Chapter 1

Systems of Linear Equations and Matrices

Chapter 1.7

Diagonal, Triangular, and Symmetric Matrices

Diagonal Matrices

A general $n \times n$ *diagonal matrix* D can be written as

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

E.g.

$$\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Inverses and Powers of Diagonal Matrices

A diagonal matrix is invertible if and only if all of its diagonal entries are nonzero;

in this case

$$D^{-1} = \begin{bmatrix} 1/d_1 & 0 & \cdots & 0 \\ 0 & 1/d_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1/d_n \end{bmatrix}$$

Powers of diagonal matrices are easy to compute;

$$D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

EXAMPLE 1 Inverses and Powers of Diagonal Matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \quad A^{5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -243 & 0 \\ 0 & 0 & 32 \end{bmatrix}, \quad A^{-5} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{243} & 0 \\ 0 & 0 & \frac{1}{32} \end{bmatrix}$$

Compute Matrix Products Involving Diagonal Matrices

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} \\ d_1a_{41} & d_2a_{42} & d_3a_{43} \end{bmatrix}$$

Triangular Matrices

EXAMPLE 2 Upper and Lower Triangular Matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

A general 4×4 upper triangular matrix

A general 4×4 lower triangular matrix

Upper triangular matrix

Lower triangular matrix

Upper Triangular

$$a) \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 9 & 6 & 4 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad b) \begin{pmatrix} 1 & 3 & 7 & 9 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 9 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lower Triangular

$$a) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 5 & 9 & 2 & 0 \\ 7 & 6 & 4 & 8 \end{pmatrix} \qquad b) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 7 & 6 & 4 & 8 \end{pmatrix}$$

Properties of Triangular Matrices

THEOREM 1.7.1

(a) The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.

(b) The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular.

(Note: The product/transpose of diagonal matrices is diagonal.)

Properties of Triangular Matrices

THEOREM 1.7.1

(c) A triangular matrix is invertible if and only if its diagonal entries are all nonzero.

(d) Inverse of an invertible triangular matrix (either upper or lower) is triangular of the same kind.

EXAMPLE 3 Computations with Triangular Matrices

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

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EXAMPLE 3 Computations with Triangular Matrices

Consider the upper triangular matrices

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows from part (*c*) of Theorem 1.7.1 that the matrix A is invertible but the matrix B is not. Moreover, the theorem also tells us that A^{-1}, AB, and BA must be upper triangular. We leave it for you to confirm these three statements by showing that

$$A^{-1} = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{7}{5} \\ 0 & \frac{1}{2} & -\frac{2}{5} \\ 0 & 0 & \frac{1}{5} \end{bmatrix}, \quad AB = \begin{bmatrix} 3 & -2 & -2 \\ 0 & 0 & 2 \\ 0 & 0 & 5 \end{bmatrix}, \quad BA = \begin{bmatrix} 3 & 5 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 5 \end{bmatrix}$$

Symmetric Matrix

DEFINITION 1

A square matrix A is said to be *symmetric* if $A = A^T$.

EXAMPLE 4

E.g. $\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix}$, $\begin{bmatrix} 1 & 4 & 5 \\ 4 & -3 & 0 \\ 5 & 0 & 7 \end{bmatrix}$

a square matrix A is symmetric
if and only if

$$(A)_{ij} = (A)_{ji} \qquad \text{for all values of } i \text{ and } j.$$

THEOREM 1.7.2

If A and B are symmetric matrices with the same size, and if k is any scalar, then:

(a) A^T is symmetric.

(b) $A + B$ and $A - B$ are symmetric.

(c) kA is symmetric.

But, it is not true, in general, that the product of symmetric matrices is symmetric.

Proof

Trivial

THEOREM 1.7.3

The product of two symmetric matrices is symmetric if and only if the matrices commute.

Proof

let A and B be symmetric matrices with the same size.

Then

$$(AB)^T = B^T A^T = BA$$

Thus, $(AB)^T = AB$ if and only if $AB = BA$,

that is, if and only if A and B commute.

EXAMPLE 5 Products of Symmetric Matrices

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -5 & 2 \end{bmatrix}$$

Not symmetric

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -4 & 3 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

symmetric

Check if these two matrices commute.

THEOREM 1.7.4

If A is an invertible symmetric matrix, then A^{-1} is symmetric.

Proof

Assume that A is symmetric and invertible.

$$(A^{-1})^T = (A^T)^{-1} = A^{-1} \qquad \text{(because } A = A^T \text{)}$$

which proves that A^{-1} is symmetric.

Products AA^T and A^TA

If A is an $m \times n$ matrix, AA^T and A^TA are both square matrices-

$$(AA^T)^T = (A^T)^T A^T = AA^T \qquad \text{size } m \times m$$

$$(A^TA)^T = A^T(A^T)^T = A^TA \qquad \text{size } n \times n.$$

products are always symmetric

EXAMPLE 6 The Product of a Matrix and Its Transpose Is Symmetric

Let A be the 2×3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Then

$$A^T A = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$A A^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that $A^T A$ and $A A^T$ are symmetric as expected.

THEOREM 1.7.5

If A is an invertible matrix, then AA^T and A^TA are also invertible.

Proof Since A is invertible, so is A^T. by Theorem 1.4.9

Thus AA^T and A^TA are invertible, since they are the products of invertible matrices.

Recall:

THEOREM 1.4.6

If A and B are invertible matrices with the same size, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

Chapter 1-7 Objectives

- Determine whether a diagonal matrix is invertible with no computations.
- Compute matrix products involving diagonal matrices by inspection.
- Determine whether a matrix is triangular.
- Understand how the transpose operation affects diagonal and triangular matrices.
- Understand how inversion affects diagonal and triangular matrices.
- Determine whether a matrix is a symmetric matrix.