Digital Signal Processing

College of Communication & Information Engineering
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Chapter 5

Finite-Length Discrete Transforms

Part B: Operations on Finite-Length Sequences and DFT Properties

- **Circular Time-reversal of a Sequence （Section 2.3.1 ）**
- **Circular Shift of a Sequence（Section 2.3.2 and 5.7 ）**
- **Circular Convolution（Section 5.4 and 5.7 ）**
- **Classification of Finite-Length Sequences (Section 5.5)**
- **DFT Symmetry Relations and Theorems (Section 5.6 and 5.7)**
- **Fourier-Domain Filtering (Section 5.8)**
- **Computation of the DFT of Real Sequences (Section 5.9)**
- **Linear Convolution Using the DFT(Section 5.10)**

1、Circular Time-Reversal Operation

See in textbook Section 2.3.1

Modulo Operation(取模运算)

- **The time-reversal operation on a finite-length sequence is obtained by the modulo operation**

$$\langle m \rangle_N = m \text{ modulo } N$$

where m and N is any integer, and 0,1,…,N-1 be a set of N positive integers

- **$r = \langle m \rangle_N$ is called the residue(余数), which is an integer with a value between 0 and $N-1$**

$r=m+lN$, where l is a positive or negative integer chosen to make $m+lN$ an integer between 0 and $N-1$

1、Circular Time-Reversal Operation

- **The circular time-reversal version {*y*[*n*]} of a length-N sequence {*x*[*n*]} defined for $0 \le n \le N-1$ is given by** $\{y[n]\}=\{x[\langle -n \rangle_N]\}$

2、Circular Shift of a Sequence

See in Section 2.3.2 and 5.7

2、Circular Shift of a Sequence

- The desired shift, called the *circular shift*, is defined using a modulo operation:

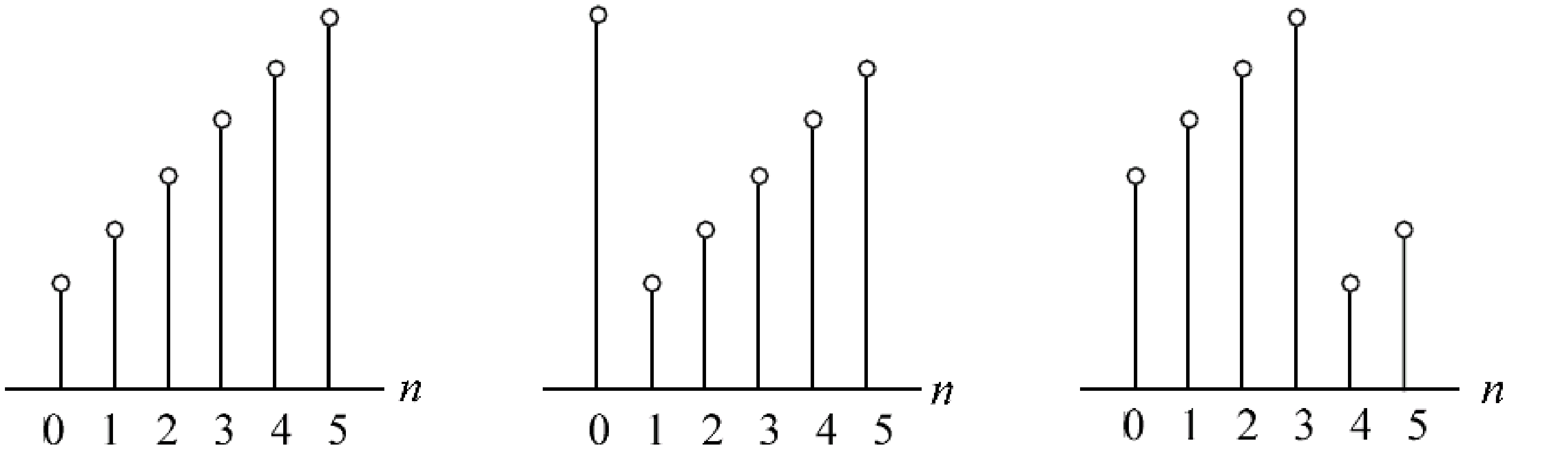
$$x_c(n) = x\left(\langle n - n_0 \rangle_N\right)$$

- For $n_0>0$ (*right circular shift*), the above equation implies

$$x_c(n) = \begin{cases} x(n-n_0), & for \;\; n_0 \le n \le N-1 \\ x(N-n_0+n), & for \;\; 0 \le n < n_0 \end{cases}$$

2、Circular Shift of a Sequence

Illustration of the concept of a circular shift

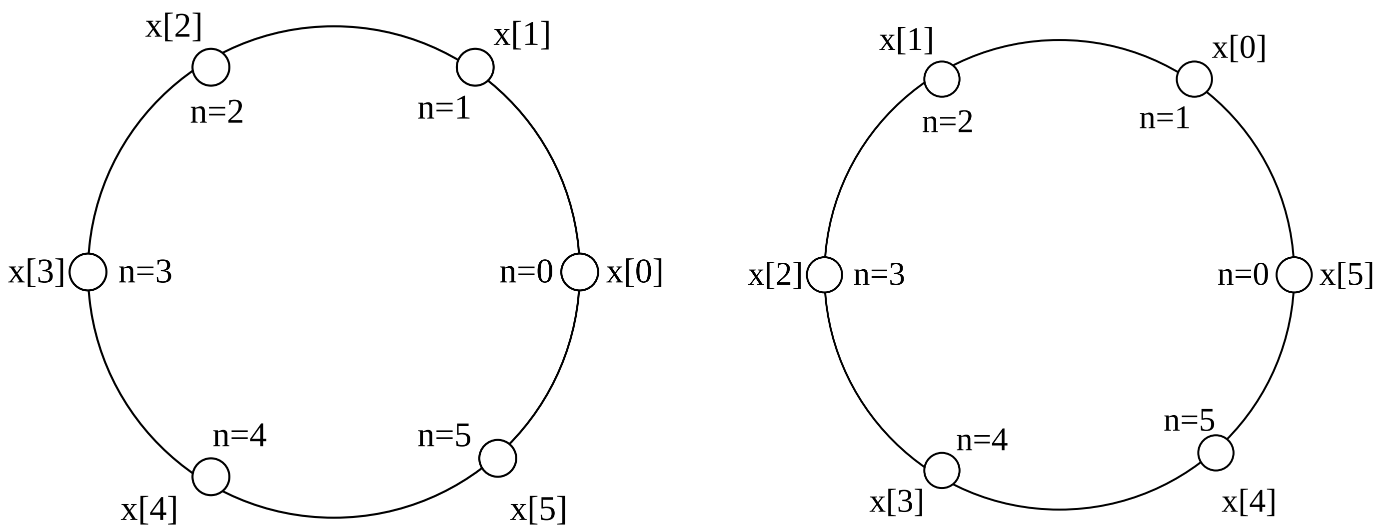


$x(n)$ $\quad x(\langle n-1\rangle_6)=x(\langle n+5\rangle_6)$ $\quad x(\langle n-4\rangle_6)=x(\langle n+2\rangle_6)$

- ***A right circular shift by n_0 is equivalent to a left circular shift by $N-n_0$ sample periods.***

2、Circular Shift of a Sequence

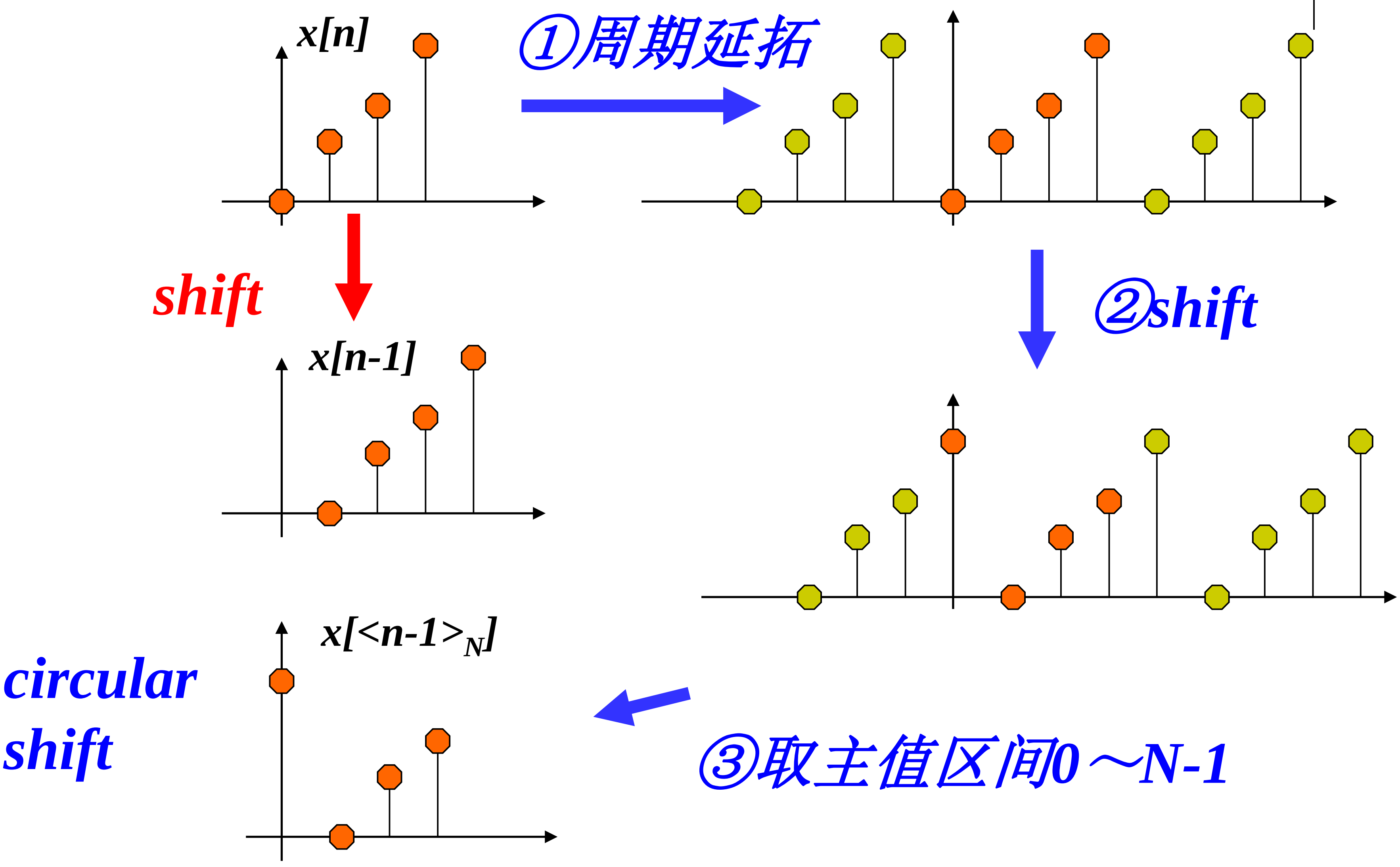
- **The length-*N* sequence is displayed on a circle at *N* equally spaced points**
- **The circular shift operation can be viewed as a clockwise or anti-clockwise rotation of the sequence by n_0 sample spacings**



(a) x[n]

(b) $x[\langle n-1\rangle_6]=x[\langle n+5\rangle_6]$

Compare the shift and circular shift



2、Circular Shift of a Sequence

- DFT of the circular shift sequence

$$y(n) = x\left(\langle n+m \rangle_N\right) R_N\left(\langle n+m \rangle_N\right)$$

$$Y(k) = DFT\left[y(n)\right]$$

$$= \sum_{n=0}^{N-1} x\left(\langle n+m \rangle_N\right) R_N(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} x\left(\langle n+m \rangle_N\right) W_N^{kn}$$

2、Circular Shift of a Sequence

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2、Circular Shift of a Sequence

$$Y(k) = \sum_{n'=m}^{N-1+m} x\left(\langle n' \rangle_N\right) W_N^{k(n'-m)}$$

$$= W_N^{-km} \sum_{n'=m}^{N-1+m} x\left(\langle n' \rangle_N\right) W_N^{kn'}$$

$$= W_N^{-km} \left(\sum_{n'=0}^{N-1} (.) - \underline{\sum_{n'=0}^{m-1} (.) + \sum_{n'=N}^{N-1+m} (.)} \right)$$

$$= W_N^{-km} \sum_{n'=0}^{N-1} \underline{x\left(\langle n' \rangle_N\right)} W_N^{kn'}$$

$$= W_N^{-km} \sum_{n'=0}^{N-1} x(n') W_N^{kn'} = W_N^{-km} X(k)$$

3、Circular Convolution

See in Section 5.4 and 5.7

3、Circular Convolution

- To develop a convolution-like operation resulting in a length-N sequence $y_C(n)$, we need to define a circular time-reversal, and then apply a circular time-shift.
- Resulting operation, called a *circular convolution*, is defined by

$$y_C(n) = \sum_{m=0}^{N-1} g(m) h\left(\langle n-m \rangle_N\right), \quad 0 \le n \le N-1$$

3、Circular Convolution

- Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as

$$y_C(n)=g(n) \circledN h(n)$$

- The circular convolution is commutative, i.e.

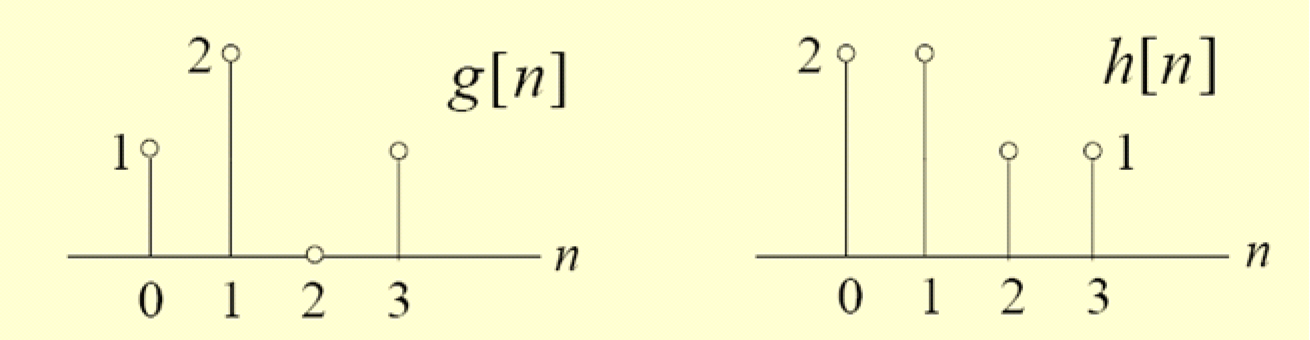
$$g(n) \circledN h(n)=h(n) \circledN g(n)$$

3、Circular Convolution

- **Example** Determine the 4-point circular convolution of the two length-4 sequences:

$$g[n]=\{\underset{\uparrow}{1}\ \ 2\ \ 0\ \ 1\},\qquad h[n]=\{\underset{\uparrow}{2}\ \ 2\ \ 1\ \ 1\}$$

as skecthed below



3、Circular Convolution

- **The result is a length-4 sequence $y_C[n]$**

$$y_C(n) = g(n) \circledN h(n) = \sum_{m=0}^{N-1} g(m) h\left(\langle n-m \rangle_N\right), 0 \le n \le N-1;$$

- **From the above we observe**

$$y_C(0) = \sum_{m=0}^{3} g(m) h\left(\langle -m \rangle_4\right)$$

$$= g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1]$$

$$= (1 \times 2) + (2 \times 1) + (0 \times 1) + (1 \times 2) = 6$$

3、Circular Convolution

- **Likewise,** $y_C(1)=\sum_{m=0}^{3} g(m)h\left(\langle 1-m\rangle_4\right)$

$$= g[0]h[1]+g[1]h[0]+g[2]h[3]+g[3]h[2]$$

$$=\left(1\times 2\right)+\left(2\times 2\right)+\left(0\times 1\right)+\left(1\times 1\right)=7$$

$$y_C(2)=\sum_{m=0}^{3} g(m)h\left(\langle 2-m\rangle_4\right)$$

$$= g[0]h[2]+g[1]h[1]+g[2]h[0]+g[3]h[3]$$

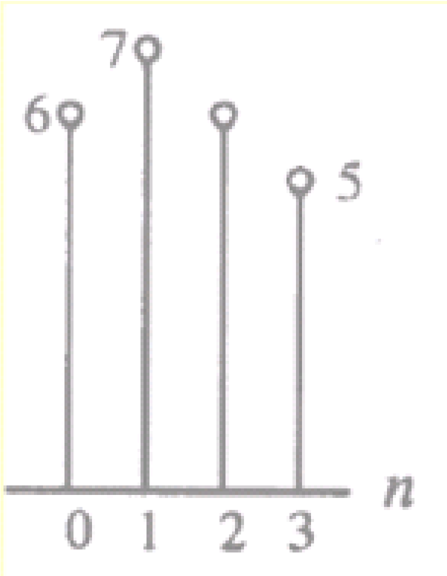
$$=\left(1\times 1\right)+\left(2\times 2\right)+\left(0\times 2\right)+\left(1\times 1\right)=6$$

3、Circular Convolution

$$y_C(3) = \sum_{m=0}^{3} g(m) h\left(\langle 3-m \rangle_4\right)$$

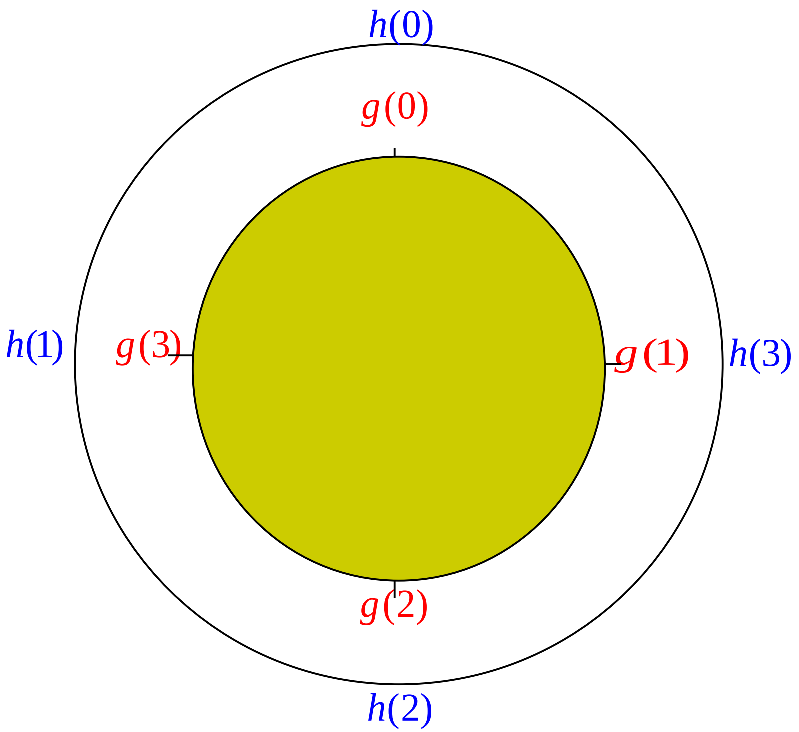
$$= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5$$

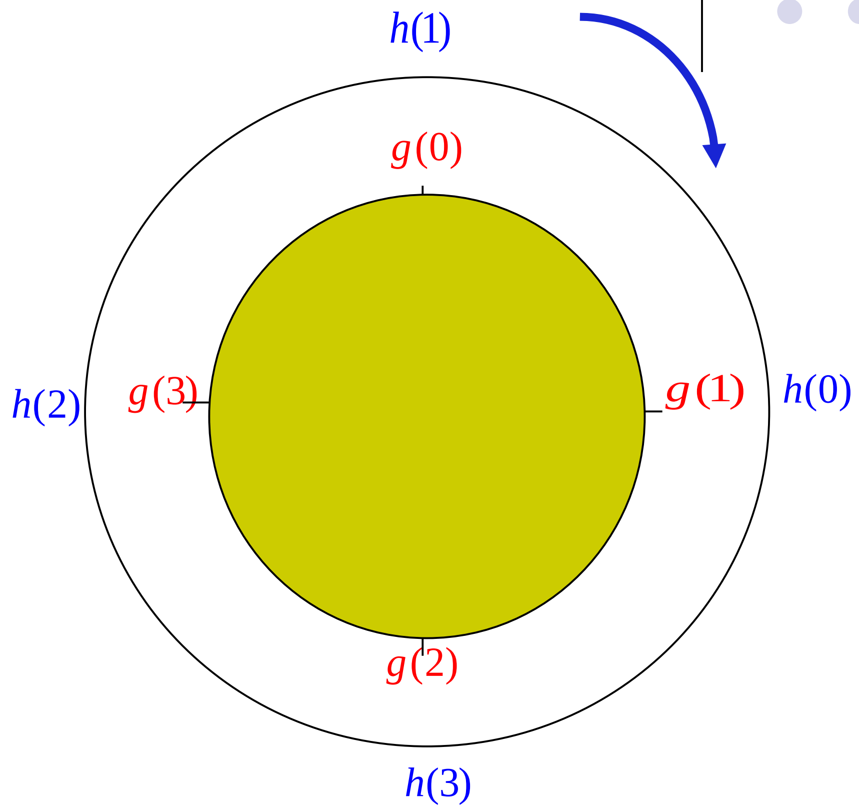


$y_C[n]$

循环卷积过程图解



(a) $y_c[0]$



(b) $y_c[1]$

3、Circular Convolution

- **The circular convolution can also be computed using a DFT-based approach**
- **The N-point circular convolution can be written in matrix form as**

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ \vdots \\ y_C[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

3、Circular Convolution

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ \vdots \\ y_C[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

Note: 1、The element in each row of the matrix are obtained by circularly rotating the elements of the previous row to the right by one position. Such a matrix is called a circulant matrix(轮换矩阵、 循环行列式矩阵)

2、使用矩阵形式计算循环卷积前，需要通过补零把参与循环卷积的两个输入序列扩充成相同长度，且此长度等于DFT的点数

3、Circular Convolution

- **Example** **Now let us extend the two length-4 sequences to length 7 by appending each with three zero-valued samples, i.e.,**

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

3、Circular Convolution

$$g_e(n) = \{1, 2, 0, 1, 0, 0, 0\} \qquad 0 \le n \le 6$$

$$h_e(n) = \{2, 2, 1, 1, 0, 0, 0\} \qquad 0 \le n \le 6$$

- **We next determine the 7-point circular convolution of $g_e[n]$ and $h_e[n]$:**

$$y_C(n) = \sum_{m=0}^{6} g_e(m) h_e\left(\langle n-m \rangle_7\right), 0 \le n \le 6$$

Matrix method:

$$Y = \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ y(N-1) \end{pmatrix} = H_e G_e$$

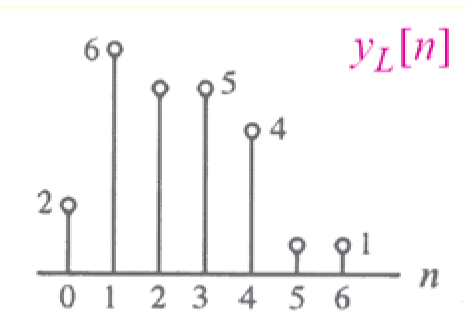
$$G_e = \begin{pmatrix} g_e(0) \\ g_e(1) \\ g_e(2) \\ \vdots \\ g_e(N-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

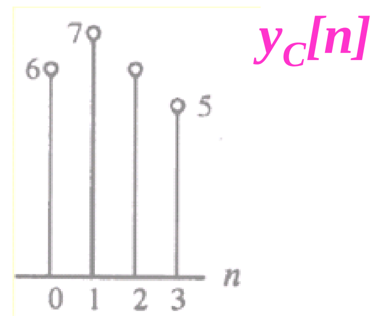
$$H_e = \begin{pmatrix} h_e(0) & h_e(N-1) & h_e(N-2) \cdots h_e(1) \\ h_e(1) & h_e(0) & h_e(N-1) \cdots h_e(2) \\ h_e(2) & h_e(1) & h_e(0) \quad \cdots h_e(3) \\ \vdots & \vdots & \vdots \quad\quad \vdots \\ h_e(N-1) & h_e(N-2) & h_e(N-3) \cdots h_e(0) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{pmatrix}$$

$$\therefore \begin{pmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \\ y(5) \\ y(6) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 5 \\ 5 \\ 4 \\ 1 \\ 1 \end{pmatrix}$$

3、Circular Convolution

- **As can be seen from the above that y[n] is precisely the sequence $y_L[n]$ obtained by a linear convolution of g[n] and h[n]**





- **Try to think:**

What is the relation between the circular convolution and the linear convolution?

线性卷积的运算过程

假设g(n)和 h(n)为有限长序列，长度分别为L和K，那么它们的线性卷积

$$y(n)=g(n)*h(n)=\sum_{m=-\infty}^{\infty}g(m)h(n-m)$$

是一个长度为L+K-1的有限长序列。

注意：对卷积序列的长度没有严格要求

卷积过程：反转平移法

序列x(n)={1,2,3}, h(n)={1,2}

求两个序列的线性卷积y(n)=x(n)*h(n)

x(n)		1	2	3		
h(n)		1	2			
反转 h(-n)	2	1				y(0)=1
平移 h(1-n)		2	1			y(1)=4
平移 h(2-n)			2	1		y(2)=7
平移 h(3-n)				2	1	y(3)=6

周期卷积 periodic convolution

- 周期卷积（periodic convolution）定义（P234）

$$\tilde{y}(n)=\sum_{r=0}^{N-1}\tilde{x}[r]\tilde{h}[n-r]$$

- 注意：三个序列均为周期序列，周期为N
- 卷积过程：反转平移法，过程同线性卷积

序列x(n)={1,2,3}, h(n)={1,2,0} 周期N＝3

求两个序列的周期卷积g(n)

x’(n)	1	2	3	1	2	3	1	2	3	
h’(n)	1	2	0	1	2	0	1	2	0	
反转 h’(-n)	1	0	2	1	0	2	1	0	2	g(0)=7
平移 h’(1-n)	2	1	0	2	1	0	2	1	0	g(1)=4
平移 h’(2-n)	0	2	1	0	2	1	0	2	1	g(2)=7

g(n)是周期序列，主值区间的序列值为{7,4,7}

借助周期卷积求循环卷积

重新审视循环卷积的过程，所有的运算都是在一个主值区间里完成，根据DFT运算的周期性，可以借助周期卷积来实现循环卷积运算：

1）由有限长序g(n)和 h(n)构造周期序列 $\tilde{g}(n)$ $\tilde{h}(n)$

2）计算周期卷积： $$\tilde{y}(n)=\sum_{m=0}^{N-1}\tilde{g}(m)\tilde{h}(n-m)$$

3）卷积结果取主值： $$y_c(n)=\tilde{y}(n)R_N(n)$$

序列x(n)={1,2,3}, h(n)={1,2} N＝3

求两个序列的循环卷积y1(n)

x’(n)	1 2 3	1 2 3	1 2 3	
h’(n)	1 2 0	1 2 0	1 2 0	
反转 h’(-n)	1 0 2	1 0 2	1 0 2	y1(0)=7
平移 h’(1-n)	2 1 0	2 1 0	2 1 0	y1(1)=4
平移 h’(2-n)	0 2 1	0 2 1	0 2 1	y1(2)=7

y1(n)是有限长序列，序列值为{7,4,7}

序列x(n)={1,2,3}, h(n)={1,2} N＝4

求两个序列的循环卷积y2(n)

	x'(n)	2	3	0	1	2	3	0	1	
	h'(n)	2	0	0	1	2	0	0	1	
反转	h'(-n)	0	0	2	1	0	0	2	1	y2(0)=1
平移	h'(1-n)	1	0	0	2	1	0	0	2	y2(1)=4
平移	h'(2-n)	2	1	0	0	2	1	0	0	y2(2)=7
平移	h'(3-n)	0	2	1	0	0	2	1	0	y2(3)=6

*y2(n)*是有限长序列，序列值为{1,4,7,6}

借助周期卷积求循环卷积

对于重新构造两个有限长序列$g_e(n)$、$h_e(n)$来说，其周期延拓后的序列为：

$$\tilde{g}_e(n)=\sum_{q=-\infty}^{\infty} g_e(n+qN)$$

$$\tilde{h}_e(n)=\sum_{r=-\infty}^{\infty} h_e(n+rN)$$

借助周期卷积求循环卷积

它们的周期卷积序列为：

$$\tilde{y}(n)=\sum_{m=0}^{N-1}\tilde{g}_e(m)\tilde{h}_e(n-m)=\sum_{m=0}^{N-1}g_e(m)\tilde{h}_e(n-m)$$

$$=\sum_{m=0}^{N-1}g_e(m)\sum_{r=-\infty}^{+\infty}h_e(n+rN-m)$$

$$=\sum_{r=-\infty}^{\infty}\sum_{m=0}^{N-1}g_e(m)h_e(n+rN-m)$$

$$=\sum_{r=-\infty}^{\infty}y(n+rN)$$

从中可以看出，$g_e(n)$、$h_e(n)$周期延拓后的周期卷积是$g_e(n)$、$h_e(n)$线性卷积的周期延拓，周期为N。

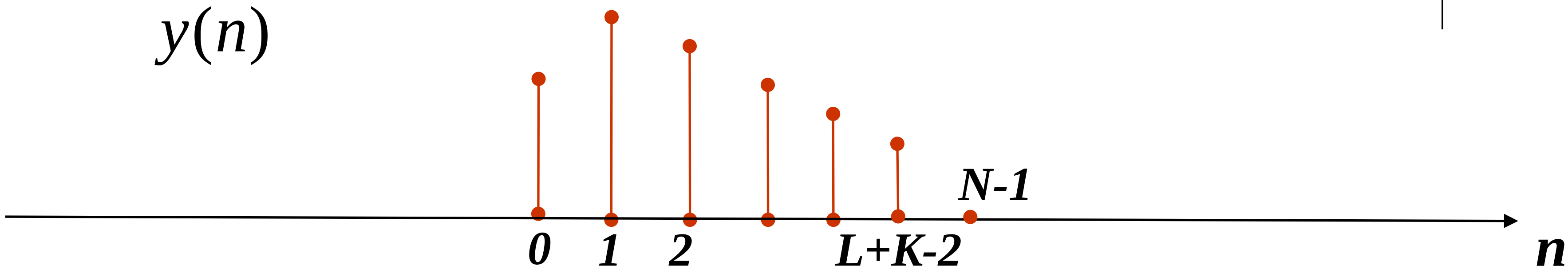
借助周期卷积求循环卷积

- y(n)具有L+K-1个非零序列值，如果周期卷积的周期N<L+K-1，那么，y(n)周期延拓后，必然有一部分非零序列值要重叠，出现混淆现象。
- 只有N≥L+K-1时，才不会产生交叠，这时y(n)的周期延拓中每一个周期N内，前L+K-1个序列值是y(n)的全部非零序列值，而剩下的N-(L+K-1)点的序列则是补充的零值。
- 循环卷积正是周期卷积取主值序列：

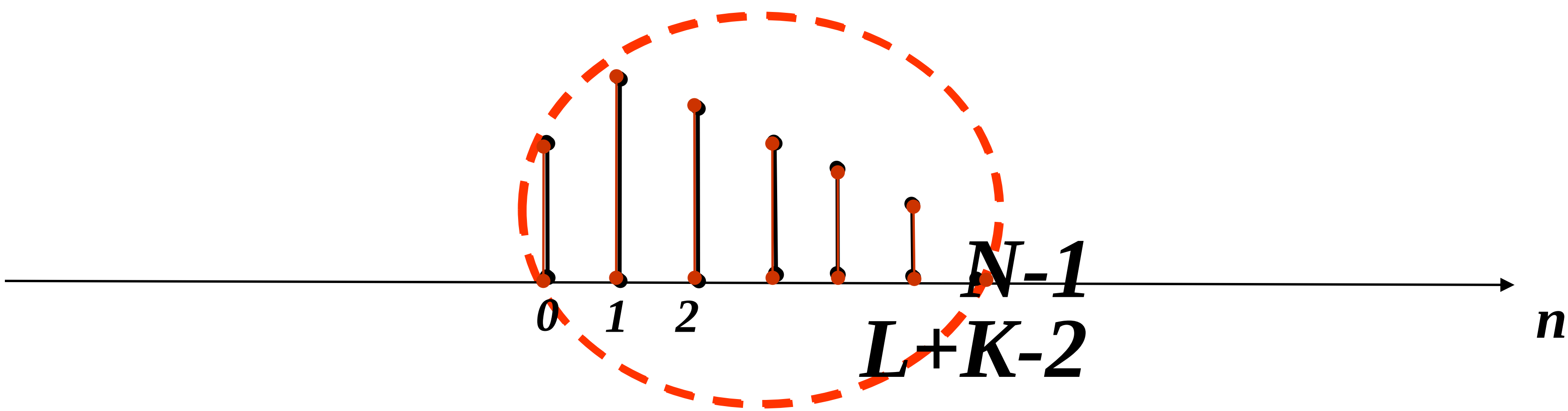
$$y_c(n)=\tilde{y}(n)R_N(n)=\left[\sum_{r=-\infty}^{\infty}y(n+rN)\right]R_N(n)$$

$$\tilde{y}(n) = \sum_{r=-\infty}^{\infty} y(n+rN) \qquad N \geq L+K-1$$

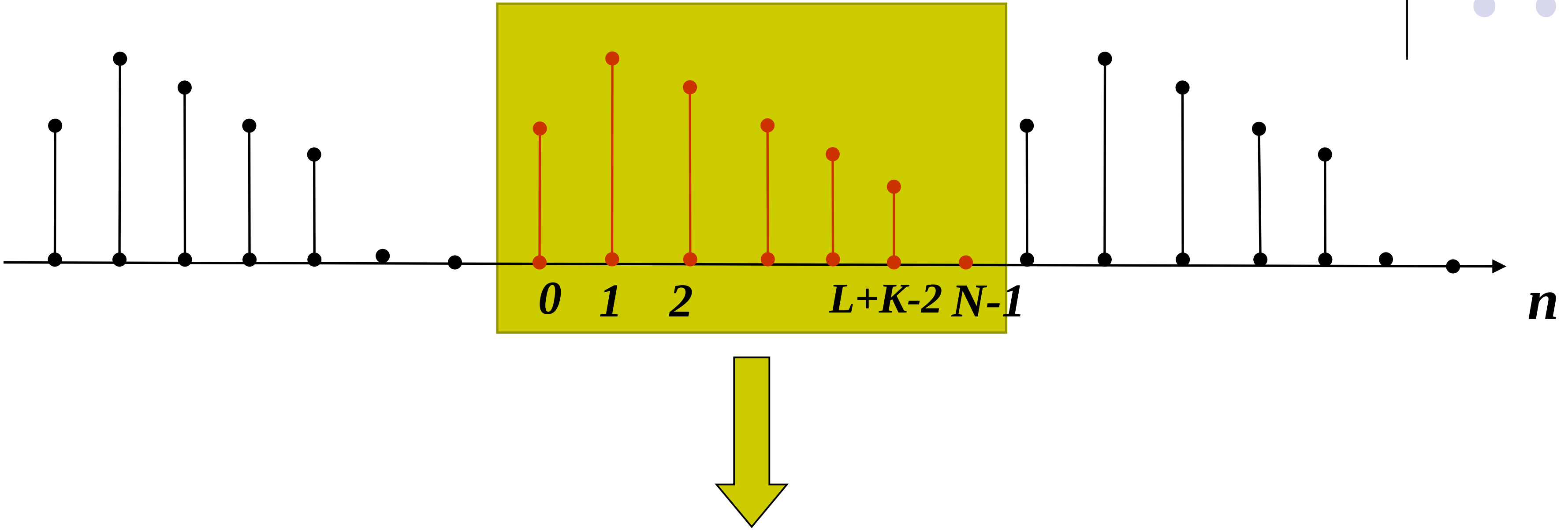
$y(n)$

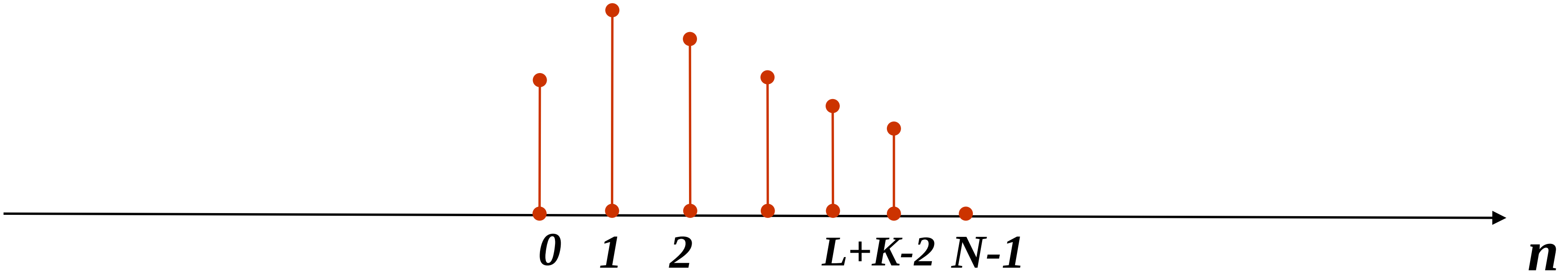


$\tilde{y}(n)$



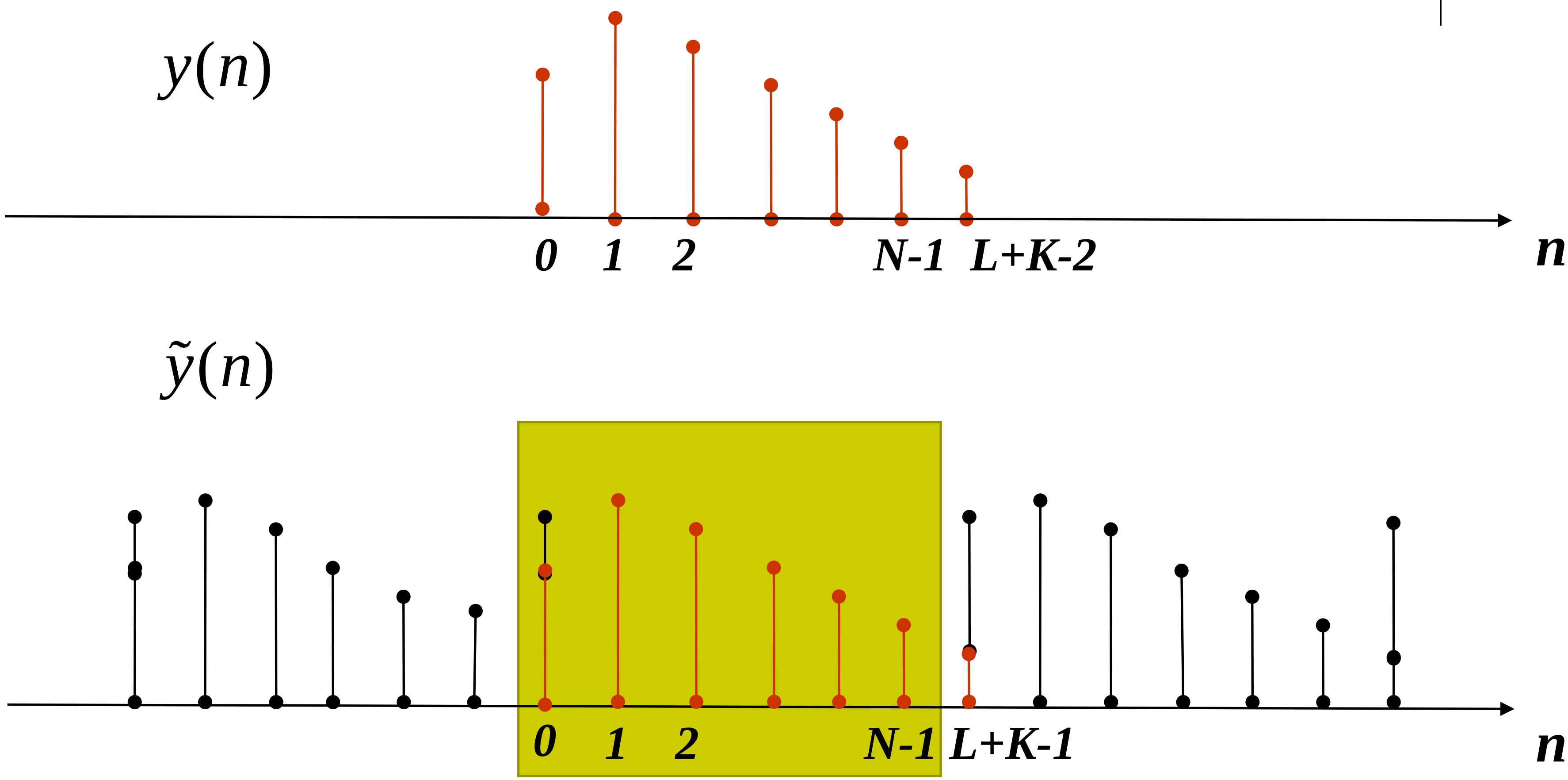
$$y_c(n) = \tilde{y}(n) R_N(n) = \left[\sum_{r=-\infty}^{\infty} y(n+rN) \right] R_N(n)$$





$$y_c(n) = y(n), \quad 0 \le n \le N\text{-}1$$

$$\tilde{y}(n) = \sum_{r=-\infty}^{\infty} y(n+rN) \qquad N < L+K-1$$



$$y_c(n) \neq y(n), \quad 0 \leq n \leq N\text{-}1$$

循环卷积与线性卷积的关系

- 循环卷积等于线性卷积而不产生混淆的必要条件是：

$$N \ge L + K - 1$$

循环卷积的点数 (N)　　线性卷积结果的长度 ($L + K - 1$)

循环卷积比起线性卷积，在运算速度上有很大的优越性，可采用快速傅里叶变换（FFT）技术。若能利用循环卷积求线性卷积，会带来很大的方便。

4、Classification of Finite-Length Sequences

See in textbook section 5.5

4、Classification of Finite-Length Sequences

- **Based on Conjugate Symmetry** （see in Section 5.5.1）

A complex DFT $X[k]$ can be expressed as a sum of a circular conjugate symmetric part $X_{cs}[k]$ and a circular conjugate anti-symmetric $X_{ca}[k]$ part

$$X[k] = X_{cs}[k] + X_{ca}[k],\ 0 \le k \le N-1$$

Where

$$X_{cs}[k] = \tfrac{1}{2}(X[k] + X^*[\langle -k \rangle_N])\quad 0 \le k \le N-1$$

$$X_{ca}[k] = \tfrac{1}{2}(X[k] - X^*[\langle -k \rangle_N])\quad 0 \le k \le N-1$$

4、Classification of Finite-Length Sequences

- **Based on Conjugate Symmetry（see in Section 5.5.1）**
  - ✓ **An N-point DFT $X[k]$ is said to be a circular conjugate-symmetric sequence if**

$$X[k]= X^*[\langle -k\rangle_N]=X^*[\langle N-k\rangle_N]$$

  - ✓ **An N-point DFT $X[k]$ is said to be a circular conjugate-anti-symmetric sequence if**

$$X[k]= -X^*[\langle -k\rangle_N]=-X^*[\langle N-k\rangle_N]$$

4、Classification of Finite-Length Sequences

- **Based on Geometric Symmetry** （see in Section 5.5.2）
  - **A length-*N* *symmetry* sequence *x*(*n*) satisfies the condition**

$$x(n) = x(N-1-n)$$

  - **A length-*N* *antisymmetry* sequence *x*(*n*) satisfies the condition**

$$x(n) = -x(N-1-n)$$

5、DFT Symmetry Relations and DFT Theorems

See in textbook section 5.6 and 5.7

Table 5.1: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{pcs}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$j\,\mathrm{Im}\{x[n]\}$	$X_{\mathrm{pca}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{\mathrm{pcs}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{pca}}[n]$	$j\,\mathrm{Im}\{X[k]\}$

Note: $x_{\mathrm{pcs}}[n]$ and $x_{\mathrm{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\mathrm{pcs}}[k]$ and $X_{\mathrm{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

$x[n]$ is a complex sequence

用DFT定义证明：

$$DFT\left(x[n]\right) = X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}, 0 \le k \le N-1$$

$$DFT\left(x^*[n]\right) = \sum_{n=0}^{N-1} x^*[n] W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \left(x[n] W_N^{-nk}\right)^* = \left(\sum_{n=0}^{N-1} x[n] W_N^{-nk}\right)^*$$

$$= X^*\left[\langle -k \rangle_N\right], 0 \le k \le N-1$$

Table 5.1: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$\text{Re}\{x[n]\}$	$X_{\text{pcs}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$j\,\text{Im}\{x[n]\}$	$X_{\text{pca}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{\text{pcs}}[n]$	$\text{Re}\{X[k]\}$
$x_{\text{pca}}[n]$	$j\,\text{Im}\{X[k]\}$

Note: $x_{\text{pcs}}[n]$ and $x_{\text{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\text{pcs}}[k]$ and $X_{\text{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

$x[n]$ is a complex sequence

用DFT定义证明：

$$DFT\left(x^*[<-n>_N]\right) = \sum_{n=0}^{N-1} x^*[<-n>_N]W_N^{nk}$$

$$= \sum_{n=0}^{N-1} \left(x[<-n>_N]W_N^{-nk}\right)^*$$

$$= \left(\sum_{n=0}^{N-1} x[<N-n>_N]W_N^{(N-n)k}\right)^*$$

$$= \left(\sum_{m=1}^{N} x[m_N]W_N^{mk}\right)^* = \left(\sum_{m=0}^{N-1} x[m_N]W_N^{mk}\right)^* = \left(\sum_{m=0}^{N-1} x[m]W_N^{mk}\right)^*$$

$$= X^*[k], 0 \le k \le N-1$$

Table 5.1: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$\text{Re}\{x[n]\}$	$X_{\text{pcs}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$j\,\text{Im}\{x[n]\}$	$X_{\text{pca}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{\text{pcs}}[n]$	$\text{Re}\{X[k]\}$
$x_{\text{pca}}[n]$	$j\,\text{Im}\{X[k]\}$

Note: $x_{\text{pcs}}[n]$ and $x_{\text{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\text{pcs}}[k]$ and $X_{\text{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

$x[n]$ is a complex sequence

用DFT定义证明：

$$DFT\left(\mathrm{Re}\{x[n]\}\right) = DFT\left(\frac{1}{2}\{x[n]+x^*[n]\}\right)$$

$$=\frac{1}{2}\left\{X[k]+X^*[<-k>_N]\right\}$$

Table 5.2: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\}$
$x_{pe}[n]$	$\text{Re}\{X[k]\}$
$x_{po}[n]$	$j\,\text{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k \rangle_N]$ $\text{Re}\,X[k] = \text{Re}\,X[\langle -k \rangle_N]$ $\text{Im}\,X[k] = -\text{Im}\,X[\langle -k \rangle_N]$ $\lvert X[k] \rvert = \lvert X[\langle -k \rangle_N] \rvert$ $\arg X[k] = -\arg X[\langle -k \rangle_N]$

Note: $x_{pe}[n]$ and $x_{po}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

$x[n]$ is a real sequence

5.43 Since $x[n]$ is a length-11 real sequence, its DFT satisfies $X[k] = X^*[\langle -k \rangle_{11}]$. Thus:

$$X[1] = X^*[\langle -1 \rangle_{11}] = X^*[10] = 1.5 + j5.31,$$
$$X[3] = X^*[\langle -3 \rangle_{11}] = X^*[8] = -3.34 - j3.69,$$
$$X[5] = X^*[\langle -5 \rangle_{11}] = X^*[6] = -7.55 - j13.69,$$
$$X[7] = X^*[\langle -7 \rangle_{11}] = X^*[4] = -12.44 - j12.7,$$
$$X[9] = X^*[\langle -9 \rangle_{11}] = X^*[2] = 2.49 + j19.12.$$

Table 5.2: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = \text{Re}\{X[k]\} + j\,\text{Im}\{X[k]\}$
$x_{pe}[n]$	$\text{Re}\{X[k]\}$
$x_{po}[n]$	$j\,\text{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k \rangle_N]$ $\text{Re}\, X[k] = \text{Re}\, X[\langle -k \rangle_N]$ $\text{Im}\, X[k] = -\text{Im}\, X[\langle -k \rangle_N]$ $\lvert X[k] \rvert = \lvert X[\langle -k \rangle_N] \rvert$ $\arg X[k] = -\arg X[\langle -k \rangle_N]$

Note: $x_{pe}[n]$ and $x_{po}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

$x[n]$ is a real sequence

5.45 Since the DFT $X[k]$ is real-valued, $x[n]$ is circularly even: $x[n] = x[\langle -n \rangle_{10}]$. Therefore:

$$x[2] = x[\langle -2 \rangle_{10}] = x[8] = 6.26,$$
$$x[6] = x[\langle -6 \rangle_{10}] = x[4] = -3.1,$$
$$x[7] = x[\langle -7 \rangle_{10}] = x[3] = 8.58,$$
$$x[9] = x[\langle -9 \rangle_{10}] = x[1] = 6.2.$$

Table 5.3: General Properties of DFT

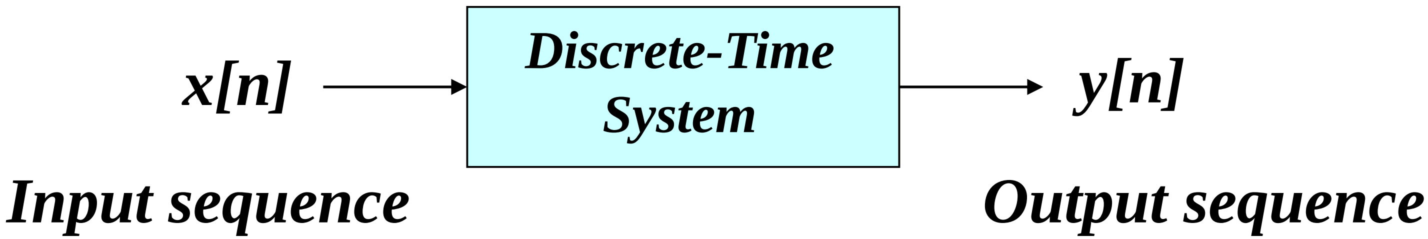
Type of Property	Length-N Sequence	N-point DFT
	$g[n]$ $h[n]$	$G[k]$ $H[k]$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G[k] + \beta H[k]$
Circular time-shifting	$g[\langle n - n_o \rangle_N]$	$W_N^{kn_o} G[k]$
Circular frequency-shifting	$W_N^{-k_o n} g[n]$	$G[\langle k - k_o \rangle_N]$
Duality	$G[n]$	$N g[\langle -k \rangle_N]$
N-point circular convolution	$\sum_{m=0}^{N-1} g[m] h[\langle n - m \rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N} \sum_{m=0}^{N-1} G[m] H[\langle k - m \rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2 = \frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$	

6、Fourier-Domain Filtering

See in textbook section 5.8

6、Fourier-Domain Filtering

- **Often one is interested in removing the components of a finite-length discrete-time signal in one or more frequency bands.**



$$y[n] = x[n] \circledast h[n]$$

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$Y[k] = X[k] \cdot H[k]$$

length L *length K* *length L+K-1*

$$x(n) \circledast h(n) = y_L(n) \xleftrightarrow{FT} Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega})$$

$$\updownarrow \qquad \updownarrow \text{ Sampling (N samples in a period)}$$

$$x(n) \, Ⓝ \, h(n) = y_c(n) \xleftrightarrow[N \;\; points]{DFT} Y(k) = X(k) \cdot H(k)$$

N points *N points* *length N*

7、Linear Convolution Using the DFT

See in textbook section 5.10

7、Linear Convolution Using the DFT

- ***Linear convolution*** **is a key operation in many signal processing applications.**
- **Since a DFT can be efficiently implemented using FFT algorithms, it is of interest to develop methods for the implementation of linear convolution using the DFT**

Linear Convolution of Two Finite-Length Sequences

- Let $g(n)$ and $h(n)$ be two finite-length sequences of length N and M, respectively
- Denote $L = N + M - 1$
- Define two length-L sequences

$$g_e(n) = \begin{cases} g(n), & 0 \le n \le N-1 \\ 0, & N \le n \le L-1 \end{cases}$$

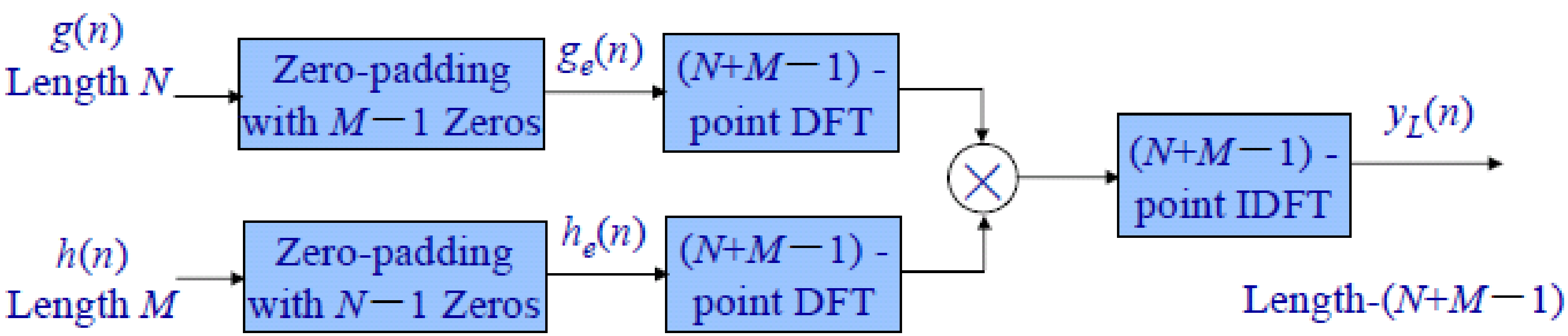
$$h_e(n) = \begin{cases} h(n), & 0 \le n \le M-1 \\ 0, & M \le n \le L-1 \end{cases}$$

Linear Convolution of Two Finite-Length Sequences

- Then

$$y_L(n) = g(n) \circledast h(n) = g(n) \, \textcircled{\scriptsize L} \, h(n)$$

- The corresponding implementation scheme is illustrated below



Linear Convolution of a Finite Sequence with an Infinite Sequence

Overlap-Add Method

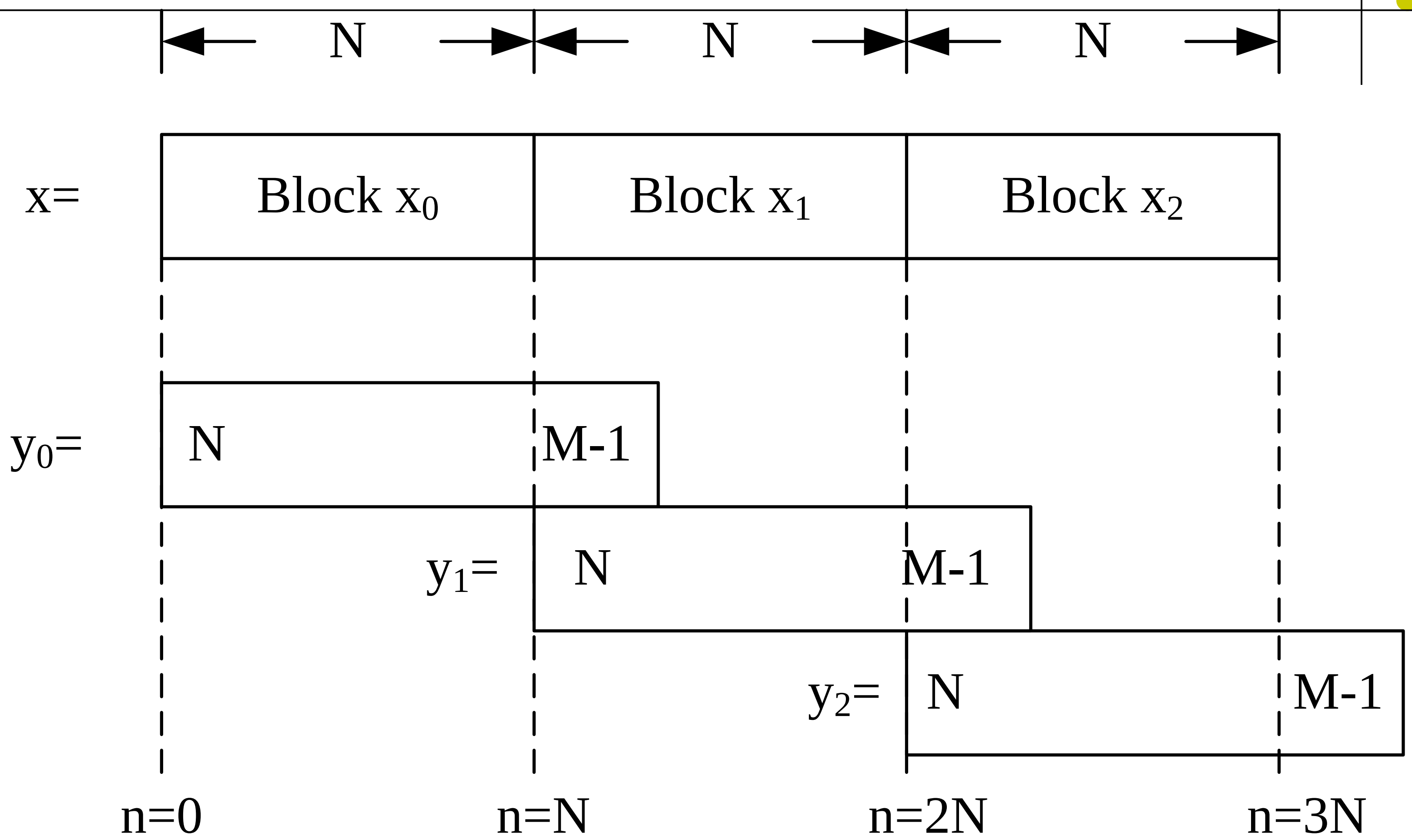
Overlap-Save Method

Overlap-Add Method

Overlap-add method: When the input x is infinite or extremely long, divide the long input into contiguous non-overlapping blocks $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ **of manageable length, then filter each block and add the output overlapped blocks**

$$\mathbf{y}_0 = \mathbf{h} * \mathbf{x}_0,$$
$$\mathbf{y}_1 = \mathbf{h} * \mathbf{x}_1,$$
$$\mathbf{y}_2 = \mathbf{h} * \mathbf{x}_2,$$
$$\ldots$$

to obtain the overall output.


N
N
N
x=
Block x0
Block x1
Block x2
y0=
N
M-1
y1=
N
M-1
y2=
N
M-1
n=0
n=N
n=2N
n=3N

$$\mathbf{x} = [\underbrace{1,1,2}_{\mathbf{x}_0},\underbrace{1,2,2}_{\mathbf{x}_1},\underbrace{1,1,0}_{\mathbf{x}_2}]$$

	block 0			block 1			block 2		
h\x	1	1	2	1	2	2	1	1	0
1	1	1	2	1	2	2	1	1	0
2	2	2	4	2	4	4	2	2	0
-1	-1	-1	-2	-1	-2	-2	-1	-1	0
1	1	1	2	1	2	2	1	1	0

$$\mathbf{y}_0 = \mathbf{h} * \mathbf{x}_0 = [1,3,3,4,-1,2]$$

$$\mathbf{y}_1 = \mathbf{h} * \mathbf{x}_1 = [1,4,5,3,0,2]$$

$$\mathbf{y}_2 = \mathbf{h} * \mathbf{x}_2 = [1,3,1,0,1,0]$$

Overlap-add, M=4

n	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{y}_0$	1	3	3	4	-1	2					
$\mathbf{y}_1$				1	4	5	3	0	2		
$\mathbf{y}_2$							1	3	1	0	1
$\mathbf{y}$	1	3	3	5	3	7	4	3	3	0	1

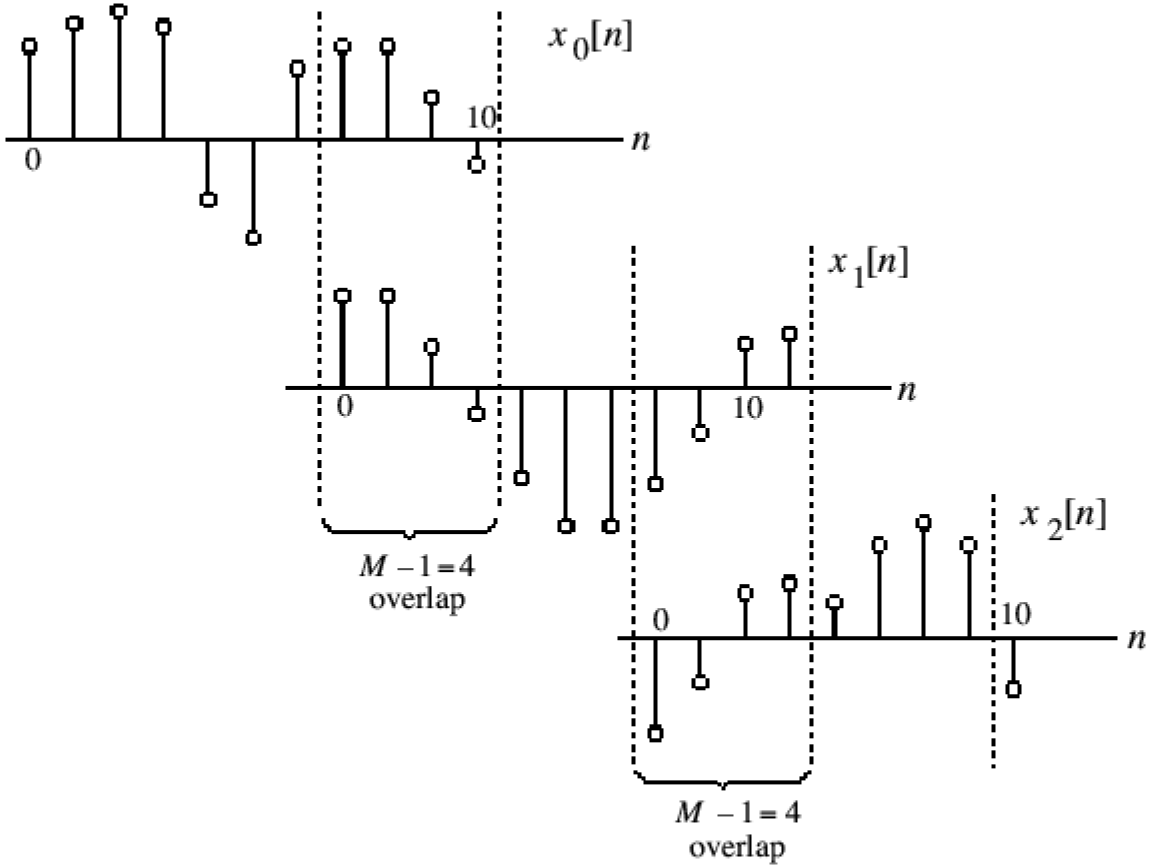
Overlap-Save Method

- 基本思想

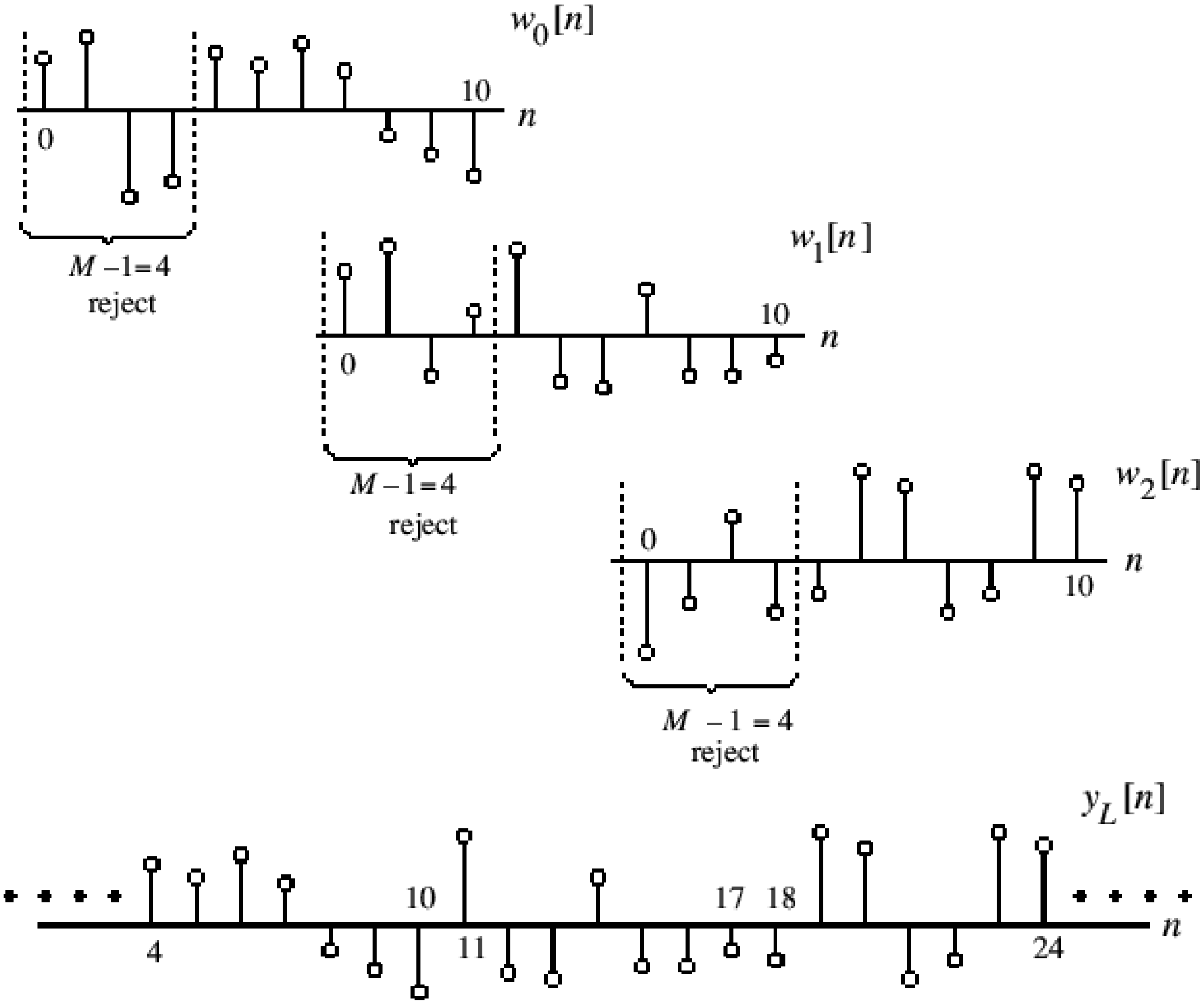
segment $x[n]$ into overlapping blocks , keep the $y_m[n]$ terms of the circular convolution of $h[n]$ with $x_m[n]$ that corresponds to the terms obtained by a linear convolution of $h[n]$ and $x_m[n]$, and throw away the other parts of the circular convolution

Overlap-Save Method

- **Process is illustrated next**



Overlap-Save Method



本章重点

- DFT的定义、性质及其证明
- 循环卷积、循环卷积与线性卷积的关系
- 实序列的DFT
- 线性卷积的DFT实现
- 重叠相加法、重叠保留法

Homework

Problems:

5.2(a), 5.9(a), 5.25, 5.28, 5.43, 5.45, 5.55, 5.68, 5.76(a,b,c,d)