Digital Signal Processing

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Chapter 3
Discrete-Time Signals in the Frequency-Domain

Review of DTFT and IDTFT

$$x[n] \leftrightarrow X(e^{j\omega})$$

- 复数性
- 连续性
- 周期性

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Effect of Sampling in the Frequency Domain

What is the relationship between $G_a(j\Omega)$ and $G(e^{j\omega})$?

3.8 Sampling of Continuous-time Signals

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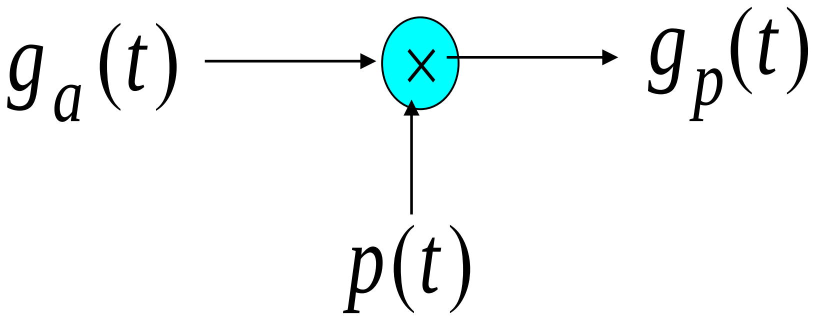
- **The frequency-domain representation of $g_a(t)$ is given by CTFT:**

$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} dt$$

- **The frequency-domain representation of *g*[*n*] is given by DTFT:**

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$

Review of Sampling



$$\boldsymbol{g_p(t) = g_a(t)\,p(t)}$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$g_p(t) = g_a(t)\,p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\,\delta(t - nT)$$

采样的频域分析

$g_a(t)$		$g_p(t)$		$g[n]$
↓		↓		↓
$G_a(j\Omega)$	→	$G_p(j\Omega)$	→	$G(e^{j\omega})$

$$g_p(t) = g_a(t)\,p(t)$$

3.8.1 Effect of Sampling in the Frequency Domain

- **There are two different forms of $G_p(j\Omega)$:**

1) One form is given by the weighted sum of the CTFTs of $\delta(t-nT)$:

$$G_p(j\Omega)=\int_{-\infty}^{\infty} g_p(t)e^{-j\Omega t}dt$$

$$=\int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} g_a(nT)\delta(t-nT)\right]e^{-j\Omega t}dt$$

$$=\sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Omega nT}\int_{-\infty}^{\infty}\left[\delta(t-nT)\right]dt$$

$$=\sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Omega nT}$$

3.8.1 Effect of Sampling in the Frequency Domain

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(2) note that *p*(*t*) can be expressed as a Fourier series:

$$p(t)=\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{j(2\pi/T)kt}=\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{j\Omega_T kt}$$

where $\Omega_T=2\pi/T$

- **The impulse train $g_p(t)$ therefore can be expressed as**

$$g_p(t)=\left(\frac{1}{T}\sum_{k=-\infty}^{\infty}e^{j\Omega_T kt}\right)\cdot g_a(t)$$

调制关系

3.8.1 Effect of Sampling in the Frequency Domain

- From the frequency-shifting property of the CTFT, an alternative form of the CTFT of $g_p(t)$ is given by

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j(\Omega - k\Omega_T)\right)$$

- $G_p(j\Omega)$ is a periodic function of Ω consisting of a sum of shifted and scaled replicas of $G_a(j\Omega)$, shifted by integer multiples of Ω_T and scaled by 1/T

3.8.1 Effect of Sampling in the Frequency Domain

- The term on the RHS of the previous equation for k = 0 is the baseband portion（基带部分）of $G_p(j\Omega)$, and each of the remaining terms are the frequency translated portions (频移部分)
- The frequency range

$$-\frac{\Omega_T}{2} \leq \Omega \leq \frac{\Omega_T}{2}$$

is called the baseband or Nyquist band

采样的频域分析

$g_a(t)$ → $G_a(j\Omega)$

$g_p(t)$ → $G_p(j\Omega)$

$g[n]$ → $G(e^{j\omega})$

$G_a(j\Omega)$ → $G_p(j\Omega)$ → $G(e^{j\omega})$

采样的频域分析

$G_p(j\Omega)$ 与 $G(e^{j\omega})$ 之间的关系

$$G_p(j\Omega)=\sum_{n=-\infty}^{\infty} g_a(nT)e^{-j\Omega nT}$$

$g[n]=g_a(nT)$

$\omega=\Omega T$

$$G_p(j\Omega)=\sum_{n=-\infty}^{\infty} g[n]e^{-j\omega n}=G(e^{j\omega})$$

所以 $G_p(j\Omega)=G(e^{j\omega})\big|_{\omega=\Omega T}$ 或 $G(e^{j\omega})=G_p(j\Omega)\big|_{\Omega=\omega/T}$

对频率轴Ω进行尺度缩放，可从$G_p(j\Omega)$ 得到$G(e^{j\omega})$

采样的频域分析

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$g_a(t)$ → $G_a(j\Omega)$

$g_p(t)$ → $G_p(j\Omega)$

$g[n]$ → $G(e^{j\omega})$

$G_a(j\Omega) \longrightarrow G_p(j\Omega) \longrightarrow G(e^{j\omega})$

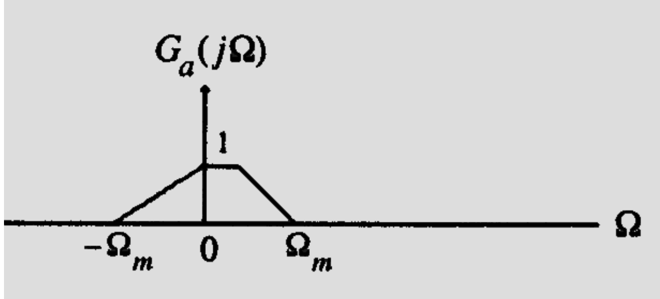
$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a(j(\Omega - k\Omega_T)) \qquad G(e^{j\omega}) = G_p(j\Omega)\Big|_{\Omega=\omega/T}$$

时域采样带来了频谱的周期复制
以采样频率为周期进行周期延拓

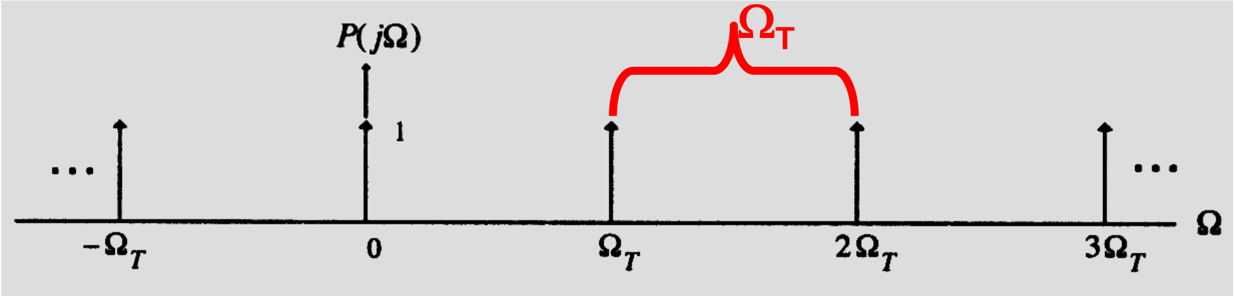
3.8.1 Effect of Sampling in the Frequency Domain

- **Assume $g_a(t)$ is a band-limited signal with a CTFT $G_a(j\Omega)$ as shown below**

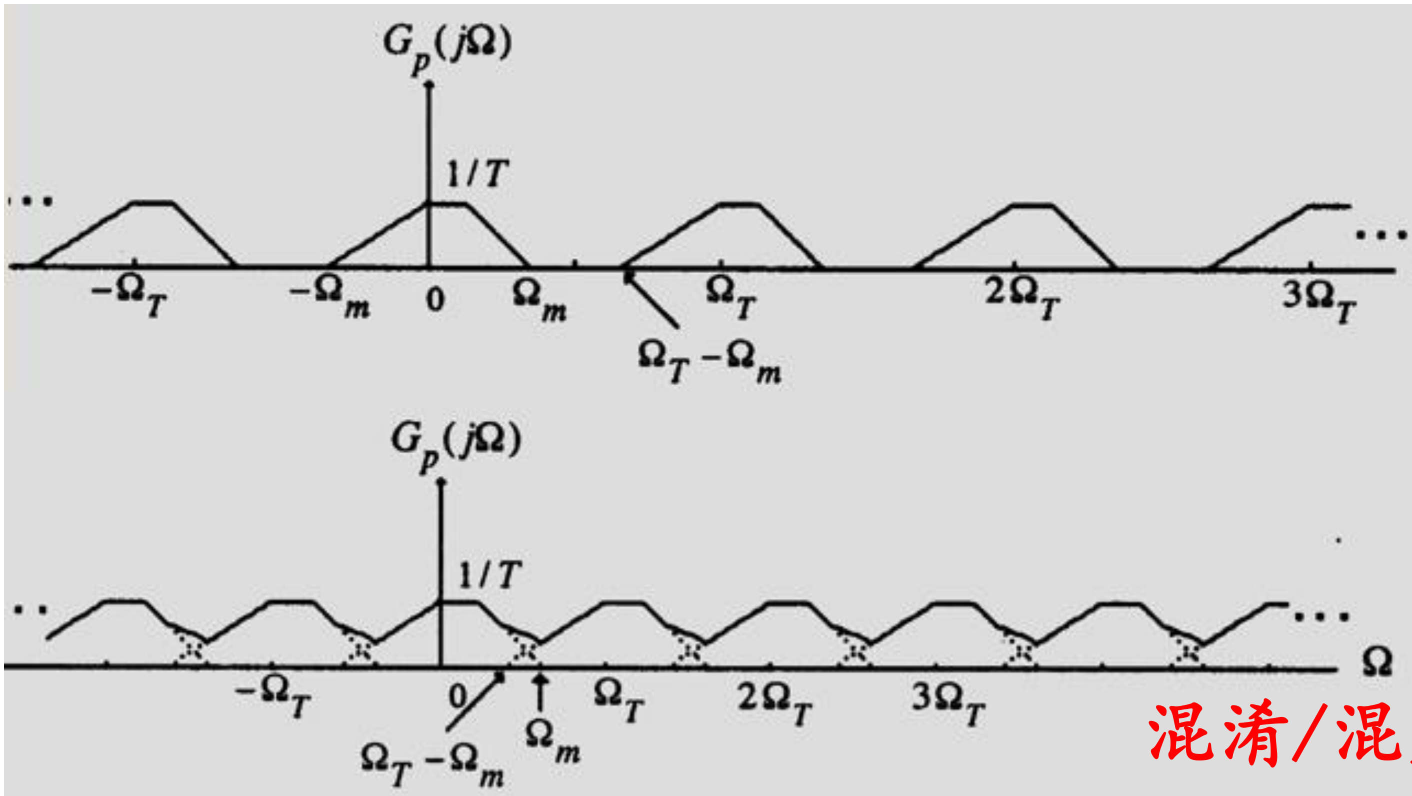
卷积关系



- **The spectrum $P(j\Omega)$ of p(t) having a sampling period $T=2\pi/\Omega_T$ is indicated below**



采样的频域分析



混淆/混叠

从频域看不产生混叠的条件：$\Omega_T \geq 2\Omega_m$

Sampling theorem

Let $g_a(t)$ be a band-limited signal with CTFT $G_a(j\Omega)=0$ for $|\Omega|> \Omega_m$

Then $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, $-\infty\leq n\leq\infty$, if

$$\Omega_T \geq 2\,\Omega_m$$

where $\Omega_T=2\pi/T$

3.8.1 Effect of Sampling in the Frequency Domain

- **The condition $\Omega_T \geq 2\Omega_m$ is often referred to as the Nyquist condition**

- **The frequency $\Omega_T/2$ is usually referred to as the folding frequency**

3.8.1 Effect of Sampling in the Frequency Domain

- **The highest frequency Ω_m contained in $g_a(t)$ is usually called the Nyquist frequency since it determines the minimum sampling frequency $\Omega_T = 2\Omega_m$ that must be used to fully recover $g_a(t)$ from its sampled version**
- **The frequency $2\Omega_m$ is called the Nyquist rate**

3.8.1 Effect of Sampling in the Frequency Domain

- **Oversampling - The sampling frequency is higher than the Nyquist rate**
- **Undersampling-The sampling frequency is lower than the Nyquist rate**
- **Critical sampling（临界采样）- The sampling frequency is equal to the Nyquist rate**
- **Note: A pure sinusoid may not be recoverable from its critically sampled version**

3.8.1 Aliasing in the Frequency Domain

- **Example - Consider the three continuous-time sinusoidal signals:**

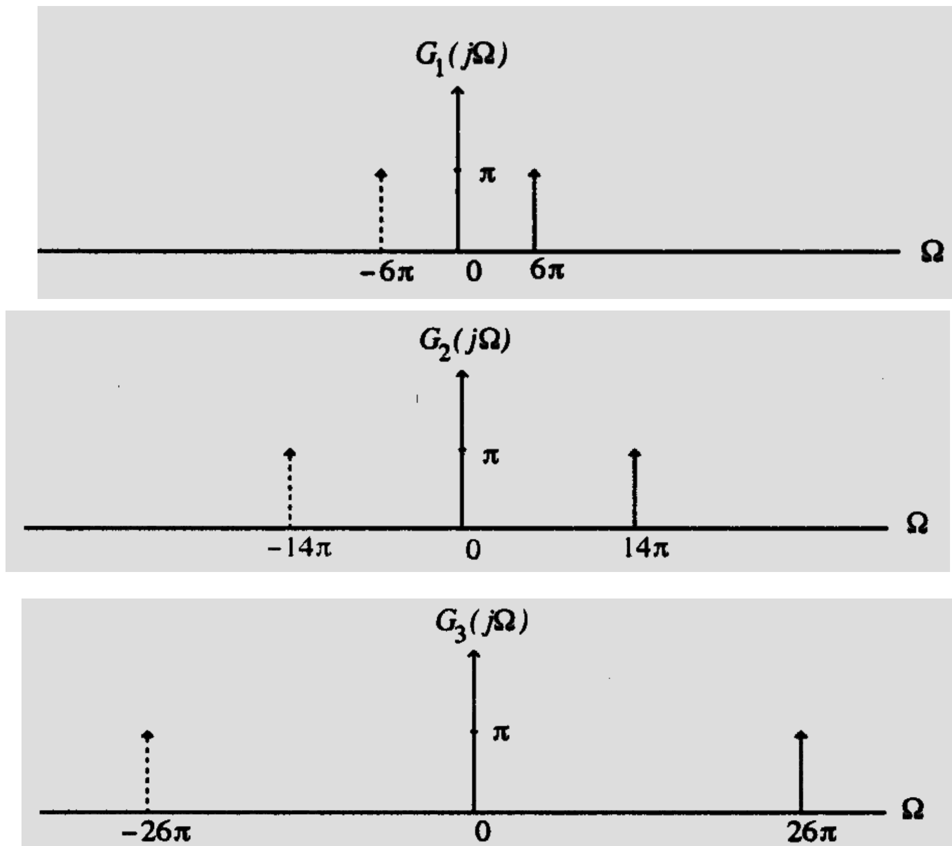
$$g_1(t) = \cos(6\pi t)$$
$$g_2(t) = \cos(14\pi t)$$
$$g_3(t) = \cos(26\pi t)$$

- **Their corresponding CTFTs are:**

$$G_1(j\Omega) = \pi[\delta(\Omega - 6\pi) + \delta(\Omega + 6\pi)]$$
$$G_2(j\Omega) = \pi[\delta(\Omega - 14\pi) + \delta(\Omega + 14\pi)]$$
$$G_3(j\Omega) = \pi[\delta(\Omega - 26\pi) + \delta(\Omega + 26\pi)]$$

§ 3.8.1 Aliasing in the Frequency Domain

- These three transforms are plotted below



§ 3.8.1 Aliasing in the Frequency Domain

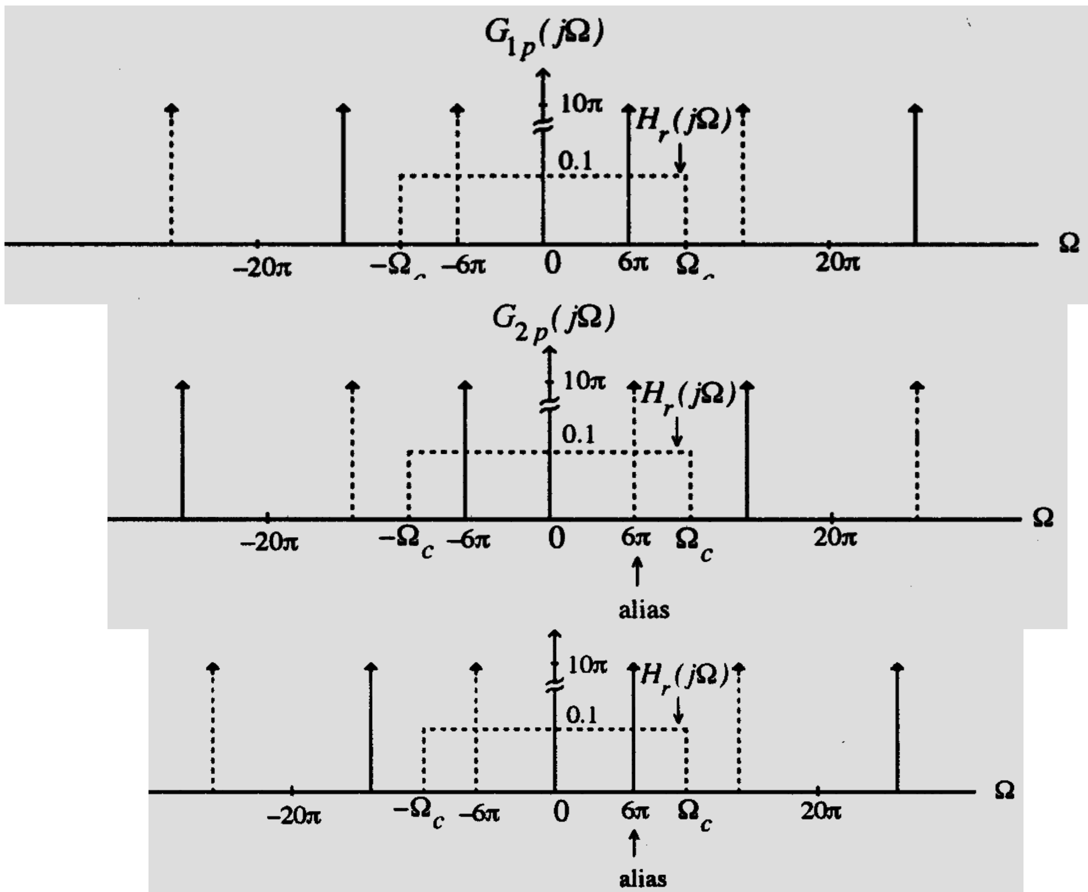
Digital Signal Processing

- These continuous-time signals sampled at a rate of T = 0.1 sec, i.e., with a sampling frequency Ω_T =20π rad/sec
- The sampling process generates the continuous-time impulse trains, $g_{1p}(t)$, $g_{2p}(t)$, and $g_{3p}(t)$
- Their corresponding CTFTs are given by

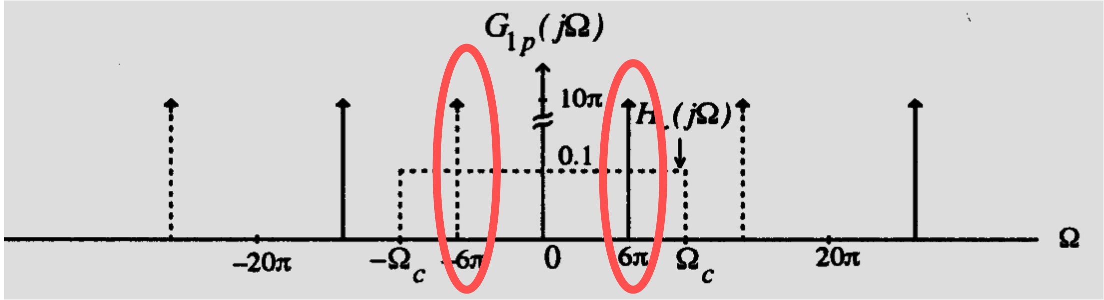
$$G_{\ell p}(j\Omega) = 10\sum_{k=-\infty}^{\infty} G_{\ell}\left(j(\Omega - k\Omega_T)\right), \quad 1 \le \ell \le 3$$

§ 3.8.1 Aliasing in the Frequency Domain

- **Plots of the 3 CTFTs are shown below**

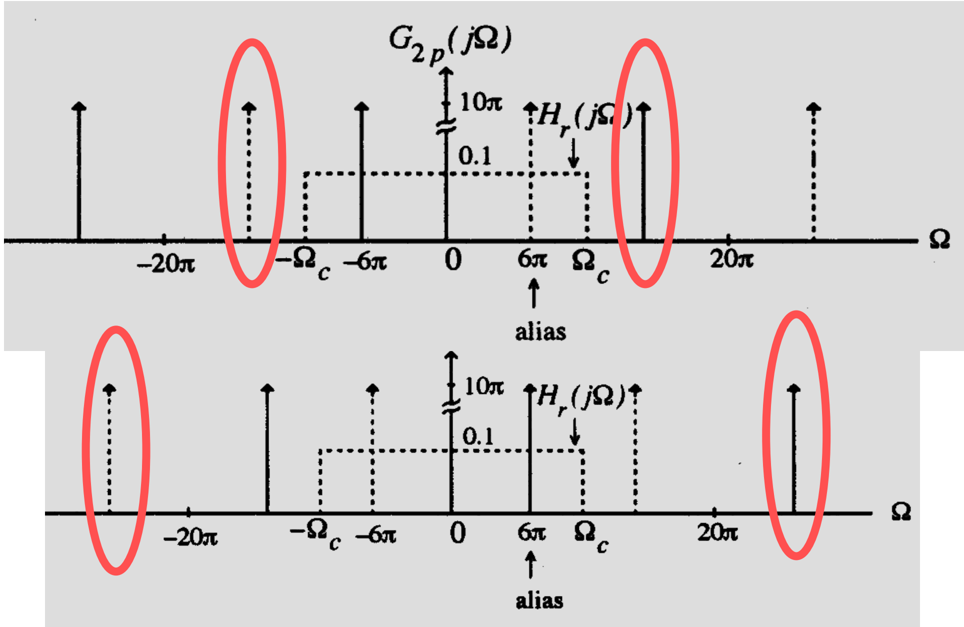


§ 3.8.1 Aliasing in the Frequency Domain



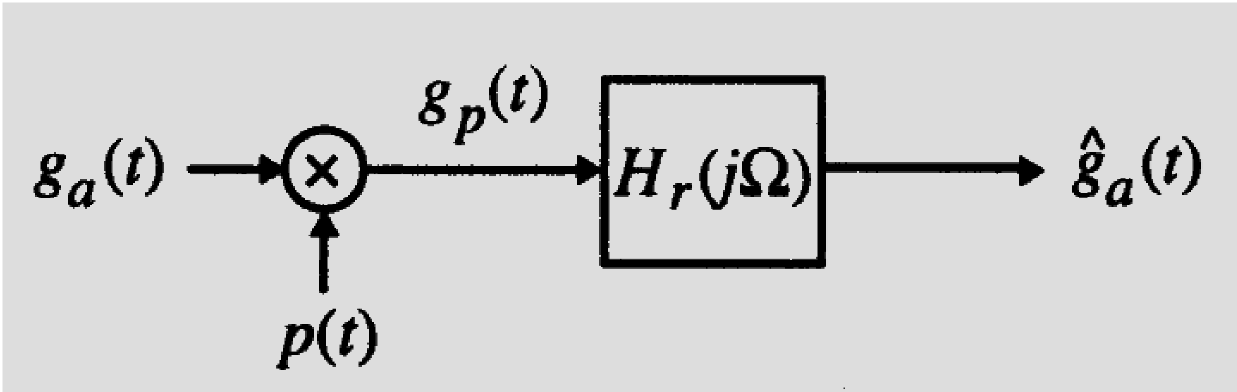
- **Dotted lines indicate the frequency response of an ideal lowpass filter with a cutoff at $\Omega_c=\Omega_T/2=10\pi$ and a gain T=0.1**
- **In the case of $g_1(t)$, the sampling rate satisfies the Nyquist condition, hence no aliasing. Moreover, the reconstructed output is precisely the original continuous-time signal**

§ 3.8.1 Aliasing in the Frequency Domain



- **In the other two cases, the sampling rate does not satisfy the Nyquist condition, resulting in aliasing and the filter outputs are all equal to cos(6pt)**

3.8.2 Recovery of the Analog Signal



$$\hat{G}_a(j\Omega)=G_p(j\Omega)H_r(j\Omega)$$

$$g_a(t)=g_p(t)\circledast h_r(t)$$

3.8.2 Recovery of the Analog Signal

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

- **The impulse response $h_r(t)$ of the lowpass reconstruction filter is obtained by taking the inverse DTFT of $H_r(j\Omega)$**

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega) e^{j\Omega t} d\Omega = \frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t} d\Omega$$

$$= \frac{\sin(\Omega_c t)}{\Omega_T t / 2}, \qquad -\infty \le t \le \infty$$

3.8.2 Recovery of the Analog Signal

- **The input to the lowpass filter is the impulse train $g_p(t)$:**

$$g_p(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT)$$

$$\hat{g}_a(t) = h_r(t) \circledast g_p(t) = \sum_{n=-\infty}^{\infty} g[n] h_r(t - nT)$$

Substituting $h_r(t)=\sin(\Omega_c t)/(\Omega_T t/2)$ in the above and assuming for simplicity $\Omega_c= \Omega_T/2= \pi/T$,

3.8.2 Recovery of the Analog Signal

we get

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

which is called Poisson sum formula

3.8.2 Recovery of the Analog Signal

- **The ideal bandlimited interpolation process is illustrated below**

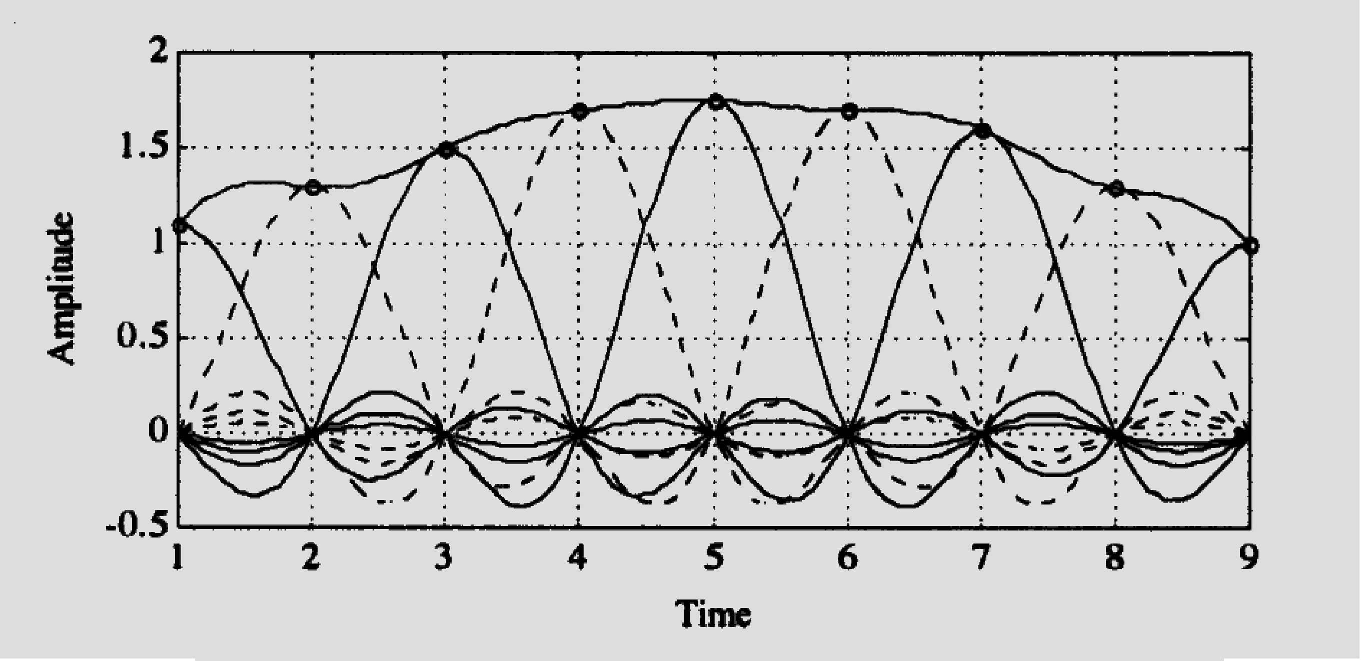


Illustration of Poisson sum formula

§ 3.8 Digital Processing of Continuous-Time Signals

Digital processing of a continuous-time signal involves the following basic steps:

(1) Conversion of the continuous-time signal into a discrete-time signal（A/D converter）

(2) Processing of the discrete-time signal

(3) Conversion of the processed discrete-time signal back into a continuous-time signal （D/A converter）

§ 3.8 Digital Processing of Continuous-Time Signals

Complete block-diagram

- **Both the anti-aliasing filter and the reconstruction filter are analog lowpass filters**
- **Also, the most widely used IIR digital filter design method is based on the conversion of an analog lowpass prototype**

本章重点

1.DTFT定义：正反变换公式

2.会用定义式计算序列的**DTFT**

3. $X(e^{j\omega})$ 的周期性、复数性、存在性条件

4. DTFT的对称性

5. DTFT的运算性质

6. 采样定理

Homework

- **Read textbook from p.81 to 107**
- **Problems**

3.16 (b)(c) , 3.21(a)(b)(d), 3.23(a), 3.29, 3.30, 3.31(c), 3.46, 3.48(a)(b)(c)(d), 3.60， 3.61

- **MATLAB Exercise M 3.3**
- 课后自学并练习《实验指导书》**p26 to 35**