

02. Convex Sets – 凸集合

By Yang Lin¹ (2023 秋季, @NJU)

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¹Institute: Nanjing University. Email: linyang@nju.edu.cn.

Preliminaries: linear space

Define set with closed operations ...

Definition: For set V (e.g. \mathbf{R}^n) and field F (e.g. \mathbf{R}), define addition on V

$$\forall x, y \in V \Rightarrow x + y \in V,$$

and scalar multiplication

$$\forall x \in V, \forall \alpha \in F \Rightarrow \alpha x \in V,$$

Also, satisfies (ignored on real numbers)

- $x + y = y + x$
- $1x = x$
- $\alpha(x + y) = \alpha x + \alpha y$
- $(\alpha + \beta)x = \alpha x + \beta x$
- $\alpha(\beta x) = (\alpha\beta)x$

Preliminaries: linear space

examples.

1. 实的 n 重有序数组集合是实数域上的线性空间。
2. 全体 $m \times n$ 阶实矩阵左乘的集合 $\mathbf{R}^{m \times n}$ 按通常的矩阵加法以及实数与矩阵的乘法构成实数域上的线性空间。(复矩阵类似)
3. 区间 $[a, b]$ 上的连续实函数集合 $C[a, b]$, 按函数普通加法与数乘构成实数域上的线性空间。

Preliminaries: linear space

Linear subspace: if a subset of linear space $V(F)$ is a linear space, then it is a subspace of $V(F)$

Theorem 1 $W \neq \emptyset$ and $W \subset V$. W is a subspace of $V(F)$ iff

$$\forall x, y \in W \Rightarrow x + y \in W,$$

$$\forall x \in W, \forall \alpha \in F \Rightarrow \alpha x \in W$$

Or equivalently

$$\forall x, y \in W, \forall \alpha, \beta \in F \Rightarrow \alpha x + \beta y \in W$$

(only justify it is closed on sums and scalar multiplication)

Theorem 2 $S, T \subset V(F)$ are subspaces, then $S \cap T$ is a subspace, $S \cup T$ is usually not a subspace, $S + T$ is a subspace

$$S + T := \{z \mid z = x + y, x \in S, y \in T\}$$

Preliminaries: linear space

Theorem 3 a subspace spanned by a set of vectors x_1, \dots, x_m

$$\text{Span}\{x_1, \dots, x_m\} = \left\{ x \mid x = \sum_{i=1}^m \alpha_i x_i, \alpha_i \in F, i = 1, \dots, m \right\}$$

is the minimum subspace containing x_1, \dots, x_m

Element in a linear space

$$x = \sum_{i=1}^m \alpha_i x_i, \alpha_i \in F, i = 1, \dots, m$$

Linear space is closed on **linear combination**

Linearly dependent there are α_i some of which are nonzeros, such that $\sum_{i=1}^m \alpha_i x_i = 0$

Linearly independent $\sum_{i=1}^m \alpha_i x_i = 0$ iff $\alpha_i = 0, \forall i$

Preliminaries: linear space

Dimension of a linear space In space V , there exists $\{x_1, \dots, x_m\}$ which are linearly independent; and for any set of $m + 1$ vectors, the elements are linearly dependent, we call $\{x_1, \dots, x_m\}$ as the maximal linearly independent system.

$$\dim V = m$$

Base of a linear space

Important fact on linear dependency and independency:

Vector $y \in \mathbf{Span}\{x_1, \dots, x_m\}$ has unique coefficients in $y = \sum_{i=1}^m \alpha_i x_i$ iff x_1, \dots, x_m are linearly independent

Preliminaries: linear space

Theorem subspaces W_1 and W_2 , there is

$$\dim(W_1 \cap W_2) \leq \min\{\dim(W_1), \dim(W_2)\}$$

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Preliminaries: linear map

Definition If V, V' are linear spaces with the same domain F , if $\sigma : V \rightarrow V'$ satisfies

$$\sigma(x + y) = \sigma(x) + \sigma(y), \quad \forall x, y \in V$$

$$\sigma(\alpha x) = \alpha \sigma(x), \quad \forall x \in V, \forall \alpha \in F$$

σ is a linear map from V to V'

A linear map is linear isomorphism map if it is a one-to-one map

A linear map is called linear transformation if $\sigma : V \rightarrow V$

Theorem Assume V, V' are linear spaces, all linear maps from V to V' consists of set $\mathcal{L}(V, V')$. $\mathcal{L}(V, V')$ is also a linear space, with the following equations holding

$$(\sigma + \tau)(x) = \sigma(x) + \tau(x), \quad \forall x \in V$$

$$(\alpha \sigma)(x) = \alpha \sigma(x), \quad \forall x \in V, \forall \alpha \in F$$

Preliminaries: linear map

线性映射 $\sigma : V \rightarrow V'$ 的核和像分别是

$$\text{Ker}\sigma = \{x \in V : \sigma(x) = 0\}$$

$$\text{Im}\sigma = \{y \in V' : y = \sigma(x), x \in V\}.$$

一个矩阵 $A \in F^{m \times n}$, 按矩阵与向量的乘法可以作为一个线性映射 $A : F^n \rightarrow F^m$, 其核空间常称为零空间 $\mathcal{N}(A)$, 其像空间常称为列空间 $\mathcal{R}(A)$, 它可以由矩阵 A 的全部列向量张成。

Preliminaries: linear map

What is matrix (understand matrix thru linear space)

(1) A set of numbers according to an order (like vector, certainly a linear space)

(2) A set of (row/column) vectors/first-order equations (rank)

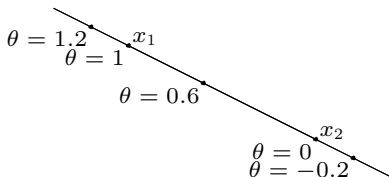
The rank of matrix determines the dimension of the **linear space** that those vectors can span, the solution vector spans the rest (垂直于行向量)

(3) Linear operator/map (linear space)

1 Affine and convex sets

Line Through x_1 and x_2 , all points:

$$x = \theta x_1 + (1 - \theta)x_2, \quad (\theta \in \mathbf{R})$$



Affine set (仿射集合): contains the line through any two distinct points in the set

- Example (look at analysis in next slide): solution set of linear equations $\{x | Ax = b\}$ (conversely, every affine set can be expressed as solution set of system of linear equations)

1 Affine and convex sets

Alternative explanation of affine set

Assume C is affine, then

$$C = V + x_0 = \{v + x_0 | v \in V\}$$

for some x_0 , where V is a subspace

Analysis: For an affine set C and $x_0 \in C$

$$V = C - x_0 = \{x - x_0 | x \in C\}$$

is a subspace (closed under sums and scalar multiplication)

proof ?

1 Affine and convex sets

proof.

Assume $x_0 \in C$ and C is affine, there is:

(1) $0 \in C - x_0$ (do we need that?)

(2) Closed under scalar multiplication

$$x_1 \in C - x_0 \Leftrightarrow x_1 + x_0 \in C$$

$$ax_1 + x_0 = a(x_1 + x_0) + (1 - a)x_0 \in C \Rightarrow ax_1 \in C - x_0$$

(3) Closed under sums

$$x_1 \in C - x_0, x_2 \in C - x_0 \Rightarrow 2x_1 \in C - x_0, 2x_2 \in C - x_0$$

$$x_1 + x_2 + x_0 = \frac{1}{2}(2x_1 + x_0) + \frac{1}{2}(2x_2 + x_0) \in C$$

1 Affine and convex sets

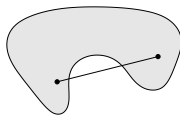
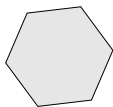
Line segment between x_1 and x_2 : all points

$$x = \theta x_1 + (1 - \theta)x_2, \quad (\theta \in [0, 1])$$

Convex set: contains line segment between any two points in the set

$$x_1, x_2 \in C, \quad \theta \in [0, 1] \Rightarrow \theta x_1 + (1 - \theta)x_2 \in C$$

Examples: one convex, two non-convex



1 Affine and convex sets

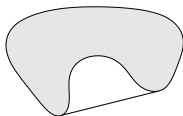
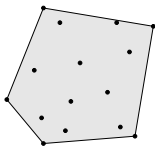
Convex Combination (凸组合) of x_1, \dots, x_k : any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2 + \dots + \theta_k x_k,$$

with $\theta_1 + \dots + \theta_k = 1$, and $\theta_i \geq 0$

(Convex set: convex combination lies in the set)

Convex hull (凸包) $\text{conv}S$: set of all convex combinations of points in S

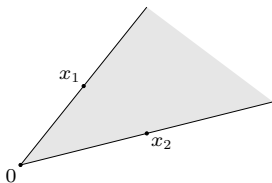


1 Affine and convex sets

Conic (nonnegative) combination of x_1 and x_2 : any point of the form

$$x = \theta_1 x_1 + \theta_2 x_2,$$

with $\theta_1 \geq 0$, and $\theta_2 \geq 0$



Convex cone: set that contains all conic combinations of points in the set

Preliminaries: Open and Closed Sets

Definition 1 The set $B(x, r) := \{y \in \mathbf{R}^n : \|y - x\| < r\}$ is called the open ball centred at x with radius $r > 0$.

Definition 2 $S \subset \mathbf{R}^n$ is called open if either $S = \emptyset$ or for every $x \in S$ if there exists $r > 0$ such that $B(x, r) \subset S$; $U \subset \mathbf{R}^n$ is called closed if the complement $U^c = \{x \in \mathbf{R}^n : x \notin U\}$ is open.

Theorem A nonempty set is closed if and only if it contains all limit points. (对于极限运算封闭)

Preliminaries: Open and Closed Sets

proof.

\Leftarrow Let U be a closed set. Assume there exist a series of points $\{x_1, x_2, \dots, x_i, \dots\} \subset U$ converging to a limit point $x \notin U$. That is to say, $x \in U^c$, and there is an open ball $B(x, r) \not\subset U$ for some $r > 0$. Apparently, $x_i \notin B(x, r)$, contradicting the assumption that $\{x_1, x_2, \dots, x_i, \dots\} \subset U$ converges to x . Hence, U contains all limit points if U is a closed set.

\Rightarrow Let U be a set which contains all limit points. Now we prove U^c is an open set. For some $x \in U^c$, we assume there does not exist an open ball $B(x, r) \subset U$ for $r > 0$. Then, for some $r_1 > 0$, we can find a point $x_1 \in B(x, r_1) \cap U$. For $r_2 = r_1/2$, we have some $x_2 \in B(x, r_2) \cap U$. In this way, we construct a point series $\{x_1, x_2, \dots, x_i, \dots\} \subset U$, while its limit $x \in U^c$, contradicting our assumption. As a result U^c is open and U is a closed set.

Preliminaries: Open and Closed Sets

example.

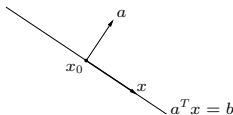
Let A and B are two closed sets. $S = A \cap B$ and $S \neq \emptyset$. Is S a closed set? Why?

Is an arbitrary union of open sets an open set? Why?

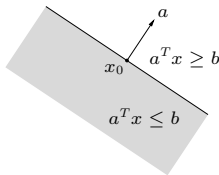
Is an arbitrary intersection of open sets an open set? Why?

2 Important examples

Hyperplane: set of the form $\{x | a^T x = b\}$, $a \neq 0$



Halfspace: set of the form $\{x | a^T x \leq b\}$ ($a \neq 0$)



a is the normal vector (确定了法线的方向)

hyperplanes are affine and convex; halfspaces are convex

2 Important examples

examples.

1. 将闭凸集 $\{x \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1\}$ 表示为半空间的交集。
2. 设 $K \subseteq \mathbb{R}^n, K \neq \emptyset$ 。证明 K 是一个闭凸集当且仅当它是包含 K 的所有闭半空间的交集。(闭集合：如果一个集合包含它所有的边界点，那么这个集合叫做闭集。另一种说法，对极限运算是封闭的，包含了所有的极限点。)

2 Important examples

examples.

1. 对任何满足 $mn = 1$ 的 (m, n) , $(m, n > 0)$ 有 $x_2 - n \geq -\frac{1}{m^2} (x_1 - m)$ 。

即 $-m^2 x_2 - x_1 \leq -m^2 n - m$

则原集合可表示为

$$\bigcap_{\substack{m \in \mathbb{R}^2 \\ mn=1}} \left\{ (x_1, x_2) \mid (-1, -m^2)^\top x \leq (-m^2 n - m) \right\}$$

即半空间交集。

2 Important examples

examples.

2. 一方面, 若 K 是包含 K 的所有闭半空间的交集。由交集是保凸运算, 故 K 凸。又 K 是半空间交集, 知 K 闭, 故 K 闭凸集。

另一方面, 若 K 是一个闭凸集, 去证

$$K = \bigcap_{\substack{K \subseteq A \\ A \text{ 是闭半空间}}} A$$

任取 $x \in K$, 由 $K \subseteq A$ 知 $x \in \bigcap_{\substack{K \subseteq A \\ A \text{ 是闭半空间}}} A$ 。

任取 $x \in \bigcap_{\substack{K \subseteq A \\ A \text{ 是闭半空间}}} A$, 若 $x \notin K$ 可知存在一个半空间 A_0 包含 K , 但 $x \notin A_0$ 。与 $x \in \bigcap A$ 即对任意 A_0 ($K \subseteq A_0$, A_0 是闭半空间) $x \in A_0$ 矛盾。

则 $K = \bigcap_{\substack{K \subseteq A \\ A \text{ 是闭半空间}}} A$ 。

综合两方面得证。

2 Important examples

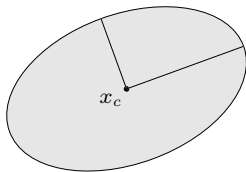
(Euclidean) ball: with center x_c and radius r :

$$B(x_c, r) = \{x \mid \|x - x_c\|_2 \leq r\} = \{x_c + ru \mid \|u\|_2 \leq 1\}$$

Ellipsoid (椭球) : set of the form

$$\{x \mid \|(x - x_c)^T P^{-1}(x - x_c) \leq 1\}$$

with $P \in \mathbf{S}_{++}^n$ (i.e., P symmetric positive definite)



Other representation: $\{x_c + Au \mid \|u\| \leq 1\}$ with A square and nonsingular (非奇异方阵)

2 Important examples

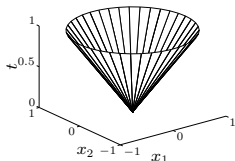
Norm: a function $\|\cdot\|$ that satisfies:

- (1) $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$
- (2) $\|tx\| = |t|\|x\|, t \in \mathbf{R}$
- (3) $\|x + y\| \leq \|x\| + \|y\|$

notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\text{symb}}$ is particular norm

Norm ball (范数球): with center x_c and radius r : $\{x \mid \|x - x_c\| \leq r\}$

Norm cone (范数锥): $\{(x, t) \mid \|x\| \leq t\}$ (Euclidean norm cone is called second-order cone)



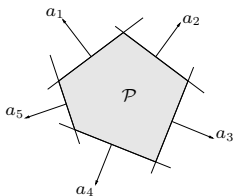
Norm balls and cones are convex

2 Important examples

Polyhedra (多面体): solution set of finitely many linear inequalities and equalities

$$Ax \preceq b, \quad Cx = d$$

($A \in R^{m \times n}$, $C \in R^{p \times n}$, \preceq is componentwise inequality)



Polyhedron is intersection of finite number of halfspaces and hyperplanes

2 Important examples

Notations:

\mathbf{S}^n is set of symmetric $n \times n$ matrices

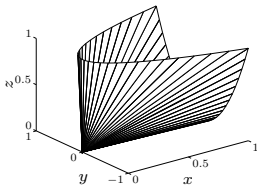
$\mathbf{S}_+^n = \{X \in \mathbf{S}^n \mid X \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$X \in \mathbf{S}_+^n \iff z^T X z \geq 0 \text{ for all } z$$

\mathbf{S}_+^n is a convex cone

$\mathbf{S}_{++}^n = \{X \in \mathbf{S}^n \mid X \succ 0\}$: positive definite $n \times n$ matrices

Example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbf{S}_+^2$



2 Important examples

proof.

Assume $A \succeq 0$ and $B \succeq 0$

For any $\theta_1, \theta_2 \geq 0$

$$x^T(\theta_1 A + \theta_2 B)x = \theta_1 x^T A x + \theta_2 x^T B x \geq 0$$

That is $\theta_1 A + \theta_2 B \in \mathbf{S}_+^n$

3 Operations that preserve convexity

Practical methods for establishing convexity of a set C :

1. Apply definition

$$x_1, x_2 \in C, 0 \leq \theta \leq 1 \Rightarrow \theta x_1 + (1-\theta)x_2 \in C$$

2. Show that C is obtained from simple convex sets (hyperplanes, halfspaces, norm balls, ...) by operations that preserve convexity

- ☐ Intersection
- ☐ Affine functions
- ☐ Perspective function – 透视函数
- ☐ Linear-fractional functions – 线性分式

3 Operations that preserve convexity

Intersection

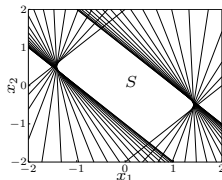
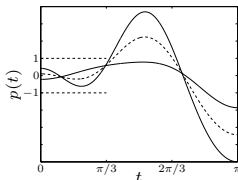
The intersection of (any number of) convex sets is convex

Example

$$S = \{x \in \mathbf{R}^m \mid |p(t)| \leq 1 \text{ for } |t| \leq \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$

for $m = 2$



3 Operations that preserve convexity

Affine function

Suppose $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$ is affine ($f(x) = Ax + b$ with $A \in \mathbf{R}^{m \times n}$, $b \in \mathbf{R}^m$), then:

1. The image of a convex set under f is convex

$$S \subseteq \mathbf{R}^n \text{ convex} \Rightarrow f(S) = \{f(x) \mid x \in S\} \text{ convex}$$

2. The inverse image $f^{-1}(C)$ of a convex set under f is convex

$$C \subseteq \mathbf{R}^m \text{ convex} \Rightarrow f^{-1}(C) = \{x \in \mathbf{R}^n \mid f(x) \in C\} \text{ convex}$$

Examples:

- Scaling, translation, projection
- Solution set of linear matrix inequality $\{x \mid x_1 A_1 + \cdots + x_m A_m \preceq B\}$
(with $A_i, B \in \mathbf{S}^p$)
- Hyperbolic cone $\{x \mid x^T P x \leq (c^T x)^2, c^T x \geq 0\}$ (with $P \in \mathbf{S}_+^n$)

3 Operations that preserve convexity

proof. of Example 2

Define $f(x) := B - A(x) = B - (x_1 A_1 + \cdots + x_m A_m)$

It is affine

Prove set of images $\{f(x) | x \in \mathbf{R}^n, f(x) \succeq 0\}$ is convex

proof. of Example 3

The above set is obtained from the inverse image of $\{(z, t) | z^T z \leq t^2, t \geq 0\}$ through affine function $f(x) = (P^{1/2}x, c^T x)$

3 Operations that preserve convexity

Perspective function $P: \mathbf{R}^{n+1} \rightarrow \mathbf{R}^n$:

$$P(x, t) = x/t, \text{ dom } P = \{(x, t) \mid t > 0\}$$

Images and inverse images of convex sets under perspective are convex

Linear-fractional function: $f: \mathbf{R}^n \rightarrow \mathbf{R}^m$:

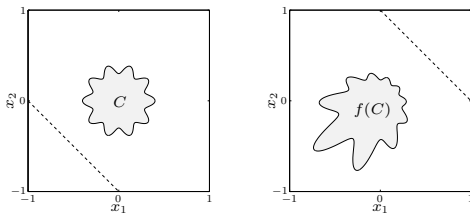
$$f(x) = \frac{Ax + b}{c^T x + d}, \text{ dom } f = \{x \mid c^T x + d > 0\}$$

Images and inverse images of convex sets under linear-fractional functions are convex

3 Operations that preserve convexity

Example of a linear-fractional function:

$$f(x) = \frac{1}{x_1 + x_2 + 1} x$$



4 Generalized inequalities – 广义不等式

A convex cone $K \subseteq \mathbf{R}^n$ is a proper cone (正常锥) if

- K is closed (contains its boundary)
- K is solid (has nonempty interior)
- K is pointed (contains no line)

Examples:

1. Nonnegative orthant $K = \mathbf{R}_+^n = \{x \in \mathbf{R}^n \mid x_i \geq 0, i = 1, \dots, n\}$
2. Positive semidefinite cone $K = \mathbf{S}_+^n$
3. Nonnegative polynomials on $[0, 1]$:

$$K = \{x \in \mathbf{R}^n \mid x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

4 Generalized inequalities – 广义不等式

Generalized inequality defined by a proper cone K :

$$x \preceq_K y \iff y - x \in K, \quad x \prec_K y \iff y - x \in \text{int } K$$

Examples:

1. Componentwise inequality ($K = \mathbf{R}_+^n$)

$$x \preceq_{\mathbf{R}_+^n} y \iff x_i \leq y_i, \quad i = 1, \dots, n$$

2. Matrix inequality ($K = \mathbf{S}_+^n$)

$$X \preceq_{\mathbf{S}_+^n} Y \iff Y - X \text{ positive semidefinite}$$

These two types are so common that we drop the subscript in \preceq_K

Many properties of \preceq_K are similar to \leq on \mathbf{R} , *e.g.*,

$$x \preceq_K y, \quad u \preceq_K v \Rightarrow x + u \preceq_K y + v$$

4 Generalized inequalities – 广义不等式

Properties on generalized inequalities:

1. 对于加法是保序的: 如果 $x \preceq_K y$ 并且 $u \preceq_K v$, 那么 $x+u \preceq_K y+v$.
2. 具有传递性: 如果 $x \preceq_K y$ 并且 $y \preceq_K z$, 那么 $x \preceq_K z$.
3. 对于非负数乘是保序的: 如果 $x \preceq_K y$ 并且 $\alpha \geq 0$, 那么 $\alpha x \preceq_K \alpha y$.
4. 是自反的: $x \preceq_K x$.
5. 是反对称的: 如果 $x \preceq_K y$ 并且 $y \preceq_K x$, 那么 $x = y$.
6. 对于极限运算是保序的: 如果对于 $i = 1, 2, \dots$ 均有 $x_i \preceq_K y_i$, 当 $i \rightarrow \infty$ 时, 有 $x_i \rightarrow x$ 和 $y_i \rightarrow y$, 那么 $x \preceq_K y$.

4 Generalized inequalities – 广义不等式

Minimum and minimal elements

\preceq_K is not in general a linear ordering: we can have $x \not\preceq_K y$ and $y \not\preceq_K x$

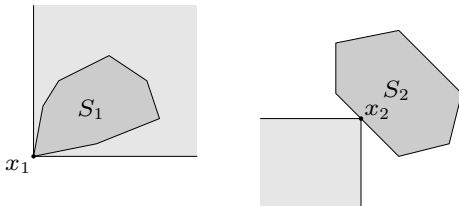
$x \in S$ is the minimum element (最小元) of S with respect to \preceq_K if

$$y \in S \Rightarrow x \preceq_K y$$

$x \in S$ is the minimal element (极小元) of S with respect to \preceq_K if

$$y \in S, y \preceq_K x \Rightarrow y = x$$

Example:

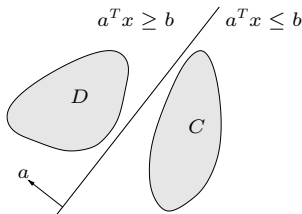


5 Separating and supporting hyperplanes

Separating hyperplane theorem

If C and D are disjoint convex sets, then there exists a $a \neq 0, b$ such that

$$a^T x \leq b \text{ for } x \in C, \quad a^T x \geq b \text{ for } x \in D$$



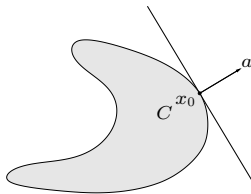
The hyperplane $\{x | a^T x = b\}$ separates C and D

5 Separating and supporting hyperplanes

Supporting hyperplane to set C at boundary point x_0 :

$$\{x | a^T x = a^T x_0\}$$

where $a \neq 0$ and $a^T x \leq a^T x_0$ for all $x \in C$



Supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

6 Dual cones and generalized inequalities

Dual cone of a cone K :

$$K^* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}$$

Examples:

- $K = \mathbf{R}_+^n$: $K^* = \mathbf{R}_+^n$
- $K = \mathbf{S}_+^n$: $K^* = \mathbf{S}_+^n$
- $K = \{(x, t) \mid \|x\|_2 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_2 \leq t\}$
- $K = \{(x, t) \mid \|x\|_1 \leq t\}$: $K^* = \{(x, t) \mid \|x\|_\infty \leq t\}$

First three examples are **self-dual cones**

Dual cones of proper cones are proper, hence define generalized inequalities:

$$y \succeq_{K^*} 0 \iff y^T x \geq 0 \text{ for all } x \succeq_K 0$$

6 Dual cones and generalized inequalities

proof. of example 3

If $(y, z) \in K^*$, we have $y^T x + zt \geq 0$ for $\forall (x, t) \in K$

I.e., $y^T \frac{x}{t} + z \geq 0$ for $\forall (x, t) \in K$

Define $u = x/t$: $y^T u + z \geq 0$ for $\forall u : \|u\|_2 \leq 1$

Take supremum: $-\|y\|_2 + z \geq 0$ or $\|y\|_2 \leq z$

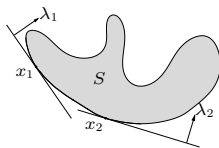
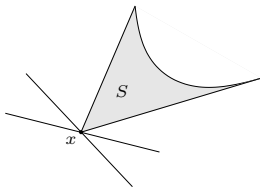
Define $\|u\|_* = \sup\{u^T x \mid \|x\| \leq 1\}$, then $K^* = \{(y, z) \mid \|y\|_* \leq z\}$

6 Dual cones and generalized inequalities

Minimum and minimal elements via dual inequalities

Minimum element with respect to \preceq_K : x is minimum element of S iff for all $\lambda \succ_{K^*} 0$, x is the unique minimizer of $\lambda^T z$ over S

$$K^* = \{y | y^T x \geq 0 \text{ for all } x \in K\}$$



Minimal element with respect to \preceq_K : if x minimizes $\lambda^T z$ over S for some $\lambda \succ_{K^*} 0$, then x is minimal

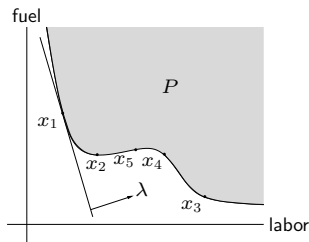
If x is a minimal element of a convex set S , then there exists a nonzero $\lambda \succeq_{K^*} 0$ such that x minimizes $\lambda^T z$ over S

6 Dual cones and generalized inequalities

Optimal production frontier

- Different production methods use different amounts of resources $x \in \mathbf{R}^n$
- Production set P : resource vectors x for all possible production methods
- Efficient (Pareto optimal) methods correspond to resource vectors x that are minimal w.r.t. \mathbf{R}_+^n

example (n = 2): x_1, x_2, x_3 are efficient; x_4, x_5 are not



* Homework

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