Digital Signal Processing

College of Communication & Information Engineering

Nanjing University of Posts and Telecommunications

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Chapter 4

Discrete-Time System

上讲回顾

- **How to describe the input-output relationship in a LTI Discrete Time System in time domain?**
- **How to describe a stable LTI Discrete Time System？**
- **How to describe a causal LTI Discrete Time System？**

4.4.3 Stability Condition

BIBO Stability Condition --

- **A discrete-time system is BIBO stable if the output sequence {*y*(*n*)} remains bounded for all bounded input sequence {*x*(*n*)}**
- **An LTI discrete-time system is BIBO stable if and only if its impulse response sequence {*h*(*n*)} is absolutely summable, i.e.**

$$S = \sum_{n=-\infty}^{\infty} \left| h(n) \right| < \infty$$

4.4.4 Causality Condition

Causality Condition ——

- **An LTI discrete-time system is causal if and only if its impulse response {*h*(*n*)} is a causal sequence, i.e., h[n]=0, for all n<0.**

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

$$= \cdots + \underline{x[n+2]}h[-2] + \underline{x[n+1]}h[-1] + x[n]h[0]$$

$$+ x[n-1]h[1] + x[n-2]h[2] + \cdots$$

4.4.4 Causality Condition

Causality Condition ——

- **A non-causal LTI discrete-time system with a finite-length impulse response can often be realized as a causal system by inserting an appropriate amount of delay.**

- **Clipping+delaying**

4.4.4 Causality Condition

- Example - The discrete-time accumulator defined by

$$y[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

is a causal system as it has a causal impulse response given by

$$h[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

- 系统稳定性的时域充要条件：**h(n)**绝对可和，即

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- 系统因果性的时域充要条件：

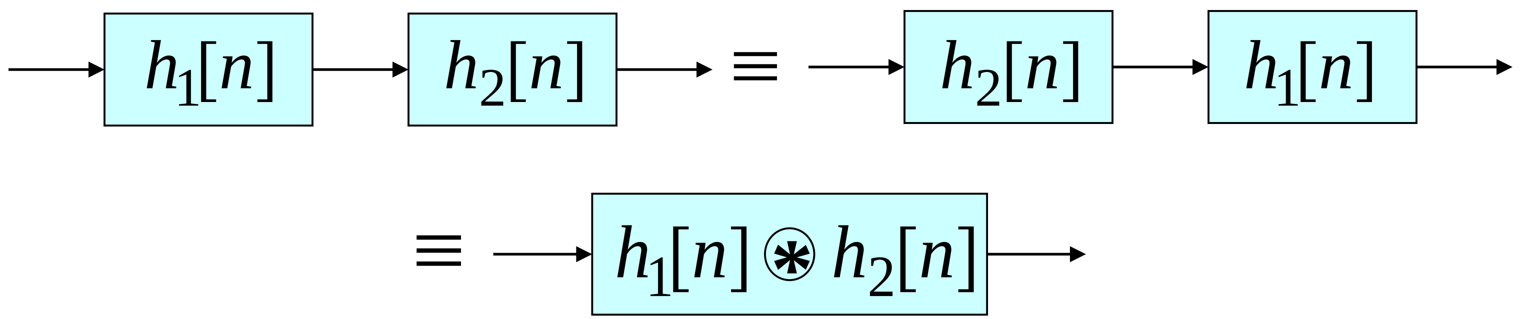
$$h[n] \equiv 0, n < 0$$

Chapter 4 Discrete-Time Systems

- 4.1 Discrete-Time System Examples
- 4.2 Classification of Discrete-Time Systems
- 4.3 Impulse and Step Responses
- 4.4 Time-Domain Characterization of LTI Discrete-Time Systems
- 4.5 Simple Interconnection Schemes
- 4.6 Finite-Dimensional LTI Discrete-Time Systems
- 4.7 Classification of LTI Discrete-Time Systems
- 4.8 Frequency-Domain Representations of LTI Discrete-Time Systems
- 4.9 Phase and Group Delays

§ 4.5 Simple Interconnection Schemes

- **Cascade Connection**

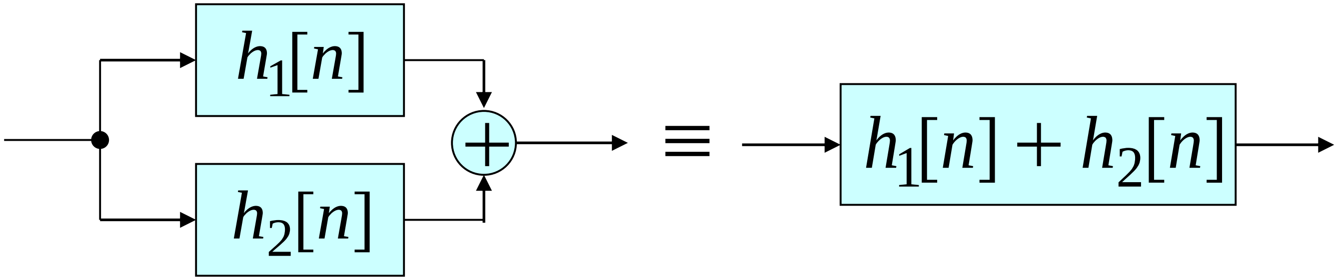


Impulse response h[n] of the cascade of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by

$$h[n] = h_1[n] \circledast h_2[n]$$

§ 4.5 Simple Interconnection Schemes

- **Parallel Connection**

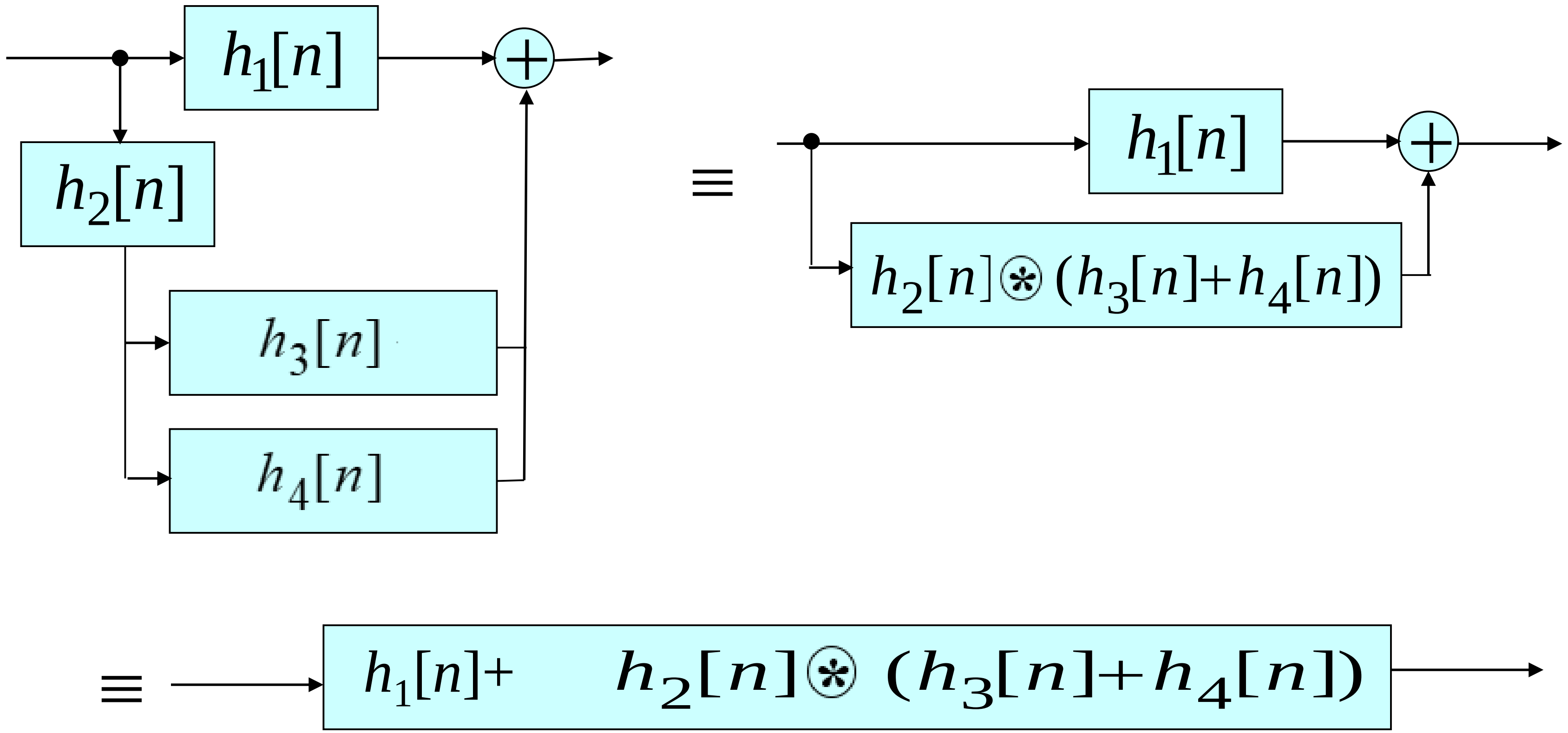


•Impulse response h[n] of the parallel connection of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by

$$h[n] = h_2[n] + h_1[n]$$

§ 4.5 Simple Interconnection Schemes

- **Simplifying the block-diagram we obtain**



Chapter 4 Discrete-Time Systems

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§ 4.6 Finite-Dimensional LTI Discrete-Time Systems

- **A linear constant coefficient difference equation**

$$\sum_{k=0}^{N} d_k\, y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- **$\{d_k\}$and$\{p_k\}$ are constants characterizing the system**
- **The order of the system is given by max(N,M)**

§ 4.6 Finite-Dimensional LTI Discrete-Time Systems

- **If we assume the system to be causal, then the output $y[n]$ can be recursively computed using**

$$y(n) = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

Provided $d_0 \neq 0$

- **$y[n]$ can be computed for all $n \geq n_0$, knowing $x[n]$ and the initial conditions**

$$y[n_0-1], y[n_0-2], \ldots y[n_0-N]$$

§ 4.7 Classification of LTI Discrete-Time Systems

Based on Impulse Response Length -

- **If the impulse response h[n] is of finite length, then it is known as a finite impulse response (FIR) discrete-time system**
- **The convolution sum description here is**

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k]$$

§ 4.7 Classification of LTI Discrete-Time Systems

The linear constant coefficient difference equation of FIR system

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k] \qquad d_0 \neq 0$$

$$y[n] = \boxed{\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k]} + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

$$y[n] = \quad 0 \quad + \sum_{k=N_1}^{N_2} \underline{h[k]} x[n-k]$$

§ 4.7 Classification of LTI Discrete-Time Systems

- **If the impulse response is of infinite length, then it is known as an infinite impulse response (IIR) discrete-time system**
- **The class of IIR systems we are concerned with in this course are characterized by linear constant coefficient difference equations**

§ 4.7 Classification of LTI Discrete-Time Systems

$$y[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

The linear constant coefficient difference equation of IIR system

$$\sum_{k=0}^{N}d_k y[n-k]=\sum_{k=0}^{M}p_k x[n-k] \qquad d_0\neq 0$$

$$y[n]=\sum_{k=1}^{N}\frac{d_k}{d_0}y[n-k]+\sum_{k=0}^{M}\frac{p_k}{d_0}x[n-k]$$

§ 4.7 Classification of LTI Discrete-Time Systems

Based on the Output Calculation Process

- **Nonrecursive System - Here the output can be calculated sequentially, knowing only the present and past input samples**
- **Recursive System - Here the output computation involves past output samples in addition to the present and past input samples**

§ 4.7 Classification of LTI Discrete-Time Systems

- **Example - The discrete-time accumulator defined by**

$$y[n]=y[n-1]+x[n]$$

is seen to be an IIR system. It is also a recursive system.

Chapter 4 Discrete-Time Systems

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- **4.9 Phase and Group Delays**

Response to an Exponential Sequence

- 当输入为双边复指数序列时

$$x[n] = e^{j\omega n}$$

- 输出为

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)}$$

$$= \left[\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right]e^{j\omega n}$$

$$= H(e^{j\omega})e^{j\omega n}$$

- **Page167 Eq.(4.66-4.70)**

A signal $A_1 e^{j\omega_1 n}$ $(-\infty < n < +\infty)$ is the input of a LTI system whose frequency response is $H(e^{j\omega})$, then the response of this system is ( ).

A. $A_1 e^{j\omega_1 n}$ **B.** $A_1 e^{j\omega_1 n} H(e^{j\omega})$ **C.** $2\pi H(e^{j\omega_1})\mathrm{A}_1\delta(\omega-\omega_1)$ **D.** $A_1 e^{j\omega_1 n} H(e^{j\omega_1})$

§ 4.8.1 The Frequency Response

frequency response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

Complex

Periodic

$$H(e^{j\omega}) = H_{re}(e^{j\omega}) + jH_{im}(e^{j\omega})$$

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\theta(\omega)}$$

magnitude response

real

$$\theta(\omega) = \arg H\left(e^{j\omega}\right)$$

phase response

real

§ 4.8.1 The Frequency Response

- **Gain function**

$$G(\omega) = 20\log_{10}|H(e^{j\omega})| \quad dB$$

- **Attenuation or loss function**

$$A(\omega) = -G(\omega)$$

§ 4.8.1 The Frequency Response

- **If the impulse response h[n] is real then the magnitude function is an even function**

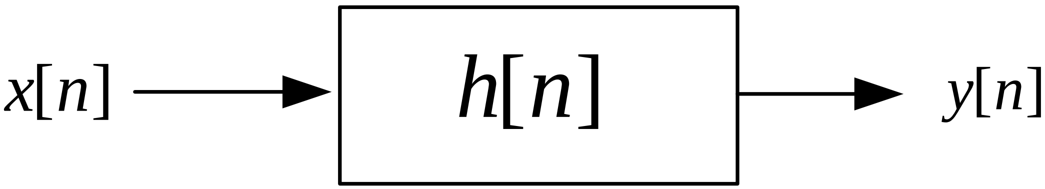
$$\left|H\left(e^{j\omega}\right)\right| = \left|H\left(e^{-j\omega}\right)\right|$$

and the phase function is an odd function

$$\theta\left(\omega\right) = -\theta\left(-\omega\right)$$

$H_{re}(e^{j\omega})$ **is even and** $H_{im}(e^{j\omega})$ **is odd**

4.8.2 Frequency-Domain Characterization of the LTI Discrete-Time System



$$y[n] = h[n] \circledast x[n]$$

DTFT DTFT DTFT

$$Y(e^{j\omega}) = H(e^{j\omega}) \bullet X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

4.8.3 Frequency Response of LTI Discrete-Time Systems

- ***FIR systems***

$$y[n] = \sum_{k=N_1}^{N_2} h[k]x[n-k] \qquad N_1 < N_2$$

$$H(e^{j\omega}) = \sum_{k=N_1}^{N_2} h[k]e^{-j\omega k}$$

4.8.3 Frequency Response of LTI Discrete-Time Systems

- ***IIR systems***

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

$$\sum_{k=0}^{N} d_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} p_k e^{-j\omega k} X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} p_k e^{-j\omega k}}{\sum_{k=0}^{N} d_k e^{-j\omega k}}$$

which is a rational function in $e^{-j\omega}$

4.8.4 Frequency Response Computation Using Matlab

pp.170 Example 4.31

例：一个滑动平均滤波器的单位脉冲响应为

$$h[n]=\begin{cases}1/M, & 0\le n\le M-1\\ 0, & \text{其它}\end{cases}$$

其频率响应为

$$H\left(e^{j\omega}\right)=\frac{1}{M}\sum_{n=0}^{M-1}e^{-j\omega n}=\frac{1}{M}\frac{1-e^{-j\omega M}}{1-e^{-j\omega}}$$

根据有限项等比级数求和公式可得

$$\frac{a_0\left(1-q^N\right)}{1-q}$$

首项 (a_0)，项数 (N)，公比 (q)

4.8.4 Frequency Response Computation Using Matlab

$$H\left(e^{j\omega}\right)=\frac{1}{M}\frac{1-e^{-j\omega M}}{1-e^{-j\omega}}$$

$$=\frac{1}{M}\frac{e^{-j\omega M/2}\left(e^{j\omega M/2}-e^{-j\omega M/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}$$

$$=\frac{1}{M}\frac{\sin\left(\omega M/2\right)}{\sin\left(\omega/2\right)}e^{-j\omega(M-1)/2}$$

根据欧拉公式 $\sin\omega=\frac{1}{2j}\left(e^{j\omega}-e^{-j\omega}\right)$

4.8.4 Frequency Response Computation Using Matlab

因此，M点滑动平均滤波器的幅频响应为

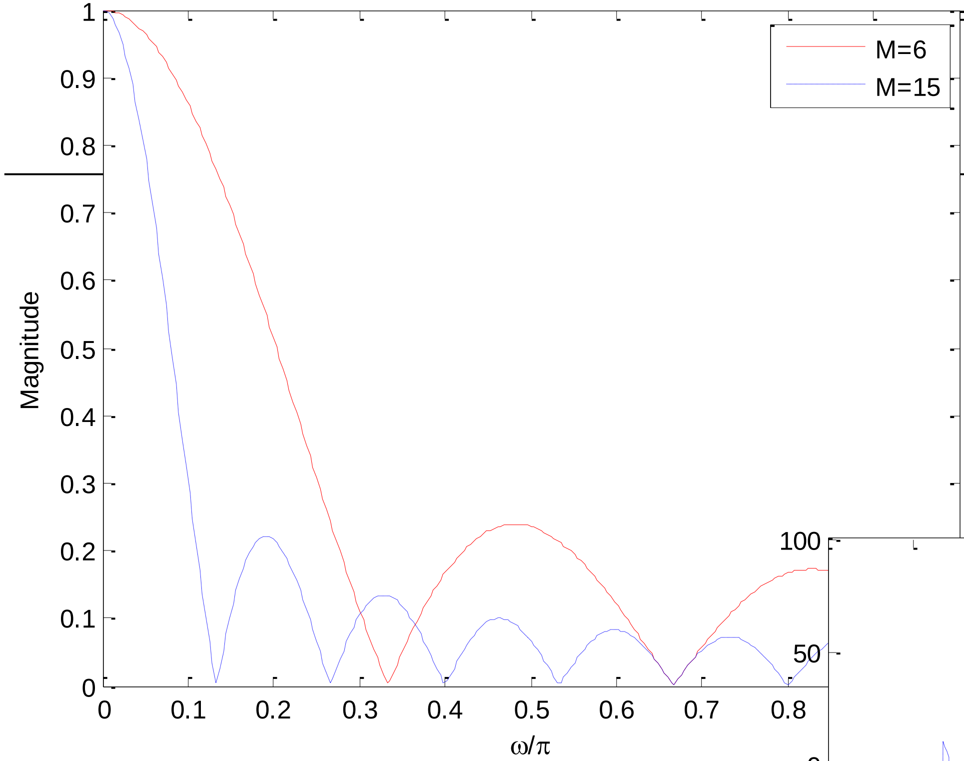
$$H\left(e^{j\omega}\right)=\left|\frac{1}{M}\frac{\sin\left(\omega M/2\right)}{\sin\left(\omega/2\right)}\right|$$

相频响应为

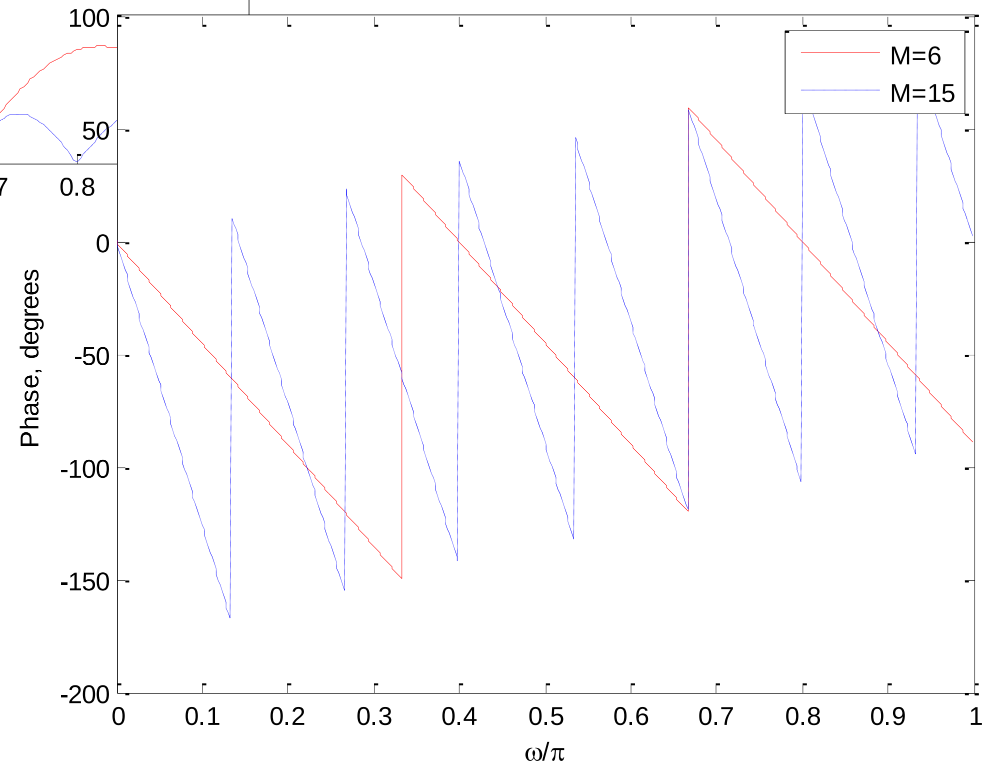
$$\theta\left(\omega\right)=-\frac{\left(M-1\right)\omega}{2}+\pi\sum_{k=0}^{\lfloor M/2\rfloor}\mu(\omega-\frac{2\pi k}{M})$$

§4.8.4 Frequency Response Computation Using MATLAB

- **The function freqz(h,w) can be used to determine the values of the frequency response vector h at a set of given frequency points w**
- **From h, the real and imaginary parts can be computed using the functions real and imag, and the magnitude and phase functions using the functions abs and angle**



M=6
M=15
Magnitude
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
ω/π



M=6
M=15
Phase, degrees
100
50
0
-50
-100
-150
-200
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
ω/π

Response to an Exponential Sequence

- Page167 Eq.(4.66-4.70)

  当输入为双边复指数序列时

$$x[n] = e^{j\omega n}$$

- 输出为

$$y[n] = H(e^{j\omega})e^{j\omega n}$$

How about the response to a Causal Exponential Sequence?

4.8.6 Response to a Causal Exponential Sequence

- **From the input-output relation**

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

we observe that for an input

$$x[n] = e^{j\omega n}\mu[n]$$

The output is

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)}\mu[n-k] = \left(\sum_{k=0}^{n} h[k]e^{j\omega(n-k)}\right)\mu[n]$$

4.8.6 Response to a Causal Exponential Sequence

Or, $$y[n]=(\sum_{k=0}^{n} h[k]e^{-j\omega k})e^{j\omega n}u(n)$$

The output for *n* < 0 is *y*[*n*] = *0*

The output for *n*≥0 is given by:

$$y[n]=(\sum_{k=0}^{n} h[k]e^{-j\omega k})e^{j\omega n}$$

$$=(\sum_{k=0}^{\infty} h[k]e^{-j\omega k})e^{j\omega n}-(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k})e^{j\omega n}$$

$$=H(e^{j\omega})e^{j\omega n}-(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k})e^{j\omega n}$$

4.8.6 Response to a Causal Exponential Sequence

$$y[n] = H(e^{j\omega})e^{j\omega n} - \left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$$

steady-state response: $y_{sr}[n] = H(e^{j\omega})e^{j\omega n}$

transient response: $y_{tr}[n] = -\left(\sum_{k=n+1}^{\infty} h[k]e^{-j\omega k}\right)e^{j\omega n}$

4.8.6 Response to a Causal Exponential Sequence

$$|y_{tr}[n]| = \left|\sum_{k=n+1}^{\infty} h[k]e^{-j\omega(k-n)}\right| \le \sum_{k=n+1}^{\infty} |h[k]| \le \sum_{k=0}^{\infty} |h[k]|$$

- **For a causal, stable LTI IIR discrete-time system, *h*[*n*] is absolutely summable**
- **As a result, the transient response is a bounded sequence**
- **As** $n \to \infty, \sum_{k=n+1}^{\infty} |h[k]| \to 0,$

4.8.6 Response to a Causal Exponential Sequence

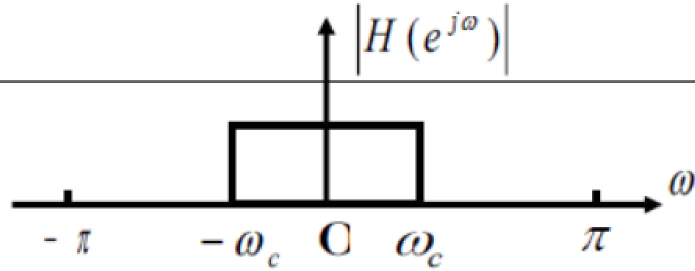
- **For a causal FIR LTI discrete-time system with an impulse response $h[n]$ of length $N + 1$, and $h[n] = 0$, for $n > N$**
- **Hence, $y_{tr}[n]=0$, for $n > N$**
- **Here the output reaches the steady-state value $y_{sr}[n] = H(e^{j\omega})e^{j\omega n}$ at $n = N$**

§ 4.8.7 The Concept of Filtering

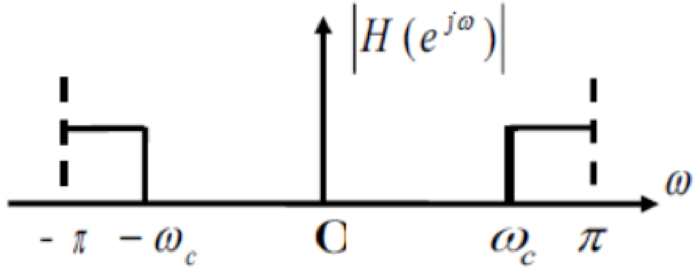
- **One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components**
- **Such systems are called digital filters and one of the main subjects of discussion in this course**

理想滤波器按幅频响应分类

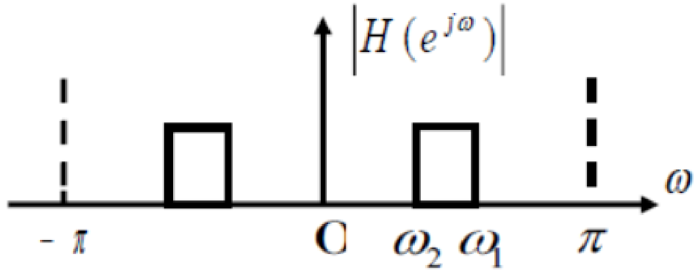
(1)理想低通



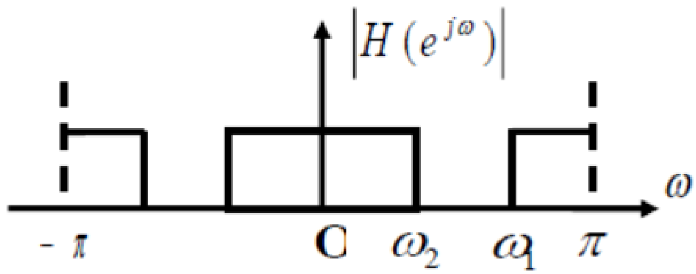
(2)理想高通



(3)理想带通



(4)理想带阻



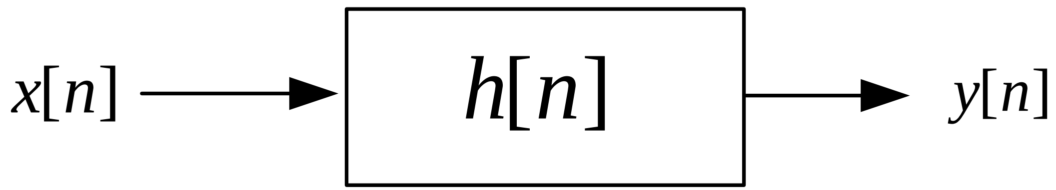
§ 4.8.7 The Concept of Filtering

- **The key to the filtering process is**

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

- **It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences**

§ 4.8.7 The Concept of Filtering



$$y[n] = h[n] \circledast x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

§ 4.8.7 The Concept of Filtering

$$Y(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{\arg(H(e^{j\omega}))} \left|X(e^{j\omega})\right| e^{\arg(X(e^{j\omega}))}$$

$$= \left|H(e^{j\omega})\right| \left|X(e^{j\omega})\right| e^{\arg(H(e^{j\omega}))+\arg(X(e^{j\omega}))}$$

$$\left|Y(e^{j\omega})\right| = \left|H(e^{j\omega})\right| \left|X(e^{j\omega})\right|$$

By appropriately choosing the values of $\left|H(e^{j\omega})\right|$ at certain frequencies, some of these components can be selectively heavily attenuated or filtered with respect to the others

§ 4.8.7 The Concept of Filtering

Consider a real-coefficient LTI discrete-time system characterized by a magnitude function

$$\left|H(e^{j\omega})\right| \cong \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

§ 4.8.7 The Concept of Filtering

- We apply an input *x*[n] to this system

$$x[n] = A\cos(\omega_1 n) + B\cos(\omega_2 n),$$

$$where, \quad 0 < \omega_1 < \omega_c < \omega_2 < \pi$$

- Because of linearity, the output of this system is of the form

$$y[n] = A\left|H(e^{j\omega_1})\right|\cos(\omega_1 n + \theta(\omega_1))$$

$$+ B\left|H(e^{j\omega_2})\right|\cos(\omega_2 n + \theta(\omega_2))$$

§ 4.8.7 The Concept of Filtering

- As $\left|H(e^{j\omega_1})\right| \cong 1, \quad \left|H(e^{j\omega_2})\right| \cong 0$

the output reduces to

$$y[n] \cong A\left|H(e^{j\omega_1})\right| \cos\left(\omega_1 n + \theta(\omega_1)\right)$$

- Thus, the system acts like a lowpass filter

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§ 4.9 Phase and Group Delays

$$Y(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{\arg(H(e^{j\omega}))} \left|X(e^{j\omega})\right| e^{\arg(X(e^{j\omega}))}$$

$$= \left|H(e^{j\omega})\right| \left|X(e^{j\omega})\right| e^{\arg(H(e^{j\omega}))+\arg(X(e^{j\omega}))}$$

$$\arg\ Y(e^{j\omega}) = \arg(H(e^{j\omega})) + \arg(X(e^{j\omega}))$$

- **The output y[n] exhibits some delay relative to the input x[n] caused by the nonzero phase response $\theta(\omega)=\arg\{H(e^{j\omega})\}$ of the system**

§ 4.9 Phase and Group Delays

- **For an input**

$$x[n] = A\cos(\omega_o n + \phi), \qquad -\infty < n < \infty$$

the output is

$$y[n] = A\left|H(e^{j\omega_o})\right|\cos(\omega_o n + \theta(\omega_o) + \phi)$$

- **The output lags in phase by $\theta(\omega_0)$ radians**

$$y[n] = A\left|H(e^{j\omega_o})\right|\cos\left(\omega_o\left(n + \frac{\theta(\omega_o)}{\omega_o}\right) + \phi\right)$$

§ 4.9 Phase and Group Delays

- **This expression indicates a time delay, known as phase delay, at $\omega = \omega_0$ given by**

$$\tau_p(\omega_o) = -\frac{\theta(\omega_o)}{\omega_o}$$

- **The output y[n] is a time-delayed version of the input x[n] when** $\tau_p(\omega_o)$ **is an integer**

§ 4.9 Phase and Group Delays

When the input is composed of many sinusoidal components with different frequencies that are not harmonically related, each component will go through different phase delays.

The signal delay is defined as group delay

$$\tau_g(\omega) = -\frac{d\theta(\omega)}{d\omega}$$

本章重点

- 四个性质:线性，移不变性，因果性，稳定性
- 掌握上述性质的定义与时域判决方法
- 简单的系统互连方法
- FIR和IIR系统的时域表示
- 频率响应的求解方法

Homework

Problems:

4.3(b), 4.20(a), 4.23(a)(解卷积)，4.30(a)(互联结构), 4.67（第一问）

Matlab Exercises: M4.1