

Chapter 1

Systems of Linear Equations and Matrices

- 1.1. Introduction to Systems of Linear Equations
- 1.2. Gaussian Elimination
- 1.3. Matrices and Matrix Operations
- 1.4. Inverses; Algebraic Properties of Matrices
- 1.5. Elementary Matrices and a Method for Finding Inverse
- 1.6. More on Linear Systems and Invertible Matrices
- 1.7. Diagonal, Triangular, and Symmetric Matrices
- 1.8. Introduction to Linear Transformations

Chapter 1.3

Matrices and Matrix Operations

Matrix and its Size

DEFINITION 1

A *matrix* is a rectangular array of numbers. The numbers in the array are called the *entries* in the matrix.

A *m x n matrix* has *m* rows and *n* columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \begin{array}{l} \text{a compact notation} \\ A = [a_{ij}] \end{array}$$

The entry in row *i* and column *j* of a matrix *A* $(A)_{ij} = a_{ij}$

Row Vectors and Column Vectors

$1 \times n$ row vector **a**
(or row matrix)

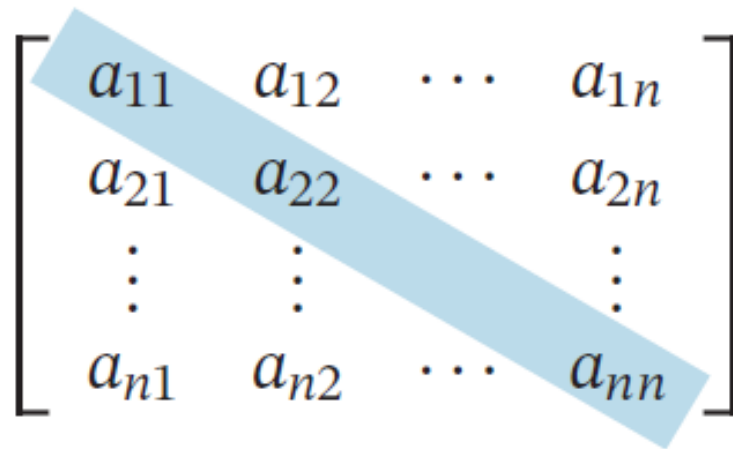
$m \times 1$ column vector **b**
(or column matrix)

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n] \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

Square Matrix & Main Diagonal

A matrix A with n rows and n columns is called
a square matrix of order n ,

and the shaded entries are said to be on the *main diagonal* of A .


$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Matrix Operations

DEFINITION 2 *equal* $A = B$ *same size and corresponding entries are equal*

DEFINITION 3 *sum* $A + B$
difference $A - B$

DEFINITION 4 *product* cA *a scalar multiple of A*

DEFINITION 5 *product* AB

EXAMPLE 1 ~ EXAMPLE 4

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EXAMPLE 1

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 3 & x \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 0 \end{bmatrix}$$

If $x = 5$, then $A = B$, but for all other values of x the matrices A and B are not equal, since not all of their corresponding entries are the same. There is no value of x for which $A = C$ since A and C have different sizes.

EXAMPLE 2

Consider the matrices

$$A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ -1 & 0 & 2 & 4 \\ 4 & -2 & 7 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 3 & 5 & 1 \\ 2 & 2 & 0 & -1 \\ 3 & 2 & -4 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} -2 & 4 & 5 & 4 \\ 1 & 2 & 2 & 3 \\ 7 & 0 & 3 & 5 \end{bmatrix} \quad \text{and} \quad A - B = \begin{bmatrix} 6 & -2 & -5 & 2 \\ -3 & -2 & 2 & 5 \\ 1 & -4 & 11 & -5 \end{bmatrix}$$

The expressions $A + C$, $B + C$, $A - C$, and $B - C$ are undefined.

EXAMPLE 3

For the matrices

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 7 \\ -1 & 3 & -5 \end{bmatrix}, \quad C = \begin{bmatrix} 9 & -6 & 3 \\ 3 & 0 & 12 \end{bmatrix}$$

we have

$$2A = \begin{bmatrix} 4 & 6 & 8 \\ 2 & 6 & 2 \end{bmatrix}, \quad (-1)B = \begin{bmatrix} 0 & -2 & -7 \\ 1 & -3 & 5 \end{bmatrix}, \quad \frac{1}{3}C = \begin{bmatrix} 3 & -2 & 1 \\ 1 & 0 & 4 \end{bmatrix}$$

It is common practice to denote $(-1)B$ by $-B$.

EXAMPLE 4

Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix}$$

Since A is a 2×3 matrix and B is a 3×4 matrix, the product AB is a 2×4 matrix. To determine, for example, the entry in row 2 and column 3 of AB , we single out row 2 from A and column 3 from B . Then, as illustrated below, we multiply corresponding entries together and add up these products.

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & \square \\ \square & \square & 26 & \square \end{bmatrix}$$

$$(2 \cdot 4) + (6 \cdot 3) + (0 \cdot 5) = 26$$

The entry in row 1 and column 4 of AB is computed as follows:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square & \square & 13 \\ \square & \square & \square & \square \end{bmatrix}$$

$$(1 \cdot 3) + (2 \cdot 1) + (4 \cdot 2) = 13$$

The computations for the remaining entries are

$$(1 \cdot 4) + (2 \cdot 0) + (4 \cdot 2) = 12$$

$$(1 \cdot 1) - (2 \cdot 1) + (4 \cdot 7) = 27$$

$$(1 \cdot 4) + (2 \cdot 3) + (4 \cdot 5) = 30$$

$$(2 \cdot 4) + (6 \cdot 0) + (0 \cdot 2) = 8$$

$$(2 \cdot 1) - (6 \cdot 1) + (0 \cdot 7) = -4$$

$$(2 \cdot 3) + (6 \cdot 1) + (0 \cdot 2) = 12$$

$$AB = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

Matrix Multiplications (Products)

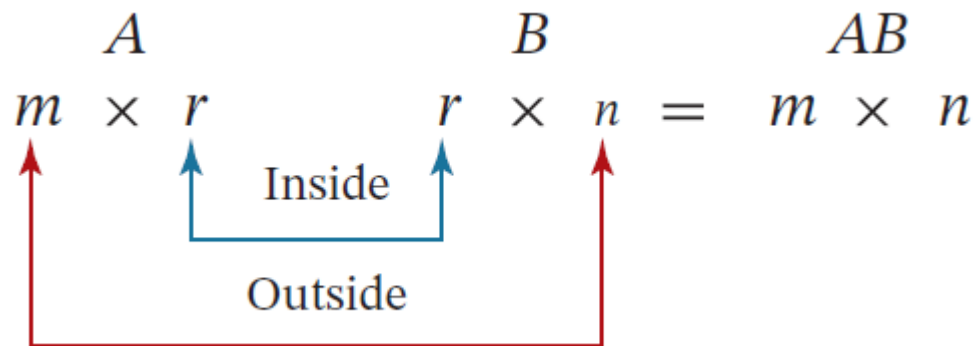
Animated Example

https://www.youtube.com/watch?v=1hf_cHNbgCk

$$\begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} =$$

?

Matrix Multiplications (Products)



EXAMPLE 5 Determining Whether a Product Is Defined

$$\begin{array}{ccc} A & B & C \\ 3 \times 4 & 4 \times 7 & 7 \times 3 \end{array}$$

AB is defined and is a 3×7 matrix; BC is defined and is a 4×3 matrix;
 CA is defined and is a 7×4 matrix.

The products AC , CB , and BA are all undefined.

Matrix Multiplications (Products)

In general, if $A = [a_{ij}]$ is an $m \times r$ matrix and $B = [b_{ij}]$ is an $r \times n$ matrix, illustrated by the shading in the following display,

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ir} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rj} & \cdots & b_{rn} \end{bmatrix}$$

the entry $(AB)_{ij}$ in row i and column j of AB is given by

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{ir}b_{rj}$$

is called the *row-column rule* for matrix multiplication.

Partitioned Matrices

$$A = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \textit{submatrices}$$

$$A = \left[\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix} \quad \textit{row vectors}$$

$$A = \left[\begin{array}{c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 \quad \mathbf{c}_4] \quad \textit{column vectors}$$

Matrix Multiplication by Columns/Rows

$$AB = A[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n] = [A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_n]$$

(AB computed column by column)

j th column vector of $AB = A[j\text{th column vector of } B]$

$$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \vdots \\ \mathbf{a}_m B \end{bmatrix}$$

(AB computed row by row)

i th row vector of $AB = [i\text{th row vector of } A]B$

Matrix Multiplication by Columns/Rows

EXAMPLE 6 Example 4 Revisited

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix} = \begin{bmatrix} 27 \\ -4 \end{bmatrix}$$



Second column
of B



Second column
of AB

$$\begin{bmatrix} 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \end{bmatrix}$$

First row of A First row of AB

Linear Combinations

DEFINITION 6

If A_1, A_2, \dots, A_r are matrices of the same size, and if c_1, c_2, \dots, c_r are scalars, then an expression of the form

$$c_1 A_1 + c_2 A_2 + \dots + c_r A_r$$

is called a *linear combination* of A_1, A_2, \dots, A_r

with *coefficients* c_1, c_2, \dots, c_r

Product of a Matrix and a Column Vector as Linear Combinations

THEOREM 1.3.1

Matrix-Vector Products Let $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n]$ be an $m \times n$ matrix, written in terms of its columns $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is any n -vector, the product $A\mathbf{x}$ is defined to be the m -vector given by:

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

In other words, the product $A\mathbf{x}$ can be expressed as a **linear combination** of the column vectors of A in which the coefficients are the entries of \mathbf{x} .

EXAMPLE 7 Matrix Products as Linear Combinations

The matrix product

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

can be written as the following linear combination of column vectors:

$$2 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

EXAMPLE 8 Columns of a Product AB as Linear Combinations

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EXAMPLE 8

$$AB = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 6 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 & 4 & 3 \\ 0 & -1 & 3 & 1 \\ 2 & 7 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 27 & 30 & 13 \\ 8 & -4 & 26 & 12 \end{bmatrix}$$

It follows from Formula (6) and Theorem 1.3.1 that the j th column vector of AB can be expressed as a linear combination of the column vectors of A in which the coefficients in the linear combination are the entries from the j th column of B . The computations are as follows:

$$\begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 27 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 7 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 30 \\ 26 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ 12 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 2 \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Column-Row Expansion

$$A = [\mathbf{c}_1 \quad \mathbf{c}_2 \quad \cdots \quad \mathbf{c}_r]_{m \times r}$$

$$B = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_r \end{bmatrix}_{r \times n}$$

$$AB = \mathbf{c}_1 \mathbf{r}_1 + \mathbf{c}_2 \mathbf{r}_2 + \cdots + \mathbf{c}_r \mathbf{r}_r$$

Column-Row Expansion

EXAMPLE 9 Find the column-row expansion of the product

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ -3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} -7 & 15 & 7 \\ 7 & -5 & 7 \end{bmatrix}$$

The column vectors of A and the row vectors of B are, respectively,

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{c}_2 = \begin{bmatrix} 3 \\ -1 \end{bmatrix}; \quad \mathbf{r}_1 = [2 \quad 0 \quad 4], \quad \mathbf{r}_2 = [-3 \quad 5 \quad 1]$$

Thus, the column-row expansion of AB is

$$\begin{aligned} AB &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} [2 \quad 0 \quad 4] + \begin{bmatrix} 3 \\ -1 \end{bmatrix} [-3 \quad 5 \quad 1] \\ &= \begin{bmatrix} 2 & 0 & 4 \\ 4 & 0 & 8 \end{bmatrix} + \begin{bmatrix} -9 & 15 & 3 \\ 3 & -5 & -1 \end{bmatrix} \end{aligned}$$

Summarizing Matrix Multiplication

Four ways [Note]

Matrix Form of a Linear System

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\&\vdots \\a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m\end{aligned}$$

A is called
the *coefficient
matrix*

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$[A \mid \mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array} \right]$$

This is the *augmented
matrix* for the system.

Transpose of a Matrix

DEFINITION 7

If A is any matrix, then the *transpose of A* , denoted by A^T , is defined to be the matrix that results by interchanging the rows and columns of A

$$(A^T)_{ij} = (A)_{ji}$$

E.g.

$$A = \begin{bmatrix} 1 & 6 & -4 \\ -8 & -2 & 9 \end{bmatrix} \text{ and } A^T = \begin{bmatrix} 1 & -8 \\ 6 & -2 \\ -4 & 9 \end{bmatrix}$$

Transpose of a Matrix

EXAMPLE 10 Some Transposes

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \\ 5 & 6 \end{bmatrix}, \quad C = [1 \quad 3 \quad 5], \quad D = [4]$$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}, \quad B^T = \begin{bmatrix} 2 & 1 & 5 \\ 3 & 4 & 6 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \quad D^T = [4]$$

Trace of a Matrix

DEFINITION 8

If A is a square matrix, then the *trace of A* , denoted by $\text{tr}(A)$, is defined to be the sum of the entries on the main diagonal of A . The trace of A is undefined if A is not a square matrix.

EXAMPLE 11 Trace of a Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 2 & 7 & 0 \\ 3 & 5 & -8 & 4 \\ 1 & 2 & 7 & -3 \\ 4 & -2 & 1 & 0 \end{bmatrix}$$

$$\text{tr}(A) = a_{11} + a_{22} + a_{33}$$

$$\text{tr}(B) = -1 + 5 + 7 + 0 = 11$$

Chapter 1-3 Objectives

- ❑ Determine the size of a given matrix.
- ❑ Identify the row vectors and column vectors of a given matrix.
- ❑ Perform the arithmetic operations of matrix addition, subtraction, scalar multiplication, and multiplication.
- ❑ Determine whether the product of two given matrices is defined.
- ❑ Compute matrix products using the row-column, the column by column, the row by row, and column-row expansion methods, respectively.
- ❑ Express the product of a matrix and a column vector as a linear combination of the columns of the matrix.

Chapter 1-3 Objectives

- ❑ Express a linear system as a matrix equation, and identify the coefficient matrix.
- ❑ Compute the transpose of a matrix.
- ❑ Compute the trace of a square matrix.