04. Convex Optimization Problems

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Contents

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1 Optimization problem in standard form

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, 2, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, 2, \ldots, p \end{array}$$

$x \in \mathbf{R}^n$ is the optimization variable

$f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function

$f_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, m$ are the inequality constraint functions

$h_i : \mathbf{R}^n \to \mathbf{R}$, $i = 1, \ldots, p$ are the equality constraint functions

Optimal value:

$$p^* = \inf\{f_0(x) | f_i(x) \leq 0, i = 1, \ldots, m, h_i(x) = 0, \;\; i = 1, \ldots, p\}$$

$p^* = \infty$ if problem is infeasible (no x satisfies the constraints)

$p^* = -\infty$ if problem is unbounded below

1 Optimization problem in standard form

Optimal and locally optimal points:

x is feasible if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints

Feasible x is optimal if $f_0(x) = p^*$; X_{opt} the set of optimal points

x is locally optimal if there is an $R > 0$ such that x is optimal for

$$
\begin{array}{ll}
\texttt{minimize (over } z\texttt{)} & f_0(z) \\
\texttt{subject to} & f_i(z) \leq 0, \;\; i = 1, 2, \ldots, m \\
 & h_i(z) = 0, \;\; i = 1, 2, \ldots, p \\
 & \|z - x\|_2 \leq R
\end{array}
$$

Examples (with $n = 1$, $m = p = 0$)

$f_0(x) = 1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^* = 0$, no optimal point

$f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^* = -\infty$

$f_0(x) = x \log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^* = -1/e$, $x = 1/e$ is optimal

$f_0(x) = x^3 - 3x$, $p^* = -\infty$, local optimum at $x = 1$

1 Optimization problem in standard form

Implicit constraints

The standard form optimization problem has an implicit constraint

$$x \in \mathcal{D} = \bigcap_{i=0}^{m} \mathbf{dom}\ f_i \cap \bigcap_{i=0}^{p} \mathbf{dom}\ h_i$$

We call $\mathcal{D}$ the domain of the problem

The constraints $f_i(x) \leq 0, h_i(x) = 0$ are the explicit constraints

A problem is unconstrained if it has no explicit constraints ($m = p = 0$)

Examples

$$\texttt{minimize}\ f_0(x) = -\sum_{i=1}^{k} \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

1 Optimization problem in standard form

Feasibility problem

$$\begin{array}{ll} \texttt{find} & x \\ \texttt{subject to} & f_i(x) \le 0, \;\; i = 1, 2, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, 2, \ldots, p \end{array}$$

Can be considered a special case of the general problem with $f_0(x) = 0$:

$$\begin{array}{ll} \texttt{minimize} & 0 \\ \texttt{subject to} & f_i(x) \le 0, \;\; i = 1, 2, \ldots, m \\ & h_i(x) = 0, \;\; i = 1, 2, \ldots, p \end{array}$$

$p^* = 0$, if constraints are feasible; any feasible x is optimal

$p^* = \infty$ if constraints are infeasible

2 Convex Optimization Problem

Standard form convex optimization problem

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, 2, \ldots, m \\ & a_i^T x = b_i, \;\; i = 1, 2, \ldots, p \end{array}$$

$f_0, f_1, \ldots, f_m$ are convex; equality constraints are affine

problem is *quasiconvex* if f_0 is quasiconvex (and $f_1, \ldots, f_m$ convex)

often written as

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, 2, \ldots, m \\ & Ax = b, \;\; i = 1, 2, \ldots, p \end{array}$$

important property: feasible set of a convex optimization problem is convex

2 Convex Optimization Problem

Local and global optimal

Any locally optimal point of a convex problem is (globally) optimal

proof.

Suppose x is locally optimal and y is optimal with $f_0(y) < f_0(x)$

x locally optimal means there is an $R > 0$ such that

$$z \text{ feasible}, \ \|z - x\|_2 \leq R \Longrightarrow f_0(z) \geq f_0(x)$$

consider $z = \theta y + (1 - \theta)x$ with $\theta = R/(2\|y - x\|_2)$

$\|y - x\|_2 > R$, so $0 < \theta < 1/2$

z is a convex combination of two feasible points, hence also feasible

$\|z - x\|_2 = R/2$ and

$$f_0(z) \leq \theta f_0(x) + (1 - \theta) f_0(y) < f_0(x)$$

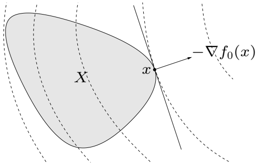
which contradicts our assumption that x is locally optimal

2 Convex Optimization Problem

Optimality criterion for differentiable f_0

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T(y - x) \geq 0 \text{ for all feasible } y$$



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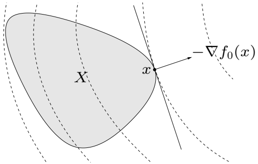


proof. if nonzero, $\nabla f_0(x)$ defines a supp. hyperplane to X at x

$\Rightarrow$: $f(y) \geq f(x) + (\nabla f_0(x))^T(y - x) \geq f(x)$

$\Leftarrow$: if $\exists y$, $(\nabla f_0(x))^T(y - x) < 0$, the hyperplane determined by $(\nabla f_0(x))^T$ and x intersects the interior of set X, as well as the contour

2 Convex Optimization Problem

Unconstrained problem: x is optimal if and only if

$$x \in \mathbf{dom} f_0, \nabla f_0(x) = 0$$

Equality constrained problem:

$$\texttt{minimize}\ f_0(x) \quad \texttt{subject to}\ Ax = b$$

x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom}\ f_0,\ Ax = b, \nabla f_0(x) + A^T \nu = 0$$

2 Convex Optimization Problem

Unconstrained problem: x is optimal if and only if

$$x \in \mathbf{dom} f_0, \nabla f_0(x) = 0$$

Equality constrained problem:

$$\texttt{minimize } f_0(x) \quad \texttt{subject to } Ax = b$$

x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom}\ f_0,\ \ Ax = b, \nabla f_0(x) + A^T \nu = 0$$

proof. x is optimal is equivalent to $\nabla f_0(x)^T(y - x) \geq 0$ for all y satisfying $Ay = b$. Note that $y = x + v$ where $v \in \mathcal{N}(A)$. Hence, the above condition is equivalent to $\nabla f_0(x)^T v \geq 0$, $\forall v \in \mathcal{N}(A)$, or $\nabla f_0(x)^T v = 0$, $\forall v \in \mathcal{N}(A)$. In other words, $\nabla f_0(x)^T \in \mathcal{R}(A)$, or $\exists \nu \in \mathcal{R}(A)$, $\nabla f_0(x) + A^T \nu = 0$.

2 Convex Optimization Problem

Minimization over nonnegative orthant:

$$\texttt{minimize}\ f_0(x) \quad \texttt{subject to}\ x \succeq 0$$

x is optimal if and only if

$$x \in \textbf{dom}\ f_0,\ \ x \succeq 0, \begin{cases} \nabla f_0(x)_i \geq 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

proof. Optimality condition: $x \succeq 0,\ \nabla f_0(x)^T(y-x) \geq 0,\ \forall y \succeq 0$.

Second condition yields $\nabla f_0(x)^T \succeq 0$, since we can always find $y \succ x$. Moreover, let $y = 0$, the second condition is converted to $-\nabla f_0(x)^T x \geq 0$. Consider $x \succeq 0$ and $\nabla f_0(x)^T \succeq 0$, we have $\nabla f_0(x)^T x = 0$, or $(\nabla f_0(x))_i x_i = 0$, for $\forall i$.

This completes the proof.

2 Convex Optimization Problem

Equivalent problems

Example

$$\begin{array}{ll} \texttt{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \texttt{subject to} & f_1(x) = x_1/(1 + x_2^2) \le 0 \\ & h_1(x) = (x_1 + x_2)^2 = 0 \end{array}$$

f_0 is convex; feasible set $\{(x_1, x_2) | x_1 = -x_2 \le 0\}$ is convex

Not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine

Equivalent (but not identical) to the convex problem

$$\begin{array}{ll} \texttt{minimize} & x_1^2 + x_2^2 \\ \texttt{subject to} & x_1 \le 0 \\ & x_1 + x_2 = 0 \end{array}$$

2 Convex Optimization Problem

Equivalent convex problems

Two problems are (informally) equivalent if the solution of one is readily obtained from the solution of the other, and vice-versa

Some common transformations that preserve convexity:

1. Eliminating equality constraints

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & Ax = b \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } z\texttt{)} & f_0(Fz + x_0) \\ \texttt{subject to} & f_i(Fz + x_0) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

where F and x_0 are such that

$$Ax = b \Longleftrightarrow x = Fz + x_0 \text{ for some } z$$

2 Convex Optimization Problem

Equivalent convex problems

2. Introducing equality constraints

$$\begin{array}{ll} \texttt{minimize} & f_0(A_0x + b_0) \\ \texttt{subject to} & f_i(A_ix + b_i) \leq 0, \;\; i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x \; y_i\texttt{)} & f_0(y_0) \\ \texttt{subject to} & f_i(y_i) \leq 0, \;\; i = 1, \ldots, m \\ & y_i = A_ix + b_i, \;\; i = 0, 1, \ldots, m \end{array}$$

2 Convex Optimization Problem

Equivalent convex problems

3. Introducing slack variables for linear inequalities

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & a_i^T x \leq b_i, \ \ i = 1, \ldots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x \ s\texttt{)} & f_0(x) \\ \texttt{subject to} & a_i^T x + s_i = b_i, \ \ i = 1, \ldots, m \\ & s_i \geq 0, \ \ i = 1, \ldots, m \end{array}$$

2 Convex Optimization Problem

Equivalent convex problems

4. Epigraph form: standard form convex problem is equivalent to

$$
\begin{array}{ll}
\texttt{minimize (over } x\ t\texttt{)} & t \\
\texttt{subject to} & f_0(x) - t \leq 0 \\
& f_i(x) \leq 0, \;\; i = 1, \ldots, m \\
& Ax = b
\end{array}
$$

5. Minimizing over some variables:

$$
\begin{array}{ll}
\texttt{minimize (over } x\texttt{)} & f_0(x_1, x_2) \\
\texttt{subject to} & f_i(x_1) \leq 0, \;\; i = 1, \ldots, m
\end{array}
$$

is equivalent to

$$
\begin{array}{ll}
\texttt{minimize (over } x_1\texttt{)} & \tilde{f}_0(x_1) \\
\texttt{subject to} & f_i(x_1) \leq 0, \;\; i = 1, \ldots, m
\end{array}
$$

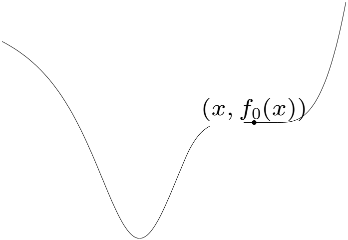
where $\tilde{f}_0(x_1) = \inf_{x_2} f_0(x_1, x_2)$

3 Quasiconvex Optimization

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \leq 0, \;\; i = 1, \ldots, m \\ & Ax = b \end{array}$$

with $f_0 : \boldsymbol{R}^n \rightarrow R$ quasiconvex, $f_1, \ldots, f_m$ convex

can have locally optimal points that are not (globally) optimal



3 Quasiconvex Optimization

Convex representation of sublevel sets of f_0

if f_0 is quasiconvex, there exists a family of functions ϕ_t such that:

1. $\phi_t(x)$ is convex in x for fixed t
2. t-sublevel set of f_0 is 0-sublevel set of ϕ_t, i.e.,

$$f_0(x) \leq t \Longleftrightarrow \phi_t(x) \leq 0$$

Example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $p(x) \geq 0$, $q(x) > 0$ on **dom** f_0

can take $\phi_t(x) = p(x) - tq(x)$:

1. for $t \geq 0$, ϕ_t convex in x
2. $p(x)/q(x) \leq t$ if and only if $\phi_t(x) \leq 0$

3 Quasiconvex Optimization

Quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \le 0, \qquad f_i(x) \le 0, \;\; i = 1, \ldots, m, \qquad Ax = b$$

1. For fixed t, a convex feasibility problem in x
2. If feasible, we can conclude that $t \ge p^*$; if infeasible, $t \le p^*$

Bisection method for quasiconvex optimization

given $l \le p^\star$, $u \ge p^\star$, tolerance $\epsilon > 0$.
repeat
  1. $t := (l + u)/2$.
  2. Solve the convex feasibility problem (1).
  3. **if** (1) is feasible, $u := t$; **else** $l := t$.
until $u - l \le \epsilon$.

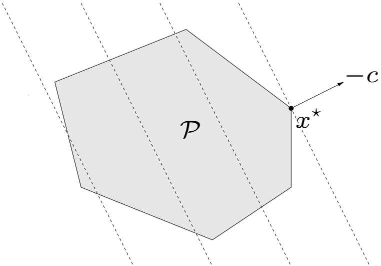
---

requires exactly $\lceil \log 2((u - l)/\epsilon) \rceil$ iterations (where u, l are initial values)

4 Examples: Linear Program (LP)

$$\begin{array}{ll} \texttt{minimize} & c^T x + d \\ \texttt{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

1. Convex problem with affine objective and constraint functions
2. Feasible set is a polyhedron



4 Examples: Linear Program (LP)

Standard form for LP:

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

Standard LP without equality constraints:

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax \preceq b \end{array}$$

4 Examples: Linear Program (LP)

Convert the following LP to the standard form:

$$
\begin{array}{ll}
\texttt{minimize} & c^T x + d \\
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 & Ax = b
\end{array}
$$

4 Examples: Linear Program (LP)

Convert the following LP to the standard form:

$$\begin{array}{ll} \texttt{minimize} & c^T x + d \\ \texttt{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Introducing the slack variables

$$\begin{array}{ll} \texttt{minimize} & c^T x + d \\ \texttt{subject to} & Gx + s = h \\ & Ax = b \\ & s \succeq 0 \end{array}$$

let $x = x^+ - x^-$:

$$\begin{array}{ll} \texttt{minimize} & c^T x^+ + c^T x^- + d \\ \texttt{subject to} & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & x^+ \succeq 0, x^- \succeq 0, s \succeq 0 \end{array}$$

4 Examples: Linear Program (LP)

Examples

Diet problem: choose quantities $x_1, \ldots, x_n$ of n foods

1. One unit of food j costs c_j, contains amount a_{ij} of nutrient i
2. Healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

Piecewise-linear minimization

$$\texttt{minimize} \quad \max_{i=1,\ldots,m} \, (a_i^T x + b_i)$$

equivalent to an LP

4 Examples: Linear Program (LP)

Examples

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Piecewise-linear minimization

$$\texttt{minimize} \quad \max_{i=1,\dots,m} \; (a_i^T x + b_i)$$

equivalent to an LP

$$\begin{array}{ll} \texttt{minimize} & t \\ \texttt{subject to} & a_i^T x + b_i \le t, \;\; i = 1, \dots, m \end{array}$$

4 Examples: Linear Program (LP)

Chebyshev center of a polyhedron

Chebyshev center of $\mathcal{P} = \{x | a_i^T x \le b_i, \ i = 1, \dots, m\}$ is center of largest inscribed ball $\mathcal{B} = \{x_c + u | \|u\|_2 \le r\}$

1. $a_i^T x \le b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c + u) | \|u\|_2 \le r\} = a_i^T x_c + r\|a_i\|_2 \le b_i$$

2. hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \texttt{maximize} & r \\ \texttt{subject to} & a_i^T x_c + r\|a_i\|_2 \le b_i, i = 1, \dots, m \end{array}$$

4 Examples: (Generalized) Linear-fractional Program

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & Gx \preceq h, \qquad Ax = b \end{array}$$

linear-fractional program

$$f_0(x) = \frac{c^T x + d}{e^T x + f}, \qquad \textbf{dom } f_0(x) = \{x | e^T x + f > 0\}$$

a quasiconvex optimization problem; can be solved by bisection

also equivalent to the LP(variables y, z)

$$\begin{array}{ll} \texttt{minimize} & c^T y + dz \\ \texttt{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^T y + fz = 1 \\ & z \geq 0 \end{array}$$

4 Examples: (Generalized) Linear-fractional Program

exercise. **Relaxation of Boolean LP.** In a Boolean linear program, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \ \ i = 1, \ldots, n \end{array}$$

The above problem can be related to:

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax \preceq b \\ & 0 \leq x_i \leq 1, \ \ i = 1, \ldots, n \end{array}$$

4 Examples: (Generalized) Linear-fractional Program

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The above problem can be related to:

$$\begin{array}{ll} \texttt{minimize} & c^T x \\ \texttt{subject to} & Ax \preceq b \\ & 0 \le x_i \le 1, \ i = 1, \ldots, n \end{array}$$

Optimal solution of relaxation is also optimal for Boolean LP.

4 Examples: (Generalized) Linear-fractional Program

A more general theorem:

试证明在一个集合上最小化一个线性函数等价于在其凸包上最小化该函数, 即:

$$\inf_{x \in \mathbf{conv}\ X} c^T x = \inf_{x \in X} c^T x,$$

其中 $X \subset R^m$ 和 $c \in R^m$。此外，当且仅当等式右侧的下确界可以达到时，等式左侧的下确界才能达到。

4 Examples: (Generalized) Linear-fractional Program

proof.

$X \subset \mathbf{conv}\ X$ yields $\inf_{x \in \mathbf{conv}\ X} c^T x \le \inf_{x \in X} c^T x$.

Any $\bar{x} \in \mathbf{conv}\ X$ can be written as $\bar{x} = \sum_{i=1}^m \alpha_i x_i$, where $x_1, \ldots, x_m \in X$ and $\alpha_1, \ldots, \alpha_m \ge 0$ with $\sum_{i=1}^m \alpha_i = 1$. Since $c^T x_i \ge \inf_{x \in X} c^T x$,

$$c^T \bar{x} = \sum_{i=1}^m \alpha_i c^T x_i \ge \left(\sum_{i=1}^m \alpha_i\right) \inf_{x \in X} c^T x = \inf_{x \in X} c^T x, \qquad \forall \bar{x} \in \mathbf{conv}\ X.$$

Take the infimum for the L.H.S. for $\bar{x} \in \mathbf{conv}\ X$

$$\inf_{\bar{x} \in \mathbf{conv}\ X} c^T \bar{x} \ge \inf_{x \in X} c^T x.$$

To summarize,

$$\inf_{x \in \mathbf{conv}\ X} c^T x = \inf_{x \in X} c^T x.$$

4 Examples: (Generalized) Linear-fractional Program

contd. Since $X \subset \textbf{conv}\ X$ and $\inf\limits_{x \in \textbf{conv}\ X} c^T x = \inf\limits_{x \in X} c^T x$, the optimal point in X with respect to $c^T x$ can be found in **conv** X.

On the other hand, assume function $c^T x$ reaches the minimum in **conv** X, which is $\bar{x}$. For $x_1, \ldots, x_m$ and $\sum_{i=1}^m \alpha_i = 1$ with $\alpha_1, \ldots, \alpha_m \geq 0$ we construct $\bar{x} = \sum_{i=1}^m \alpha_i x_i$, there is

$$\inf_{x \in X} c^T x = (\sum_{i=1}^m \alpha_i) \inf_{x \in X} c^T x \leq \sum_{i=1}^m \alpha_i c^T x_i = c^T \bar{x} = \inf_{x \in \textbf{conv}\ X} c^T x = \inf_{x \in X} c^T x.$$

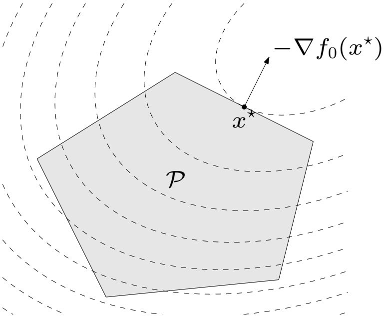
L.H.S. equals to R.H.S., hence, for all i and $\alpha_i > 0$, $c^T x_i = \inf\limits_{x \in \textbf{conv}\ X} c' x$ must hold. As a result, the minimum of $c' x$ is attainable in X.

4 Examples: Quadratic Program (QP)

$$\begin{array}{ll} \texttt{minimize} & (1/2)x^T P x + q^T x + r \\ \texttt{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

$P \in \boldsymbol{S}^n$, so objective is convex quadratic

minimize a convex quadratic function over a polyhedron



4 Examples: Quadratic Program (QP)

Examples: least-squares

$$\texttt{minimize} \quad \|Ax - b\|_2^2$$

analytical solution

can add linear constraints, e.g., $l \succeq x \succeq u$

Linear program with random cost

$$\begin{array}{ll} \texttt{minimize} & \bar{c}^T x + \gamma x^T \Sigma x = \mathbf{E}\, c^T x + \gamma \mathbf{var}(c^T x) \\ \texttt{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

1. c is random vector with mean $\bar{c}$ and covariance Σ
2. hence, $c^T x$ is random variable with mean $\bar{c}^T x$ and variance $x^T \Sigma x$
3. $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

Examples: Geometric Programming

单项式函数或单项式：$f : \mathbf{R}^n \to \mathbf{R}$, $\mathbf{dom}\ f = \mathbf{R}^n_{++}$,

$$f(x) = c x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}, \quad c > 0, \quad a_i \in \mathbf{R}$$

正项式函数：

$$f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

其中 $c_k > 0$

4 Examples: Geometric Programming

具有下列形式的优化问题称为几何规划（GP）:

$$
\begin{array}{ll}
\texttt{minimize} & f_0(x) \\
\texttt{subject to} & f_i(x) \le 1, \ \ i = 1, \ldots, m \\
 & h_i(x) = 1, \ \ i = 1, \ldots, p
\end{array}
$$

其中 $f_0, \ldots, f_m$ 为正项式， $h_1, \ldots, h_p$ 为单项式

几何规划的拓展

如果 f 是一个正项式， 而 h 为单项式， 那么 $f(x) \le h(x)$ 表示为 $f(x)/h(x) \le 1$, 而等式约束 $h_1(x) = h_2(x)$ 可以表示为 $h_1(x)/h_2(x) = 1$

4 Examples: Geometric Programming

Examples

考虑下面的问题:

$$
\begin{array}{ll}
\texttt{minimize} & x/y \\
\texttt{subject to} & 2 \leq x \leq 3 \\
& x^2 + 3y/z \leq \sqrt{y} \\
& x/y = z^2
\end{array}
$$

4 Examples: Geometric Programming

Examples

考虑下面的问题:

$$
\begin{array}{ll}
\texttt{minimize} & x/y \\
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& x/y = z^2
\end{array}
$$

可以转化为

$$
\begin{array}{ll}
\texttt{minimize} & x^{-1}y \\
\texttt{subject to} & 2x^{-1} \leq 1, \ (1/3)x \leq 1 \\
& x^2 y^{-1/2} + 3y^{1/2} z^{-1} \leq 1 \\
& xy^{-1} z^{-2} = 1
\end{array}
$$

4 Examples: Geometric Programming

凸形式的集合规划

定义 $y_i = \log x_i$，因此 $x_i = e^{y_i}$

如果 $f(x) = cx_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$，那么

$$\begin{aligned} f(x) =& f(e^{y_1}, e^{y_2}, \ldots, e^{y_n}) \\ =& c(e^{y_1})^{a_1} \cdot (e^{y_n})^{a_n} \\ =& e^{a^T y + b} \end{aligned}$$

其中，$b = \log c$

类似地，正项式 $f(x) = \sum_{k=1}^{K} c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ 可以转化为相似形式:

$$f(x) = \sum_{k=1}^{K} e^{a_k^T y + b_k}$$

4 Examples: Geometric Programming

几何规划可以用新的变量 y 来表示:

$$\begin{array}{ll} \texttt{minimize} & \sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}} \\ \texttt{subject to} & \sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}} \leq 1, \; i = 1, \ldots, m \\ & e^{g_i^T y + h_i} = 1, \; i = 1, \ldots, p \end{array}$$

采用对数函数对目标函数和约束条件进行转换:

$$\begin{array}{ll} \texttt{minimize} & \tilde{f}_0(y) = \log\left(\sum_{k=1}^{K_0} e^{a_{0k}^T y + b_{0k}}\right) \\ \texttt{subject to} & \tilde{f}_i(y) = \log\left(\sum_{k=1}^{K_i} e^{a_{ik}^T y + b_{ik}}\right) \leq 0, \; i = 1, \ldots, m \\ & \tilde{h}_i(y) = g_i^T y + h_i = 0, \; i = 1, \ldots, p \end{array}$$