

2022 Midterm Exam

Algorithm Design and Analysis

Student No. _____ Name _____ Score _____

Problem No.	1	2	3	4	5	6
Points	15	20	15	20	15	15

Problem 1

Solve the recurrence $T(n) = 3T(\sqrt{n}) + \log n$.

Problem 2

Given two arrays $A[1, \dots, n]$ and $B[1, \dots, n]$ containing n and n integers, respectively, compute the pair of values $\langle A[i], B[j] \rangle$ (one value in each array) with the smallest (non-negative, i.e., $|A[i] - B[j]|$) difference. Return the difference. Your algorithm should use $O(n \log n)$ time. For example,



- Input: $A[] = \{1, 3, 15, 11, 2\}$ and $B[] = \{23, 127, 235, 19, 8\}$
- Output: 3. That is, the pair $(11, 8)$

Problem 3

Describe an algorithm that, given n integers (a_1, \dots, a_n) in the range 1 to k , preprocesses its input and then answers any query about what is the l -th $(1 \leq l \leq n)$ smallest element in $O(1)$ time. Your algorithm should use $\Theta(n + k)$ preprocessing time.

Problem 4

Recall that a standard (FIFO) queue ^{队列} maintains a sequence of items subject to the following operations.

- **Push(x):** Add item x to the end of the sequence. 
- **Pull():** Remove and return the item at the beginning of the sequence. 
- **Size():** Return the current number of items in the sequence.

(a) Design an algorithm with proper data structure to implement a queue, satisfying that it uses $O(n)$ space (where n is the number of items in the queue) and the worst-case time for each of these operations is $O(1)$.

(b) Consider the following new operation, which removes every tenth element from the queue, starting at the beginning, in $\Theta(n)$ worst-case time.

```
function Decimate( )  
   $n \leftarrow \text{Size}();$   
  for  $i \leftarrow 0$  to  $n - 1$  do  
    if  $i \bmod 10 = 0$  then  
      Pull();  
       $\text{Size}() \leftarrow \text{Size}() - 1;$   
    else  
      Push(Pull());  
    end if  
  end for  
end function
```

Prove that in any intermixed sequence of Push, Pull, and Decimate operations, the amortized cost of each operation is $O(1)$.

Problem 5

Given the root r of a binary tree, return the length of the diameter of the tree. The diameter of a binary tree is the length of the longest path between any two nodes in a tree. This path may or may not pass through the root r . The length of a path between two nodes is represented by the number of edges between them.