一．作业

1.

$$P(X_1 = 0, \ X_2 = 0) = P(抽到三等品) = 0.1,$$

$$P(X_1 = 0, \ X_2 = 1) = P(抽到二等品) = 0.2,$$

$$P(X_1 = 1, \ X_2 = 0) = P(抽到一等品) = 0.7,$$

$$P(X_1 = 1, \ X_2 = 1) = P(\varnothing) = 0.$$

所以$(X_1, \ X_2)$的联合分布律为

X_1 \ X_2	0	1
0	0.1	0.2
1	0.7	0

$$P(X_1 = 0, \ X_2 = 0) = P(抽到三等品) = 0.1,$$

$$P(X_1 = 0, \ X_2 = 1) = P(抽到二等品) = 0.2,$$

$$P(X_1 = 1, \ X_2 = 0) = P(抽到一等品) = 0.7,$$

$$P(X_1 = 1, \ X_2 = 1) = P(\varnothing) = 0,$$

故$(X_1, \ X_2)$的联合分布律为

X_1 \ X_2	0	1
0	0.1	0.2
1	0.7	0

求联合分布律表格中的行和及列和得到X_1和X_2的边缘分布律分别为

X_1	0	1
P	0.3	0.7

X_2	0	1
P	0.8	0.2

2.

解 (1) 因为 $1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,\,y)\mathrm{d}x\mathrm{d}y$，所以

$$1 = \int_0^2 \int_2^4 c(6-x-y)\mathrm{d}y = c\int_0^2 6-2x\mathrm{d}x = 8c,$$

故 $c = \dfrac{1}{8}$.

(2) 由已知得

$$P(X+Y<4) = \iint\limits_{D} \frac{1}{8}(6-x-y)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^2 \mathrm{d}x\int_2^{4-x} \frac{1}{8}(6-x-y)\mathrm{d}y = \int_0^2 \frac{1}{8}\left[(6-x)\cdot y - \frac{y^2}{2}\right]_2^{4-x}\mathrm{d}x$$

$$= \int_0^2 \frac{1}{16}(x^2-8x+12)\mathrm{d}x = \frac{1}{16}\left[\frac{x^3}{3}-4x^2+12x\right]_0^2 = \frac{2}{3}.$$

(3) 区域 D_1 与 D 如图 3.10 所示，由已知及条件概率公式得

$$P(X<1\,|\,X+Y<4) = \frac{P(X<1,\ X+Y<4)}{P(X+Y<4)}$$

图 3.10

$$= \frac{\displaystyle\iint\limits_{D_1} \frac{1}{8}(6-x-y)\mathrm{d}x\mathrm{d}y}{\displaystyle\iint\limits_{D} \frac{1}{8}(6-x-y)\mathrm{d}x\mathrm{d}y}$$

$$= \frac{\displaystyle\int_0^1 \mathrm{d}x\int_2^{4-x} \frac{1}{8}(6-x-y)\mathrm{d}y}{\dfrac{2}{3}}$$

$$= \frac{\displaystyle\int_0^1 \frac{1}{16}(x^2-8x+12)\mathrm{d}x}{\dfrac{2}{3}} = \frac{\dfrac{25}{48}}{\dfrac{2}{3}} = \frac{25}{32}.$$
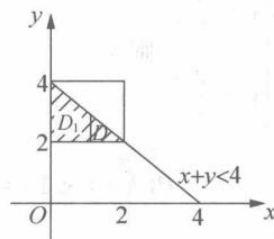
解 (1) 因为 $\Omega_X = (0, 2)$，$\Omega_Y = (2, 4)$，所以当 $0 < x < 2$ 时，有

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\,dy = \int_2^4 \frac{1}{8}(6-x-y)\,dy = \frac{1}{8}\left[(6-x)\cdot 2 - \frac{12}{2}\right] = \frac{1}{4}(3-x).$$

故

$$f_X(x) = \begin{cases} \dfrac{1}{4}(3-x), & 0 < x < 2, \\ 0, & \text{其他}. \end{cases}$$

当 $2 < y < 4$ 时，有

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\,dx = \int_0^2 \frac{1}{8}(6-x-y)\,dx$$

$$= \frac{1}{8}\left[(6-y)\cdot 2 - \frac{4}{2}\right] = \frac{1}{4}(5-y).$$

故

$$f_Y(y) = \begin{cases} \dfrac{1}{4}(5-y), & 2 < y < 4, \\ 0, & \text{其他}. \end{cases}$$

3.

解 (1) 如图 3.13 所示，由已知得 $S_G = 4$，而 (X, Y) 是 G 上的均匀分布，所以

$$f(x, y) = \begin{cases} \dfrac{1}{4}, & (x, y) \in G, \\ 0, & \text{其他}. \end{cases}$$

(2) 由已知得

$$P(X+Y < 2) = 1 - P(X+Y \geqslant 2)$$

$$= 1 - \iint_{x+y \geqslant 2} f(x, y)\,dx\,dy$$

$$= 1 - \int_1^2 dx \int_{2-x}^x \frac{1}{4}\,dy = \frac{3}{4}.$$

或如图 3.13 所示，

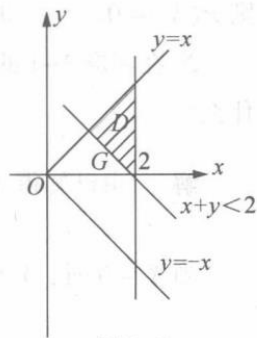$$P(X+Y < 2) = 1 - P(X+Y \geqslant 2) = 1 - \frac{S_D}{S_G} = \frac{3}{4}.$$

图 3.13

解 （1）因为(X, Y)的联合密度函数为

$$f(x, y) = \begin{cases} \dfrac{1}{4}, & (x, y) \in G, \\ 0, & \text{其他.} \end{cases}$$

而$\Omega_X = (0, 2)$，$\Omega_Y = (-2, 2)$，如图3.17所示．则当$0 < x < 2$时，有

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}y = \int_{-x}^{x} \frac{1}{4}\mathrm{d}y = \frac{x}{2},$$

所以

$$f_X(x) = \begin{cases} \dfrac{x}{2}, & 0 < x < 2, \\ 0, & \text{其他.} \end{cases}$$

当$-2 < y < 2$时，有

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}x = \int_{|y|}^{2} \frac{1}{4}\mathrm{d}x = \frac{2 - |y|}{4}.$$
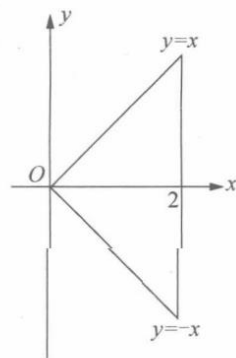
则

图 3.17

$$f_Y(y) = \begin{cases} \dfrac{2 - |y|}{4}, & |y| < 2, \\ 0, & \text{其他.} \end{cases}$$

4.

解 （1）如图3.15所示，因为$S_G = \dfrac{1}{2} - \dfrac{1}{2} \times \dfrac{1}{4} + \dfrac{1}{2} \times \dfrac{1}{4} = \dfrac{1}{2}$，所以

$$f(x, y) = \begin{cases} 2, & (x, y) \in G, \\ 0, & \text{其他.} \end{cases}$$

（2）因为$\Omega_X = (0, 1)$，$\Omega_Y = (0, 1)$，所以

当$0 < x < 1$时，$f_X(x) = \displaystyle\int_{-\infty}^{+\infty} f(x, y)\mathrm{d}y$；

图 3.15

当$0 < x < \dfrac{1}{2}$时，$f_X(x) = \displaystyle\int_{x}^{x+\frac{1}{2}} 2\mathrm{d}y = 1$；

当$\dfrac{1}{2} < x < 1$时，$f_X(x) = \displaystyle\int_{0}^{x-\frac{1}{2}} 2\mathrm{d}y + \int_{x}^{1} 2\mathrm{d}y = 2\left[\left(x - \dfrac{1}{2}\right) + 1 - x\right] = 1$；
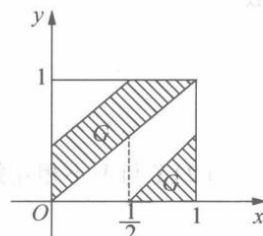
故

$$f_X(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{其他}. \end{cases}$$

当 $0 < y < 1$ 时, $f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\, dx$;

当 $0 < y < \dfrac{1}{2}$ 时, $f_Y(y) = \int_0^y 2\, dx + \int_{y+\frac{1}{2}}^1 2\, dx = 2\left[y + \left(1 - y - \dfrac{1}{2}\right)\right] = 1$;

当 $\dfrac{1}{2} < y < 1$ 时, $f_Y(y) = \int_{y-\frac{1}{2}}^y 2\, dx = 1$.

故

$$f_Y(y) = \begin{cases} 1, & 0 < y < 1, \\ 0, & \text{其他}. \end{cases}$$

二．练习

1.

$$X \sim B\left(3, \frac{1}{2}\right), \quad Y = |X - (3 - X)| = |2X - 3|.$$

当 $X = 0$ 时，必有 $Y = 3$，所以

$$P(X = 0, \ Y = 3) = P(X = 0) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

同理，有

$$P(X = 1, \ Y = 1) = P(X = 1) = C_3^1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = \frac{3}{8},$$

$$P(X = 2, \ Y = 1) = P(X = 2) = C_3^2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^1 = \frac{3}{8},$$

$$P(X = 3, \ Y = 3) = P(X = 3) = C_3^3 \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

所以 (X, Y) 的联合分布律为

X \ Y	1	3
0	0	$\frac{1}{8}$
1	$\frac{3}{8}$	0
2	$\frac{3}{8}$	0
3	0	$\frac{1}{8}$

解 由已知得 $X \sim B\left(3, \dfrac{1}{2}\right)$，$Y = |X-(3-X)| = |2X-3|$.

当 $X=0$ 时，$Y=3$，而 $P(X=0) = C_3^0\left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$.

当 $X=1$ 时，$Y=1$，$P(X=1) = C_3^1\left(\dfrac{1}{2}\right)^3 = \dfrac{3}{8}$.

当 $X=2$ 时，$Y=1$，$P(X=2) = C_3^2\left(\dfrac{1}{2}\right)^3 = \dfrac{3}{8}$.

当 $X=3$ 时，$Y=3$，$P(X=3) = C_3^3\left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$.

故 (X, Y) 的联合分布律为

X \ Y	1	3	$p_{i\cdot}$
0	0	$\dfrac{1}{8}$	$\dfrac{1}{8}$
1	$\dfrac{3}{8}$	0	$\dfrac{3}{8}$
2	$\dfrac{3}{8}$	0	$\dfrac{3}{8}$
3	0	$\dfrac{1}{8}$	$\dfrac{1}{8}$
$p_{\cdot j}$	$\dfrac{6}{8}$	$\dfrac{2}{8}$	1

（1）由联合分布律表格计算行和、列和，得 X 和 Y 的边缘分布律分别为

X	0	1	2	3
概率	$\dfrac{1}{8}$	$\dfrac{3}{8}$	$\dfrac{3}{8}$	$\dfrac{1}{8}$

Y	1	3
概率	$\dfrac{3}{4}$	$\dfrac{1}{4}$

2.

解 （1）因为 $1 = \displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x, y)\,\mathrm{d}x\mathrm{d}y$，所以

$$1 = \int_0^{+\infty}\mathrm{d}x\int_0^{+\infty} c e^{-(x+2y)}\,\mathrm{d}y = \int_0^{+\infty}\frac{c}{2}e^{-x}\,\mathrm{d}x = \frac{c}{2},$$

有 $c = 2$.

（2）由已知得

$$P(X<1, Y>2) = \iint\limits_{D} f(x, y)\,\mathrm{d}x\mathrm{d}y = \int_0^1\mathrm{d}x\int_2^{+\infty} 2e^{-(x+2y)}\,\mathrm{d}y$$

$$= \int_0^1\left[-e^{-2y}\right]_2^{+\infty} e^{-x}\,\mathrm{d}x = \left[1-e^{-1}\right]\cdot e^{-4} = e^{-4} - e^{-5}.$$

解 (1) 因为 $\Omega_X = \Omega_Y = (0, +\infty)$，所以当 $x > 0$ 时，有

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\,dy = \int_0^{+\infty} 2e^{-(x+2y)}\,dy = e^{-x}.$$

故

$$f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{其他}. \end{cases}$$

当 $y > 0$ 时，有

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\,dx = \int_0^{+\infty} 2e^{-(x+2y)}\,dx = 2e^{-2y}.$$

故

$$f_Y(y) = \begin{cases} 2e^{-2y}, & y > 0, \\ 0, & \text{其他}. \end{cases}$$

3.

解 由已知得 $F_Y(y) = \begin{cases} 1-e^{-y}, & y \geqslant 0, \\ 0, & \text{其他}. \end{cases}$ 则

$$P(X_1 = 0, X_2 = 0) = P(Y \leqslant 1, Y \leqslant 2) = P(Y \leqslant 1) = F_Y(1) = 1-e^{-1},$$
$$P(X_1 = 0, X_2 = 1) = P(Y \leqslant 1, Y > 2) = P(\varnothing) = 0,$$
$$P(X_1 = 1, X_2 = 0) = P(Y > 1, Y \leqslant 2) = P(1 < Y \leqslant 2) = F_Y(2) - F_Y(1)$$
$$= (1-e^{-2}) - (1-e^{-1}) = e^{-1} - e^{-2},$$
$$P(X_1 = 1, X_2 = 1) = P(Y > 1, Y > 2) = P(Y > 2) = 1-F_Y(2) = 1-(1-e^{-2}) = e^{-2},$$

所以，(X_1, X_2) 的联合分布律为

X_1 \\ X_2	0	1
0	$1-e^{-1}$	0
1	$e^{-1}-e^{-2}$	e^{-2}

4.

解 (1) 由联合密度函数的规范性知

$$1 = \iint_G cxy\,dx\,dy = \int_0^2 dx \int_0^{2x} cxy\,dy = 8c,$$

于是，$c = \dfrac{1}{8}$.

（2）如图 3.11 所示，由 $\begin{cases} x+y=1, \\ y=2x, \end{cases}$ 得 $x+y=1$ 与 $y=2x$ 的交点为

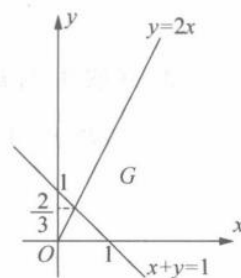$\begin{cases} x = \dfrac{1}{3}, \\ y = \dfrac{2}{3}. \end{cases}$ 则

图 3.11

$$P(X+Y<1) = \iint\limits_{x+y<1} f(x,\,y)\mathrm{d}x\mathrm{d}y = \int_0^{\frac{2}{3}} \mathrm{d}y \int_{\frac{y}{2}}^{1-y} \frac{1}{8}xy\mathrm{d}x$$

$$= \int_0^{\frac{2}{3}} \frac{1}{8}y \left[\frac{x^2}{2}\right]_{\frac{y}{2}}^{1-y} \mathrm{d}y = \int_0^{\frac{2}{3}} \frac{1}{64}(3y^3 - 8y^2 + 4y)\,\mathrm{d}y$$

$$= \frac{1}{64}\left[\frac{3}{4}y^4 - \frac{8}{3}y^3 + 2y^2\right]_0^{\frac{2}{3}} = \frac{5}{1296}.$$

解 （1）由已知得，$\varOmega_X = (0,\,2)$，$\varOmega_Y = (0,\,4)$，所以，当 $0 < x < 2$ 时，有

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,\,y)\mathrm{d}y = \int_0^{2x} \frac{1}{8}xy\mathrm{d}y = \frac{1}{16}x \cdot 4x^2 = \frac{1}{4}x^3.$$

则

$$f_X(x) = \begin{cases} \dfrac{1}{4}x^3, & 0 < x < 2, \\ 0, & \text{其他}. \end{cases}$$

所以，当 $0 < y < 4$ 时，有

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x,\,y)\mathrm{d}x = \int_{\frac{y}{2}}^2 \frac{1}{8}xy\mathrm{d}x = \frac{1}{16}y \cdot \left(4 - \frac{y^2}{4}\right) = \frac{1}{4}y\left(1 - \frac{1}{16}y^2\right).$$

则

$$f_Y(y) = \begin{cases} \dfrac{1}{4}y\left(1 - \dfrac{1}{16}y^2\right), & 0 < y < 4, \\ 0, & \text{其他}. \end{cases}$$

5.

1. 设 (X, Y) 服从以原点为圆心的单位圆上的均匀分布，记

$$U = \begin{cases} 1, & X+Y \leqslant 0, \\ 0, & X+Y > 0, \end{cases}$$

$$V = \begin{cases} 1, & X-Y \leqslant 0, \\ 0, & X-Y > 0. \end{cases}$$

试求 (U, V) 的联合分布律.

解 如图 3.12 所示，由已知得

$$P(U = 1, V = 1) = P(U = 0, V = 1) = P(U = 1, V = 0)$$
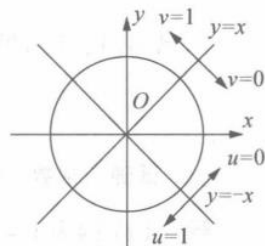
$$= P(U = 0, V = 0) = \frac{1}{4},$$

图 3.12

所以，(U, V) 的联合分布律为

V U	0	1
0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{4}$

6.

解 $f(x,y)$ 为密度函数 $\Leftrightarrow f(x,y) \geqslant 0$ 且 $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x\mathrm{d}y = 1$,

由此可推得， $1 = a+b$, 且 $ap(x,y)+bg(x,y) \geqslant 0$ $(\forall x,y \in \mathbf{R})$.

7.

解 (X,Y) 的所有可能取值为 $(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1)$, $(2,2),(3,0),(3,1),(3,2)$.

按古典概型，显有

$$P\{X=0, Y=2\} = \frac{C_3^0 \times C_2^2 \times C_2^2}{C_7^4} = \frac{1}{35}$$

$$P\{X=1, Y=1\} = \frac{C_3^1 \times C_2^1 \times C_2^2}{C_7^4} = \frac{6}{35}$$

$$P\{X=1, Y=2\} = \frac{C_3^1 \times C_2^2 \times C_2^1}{C_7^4} = \frac{6}{35}$$

$$P\{X=2, Y=1\} = \frac{C_3^2 \times C_2^1 \times C_2^1}{C_7^4} = \frac{12}{35}$$

$$P\{X=2, Y=0\} = \frac{C_3^2 \times C_2^0 \times C_2^2}{C_7^4} = \frac{3}{35}$$

$$P\{X=2, Y=2\} = \frac{C_3^2 \times C_2^2 \times C_2^0}{C_7^4} = \frac{3}{35}$$

$$P\{X=3, Y=0\} = \frac{C_3^3 \times C_2^0 \times C_2^1}{C_7^4} = \frac{2}{35}$$

$$P\{X=3, Y=1\} = \frac{C_3^3 \times C_2^1 \times C_2^0}{C_7^4} = \frac{2}{35}$$

则 X 和 Y 的联合分布律为：

Y \ X	0	1	2	3
0	0	0	$\frac{3}{35}$	$\frac{2}{35}$
1	0	$\frac{6}{35}$	$\frac{12}{35}$	$\frac{2}{35}$
2	$\frac{1}{35}$	$\frac{6}{35}$	$\frac{3}{35}$	0

8.

$$P\{X+Y\leqslant 1\} = \iint\limits_{x+y\leqslant 1} f(x,y)\mathrm{d}x\mathrm{d}y$$

$$= \int_0^{\frac{1}{2}} \mathrm{d}x \int_x^{1-x} 6x\mathrm{d}y$$

$$= \int_0^{\frac{1}{2}} 6x(1-2x)\mathrm{d}x = \frac{1}{4}.$$

9.

解 由 $P\{X_1 X_2 = 0\} = 1 \Rightarrow P\{X_1 X_2 \neq 0\} = 0$，即

$P\{X_1=-1, X_2=-1\}, P\{X_1=-1, X_2=1\}, P\{X_1=1, X_2=-1\}, P\{X_1=1, X_2=1\}$

均为 0.

由以上条件求出，X_1, X_2 的联合概率分布如下表所示

X_1 \ X_2	-1	0	1	$p_i.$
-1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$
1	0	$\frac{1}{4}$	0	$\frac{1}{4}$
$p._j$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

那么 $P\{X_1=X_2\} = P\{X_1=-1, X_2=-1\} + P\{X_1=0, X_2=0\} + P\{X_1=1, X_2=1\} = 0.$

10.

解 (X_1, X_2) 有四个可能值:$(0,0),(0,1),(1,0),(1,1)$.

易见

$$P\{X_1=0, X_2=0\} = P\{Y \leqslant 1, Y \leqslant 2\} = P\{Y \leqslant 1\} = \frac{1}{3}$$

$$P\{X_1=0, X_2=1\} = P\{Y \leqslant 1, Y > 2\} = 0$$

$$P\{X_1=1, X_2=0\} = P\{Y > 1, Y \leqslant 2\} = P\{1 < Y \leqslant 2\} = \frac{1}{3}$$

$$P\{X_1=1, X_2=1\} = P\{Y > 1, Y > 2\} = P\{Y > 2\} = \frac{1}{3}$$

于是,X_1 和 X_2 联合概率分布表如下:

P \ X_1 / X_2	0	1
0	$\frac{1}{3}$	$\frac{1}{3}$
1	0	$\frac{1}{3}$

由联合分布可求得 X_1, X_2 的边缘分布,合并列表为:

X_2 \ X_1	0	1	$p_{\cdot j}$
0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$
1	0	$\frac{1}{3}$	$\frac{1}{3}$
$p_{i \cdot}$	$\frac{1}{3}$	$\frac{2}{3}$	1

三 . 思考题

1.

解：多项分布的第 i 计类别概率为 P_i $(i=1,2,\cdots n)$

且 $X_1+X_2+\cdots+X_n=n$，$P_1+P_2+\cdots+P_n=1$

由定义，X_1 的边缘分布为

$$P(X_1=k_1)=\sum_{\substack{k_2+k_3\\+\cdots+k_n\\=n-k_1}} P(X_1=k_1,X_2=k_2,\cdots X_n=k_n)$$

$$=\sum_{\substack{k_2+k_3\\+\cdots+k_n\\=n-k_1}} C_n^{k_1} C_{n-k_1}^{k_2}\cdots C_{n-k_1-k_2-\cdots-k_{n-1}}^{k_n} P_1^{k_1} P_2^{k_2}\cdots P_n^{k_n}$$

$$=C_n^{k_1}P_1^{k_1}\sum_{\substack{k_2+k_3\\+\cdots+k_n\\=n-k_1}} C_{n-k_1}^{k_2}\cdots C_{n-k_1-k_2-\cdots-k_{n-1}}^{k_n} P_1^{k_1} P_2^{k_2}\cdots P_n^{k_n} \quad ①$$

$\because P_1+P_2+\cdots+P_n=1$

$\therefore (P_2+P_3+\cdots+P_n)^{n-k_1}=(1-P_1)^{n-k_1}$

$$左=\sum_{\substack{k_2+k_3\\+\cdots+k_n\\=n-k_1}} C_{n-k_1}^{k_2} C_{n-k_1-k_2}^{k_3}\cdots C_{n-k_1-k_2-\cdots-k_{n-1}}^{k_n} P_2^{k_2} P_3^{k_3}\cdots P_n^{k_n} \quad ②$$

把②式入①得：

$$P(X_1=k_1)=C_n^{k_1} P_1^{k_1}(1-P_1)^{n-k_1}$$

$\therefore$ 多项分布的边缘分布是二项分布。

由Copula函数：$C(u,v) = H(F^{-1}(u), G^{-1}(v))$

其中H是 u,v 联合分布函数，$F(u)$ 是 u 的分布函数，$G(v)$ 同理

设 $N(0,1)$ 原函数即标准正态分布函数为 $\phi(x)$

两个相关系数为 ρ 的标准正态变量联合分布函数为 $\phi_\rho(x,y)$

解：$C(x,y) = W_\rho(h^{-1}(x), g^{-1}(y))$

W 为 x,y 联合分布函数，$h(x),g(y)$ 为 x,y 分布函数

∴有 $W_\rho(x,y) = \phi_\rho\left(\dfrac{x-u_x}{\sigma_x}, \dfrac{y-u_y}{\sigma_y}\right)$

$h(x) = \phi\left(\dfrac{x-u_x}{\sigma_x}\right)$ ①

①式两边同时取 $h^{-1}(x)$

∴ $x = h^{-1}\left[\phi\left(\dfrac{x-u_x}{\sigma_x}\right)\right]$ 令 $\phi\left(\dfrac{x-u_x}{\sigma_x}\right) = t$

∴ $\dfrac{x-u_x}{\sigma_x} = \phi^{-1}(t)$

∴ $u_x + \sigma_x\phi^{-1}(t) = h^{-1}(t)$，$y$ 同理

∴ $C(x,y) = W_\rho\left(\sigma_x\phi^{-1}(x)+u_x, \sigma_y\phi^{-1}(y)+u_y\right)$

$\qquad\qquad = \phi_\rho\left(\phi^{-1}(x), \phi^{-1}(y)\right)$

hw8

一．作业

1.

解 由已知得 $a+b=0.5$ 且 $P(X=0, X+Y=1)=P(X=0, Y=1)=a$，又因为

$$P(X=0)P(X+Y=1)=(0.4+a)(a+b)=(0.4+a)0.5,$$

由 $\{X=0\}$ 与 $\{X+Y=1\}$ 相互独立得

$$P(X=0, X+Y=1)=P(X=0)P(X+Y=1),$$

即

$$a=(0.4+a)0.5,$$

于是 $a=0.4$，$b=0.1$.

2.

$(2)X_1$ 与 X_2 不相互独立. 因为

$$P(X_1=1, X_2=1)=0, \quad P(X_1=1)P(X_2=1)=0.14,$$

解 (1) 因为 $P(X_2=y_j|X_1=x_i)=\dfrac{p_{ij}}{p_{i\cdot}}$，$y_j \in \Omega_{X_2|X_1=x_i}$，而 $\Omega_{X_2|X_1=1}=\{0\}$，所以

$$P(X_2=0|X_1=1)=\frac{P(X_1=1, X_2=0)}{P(X_1=1)}=1,$$

故给定条件 $\{X_1=1\}$ 下 X_2 的条件分布律为

$X_2|X_1=1$	0
概率	1

(2) 因为 $\Omega_{X_1|X_2=0}=\{0, 1\}$，所以有

$$P(X_1=0|X_2=0)=\frac{P(X_1=0, X_2=0)}{P(X_2=0)}=\frac{1}{8},$$

$$P(X_1=1|X_2=0)=\frac{P(X_1=1, X_2=0)}{P(X_2=0)}=\frac{7}{8}.$$

则给定条件 $\{X_2=0\}$ 下 X_1 的条件分布律为

$X_1|X_2=0$	0	1
概率	$\dfrac{1}{8}$	$\dfrac{7}{8}$

(3) 由条件分布函数的定义知

$$F_{X_1|X_2}(x_1|0)=P(X_1\leqslant x_1|X_2=0)=\begin{cases}0, & x_1<0, \\ \dfrac{1}{8}, & 0\leqslant x_1<1, \\ 1, & x_1\geqslant 1.\end{cases}$$

3.

解 （1）如图 3.14 所示，因为 $S_G = \int_1^{e^2} \mathrm{d}x \int_0^{\frac{1}{x}} 1 \mathrm{d}y = \int_1^{e^2} \frac{1}{x}\mathrm{d}x = 2$，所以

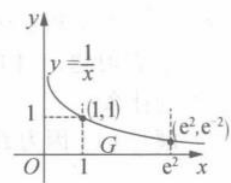$$f(x, y) = \begin{cases} \dfrac{1}{2}, & (x, y) \in G, \\ 0, & 其他. \end{cases}$$

图 3.14

（2）因为 $\Omega_X = (1, e^2)$，$\Omega_Y = (0, 1)$，所以，当 $1 < x < e^2$ 时，有

$$f_X(x) = \int_0^{\frac{1}{x}} \frac{1}{2}\mathrm{d}y = \frac{1}{2x}$$

故

$$f_X(x) = \begin{cases} \dfrac{1}{2x}, & 1 < x < e^2, \\ 0, & 其他. \end{cases}$$

当 $0 < y < 1$ 时，$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}x$，

当 $0 < y < e^{-2}$ 时，$f_Y(y) = \int_1^{e^2} \frac{1}{2}\mathrm{d}x = \frac{e^2-1}{2}$，

当 $e^{-2} < y < 1$ 时，$f_Y(y) = \int_1^{\frac{1}{y}} \frac{1}{2}\mathrm{d}x = \frac{1}{2}\left(\frac{1}{y} - 1\right)$，

故

$$f_Y(y) = \begin{cases} \dfrac{1}{2}(e^2-1), & 0 < y < e^{-2}, \\ \dfrac{1}{2}\left(\dfrac{1}{y} - 1\right), & e^{-2} < y < 1, \\ 0, & 其他. \end{cases}$$

（3）X 与 Y 不相互独立．因为

$$f(1.5, 0.12) = \frac{1}{2} \neq f_X(1.5)f_Y(0.12) = \frac{1}{3} \times \frac{1}{2}(e^2-1) = \frac{1}{6}(e^2-1).$$

4.

解 （1）因为(X, Y)的联合密度函数为

$$f(x, y) = \begin{cases} \dfrac{1}{4}, & (x, y) \in G, \\ 0, & \text{其他.} \end{cases}$$

而$\Omega_X = (0, 2)$，$\Omega_Y = (-2, 2)$，如图3.17所示. 则当$0 < x < 2$时，有

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}y = \int_{-x}^{x} \frac{1}{4}\mathrm{d}y = \frac{x}{2},$$

所以

$$f_X(x) = \begin{cases} \dfrac{x}{2}, & 0 < x < 2, \\ 0, & \text{其他.} \end{cases}$$

当$-2 < y < 2$时，有

$$f_Y(y) = \int_{-\infty}^{+\infty} f(x, y)\mathrm{d}x = \int_{|y|}^{2} \frac{1}{4}\mathrm{d}x = \frac{2 - |y|}{4}.$$
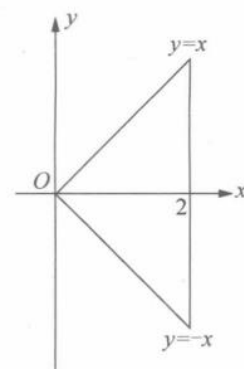
则

图 3.17

$$f_Y(y) = \begin{cases} \dfrac{2-|y|}{4}, & |y| < 2, \\ 0, & \text{其他.} \end{cases}$$

由已知, $\Omega_{X|Y=1} = (1, 2)$, 则当 $1 < x < 2$ 时, 有

$$f_{X|Y}(x|1) = \frac{f(x, 1)}{f_Y(1)} = \frac{\dfrac{1}{4}}{\dfrac{1}{4}} = 1.$$

则

$$f_{X|Y}(x|1) = \begin{cases} 1, & 1 < x < 2, \\ 0, & \text{其他.} \end{cases}$$

由已知得 $\Omega_{X|Y=y} = (|y|, 2)$. 则当 $|y| < 2$ 且 $|y| < x < 2$ 时, 有

$$f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \frac{\dfrac{1}{4}}{\dfrac{2-|y|}{4}} = \frac{1}{2-|y|}.$$

故当 $|y| < 2$ 时, $f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{2-|y|}, & |y| < x < 2, \\ 0, & \text{其他.} \end{cases}$

(2) 由 (1) 知 $f_{X|Y}(x|1) = \begin{cases} 1, & 1 < x < 2, \\ 0, & \text{其他.} \end{cases}$ 则

$$P(X \leqslant \sqrt{2} \mid Y = 1) = \int_{-\infty}^{\sqrt{2}} f_{X|Y}(x|1) \mathrm{d}x = \int_1^{\sqrt{2}} 1 \mathrm{d}y = \sqrt{2} - 1.$$

(3) 由 (1) 知, $f_{X|Y}(x|1) = \begin{cases} 1, & 1 < x < 2, \\ 0, & \text{其他} \end{cases}$, $\Omega_{X|Y=1} = (1, 2)$, 所以, 当 $1 \leqslant x < 2$ 时,

有

$$F_{X|Y}(x|1) = \int_{-\infty}^x f_{X|Y}(u|1) \mathrm{d}u = \int_1^x 1 \mathrm{d}u = x - 1.$$

故

$$F_{X|Y}(x|1) = \begin{cases} 0, & x < 1, \\ x - 1, & 1 \leqslant x < 2, \\ 1, & x \geqslant 2. \end{cases}$$

(4) 由 (1) 知当 $|y| < 2$ 时,

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{1}{2-|y|}, & |y| < x < 2, \\ 0, & \text{其他.} \end{cases}$$

$\Omega_{X|Y=y} = (|y|, 2)$. 所以当 $|y| \leqslant x < 2$ 时, 有

$$F_{X|Y}(x|y) = \int_{-\infty}^x f_{X|Y}(u|y) \mathrm{d}u = \int_{|y|}^x \frac{1}{2-|y|} \mathrm{d}u = \frac{x-|y|}{2-|y|}.$$

故当 $|y| < 2$ 时，$F_{X|Y}(x|y) = \begin{cases} 0, & x < |y|, \\ \dfrac{x - |y|}{2 - |y|}, & |y| \leqslant x < 2, \\ 1, & x \geqslant 2. \end{cases}$

5.

解 （1）由已知得，当 $n = 0, 1, 2, \cdots$ 时，$P(Y = m|X = n) = C_n^m p^m (1-p)^{n-m}$，$m = 0,$
$1, \cdots, n$.

（2）由已知及乘法公式得

$$P(X = n, \ Y = m) = P(X = n)P(Y = m|X = n) = e^{-\lambda} \frac{\lambda^n}{n!} \cdot C_n^m p^m (1-p)^{n-m},$$

$$n = 0, 1, 2, \cdots, m = 0, 1, \cdots, n.$$

（3）由全概率公式得

$$P(Y = m) = \sum_{n=m}^{\infty} P(X = n)P(Y = m|X = n)$$

$$= \sum_{n=m}^{\infty} e^{-\lambda} \frac{\lambda^n}{n!} \cdot C_n^m p^m (1-p)^{n-m}$$

$$= e^{-\lambda} \frac{(\lambda p)^m}{m!} \sum_{n=m}^{\infty} \frac{[\lambda(1-p)]^{n-m}}{(n-m)!}$$

$$= e^{-\lambda} \frac{(\lambda p)^m}{m!} e^{\lambda(1-p)} = e^{-\lambda p} \frac{(\lambda p)^m}{m!}, \ m = 0, 1, 2, \cdots.$$

得证.

二．练习
1.

3. 已知随机变量(X, Y)的联合分布律如下. 当α, β取何值时X与Y相互独立?

X \ Y	1	2	3
1	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$
2	$\frac{1}{3}$	α	β

解 由已知得

X \ Y	1	2	3	$P_{i\cdot}$
0	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{3}$
1	$\frac{1}{3}$	α	β	$\frac{2}{3}$
$p_{\cdot j}$	$\frac{1}{2}$	$\frac{1}{9}+\alpha$	$\frac{1}{18}+\beta$	1

又因为X与Y相互独立，所以有

$$P(X = 1, Y = 2) = P(X = 1)P(Y = 2)，即\frac{1}{9} = \frac{1}{3} \cdot \left(\frac{1}{9} + \alpha\right) \Rightarrow \alpha = \frac{2}{9}.$$

$$P(X = 1, Y = 3) = P(X = 1)P(Y = 3)，即\frac{1}{18} = \frac{1}{3} \cdot \left(\frac{1}{18} + \beta\right) \Rightarrow \beta = \frac{1}{9}.$$

2.

(2)X与Y不相互独立. 因为

$$P(X = 0, Y = 1) = 0, \quad P(X = 0)P(Y = 1) = \frac{1}{8} \times \frac{6}{8},$$

则$P(X = 0, Y = 1) \neq P(X = 0)P(Y = 1)$，所以$X$与$Y$不相互独立.

解　因为(X, Y)的联合分布律为

X \ Y	1	3	$p_{i\cdot}$
0	0	$\dfrac{1}{8}$	$\dfrac{1}{8}$
1	$\dfrac{3}{8}$	0	$\dfrac{3}{8}$
2	$\dfrac{3}{8}$	0	$\dfrac{3}{8}$
3	0	$\dfrac{1}{8}$	$\dfrac{1}{8}$
$p_{\cdot j}$	$\dfrac{6}{8}$	$\dfrac{2}{8}$	1

由条件分布律定义 $P(X = a_i \mid Y = b_j) = \dfrac{p_{ij}}{p_{\cdot j}}$ 及 $P(Y = b_j \mid X = a_i) = \dfrac{p_{ij}}{p_{i\cdot}}$ 得下列结果.

$(1)\, P(X = 1 \mid Y = 1) = \dfrac{P(X = 1,\ Y = 1)}{P(Y = 1)} = \dfrac{\dfrac{3}{8}}{\dfrac{6}{8}} = \dfrac{1}{2}.$

同理，$P(X = 2 \mid Y = 1) = \dfrac{1}{2}.$　则

$X \mid Y = 1$	1	2
概率	0.5	0.5

$(2)\, P(Y = 1 \mid X = 1) = \dfrac{P(X = 1,\ Y = 1)}{X = 1} = \dfrac{\dfrac{3}{8}}{\dfrac{3}{8}} = 1,$　则

$Y \mid X = 1$	1
概率	1

3.

解 (1) 由 $P(XY = 0) = 1$，知 $P(XY \neq 0) = 0$，所以有

$$P(X = -1, Y = 1) = P(X = 1, Y = 1) = 0,$$

而 $P(X = -1) = P(X = -1, Y = 0) + P(X = -1, Y = 1) = \dfrac{1}{4}$，故 $P(X = -1, Y = 0) = \dfrac{1}{4}$，同理

$$P(X = 1, Y = 0) = \dfrac{1}{4},$$

而 $P(Y = 0) = P(X = -1, Y = 0) + P(X = 0, Y = 0) + P(X = 1, Y = 0) = \dfrac{1}{2}$，得 $P(X = 0, Y = 0) = 0$，有 $P(X = 0, Y = 1) = P(X = 0) - P(X = 0, Y = 0) = \dfrac{1}{2}$.

所以 (X, Y) 的联合分布律如下

X \ Y	0	1	$p_{i\cdot}$
-1	$\dfrac{1}{4}$	0	$\dfrac{1}{4}$
0	0	$\dfrac{1}{2}$	$\dfrac{1}{2}$
1	$\dfrac{1}{4}$	0	$\dfrac{1}{4}$
$p_{\cdot j}$	$\dfrac{1}{2}$	$\dfrac{1}{2}$	1

(2) X 与 Y 不相互独立. 因为由 (X, Y) 的联合分布律可知

$$P(X = 0, Y = 0) = 0, \quad P(X = 0)P(Y = 0) = \dfrac{1}{2} \times \dfrac{1}{2} = \dfrac{1}{4},$$

则 $P(X = 0, Y = 0) \neq P(X = 0)P(Y = 0)$，所以 X 与 Y 不相互独立.

4.

4. 已知 (X, Y) 的联合密度函数为

$$f(x, y) = \begin{cases} 2e^{-(x+2y)}, & x > 0, y > 0, \\ 0, & \text{其他}. \end{cases} \quad 求:$$

(1) 条件密度函数 $f_{X|Y}(x|1)$ 与 $f_{X|Y}(x|y)$，其中 $y > 0$；

(2) (X, Y) 的联合分布函数；

(3) 概率 $P(X < 1, Y > 2)$.

解 (1) 因为 $\Omega_X = (0, +\infty)$，所以当 $x > 0$ 时，有

$$f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\,dy = \int_0^{+\infty} 2e^{-(x+2y)}\,dy = \left[-e^{-(x+2y)}\right]_0^{+\infty} = e^{-x}.$$

故 $f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{其他}. \end{cases}$ 同理，$f_Y(y) = \begin{cases} 2e^{-2y}, & y > 0, \\ 0, & \text{其他}. \end{cases}$ 显然对任意 $x, y \in \mathbf{R}$，都有

$f(x, y) = f_X(x)f_Y(y)$ 成立，所以，X 与 Y 相互独立．因此

$$f_{X|Y}(x|1) = f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{其他}. \end{cases}$$

同理，当 $y > 0$ 时，$f_{X|Y}(x|y) = f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{其他}. \end{cases}$

(2) 因为 X 与 Y 相互独立，所以，对任意 $x, y \in \mathbf{R}$，都有 $F(x, y) = F_X(x)F_Y(y)$，

而 $F_X(x) = \begin{cases} 1-e^{-x}, & x \geq 0, \\ 0, & \text{其他}, \end{cases}$ $F_Y(y) = \begin{cases} 1-e^{-2y}, & y \geq 0 \\ 0, & \text{其他}. \end{cases}$ 那么

$$F(x, y) = \begin{cases} (1-e^{-x})(1-e^{-2y}), & x \geq 0, y \geq 0, \\ 0, & \text{其他}. \end{cases}$$

(3) 因为 X 与 Y 相互独立，所以

$$\begin{aligned}
P(X < 1, Y > 2) &= P(X < 1)P(Y > 2) \\
&= F(1)[1 - F(2)] \\
&= (1-e^{-1})e^{-4} = e^{-4} - e^{-5}.
\end{aligned}$$

5.

解 由已知得

$$f_X(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{其他}, \end{cases} \quad f_{Y|X}(y|x) = \begin{cases} \dfrac{1}{x}, & 0 < y < x, \\ 0, & \text{其他}. \end{cases}$$

而 $\Omega_X = (0, +\infty)$，$\Omega_{Y|X=x} = (0, x)$，则 $\Omega_Y = (0, +\infty)$．所以当 $x > 0$ 且 $0 < y < x$ 时，

有

$$f(x, y) = f_X(x)f_{Y|X}(y|x) = e^{-x}\frac{1}{x}.$$

故

$$f(x, y) = \begin{cases} e^{-x}\dfrac{1}{x}, & 0 < y < x, \\ 0, & \text{其他}. \end{cases}$$

6.

解 由于 (X,Y) 服从均匀分布,易知

$$f(x,y)=\begin{cases} \dfrac{1}{\pi}, & x^2+y^2\leqslant 1 \\ 0, & \text{其他} \end{cases}$$

由 $f_X(x)=\displaystyle\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y$,求得

$$f_X(x)=\begin{cases} \dfrac{2\sqrt{1-x^2}}{\pi}, & -1\leqslant x\leqslant 1 \\ 0, & \text{其他} \end{cases}$$

同理可得

$$f_Y(y)=\begin{cases} \dfrac{2\sqrt{1-y^2}}{\pi}, & -1\leqslant y\leqslant 1 \\ 0, & \text{其他} \end{cases}$$

当 $-1<y<1$ 时

$$f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\begin{cases} \dfrac{1}{2\sqrt{1-y^2}}, & -\sqrt{1-y^2}\leqslant x\leqslant\sqrt{1-y^2} \\ 0, & \text{其他} \end{cases}$$

同理,当 $-1<x<1$ 时

$$f_{Y|X}(y|x)=\frac{f(x,y)}{f_X(x)}=\begin{cases} \dfrac{1}{2\sqrt{1-x^2}}, & -\sqrt{1-x^2}\leqslant y\leqslant\sqrt{1-x^2} \\ 0, & \text{其他} \end{cases}$$

当 $y=0$ 时, $f_{X|Y}(x|0)=\begin{cases} \dfrac{1}{2}, & -1\leqslant x\leqslant 1 \\ 0, & \text{其他} \end{cases}$

$$P\left\{X>\frac{1}{2}\,\Big|\,Y=0\right\}=\int_{\frac{1}{2}}^{+\infty} f_{X|Y}(x|0)\mathrm{d}x=\int_{\frac{1}{2}}^{1}\frac{1}{2}\mathrm{d}x=\frac{1}{4}.$$

7.

解 (1)由性质 $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x\mathrm{d}y=1$, 可知 $\dfrac{A}{6}=1$, 则 $A=6$.

(2)边缘密度为

$$f_X(x)=\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y=\begin{cases} 2x, & 0<x<1 \\ 0, & \text{其他} \end{cases}$$

$$f_Y(y)=\int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x=\begin{cases} 3y^2, & 0<y<1 \\ 0, & \text{其他} \end{cases}$$

显然, $f(x,y)=f_X(x)\cdot f_Y(y)$.

故 X,Y 相互独立.

8.

X 与 Y 具有相同的概率密度 $f(x)=\begin{cases}\dfrac{1}{3}, & 0\leqslant x\leqslant 3 \\ 0, & \text{其他.}\end{cases}$

则 $P\{X\leqslant 1\}=P\{Y\leqslant 1\}=\dfrac{1}{3}$.

由 X,Y 独立性可知:
$$P\{\max\{X,Y\}\leqslant 1\}=P\{X\leqslant 1,\ Y\leqslant 1\}$$
$$=P\{X\leqslant 1\}P\{Y\leqslant 1\}=\dfrac{1}{9}.$$

三．思考题

$$f(x,y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(\frac{(x-\mu_x)^2}{\sigma_x^2} - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \frac{(y-\mu_y)^2}{\sigma_y^2}\right)\right)$$

$$\therefore f(y) = \int_{-\infty}^{+\infty} f(x,y)\, dx$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x} - \rho\frac{y-\mu_y}{\sigma_y}\right)^2 + (1-\rho^2)\frac{(y-\mu_y)^2}{\sigma_y^2}\right]\right) dx$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right) \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{x-\mu_x}{\sigma_x} - \rho\frac{y-\mu_y}{\sigma_y}\right]^2\right) dx$$

$$t = \frac{x-\mu_x}{\sigma_x} - \rho\frac{y-\mu_y}{\sigma_y} \qquad \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right) \int_{-\infty}^{+\infty} \sigma_x \exp\left(-\frac{t^2}{2(1-\rho^2)}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left(-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right)$$

$$\therefore \frac{f(x,y)}{f(y)} \sim N\left(\mu_x + \rho(y-\mu_y)\frac{\sigma_x}{\sigma_y},\ (1-\rho^2)\sigma_x^2\right)$$

$$= f(x|y)$$

$f(y|x)$ 同理

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hw9

一. 作业

1.

解 (1) 在 (X, Y) 的联合分布律表中每格左上角、左下角格分别给出 U、V 的取值，有

X\Y	-2	-1	0	1	4
0	$^0_{-2}$0.2	$^0_{-1}$0	$^0_0$0.1	$^1_0$0.2	$^4_0$0
1	$^1_{-2}$0	$^1_{-1}$0.2	$^1_0$0.1	$^1_0$0	$^4_1$0.2

所以有

U	0	1	4
概率	0.3	0.5	0.2

V	-2	-1	0	1
概率	0.2	0.2	0.4	0.2

(2) 在 (X, Y) 的联合分布律表中，由 U、V 的取值及相应的概率得

U\V	-2	-1	0	1
0	0.2	0	0.1	0
1	0	0.2	0.3	0
4	0	0	0	0.2

2.

解 由已知得

$$f(x, y) = \begin{cases} e^{-x} \cdot 2e^{-2y}, & x > 0, \ y > 0, \\ 0, & \text{其他}. \end{cases}$$

而 $\Omega_Z = (0, +\infty)$. 所以当 $z \geqslant 0$ 时，有

$$F_Z(z) = P\left(\frac{X}{Y} \leqslant z\right) = \iint\limits_{\frac{x}{y} \leqslant z} f(x, y) \mathrm{d}x\mathrm{d}y = \int_0^{+\infty} \mathrm{d}x \int_{\frac{x}{z}}^{+\infty} 2e^{-(x+2y)} \mathrm{d}y$$

$$= \int_0^{+\infty} e^{-x} \left[-e^{-2y}\right]_{\frac{x}{z}}^{+\infty} \mathrm{d}x = \int_0^{+\infty} e^{-x\left(1+\frac{2}{z}\right)} \mathrm{d}x = \frac{z}{z+2}.$$

故

$$F_Z(z) = \begin{cases} \dfrac{z}{z+2}, & z \geqslant 0, \\ 0, & \text{其他}, \end{cases} \qquad f_Z(z) = \begin{cases} \dfrac{2}{(z+2)^2}, & z > 0, \\ 0, & \text{其他}. \end{cases}$$

3.

解 由已知得 $f(x,\ y)=\begin{cases}\dfrac{1}{4}, & (x,\ y)\in D, \\[2mm] 0, & 其他.\end{cases}$

而 $\Omega_Z=(0,\ 2)$，所以当 $0\leqslant z\leqslant 2$ 时(见图 3.18)，有

$$\begin{aligned}
F_Z(z)&=P(|X-Y|\leqslant Z)\\
&=P(-z\leqslant X-Y\leqslant z)\\
&=1-\iint\limits_{D_1\cup D_2}\frac{1}{4}\mathrm{d}x\mathrm{d}y=1-\frac{1}{4}(S_{D_1}+S_{D_2})\\
&=1-\frac{1}{4}\left[\frac{1}{2}(2-z)^2+\frac{1}{2}(2-z)^2\right]\\
&=1-\frac{1}{4}(2-z)^2=1-\frac{1}{4}(z-2)^2.
\end{aligned}$$

有

$$F_Z(z)=\begin{cases}0, & z<0, \\[2mm] 1-\dfrac{1}{4}(z-2)^2, & 0\leqslant z<2, \\[2mm] 1, & z\geqslant 2.\end{cases}$$

求导得

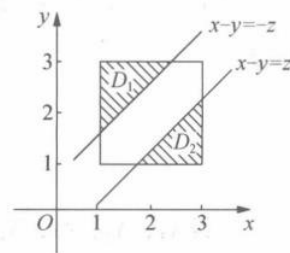$$f_Z(z)=\begin{cases}1-\dfrac{1}{2}z, & 0<z<2, \\[2mm] 0, & 其他.\end{cases}$$

4.

图 3.18

解 设第一天和第二天的需求量分别用 X_1 和 X_2 表示，则 X_1 与 X_2 相互独立同分布，且

$$X_1 \sim f_{X_1}(x_1) = \begin{cases} x_1 e^{-x_1}, & x_1 > 0, \\ 0, & \text{其他}, \end{cases}$$

则 (X_1, X_2) 的联合密度函数为

$$f(x_1, x_2) = \begin{cases} x_1 x_2 e^{-(x_1+x_2)}, & x_1, x_2 > 0, \\ 0, & \text{其他}. \end{cases}$$

方法一　分布函数法

两天的需求量为 Y，则 $Y = X_1 + X_2$，$\Omega_Y = (0, +\infty)$. 所以当 $y \geqslant 0$ 时，有

$$F_Y(y) = P(X_1 + X_2 \leqslant y) = \iint\limits_{x_1+x_2 \leqslant y} f(x_1, x_2)\,dx_1 dx_2 = \int_0^y dx_1 \int_0^{y-x_1} x_1 x_2 e^{-(x_1+x_2)}\,dx_2$$

$$= \int_0^y x_1 e^{-x_1} \left[-x_2 e^{-x_2} - e^{-x_2} \right]_0^{y-x_1} dx_1 = \int_0^y x_1^2 e^{-y} - x_1 y e^{-y} - x_1 e^{-y} + x_1 e^{-x_1}\,dx_1$$

$$= 1 - e^{-y}\left(\frac{y^3}{6} + \frac{y^2}{2} + y + 1 \right),$$

故

$$F_Y(y) = \begin{cases} 1 - e^{-y}\left(\dfrac{y^3}{6} + \dfrac{y^2}{2} + y + 1 \right), & y \geqslant 0, \\ 0, & \text{其他}, \end{cases} \qquad f_Y(y) = \begin{cases} \dfrac{y^3}{6} e^{-y}, & y > 0, \\ 0, & \text{其他}. \end{cases}$$

方法二　卷积公式法

$\Omega_Y = (0, +\infty)$，当 $y > 0$ 时，由卷积公式得

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{X_1}(x_1) f_{X_2}(y-x_1)\,dx_1,$$

x_1 的积分区间由 $\begin{cases} x_1 > 0 \\ y - x_1 > 0 \end{cases} \Rightarrow 0 < x_1 < y$ 确定，故

$$f_Y(y) = \int_0^y x_1 e^{-x_1} (y-x_1) e^{-(y-x_1)}\,dx_1 = \int_0^y (x_1 y - x_1^2) e^{-y}\,dx_1$$

$$= \left[\left(\frac{x_1^2}{2} y - \frac{x_1^3}{3} \right) e^{-y} \right]_0^y = \frac{y^3}{6} e^{-y},$$

求导得

$$f_Y(y) = \begin{cases} \dfrac{y^3}{6} e^{-y}, & y > 0, \\ 0, & \text{其他}. \end{cases}$$

5.

解 (1) 由已知得 $\Omega_Y = (0, 4)$，当 $0 \leqslant y < 4$ 时，有

$$F_Y(y) = P(X^2 \leqslant y) = P(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,\mathrm{d}x.$$

当 $0 \leqslant y < 1$ 时，$F_Y(y) = \int_{-\sqrt{y}}^{0} \dfrac{1}{2}\mathrm{d}x + \int_{0}^{\sqrt{y}} \dfrac{1}{4}\mathrm{d}x = \dfrac{1}{2}\sqrt{y} + \dfrac{1}{4}\sqrt{y} = \dfrac{3}{4}\sqrt{y}.$

当 $1 \leqslant y < 4$ 时，$F_Y(y) = \int_{-1}^{0} \dfrac{1}{2}\mathrm{d}x + \int_{0}^{\sqrt{y}} \dfrac{1}{4}\mathrm{d}x = \dfrac{1}{2} + \dfrac{1}{4}\sqrt{y},$

所以

$$F_Y(y) = \begin{cases} 0, & y < 0, \\[2mm] \dfrac{3}{4}\sqrt{y}, & 0 \leqslant y < 1, \\[2mm] \dfrac{1}{2} + \dfrac{1}{4}\sqrt{y}, & 1 \leqslant y < 4, \\[2mm] 1, & y \geqslant 4. \end{cases}$$

所以

$$f_Y(y) = \begin{cases} \dfrac{3}{8\sqrt{y}}, & 0 < y < 1, \\[2mm] \dfrac{1}{8\sqrt{y}}, & 1 < y < 4, \\[2mm] 0, & \text{其他}. \end{cases}$$

$(2)\, F\left(-\dfrac{1}{2}, 4\right) = P\left(X \leqslant -\dfrac{1}{2}, Y \leqslant 4\right) = P\left(X \leqslant -\dfrac{1}{2}, X^2 \leqslant 4\right)$

$\quad = P\left(X \leqslant -\dfrac{1}{2}, -2 \leqslant X \leqslant 2\right) = P\left(-2 \leqslant X \leqslant -\dfrac{1}{2}\right)$

$\quad = \int_{-1}^{-\frac{1}{2}} \dfrac{1}{2}\mathrm{d}x = \dfrac{1}{4}.$

二．练习

1.

X \ Y	0	1	$p_{i\cdot}$
0	$^1(1-p)^2$	$^0p(1-p)$	$1-p$
1	$^0p(1-p)$	$^1p^2$	p
$p_{\cdot j}$	$1-p$	p	1

在表格中每格左上角标出 Z 的取值，将 Z 的取值相同的格子中的概率相加，得

Z	0	1
概率	$2p(1-p)$	$p^2+(1-p)^2$

(2) 由(1)知 (X, Z) 的联合分布律为

X \ Z	0	1	$p_{i\cdot}$
0	$p(1-p)$	$(1-p)^2$	$1-p$
1	$p(1-p)$	p^2	p
$p_{\cdot j}$	$2p(1-p)$	$p^2+(1-p)^2$	1

(3) 在(2)中 (X, Z) 的联合分布律表中写出 X 与 Z 的边缘分布律. 要使 X 与 Z 相互独立, 对任意的 $i, j = 1, 2$ 有 $p_{ij} = p_{i\cdot}p_{\cdot j}$, 特别地

$$P(X = 0, Z = 0) = P(X = 0)P(Z = 0),$$

$$p(1-p) = (1-p) \cdot 2p(1-p) \Rightarrow p = \frac{1}{2}.$$

则 (X, Z) 的联合分布律为

X \ Z	0	1
0	$\dfrac{1}{4}$	$\dfrac{1}{4}$
1	$\dfrac{1}{4}$	$\dfrac{1}{4}$

可以验证此时对任意的 $i, j = 1, 2$, 都有 $p_{ij} = p_{i\cdot}p_{\cdot j}$. 故当 $p = \frac{1}{2}$ 时, X 与 Z 相互独立.

2.

解　由已知得 $f(x, y) = \begin{cases} \dfrac{1}{2}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{其他}, \end{cases}$ 　　$\Omega_z = (0,\ 2).$

当 $0 \leqslant z < 2$ 时, 有

$$F_Z(z) = P(XY \leqslant z) = \iint\limits_{xy \leqslant z} f(x,\ y)\,\mathrm{d}x\mathrm{d}y = \int_0^z \mathrm{d}x \int_0^1 \frac{1}{2}\mathrm{d}y + \int_z^2 \mathrm{d}x \int_0^{\frac{z}{x}} \frac{1}{2}\mathrm{d}y$$

$$= \frac{z}{2} + \int_z^2 \frac{1}{2} \frac{z}{x}\mathrm{d}x = \frac{z}{2} + \frac{z}{2}(\ln 2 - \ln z)$$

$$= \frac{z}{2}(1 + \ln 2 - \ln z).$$

所以

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ \dfrac{z}{2}(1 + \ln 2 - \ln z), & 0 \leqslant z < 2, \\ 1, & z \geqslant 2. \end{cases}$$

故

$$f_Z(z) = \begin{cases} \dfrac{1}{2}(\ln 2 - \ln z), & 0 < z < 2, \\ 0, & \text{其他}. \end{cases}$$

3.

解　由已知及 X 与 Y 相互独立得

$$F_Z(z) = P(X + Y \leqslant z)$$

$$= P(X=1)P(X+Y \leqslant z \mid X=1) + P(X=2)P(X+Y \leqslant z \mid X=2)$$

$$= 0.3 \cdot P(Y \leqslant z-1 \mid X=1) + 0.7 \cdot P(Y \leqslant z-2 \mid X=2)$$

$$= 0.3 \cdot P(Y \leqslant z-1) + 0.7 \cdot P(Y \leqslant z-2)$$

$$= 0.3 \cdot \int_{-\infty}^{z-1} f_Y(y)\,\mathrm{d}y + 0.7 \cdot \int_{-\infty}^{z-2} f_Y(y)\,\mathrm{d}y,$$

求导得 Z 的密度函数为

$$f_Z(z) = 0.3 \cdot f_Y(z-1) + 0.7 \cdot f_Y(z-2).$$

4.

解 (1) $P(X - Y < 2) = \iint\limits_{x-y<2} f(x, y)\,dxdy = \int_0^{+\infty} dy \int_0^{y+2} e^{-(x+y)}\,dx$

$$= \int_0^{+\infty} [1 - e^{-(y+2)}] \cdot e^{-y}dy = \left[-e^{-y} + \frac{1}{2}e^{-2y-2}\right]_0^{+\infty}$$

$$= 1 - \frac{1}{2}e^{-2}.$$

(2) 由已知得 $\Omega_Z = (-\infty, +\infty)$，所以当 $-\infty < z < +\infty$，且 $z \geq 0$ 时，有

$$F_Z(z) = P(X - Y \leq z) = \int_0^{+\infty} dy \int_0^{y+z} e^{-(x+y)}\,dx$$

$$= \int_0^{+\infty} [1 - e^{-(y+z)}] \cdot e^{-y}dy = 1 + \left[\frac{1}{2}e^{-2y-z}\right]_0^{+\infty}$$

$$= 1 - \frac{1}{2}e^{-z}.$$

当 $z < 0$ 时，有

$$F_Z(z) = P(X - Y \leq z) = \int_{-z}^{+\infty} dy \int_0^{y+z} e^{-(x+y)}\,dx$$

$$= \int_{-z}^{+\infty} [1 - e^{-(y+z)}] \cdot e^{-y}dy = \left[-e^{-y} + \frac{1}{2}e^{-2y-z}\right]_{-z}^{+\infty}$$

$$= e^z - \frac{1}{2}e^z = \frac{1}{2}e^z.$$

所以

$$F_Z(z) = \begin{cases} \dfrac{1}{2}e^z, & z < 0, \\[2mm] 1 - \dfrac{1}{2}e^{-z}, & z \geq 0. \end{cases}$$

则 $Z = X - Y$ 的密度函数为

$$f_Z(z) = \begin{cases} \dfrac{1}{2}e^z, & z < 0, \\[2mm] \dfrac{1}{2}e^{-z}, & z \geq 0. \end{cases}$$

即 $f_Z(z) = \dfrac{1}{2}e^{-|z|}, z \in \mathbf{R}.$

5.

解 由已知得 $f(x, y) = \begin{cases} e^{-y}, & 0 < x < 1, \ y > 0, \\ 0, & \text{其他,} \end{cases}$ $\Omega_z = (0, +\infty).$

当 $z \geq 0$ 时，有

$$F_Z(z) = P(2X + Y \leq Z) = \iint\limits_{2x+y \leq z} f(x, y) \, dx dy.$$

当 $0 \leq z < 2$ 时，有

$$F_Z(z) = \int_0^{\frac{z}{2}} dx \int_0^{z-2x} e^{-y} dy = \int_0^{\frac{z}{2}} 1 - e^{-(z-2x)} dx = \frac{z}{2} - \frac{1}{2}(1 - e^{-z}).$$

当 $z \geq 2$ 时，有

$$F_Z(z) = \int_0^1 dx \int_0^{z-2x} e^{-y} dy = \int_0^1 1 - e^{-(z-2x)} dx = 1 - \frac{1}{2}(e^{2-z} - e^{-z}).$$

所以有

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ \dfrac{z}{2} - \dfrac{1}{2}(1 - e^{-z}), & 0 \leq z < 2, \\ 1 - \dfrac{1}{2}(e^{2-z} - e^{-z}), & z \geq 2. \end{cases}$$

求导得

$$f_Z(z) = \begin{cases} \dfrac{1}{2}(1 - e^{-z}), & 0 < z < 2, \\ \dfrac{1}{2}(e^2 - 1)e^{-z}, & z > 2, \\ 0, & \text{其他.} \end{cases}$$

6.

解 由于 ξ 与 η 相互独立，因此有 $p_{ij} = p_i \cdot p_j$.
得到二维随机变量的联合分布：

ξ \ η	2	4
1	0.18	0.12
3	0.42	0.28

因为 $Z=\xi+\eta$,易知 Z 的分布为

p_{ij}	(ξ,η)	Z
0.18	(1,2)	3
0.12	(1,4)	5
0.42	(3,2)	5
0.28	(3,4)	7

由离散型随机变量函数的定义 $P\{Z=z_k\}=\sum\limits_{x_i+y_j=z_k}P\{X=x_i,Y=y_j\}$,得到 Z 的分布律为

Z	3	5	7
p	0.18	0.54	0.28

7.

解 由于 X 与 Y 相互独立,所以

$$P\{X=i,Y=j\}=P\{X=i\}P\{X=j\}$$

于是

$$P\{Z=0\}=P\{\max(X,Y)=0\}=P\{X=0,Y=0\}=\frac{1}{2^2}$$

$$P\{Z=1\}=1-P\{Z=0\}=\frac{3}{2^2}$$

故 $Z=\max\{X,Y\}$ 的分布律为

Z	0	1
P	$\frac{1}{4}$	$\frac{3}{4}$

8.

解　记 $Y_1 = X_1 X_4$，$Y_2 = X_2 X_3$，则 $X = Y_1 - Y_2$，随机变量 Y_1 和 Y_2 独立同分布.

$$P\{Y_1 = 1\} = P\{Y_2 = 1\} = P\{X_2 = 1, X_3 = 1\} = 0.16$$
$$P\{Y_1 = 0\} = P\{Y_2 = 0\} = 1 - 0.16 = 0.84$$

随机变量 $X = Y_1 - Y_2$ 有三个可能值 -1，0，1，易见

$$P\{X = -1\} = P\{Y_1 = 0, Y_2 = 1\} = 0.84 \times 0.16 = 0.1344$$
$$P\{X = 1\} = P\{Y_1 = 1, Y_2 = 0\} = 0.16 \times 0.84 = 0.1344$$
$$P\{X = 0\} = 1 - 2 \times 0.1344 = 0.7312$$

于是行列式的概率分布为

$$X = \begin{vmatrix} X_1 & X_2 \\ X_3 & X_4 \end{vmatrix} \sim \begin{bmatrix} -1 & 0 & 1 \\ 0.1344 & 0.7312 & 0.1344 \end{bmatrix}$$

9.

解　由条件知 X 和 Y 的联合密度为　$f(x,y) = \begin{cases} \dfrac{1}{4}, & 1 \leqslant x \leqslant 3,\ 1 \leqslant y \leqslant 3 \\ 0, & \text{其他} \end{cases}$

以 $F(u) = P\{U \leqslant u\}$（$-\infty < u < +\infty$）表示随机变量 U 的分布函数.

显然，当 $u \leqslant 0$ 时，$F(u) = 0$；

当 $u \geqslant 2$ 时，$F(u) = 1$.

设 $0 < u < 2$，如图 $3-5.9$ 所示，则

$$F(u) = \iint\limits_{|x-y| \leqslant u} f(x,y)\,dxdy = \iint\limits_{|x-y| \leqslant u} \frac{1}{4}\,dxdy$$
$$= \frac{1}{4}[4 - (2-u)^2] = 1 - \frac{1}{4}(2-u)^2$$

于是，随机变量 U 的密度为

$$p(u) = \begin{cases} \dfrac{1}{2}(2-u), & 0 < u < 2 \\ 0, & \text{其他} \end{cases}$$

图 $3-5.9$

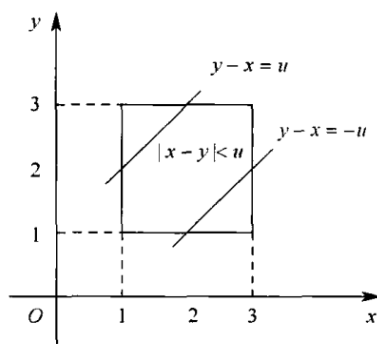10.

解　设 $F(y)$ 是 Y 的分布函数，则由全概率公式，知 $U = X + Y$ 的分布函数为

$$G(u) = P\{X + Y \leqslant u\}$$
$$= P\{X = 1\}P\{X + Y \leqslant u \mid X = 1\} + P\{X = 2\}P\{X + Y \leqslant u \mid X = 2\}$$
$$= 0.3 P\{X + Y \leqslant u \mid X = 1\} + 0.7 P\{X + Y \leqslant u \mid X = 2\}$$
$$= 0.3 P\{Y \leqslant u - 1 \mid X = 1\} + 0.7 P\{Y \leqslant u - 2 \mid X = 2\}.$$

由于 X 和 Y 独立，可见

$$G(u) = 0.3 P\{Y \leqslant u - 1\} + 0.7 P\{Y \leqslant u - 2\}$$
$$= 0.3 F(u-1) + 0.7 F(u-2).$$

由此，得 U 的概率密度

$$g(u) = G'(u) = 0.3 F'(u-1) + 0.7 F'(u-2)$$
$$= 0.3 f(u-1) + 0.7 f(u-2).$$

11.

解 随机地取 4 只，记其寿命分别为 X_1, X_2, X_3, X_4，由题设知，它们独立同分布，且

$$X_i \sim N(160, 20^2), \qquad i = 1, 2, 3, 4$$

记 $X = \min(X_1, X_2, X_2, X_4)$，事件"没有一只寿命小于 180"就是 $\{X \geq 180\}$，从而

$$P\{X \geq 180\} = 1 - P\{X < 180\} = [1 - F(180)]^4 = \left[1 - \Phi\left(\frac{180 - 160}{20}\right)\right]^4$$

$$= (1 - 0.8413)^4 = 0.000634.$$

补充题
1.

1.每次独立地随机选取小数,平均选几次才能使得小数的和大于1?

设 A_n 为取了 n 次和仍小于 1，X 为刚好大于 1 时取的次数

$P(X=n) = P(n$ 次大于 1，$n-1$ 次小于 $1)$

$= P(\overline{A_n} A_{n-1}) = P(A_{n-1}) - P(A_n)$

下面求 $P(A_n)$

$P(A_1) = 0$，$P(A_2) = \dfrac{1}{2}$ $P(A_3) = \dfrac{1}{6}$

设第 k 次取的数为 X_k，X_k，X_l $(k \neq l)$ 相互独立

$P(X_1 + X_2 + \cdots + X_n < 1) = P(A_n) = \iint \cdots \int\limits_{\substack{X_1 + X_2 + \\ \cdots + X_n < 1}} f(X_1, X_2, \cdots X_n) dX_1 dX_2 \cdots dX_n$

$f(X_1, X_2, \cdots X_n) = \begin{cases} 1 & X_1, X_2, \cdots X_n \in (0,1) \\ 0 & 其它 \end{cases}$

$\therefore P(X_1 + X_2 + \cdots + X_n < 1) = \int_0^1 dX_n \int_0^{1-X_n} dX_{n-1} \cdots \int_0^{\underline{1-X_n X_{n-1} \cdots X_2}} 1 \, dX_1$

注意到红线部分为 $P(X_1 + X_2 + \cdots + X_{n-1} < 1 - X_n | X_n)$

故设 $P(X_1 + X_2 + \cdots + X_n < 1 - k) = g(k)$

$\therefore P(X_1 < 1-k) = 1-k$，$P(X_2 < 1-k) = \dfrac{(1-k)^2}{2}$，$P(X_3 < 1-k) = \dfrac{(1-k)^3}{6}$

设 $P(X_1 + X_2 + \cdots + X_n < 1-k) = \dfrac{(1-k)^{n-1}}{(n-1)!}$

$\therefore P(X_1 + X_2 + \cdots + X_{n-1} + X_n < 1-k)$

$= \int_0^1 \dfrac{(1-k-X_n)^{n-1}}{(n-1)!} dX_n = \dfrac{(1-k)^n}{n!}$

$\therefore P(X_1 + X_2 + \cdots + X_n < 1) = \dfrac{1}{n!} = P(A_n)$

$\therefore P(X=n) = \dfrac{1}{n!}$

$\therefore E(X) = \sum_{n=1}^{\infty} \dfrac{1}{(n-1)!} = e$

2.

2.设随机变量 X 与 Y 独立同分布, 且都服从 $N(0,1)$.试证明 $U = X^2 + Y^2$ 和 $V = \dfrac{X}{Y}$ 相互独立.

17. 设随机变量 X 与 Y 独立同分布,且都服从标准正态分布 $N(0,1)$.试证:$U = X^2 + Y^2$ 与 $V = X/Y$ 相互独立.

证 设 $u = x^2 + y^2, v = x/y$,则

$$J^{-1} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 1/y & -x/y^2 \end{vmatrix} = -2\left(\dfrac{x^2}{y^2} + 1\right) = -2(v^2 + 1),$$

所以 $|J| = \dfrac{1}{2(1 + v^2)}$.由此得 $U = X^2 + Y^2$ 和 $V = X/Y$ 的联合密度为

$$p_{U,V}(u,v) = p_{X,Y}(x,y)|J| \cdot I_{|y \leqslant 0|} + p_{X,Y}(x,y)|J| \cdot I_{|y > 0|}$$

$$= 2 \cdot \dfrac{1}{2\pi} \exp\left\{-\dfrac{x^2 + y^2}{2}\right\} \cdot \dfrac{1}{2(1 + v^2)}$$

$$= \dfrac{1}{2} e^{-\frac{u}{2}} \cdot \dfrac{1}{\pi(1 + v^2)}, \quad u > 0, -\infty < v < \infty.$$

所以 $p_{U,V}(u,v)$ 可分离变量,即 U 与 V 相互独立.