概率论与数理统计第12次作业

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习题七

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先求矩估计：

$$\begin{aligned}\mu_1 &= EX\\ &= \int_0^1 p(x;a)xdx\\ &= \int_0^1 (\alpha+1)x^{\alpha+1}dx\\ &= \frac{\alpha+1}{\alpha+2}x^{\alpha+2}|0^1\\ &= \frac{\alpha+1}{\alpha+2}\end{aligned}$$

又因为

$$A_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$$

令$EX = X$，则有

$$\frac{\hat{\alpha}+1}{\hat{\alpha}+2} = \overline{X}$$

即矩估计值为

$$\hat{\alpha} = \frac{1-2\overline{X}}{\overline{X}-1}$$

另一方面，似然函数为

$$\begin{aligned}L(\alpha) &= \prod_{i=1}^{n}(\alpha+1)x_i^{\alpha}\\ &= (\alpha+1)^n(\prod_{i=1}^{n}x_i)^{\alpha}\end{aligned}$$

$$\ln L(\alpha) = n\ln(\alpha+1) + \alpha\sum_{i=1}^{n}\ln x_i$$

令$\frac{d\ln L(\alpha)}{d\alpha} = 0$，则有

$$\frac{n}{\alpha+1} + \sum_{i=1}^{n}\ln x_i = 0$$

即可求得α的极大似然估计值为

$$\hat{\alpha} = \frac{n}{-\sum_{i=1}^{n}\ln x_i} - 1$$

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先求矩估计：

$$\begin{aligned}\mu_1 &= EX\\ &= \int_0^{+\infty}(2\pi\sigma^2)^{-\frac{1}{2}}\exp\{-\frac{1}{2\sigma^2}(\ln x-\mu)^2\}dx\\ &= (2\pi\sigma^2)^{-\frac{1}{2}}\int_0^{+\infty}\exp\{-\frac{1}{2\sigma^2}(\ln x-\mu)^2\}dx\end{aligned}$$

令$\frac{\ln x-\mu}{\sqrt{2}\sigma} = t$，则有

$$\begin{aligned}EX &= \int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{-t^2}\cdot\sqrt{2}\sigma e^{\sqrt{2}\sigma t+\mu}dt\\ &= \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-(t-\frac{\sqrt{2}}{2}\sigma)^2}dt\end{aligned}$$

令$m = t - \frac{\sqrt{2}}{2}\sigma$，则有

$$EX = \frac{e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-m^2}dm$$

$$= e^{\mu+\frac{\sigma^2}{2}}$$

类似地通过积分可以求得

$$\mu_2 = E(X^2) = e^{2\sigma^2+2\mu}$$

所以可得方差

$$D(X) = (e^{\sigma^2} - 1)e^{2\mu+\sigma^2}$$

又因为

$$A_1 = \overline{X} \quad A_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

令

$$\mu_1 = A_1 \quad \mu_2 = A_2$$

可以解得矩估计

$$\hat{\mu} = 2\ln\overline{X} - \frac{1}{2}\ln(\frac{1}{n}\sum_{i=1}^{n} X_i^2)$$

$$\hat{\sigma^2} = \ln(\frac{1}{n}\sum_{i=1}^{n} X_i^2) - 2\ln\overline{X}$$

另一方面，其似然函数为

$$\begin{aligned} L(\mu,\sigma^2) &= \prod_{i=1}^{n}(2\pi\sigma^2)^{-\frac{1}{2}}\exp\{-\frac{1}{2\sigma^2}(\ln x_i - \mu)^2\} \\ &= (2\pi\sigma^2)^{-\frac{n}{2}}\exp\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu)^2\} \end{aligned}$$

$$\ln L(\mu,\sigma^2) = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln x_i - \mu)^2$$

令

$$\frac{\partial \ln L(\mu,\sigma^2)}{\partial\mu} = 0$$

$$\frac{\partial \ln L(\mu,\sigma^2)}{\partial\sigma^2} = 0$$

可以解得极大似然估计值

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\ln X_i$$

$$\hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(\ln X_i - \frac{1}{n}\sum_{i=1}^{n}\ln X_i)^2$$

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先求矩估计：

$$\begin{aligned} \mu_1 &= EX \\ &= \int_{\mu}^{+\infty}\frac{1}{\theta}xe^{-(x-\mu)/\theta}dx \\ &= -\int_{\mu}^{+\infty}xd(e^{-(x-\mu)/\theta}) \\ &= -xe^{-(x-\mu)/\theta})|_{\mu}^{\infty} + \int_{\mu}^{+\infty}e^{-(x-\mu)/\theta}dx \\ &= \mu+\theta \\ \mu_2 &= E(X^2) \\ &= \int_{\mu}^{+\infty}\frac{1}{\theta}x^2e^{-(x-\mu)/\theta}dx \\ &= \mu^2 + 2\int_{\mu}^{+\infty}xe^{-(x-\mu)/\theta}dx \\ &= \mu^2 + 2\theta(\mu+\theta) \end{aligned}$$

令

$$\overline{X} = \mu_1 \quad \frac{1}{n}\sum_{i=1}^{n}X_i^2 = \mu_2$$

解得矩估计值为

$$\hat{\mu} = \overline{X} - S_n \quad \hat{\theta} = S_n$$

另一方面，其似然函数为

$$\begin{aligned} L(\mu,\theta) &= \prod_{i=1}^{n}\frac{1}{\theta}e^{-(x_i-\mu)/\theta} \\ &= (\theta)^{-n}e^{-\frac{1}{\theta}\sum_{i=1}^{n}(x_i-\mu)} \\ &= (\theta)^{-n}e^{-\frac{1}{\theta}(\sum_{i=1}^{n}x_i-n\mu)} \\ &= (\theta)^{-n}e^{-\frac{1}{\theta}(n\overline{X}-n\mu)} \end{aligned}$$

$$\ln L(\mu,\theta) = -n\ln\theta - \frac{1}{\theta}(n\overline{X} - n\mu)$$

令

$$\frac{\partial \ln L(\mu,\theta)}{\partial\mu} = \frac{\partial \ln L(\mu,\theta)}{\partial\theta} = 0$$

然而

$$\frac{\partial \ln L(\mu,\theta)}{\partial\mu} = \frac{n}{\theta} > 0$$

由后者可以得到
$$\overline{X}=\theta+\mu$$
而前者的式子恒大于0，说明似然函数随μ单调递增，取$\mu=X_{(1)}=\min\{x_1,x_2,L,x_n\}$，即有极大似然估计值
$$\hat{\mu}=X_{(1)}\quad \hat{\theta}=\overline{X}-X_{(1)}$$

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由题意可知总体为$[\theta,2\theta]$上的均匀分布满足$X\sim U[\theta,2\theta]$.于是
$$EX=\frac{3}{2}\theta$$
令$\overline{X}=EX$，则有
$$\theta=\frac{2}{3}\overline{X}$$
另一方面，其似然函数
$$L(\theta)=\frac{1}{\theta^n}\quad \frac{1}{2}X_{(n)}\le\theta\le X_{(1)}$$
又因为
$$\frac{d\ln L(\theta)}{d\theta}=-\frac{n}{\theta}<0$$
所以最大似然估计值为$\hat{\theta}=\frac{1}{2}X_{(n)}$

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首先计算矩估计如下
$$EX=2\theta(1-\theta)+2\theta^2+3(1-2\theta)=3-4\theta$$
又因为
$$\overline{X}=\frac{1}{8}(3+1+3+0+3+1+2+3)=2$$
令$EX=\overline{X}$，可得
$$\theta=\frac{1}{4}$$
另一方面，其似然函数
$$\begin{aligned}L(\theta)&=(1-2\theta)^4\cdot[2\theta(1-\theta)]^2\cdot\theta^4\\&=4\theta^6(1-\theta)^2(1-2\theta)^4\\ \ln L(\theta)&=2\ln 2+6\ln\theta+2\ln(1-\theta)+4\ln(1-2\theta)\end{aligned}$$
令$\frac{d\ln L(\theta)}{d\theta}=0$可得
$$\theta=\frac{7\pm\sqrt{13}}{12}$$
又因为$0<\theta<1/2$，所以极大似然估计值为
$$\hat{\theta}=\frac{7-\sqrt{13}}{12}$$

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$\hat{\theta}$的方差如下计算：
$$\begin{aligned}D(\hat{\theta})&=c^2D(\hat{\theta}_1)+(1-c)^2\hat{\theta}_2\\&=c^2\sigma_1^2+(1-c)^2\sigma_2^2\\&=(\sigma_1^2+\sigma_2^2)c^2-2\sigma_2^2c+\sigma_2^2\end{aligned}$$
有二次函数性质易知当$c=\frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}$时，方差最小.

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因为$X_1,X_2,\cdots,X_n$是来自总体$N(\mu,\sigma^2)$的一个样本，所以
$$EX_i=EX=\mu\quad DX_i=DX=\sigma^2$$
所以
$$\begin{aligned}E[c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2]&=c\sum_{i=1}^{n-1}[E(X_{i+1})^2+E(X_i)^2-2E(X_iX_{i+1})]\\&=c(n-1)[\mu^2+\sigma^2+\mu^2+\sigma^2-2\mu^2]\\&=2c(n-1)\sigma^2\end{aligned}$$
令$E[c\sum_{i=1}^{n-1}(X_{i+1}-X_i)^2]=\sigma^2$，可得
$$c=\frac{1}{2(n-1)}$$

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正态分布的似然函数已经在前面的题目中求过，过程不再赘述.令
$$\frac{\partial\ln L(\sigma^2)}{\partial\sigma^2}=0$$

得到

$$-\frac{n}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{i=1}^{n}(X_i-\mu)^2$$

解之得

$$\hat{\sigma^2}=\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$$

由课堂内容易知S^2为无偏估计，下面证明此时$\hat{\sigma^2}$是无偏估计.

$$\begin{aligned}E(\hat{\sigma^2})&=\frac{1}{n}\sum_{i=1}^{n}E(X_i-\mu)^2\\&=\frac{1}{n}\sum_{i=1}^{n}(EX_i^2-2\mu EX_i+\mu^2)\\&=(EX_i)^2+DX_i-\mu^2\\&=\sigma^2\end{aligned}$$

所以$\hat{\sigma^2}$是σ^2的无偏估计.此时对于$\hat{\sigma^2}$，因为$X_i \sim N(\mu,\sigma^2)$，所以$\frac{X_i-\mu}{\sigma}\sim N(0,1)$，由此可以得到

$$\frac{\sum_{i=1}^{n}(X_i-\mu)^2}{n}\cdot\frac{n}{\sigma^2}\sim\chi^2(n)$$

又$D(\chi^2(n))=2n$，所以

$$D(\hat{\sigma^2})=\frac{\sigma^4}{n^2}D(\chi^2(n))=\frac{2\sigma^4}{n}=MSE(\hat{\sigma^2})$$

又因为

$$D(S^2)=\frac{\sigma^4}{(n-1)^2}D(\chi^2(n-1))=\frac{2\sigma^4}{n-1}=MSE(S^2)$$

因此

$$MSE(\hat{\sigma^2})<MSE(S^2)$$