Chapter 6 z-Transform

Part A: z-Transform

Part B: The Inverse z-Transform

and z-Transform Theorems

Part C: Convolution(卷积)

Part D: The Transfer Function

6.7 The Transfer Function

- **The convolution sum description of an LTI discrete-time system with an impulse response *h*[*n*] is given by**

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]\,x[n-k]$$

6.7 The Transfer Function

- **Taking the z-transforms of both sides we get**

$$Y(z)=\sum_{n=-\infty}^{\infty} y[n]z^{-n}=\sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\right)z^{-n}$$

$$=\sum_{k=-\infty}^{\infty} h[k]\left(\sum_{n=-\infty}^{\infty} x[\underline{n-k}]z^{-n}\right) \quad \text{变量代换}$$

$$=\sum_{k=-\infty}^{\infty} h[k]\left(\sum_{\ell=-\infty}^{\infty} x[\ell]\underline{z^{-(\ell+k)}}\right)$$

6.7 The Transfer Function

Or, $$Y(z) = \sum_{k=-\infty}^{\infty} h[k] \underbrace{\left(\sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} \right)}_{X(z)} z^{-k}$$

Therefore,

$$Y(z) = \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right)}_{H(z)} X(z)$$

Thus, $\boldsymbol{Y(z) = H(z)X(z)}$

6.7.1 Definition

- **Transfer function or the system function**

$$H(z) = \frac{Y(z)}{X(z)}$$

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

6.7.2 Transfer Function Expression

- **Consider an LTI discrete-time system characterized by a difference equation**

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- **Its transfer function is obtained by taking the *z*-transform of both sides of the above equation**
- **Thus**

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

6.7.2 Transfer Function Expression

- **Or, equivalently as**

$$H(z) = z^{(N-M)} \frac{\sum_{k=0}^{M} p_k z^{M-k}}{\sum_{k=0}^{N} d_k z^{N-k}}$$

- **An alternate form of the transfer function is given by**

$$H(z) = \frac{p_0}{d_0} \cdot \frac{\prod_{k=1}^{M} (1 - \xi_k z^{-1})}{\prod_{k=1}^{N} (1 - \lambda_k z^{-1})}$$

IIR数字滤波器传递函数的表达式

- For a causal IIR digital filter, the impulse response is a causal sequence.
- The ROC of the causal transfer function

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

is thus exterior to a circle going through the pole furthest from the origin

- Thus the ROC is given by $|z| > \max_k |\lambda_k|$

IIR数字滤波器传递函数的表达式

- **Example** - A causal LTI IIR digital filter is described by a constant coefficient difference equation given by

$$y[n]=x[n-1]-1.2x[n-2]+x[n-3]+1.3y[n-1]$$
$$-1.04y[n-2]+0.222y[n-3]$$

- Its transfer function is therefore given by

$$H(z)=\frac{z^{-1}-1.2z^{-2}+z^{-3}}{1-1.3z^{-1}+1.04z^{-2}-0.222z^{-3}}$$

IIR数字滤波器传递函数的表达式

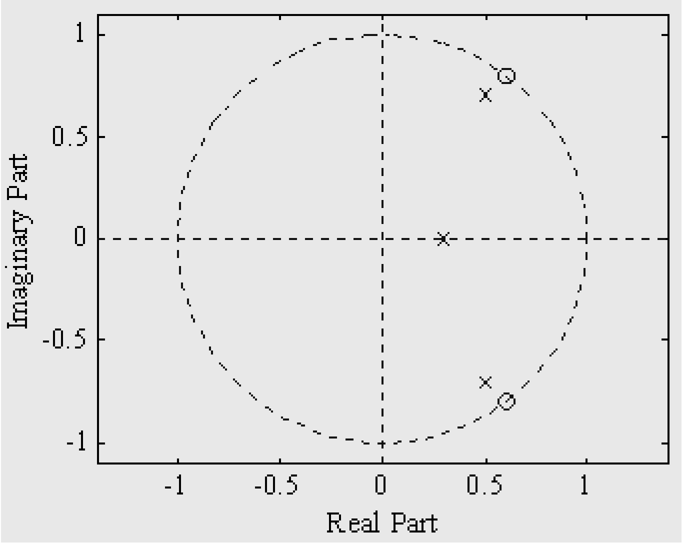
- **Alternate forms:**

$$H(z)=\frac{z^2-1.2z+1}{z^3-1.3z^2+1.04z-0.222}$$

$$=\frac{(z-0.6+j0.8)(z-0.6-j0.8)}{(z-0.3)(z-0.5+j0.7)(z-0.5-j0.7)}$$

- **Note: Poles farthest from *z*=0 have a magnitude** $\sqrt{0.74}$

ROC: $|z|>\sqrt{0.74}$



FIR数字滤波器传递函数的表达式

- FIR数字滤波器的卷积和方程可表达为

$$y[n]=\sum_{k=N_1}^{N_2} h[k]x[n-k] \qquad N_1<N_2$$

$$\sum_{k=0}^{N} d_k y[n-k]=\sum_{k=0}^{M} p_k x[n-k], d_0=1, d_k=0, \quad k=1,\cdots,N$$

- FIR数字滤波器的传递函数可表达为

$$H(z)=\sum_{n=N_1}^{N_2} h[n]z^{-n}=\sum_{k=N_1}^{N_2} p_k z^{-k}$$

- 注意，因果FIR滤波器的$H(z)$的所有极点均在z平面的原点处. 因此, $H(z)$的收敛域在除了z=0之外的整个z平面上。

FIR数字滤波器传递函数的表达式

- <u>**Example**</u> **- Consider the *M*-point moving-average FIR filter with an impulse response**

$$h[n] = \begin{cases} 1/M, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

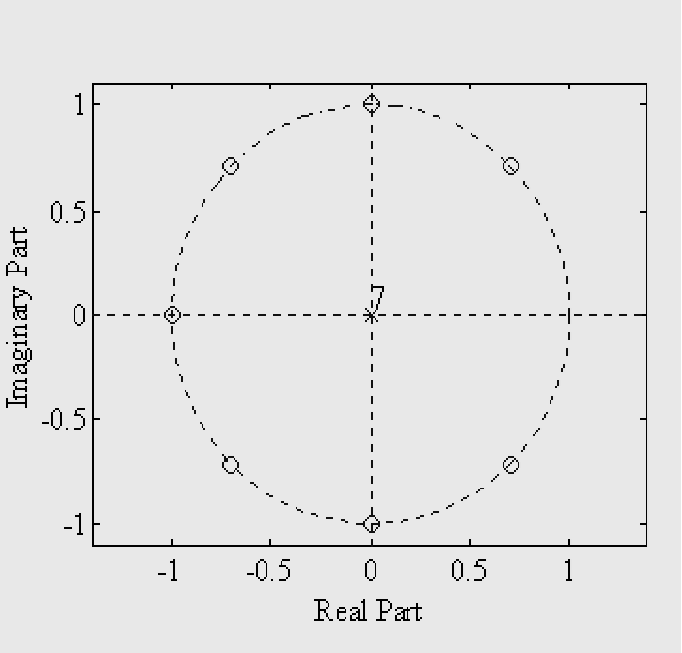
- **Its transfer function is then given by**

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^{M}-1}{M[z^{M-1}(z-1)]}$$

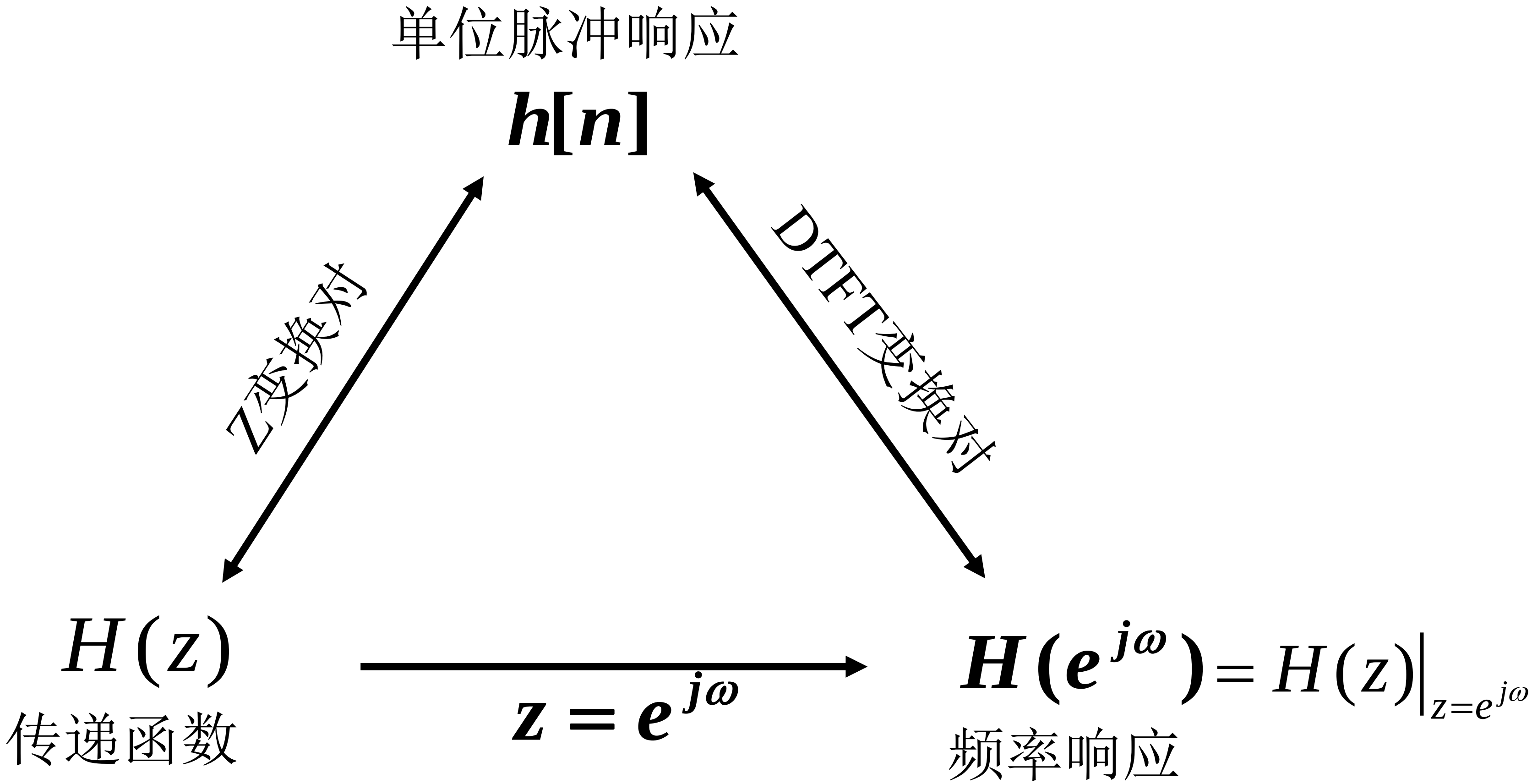
FIR数字滤波器传递函数的表达式

- **M zeros on the unit circle at $z=e^{j2\pi k/M}$, $0 \le k \le M-1$**
- **M-1 poles at z = 0 and 1 pole at z = 1**
- **The pole at z = 1 exactly cancels the zero at z = 1**
- **The ROC is the entire z-plane except z = 0**

M = 8



从传递函数得到频率响应



6.7.3 Frequency Response from Transfer Function

- **For a stable rational transfer function in the form**

$$H(z)=\frac{p_0}{d_0}z^{(N-M)}\frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

the factored form of the frequency response is given by

$$H(e^{j\omega})=\frac{p_0}{d_0}e^{j\omega(N-M)}\frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)}$$

6.7.3 Frequency Response from Transfer Function

- **The magnitude function is given by**

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \left|e^{j\omega(N-M)}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-\lambda_k\right|} = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-\lambda_k\right|}$$

- **The phase response for a rational transfer function is of the form**

$$\arg H(e^{j\omega}) = \arg\left(p_0/d_0\right) + \omega(N-M) + \sum_{k=1}^{M}\arg\left(e^{j\omega}-\xi_k\right) - \sum_{k=1}^{N}\arg\left(e^{j\omega}-\lambda_k\right)$$

6.7.4 Geometric Interpretation of Frequency Response Computation

- **The factored form of the frequency response**

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)}$$

is convenient to develop a geometric interpretation of the frequency response computation from the pole-zero plot as ω varies from 0 to 2π on the unit circle

6.7.4 Geometric Interpretation of Frequency Response Computation

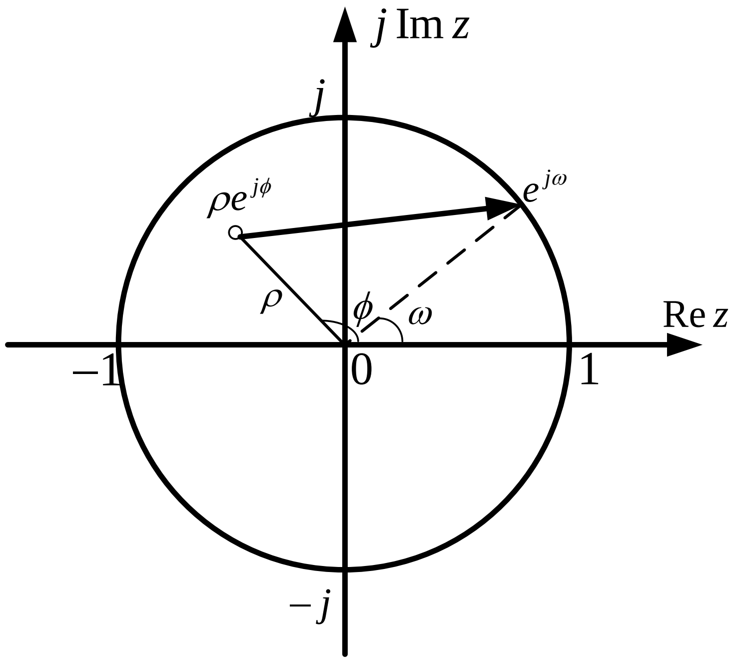
- **The geometric interpretation can be used to obtain a sketch of the response as a function of the frequency**
- **A typical factor in the factored form of the frequency response is given by**

$$(e^{j\omega} - \rho e^{j\Phi})$$

where $\rho e^{j\Phi}$ is a zero if it is zero factor or is a pole if it is a pole factor

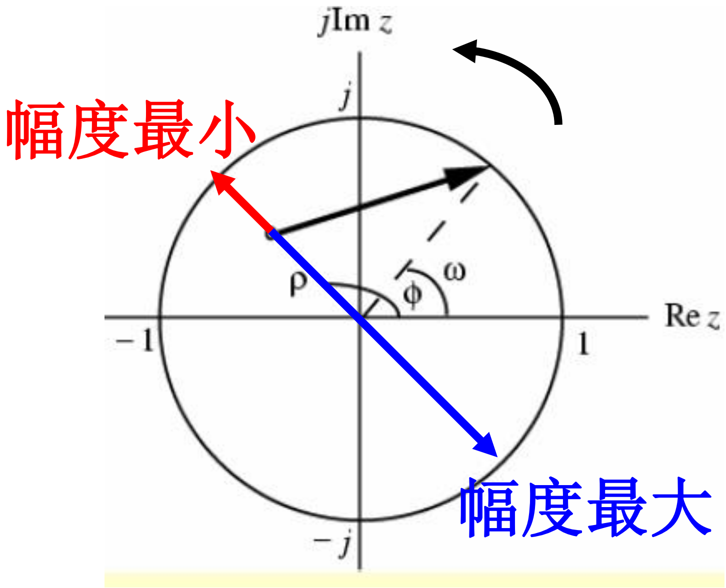
6.7.4 Geometric Interpretation of Frequency Response Computation

- **As shown below in the z-plane the factor ($e^{j\omega}$- $\rho e^{j\Phi}$) represents a vector starting at the point $z=\rho e^{j\Phi}$ and ending on the unit circle at $z= e^{j\omega}$**



6.7.4 Geometric Interpretation of Frequency Response Computation

- As ω is varied from 0 to 2π, the tip of the vector moves counterclockise (逆时针) from the point $z = 1$ tracing the unit circle and back to the point $z = 1$



6.7.4 Geometric Interpretation of Frequency Response Computation

- To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range（零点——谷值）
- Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to or on the unit circle in this range（极点——峰值）

Question：

1、极点的位置对系统的稳定性会有影响吗？

2、系统在Z域的稳定性条件是什么？

6.7.5 Stability Condition in Terms of the Pole Locations

- **A causal LTI digital filter is BIBO stable if and only if its impulse response $h[n]$ is absolutely summable, i.e.,**

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- **An FIR digital filter with bounded impulse response is always stable**
- **On the other hand, an IIR filter may be unstable if not designed properly**

6.7.5 Stability Condition in Terms of the Pole Locations

- **The ROC of the z-transform $H(z)$ of the impulse response sequence $h[n]$ is defined by values of $|z| = r$ for which $h[n]r^{-n}$ is absolutely summable**

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} \overset{z=re^{j\omega}}{=} \sum_{n=-\infty}^{\infty} h[n] r^{-n} e^{-j\omega n}$$

$$S' = \sum_{n=-\infty}^{\infty} |h[n] r^{-n} e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |h[n] r^{-n}| < \infty$$

$$\overset{|r|=1}{\Rightarrow} \sum_{n=-\infty}^{\infty} |h[n] r^{-n}| = \boxed{\sum_{n=-\infty}^{\infty} |h[n]| < \infty}$$

BIBO stable

6.7.5 Stability Condition in Terms of the Pole Locations

- **Thus, if the ROC includes the unit circle $|z| = 1$, then the digital filter is stable, and vice versa**
- **This in turn implies that the DTFT $H(e^{j\omega})$ of {h[n]} exists**
- **Now, if the ROC of the z-transform $H(z)$ includes the unit circle, then**

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

6.7.5 Stability Condition in Terms of the Pole Locations

- **Consider the causal IIR digital filter with a rational transfer function *H(z)* given by**

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

- **Its impulse response $\{h[n]\}$ is a right-sided sequence**
- **The ROC of $H(z)$ is exterior to a circle going through the pole furthest from $z = 0$**

6.7.5 Stability Condition in Terms of the Pole Locations

- **For a stable and causal digital filter for which $h[n]$ is a right-sided sequence, the ROC will include the unit circle and entire z-plane including the point $z=\infty$**
- **Conclusion: All poles of a causal stable transfer function $H(z)$ must be strictly inside the unit circle.**

6.7.5 Stability Condition in Terms of the Pole Locations

- **Example - The factored form of**

$$H(z) = \frac{1}{1 - 1.845z^{-1} + 0.850856z^{-2}}$$

is

$$H(z) = \frac{1}{(1 - 0.902z^{-1})(1 - 0.943z^{-1})}$$

which has a real pole at z = 0.902 and a real pole at z = 0.943

- **Since both poles are inside the unit circle, *H(z)* is BIBO stable**

6.7.5 Stability Condition in Terms of the Pole Locations

- **<u>Example</u> - The factored form of**

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

is

$$\hat{H}(z) = \frac{1}{(1 - z^{-1})(1 - 0.85z^{-1})}$$

which has a real pole on the unit circle at z = 1 and the other pole inside the unit circle

- **Since one pole is not inside the unit circle, $H(z)$ is unstable**

系统	时域条件	Z域条件
因果	**h(n)≡0 (n<0)**	**ROC: R1 < ∣Z∣ ≤∞**
稳定	$\sum_{n=-\infty}^{\infty} \lvert h(n) \rvert < \infty$	**ROC: 包含单位圆**
因果稳定	所有极点全在单位圆内部	

Homework

- **Problems: 6.2(a,b), 6.5, 6.7, 6.8(a)(i,iv), 6.13(a), 6.42, 6.44, 6.81**
- **Matlab Exercises: M6.1(a), M6.5**

Consider a LTI *causal* system whose I/O difference equation is $y(n)=\frac{5}{2}y(n-1)-y(n-2)+x(n-1)$

1) Compute the transform function.

2) Determine the corresponding pole/zero pattern and the ROC.

3) Compute the impulse response.

4) It is easy to know this system is not stable. Determine another stable (but anticausal) system satisfying the same I/O difference equation.

Solution:

a) $$H(z)=\frac{z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}}=\frac{z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})}$$

b) zeroes： z＝0；

poles： z=2, z=1/2;

ROC: |z|>2

c)

$$H(z)=\frac{2/3}{1-2z^{-1}}+\frac{-2/3}{1-\frac{1}{2}z^{-1}}$$

$$h[n]=\frac{2}{3}2^{n}\mu[n]-\frac{2}{3}(\frac{1}{2})^{n}\mu[n]$$

d) ROC: 1/2<|z|<2,

$$h[n] = -\frac{2}{3}2^n \mu[-n-1] - \frac{2}{3}(\frac{1}{2})^n \mu[n]$$