概率论与数理统计第11次作业

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设总体的均值为μ，方差为σ^2，则有

$$\mu = \frac{1}{2} \quad \sigma^2 = \frac{1}{12}$$

因此样本均值$\overline{X}$的期望和方差分别为

$$E(\overline{X}) = \frac{1}{12} \quad D(\overline{X}) = \frac{1}{12n}$$

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易知$\frac{X_i}{\sqrt{5}}] \sim N(0,1)$，于是有

$$Y^2 = \sum_{i=1}^{10}(\frac{X_i}{\sqrt{5}})^2 \sim \chi^2(10)$$

因此

$$P(\sum_{i=1}^{10} X_i^2 > 80) = P(\frac{1}{5}\sum_{i=1}^{10} X_i^2 > 16) = P(Y^2 > 16)$$

查表可知

$$\chi^2_{0.1}(10) = 16$$

因此可求得

$$P(\sum_{i=1}^{10} X_i^2 > 80) = 0.1$$

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因为$X_1, X_2, X_3, \cdots, X_{15}$是来自正态总体$N(0,4)$的样本，所以$X_1, X_2, \cdots, X_{15}$两两相互独立，因此$\sum_{i=1}^{10} X_i^2$和$\sum_{i=11}^{15} X_i^2$独立.此时令$n_1 = 10, n_2 = 5$，则有

$$Y = \frac{\sum_{i=1}^{10} X_i^2/10}{\sum_{i=11}^{15} X_i^2/5} = \frac{\sum_{i=1}^{10}(\frac{X_i}{2})^2/10}{\sum_{i=11}^{15}(\frac{X_i}{2})^2/5} = F(10,5)$$

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设$Z_i = X_i + X_{n+i} - 2\mu$.由于$X_1, X_2, \cdots, X_{2n}$是来自正态总体$N(\mu, \sigma^2)$的样本，所以$Z_1, Z_2, \cdots, Z_n$两两独立，且有

$$Z \sim Z_i \sim N(0, 2\sigma^2)$$

由此有

$$\overline{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i = \overline{X} - 2\mu$$

所以

$$Y = \sum_{i=1}^{n}(Z_i - \overline{Z})^2 = \sum_{i=1}^{n} Z_i^2 - 2\overline{Z}\sum_{i=1}^{n} Z_i + n\overline{Z}^2$$

又因为

$$\sum_{i=1}^{n} Z_i = n\overline{Z}$$

所以

$$Y = \sum_{i=1}^{n} Z_i^2 - n\overline{Z}^2$$

因此

$$\begin{aligned} E(Y) &= \sum_{i=1}^{n} E(Z_i^2) - nE(\overline{Z}^2) \\ &= nE(Z^2) - nE(\overline{Z}^2) \end{aligned}$$

$$
\begin{aligned}
&= n \cdot 2\sigma^2 - n(D(\overline{Z}^2) + E^2(\overline{Z})) \\
&= 2n\sigma^2 - n(\frac{2\sigma^2}{n} + 0) \\
&= 2(n-1)\sigma^2
\end{aligned}
$$

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因为$X_1, X_2, \cdots, X_{n+1}$是来自正态总体$N(\mu, \sigma^2)$的样本，所以$X_{n+1} - \overline{X}$也服从正态分布.与此同时，

$$E(X_{n+1} - \overline{X}) = 0 \quad D(X_{n+1} - \overline{X}) = \sigma^2 + \frac{\sigma^2}{n}$$

因此

$$\frac{X_{n+1} - \overline{X}}{\sigma\sqrt{\frac{n+1}{n}}} \sim N(0,1)$$

又因为

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$$

所以

$$\frac{X_{n+1} - \overline{X}}{S}\sqrt{\frac{n}{n+1}} = \frac{\frac{X_{n+1} - \overline{X}}{\sigma\sqrt{\frac{n+1}{n}}}}{\sqrt{\frac{(n-1)S^2/\sigma^2}{n-1}}} \sim t(n-1)$$

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(1) 首先易知

$$\frac{\overline{X} - 12}{2/\sqrt{5}} \sim N(0,1)$$

所以

$$P(\overline{X} > 13) = P(\frac{\overline{X} - 12}{2/\sqrt{5}} > \frac{\sqrt{5}}{2}) = 1 - \Phi(1.118)$$

所以

$$P(\overline{X} > 13) = 0.1314$$

(2) $\min_{1 \le i \le 5} X_i < 10$等价于存在$i \in [1,5]$且$i \in N$使得$X_i < 10$.设该事件为$A$，则有

$$
\begin{aligned}
P(A) &= 1 - P(\overline{A}) \\
&= 1 - (P(X_1 \geq 10))^5 \\
&= 1 - (\phi(1))^5 \\
&= 0.5785
\end{aligned}
$$

(3) 同理，设事件B为$\max_{1 \le i \le 5} X_i > 15$.则有

$$
\begin{aligned}
P(B) &= 1 - P(\overline{B}) \\
&= 1 - (P(X_1 \leq 15))^5 \\
&= 1 - (\Phi(1.5))^2 \\
&= 0.2923
\end{aligned}
$$

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不 妨 设 第 一 组 样 本 为$X_1, X_2, \cdots, X_{n_1}$，第 二 组 样 本 为$Y_1, Y_2, \cdots, Y_{n_2}$.设 联 合 样 本 为$Z_1, Z_2, \cdots, Z_{n_1+n_2}$.于是

$$\overline{Z} = \frac{\sum_{i=1}^{n_1+n_2} Z_i}{n_1 + n_2} = \frac{\sum_{i=1}^{n_1} X_i + \sum_{j=1}^{n_2} Y_j}{n_1 + n_2} = \frac{n_1\overline{X} + n_2\overline{Y}}{n_1 + n_2}$$

并且

$$
\begin{aligned}
S_3^2 &= \frac{1}{n_1 + n_2 - 1}\sum_{i=1}^{n_1+n_2}(Z_i - \overline{Z})^2 \\
&= \frac{\sum_{i=1}^{n_1}(X_i - \overline{Z})^2 + \sum_{j=1}^{n_2}(Y_j - \overline{Z})^2}{n_1 + n_2 - 1}
\end{aligned}
$$

又因为

$$S_1^2 = \frac{\sum_{i=1}^{n_1}(X_i - \overline{X})^2}{n_1 - 1} \quad S_2^2 = \frac{\sum_{j=1}^{n_2}(Y_j - \overline{Y})^2}{n_2 - 1}$$

由此可知

$$S_3^2 = \frac{\sum_{i=1}^{n_1}(X_i - \overline{X} + \overline{X} - \overline{Z})^2 + \sum_{j=1}^{n_2}(Y_j - \overline{Y} + \overline{Y} - \overline{Z})^2}{n_1 + n_2 - 1}$$

于是设

$$S_3^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2 + K}{n_1 + n_2 - 1}$$

其中

$$K = \sum_{i=1}^{n_1}(\overline{X} - \overline{Z})^2 + \sum_{i=1}^{n_1} 2(X_i - \overline{X})(\overline{X} - \overline{Z})$$

$$
\begin{aligned}
&+\sum_{j=1}^{n_2}(\overline{Y}-\overline{Z})^2+\sum_{j=1}^{n_2}2(Y_j-\overline{Y})(\overline{Y}-\overline{Z}) \\
&=n_1(\overline{X}-\overline{Z})^2+n_2(\overline{Y}-\overline{Z})^2 \\
&=\frac{n_1 n_2}{n_1+n_2}(\overline{X}-\overline{Y})^2
\end{aligned}
$$

因此

$$
S_3^2=\frac{(n_1-1)S_1^2+(n_2-1)S_2^2+\frac{n_1 n_2}{n_1+n_2}(\overline{X}-\overline{Y})^2}{n_1+n_2-1}
$$