2.15.

练习

1. $$(A|E)=\left(\begin{array}{ccc|ccc} 2 & 1 & -5 & 1 & 0 & 0 \\ 3 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 2 & -5 & -3 & 1 & 0 \\ 0 & 1 & -11 & -2 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 0 & 1 \\ 0 & 2 & -5 & -3 & 1 & 0 \\ 0 & 0 & -\frac{17}{2} & -\frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{17} & -\frac{3}{17} & \frac{6}{17} \\ 0 & 1 & 0 & \frac{-23}{17} & \frac{11}{17} & -\frac{5}{17} \\ 0 & 0 & 1 & \frac{1}{17} & \frac{1}{17} & -\frac{2}{17} \end{array}\right)$$

∴A可逆 $A^{-1}=\begin{pmatrix} -\frac{3}{17} & \frac{3}{17} & \frac{6}{17} \\ \frac{-23}{17} & \frac{11}{17} & \frac{-5}{17} \\ \frac{1}{17} & \frac{1}{17} & -\frac{2}{17} \end{pmatrix}$

2. $$\begin{cases} 3x_1-x_2-x_3=4 \\ 3x_1+4x_2-2x_3=11 \\ 3x_1-2x_2+4x_3=11 \end{cases}$$

解: 设: $AX=B$ $A=\begin{bmatrix} 3 & -1 & -1 \\ 3 & 4 & -2 \\ 3 & -2 & 4 \end{bmatrix}$ $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $B=\begin{pmatrix} 4 \\ 11 \\ 11 \end{pmatrix}$

∴ $X=A^{-1}B$ $(A\,\vdots\,B)=\left(\begin{array}{ccc|c} 3 & -1 & -1 & 4 \\ 3 & 4 & -2 & 11 \\ 3 & -2 & 4 & 11 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & \frac{7}{4} \\ 0 & 0 & 1 & \frac{7}{4} \end{array}\right)$

∴ $X=\begin{pmatrix} \frac{5}{2} \\ \frac{7}{4} \\ \frac{7}{4} \end{pmatrix}$

No.

Date

3. $\begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} X \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$

设 $A=\begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix}$ $B=\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$ $C=\begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}$

$\therefore X = A^{-1}CB^{-1}$

$$(A \vdots C) = \left(\begin{array}{cc|cc} 1 & 4 & 3 & 1 \\ -1 & 2 & 0 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right)$$

$\therefore A^{-1}C = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix}$

$$\left(\frac{B}{A^{-1}C}\right) = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 1 & 1 \\ \frac{1}{2} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ \frac{1}{4} & 0 \end{pmatrix}$$

$\therefore X = A^{-1}CB^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{1}{4} & 0 \end{pmatrix}$

3).

$$(A \vdots E) = \left(\begin{array}{ccc|ccc} 2 & 2 & 3 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 3 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 6 & 4 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -4 & -3 \\ 0 & 1 & 0 & 1 & -5 & -3 \\ 0 & 0 & 1 & -1 & 6 & 4 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$$

7)

$$(A \vdots E) = \left(\begin{array}{cccc|cccc} 1 & 3 & -5 & 7 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 3 & -5 & 0 & 1 & 0 & 0 & -7 \\ 0 & 1 & 2 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -3 & 11 & -38 \\ 0 & 1 & 0 & 0 & 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & -3 & 11 & -38 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

10).

$$(A \vdots E) = \left(\begin{array}{ccccc|ccccc} 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{32} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} & \frac{1}{32} \\ 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} & -\frac{1}{16} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

22. $X = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$ $a_i \neq 0.$

$$(X \vdots E) = \left(\begin{array}{cccccc|cccccc} 0 & a_1 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} & 0 & 0 & 0 & \cdots & 1 & 0 \\ a_n & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|cccccc} a_n & & & & 0 & 0 & \cdots & 0 & 1 \\ & a_1 & & & 1 & 0 & \cdots & 0 & 0 \\ & & \ddots & & \vdots & \vdots & & \vdots & \vdots \\ & & & a_{n-1} & 0 & 0 & \cdots & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{cccc|ccccc} 1 & & & & 0 & 0 & \cdots & 0 & \frac{1}{a_n} \\ & 1 & & & \frac{1}{a_1} & 0 & \cdots & 0 & 0 \\ & & \ddots & & \vdots & \vdots & & \vdots & \vdots \\ & & & 1 & 0 & 0 & \cdots & \frac{1}{a_{n-1}} & 0 \end{array}\right)$$

$$\therefore X^{-1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{1}{a_n} \\ \frac{1}{a_1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{n-1}} & 0 \end{pmatrix}$$

No.

Date

23. 2) $\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix} X = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

解：设 $A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}$，$B = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

$\therefore AX = B \Rightarrow X = A^{-1}B$

$$(A \vdots B) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & -1 & 1 \\ 0 & 2 & 2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 2 & 1 & 1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{11}{6} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -\frac{1}{6} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{2}{3} & 1 & 0 \end{array}\right)$$

4) $X \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

解：$XA = B$

$X = BA^{-1}$

$$\begin{pmatrix} A \\ \hdashline B \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \\ \hdashline 1 & -1 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & -2 & 1 \\ 1 & -2 & 2 \\ 1 & 0 & 1 \\ 2 & -1 & 3 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hdashline -\frac{1}{3} & \frac{1}{3} & \frac{5}{3} \\ [illegible] & [illegible] & [illegible] \\ [illegible] & [illegible] & [illegible] \end{pmatrix}$$

$$\therefore X = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{5}{6} & \frac{4}{3} \end{pmatrix}$$

证：1）如果A可逆对称（反称），那么A^{-1}也对称（反称）

解：∵ A对称

∴ $A = A^T$

∴ $A^{-1} = (A^T)^{-1}$

$A^{-1} = (A^{-1})^T$

即 A^{-1}也对称.

2.17

1) 设上三角矩阵 $A=(a_{ij})_{n\times n}$ 则 $i>j$ 时 $a_{ij}=0$

上三角矩阵 $B=(b_{ij})_{n\times n}$ 则 $i>j$ 时 $b_{ij}=0$

又设 $C=AB=(c_{ij})_{n\times n}$

则 $c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+\cdots+a_{ii}b_{ij}+a_{i,i+1}b_{i+1,j}+\cdots$

当 $i>j$ 时，$a_{ij}=0$，$b_{ij}=0$

$\therefore c_{ij}=a_{ii}b_{ij}+a_{i,i+1}b_{i+1,j}+\cdots+a_{in}b_{nj}$

又 $b_{ij}=0\ (i>j)$ $\therefore b_{ij}, b_{i+1,j},\cdots,b_{nj}=0$

$\therefore c_{ij}=0$

$\therefore$ AB仍是上三角矩阵

同理下三角矩阵的乘积仍为下三角矩阵

2). 设可逆上三角矩阵 $A=(a_{ij})_{n\times n}$

则 $i>j$ 时 $a_{ij}=0$

其可逆矩阵 $A^{-1}=\dfrac{1}{|A|}\begin{pmatrix}A_{11} & A_{21} & \cdots & A_{n1}\\ A_{12} & A_{22} & \cdots & A_{n2}\\ \vdots & \vdots & & \vdots\\ A_{1n} & A_{2n} & \cdots & A_{nn}\end{pmatrix}$ A_{ij} 为 a_{ij} 代数

只需证 $i<j$ 时，$A_{ij}=0$ 即可

$i<j$ 时，$A_{ij}=\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1,j-1} & a_{1,j+1} & \cdots & a_{1n}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ a_{i-1,1} & a_{i-1,2} & \cdots & a_{i-1,j-1} & a_{i-1,j+1} & \cdots & a_{i-1,n}\\ a_{i+1,1} & a_{i+1,2} & \cdots & a_{i+1,j-1} & a_{i+1,j+1} & \cdots & a_{i+1,n}\\ \vdots & \vdots & & \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{n,j-1} & a_{n,j+1} & \cdots & a_{n,n}\end{vmatrix}$

A_{ij} 也是上三角矩阵，$|A_{ij}|=a_{11}a_{22}a_{33}\cdots a_{i-1,i-1}a_{i+1,i-1}\cdots a_{nn}$

$\because a_{i+1,i-1}=0$ $\therefore A_{ij}=0$

$\therefore$ 上三角可逆矩阵的逆矩阵仍是上三角矩阵

No.

Date

1). $(A\vdots E)=\left(\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\1&-1&1&-1&0&1&0&0\\1&1&-1&-1&0&0&1&0\\1&-1&-1&1&0&0&0&1\end{array}\right)\left(\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&-2&0&-2&-1&1&0&0\\0&0&-2&-2&-1&0&1&0\\0&-2&-2&0&-1&0&0&1\end{array}\right)$

$\rightarrow\left(\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&-2&0&-2&-1&1&0&0\\0&0&-2&-2&-1&0&1&0\\0&0&-2&2&0&-1&0&1\end{array}\right)\rightarrow\left(\begin{array}{cccc|cccc}1&1&1&1&1&0&0&0\\0&-2&0&-2&-1&1&0&0\\0&0&-2&-2&-1&0&1&0\\0&0&0&4&1&-1&-1&1\end{array}\right)$

$\rightarrow\left(\begin{array}{cccc|cccc}1&0&0&0&\frac14&\frac14&\frac14&\frac14\\0&1&0&0&\frac14&-\frac14&\frac14&-\frac14\\0&0&1&0&\frac14&\frac14&-\frac14&-\frac14\\0&0&0&1&\frac14&-\frac14&-\frac14&\frac14\end{array}\right)$

). $A=\begin{pmatrix}X&X\\X&-X\end{pmatrix}$, $X=\begin{pmatrix}1&1\\1&-1\end{pmatrix}$, $X^{-1}=\begin{pmatrix}\frac12&\frac12\\\frac12&-\frac12\end{pmatrix}$

∵ $\begin{pmatrix}E&\frac12E\\0&E\end{pmatrix}\begin{pmatrix}E&0\\-E&E\end{pmatrix}A=\begin{pmatrix}X&0\\0&-2X\end{pmatrix}$

又 $\begin{pmatrix}X&0\\0&-2X\end{pmatrix}^{-1}=\begin{pmatrix}X^{-1}&0\\0&-\frac12X^{-1}\end{pmatrix}$

∴ $A^{-1}=\begin{pmatrix}X^{-1}&0\\0&-\frac12X^{-1}\end{pmatrix}\begin{pmatrix}E&\frac12E\\0&E\end{pmatrix}\begin{pmatrix}E&0\\-E&E\end{pmatrix}$

$=\begin{pmatrix}\frac12X^{-1}&\frac12X^{-1}\\\frac12X^{-1}&-\frac12X^{-1}\end{pmatrix}=\begin{pmatrix}\frac14&\frac14&\frac14&\frac14\\\frac14&-\frac14&\frac14&-\frac14\\\frac14&\frac14&-\frac14&-\frac14\\\frac14&-\frac14&-\frac14&\frac14\end{pmatrix}$

No.

Date

29. 证：$\begin{vmatrix} E_m & B \\ A & E_n \end{vmatrix} \xrightarrow{-A\times R_1+R_2} \begin{vmatrix} E_m & B \\ 0 & E_n-AB \end{vmatrix} = |E_m|\,|E_n-AB|$

$= |E_n-AB|$

$\begin{vmatrix} E_m & B \\ A & E_n \end{vmatrix} \xrightarrow{-B\times R_2+R_1} \begin{vmatrix} E_m-BA & 0 \\ A & E_n \end{vmatrix} = |E_m-BA|$

30. 证：由29题 $\begin{vmatrix} \lambda E_m & B \\ A & \lambda E_n \end{vmatrix} = |\lambda E_n-AB|$

又 $\begin{vmatrix} E_m & B \\ A & \lambda E_n \end{vmatrix} \xrightarrow{-\frac{1}{\lambda}B R_2+R_1} \begin{vmatrix} E_m-\frac{1}{\lambda}BA & 0 \\ A & \lambda E_n \end{vmatrix}$

$= \left|E_m-\frac{1}{\lambda}BA\right| |\lambda E_n| = \lambda^{n-m}|\lambda E_m-BA|$

即 $|\lambda E_n-AB| = \lambda^{n-m}|\lambda E_m-BA|$.