01. Introduction to Convex Optimization –最优化导论
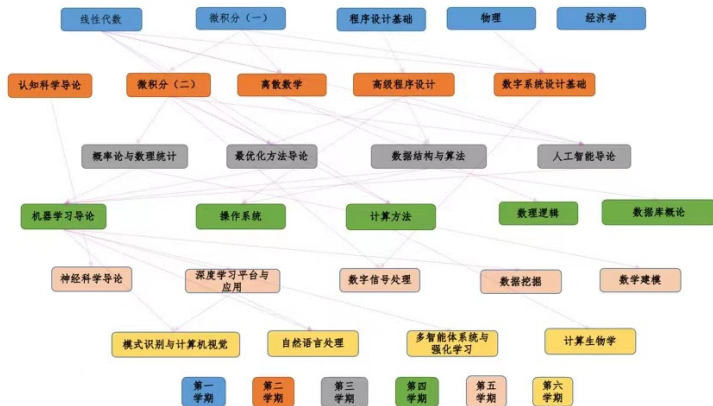
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0 About this course

1 助教

李紫微、张浩然、陈浩旭、吴光华

2 课程时间

周五 2-4 节

3 课程考核（适当微调）

- □ 期末考试（闭卷）60%
- □ 作业（两周一次，optimize2023@163.com）、课堂或期中考核 30%
- □ 课程项目（TBD，文献阅读等）10%

4 参考书

"Convex Optimization" Stephen Boyd and Lieven Vandenberghe
关键字"最优化理论与方法"相关教材

1 Mathematical Optimization

"Optimization" as a tool in engineering:

"在给定的决策集中（通过某种方式）寻找最优解"

$$\begin{aligned}\text{minimize} \quad & f_0(x)\\ \text{subject to} \quad & f_i(x) \le b_i, \ i = 1, 2, \ldots, m\end{aligned}$$

- □ $x = (x_1, x_2, \ldots, x_n)$: optimization variables
- □ $f_0 : \boldsymbol{R}^n \to \boldsymbol{R}$: Objective function (objective)
- □ $f_i : \boldsymbol{R}^n \to \boldsymbol{R}, \ i = 1, \ldots, m$: Constraint functions (constraints)

Optimal solution x^* has smallest value of f_0 among all vectors that satisfy the constraints

Preliminaries: norms

Norm: a function $\|\cdot\|$ that satisfies:

(1) $\|x\| \geq 0$; $\|x\| = 0$ if and only if $x = 0$

(2) $\|tx\| = |t|\|x\|, t \in \boldsymbol{R}$

(3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Notation: $\|\cdot\|$ is general (unspecified) norm; $\|\cdot\|_{\texttt{symb}}$ is particular norm

Distance: between vectors x and y as the length of their difference, *i.e.*,

$$\texttt{dist}(x, y) = \|x - y\|$$

Preliminaries: norms

Some common norms on $\boldsymbol{R}^n$:

1. Sum-absolute-value, or l_1-norm:

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|, \ x \in \boldsymbol{R}^n$$

2. l_2-norm

$$\|x\|_2 = (x_1^2 + x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}, \ x \in \boldsymbol{R}^n$$

3. l_p-norm $(p \geq 1)$:

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}}, \ x \in \boldsymbol{R}^n$$

4. l_∞-norm (Chebyshev norm):

$$\|x\|_\infty = \lim_{p \to \infty} \|x\|_p = \max\{|x_1|, \ldots, |x_n|\}$$

5. For $P \in \boldsymbol{S}_{++}^n$, P-quadratic norm (P-二次范数) is

$$\|x\|_P = (x^T P x)^{\frac{1}{2}} = \|P^{\frac{1}{2}} x\|_2$$

Preliminaries: norms

proof.

$$\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{\frac{1}{p}} \geq |x_{\max}|$$

$$\|x\|_p \leq (n \cdot |x_{\max}|^p)^{\frac{1}{p}} = n^{\frac{1}{p}} |x_{\max}|$$

$$n^{\frac{1}{p}} |x_{\max}| \to |x_{\max}| \quad \text{when} \quad p \to \infty$$

$$\cdots$$

Preliminaries: norms

Equivalence between norms:

1. Suppose that $\|\cdot\|_a$ and $\|\cdot\|_b$ are norms on $\boldsymbol{R}^n$, there exist positive constants α and β, for all $x \in \boldsymbol{R}^n$:

$$\alpha\|x\|_a \leq \|x\|_b \leq \beta\|x\|_a$$

2. If $\|\cdot\|$ is any norm on $\boldsymbol{R}^n$, then there exists a quadratic norm $\|\cdot\|_P$ for which

$$\|x\|_P \leq \|x\| \leq \sqrt{n}\|x\|_P$$

holds for all x

Preliminaries: norms

proof. of first part:

It is sufficient to prove $\alpha\|x\|_1 \leq \|x\|_b \leq \beta\|x\|_1$

Or $\alpha \leq \|u\|_b \leq \beta$, where $u = x/\|x\|_1$ has norm $\|u\|_1 = 1$

Let

$$\alpha = \min_{\|u\|_1=1} \|u\|_b$$

$$\beta = \max_{\|u\|_1=1} \|u\|_b$$

α, β exist for given b

Proof completes

examples.

1. 假设 $\|\cdot\|$ 是一个定义在 $\mathbb{R}^m$ 上的范数，定义 $\|x\|_A = \|Ax\|$，$A \in \mathbb{R}^{m \times n}$。证明如果 $\text{rank}(A) = n$，则 $\|\cdot\|_A$ 也是一个范数。

2. 定义函数 $f \colon \mathbb{C}^n \to \mathbb{R}, f(x) = \sum_{i=1}^{n} (|\text{Re}\, x_i| + |\text{Im}\, x_i|)$，$f$ 是范数吗？试证明。

Preliminaries: norms

proof.

(1) $\|x\|_A = \|Ax\| \geqslant 0$ (非负性)

(2) 因为 $\text{rank}(A) = n$，所以 $\|x\|_A = \|Ax\| = 0$ 当且仅当 $x = 0$

(3) $\|tx\|_A = \|tAx\| = |t| \cdot \|Ax\| = |t|\|x\|_A$, $t \in R$（齐次性）

(4) $\|x + y\|_A = \|Ax + Ay\| \leq \|Ax\| + \|Ay\| = \|x\|_A + \|y\|_A$
$(x \in R^n, y \in R^n)$ 满足三角不等式。

综上，$\|\cdot\|_A$ 是一个范数

Preliminaries: norms

proof.

(1) $f(x) = \sum_{i=1}^{n} (|\mathrm{Re}\, x_i| + |\mathrm{Im}\, x_i|) \geqslant 0$ （非负性）

(2) 满足齐次性:

$$f(tx) = \sum_{i=1}^{n} (|\mathrm{Re}\, tx_i| + |\mathrm{Im}\, tx_i|)$$

$$= |t| \sum_{i=1}^{n} (|\mathrm{Re}\, x_i| + |\mathrm{Im}\, x_i|) = |t| f(x), \forall t \in R.$$

(3) 满足三角不等式:

$$f(x + y) = \sum_{i=1}^{n} (|\mathrm{Re}\, (x_i + y_i)| + |\mathrm{Im}\, (x_i + y_i)|)$$

$$\leqslant \sum_{i=1}^{n} (|\mathrm{Re}\, x_i| + |\mathrm{Im}\, x_i| + |\mathrm{Re}\, y_i| + |I_m y_i|)$$

$$= f(x) + f(y), \forall x, y \in C^n$$

综上，f 是范数。

2 Examples: least squares

Least squares formulation:

$$\texttt{minimize} \quad f_0(x) = \|Ax - b\|_2^2 = \sum_{i=1}^{k}(a_i^T x - b_i)^2$$

- $\square$ $A \in \boldsymbol{R}^{k \times n}(k \geq n)$, a_i^T 是矩阵 A 的行向量
- $\square$ Solving the above problem is equivalent to solving $(A^T A)x = A^T b$. Hence, $x = (A^T A)^{-1} A^T b$
- $\square$ Computation time: proportional to $n^2 k$

Properties of Least-squares problems:
- $\square$ Easy to recognize and standard techniques
- $\square$ Increase flexibility through weights, regularization terms

2 Examples: linear programming

Linear programming formulation:

$$\begin{aligned}
\texttt{minimize} \quad & f_0(x) = c^T x \\
\texttt{subject to} \quad & a_i^T x \le b_i, \ i = 1, \ldots, m.
\end{aligned}$$

□ 向量 $c, a_1, \ldots, a_m \in \boldsymbol{R}^n, b_1, \ldots, b_m \in \boldsymbol{R}$ 为问题参数（parameter）
□ No analytical solution (closed-form equation)
□ Computation time: proportional to $n^2 m$ if $m \ge n$, less if structured

Properties of Least-squares problems:

□ not as easy to recognize as least-squares problems
□ A few standard tricks used to convert problems into linear programs
(piecewise-linear functions)

2 Examples: linear programming

examples.

（食谱问题）一份健康的饮食包含 m 种不同的营养，每种至少需要 $b_1, \ldots, b_m$。我们可以从 n 种食物种选择非负的量构成一份食谱。单位第 j 种食品含有营养 i 的量为 a_{ij}，而价格为 c_j。我们希望设计出一份最便宜的满足所有营养需求的食谱。如何将这一问题描述为线性规划问题？

2 Examples: linear programming

examples.

（食谱问题）一份健康的饮食包含 m 种不同的营养，每种至少需要 $b_1, \ldots, b_m$。我们可以从 n 种食物种选择非负的量构成一份食谱。单位第 j 种食品含有营养 i 的量为 a_{ij}，而价格为 c_j。我们希望设计出一份最便宜的满足所有营养需求的食谱。如何将这一问题描述为线性规划问题？

$$
\begin{aligned}
\text{minimize} \quad & c^T x \\
\text{subject to} \quad & Ax \succeq b, \\
& x \succeq 0.
\end{aligned}
$$

2 Examples: linear programming

examples.

(多面体的 Chebyshev 中心) 考虑在多面体中寻找最大的 Euclid 球的问题，多面体由线性不等式表示为

$$\mathcal{P} = \{x \in \mathbf{R}^n | a_i^T x \le b_i, \ i = 1, \ldots, m\}$$

最优球的中心成为多面体的 Chebyshev 中心，它是多面体内部最深的点，即离边界最远的点。

2 Examples: linear programming

examples.

可以将球表述为
$$\mathcal{B} = \{x_c + u\|u\|_2 \le r\},$$
其中变量为 x_c 和 r。

条件"球在半空间 $a_i^T x \le b_i$ 中"可以表达为:
$$\|u\|_2 \le r \Rightarrow a_i^T(x_c + u) \le b_i, (整个公式是一个条件)$$
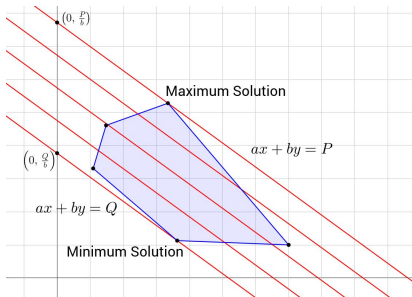
其中
$$\sup\{a_i^T u\|u\|_2 \le r\} = r\|a_i\|_2$$

因此,条件"球在半空间 $a_i^T x \le b_i$ 中"可以写作为 $a_i^T x_c + r\|a_i\|_2 \le b_i$。

$$\begin{aligned}
\texttt{maximize} \quad & r \\
\texttt{subject to} \quad & a_i^T x_c + r\|a_i\|_2 \le b_i, i = 1, \ldots, m.
\end{aligned}$$

2 Examples: linear programming

examples.

线性规划问题的求解，图解法

Q: Computation complexity (stop criterion is ϵ)?

2 Examples: convex optimization

Convex optimization formulation:

$$\begin{aligned} \texttt{minimize} \quad & f_0(x) \\ \texttt{subject to} \quad & f_i(x) \leq b_i, \ i = 1, \ldots, m. \end{aligned}$$

☐ Object and constraint functions are **both** convex

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y),$$

if $\alpha + \beta = 1,\ \alpha \geq 0,\ \beta \geq 0$

☐ Includes least-squares problems and linear programming as special cases

2 Examples: solving CO problems

- □ No analytical solution
- □ Reliable and efficient algorithms
- □ Computation time(roughly) proportional to $\max\{n^3, n^2m, F\}$, where F is cost of evaluating f_i' s and their first and second derivatives
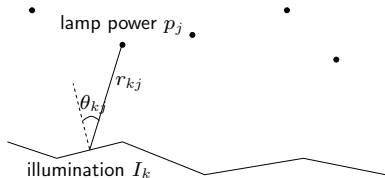
When using convex optimization (why is it popular?)

- □ Often difficult to recognize
- □ Many tricks for transforming problems into convex form
- □ Surprisingly many problems can be solved via convex optimization

"In fact the great watershed in optimization isn't between linearity and nonlinearity, but convexity and nonconvexity."

——*Rockafellar*（洛克菲拉），*1993*

2 Examples: lamps illuminating patches

m lamps illuminating n (small, flat) patches

- intensity I_k at patch k depends linearly on lamp powers p_j:

$$I_k = \sum_{j=1}^{m} a_{k,j} p_j, \quad a_{kj} = r_{kj}^{-2} \max\{\cos \theta_{kj}, 0\}$$

- Achieve desired illumination $I_{\mathtt{des}}$ with bounded lamp powers

$$
\begin{aligned}
&\mathtt{minimize} &&\max_k |\log I_k - \log I_{\mathtt{des}}| \\
&\mathtt{subject\ to} &&0 \le p_j \le p_{\max}, \ j = 1, \dots, m.
\end{aligned}
$$

2 Examples: lamps illuminating patches

How to solve?

☐ Use uniform power $p_j = p$, vary p

☐ Use least-squares

$$\texttt{minimize} \quad \sum_{k=1}^{n}(I_k - I_{\texttt{des}})^2$$

round p_j if $p_j < 0$ or $p_j > p_{\max}$

☐ Use weighted least-squares

$$\texttt{minimize} \quad \sum_{k=1}^{n}(I_k - I_{\texttt{des}})^2 + \sum_{j=1}^{m} w_j(p_j - p_{\max}/2)^2$$

iteratively adjust weights w_j until $0 \leq p_j \leq p_{\max}$

☐ Use linear programming

$$\begin{aligned} \texttt{minimize} \quad & \max_{k=1,2,\ldots,n} |I_k - I_{\texttt{des}}| \\ \texttt{subject to} \quad & 0 \leq p_j \leq p_{\max}, j = 1, \ldots, m \end{aligned}$$
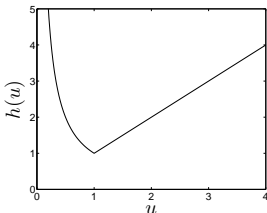
How to solve?

□ Use convex optimization: problem is equivalent to

$$\begin{array}{ll} \texttt{minimize} & f_0(p) = \max_{k=1,\ldots,n} h(I_k/I_{\texttt{des}}) \\ \texttt{subject to} & 0 \leq p_j \leq p_{\max}, \; j = 1, \ldots, m \end{array}$$

with $h(u) = \max\{u, 1/u\}$

f_0 is convex because maximum of convex functions is convex

3 Examples in applications

Portfolio optimization
- ☐ Variables: amounts invested in different assets
- ☐ Constraints: budget, max./min. investment per asset, minimum return
- ☐ Objective: overall risk or return variance
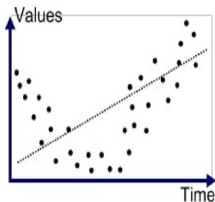
Device sizing in electronic circuits
- ☐ Variables: device widths and lengths
- ☐ Constraints: manufacturing limits, timing requirements, maximum area
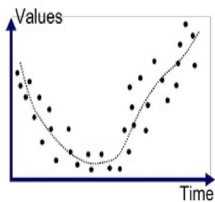- ☐ Objective: power consumption

Data fitting
- ☐ Variables: model parameters
- ☐ Constraints: prior information, parameter limits
- ☐ Objective: measure of misfit or prediction error
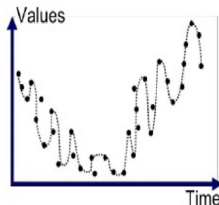
3 Examples in applications
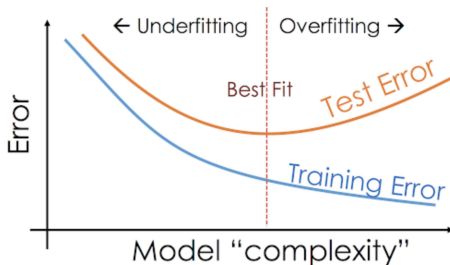
Overfitting/Underfitting

Underfitted   Good Fit/Robust   Overfitted

Overfitting/Underfitting with different levels of complexity/training

4 Brief history of convex optimization

Theory(convex analysis): ca1900–1970

Algorithms

- ☐ 1947: simplex algorithm for linear programming (Dantzig)
- ☐ 1960s: early interior-point methods (Fiacco and McCormick, Dikin, ...)
- ☐ 1970s: ellipsoid method and other subgradient methods
- ☐ 1980s: polynomial-time interior-point methods for linear programming (Karmarkar1984)
- ☐ late 1980s–now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov and Nemirovski1994)

Applications

- ☐ Before 1990: mostly in operations research; few in engineering
- ☐ Since 1990: many new applications in engineering (control, signal processing, communications, circuit design, ...); new problem classes (semidefinite and second-order cone programming, robust optimization)