第一章作业

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第一章

2.

由泰勒展开可得

$$e(x^n) \approx nx^{n-1}(x - x^*)$$

根据相对误差的定义可得

$$\begin{aligned} e_r(x^n) &\approx \frac{nx^{n-1}(x - x^*)}{x^n} \\ &= \frac{n(x - x^*)}{x^*} \\ &= ne_r(x) \\ &= 0.02n \end{aligned}$$

5.

令 $V = \frac{3}{4}\pi R^3$，可得

$$e^*(V) \approx 4\pi R^2(R^* - R)$$

由此可得

$$e_r^*(V) = \frac{e^*(V)}{\frac{4}{3}\pi R^3} = \frac{3(R^* - R)}{R^*} = 3e_r^*(R)$$

所以

$$3e_r^*(R) \leq 1\%$$
$$e_r^*(R) \leq \frac{1}{300}$$

8.

$$\int_N^{N+1} \frac{1}{x^2+1} = \text{arctan}(N+1) - \text{arctan}(N) = \text{arctan}(\frac{1}{1+N(N+1)})$$

13.

令 $y = x - \sqrt{x^2-1}$，则有

$$\xi(f(x^*)) \approx \frac{1}{|y^*|}|y^* - y|$$

由题可知，开根号使用六位有效数字，所以 $\sqrt{30^2-1} \approx 29.9833 \quad |y^*| = 0.0167$. 又由教材公式 2.2，$|y^* - y| \leq \frac{1}{2} \times 10^{-4}$，所以

$$\xi(f(x^*)) \approx \frac{|y^* - y|}{|y^*|} \leq \frac{\frac{1}{2} \times 10^{-4}}{0.0167} \approx 0.003$$

若用公式 $-\ln(x + \sqrt{x^2-1})$，同理可得 $|y*| = 59.9833$，所以

$$\xi(f(x^*)) \approx \frac{|y^* - y|}{|y^*|} \leq \frac{\frac{1}{2} \times 10^{-4}}{59.8333} \approx 8.34 \times 10^{-7}$$

第二章

1.

将右侧与函数 $f(x) = x$ 的 Lagrange 插值多项式联系

$$x \approx \sum_{i=0}^{n} x_i l_i(x) = \sum_{i=0}^{n} \left(\prod_{k=0,k\neq i} \frac{x-k}{i-k} \right) i$$

由插值余项

$$R_n(x) = f(x) - L_n(x) = \frac{x^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x) = 0$$

由此可得

$$x = \sum_{i=0}^{n} x_i l_i(x) = \sum_{i=0}^{n} \left(\prod_{k=0,k\neq i} \frac{x-k}{i-k} \right) i$$

2.

拉格朗日插值

$$\begin{aligned} L_2(x) &= \sum_{k=0}^{n} f_k(x_k) l_0(x) \\ &= 0 + (-3) \times \frac{(x-1)(x-2)}{-2 \times (-3)} + 4 \times \frac{(x-1)(x+1)}{1 \times 3} \\ &= \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3} \end{aligned}$$

4.

线性插值多项式为

$$L_1(x) = \cos x_k \frac{x - x_{k+1}}{x_k - x_{k+1}} + \cos x_{k+1} \frac{x - x_k}{x_{k+1} - x_k}$$

令 $L_1'(x)$ 为近似值线性插值多项式，$x_k = \frac{k}{60} \times \frac{\pi}{180} = \frac{k\pi}{10800}$. 由此可得误差估计

$$\begin{aligned} |\cos x - L_1'(x)| &= |\cos x - L_1(x) + L_1(x) - L_1'(x)| \\ &\leq |\cos x - L_1(x)| + |L_1(x) - L_1'(x)| \end{aligned}$$

将误差估计分为两部分分别计算

$$\begin{aligned} |\cos x - L_1(x)| &= \left| \frac{1}{2}(-\cos\xi)(x - x_k)(x - x_{k+1}) \right| \\ &\leq \frac{1}{2}|(x - x_k)(x - x_{k+1})| \\ &\leq \frac{1}{2} \times \left(\frac{1}{2} \times \frac{\pi}{10800} \right)^2 \\ &\approx 1.06 \times 10^{-8} \end{aligned}$$

$$
\begin{aligned}
|L_1(x) - L_1'(x)| &= |e(f^*(x_k))|\frac{x_{k+1}-x}{x_{k+1}-x_k} + |e(f^*(x_{k+1}))|\frac{x-x_k}{x_{k+1}-x_k} \\
&\leq \max\{|e(f^*(x_k))|, |e(f^*(x_{k+1}))|\}\left(\frac{x_{k+1}-x}{x_{k+1}-x_k} + \frac{x-x_k}{x_{k+1}-x_k}\right) \\
&= \max\{|e(f^*(x_k))|, |e(f^*(x_{k+1}))|\}
\end{aligned}
$$

由有效数字的定义可得

$$|e(f^*(x_k))| \leq \frac{1}{2} \times 10^{m_k-4}$$

所以有

$$\max\{|e(f^*(x_k))|, |e(f^*(x_{k+1}))|\} \leq \max\left\{\frac{1}{2}\times 10^{m_k-4}, \frac{1}{2}\times 10^{m_{k+1}-4}\right\} = \frac{1}{2}\times 10^{\max\{m_k, m_{k+1}\}-4}$$

综上所述

$$|\cos x - L_1'(x)| \leq 1.06\times 10^{-8} + \frac{1}{2}\times 10^{\max\{m_k, m_{k+1}\}-4}$$

在区间 $[0, \frac{\pi}{2}]$ 上可得

$$|\cos x - L_1'(x)| \leq 1.06\times 10^{-8} + \frac{1}{2}\times 10^{-5} = 0.50106\times 10^{-5}$$

6.

i)

函数 $\sum_{j=0}^{n} x_j^k l_j(x)$ 为函数 x^k 的 Lagarnge 插值多项式，同时 x^k 也为自身的插值多项式，由于插值多项式具有唯一性，所以可知 $\sum_{j=0}^{n} x_j^k l_j(x) \equiv x^k$

ii)

$$
\begin{aligned}
\sum_{j=0}^{n}(x_j - x)^k l_j(x) &= \sum_{j=0}^{n}\left[\sum_{i=0}^{n}\tbinom{k}{i}x_j^i(-x)^{(k-i)i_j(x)}\right] \\
&= \sum_{j=0}^{n}\sum_{i=0}^{k}\left[\tbinom{k}{i}x_j^i(-x)^{k-i}i_j(x)\right] \\
&= \sum_{i=0}^{k}\left[\tbinom{k}{i}(-x)^{k-i}\sum_{j=0}^{n}x_j^i l_j(x)\right]
\end{aligned}
$$

由 i) 得到的结论可知

$$\sum_{j=0}^{n}(x_j - x)^k l_j(x) = \sum_{i=0}^{k} \binom{k}{i} (-x)^{k-i} x^i = (x - x)^k \equiv 0$$

17.