1-x^2\right)+k_3\left(1+2x\right)=a+bx+cx^2 .$$

$$(2k_1+k_2+k_3)+(k_1+2k_3)x+(k_1-k_2)x^2=a+bx+cx^2 .$$

with augmented matrix $\begin{bmatrix} 2 & 1 & 1 & a \\ 1 & 0 & 2 & b \\ 1 & -1 & 0 & c \end{bmatrix}$

The coefficient matrix $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ has determinant $5 \neq 0$

So we can solve the system for all possible choices of a, b, and c.

Hence, $\mathbf{p} = 1 + x + x^2$ is in the span of $\mathbf{p}_1, \mathbf{p}_2$, and $\mathbf{p}_3$.

4-3 Suggested Exercises

1. Determine whether the following polynomials span P_2.

$$\begin{aligned} \mathbf{p}_1 &= 1 + x, & \mathbf{p}_2 &= 1 - x, \\ \mathbf{p}_3 &= 1 + x + x^2, & \mathbf{p}_4 &= 2 - x^2 \end{aligned}$$

2. Express the vector $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

as a linear combination of

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix},\ B = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$$

1. The given polynomials span P_2 if an arbitrary polynomial in P_2, $\mathbf{p} = a_0 + a_1 x + a_2 x^2$ can be expressed as a linear combination

$$a_0 + a_1 x + a_2 x^2 = k_1(1+x) + k_2(1-x) + k_3(1+x+x^2) + k_4(2-x^2)$$

Grouping the terms according to the powers of x yields

$$a_0 + a_1 x + a_2 x^2 = (k_1 + k_2 + k_3 + 2k_4) + (k_1 - k_2 + k_3)x + (k_3 - k_4)x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{ccccccccc} 1k_1 & + & 1k_2 & + & 1k_3 & + & 2k_4 & = & a_0 \\ 1k_1 & - & 1k_2 & + & 1k_3 & + & 0k_4 & = & a_1 \\ 0k_1 & + & 0k_2 & + & 1k_3 & - & 1k_4 & = & a_2 \end{array}$$

whose augmented matrix $\begin{bmatrix} 1 & 1 & 1 & 2 & a_0 \\ 1 & -1 & 1 & 0 & a_1 \\ 0 & 0 & 1 & -1 & a_2 \end{bmatrix}$ reduces

to $\begin{bmatrix} 1 & 0 & 0 & 2 & \frac{1}{2}a_0 + \frac{1}{2}a_1 - a_2 \\ 0 & 1 & 0 & 1 & \frac{1}{2}a_0 - \frac{1}{2}a_1 \\ 0 & 0 & 1 & -1 & a_3 \end{bmatrix}$ therefore the system has a solution for every choice of a_1, a_2,

and a_3. We conclude that the polynomials $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ span P_2.

2.

We need to solve the equation $k_1\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}+k_2\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}+k_3\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}+k_4\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}=\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$ to express the vector $\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$ as the desired linear combination. We can rewrite this as $\begin{bmatrix}k_1+2k_4 & -k_1+k_2+k_3\\k_4 & 2k_1+k_2-k_4\end{bmatrix}=\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix is $\begin{bmatrix}1 & 0 & 0 & 2 & 1\\-1 & 1 & 1 & 0 & 2\\0 & 0 & 0 & 1 & 2\\2 & 1 & 0 & -1 & 4\end{bmatrix}$. This matrix has reduced row echelon form $\begin{bmatrix}1 & 0 & 0 & 0 & -3\\0 & 1 & 0 & 0 & 12\\0 & 0 & 1 & 0 & -13\\0 & 0 & 0 & 1 & 2\end{bmatrix}$

hence $-3\begin{bmatrix}1 & -1\\0 & 2\end{bmatrix}+12\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}-13\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}+2\begin{bmatrix}2 & 0\\1 & -1\end{bmatrix}=\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}$.

3. Suppose that $\mathbf{v}_1 = (2, 1, 0, 3)$, $\mathbf{v}_2 = (3, -1, 5, 2)$, and $\mathbf{v}_3 = (-1, 0, 2, 1)$. Which of the following vectors are in $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

a. $(2, 3, -7, 3)$

b. $(1, 1, 1, 1)$

4. Determine whether the matrices span M_{22}.

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

3. (a)

In order for the vector $(2,3,-7,3)$ to be in $\operatorname{span}\{v_1,v_2,v_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3)+b(3,-1,5,2)+c(-1,0,2,1)=(2,3,-7,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 2 \\ 1a & - & 1b & + & 0c & = & 3 \\ 0a & + & 5b & + & 2c & = & -7 \\ 3a & + & 2b & + & 1c & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

This system is consistent (its only solution is $a=2$, $b=-1$, $c=-1$), therefore $(2,3,-7,3)$ is in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

3. (b)

In order for the vector $(1,1,1,1)$ to be in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3)+b(3,-1,5,2)+c(-1,0,2,1)=(1,1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned} 2a &+ 3b &- 1c &= 1 \\ 1a &- 1b &+ 0c &= 1 \\ 0a &+ 5b &+ 2c &= 1 \\ 3a &+ 2b &+ 1c &= 1 \end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This system is inconsistent therefore $(1,1,1,1)$ is not in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

4.

The given matrices span M_{22} if an arbitrary matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as a linear combination

$k_1\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + k_2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + k_3\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} + k_4\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We can rewrite this as

$\begin{bmatrix} k_1 + k_2 & k_2 + k_3 \\ k_1 + k_4 & k_3 + k_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Equating coefficients produces a linear system whose augmented matrix

is $\begin{bmatrix} 1 & 1 & 0 & 0 & a \\ 0 & 1 & 1 & 0 & b \\ 1 & 0 & 0 & 1 & c \\ 0 & 0 & 1 & 1 & d \end{bmatrix}$. The coefficient matrix has $\det\left(\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}\right) = 0$ which means the system is

not consistent. We conclude that the given matrices do not span M_{22}.

5. Let W be the solution space to the system $A\mathbf{x} = \mathbf{0}$. Determine whether the set $\{\mathbf{u}, \mathbf{v}\}$ spans W.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{u} = (1,\ 0,\ -1,\ 0),\ \mathbf{v} = (0,\ 1,\ 0,\ -1)$$

6. In each part, let $T_A : R^3 \to R^2$ be multiplication by A, and let $\mathbf{u}_1 = (0, 1, 1)$ and $\mathbf{u}_2 = (2, -1, 1)$ and $\mathbf{u}_3 = (1, 1, -2)$. Determine whether the set $\{T_A(\mathbf{u}_1), T_A(\mathbf{u}_2), T_A(\mathbf{u}_3)\}$ spans R^2.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

5.

The solution space W to the homogenous system $A\mathbf{x}=\mathbf{0}$ where $A=\begin{bmatrix}1&1&1&1\\1&0&1&0\\0&1&0&1\end{bmatrix}$ is obtained from the reduced row echelon form $\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&0&0\end{bmatrix}$. The general solution in vector form is $(x,y,z,w)=(s,t,-s,-t)=s(1,0,-1,0)+t(0,1,0,-1)$ therefore the solution space is spanned by the vectors $\mathbf{v}_1=(1,0,-1,0)$ and $\mathbf{v}_2=(0,1,0,-1)$. We conclude that the vectors $\mathbf{u}=(1,0,-1,0)$ and $\mathbf{v}=(0,1,0,-1)$ span the solution space W.

6.

The vectors $T_A(0,1,1)=(1,0)$, $T_A(2,-1,1)=(1,-2)$, and , $T_A(1,1,-2)=(2,3)$ span R^2 if an arbitrary vector

$\mathbf{b}=(b_1,b_2)$ can be expressed as a linear combination

$$(b_1,b_2)=k_1(1,0)+k_2(1,-2)+k_3(2,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{ccccccc} 1k_1 & + & 1k_2 & + & 2k_3 & = & b_1 \\ 0k_1 & - & 2k_2 & + & 3k_3 & = & b_2 \end{array}$$

The reduced row echelon form of the coefficient matrix of this system is $\begin{bmatrix} 1 & 0 & \frac{7}{2} \\ 0 & 1 & -\frac{3}{2} \end{bmatrix}$, therefore the system is consistent for all right hand side vectors $\mathbf{b}$.

We conclude that $T_A(\mathbf{u}_1)$, $T_A(\mathbf{u}_2)$, and, $T_A(\mathbf{u}_3)$ span R^2.