This study note synthesizes key concepts from the lecture on exponents and derivatives, incorporating visual explanations from the provided analysis blocks.

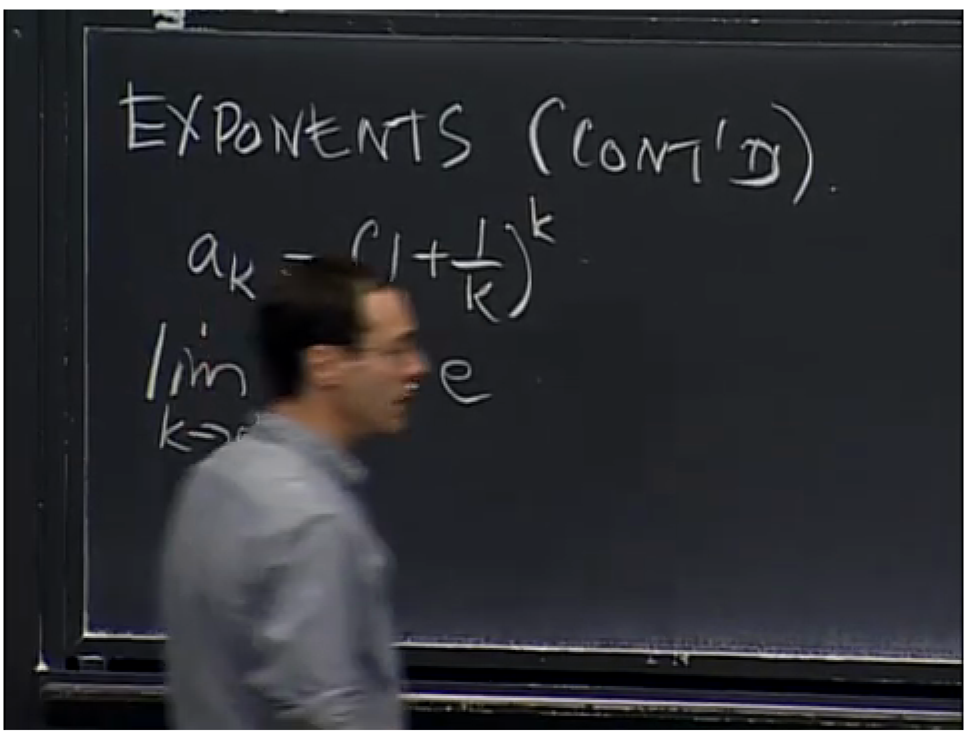
Unit 1 Review: Exponents, Derivatives, and Their Interpretations

This lecture concludes Unit 1 by reviewing fundamental concepts of exponents, differentiation rules, and the geometric and economic interpretations of derivatives. The goal is to solidify understanding for the upcoming assessment.

1. Understanding the Number *e*

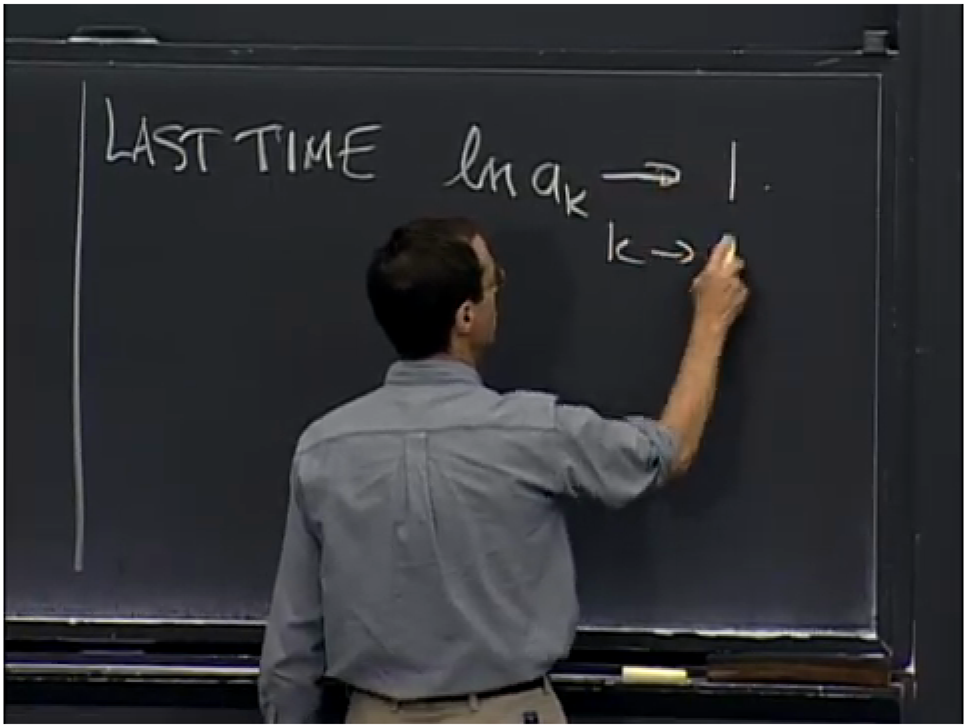
The number *e* is introduced through its limit definition.
The sequence a_k is defined as:
$a_k = \left(1 + \frac{1}{k}\right)^k$

The key insight is that the limit of a_k as k approaches infinity is *e*.

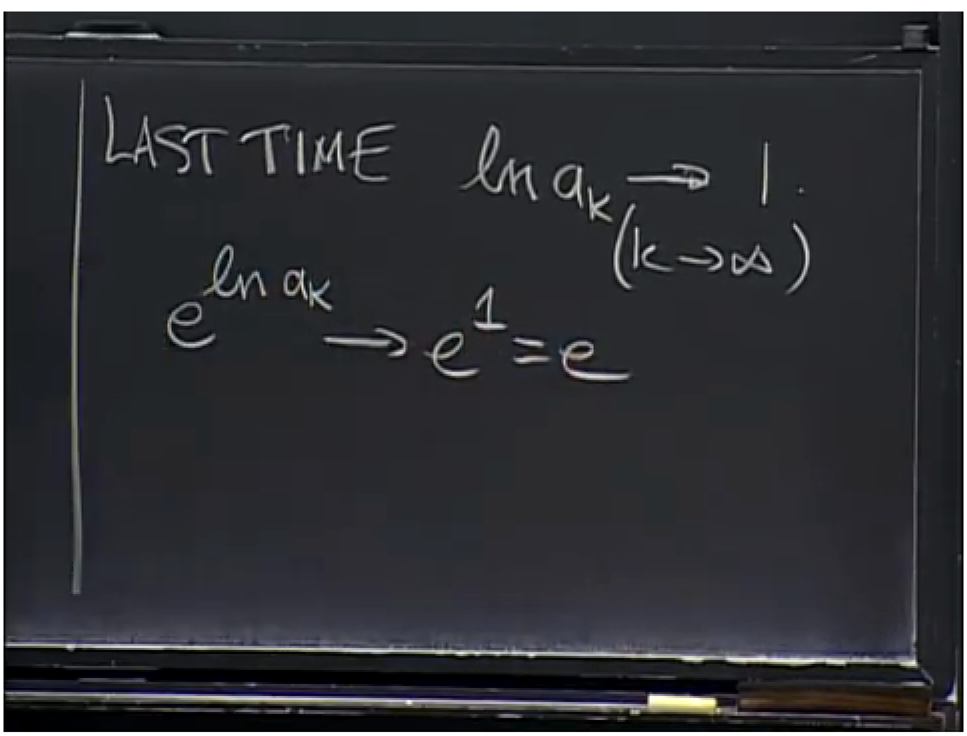


To prove this, we typically examine the natural logarithm of a_k:
$\ln(a_k) = \ln\left(\left(1 + \frac{1}{k}\right)^k\right) = k \ln\left(1 + \frac{1}{k}\right)$

As $k \to \infty$, $k \to \infty$ and $\ln\left(1 + \frac{1}{k}\right) \to \ln(1) = 0$. This results in an indeterminate form of $0 \times \infty$. Advanced techniques (like L'Hôpital's Rule, not explicitly shown but referenced in a previous lecture) demonstrate that this limit is 1.



If $\lim_{k\to\infty} \ln(a_k) = 1$, then by exponentiating both sides (using the property $e^{\ln x} = x$), we find the limit of a_k:
$e^{\lim_{k\to\infty} \ln(a_k)} = e^1 = e$
Therefore, $\lim_{k\to\infty} a_k = e$.

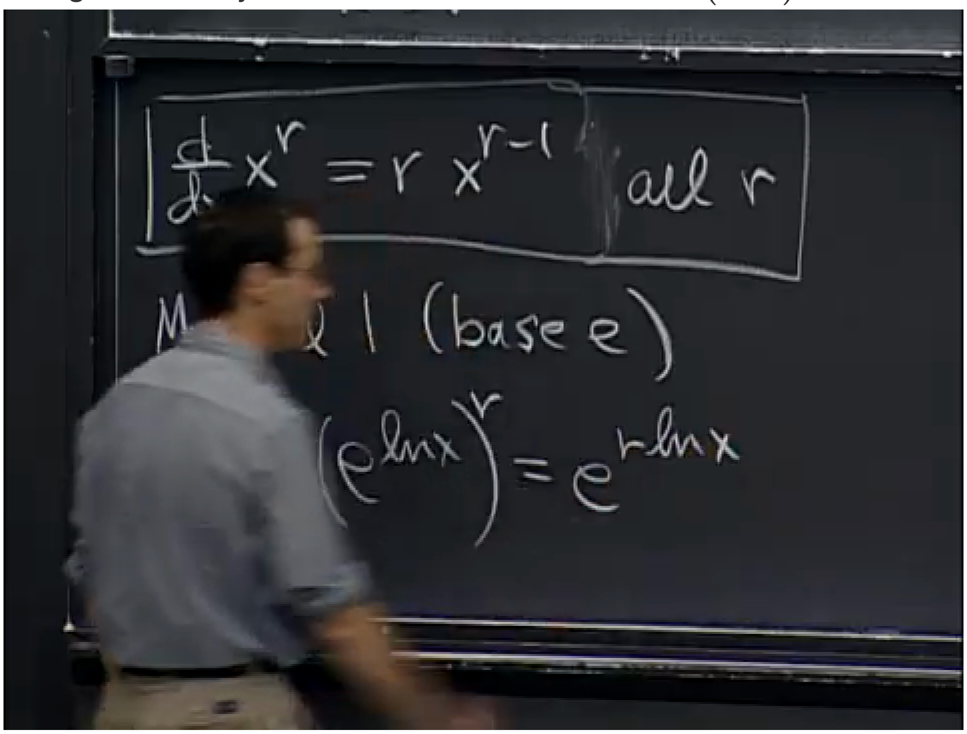


This relationship can also be read in reverse: *e* can be defined by this limit. This flexibility in perspective is crucial in mathematics.

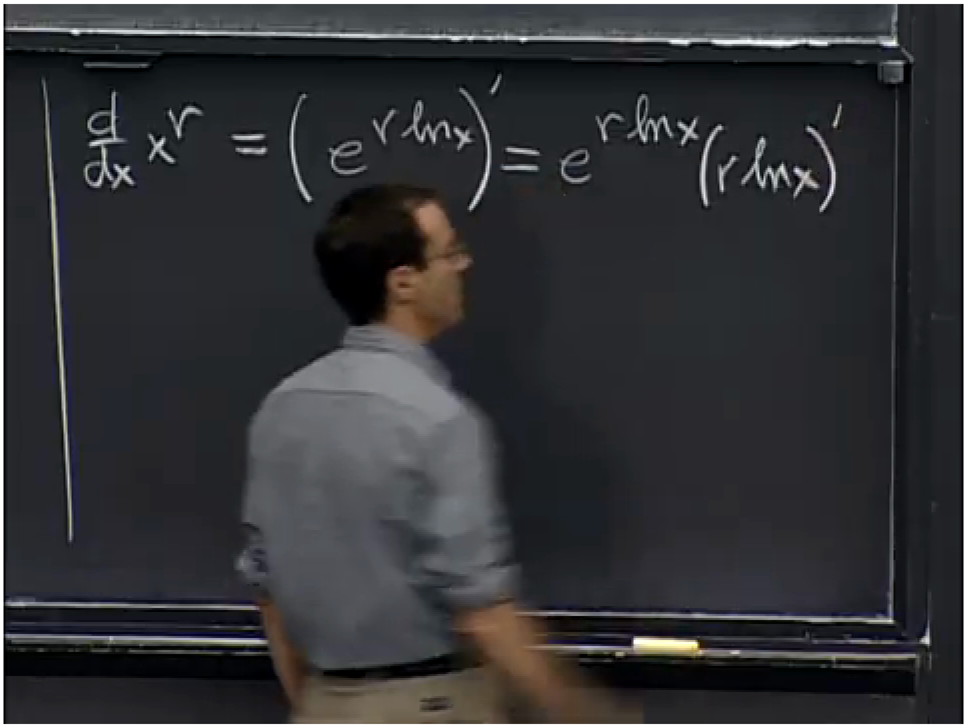
2. Deriving the Power Rule for All Real Exponents

The power rule states that $\frac{d}{dx}x^R = Rx^{R-1}$. This formula holds for all real numbers R, not just rational ones. We can prove this using two methods:

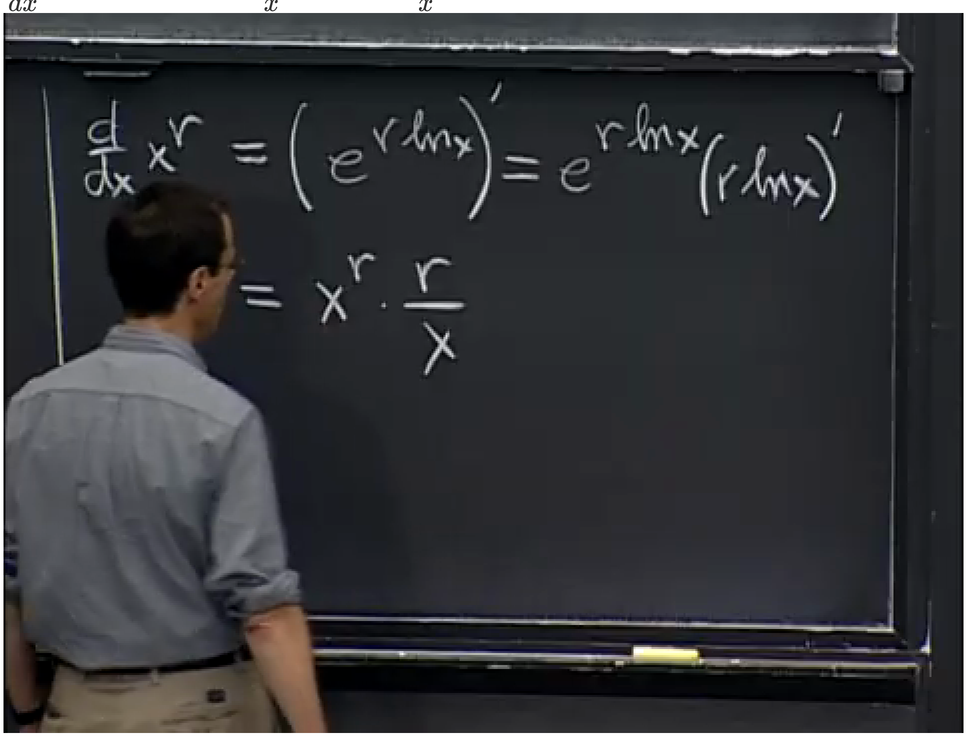
Method 1: Using Base *e* Transformation

1. **Rewrite x^R in base *e*:**
   Using the identity $x = e^{\ln x}$, we can write $x^R = (e^{\ln x})^R = e^{R \ln x}$.
   

2. **Differentiate using the Chain Rule:**
   Let $y = e^{R \ln x}$. To find $\frac{dy}{dx}$, we apply the chain rule: $\frac{d}{dx}e^{u(x)} = e^{u(x)} \cdot u'(x)$, where $u(x) = R \ln x$.
   $\frac{d}{dx}(e^{R \ln x}) = e^{R \ln x} \cdot \frac{d}{dx}(R \ln x)$

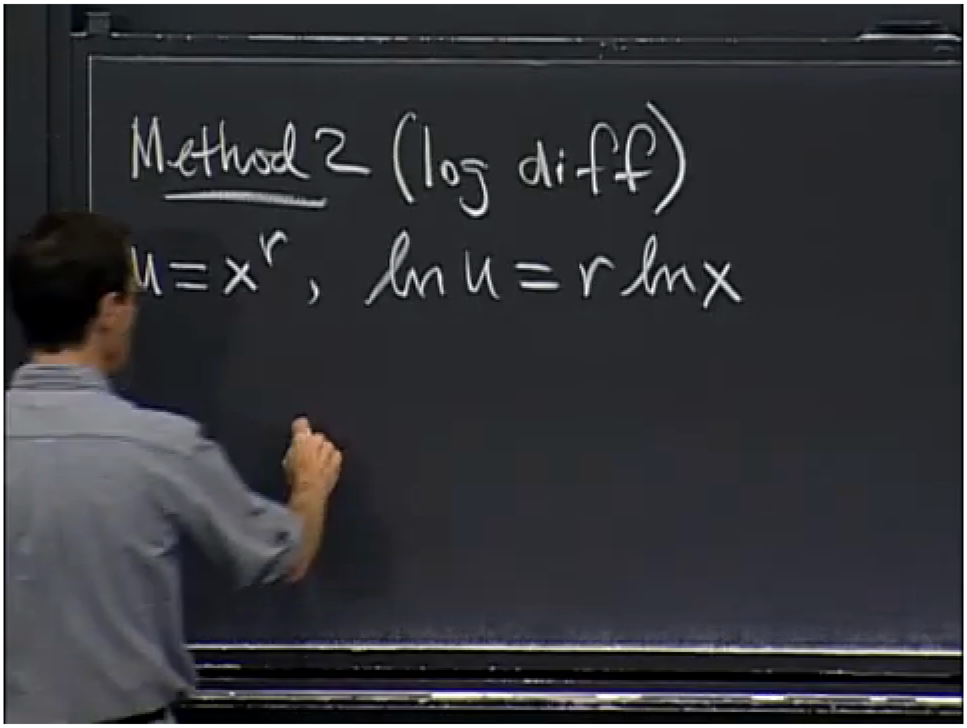


3. **Evaluate the inner derivative:**
   $\frac{d}{dx}(R \ln x) = R \cdot \frac{1}{x}$.
4. **Substitute back and simplify:**
   $\frac{d}{dx} x^R = e^{R \ln x} \cdot \frac{R}{x} = x^R \cdot \frac{R}{x} = R x^{R-1}$.

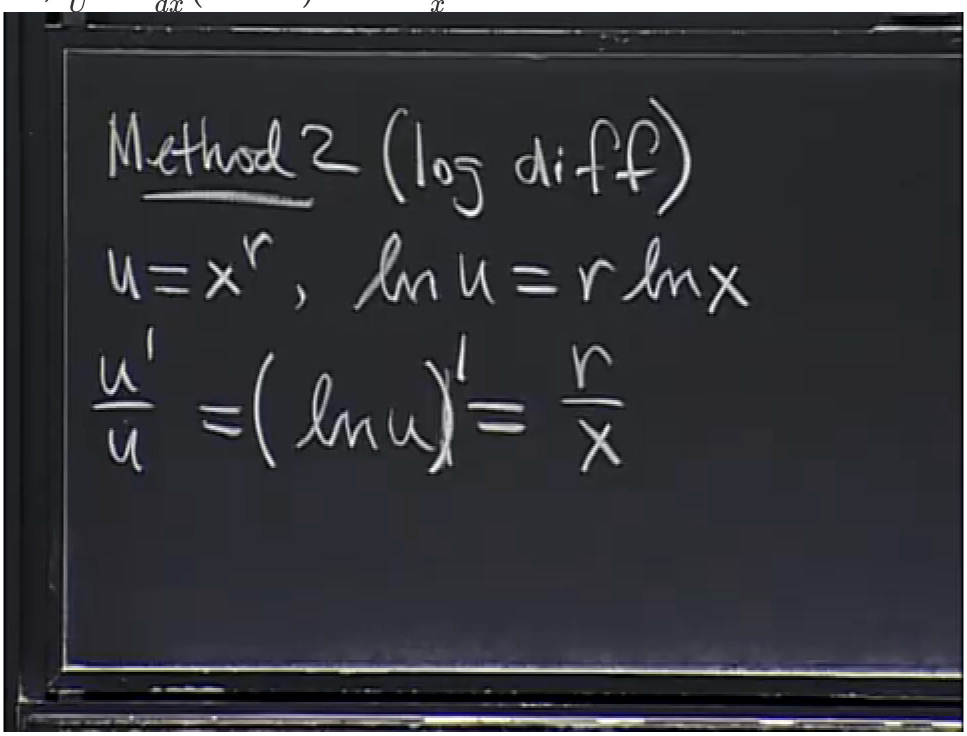


Method 2: Logarithmic Differentiation

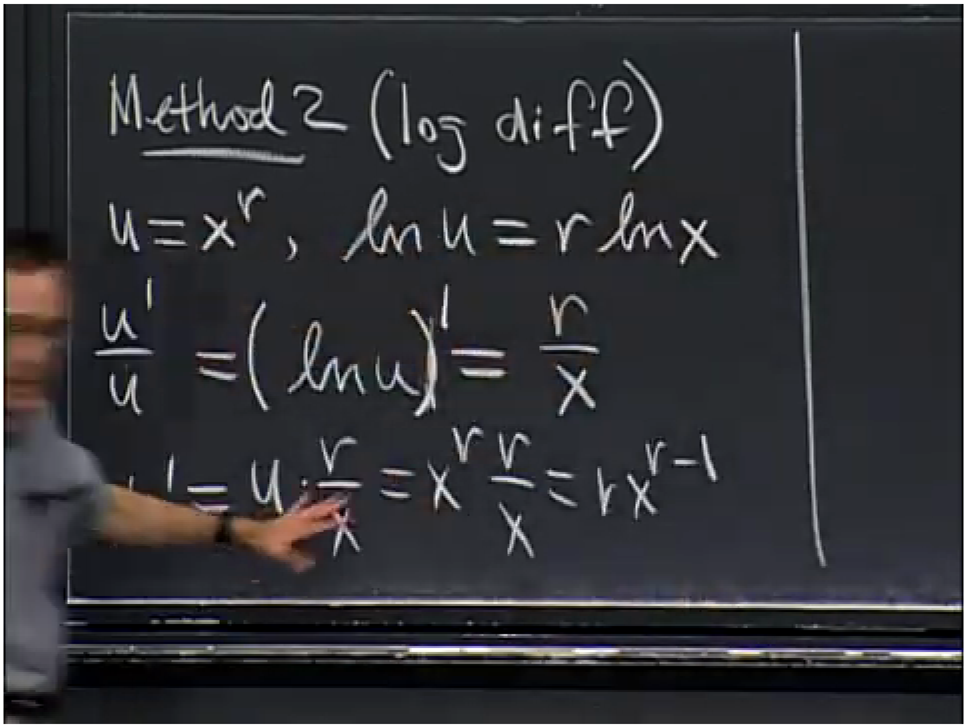
1. **Define $U = x^R$ and take the natural logarithm of both sides:**
   $\ln U = \ln(x^R) = R \ln x$.



2. **Differentiate both sides with respect to *x*:**
   Recall that $\frac{d}{dx}\ln U = \frac{U'}{U}$.
   So, $\frac{U'}{U} = \frac{d}{dx}(R\ln x) = R \cdot \frac{1}{x}$.



3. **Solve for U':**
   $U' = U \cdot \frac{R}{x}$.
   Substitute $U = x^R$ back into the equation:
   $U' = x^R \cdot \frac{R}{x} = Rx^{R-1}$.



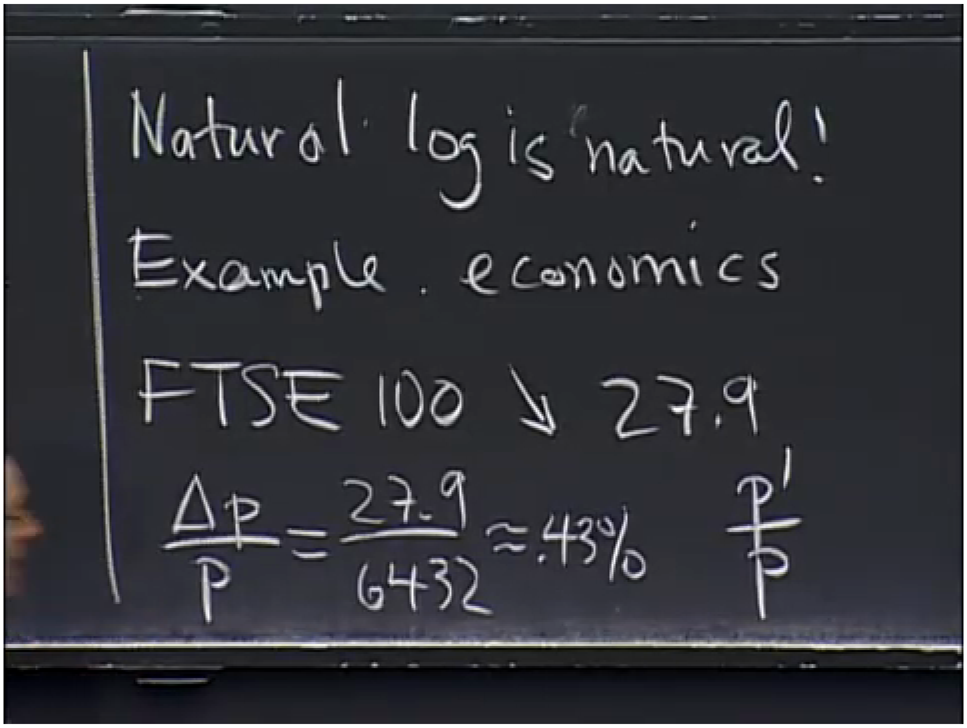
Both methods yield the same result, confirming the power rule for all real exponents.

3. The "Naturalness" of the Natural Logarithm

The natural logarithm (ln) is considered "natural" in many applications, particularly in fields like economics, because it directly relates to relative changes.

In economics, an absolute change in price (e.g., a stock price dropping by $1)isoftenmeaninglesswithoutcontext.Whatmattersisthe*relativechange*(e.g.,droppingby$1 when the price was $2vs.whenitwas$100).
The relative change is expressed as $\frac{\Delta P}{P}$.

For infinitesimal changes, this becomes $\frac{dP}{P}$ or P'/P. This expression is precisely the derivative of $\ln P$:
$\frac{d}{dP}(\ln P) = \frac{P'}{P}$



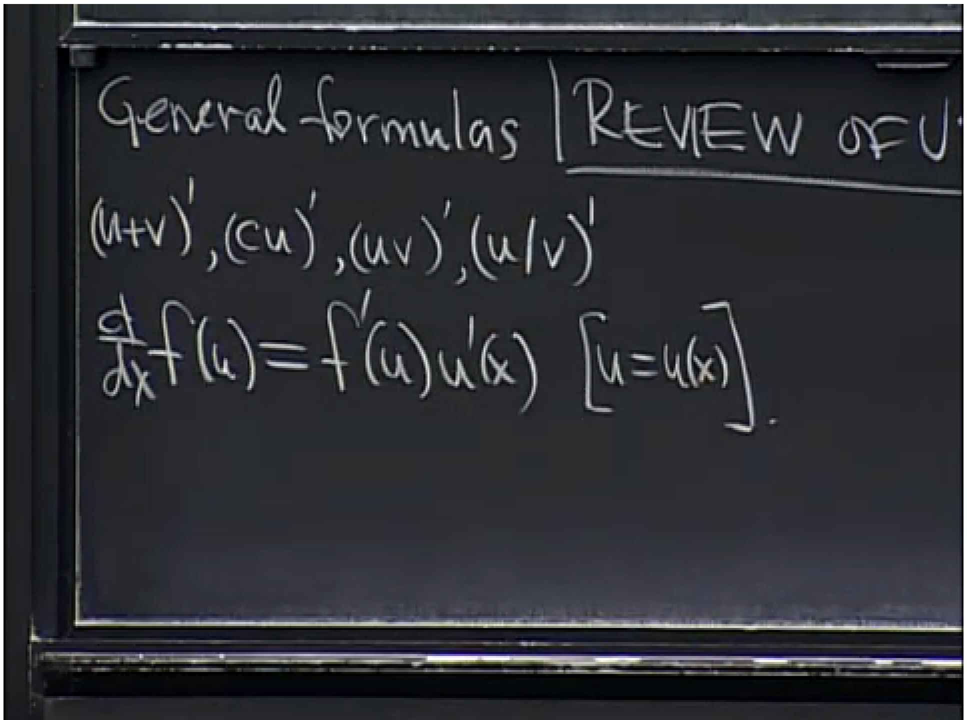
This inherent relationship makes the natural logarithm the most convenient base for analyzing proportional changes, as it avoids extra scaling factors that would arise from using other logarithm bases (like $\log_{10}$).

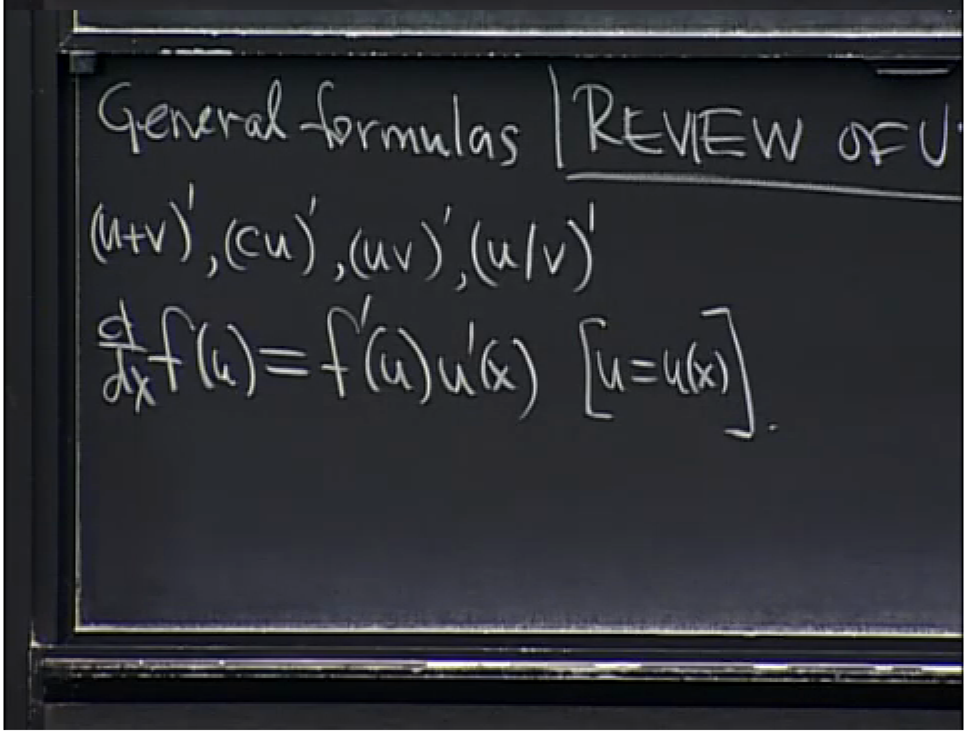
4. Review of Unit 1: Differentiation Formulas

Unit 1 covered various general rules and specific function derivatives.

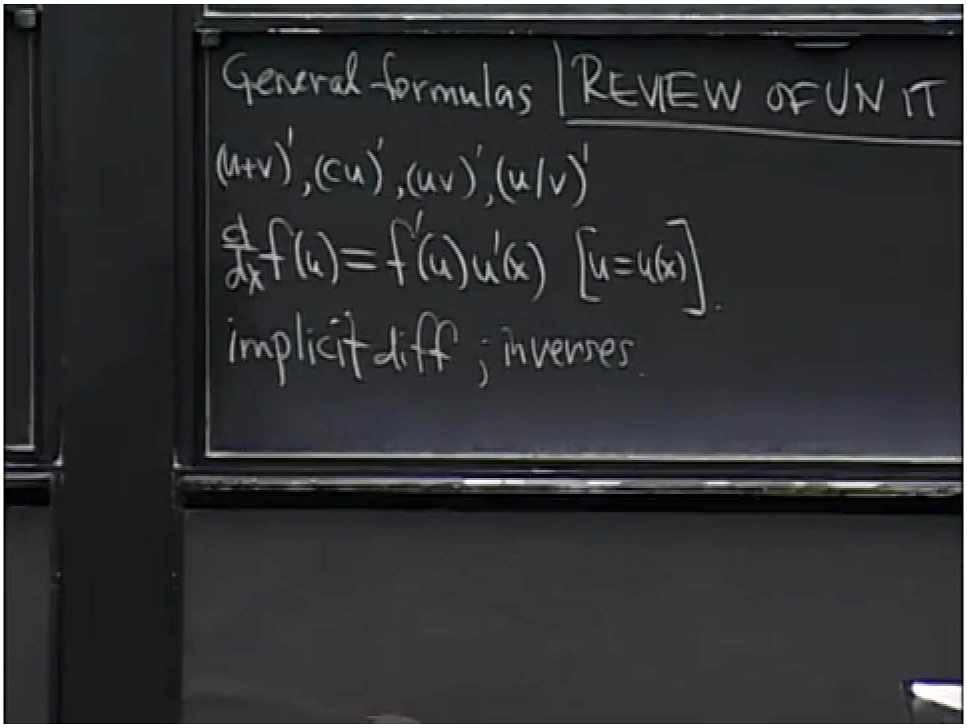
General Differentiation Formulas:

- **Sum Rule:** $(u+v)' = u' + v'$
- **Constant Multiple Rule:** $(cu)' = cu'$
- **Product Rule:** $(uv)' = u'v + uv'$
- **Quotient Rule:** $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$
- **Chain Rule:** $\frac{d}{dx} f(u(x)) = f'(u(x)) \cdot u'(x)$ or $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

- 

- 

- **Implicit Differentiation:** A technique where an equation involving x and y is differentiated with respect to x, treating y as a function of x (e.g., for finding derivatives of inverse functions like $\sin^{-1} x$ or $\ln x$). Logarithmic differentiation is a specific application.

- 

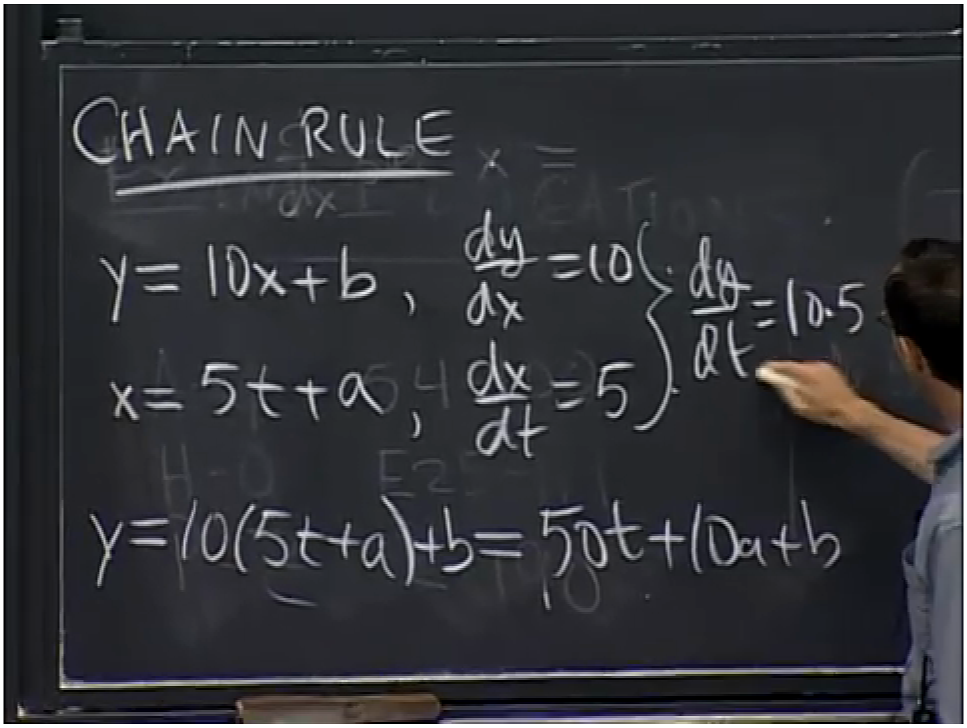
Specific Function Derivatives to Memorize:

- x^R (power rule)
- $\sin x$, $\cos x$, $\tan x$, $\sec x$
- e^x, $\ln x$
- $\arcsin x$, $\arctan x$

5. Applications and Consequences of the Chain Rule

The Chain Rule is extremely versatile and can often simplify derivations of other rules.

Example 1: Chain Rule Illustrated
Consider $y = 10x + B$, so $\frac{dy}{dx} = 10$.
Now let $x = 5t + A$, so $\frac{dx}{dt} = 5$.
If we compose them, $y = 10(5t + A) + B = 50t + 10A + B$.
Then $\frac{dy}{dt} = 50$.
This demonstrates that $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 10 \cdot 5 = 50$.



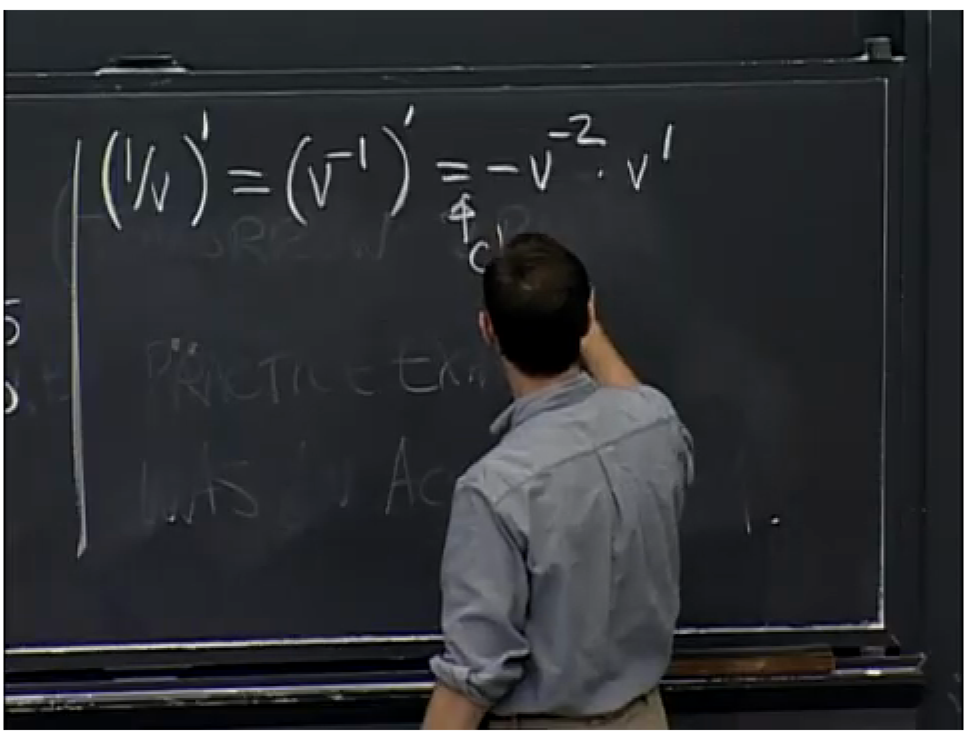
Example 2: Reciprocal Rule from Chain Rule

To differentiate $\frac{1}{V}$, rewrite it as V^{-1}.

Using the chain rule with $f(u) = u^{-1}$ and $u(x) = V(x)$:

$\frac{d}{dx}(V^{-1}) = -1 \cdot V^{-2} \cdot V'$

$= -\frac{V'}{V^2}$.

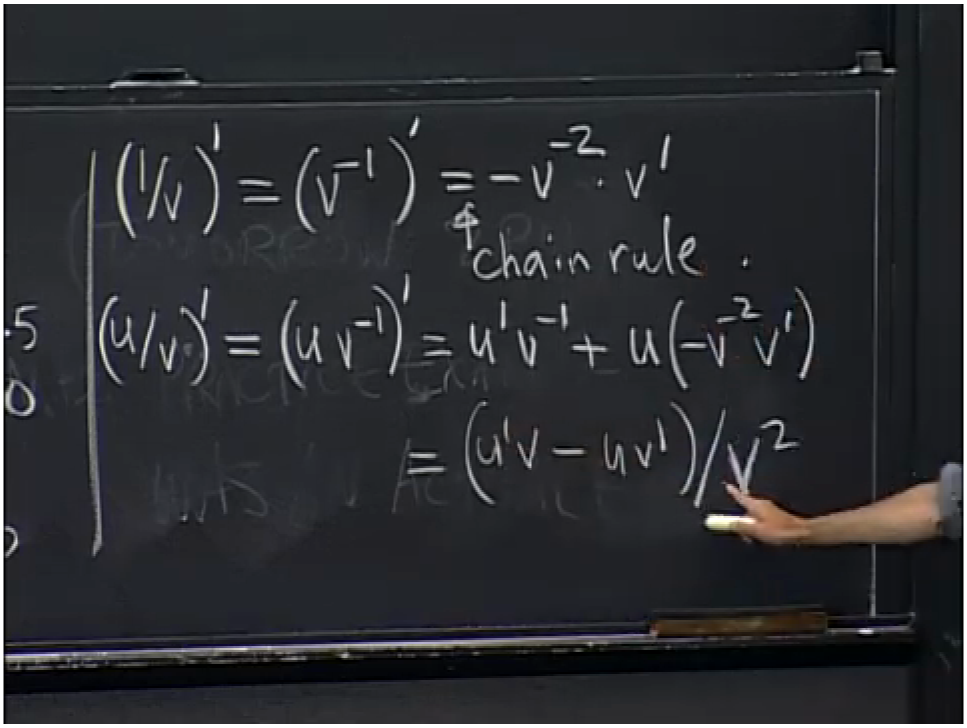


Example 3: Quotient Rule from Product Rule and Chain Rule

To differentiate $\frac{U}{V}$, rewrite it as $U \cdot V^{-1}$.

Apply the product rule: $(UV^{-1})' = U'V^{-1} + U(V^{-1})'$.

Using the reciprocal rule derived above for $(V^{-1})'$:
$= U'V^{-1} + U(-V^{-2}V')$
$= \frac{U'}{V} - \frac{UV'}{V^2}$
Find a common denominator:
$= \frac{U'V}{V^2} - \frac{UV'}{V^2} = \frac{U'V - UV'}{V^2}$.

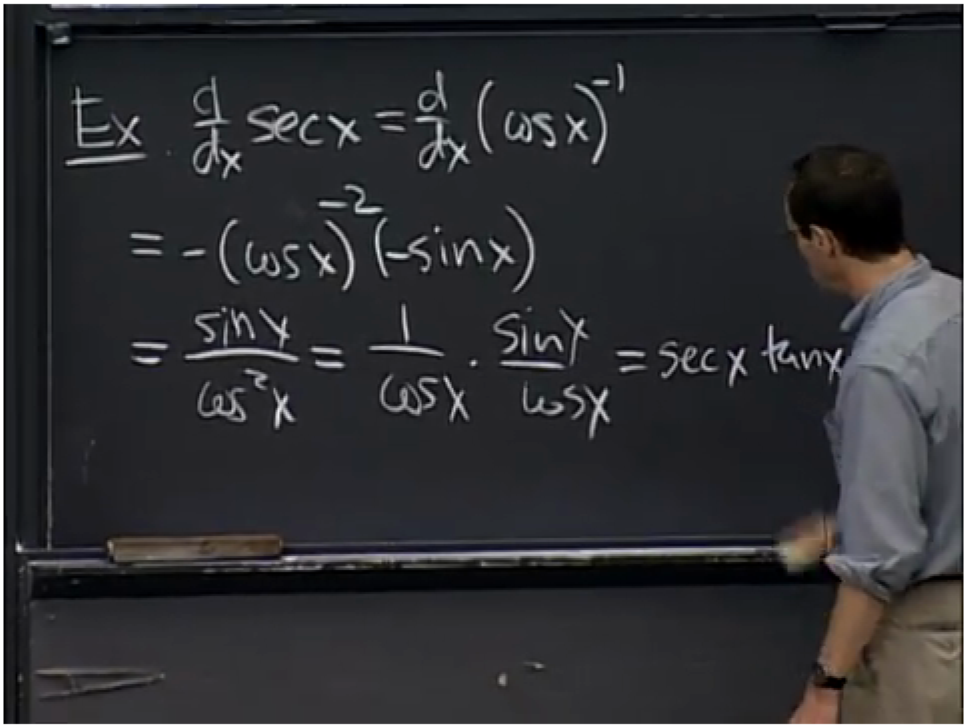


These derivations highlight how the chain rule underpins many other differentiation formulas.

6. Practice Differentiation Examples

Here are some examples demonstrating how to apply the rules:

a. Derivative of $\sec x$
Rewrite $\sec x$ as $(\cos x)^{-1}$.
Apply the chain rule:
$\frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2} \cdot \frac{d}{dx}(\cos x)$
$= -(\cos x)^{-2}(-\sin x)$
$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \tan x.$

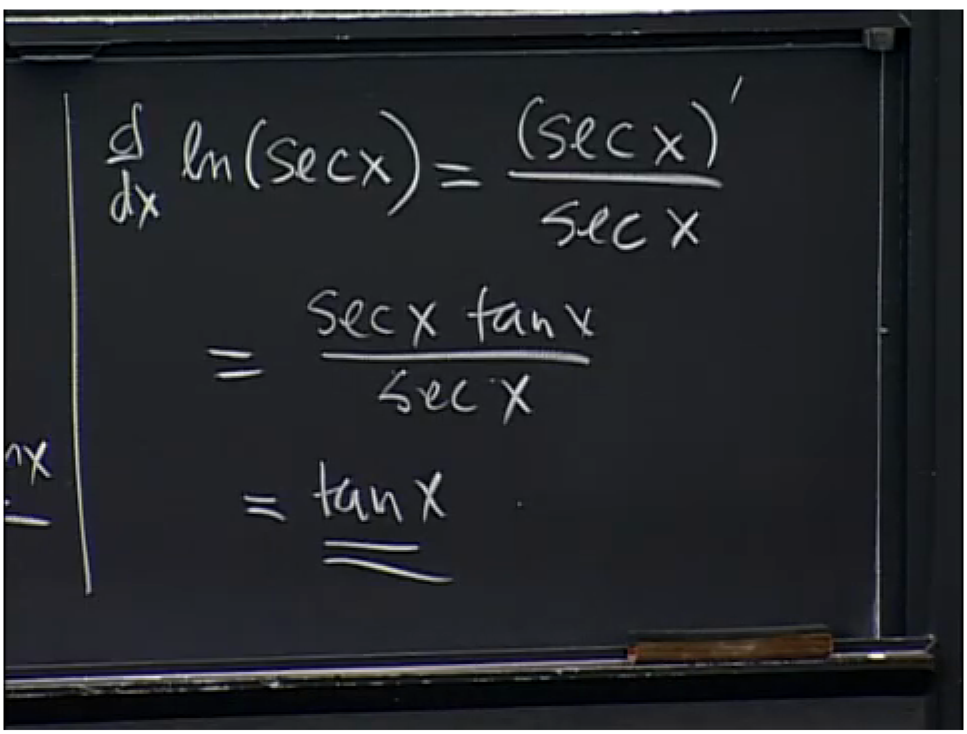


b. Derivative of $\ln(\sec x)$

Apply the chain rule:

$\frac{d}{dx}\ln(\sec x) = \frac{1}{\sec x} \cdot \frac{d}{dx}(\sec x)$

$= \frac{1}{\sec x}(\sec x \tan x) = \tan x.$

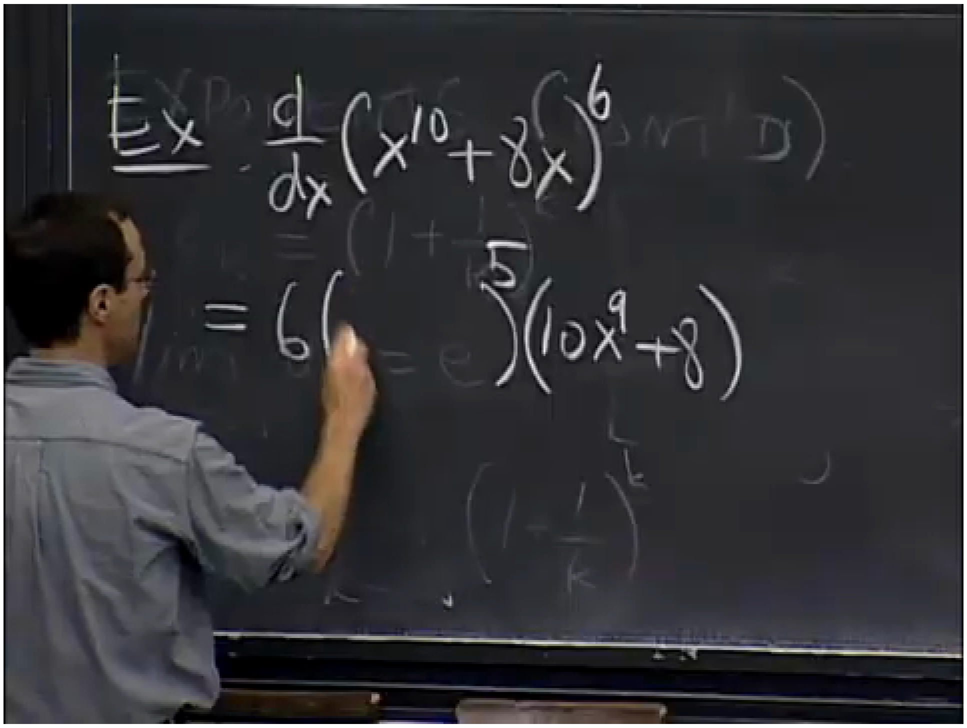


c. Derivative of $(x^{10} + 8x)^6$

Apply the chain rule: $f(u) = u^6$, $u(x) = x^{10} + 8x$.

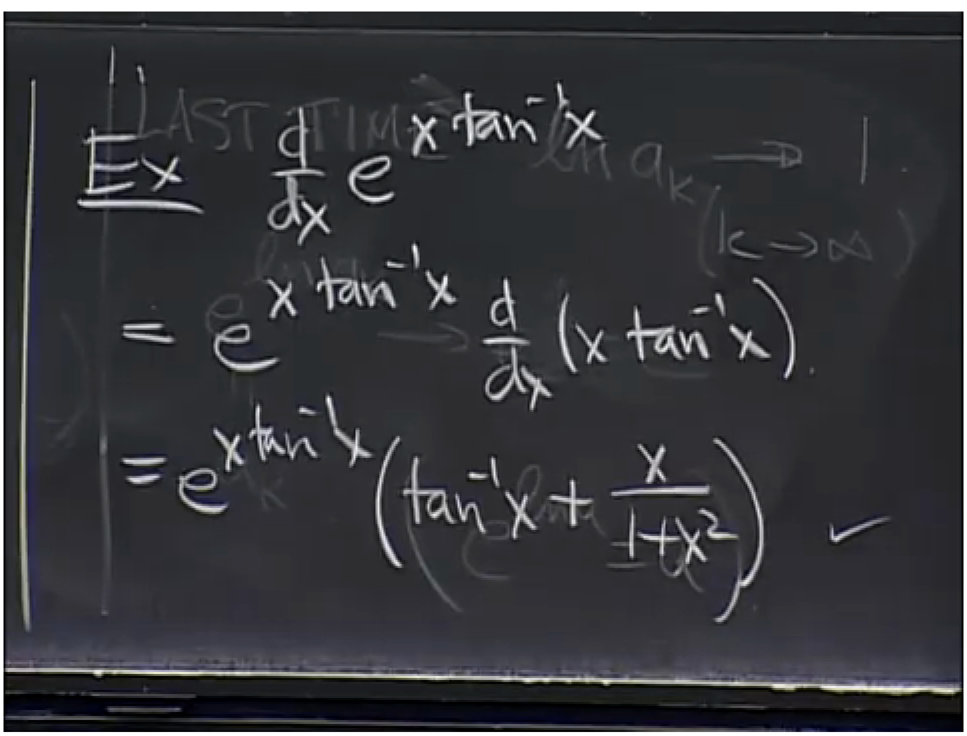
$f'(u) = 6u^5$, $u'(x) = 10x^9 + 8$.

$\frac{d}{dx}(x^{10} + 8x)^6 = 6(x^{10} + 8x)^5(10x^9 + 8).$



d. Derivative of $e^{x \arctan x}$

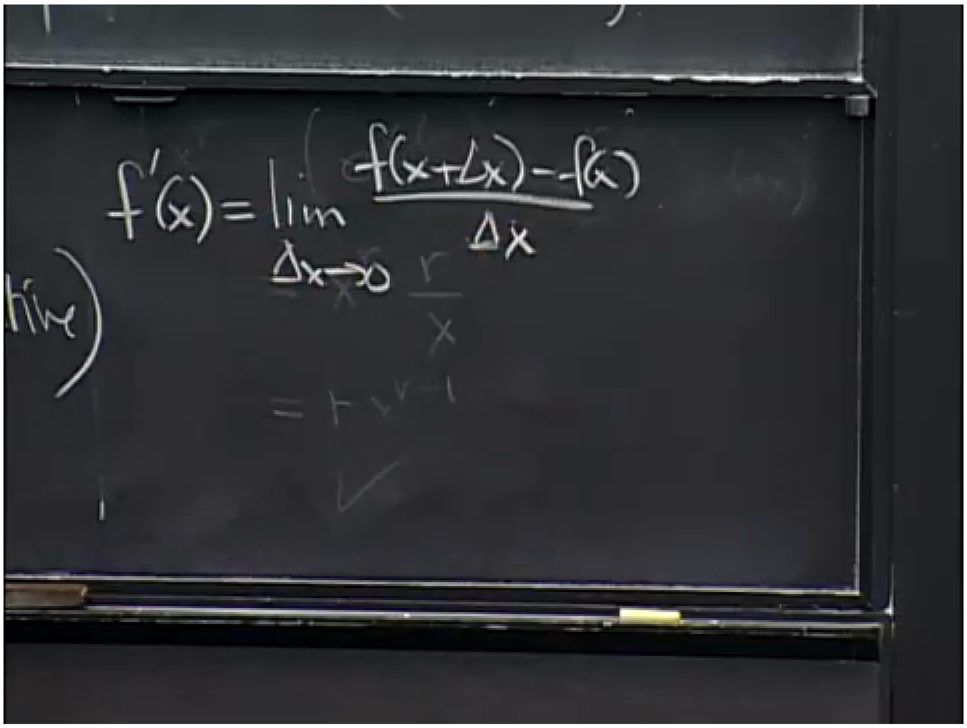
Apply the chain rule: $f(u) = e^u$, $u(x) = x \arctan x$.
$f'(u) = e^u$.
$\frac{d}{dx}(e^{x \arctan x}) = e^{x \arctan x} \cdot \frac{d}{dx}(x \arctan x)$.
Now, use the product rule for $\frac{d}{dx}(x \arctan x)$:
$\frac{d}{dx}(x \arctan x) = (1)(\arctan x) + (x)\left(\frac{1}{1+x^2}\right)$.
Combine:
$\frac{d}{dx}(e^{x \arctan x}) = e^{x \arctan x}\left(\arctan x + \frac{x}{1+x^2}\right)$.



7. Core Concepts: Definition and Geometric Meaning of the Derivative

Definition of the Derivative:

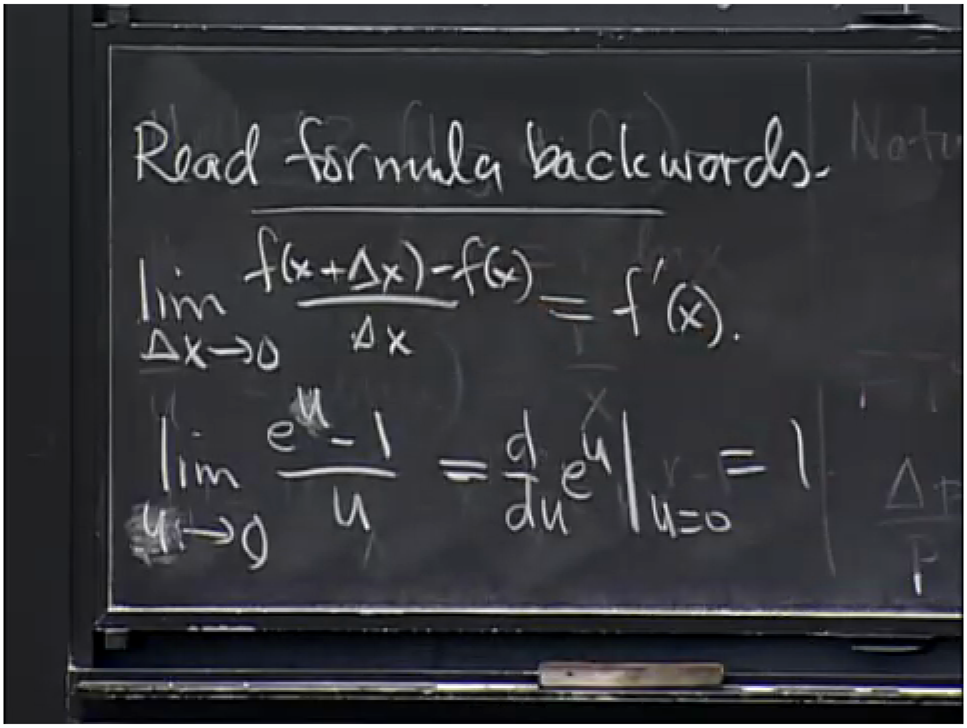
The derivative $f'(x)$ is formally defined as a limit:
$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$.



Reading the Formula Backwards (Evaluating Limits):

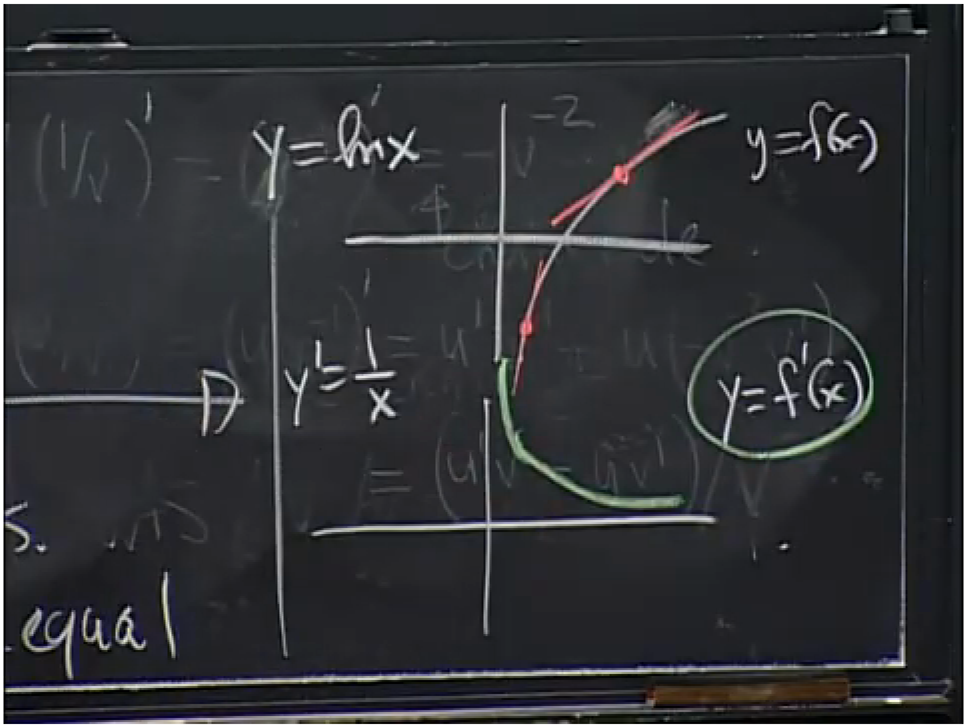
If you encounter a limit of the form $\lim_{u \to 0} \frac{f(u)-f(0)}{u}$, you should recognize it as the definition of $f'(0)$.

For example, to evaluate $\lim_{U \to 0} \frac{e^U-1}{U}$:
Recognize this as the derivative of $f(U) = e^U$ evaluated at $U = 0$, because $f(0) = e^0 = 1$.
So, $\lim_{U \to 0} \frac{e^U-1}{U} = \frac{d}{dU} e^U \Big|_{U=0} = e^U \Big|_{U=0} = e^0 = 1$.

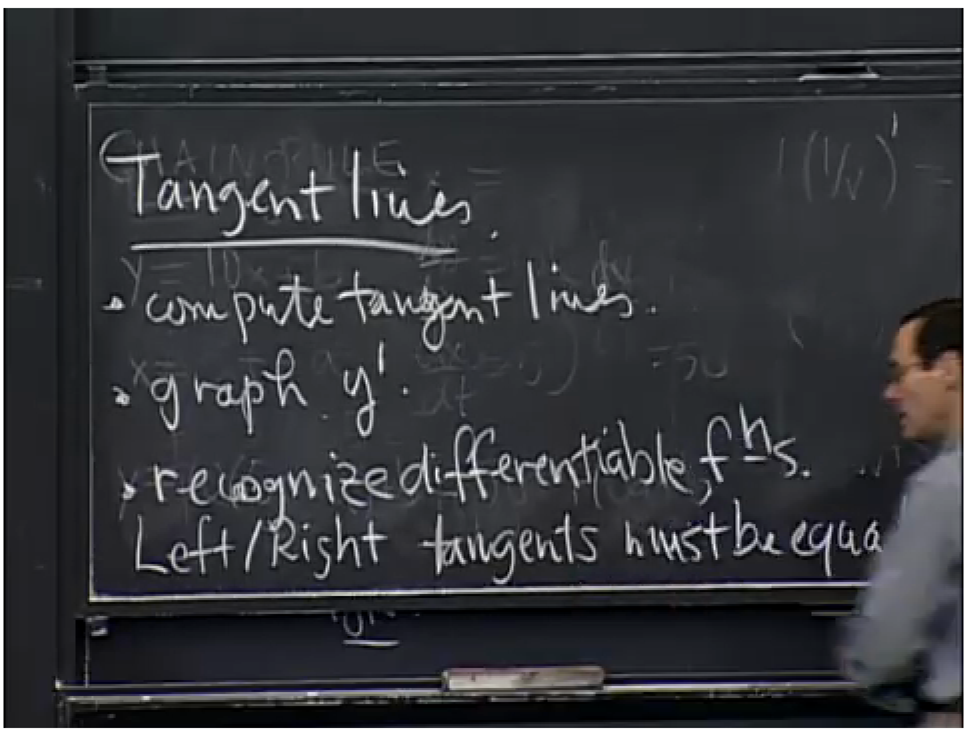


Geometric Interpretation: Tangent Lines and Graphing Derivatives

1. **Compute Tangent Lines:** The derivative $f'(x)$ gives the slope of the tangent line to the graph of $f(x)$ at point x.
2. **Graph y' from y:** You should be able to qualitatively sketch the graph of $f'(x)$ given the graph of $f(x)$ by analyzing the slopes of tangents.
   - Example: For $y = \ln x$, the graph is increasing but flattening out as x increases. This means the slope is always positive but decreasing.
   - The derivative $y' = 1/x$. The graph of $1/x$ for $x > 0$ is indeed always positive and decreases as x increases, matching the behavior of the tangent slopes of $\ln x$.



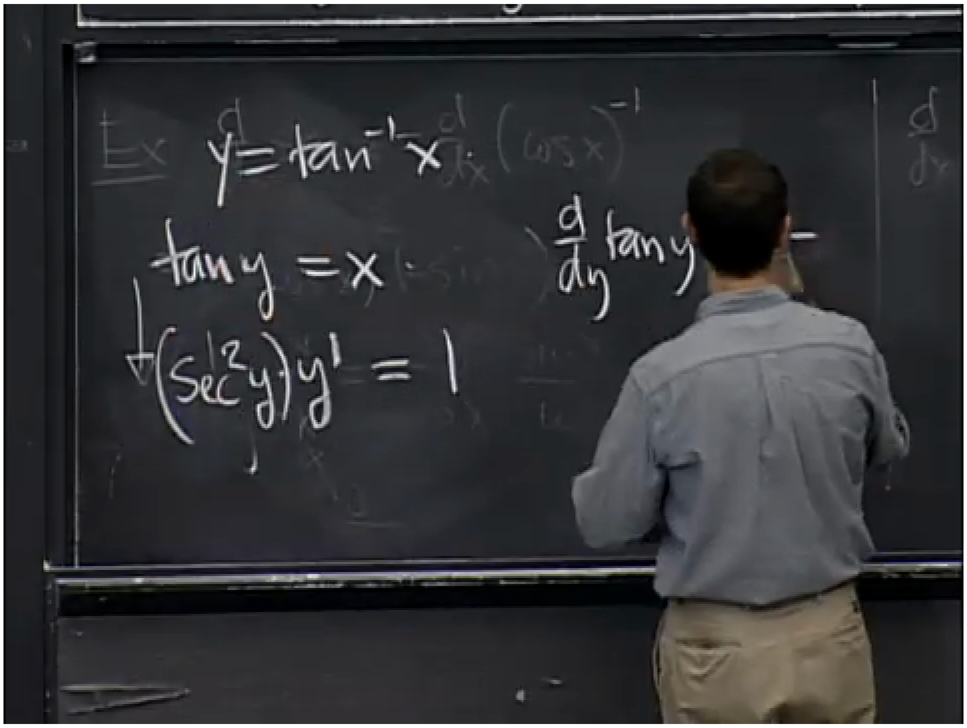
3. **Recognize Differentiable Functions:** A function is differentiable at a point if and only if the tangent line exists uniquely at that point. This means the "left tangent" (limit of slopes from the left) and "right tangent" (limit of slopes from the right) must be equal.



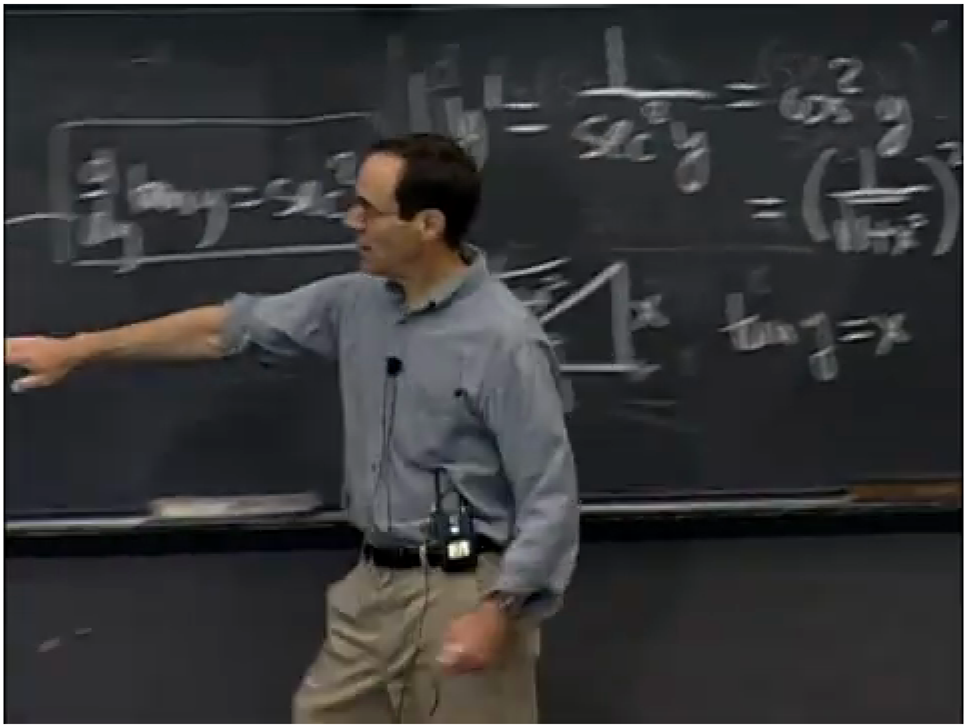
Deriving Inverse Tangent Using Implicit Differentiation:

To find $\frac{d}{dx}(\arctan x)$:

1. **Set $y = \arctan x$ and rewrite as an implicit equation:**
   $y = \arctan x \quad \Rightarrow \quad \tan y = x.$
   Setting up the inverse tangent derivation with implicit differentiation

2. **Differentiate both sides with respect to x (using the chain rule on the left side):**
   $\frac{d}{dx}(\tan y) = \frac{d}{dx}(x)$
   $\sec^2 y \cdot y' = 1.$
   

3. **Solve for y':**
   $y' = \frac{1}{\sec^2 y} = \cos^2 y.$
4. **Convert back to terms of x using a right triangle:**
   Since $\tan y = x = \frac{\text{opposite}}{\text{adjacent}}$, draw a right triangle with opposite side x and adjacent side 1.
   The hypotenuse is $\sqrt{x^2 + 1^2} = \sqrt{1 + x^2}$.
   Then $\cos y = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{1+x^2}}$.
   So, $\cos^2 y = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$.



Therefore, $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$.