# 概率论与数理统计第5次作业

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## 1.教材习题四

### (4)

(1) 由于$P(X=1)+P(X=2)=1$，故$X$可取1或2，又因为$X$与$Y$独立同分布，所以易知$(U,V)$的概率分布情况如下:

$$P(U=1,V=1)=\frac{2}{3}\cdot\frac{2}{3}=\frac{4}{9}$$
$$P(U=2,V=1)=2\cdot\frac{2}{3}\cdot\frac{1}{3}=\frac{4}{9}$$
$$P(U=2,V=2)=\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$$

(2) 由上一问求出的概率分布，及期望的定义可知，

$$E(U)=1\cdot\frac{4}{9}+2\cdot\frac{5}{9}=\frac{14}{9}$$
$$E(V)=1\cdot\frac{8}{9}+2\cdot\frac{1}{9}=\frac{10}{9}$$

(3) 根据定义可得

$$\begin{aligned}
&cov(U,V)\\
=&E(UV)-E(U)\cdot E(V)\\
=&1\cdot\frac{4}{9}+2\cdot\frac{4}{9}+4\cdot\frac{1}{9}-\frac{140}{81}\\
=&\frac{4}{81}
\end{aligned}$$

### (19)

由$EX=1,EY=2$可知$\lambda_x=1,\lambda_y=2$.因此

$$E(X^2)=\lambda_x^2+\lambda_x=2$$
$$E(Y^2)=\lambda_y^2+\lambda_y=6$$

又因为$X$和$Y$独立，所以

$$E(XY)=E(X)E(Y)$$

因此有

$$\begin{aligned}
&E(X+Y)^2\\
=&E(X^2+Y^2+2XY)\\
=&E(X^2)+E(Y^2)+2E(X)E(Y)\\
=&2+6+2\cdot 2\\
=&12
\end{aligned}$$

### (23)

*Proof.* 因为$f(x)$在$[0,+\infty]$是一个单调非减函数，所以对于任意的$|X|\geq x$，有$f(|X|)\geq f(x)$.又$f(x)>0$.由此由马尔可夫不等式可知，

$$P(|X|\geq x)=P(f(|X|)\geq f(x))\leq\frac{E[f(|X|)]}{f(x)}$$

□

## 2.

### a)

方法1:

$$\begin{aligned}
&P(max(X,Y)=n)\\
=&P(X=n,Y<n)+P(x<n,y=n)P(X=Y=n)\\
=&(1-p)^{n-1}p[1-(1-q)^{n-1}]+(1-q)^{n-1}q[1-(1-q)^{n-1}]\\
+&pq(1-q)^{n-1}(1-p)^{n-1}\\
=&p(1-p)^{n-1}+q(1-q)^{n-1}+(pq-p-q)(1-p)^{n-1}(1-q)^{n-1}
\end{aligned}$$

因此

$$E(max(X,Y)) = \sum_{n=1}^{+\infty} nP(max(X,Y))$$
$$= \sum_{n=1}^{+\infty} p(1-p)^{n-1} + q(1-q)^{n-1}$$
$$+ (pq-p-q)(1-p)^{n-1}(1-q)^{n-1}$$
$$= p\frac{1}{p^2} + q\frac{1}{q^2} + (pq-p-q)\frac{1}{(pq-p-q)^2}$$
$$= \frac{1}{p} + \frac{1}{q} + \frac{1}{pq-p-q}$$

方法2:由全期望公式可知

$$E(max(X,Y))$$
$$= \sum_{n=1}^{+\infty} P(Y=n)E[max(X,Y)|Y=n]$$
$$= \sum_{n=1}^{+\infty} P(Y=n)[\sum_{k=1}^{n} nP(X \leq n|Y=n)$$
$$+ \sum_{k=n+1}^{+\infty} kP(X > n|Y=n)]$$
$$= \frac{1}{p} + \frac{1}{q} + \frac{1}{pq-p-q}$$

## b)

$$E[X|X \leq Y]$$
$$= \sum_{n=1}^{+\infty} nP(X=n|Y \geq n)$$
$$= \sum_{n=1}^{+\infty} n(1-p)^{n-1}p(1-q)^{n-1}$$
$$= p \cdot \frac{1}{[1-(1-p)(1-q)]^2}$$
$$= \frac{p}{(pq-p-q)^2}$$

## 3.

设指示变量$X_i$如下定义:

若$\pi(i) = i$则$X_i = 1$，否则$X_i = 0$.由此可知

$$E[X] = E[\sum_{i=1}^{n} X_i]$$
$$= \sum_{i=1}^{n} E[X_i]$$
$$= \sum_{i=1}^{n} \frac{(n-1)!}{n!}$$
$$=1$$

又因为

$$E[X^2] = E[(\sum_{i=1}^{n} X_i)^2]$$
$$= \sum_{i,j \in [1,n]} E(X_iX_j)$$

由于当$i \neq j$时，$E(X_iX_j) = \frac{1}{n^2}(n > 1)$.当$i = j$时，$E(X_iX_j) = \frac{1}{n}$.因此当$n = 1$时，$E(X^2) = 1$，当$n = 2$时，

$$E(X^2) = \frac{1}{n} \cdot n + \frac{n^2-n}{n^2} = 2 - \frac{1}{n}$$

综上，$D(X) = 1 - \frac{1}{n}$.

## 4.

设$T_2$表示连续出现2个6所需实验次数，$A_{1,2}$表示出现1个6到出现连续2个6所需实验次数，于是有

$$T_2 = T_1 + A_{1,2}$$

期望值则为

$$E[T_2] = E[T_1] + E[A_{1,2}]$$
$$E[T_2] = 6 + \frac{1}{6} \cdot 1 + \frac{5}{6}(E[T_2] + 1)$$

所以可以求得

$$E[T_2] = 42$$

因此所抛次数的期望值为42.

## 5.

设$d$天后，这$d$天中涨股的天数为$X$.则$X \approx B(d,p)$.由此当$X=k$时，第$d$天的价格为

$$C = r^k(\frac{1}{r})^{d-k}$$

也即

$$P(X=k) = C_d^k p^k (1-p)^{d-k}$$

进而

$$E(C) = \sum_{k=0}^{d} r^{2k-d} C_d^k p^k (1-p)^{d-k}$$
$$= \sum_{k=0}^{d} C_d^k (pr)^k (\frac{1-p}{r})^{d-r}$$
$$= (pr + \frac{1-p}{r})^d$$

又因为

$$E(C^2) = \sum_{k=0}^{d} C_d^k (r^2 p)^k (\frac{1-p}{r^2})^{d-k}$$
$$= (pr^2 + \frac{1-p}{r^2})^d$$

因此方差

$$D(C) = E(C^2) - (E(C))^2$$
$$= (pr^2 + \frac{1-p}{r^2})^d - (pr + \frac{1-p}{r})^{2d}$$

## 6.

### a)

记异或运算结果$Y_i$得到之前的比特对为$(a_i, b_i)$.则易知

$$P(Y_i = 1) = P(a_i = 1, b_i = 0) + P(a_i = 0, b_i = 1)$$
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

由于$Y_i$的取值只能为0或1，所以

$$P(Y_i = 0) = 1 - \frac{1}{2} = \frac{1}{2}$$

### b)

*Proof.* 由于

$$\prod_{i=1}^{n(n-1)/2} P(Y_i = 1) = (\frac{1}{2})^{n(n-1)/2}$$

而当$n \geq 3$时，必有至少两个比特相同，所以

$$P(Y_1 = Y_2 = \cdots = Y_{n(n-1)/2} = 1) = 0$$

所以$Y_i$并不是相互独立. □

### c)

*Proof.* 因为

$$P(Y_i Y_j = 1) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$
$$P(Y_i Y_j = 0) = 1 - \frac{1}{4} = \frac{3}{4}$$
$$E(Y_i Y_j) = \frac{1}{4}$$
$$E(Y_i) = E(Y_j) = \frac{1}{2}$$
$$E(Y_i)E(Y_j) = \frac{1}{4}$$

所以有

$$E(Y_i Y_j) = E(Y_i)E(Y_j)$$

□

### d)

易知

$$E(Y) = E(\sum_{i=1}^{n(n-1)/2} Y_i)$$
$$= \sum_{i=1}^{n(n-1)/2} E(Y_i)$$
$$= \frac{n(n-1)}{4}$$
$$E(Y^2) = E((\sum_{i=1}^{n(n-1)/2} Y_i)^2)$$

$$= \sum_{i,j\in[1,n(n-1)/2]} E(Y_iY_j)$$

又因为当$i = j$时，$E(Y_iY_j) = \frac{1}{2}$；
当$i \neq j$时，$E(Y_iY_j) = \frac{1}{4}$.因此

$$E(Y^2) = \frac{n^4 - 4n^3 + 9n^2 - 6n}{16}$$

所以

$$D(Y) = E(Y^2) - E^2(Y) = \frac{n(n-1)(3-n)}{8}$$

## e)

由切比雪夫不等式可知

$$\begin{aligned} P(|Y - E(Y)| \geq n) &\leq \frac{D(Y)}{n^2} \\ &= \frac{1}{2} - (\frac{3}{8n} + \frac{n}{8}) \\ &\leq \frac{1}{2} - (\frac{3}{16} + \frac{2}{8}) \\ &= \frac{1}{16} \end{aligned}$$

所以上界可以是$\frac{1}{16}$.