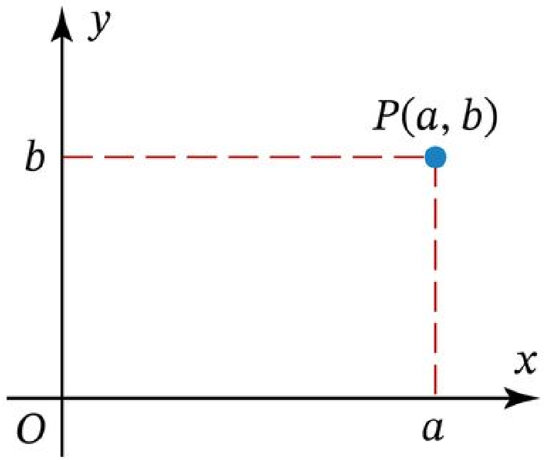
Chapter 4
General Vector Spaces

Chapter 4.5

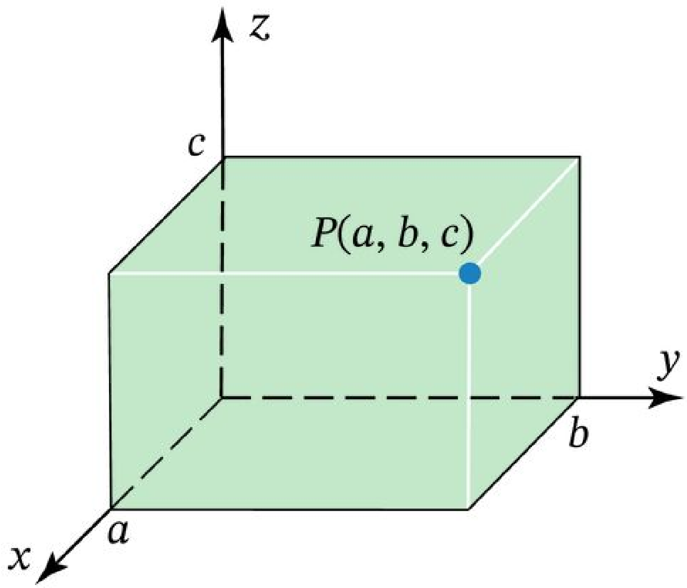
Coordinates and Basis

Coordinate Systems in Linear Algebra

Although rectangular coordinate systems are common, they are not essential.



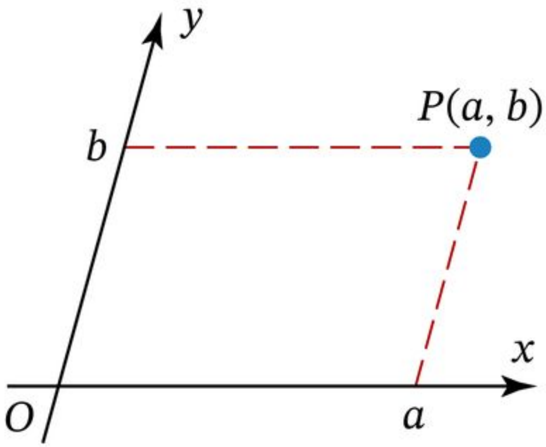
Coordinates of P in a rectangular coordinate system in 2-space.



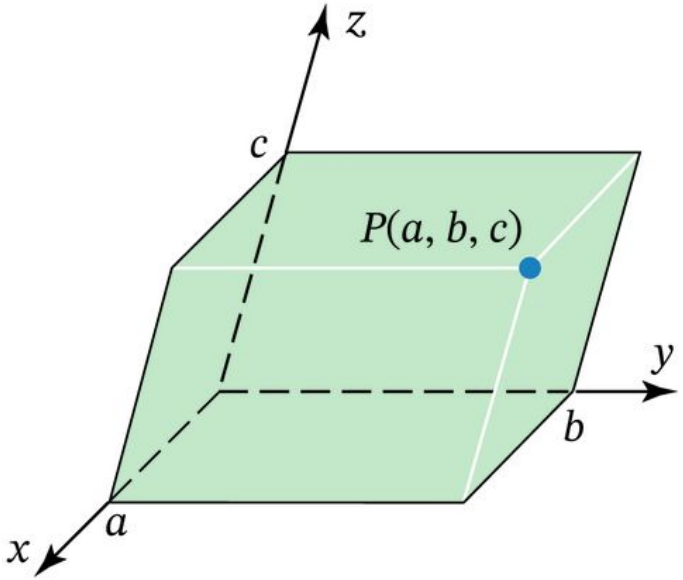
Coordinates of P in a rectangular coordinate system in 3-space.

Coordinate Systems in Linear Algebra

Although rectangular coordinate systems are common, they are not essential.

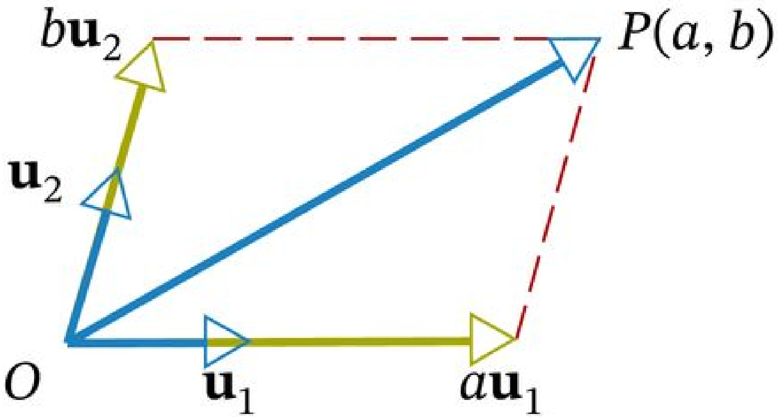


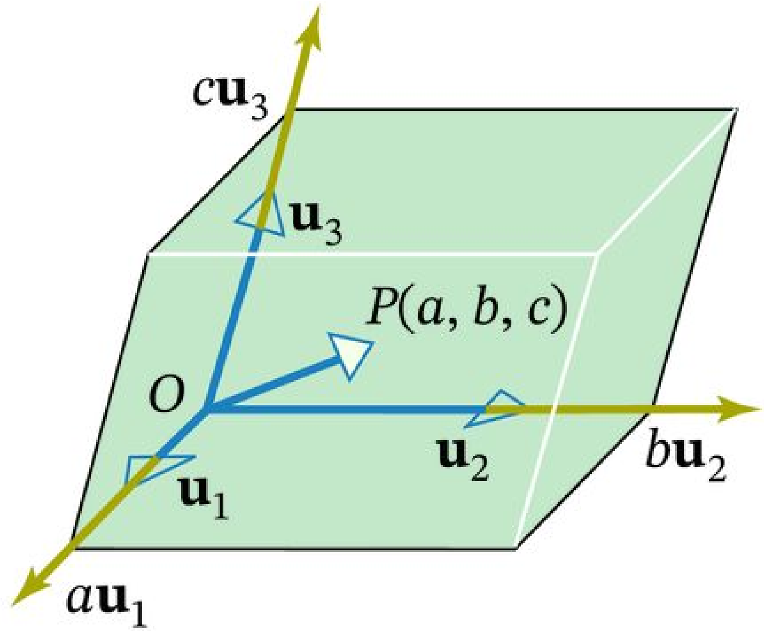
Coordinates of P in a nonrectangular coordinate system in 2-space.



Coordinates of P in a nonrectangular coordinate system in 3-space.

Coordinate Systems in Linear Algebra





$$\overrightarrow{OP} = a\mathbf{u}_1 + b\mathbf{u}_2 \quad \text{and} \quad \overrightarrow{OP} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$$

Basis for a Vector Space

DEFINITION 1

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a set of vectors in a finite-dimensional vector space V, then S is called a *basis* for V if:

(a) S is linearly independent.

(b) S spans V.

The Standard Basis

EXAMPLE 1 The standard unit vectors

$$\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \quad \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ldots, \quad \mathbf{e}_n = (0, 0, 0, \ldots, 1)$$

are call the *standard basis* for R^n

EXAMPLE 2

$S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n of polynomials

$$\mathbf{p}_0 = 1, \quad \mathbf{p}_1 = x, \quad \mathbf{p}_2 = x^2, \ldots, \quad \mathbf{p}_n = x^n$$

are call the *standard basis* for P_n

Basis for R^3

EXAMPLE 3

Show that the vectors $\mathbf{v}_1 = (1, 2, 1), \mathbf{v}_2 = (2, 9, 0)$, and $\mathbf{v}_3 = (3, 3, 4)$ form a basis for R^3.

Solution

We must show that these vectors are 1 linearly independent and 2 span R^3.

1 $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$

has only the trivial solution;

EXAMPLE 3 Cont.

2 every vector $\mathbf{b} = (b_1, b_2, b_3)$ in R^3 can be expressed as

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$$

Thus 1

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 0 \\ 2c_1 + 9c_2 + 3c_3 &= 0 \\ c_1 \qquad\;\; + 4c_3 &= 0 \end{aligned}$$

and 2

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= b_1 \\ 2c_1 + 9c_2 + 3c_3 &= b_2 \\ c_1 \qquad\;\; + 4c_3 &= b_3 \end{aligned}$$

coefficient matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix}$ $\det(A) = -1$

$\det(A) \neq 0$ implies $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ form a basis for R^3.

The Standard Basis for M_{mn}

EXAMPLE 4 Show that the matrices

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

form a basis for the vector space M_{22} of 2×2 matrices.

Solution

We must show that the matrices are linearly independent (1) and span M_{22} (2).

(1)
$$c_1M_1 + c_2M_2 + c_3M_3 + c_4M_4 = \mathbf{0}$$

has only the trivial solution, where $\mathbf{0}$ is the 2×2 zero matrix:

EXAMPLE 4 Cont.

2 every 2×2 matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ can be expressed as

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = B$$

1 $$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

2 $$c_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1 2

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

EXAMPLE 4 Cont.

1

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

2

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since the first equation has only the trivial solution

$$c_1 = c_2 = c_3 = c_4 = 0$$

the matrices are linearly independent, and since the second equation has the solution $c_1 = a, \quad c_2 = b, \quad c_3 = c, \quad c_4 = d$

the matrices span M_{22}.

This proves that the matrices M_1, M_2, M_3, M_4 form a basis for M_{22}.

They are call the *standard basis* for M_{mn}

Uniqueness of Basis Representation

THEOREM 4.5.1

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form

$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ in exactly one way.

Proof suppose that some vector $\mathbf{v}$ can be written as

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

and also as $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$

Proof (cont.)

Subtracting the second equation from the first gives

$$\mathbf{0} = (c_1 - k_1)\mathbf{v}_1 + (c_2 - k_2)\mathbf{v}_2 + \cdots + (c_n - k_n)\mathbf{v}_n$$

the linear independence of S implies that

$$c_1 - k_1 = 0, \quad c_2 - k_2 = 0, \ldots, \quad c_n - k_n = 0$$

that is, $$c_1 = k_1, \quad c_2 = k_2, \ldots, \quad c_n = k_n$$

Thus, the two expressions for $\mathbf{v}$ are the same.

Coordinates Relative to a Basis

DEFINITION 2

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an ordered basis for a vector space V, and

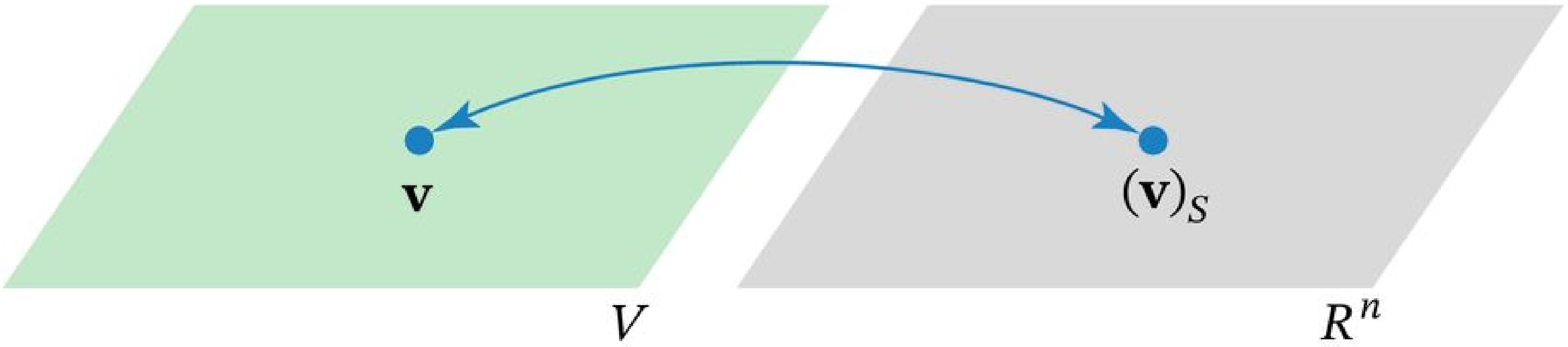
$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, \ldots, c_n$ are called the *coordinates of* $\mathbf{v}$ *relative to the basis S*. The vector $(c_1, c_2, \ldots, c_n)$ in R^n constructed from these coordinates is called the *coordinate vector of v relative to S*; it is denoted by $(\mathbf{v})_S = (c_1, c_2, \ldots, c_n)$

Coordinates Relative to a Basis

$(\mathbf{v})_S$ is a vector in R^n, so that once an ordered basis S is given for a vector space V, Theorem 4.5.1 establishes a one-to-one correspondence between vectors in V and vectors in R^n

A one-to-one correspondence



Conventionally, the order of the vectors in a basis S remains fixed.

Coordinates Relative to the Standard Bases

EXAMPLE 5

In the special case where $V = R^n$ and S is the *standard basis*, the coordinate vector $(\mathbf{v})_S$ and the vector $\mathbf{v}$ are the same

$$\mathbf{v} = (\mathbf{v})_S$$

For example, in R^3 $\quad \mathbf{v} = (a, b, c) \quad$ standard basis $S = \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

the coordinate vector relative to this basis is $(\mathbf{v})_S = (a, b, c)$, which is the same as the vector $\mathbf{v}$.

Coordinates Relative to the Standard Bases

EXAMPLE 6

(*a*) Find the coordinate vector for the polynomial

$$\mathbf{p}(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

relative to the standard basis for the vector space P_n.

(*b*) Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22}.

EXAMPLE 6 Cont.

Solution (*a*) The given formula for $\mathbf{p}(x)$ expresses this polynomial as linear combination of the standard basis vectors

$$S = \{1, x, x^2, \ldots, x^n\}.$$

$$(\mathbf{p})_S = (c_0, c_1, c_2, \ldots, c_n)$$

Solution (*b*)

We showed in Example 4 that the representation of a vector

$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as a linear combination of the standard basis vectors is

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so the coordinate vector of B relative to S is $(B)_S = (a, b, c, d)$

Coordinates Relative to an Arbitrary Basis

EXAMPLE 7 (*a*) Example 3

$$\mathbf{v}_1 = (1, 2, 1),$$
$$\mathbf{v}_2 = (2, 9, 0),$$
$$\mathbf{v}_3 = (3, 3, 4)$$

form a basis for R^3.

Find the coordinate vector of $\mathbf{v} = (5, -1, 9)$ relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(*b*) Find the vector $\mathbf{v}$ in R^3 whose coordinate vector relative to S is $(\mathbf{v})_S = (-1, 3, 2)$.

EXAMPLE 7 Cont.

Solution (*a*)

To find $(\mathbf{v})_S$ we must first express $\mathbf{v}$ as a linear combination of the vectors in S; $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$

$$(5,-1,9) = c_1(1,2,1) + c_2(2,9,0) + c_3(3,3,4)$$

$$\begin{aligned} c_1 + 2c_2 + 3c_3 &= 5 \\ 2c_1 + 9c_2 + 3c_3 &= -1 \\ c_1 \qquad\quad + 4c_3 &= 9 \end{aligned}$$

Solving this system we obtain

$$c_1 = 1, c_2 = -1, c_3 = 2$$

Therefore, $(\mathbf{v})_S = (1,-1,2)$

Solution (*b*)

Using the definition of $(\mathbf{v})_S$, we obtain

$$\begin{aligned} \mathbf{v} &= (-1)\mathbf{v}_1 + 3\mathbf{v}_2 + 2\mathbf{v}_3 \\ &= (-1)(1,2,1) + 3(2,9,0) + 2(3,3,4) = (11,31,7) \end{aligned}$$

Chapter 4-5 Objectives

- Show that a set of vectors is a basis for a vector space.
- Find the coordinates of a vector relative to a basis.
- Find the coordinate vector of a vector relative to a basis.