概率论与数理统计第7、8次作业

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教材习题三

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(1)

先求分布函数$F_Z(z)$.因为X和Y独立，所以有

$$p(x,y)=p(x)\cdot p(y)$$

因此当$0<z<1$时，有

$$\begin{aligned}F_Z(z)&=P(X+Y\le z)\\&=\int_0^z\int_0^{z-x}e^{-y}dxdy\\&=\int_0^z dx\int_0^{z-x}e^{-y}dy\\&=\int_0^z(1-e^{x-z})dx\\&=(x-e^{x-z})|_0^z\\&=z-1+e^{-z}\end{aligned}$$

此时

$$p_Z(z)=F_Z'(z)=1-e^{-z}$$

当$z\ge 1$时，有

$$\begin{aligned}F_Z(z)&=P(X+Y\le z)\\&=\int_0^1 dx\int_0^{z-x}e^{-y}dy\\&=(x-e^{x-z})|_0^1\\&=1-e^{-z}(e-1)\end{aligned}$$

此时

$$p_Z(z)=F_Z'(z)=e^{-z}(e-1)$$

综上所述，

$$p_Z(z)=\begin{cases}0 & z\le 0\\ 1-e^{-z} & 0<z<1\\ e^{-z}(e-1) & z\ge 1\end{cases}$$

(3)

首先需满足$0<x<1$，否则$X=x$事件不可能发生.若$z\le x$，则有$y\le 0$，此时有$p_Z(z)=0$.因此只考察$z>x$的情况，此时

$$\begin{aligned}F_{Z|X=x}(z)&=\int_0^{z-x}e^{-y}dy\\&=-(e^{x-z}-1)\\&=1-e^{x-z}\end{aligned}$$

求导则有

$$p_{Z|X=x}(z)=e^{x-z}$$

综上所述，

$$p_{Z|X=x}(z)=\begin{cases}0 & z\le x\\ e^{x-z} & z>x\end{cases}$$

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(1)

Proof. 首先，因为$p_1(x,y)$和$p_2(x,y)$都是密度函数，所以$p_1(x,y)\ge 0$且$p_2(x,y)\ge 0$，故$p(x,y)\ge 0$.其次，有

$$\begin{aligned}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p(x,y)dxdy=0.4\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p_1(x,y)dxdy\\+0.6\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}p_2(x,y)dxdy\end{aligned}$$

$$= 0.4 + 0.6$$
$$= 1$$

因此$p(x, y)$是一个密度函数. □

(2)

$$\begin{aligned} p_X(x) &= \int_{-\infty}^{+\infty} p(x,y)dy \\ &= \frac{0.4}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} + \frac{0.6}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} \end{aligned}$$

同理,

$$p_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(y-\mu_2)^2}{2\sigma_2^2}}$$

(3)

即使二维随机向量的边缘分布都是正态分布，但联合分布也可能不是正态分布.

教材习题四

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(1)

因为

$$P(A) = \frac{1}{4}, P(B|A) = \frac{1}{3}, P(A|B) = \frac{1}{2}$$

又

$$P(AB) = P(A|B)P(B) = P(B|A)P(A)$$

所以可以求得

$$P(B) = \frac{1}{6}, P(AB) = \frac{1}{12}$$

因此

$$P(X=1, Y=1) = \frac{1}{12}$$
$$P(X=1, Y=0) = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$$
$$P(X=0, Y=1) = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$
$$P(X=0, Y=0) = 1 - \frac{1}{12} - \frac{1}{12} - \frac{1}{6} = \frac{2}{3}$$

(2)

$$E(X) = \frac{1}{4}$$
$$E(Y) = \frac{1}{6}$$
$$E(X^2) = \frac{1}{4}$$
$$E(Y^2) = \frac{1}{6}$$
$$E(XY) = \frac{1}{12}$$

因此

$$cov(X,Y) = \frac{1}{12} - \frac{1}{24} = \frac{1}{24}$$
$$D(X) = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$
$$D(Y) = \frac{1}{6} - \frac{1}{36} = \frac{5}{36}$$
$$\rho_{XY} = \frac{cov(X,Y)}{\sqrt{D(X)D(Y)}} = \frac{1}{\sqrt{15}}$$

(3)

设随机变量X和Y如题干中所示，易见X和Y服从0-1分布，有

$$E(X) = P(A), E(Y) = P(B),$$

$$D(X) = P(A)P(\overline{A}), D(Y) = P(B)P(\overline{B})$$

$$cov(X,Y) = E(XY) - E(X)E(Y) = P(AB) - P(A)P(B)$$

因此即可知:

$$\rho_{XY} = \rho_{AB}$$

所以

$$|\rho| \le 1$$

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(1)

易知

$$
\begin{aligned}
E(X_1) &= \frac{0^2+0\cdot 6+6^2}{3} = 12 \\
D(X_1) &= \frac{(6-0)^2}{12} = 3 \\
E(X_2) &= 0, D(X_2) = 4 \\
E(X_3) &= 5, D(X_3) = 5
\end{aligned}
$$

$$
\begin{aligned}
D(Y) &= D(X_1 - 2X_2 + 3X_3) \\
&= E((X_1 - 2X_2 + 3X_3)^2) \\
&- E^2(X_1 - 2X_2 + 3X_3) \\
&= D(X_1) + 4D(X_2) + 9D(X_3) \\
&= 3 + 16 + 45 \\
&= 64
\end{aligned}
$$

(2)

因为

$$
\begin{aligned}
E(Y) &= E(X_1) - 2E(X_2) + 3E(X_3) = 27 \\
E(YX_2) &= E(X_1X_2 - 2X_2^2 + 3X_2X_3) = -8
\end{aligned}
$$

所以

$$
\begin{aligned}
\rho_{YX_2} &= \frac{E(YX_2) - E(Y)E(X_2)}{\sqrt{D(Y)D(X_2)}} \\
&= \frac{-8}{\sqrt{64\cdot 4}} \\
&= -\frac{1}{2}
\end{aligned}
$$

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因为

$$
\begin{aligned}
E(X) &= \int\int_{x^2+y^2\le 1} \frac{x}{\pi} dxdy = 0 \\
E(Y) &= \int\int_{x^2+y^2\le 1} \frac{y}{\pi} dxdy = 0
\end{aligned}
$$

$$
E(XY) = \int\int_{x^2+y^2\le 1} \frac{xy}{\pi} dxdy = 0(\text{奇偶性})
$$

所以有

$$
cov(X, Y) = E(XY) - E(X)E(Y) = 0
$$

故

$$
\rho_{XY} = 0
$$

所以X和Y不相关.而因为

$$
\begin{aligned}
p_X(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2\sqrt{1-x^2}}{\pi} \\
p_Y(y) &= \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{1}{\pi} dx = \frac{2\sqrt{1-y^2}}{\pi}
\end{aligned}
$$

此时$p(x, y) = p_X(x)p_Y(y)$不能恒成立，因此它们不独立.