# 1-7 In-Class Exercise

**1**. Find all values of $x$ for which $A$ is invertible.

$$A = \begin{bmatrix} x - \frac{1}{2} & 0 & 0 \\ x & x - \frac{1}{3} & 0 \\ x^2 & x^3 & x + \frac{1}{4} \end{bmatrix}$$

From part (c) of Theorem 1.7.1, a triangular matrix is invertible if and only if its diagonal entries are all nonzero. Therefore, the given lower triangular matrix is invertible for any real number $x$ such that

$x \neq \frac{1}{2}$, $x \neq \frac{1}{3}$, and $x \neq -\frac{1}{4}$.

# 1-7 Suggested Exercise

**1.** Compute the product by inspection.

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} (-1)(3)(5) & 0 & 0 \\ 0 & (2)(5)(-2) & 0 \\ 0 & 0 & (4)(7)(3) \end{bmatrix} = \begin{bmatrix} -15 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 84 \end{bmatrix}$$

**2.** Use what you have learned in this section about multiplying by diagonal matrices to compute the product by inspection.

**a.** $\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} u & v \\ w & x \\ y & z \end{bmatrix}$ **b.** $\begin{bmatrix} r & s & t \\ u & v & w \\ x & y & z \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

**(a)** $\begin{bmatrix} au & av \\ bw & bx \\ cy & cz \end{bmatrix}$ **(b)** $\begin{bmatrix} ra & sb & tc \\ ua & vb & wc \\ xa & yb & zc \end{bmatrix}$

**3.** Show that if $A$ is a symmetric $n \times n$ matrix and $B$ is any $n \times m$ matrix, then the following products are symmetric:

$$B^T B, \quad BB^T, \quad B^T AB$$

Because $(AB)^T = B^T A^T$. Therefore we have:

$$\left(B^T B\right)^T = B^T \left(B^T\right)^T = B^T B,$$

$$\left(BB^T\right)^T = \left(B^T\right)^T B^T = BB^T, \text{ and}$$

$$\left(B^T AB\right)^T = \left(B^T (AB)\right)^T = (AB)^T \left(B^T\right)^T = B^T A^T B = B^T AB \text{ since } A \text{ is symmetric.}$$

**4.** Find a diagonal matrix $A$ that satisfies the given condition.

$$A^{-2} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For example $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$, etc.

**5.** Let $A$ be an $n \times n$ symmetric matrix.

**a)** Show that $A^2$ is symmetric.

**b)** Show that $2A^2 - 3A + I$ is symmetric.

**a)** Because $(AB)^T = B^T A^T$

Therefore, $$\left(A^2\right)^T = (AA)^T = A^T A^T = \left(A^T\right)^2 \underset{\substack{A \text{ is} \\ \text{symmetric}}}{=} A^2$$

which shows that $A^2$ is symmetric.

**b)**

$$\left(2A^2 - 3A + I\right)^T \underset{\substack{\text{Th.} \\ 1.4.8 \\ \text{(b-d)}}}{=} 2\left(A^2\right)^T - 3A^T + I^T \underset{\substack{\text{Th.} \\ 1.4.8 \\ \text{(e)}}}{=} 2\left(A^T\right)^2 - 3A^T + I^T \underset{\substack{A \text{ and } I \\ \text{are} \\ \text{symmetric}}}{=} 2A^2 - 3A + I$$

which shows that $2A^2 - 3A + I$ is symmetric.

**6.** Let $A = [a_{ij}]$ be an $n \times n$ matrix. Determine whether $A$ is symmetric.

**a.** $a_{ij} = i^2 + j^2$ **b.** $a_{ij} = i^2 - j^2$

**c.** $a_{ij} = 2i + 2j$ **d.** $a_{ij} = 2i^2 + 2j^3$

**(a)** $a_{ji} = j^2 + i^2 = i^2 + j^2 = a_{ij}$ for all $i$ and $j$ therefore $A$ is symmetric.

**(b)** $a_{ji} = j^2 - i^2$ does not generally equal $a_{ij} = i^2 - j^2$ for $i \neq j$ therefore $A$ is not symmetric (unless $n = 1$).

**(c)** $a_{ji} = 2j + 2i = 2i + 2j = a_{ij}$ for all $i$ and $j$ therefore $A$ is symmetric.

**(d)** $a_{ji} = 2j^2 + 2i^3$ does not generally equal $a_{ij} = 2i^2 + 2j^3$ for $i \neq j$ therefore $A$ is not symmetric (unless $n = 1$).

**7.** Find an upper triangular matrix that satisfies

$$A^3 = \begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$$

For a general upper triangular $2\times 2$ matrix $A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ we have

$$A^3 = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} a^2 & ab+bc \\ 0 & c^2 \end{bmatrix}\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} a^3 & a^2b+(ab+bc)c \\ 0 & c^3 \end{bmatrix} = \begin{bmatrix} a^3 & (a^2+ac+c^2)b \\ 0 & c^3 \end{bmatrix}$$

Setting $A^3 = \begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$ we obtain the equations $a^3 = 1$, $\left(a^2 + ac + c^2\right)b = 30$, $c^3 = -8$.

The first and the third equations yield $a = 1,\ c = -2$.

Substituting these into the second equation leads to $(1 - 2 + 4)b = 30$, i.e., $b = 10$.

We conclude that the only upper triangular matrix $A$ such that $A^3 = \begin{bmatrix} 1 & 30 \\ 0 & -8 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 10 \\ 0 & -2 \end{bmatrix}$.