Digital Signal Processing

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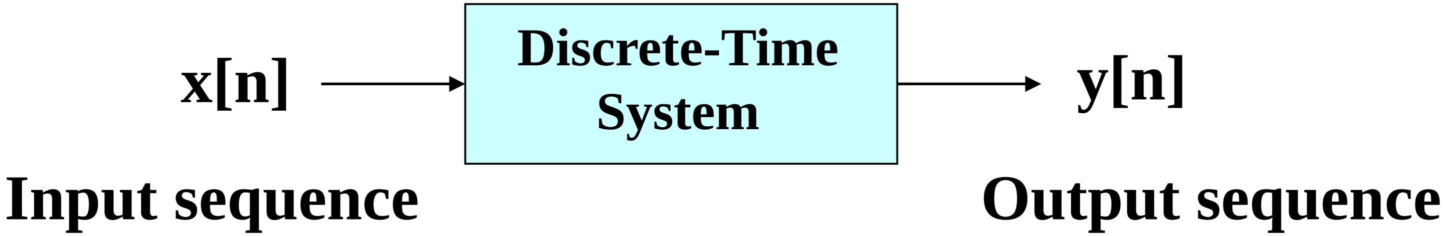
Chapter 4
Discrete-Time System

Chapter 4 Discrete-Time Systems

- **4.1 Discrete-Time System Examples**
- **4.2 Classification of Discrete-Time Systems**
- **4.3 Impulse and Step Responses**
- **4.4 Time-Domain Characterization of LTI Discrete-Time Systems**
- **4.5 Simple Interconnection Schemes**
- **4.6 Finite-Dimensional LTI Discrete-Time Systems**
- **4.7 Classification of LTI Discrete-Time Systems**
- **4.8 Frequency-Domain Representations of LTI Discrete-Time Systems**
- **4.9 Phase and Group Delays**

§ 4.1 Discrete-Time Systems



§ 4.1 Discrete-Time Systems

- **Accumulator:**

$$\boldsymbol{y[n] = \sum_{\ell=-\infty}^{n} x[\ell] = \sum_{\ell=-\infty}^{n-1} x[\ell] + x[n] = y[n-1] + x[n]}$$

递归表达式

定义式

$$y[n] = \sum_{\ell=-\infty}^{-1} x[\ell] + \sum_{\ell=0}^{n} x[\ell]$$

$$= y[-1] + \sum_{\ell=0}^{n} x[\ell], \quad n \geq 0$$

输入输出关系式

§ 4.1 Discrete-Time Systems

- **M-point Moving-Average System-**

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

- **Use:** smoothing random variations in data
- In most applications, the data x[n] is a bounded sequence, so M-point average y[n] is also bounded
- If there is no bias in the measurements, an improved estimate of the noisy data is obtained by simply increasing M

§ 4.1 Discrete-Time Systems

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

- **A more efficient implementation**

$$= \frac{1}{M}\left(\sum_{\ell=1}^{M} x[n-\ell] + x[n] - x[n-M]\right) (变量代换 \ell = \ell - 1)$$

$$= \frac{1}{M}\left(\sum_{\ell=0}^{M-1} x[n-1-\ell] + x[n] - x[n-M]\right)$$

$$= \frac{1}{M}\left(\sum_{\ell=0}^{M-1} x[n-1-\ell]\right) + \frac{1}{M}\left(x[n] - x[n-M]\right)$$

$$= y[n-1] + \frac{1}{M}\left(x[n] - x[n-M]\right)$$

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§ 4.2 Classification of Discrete-Time Systems

- **Linear System**
- **Shift-Invariant System**
- **Linear Time-Invariant System**
- **Causal System**
- **Stable System**
- **Passive and Lossless Systems(无源和无损)**

§ 4.2.1 Linear Systems

- **Definition - If $y_1[n]$ is the output due to an input $x_1[n]$ and $y_2[n]$ is the output due to an input $x_2[n]$ then for an input**

$$x[n]=ax_1[n]+bx_2[n]$$

the output is given by

$$y[n]=ay_1[n]+by_2[n]$$

- **Above property must hold for any arbitrary constants a and b and for all possible inputs $x_1[n]$ and $x_2[n]$**

§ 4.2.1 Linear Systems

Example 4.3-Property of the Accumulator

$$y_1[n]=\sum_{\ell=-\infty}^{n}x_1[\ell] \qquad y_2[n]=\sum_{\ell=-\infty}^{n}x_2[\ell]$$

$$x[n]=\alpha x_1[n]+\beta x_2[n]$$

$$y[n]=\sum_{\ell=-\infty}^{n}(\alpha x_1[\ell]+\beta x_2[\ell])$$

$$=\alpha\sum_{\ell=-\infty}^{n}x_1[\ell]+\beta\sum_{\ell=-\infty}^{n}x_2[\ell]=\alpha y_1[n]+\beta y_2[n]$$

——Linear system

§ 4.2.1 Linear Systems

Example 4.3-For Accumulator with a causal input

$$y_1[n] = y_1[-1] + \sum_{\ell=0}^{n} x_1[\ell] \qquad y_2[n] = y_2[-1] + \sum_{\ell=0}^{n} x_2[\ell]$$

$$x[n] = \alpha x_1[n] + \beta x_2[n]$$

$$y[n] = y[-1] + \sum_{\ell=0}^{n} (\alpha x_1[\ell] + \beta x_2[\ell])$$

$$= y[-1] + \alpha \sum_{\ell=0}^{n} x_1[\ell] + \beta \sum_{\ell=0}^{n} x_2[\ell]$$

$$? = \alpha y_1[n] + \beta y_2[n]$$

§ 4.2.1 Linear Systems

$$\alpha y_1[n] + \beta y_2[n]$$

$$= \alpha\left(y_1[-1] + \sum_{\ell=0}^{n} x_1[\ell] \right) + \beta\left(y_2[-1] + \sum_{\ell=0}^{n} x_2[\ell] \right)$$

$$= \left(\alpha y_1[-1] + \beta y_2[-1]\right) + \left(\alpha\sum_{\ell=0}^{n} x_1[\ell] + \beta\sum_{\ell=0}^{n} x_2[\ell] \right)$$

Thus,

$$y[n] = \alpha y_1[n] + \beta y_2[n] \quad \Big|\, if \quad y[-1] = \alpha y_1[-1] + \beta y_2[-1]$$

This condition cannot be satisfied unless accumulator is initially at rest with zero initial condition

§ 4.2.2 Shift-Invariant System

- For a shift-invariant system, if $y_1[n]$ is the response to an input $x_1[n]$, then the response to an input

$$x[n]=x_1[n-n_0]$$

is simply

$$y[n]=y_1[n-n_0]$$

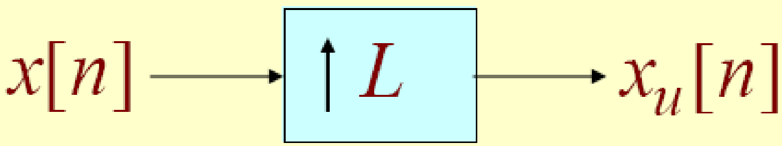
where n_0 is any positive or negative integer

- The above relation must hold for any arbitrary input and its corresponding output
- The above property is called time-invariance property, or shift-invariant proterty

§ 4.2.2 Shift-Invariant System

- **Time-invariance property ensures that for a specified input, the output is independent of the time the input is being applied**

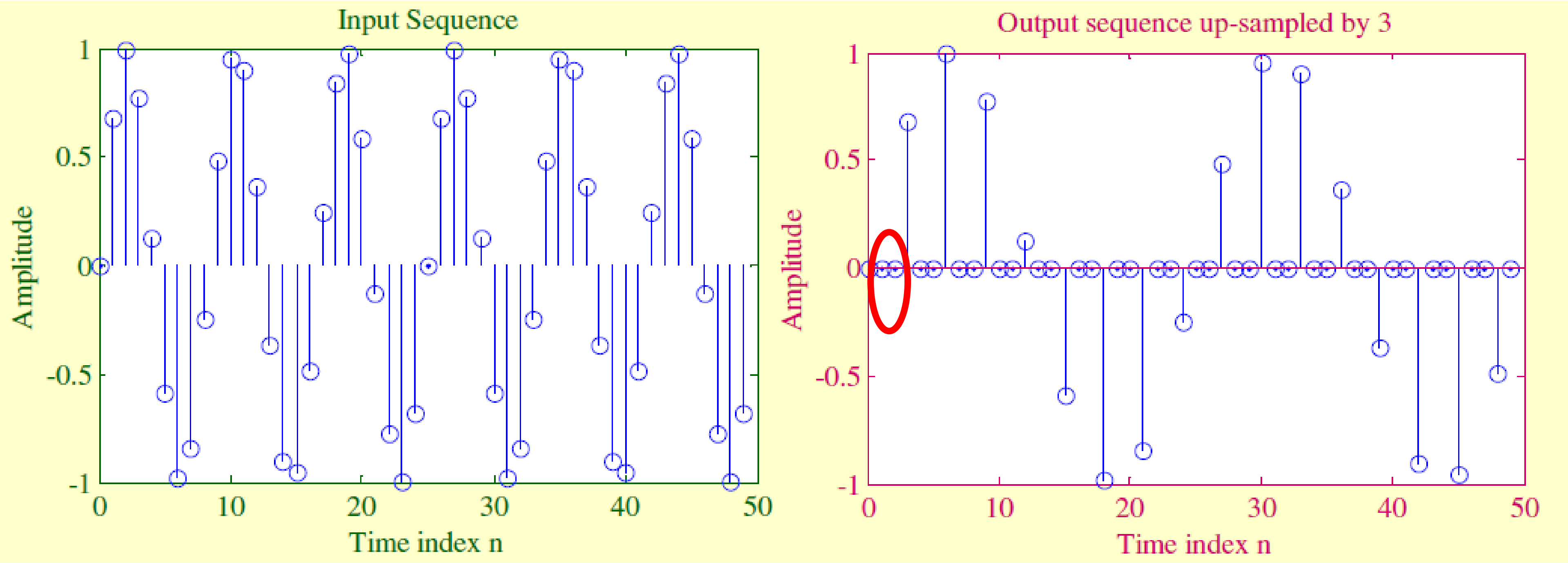
  - Example - Consider the up-sampler



with an input-output relation given by

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

- **An example of the up-sampling operation**



- For an input $x_1[n] = x[n - n_o]$ the output $x_{1,u}[n]$ is given by

$$x_{1,u}[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \ldots\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} x[(n - Ln_o)/L], & n = 0, \pm L, \pm 2L, \ldots\ldots \\ 0, & \text{otherwise} \end{cases}$$

- However from the definition of the up-sampler

$$x_u[n - n_o]$$

$$= \begin{cases} x[(n - n_o)/L], & n = n_o, n_o \pm L, n_o \pm 2L, \ldots\ldots \\ 0, & \text{otherwise} \end{cases}$$

$$\neq x_{1,u}[n]$$

- Hence, the up-sampler is a time-varying system

例题：判断线性性和时不变性

- $y(n)=2x(n)+5$　　非线性、时不变
- $y(n)=x^2(n)$　　非线性、时不变
- $y(n)=nx(n)$　　线性、时变
- $y(n)=x(n-n_0)$　　线性、时不变

$y(n)=x(n-n_0)$

$$x_1[n] \Rightarrow y_1[n] = x_1[n-n_0],$$

$$x_2[n] \Rightarrow y_2[n] = x_2[n-n_0],$$

$$ax_1[n]+bx_2[n] \Rightarrow y[n] = ax_1[n-n_0]+bx_2[n-n_0]$$

$$y[n] = ay_1[n]+by_2[n]$$

$$x[n] \Rightarrow y[n] = x[n-n_0],$$

$$x[n-1] \Rightarrow y'[n] = x[n-n_0-1],$$

$$y[n-1] = x[(n-1)-n_0],$$

$$y'[n] = y[n-1]$$

$y(n)=ax(n)+b$

$$x_1[n] \Rightarrow y_1[n] = ax_1[n] + b,$$

$$x_2[n] \Rightarrow y_2[n] = ax_2[n] + b,$$

$$x_1[n] + x_2[n] \Rightarrow y[n] = ax_1[n] + ax_2[n] + b$$

$$y[n] \neq y_1[n] + y_2[n] = ax_1[n] + ax_2[n] + 2b$$

$$x[n] \Rightarrow y[n] = ax[n] + b,$$

$$x[n - n_0] \Rightarrow y'[n] = ax[n - n_0] + b,$$

$$y[n - n_0] = y'[n],$$

例题：判断下列系统的线性和时不变性

$$y(n) = x(2n)$$

解： $$y(n-D) = x\left[2(n-D)\right] = x(2n-2D)$$

$$y_D(n) = T\left[x(n-D)\right] = x(2n-D)$$

$$\Rightarrow \qquad y(n-D) \neq y_D(n)$$

§ 4.2.3 Causal System

Definition

- **In a causal system, the n_0-th output sample $y(n_0)$ depends only on input samples $x(n)$ for $n \le n_0$ and does not depend on input samples for $n > n_0$**
- **For a causal system, changes in output samples do not precede changes in the input samples**

§ 4.2.3 Causal System

- Examples of causal systems:

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$$

$$y[n] = y[n-1] + x[n]$$

- Examples of noncausal systems:

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

§ 4.2.4 Stable System

Definition

- **There are various definitions of stability. We consider here the *bounded-input, bounded-output (BIBO)* stability**
- **If $y(n)$ is the response to an input $x(n)$ and if $x(n)$ is bounded, i.e.**

$$|x(n)|<B_x, \text{ for all values of } n$$

then $y(n)$ is bounded, i.e

$$|y(n)|<B_y, \text{ for all values of } n$$

§ 4.2.4 Stable System

Example: M-point Moving-Average filter

$$\left|x[n]\right| < B_x$$

$$\left|y[n]\right| = \left|\frac{1}{M}\sum_{k=0}^{M-1} x[n-k]\right| \le \frac{1}{M}\sum_{k=0}^{M-1}\left|x[n-k]\right| \le \frac{1}{M} \bullet M \bullet B_x \le B_x$$

§ 4.2.5 Passive and Lossless Systems

- **A discrete-time system is defined to be passive if, for every finite-energy input x[n], the output y[n] has, at most, the same energy, i.e.**

$$\sum_{n=-\infty}^{\infty} \left| y[n] \right|^2 \leq \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2 < \infty$$

- **For a lossless system, the above inequality is satisfied with an equal sign for every input**

§ 4.2.5 Passive and Lossless Systems

- **Example - Consider the discrete-time system defined by y[n]=αx[n-N] with N a positive integer**
- **Its output energy is given by**

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = |\alpha|^2 \sum_{n=-\infty}^{\infty} |x[n]|^2$$

- **Hence, it is a passive system if $|\alpha| \le 1$ and is a lossless system if $|\alpha| = 1$**

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§ 4.3 Impulse and Step Responses

- **The response of a discrete-time system to a unit sample sequence $\{\delta[n]\}$ is called the unit impulse response or simply, the impulse response, and is denoted by {h[n]}**
- **The response of a discrete-time system to a unit step sequence $\{u[n]\}$ is called the unit step response or simply, the step response, and is denoted by {s[n]}**

§ 4.3 Impulse and Step Responses

- Example - The impulse response of the system

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

is obtained by setting $x[n] = \delta[n]$ resulting in

$$h[n] = \alpha_1 \delta[n] + \alpha_2 \delta[n-1] + \alpha_3 \delta[n-2] + \alpha_4 \delta[n-3]$$

- The impulse response is thus a finite-length sequence of length 4 given by

$$\{h[n]\} = \{\underset{\uparrow}{\alpha_1}, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4\}$$

§ 4.3 Impulse and Step Responses

- **<u>Example</u> - The impulse response of the discrete-time accumulator**

$$y[n] = \sum_{\ell=-\infty}^{n} x[\ell]$$

is obtained by setting $x[n] = \delta[n]$ resulting in

$$h[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

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§ 4.4 Time-Domain Characterization of LTI Discrete-Time System

- **Linear Time-Invariant (LTI) System -**

  A system satisfying both the linearity and the time-invariance property
- **LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design**
- **Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades**

§ 4.4.1 Input-Output Relationship

- A consequence of the linear, time invariance property is that a LTI discrete time system is completely characterized by its impulse response
- Knowing the impulse response one can compute the output of the system for any arbitrary input

§ 4.4.1 Input-Output Relationship

- **Since $h(n)$ is the response of input $\delta(n)$ and the system is time invariant, we have**

$$\delta(n-k) \rightarrow h(n-k)$$

- **Likewise, as the system is linear**

$$x(k)\delta(n-k) \rightarrow x(k)h(n-k)$$

- **Note that, $x(k)$ is considered as a constant in this case**

§ 4.4.1 Input-Output Relationship

- **Taking advantage of the property of linear, we have**

$$\sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \rightarrow \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

- **Eventually, the I-O relationship of an LTI system can be written as follows**

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

4.4.3 Stability Condition

BIBO Stability Condition --

- **A discrete-time system is BIBO stable if the output sequence {*y*(*n*)} remains bounded for all bounded input sequence {*x*(*n*)}**
- **An LTI discrete-time system is BIBO stable if and only if its impulse response sequence {*h*(*n*)} is absolutely summable, i.e.**

$$S = \sum_{n=-\infty}^{\infty} \left| h(n) \right| < \infty$$

4.4.3 Stability Condition

- Example - Consider an LTI discrete-time system with an impulse response

$$h[n] = (\alpha)^n \mu[n]$$

- For this system

$$S = \sum_{n=-\infty}^{\infty} \left|\alpha^n\right| \mu[n] = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} \quad \text{if } |\alpha| < 1$$

- Therefore $S < \infty$ if $|\alpha| < 1$ for which the system is BIBO stable
- If $|\alpha| = 1$, the system is not BIBO stable

4.4.4 Causality Condition

Causality Condition ——

- **An LTI discrete-time system is causal if and only if its impulse response {*h*(*n*)} is a causal sequence, i.e., h[n]=0, for all n<0.**

$$y[n]=\sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$=\cdots+\underline{x[n+2]}h[-2]+\underline{x[n+1]}h[-1]+x[n]h[0]$$

$$+x[n-1]h[1]+x[n-2]h[2]+\cdots$$

4.4.4 Causality Condition

Causality Condition ——

- **A non-causal LTI discrete-time system with a finite-length impulse response can often be realized as a causal system by inserting an appropriate amount of delay.**

- **Clipping+delaying**

4.4.4 Causality Condition

- Example - The discrete-time accumulator defined by

$$y[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

is a causal system as it has a causal impulse response given by

$$h[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

- 系统稳定性的时域充要条件：**h(n)**绝对可和，即

$$S=\sum_{n=-\infty}^{\infty}\left|h[n]\right|<\infty$$

- 系统因果性的时域充要条件：

$$h[n]\equiv 0, n<0$$

Homework

Problems:

4.3(b), 4.20(a), 4.23(a)(解卷积)，4.30(a)(互联结构), 4.67（第一问）

Matlab Exercises: M4.1