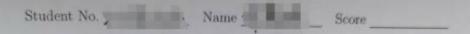
2022 Midterm Exam Algorithm Design and Analysis



Problem No.	1	2	3	4	5	6
Points	15	20	15	20	15	15

Problem 1

Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$.

Problem 2

Given two arrays $A[1,\ldots,n]$ and $B[1,\ldots,n]$ containing n and n integers, respectively, compute the pair of values $\langle A[i],B[j]\rangle$ (one value in each array) with the smallest (non-negative, i.e., |A[i]-B[j]|) difference. Return the difference. Your algorithm should use $O(n\log n)$ time. For example,

- \bullet Input: $A[\,] = \{1,3,15,11,2\}$ and $B[\,] = \{23,127,235,19,8\}$
- Output: 3. That is, the pair (11, 8)

Problem 3

Describe an algorithm that, given n integers (a_1, \ldots, a_n) in the range 1 to k, preprocesses its input and then answers any query about what is the l-th $(1 \le l \le k)$ smallest element in O(1) time. Your algorithm should use $\Theta(n+k)$ preprocessing time.

Problem 4

Recall that a standard (FIFO) queue maintains a sequence of items subject to the following operations.

- Push(x): Add item x to the end of the sequence.
- Pull(): Remove and return the item at the beginning of the sequence.
- Size(): Return the current number of items in the sequence.
- (a) Design an algorithm with proper data structure to implement a queue, satisfying that it uses O(n) space (where n is the number of items in the queue) and the worst-case time for each of these operations is O(1).
- (b) Consider the following new operation, which removes every tenth element from the queue, starting at the beginning, in $\Theta(n)$ worst-case time.

```
function Decimate()
n \leftarrow \mathsf{Size}();
\mathsf{for}\ i \leftarrow 0\ \mathsf{to}\ n-1\ \mathsf{do}
\mathsf{if}\ i \ \mathsf{mod}\ 10 = 0\ \mathsf{then}
\mathsf{Pull}();
\mathsf{Size}() \leftarrow \mathsf{Size}() - 1;
\mathsf{else}
\mathsf{Push}(\mathsf{Pull}());
\mathsf{end}\ \mathsf{if}
\mathsf{end}\ \mathsf{for}
\mathsf{end}\ \mathsf{function}
```

Prove that in any intermixed sequence of Push, Pull, and Decimate operations, the amortized cost of each operation is O(1).

Problem 5

Given the root r of a binary tree, return the length of the diameter of the tree. The diameter of a binary tree is the length of the longest path between any two nodes in a tree. This path may or may not pass through the root r. The length of a path between two nodes is represented by the number of edges between them.