Chapter4_凸优化问题

1. 标准形式的优化问题

基本形式
一个凸优化问题包括源问题的约束条件，约束条件又有不等式约束和等式约束

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \le 0, \ i=1,\dots,m \\ & h_i(x)=0, \ i=1,\dots,p \end{array}$$

$x \in \mathbf{R}^n$ is the optimization variable
$f_0 : \mathbf{R}^n \to \mathbf{R}$ is the objective or cost function
$f_i : \mathbf{R}^n \to \mathbf{R}, i=1,\dots,m$ are the inequality constraint functions
$h_i : \mathbf{R}^n \to \mathbf{R}$ are the equality constraint functions

最优值
最优值为∞表明问题不可达（没有符合约束条件的x）；最优值为-∞表明问题无下界。

Optimal value:

$$p^\star = \inf\{f_0(x) \mid f_i(x) \le 0, i=1,\dots,m, h_i(x)=0, i=1,\dots,p\}$$

$p^\star = \infty$ if problem is infeasible (no x satisfies the constraints)
$p^\star = -\infty$ if problem is unbounded below

最优和局部最优

Optimal and locally optimal points:
x is feasible if $x \in \mathbf{dom}\, f_0$ and it satisfies the constraints
Feasible x is optimal if $f_0(x) = p^\star$; $X_{\rm opt}$ the set of optimal points
x is locally optimal if there is an $R>0$ such that x is optimal for

$$\begin{array}{ll} \texttt{minimize (over } z\texttt{)} & f_0(z) \\ \texttt{subject to} & f_i(z) \le 0, \ i=1,\dots,m, \quad h_i(z)=0, \ i=1,\dots,p \\ & \|z-x\|_2 \le R \end{array}$$

可行：x可行当仅当x在目标函数定义域内且满足所有约束条件

最优：f0(x)=p*，即达到最优值的目标函数值的x

最优集：最优点的集合

局部最优：对于以自己为中心，某一个R为半径的圆中，自己是最优解，则称该点为局部最优。

例示

Examples (with $n=1, m=p=0$)
$f_0(x)=1/x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = 0$, no optimal point
$f_0(x) = -\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -\infty$
$f_0(x) = x\log x$, $\mathbf{dom}\, f_0 = \mathbf{R}_{++}$: $p^\star = -1/e$, $x=1/e$ is optimal
$f_0(x) = x^3 - 3x$, $p^\star = -\infty$, local optimum at $x=1$

隐约束
x需满足目标函数和所有约束条件的定义域。

显示约束：不等式约束和等式约束。

Implicit constraints
The standard form optimization problem has an implicit constraint

$$x \in \mathcal{D} = \bigcap_{i=0}^m \mathbf{dom}\, f_i \cap \bigcap_{i=1}^p \mathbf{dom}\, h_i$$

We call $\mathcal{D}$ the domain of the problem
The constraints $f_i(x)\le 0, h_i(x)=0$ are the explicit constraints
A problem is unconstrained if it has no explicit constraints ($m=p=0$)

例示

Examples

$$\texttt{minimize} \quad f_0(x) = -\sum_{i=1}^k \log(b_i - a_i^T x)$$

is an unconstrained problem with implicit constraints $a_i^T x < b_i$

可达性问题
判断可行性是否为空，可以视为目标函数为0的优化问题。

Feasibility problem

$$\begin{array}{ll} \texttt{find} & x \\ \texttt{subject to} & f_i(x) \le 0, \ i=1,\dots,m \\ & h_i(x)=0, \ i=1,\dots,p \end{array}$$

Can be considered a special case of the general problem with $f_0(x)=0$:

$$\begin{array}{ll} \texttt{minimize} & 0 \\ \texttt{subject to} & f_i(x) \le 0, \ i=1,\dots,m \\ & h_i(x)=0, \ i=1,\dots,p \end{array}$$

$p^\star = 0$ if constraints are feasible; any feasible x is optimal
$p^\star = \infty$ if constraints are infeasible

2. 凸优化问题

标准形式的凸优化问题
目标函数为凸；
不等式约束为凸；
等式约束仿射。

等式约束通常写成矩阵形式。

凸优化问题的可行集是凸集。

Standard form convex optimization problem

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \le 0, \ i=1,\dots,m \\ & a_i^T x = b_i, \ i=1,\dots,p \end{array}$$

$f_0, f_1, \dots, f_m$ are convex; equality constraints are affine
problem is *quasiconvex* if f_0 is quasiconvex (and $f_1,\dots,f_m$ convex)
often written as

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \le 0, \ i=1,\dots,m \\ & Ax=b \end{array}$$

important property: feasible set of a convex optimization problem is convex

一些凸优化问题的实例

重要性质：凸优化问题的任何局部最优解也是一个全局最优点。
证明：（反证法）

Suppose x is locally optimal and y is optimal with $f_0(y) < f_0(x)$
x locally optimal means there is an $R>0$ such that

$$z \text{ feasible}, \ \|z-x\|_2 \le R \Longrightarrow f_0(z) \ge f_0(x)$$

consider $z = \theta y + (1-\theta)x$ with $\theta = R/(2\|y-x\|_2)$
$\|y-x\|_2 > R$, so $0<\theta<1/2$
z is a convex combination of two feasible points, hence also feasible
$\|z-x\|_2 = R/2$ and

$$f_0(z) \le \theta f_0(y) + (1-\theta) f_0(x) < f_0(x)$$

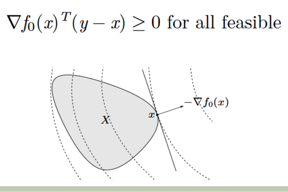
which contradicts our assumption that x is locally optimal

可微函数的最优性准则
x是最优点当且仅当x可行且对于任意y，如下不等式成立。

几何上讲，过x*且和∇f0(x)垂直的超平面是x点的一个支撑超平面。

x is optimal if and only if it is feasible and

$$\nabla f_0(x)^T (y-x) \ge 0 \text{ for all feasible } y$$



无约束问题：x是最优点当且仅当x处梯度为0。

等式约束的凸优化问题

$$\texttt{minimize} \ f_0(x) \quad \texttt{subject to} \ Ax=b$$

x is optimal if and only if there exists a ν such that

$$x \in \mathbf{dom}\, f_0, \quad Ax=b, \quad \nabla f_0(x) + A^T\nu = 0$$

非负象限上的极小化
这个最优性条件是非负约束的最优条件。对于一个最优点x，我们要么在可行域内部（即全部约束为0）；要么在边界上（某个约束为0）。

Minimization over nonnegative orthant

$$\texttt{minimize} \ f_0(x) \quad \texttt{subject to} \ x \succeq 0$$

x is optimal if and only if

$$x \in \mathbf{dom}\, f_0, \ x \succeq 0, \ \begin{cases} \nabla f_0(x)_i \ge 0 & x_i = 0 \\ \nabla f_0(x)_i = 0 & x_i > 0 \end{cases}$$

proof. Optimality condition: $x \succeq 0$, $\nabla f_0(x)^T(y-x) \ge 0$, $\forall y \succeq 0$. Second condition yields $\nabla f_0(x)^T \succeq 0$, since we can always find $y \succeq x$. Moreover, let $y=0$, the second condition is converted to: $-\nabla f_0(x)^T x \ge 0$. Consider $x\succeq 0$ and $\nabla f_0(x)^T \succeq 0$, we have $\nabla f_0(x)^T x = 0$, or $(\nabla f_0(x))_i x_i = 0$, for $\forall i$.
This completes the proof.

例示：通过等价转化，将一个非凸问题转化成一个凸问题
等价转化的原因是原问题不好求解，但是它的解可以很容易地从另一个问题中求得

Example

$$\begin{array}{ll} \texttt{minimize} & f_0(x) = x_1^2 + x_2^2 \\ \texttt{subject to} & f_1(x) = x_1/(1+x_2^2) \le 0 \\ & h_1(x) = (x_1+x_2)^2 = 0 \end{array}$$

f_0 is convex; feasible set $\{(x_1,x_2) \mid x_1 = -x_2 \le 0\}$ is convex
Not a convex problem (according to our definition): f_1 is not convex, h_1 is not affine
Equivalent (but not identical) to the convex problem

$$\begin{array}{ll} \texttt{minimize} & x_1^2 + x_2^2 \\ \texttt{subject to} & x_1 \le 0 \\ & x_1 + x_2 = 0 \end{array}$$

等价凸问题

一些常见的保持凸性的等价变换

1. 等式约束的消除
x0是非线性方程组Ax=b的特解；F是对应线性方程组Ax=0的解空间矩阵，Fz+x0构成了Ax=b的解空间。

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \le 0, \ i=1,\dots,m \\ & Ax=b \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } z\texttt{)} & f_0(Fz+x_0) \\ \texttt{subject to} & f_i(Fz+x_0) \le 0, \ i=1,\dots,m \end{array}$$

where F and x_0 are such that

$$Ax=b \Longleftrightarrow x = Fz + x_0 \text{ for some } z$$

2. 引入等式约束

2. Introducing equality constraints

$$\begin{array}{ll} \texttt{minimize} & f_0(A_0x+b_0) \\ \texttt{subject to} & f_i(A_ix+b_i) \le 0, \ i=1,\dots,m \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x \ y_i\texttt{)} & f_0(y_0) \\ \texttt{subject to} & f_i(y_i) \le 0, \ i=1,\dots,m \\ & y_i = A_ix+b_i, \ i=0,1,\dots,m \end{array}$$

3. 引入不等式约束的松弛变量

3. Introducing slack variables for linear inequalities

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & a_i^Tx \le b_i, \ i=1,\dots,m \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x \ s\texttt{)} & f_0(x) \\ \texttt{subject to} & a_i^Tx + s_i = b_i, \ i=1,\dots,m \\ & s_i \ge 0, \ i=1,\dots,m \end{array}$$

4. 标准问题的上境图形式

4. Epigraph form: standard form convex problem is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x \ t\texttt{)} & t \\ \texttt{subject to} & f_0(x) - t \le 0 \\ & f_i(x) \le 0, \ i=1,\dots,m \\ & Ax=b \end{array}$$

5. 优化部分变量

$$\begin{array}{ll} \texttt{minimize (over } x\texttt{)} & f_0(x_1,x_2) \\ \texttt{subject to} & f_i(x_1) \le 0, \ i=1,\dots,m \end{array}$$

is equivalent to

$$\begin{array}{ll} \texttt{minimize (over } x_1\texttt{)} & \tilde f_0(x_1) \\ \texttt{subject to} & f_i(x_1) \le 0, \ i=1,\dots,m \end{array}$$

where $\tilde f_0(x_1) = \inf_{x_2} f_0(x_1,x_2)$

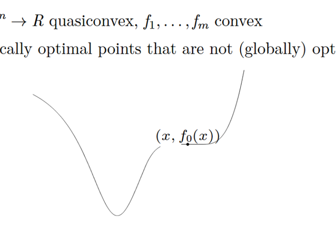
3. 拟凸问题

标准形式
目标函数是拟凸的，不等式约束是凸的

拟凸问题的局部最优点可能不是全局最优点

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \le 0, \ i=1,\dots,m \\ & Ax=b \end{array}$$

with $f_0 : \mathbf{R}^n \to \mathbf{R}$ quasiconvex, $f_1,\dots,f_m$ convex
can have locally optimal points that are not (globally) optimal



通过凸可行性问题求解拟凸优化问题
可以通过一族凸不等式来表示拟凸函数的下水平集，这里解决拟凸优化问题的一般方法。

if f_0 is quasiconvex, there exists a family of functions ϕ_t such that:
1. $\phi_t(x)$ is convex in x for fixed t
2. t-sublevel set of f_0 is 0-sublevel set of ϕ_t, i.e.,

$$f_0(x) \le t \Longleftrightarrow \phi_t(x) \le 0$$

二分法
通过求解凸可行性问题来判断拟凸问题的最优值p^*与t的大小关系。如果可行问题是可行的，那么p^*<=t；如果可行问题不可行，则有p^*>t。

Quasiconvex optimization via convex feasibility problems

$$\phi_t(x) \le 0, \quad f_i(x) \le 0, \ i=1,\dots,m, \quad Ax=b$$

1. For fixed t, a convex feasibility problem in x
2. If feasible, we can conclude that $t \ge p^\star$; if infeasible, $t \le p^\star$

Bisection method for quasiconvex optimization

given $l \le p^\star$, $u \ge p^\star$, tolerance $\epsilon > 0$.
repeat
1. $t := (l+u)/2$.
2. Solve the convex feasibility problem (1).
3. **if** (1) is feasible, $u := t$; **else** $l := t$.

until $u - l \le \epsilon$.

requires exactly $\lceil \log_2((u-l)/\epsilon) \rceil$ iterations (where u, l are initial values)

例示

Example

$$f_0(x) = \frac{p(x)}{q(x)}$$

with p convex, q concave, and $p(x) \ge 0$, $q(x) > 0$ on $\mathbf{dom}\, f_0$
can take $\phi_t(x) = p(x) - tq(x)$:
1. for $t \ge 0$, ϕ_t convex in x
2. $p(x)/q(x) \le t$ if and only if $\phi_t(x) \le 0$

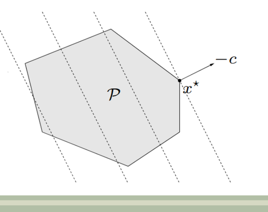
4. 例示

线性规划问题
1. 具有仿射目标和约束函数的凸问题
2. 可行集是多面体

图中给出了线性规划问题的一般形式

$$\begin{array}{ll} \texttt{minimize} & c^Tx + d \\ \texttt{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

1. Convex problem with affine objective and constraint functions
2. Feasible set is a polyhedron



续：线性规划的标准形式和无等式约束形式

标准形式要求仅有的不等式都是分量的非负约束，即x>=0

如果线性规划没有等式约束，则称为不等式形式线性规划

Standard form for LP:

$$\begin{array}{ll} \texttt{minimize} & c^Tx \\ \texttt{subject to} & Ax = b \\ & x \succeq 0 \end{array}$$

Standard LP without equality constraints:

$$\begin{array}{ll} \texttt{minimize} & c^Tx \\ \texttt{subject to} & Ax \preceq b \end{array}$$

续：将一般形式的线性规划问题转化为标准形式

1. 引入松弛变量，将不等式约束转化为等式约束；
2. 将x转化为两个正数的差，将x的定义域转换成高维非负象限

Convert the following LP to the standard form:

$$\begin{array}{ll} \texttt{minimize} & c^Tx + d \\ \texttt{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

Introducing the slack variables

$$\begin{array}{ll} \texttt{minimize} & c^Tx + d \\ \texttt{subject to} & Gx + s = h \\ & Ax = b \\ & s \succeq 0 \end{array}$$

let $x = x^+ - x^-$:

$$\begin{array}{ll} \texttt{minimize} & c^Tx^+ - c^Tx^- + d \\ \texttt{subject to} & Gx^+ - Gx^- + s = h \\ & Ax^+ - Ax^- = b \\ & x^+ \succeq 0, x^- \succeq 0, s \succeq 0 \end{array}$$

例1. 食谱问题

一份食谱中包含m中不同的营养，每种至少需要b_i，从n种食物中选择非负的量x_i构成一份食谱。单位第j中视频含有营养的量为a_{ij}，价格为c_j，要求设计出一份满足营养需求的最便宜的食谱。

Diet problem: choose quantities $x_1,\dots,x_n$ of n foods
1. One unit of food j costs c_j, contains amount a_{ij} of nutrient i
2. Healthy diet requires nutrient i in quantity at least b_i

to find cheapest healthy diet,

$$\begin{array}{ll} \texttt{minimize} & c^Tx \\ \texttt{subject to} & Ax \succeq b, \quad x \succeq 0 \end{array}$$

例2. 分片线性最小化

很熟悉了，无穷范数极小化的LP形式

Piecewise-linear minimization

$$\texttt{minimize} \quad \max_{i=1,\dots,m} (a_i^Tx + b_i)$$

equivalent to an LP

$$\begin{array}{ll} \texttt{minimize} & t \\ \texttt{subject to} & a_i^Tx + b_i \le t, \ i=1,\dots,m \end{array}$$

例3. 多面体的Chebyshev中心

多面体切比雪夫中心是多面体能容纳的最大的欧几里得球的中心。

在满足球中所有点都在多面体中的约束条件下尽量增大球的半径。

Chebyshev center of a polyhedron
Chebyshev center of $\mathcal{P} = \{x \mid a_i^Tx \le b_i, \ i=1,\dots,m\}$ is center of largest inscribed ball $\mathcal{B} = \{x_c + u \mid \|u\|_2 \le r\}$
1. $a_i^Tx \le b_i$ for all $x \in \mathcal{B}$ if and only if

$$\sup\{a_i^T(x_c+u) \mid \|u\|_2 \le r\} = a_i^Tx_c + r\|a_i\|_2 \le b_i$$

2. hence, x_c, r can be determined by solving the LP

$$\begin{array}{ll} \texttt{maximize} & r \\ \texttt{subject to} & a_i^Tx_c + r\|a_i\|_2 \le b_i, i=1,\dots,m \end{array}$$

例4. 线性分式规划

给出了一个线性分式函数的优化问题，这是一个拟凸函数（实际上也是拟线性函数）。可以转化为如下所示的线性规划问题。

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

linear-fractional program

$$f_0(x) = \frac{c^Tx + d}{e^Tx + f}, \qquad \mathbf{dom}\, f_0(x) = \{x \mid e^Tx + f > 0\}$$

a quasiconvex optimization problem; can be solved by bisection
also equivalent to the LP(variables y, z)

$$\begin{array}{ll} \texttt{minimize} & c^Ty + dz \\ \texttt{subject to} & Gy \preceq hz \\ & Ay = bz \\ & e^Ty + fz = 1 \\ & z \ge 0 \end{array}$$

续：等价性的证明

其中可行的x与y和z满足的关系挺常见的

为显示这个等价性，我们首先说明如果 x 是 (4.32) 的可行解，那么

$$\underbrace{y = \frac{x}{e^Tx+f}, \qquad z = \frac{1}{e^Tx+f}}$$

是 (4.33) 的可行解，并且具有相同的目标函数值 $c^Ty + dz = f_0(x)$。从而可知 (4.32) 的最优值大于或等于 (4.33) 的最优值。

反之，如果 (y,z) 是 (4.33) 的可行解，并且 $z \ne 0$，那么 $x = y/z$ 是 (4.32) 的可行解，并具有相同的目标函数值 $f_0(x) = c^Ty + dz$。如果 (y,z) 是 (4.33) 的可行解，$z=0$，并且 x_0 是 (4.32) 的可行解，那么对于所有 $t \geqslant 0$，$x = x_0 + ty$ 都是 (4.32) 的可行解。并且 $\lim_{t\to\infty} f_0(x_0+ty) = c^Ty + dz$，因此我们可以找到 (4.32) 的可行解使其目标函数值任意接近 (y,z) 的目标函数值。由此，我们可以得知 (4.32) 的最优解小于或等于 (4.33) 的最优解。

一个更加一般化的结论：在几何上最小化一个线性函数相当于在其凸包上最小化这个线性函数

试证明在一个集合上最小化一个线性函数等价于在其凸包上最小化该函数，即：

$$\inf_{x\in\mathbf{conv}\, X} c^Tx = \inf_{x\in X} c^Tx,$$

其中 $X \subset R^m$ 和 $c \in R^m$。此外，当且仅当等式右侧的下确界可以达到时，等式左侧的下确界才能达到。

证明

proof.
$X \subset \mathbf{conv}\, X$ yields $\inf_{x\in\mathbf{conv}\, X} c^Tx \le \inf_{x\in X} c^Tx$.
Any $\bar x \in \mathbf{conv}\, X$ can be written as $\bar x = \sum_{i=1}^m \alpha_i x_i$, where $x_1,\dots,x_m \in X$ and $\alpha_1,\dots,\alpha_m \ge 0$ with $\sum_{i=1}^m \alpha_i = 1$. Since $c^Tx_i \ge \inf_{x\in X} c^Tx$,

$$c^T\bar x = \sum_{i=1}^m \alpha_i c^Tx_i \ge (\sum_{i=1}^m \alpha_i) \inf_{x\in X} c^Tx = \inf_{x\in X} c^Tx, \quad \forall \bar x \in \mathbf{conv}\, X.$$

Take the infimum for the L.H.S. for $\bar x \in \mathbf{conv}\, X$

$$\inf_{\bar x\in\mathbf{conv}\, X} c^T\bar x \ge \inf_{x\in X} c^Tx.$$

To summarize,

$$\inf_{x\in\mathbf{conv}\, X} c^Tx = \inf_{x\in X} c^Tx.$$

续

contd. Since $X \subset \mathbf{conv}\, X$ and $\inf_{x\in\mathbf{conv}\, X} c^Tx = \inf_{x\in X} c^Tx$, the optimal point in X with respect to c^Tx can be found in $\mathbf{conv}\, X$.
On the other hand, assume function c^Tx reaches the minimum in $\mathbf{conv}\, X$, which is $\bar x$. For $x_1,\dots,x_m$ and $\sum_{i=1}^m \alpha_i = 1$ with $\alpha_1,\dots,\alpha_m \ge 0$ we construct $\bar x = \sum_{i=1}^m \alpha_i x_i$, there is

$$\inf_{x\in X} c^Tx = (\sum_{i=1}^m \alpha_i) \inf_{x\in X} c^Tx \le \sum_{i=1}^m \alpha_i c^Tx_i = c^T\bar x = \inf_{x\in\mathbf{conv}\, X} c^Tx = \inf_{x\in X} c^Tx.$$

L.H.S. equals to R.H.S., hence, for all i and $\alpha_i > 0$, $c^Tx_i = \inf_{x\in\mathbf{conv}\, X} c^Tx$ must hold. As a result, the minimum of c^Tx is attainable in X.

例5. Boolean线性规划的松弛解

松弛问题的最优解也是原问题的最优解

exercise. **Relaxation of Boolean LP.** In a Boolean linear program, the variable x is constrained to have components equal to zero or one:

$$\begin{array}{ll} \texttt{minimize} & c^Tx \\ \texttt{subject to} & Ax \preceq b \\ & x_i \in \{0,1\}, \ i=1,\dots,n \end{array}$$

The above problem can be related to:

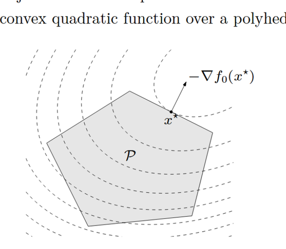
$$\begin{array}{ll} \texttt{minimize} & c^Tx \\ \texttt{subject to} & Ax \preceq b \\ & 0 \le x_i \le 1, \ i=1,\dots,n \end{array}$$

Optimal solution of relaxation is also optimal for Boolean LP.

例6. QP：在多面体可行集中优化二次函数

$$\begin{array}{ll} \texttt{minimize} & (1/2)x^TPx + q^Tx + r \\ \texttt{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

$P \in \boldsymbol{S}^n$, so objective is convex quadratic
minimize a convex quadratic function over a polyhedron



最小二乘

1. 有解析解
2. 可以添加线性约束

Examples: least-squares

$$\texttt{minimize} \quad \|Ax - b\|_2^2$$

analytical solution
can add linear constraints, e.g., $l \succeq x \succeq u$

关于随机费用的线性规划

Linear program with random cost

$$\begin{array}{ll} \texttt{minimize} & \bar c^Tx + \gamma x^T\Sigma x = \mathbf{E}c^Tx + \gamma \mathbf{var}(c^Tx) \\ \texttt{subject to} & Gx \preceq h, \quad Ax = b \end{array}$$

1. c is random vector with mean $\bar c$ and covariance Σ
2. hence, c^Tx is random variable with mean $\bar c^Tx$ and variance $x^T\Sigma x$
3. $\gamma > 0$ is risk aversion parameter; controls the trade-off between expected cost and variance (risk)

例7. 几何规划

具有下列形式的优化问题称为几何规划（GP）:

$$\begin{array}{ll} \texttt{minimize} & f_0(x) \\ \texttt{subject to} & f_i(x) \le 1, \ i=1,\dots,m \\ & h_i(x) = 1, \ i=1,\dots,p \end{array}$$

其中 $f_0,\dots,f_m$ 为正项式，$h_1,\dots,h_p$ 为单项式

几何规划的拓展
如果 f 是一个正项式，h 为单项式，那么，$f_0 \le h(x)$ 表示为 $f(x)/h(x) \le 1$，而等式约束 $h_1(x) = h_2(x)$ 可以表示为 $h_1(x)/h_2(x) = 1$

定义补充：单项式与正项式

Examples: Geometric Programming
单项式函数或单项式：$f : \mathbf{R}^n \to \mathbf{R}$, $\mathbf{dom}\, f = \mathbf{R}^n_{++}$,

$$f(x) = cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}, \quad c > 0, \quad a_i \in \mathbf{R}$$

正项式函数：

$$f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$$

其中 $c_k > 0$

例示：将一个问题转化成几何规划问题

Examples
考虑下面的问题：

$$\begin{array}{ll} \texttt{minimize} & x/y \\ \texttt{subject to} & 2 \le x \le 3 \\ & x^2 + 3y/z \le \sqrt{y} \\ & x/y = z^2 \end{array}$$

可以转化为

$$\begin{array}{ll} \texttt{minimize} & x^{-1}y \\ \texttt{subject to} & 2x^{-1} \le 1, \ (1/3)x \le 1 \\ & x^2y^{-1/2} + 3y^{1/2}z^{-1} \le 1 \\ & xy^{-1}z^{-2} = 1 \end{array}$$

凸形式的几何规划

凸形式的集合规划
定义 $y_i = \log x_i$，因此 $x_i = e^{y_i}$
如果 $f(x) = cx_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$，那么

$$\begin{aligned} f(x) &= f(e^{y_1}, e^{y_2}, \dots, e^{y_n}) \\ &= c(e^{y_1})^{a_1} \cdots (e^{y_n})^{a_n} \\ &= e^{a^Ty + b} \end{aligned}$$

其中，$b = \log c$
类似地，正项式 $f(x) = \sum_{k=1}^K c_k x_1^{a_{1k}} x_2^{a_{2k}} \cdots x_n^{a_{nk}}$ 可以转化为相似形式：

$$f(x) = \sum_{k=1}^K e^{a_k^Ty + b_k}$$

转换：GP到LP

几何规划可以用新的变量 y 来表示：

$$\begin{array}{ll} \texttt{minimize} & \sum_{k=1}^{K_0} e^{a_{0k}^Ty + b_{0k}} \\ \texttt{subject to} & \sum_{k=1}^{K_i} e^{a_{ik}^Ty + b_{ik}} \le 1, \ i=1,\dots,m \\ & e^{g_i^Ty + h_i} = 1, \ i=1,\dots,p \end{array}$$

采用对数函数对目标函数和约束条件进行转换：

$$\begin{array}{ll} \texttt{minimize} & \tilde f_0(y) = \log\left(\sum_{k=1}^{K_0} e^{a_{0k}^Ty + b_{0k}}\right) \\ \texttt{subject to} & \tilde f_i(y) = \log\left(\sum_{k=1}^{K_i} e^{a_{ik}^Ty + b_{ik}}\right) \le 0, \ i=1,\dots,m \\ & \tilde h_i(y) = g_i^Ty + h_i = 0, \ i=1,\dots,p \end{array}$$