

Homework 1

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第三章

第 6 题

$$\begin{aligned} \because f''(x) &= -\sin x \leq 0, \forall x \in [0, \frac{\pi}{2}] \\ \therefore P_1(x) &= \frac{1}{2} [f(a) + f(x_2)] + a_1 \left(x - \frac{a+x_2}{2}\right) \\ a_1 &= \frac{f(\frac{\pi}{2}) - f(0)}{\frac{\pi}{2} - 0} = \frac{2}{\pi} \\ f'(x_2) &= \cos x_2 = \frac{2}{\pi}, \text{ 解之得 } x_2 = \arccos \frac{2}{\pi} \approx 0.8807 \\ f(x_2) &= \sin(x_2) \approx 0.7712 \end{aligned}$$

$$\begin{aligned} P_1(x) &= \frac{1}{2} [f(a) + f(x_2)] + a_1 \left(x - \frac{a+x_2}{2}\right) \\ &= \frac{1}{2} (0 + 0.7712) + \frac{2}{\pi} \left(x - \frac{0.8807}{2}\right) \\ &= 0.1053 + \frac{2}{\pi} x \end{aligned} \tag{1}$$

$$\text{误差 } \|\sin(x) - P_1(x)\|_{\infty} = \max_{0 \leq x \leq \frac{\pi}{2}} |\sin(x) - P_1(x)| = |\sin(0) - P_1(0)| = 0.1053$$

第 17 题

$$\begin{aligned} \text{对于 } \varphi_1 &= \text{span}\{1, x\} \\ (\varphi_0, \varphi_0) &= \int_0^1 dx = 1, (\varphi_0, \varphi_1) = \int_0^1 x dx = \frac{1}{2} \\ (\varphi_1, \varphi_0) &= \int_0^1 x dx = \frac{1}{2}, (\varphi_1, \varphi_1) = \int_0^1 x^2 dx = \frac{1}{3} \\ (\varphi_0, f) &= \int_0^1 x^2 dx = \frac{1}{3}, (\varphi_1, f) = \int_0^1 x^3 dx = \frac{1}{4} \end{aligned}$$

法方程为:

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \tag{2}$$

$$\text{解得: } a_0 = -\frac{1}{6}, a_1 = 1$$

$$\text{最佳平方逼近: } P_1^1(x) = -\frac{1}{6} + x$$

$$\text{误差: } \|\delta_1^1(x)\|_{\infty} = \max_{0 \leq x \leq 1} |x^2 - (-\frac{1}{6} + x)| = \frac{1}{6}$$

$$\text{对于 } \varphi_2 = \text{span}\{x^{100}, x^{101}\}$$

$$\begin{aligned}
(\varphi_0, \varphi_0) &= \int_0^1 x^{200} dx = \frac{1}{201}, (\varphi_0, \varphi_1) = \int_0^1 x^{201} dx = \frac{1}{202} \\
(\varphi_1, \varphi_0) &= \int_0^1 x^{201} dx = \frac{1}{202}, (\varphi_1, \varphi_1) = \int_0^1 x^{202} dx = \frac{1}{203} \\
(\varphi_0, f) &= \int_0^1 x^{102} dx = \frac{1}{103}, (\varphi_1, f) = \int_0^1 x^{103} dx = \frac{1}{104}
\end{aligned}$$

法方程为:

$$\begin{bmatrix} \frac{1}{201} & \frac{1}{202} \\ \frac{1}{202} & \frac{1}{203} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{103} \\ \frac{1}{104} \end{bmatrix} \quad (3)$$

解得: $a_0 = \frac{2009799}{5356}, a_1 = -\frac{1004647}{2648}$

最佳平方逼近: $P_1^2(x) = \frac{2009799}{5356}x^{100} - \frac{1004647}{2648}x^{101}$

误差: $\|\delta_1^2(x)\|_\infty = \max_{0 \leq x \leq 1} |x^2 - \frac{2009799}{5356}x^{100} - \frac{1004647}{2648}x^{101}| = \frac{4851}{5356}$

$\therefore \|\delta_1^1(x)\|_\infty < \|\delta_1^2(x)\|_\infty$

第 20 题

根据勒让德多项式的性质:

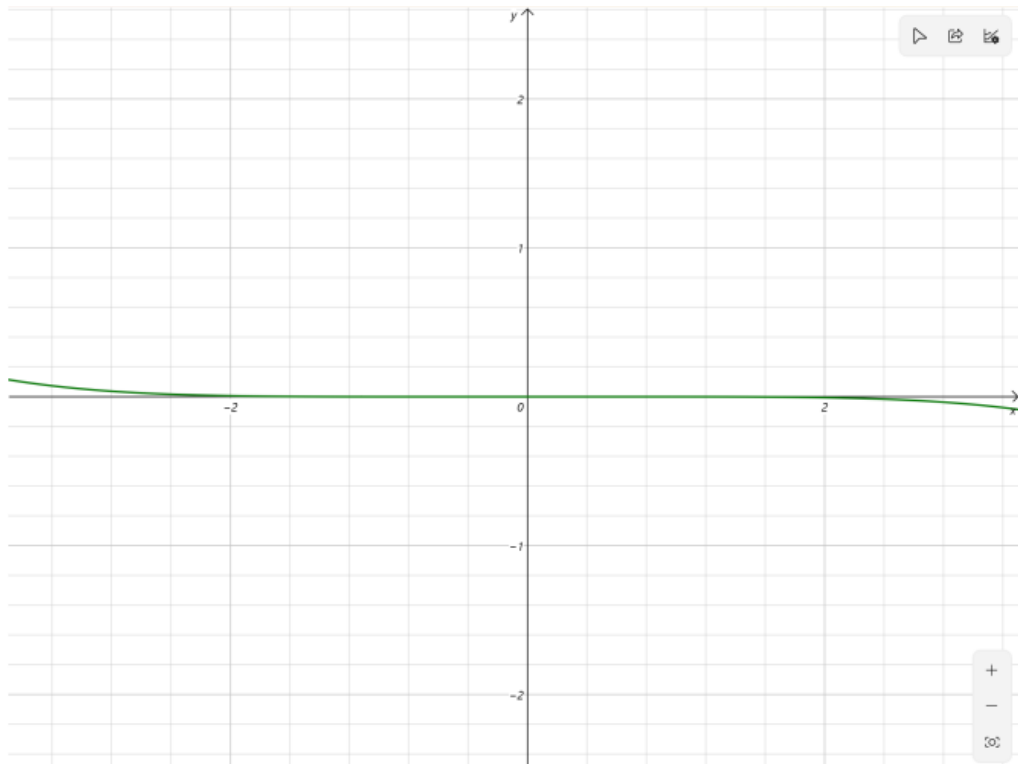
$$\begin{aligned}
a_k &= \frac{(f, P_k)}{(P_k, P_k)} = \frac{2k+1}{2} \int_{-1}^1 \sin \frac{x}{2} P_k(x) dx \\
a_0 &= \frac{1}{2} \int_{-1}^1 \sin \frac{x}{2} \cdot 1 dx = 0 \\
a_1 &= \frac{3}{2} \int_{-1}^1 \sin \frac{x}{2} \cdot x dx = 12 \sin \frac{1}{2} - 6 \cos \frac{1}{2} \\
a_2 &= \frac{5}{2} \int_{-1}^1 \sin \frac{x}{2} \cdot \frac{1}{2} (3x^2 - 1) dx = 0 \\
a_3 &= \frac{7}{2} \int_{-1}^1 \sin \frac{x}{2} \cdot \frac{1}{2} (5x^3 - 3x) dx = 826 \cos \frac{1}{2} - 1512 \sin \frac{1}{2}
\end{aligned}$$

所以:

$$\begin{aligned}
P(x) &= \left(12 \sin \frac{1}{2} - 6 \cos \frac{1}{2}\right) x + \left(826 \cos \frac{1}{2} - 1512 \sin \frac{1}{2}\right) \cdot \frac{1}{2} (5x^3 - 3x) \\
&= \left(2065 \cos \frac{1}{2} - 3780 \sin \frac{1}{2}\right) x^3 - \left(12 \sin \frac{1}{2} - 6 \cos \frac{1}{2}\right) x \\
&\approx -0.02055x^3 + 0.49994x
\end{aligned}$$

误差 $E(x) = P(x) - f(x) \approx -0.02055x^3 + 0.49994x - \sin \frac{x}{2}$

其图形:



均方误差:

$$\|\delta\|_2^2 = \int_{-1}^1 (f(x) - P(x))^2 dx \approx 1.96709 \times 10^{-10}$$

第 22 题

$$\Phi = \text{span} \{1, x^2\}$$

$$(\varphi_0, \varphi_0) = \sum_{i=1}^5 \varphi_0(x_i)^2 = 5, \quad (\varphi_0, \varphi_1) = \sum_{i=1}^5 \varphi_0(x_i) \varphi_1(x_i) = 5327$$

$$(\varphi_1, \varphi_1) = 7277699, \quad (\varphi_0, y) = 271.4, \quad (\varphi_1, y) = 369321.5$$

法方程为:

$$\begin{bmatrix} 5 & 5327 \\ 5327 & 7277699 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 271.4 \\ 369321.5 \end{bmatrix} \quad (4)$$

解得: $a \approx 0.972748, b = 0.050035$

所以经验公式: $y = 0.972748 + 0.050035x^2$

均方误差: $\|\delta\|_2^2 = \sum_{i=1}^5 (a + bx_i^2 - y_i)^2 \approx 0.015023$

第 24 题

观察源数据的图像,可以发现 y 随着 t 单调递增,但是随着 t 越大, y 的增长率越小,可以得到 $y' > 0, y'' < 0$, 可以建立拟合模型: $y = ae^{-\frac{b}{t}}, a, b > 0$

等式两边取对数:

$$\ln y = \ln a - \frac{b}{t}$$

令 $\Phi = \text{span} \{1, -\frac{1}{t}\}$

得到:

$$\begin{aligned} (1, 1) &= \sum_{i=1}^{11} 1 = 11, & (1, -\frac{1}{t}) &= -0.603975 \\ (-\frac{1}{t}, -\frac{1}{t}) &= 0.062321, & (1, \ln y) &= -87.674095, & (-\frac{1}{t}, \ln y) &= 5.032489 \end{aligned} \quad (5)$$

法方程:

$$\begin{bmatrix} 11 & -0.603975 \\ -0.603975 & 0.062321 \end{bmatrix} \begin{bmatrix} \ln a \\ b \end{bmatrix} = \begin{bmatrix} -87.674095 \\ 5.032489 \end{bmatrix} \quad (6)$$

解得: $\ln a = -7.558781, b = 7.4961692, a = 5.215148 \times 10^{-4}$

$$\therefore y = 5.215148 \times 10^{-4} \times e^{-\frac{7.4961692}{t}}$$

第四章

第 1 题

(3) 为了使求积公式

$$\int_{-1}^1 f(x) dx \approx \frac{f(-1) + 2f(x_1) + 3f(x_2)}{3} \quad (7)$$

具有尽可能高的代数精度, 只需取

$$f(x) = x^m, m = 0, 1, 2, \dots \quad (8)$$

对7式均准确成立即可

当 $f(x) = 1$ 时, $2 = \frac{1}{3}(1 + 2 + 3)$ 成立, 所以7准确成立.

当 $f(x) = x, f(x) = x^2$ 时, 代入7可得:

$$\begin{cases} -1 + 2x_1 + 3x_2 = 0 \\ 1 + 2x_1^2 + 3x_2^2 = 2 \end{cases} \quad (9)$$

解得:

$$\begin{cases} x_1 = -0.2898979 \\ x_2 = 0.6265986 \end{cases} \quad \text{或} \quad \begin{cases} x_1 = 0.6898979 \\ x_2 = -0.1265986 \end{cases} \quad (10)$$

若再将 $f(x) = x^3$ 带入已经确定的求积公式, 则:

$$\int_{-1}^1 f(x)dx \neq \frac{f(-1) + 2f(x_1) + 3f(x_2)}{3} \quad (11)$$

因此求积公式具有 2 次代数精度, 求值节点:

$$\begin{cases} x_1 = -0.2898979 \\ x_2 = 0.6265986 \end{cases} \quad \text{或} \quad \begin{cases} x_1 = 0.6898979 \\ x_2 = -0.1265986 \end{cases} \quad (12)$$

第 2 题

(1) 使用复化梯形公式 ($h = \frac{1}{8}$):

$$T_8 = \frac{h}{2} \left[f(0) + 2 \sum_{k=1}^7 f(x_k) + f(1) \right] = 0.1114024 \quad (13)$$

使用复化 Simpson 公式 ($h = \frac{1}{8}$):

$$S_8 = \frac{h}{6} \left[f(0) + 4 \sum_{k=0}^7 f\left(x_{k+\frac{1}{2}}\right) + 2 \sum_{k=1}^7 f(x_k) + f(1) \right] = 0.1115718 \quad (14)$$

第 7 题

设将区间分成 n 等份, 则 $h = \frac{b-a}{n}$ 由误差公式:

$$|R| = \left| \frac{b-a}{12} h^2 f''(\eta) \right| = \left| \frac{b-a}{12} \left(\frac{b-a}{n} \right)^2 f''(\eta) \right| \leq \frac{(b-a)^3}{12n^2} M < \varepsilon \quad (15)$$

其中, $M = \max_{a \leq x \leq b} |f''(x)|$

解得:

$$n > \sqrt{\frac{(b-a)^3 M}{12\varepsilon}}$$

第 11 题

(1) 如下表, 其中 $T_0^{(k)}$ 中下标代表加速次数, 上标代表二分次数。

表 1:					
k	$T_0^{(k)}$	$T_1^{(k-1)}$	$T_2^{(k-2)}$	$T_3^{(k-3)}$	$T_4^{(k-4)}$
0	1.3333333				
1	1.1666667	1.1111111			
2	1.1166667	1.1000000	1.0992593		
3	1.1032107	1.0987253	1.0986403	1.0986305	
4	1.0997677	1.0986200	1.0986130	1.0986126	1.0986125

取 $I = 1.0986125$ 。

(2) 做变换: $y = t + 2$, 则当 $y \in [1, 3]$ 时, $t \in [-1, 1]$. 则:

$$\int_1^3 \frac{dy}{y} = \int_{-1}^1 \frac{dt}{t+2} \quad (16)$$

三点高斯公式:

$$\begin{aligned} \int_1^3 \frac{dy}{y} &= \int_{-1}^1 \frac{dt}{t+2} \\ &\approx \frac{5}{9}f\left(-\frac{\sqrt{15}}{5}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\frac{\sqrt{15}}{5}\right) \\ &\approx 0.5555556 \left(\frac{1}{2-0.7745967} + \frac{1}{2+0.7745967} \right) \\ &\quad + 0.8888889 \times \frac{1}{2+0} \\ &= 1.0980393 \end{aligned} \quad (17)$$

五点高斯公式:

$$\begin{aligned} \int_1^3 \frac{dy}{y} &= \int_{-1}^1 \frac{dt}{t+2} \approx 0.2369269 \\ &\quad \left(\frac{1}{2-0.9061798} + \frac{1}{2+0.9061798} \right) \\ &\quad + 0.4786289 \left(\frac{1}{2-0.5384693} + \frac{1}{2+0.5384693} \right) \\ &\quad + 0.5688889 \times \frac{1}{2+0} = 1.0986093 \end{aligned} \quad (18)$$

(3) 将区间 $[1, 3]$ 分成四个小区间 $[1, 1.5], [1.5, 2], [2, 2.5], [2.5, 3]$

在第一个区间上, 做变换 $y = \frac{1}{4}t + \frac{5}{4}$.

$$\begin{aligned} I_1 &= \int_1^{1.5} \frac{dy}{y} = \int_{-1}^1 \frac{1}{5+t} dt \\ &\approx 0.5 \times \left[\frac{1}{2.5 + 0.5 \times \left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{2.5 + 0.5 \times \left(\frac{1}{\sqrt{3}}\right)} \right] \\ &= 0.4054054 \end{aligned} \quad (19)$$

在第二个区间上, 做变换 $y = \frac{1}{4}t + \frac{7}{4}$.

$$\begin{aligned} I_2 &= \int_{1.5}^2 \frac{dy}{y} = \int_{-1}^1 \frac{1}{7+t} dt \\ &\approx 0.5 \times \left[\frac{1}{3.5 + 0.5 \times \left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{3.5 + 0.5 \times \left(\frac{1}{\sqrt{3}}\right)} \right] \\ &= 0.2876712 \end{aligned} \quad (20)$$

在第三个区间上, 做变换 $y = \frac{1}{4}t + \frac{9}{4}$.

$$\begin{aligned} I_3 &= \int_2^{2.5} \frac{dx}{y} = \int_{-1}^1 \frac{1}{9+t} dt \\ &\approx 0.5 \times \left[\frac{1}{4.5 + 0.5 \times \left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{4.5 + 0.5 \times \left(\frac{1}{\sqrt{3}}\right)} \right] \\ &= 0.2231405 \end{aligned} \quad (21)$$

在第四个区间上, 做变换 $y = \frac{1}{4}t + \frac{11}{4}$.

$$\begin{aligned} I_4 &= \int_{2.5}^3 \frac{dy}{y} = \int_{-1}^1 \frac{1}{11+t} dt \\ &\approx 0.5 \times \left[\frac{1}{5.5 + 0.5 \times \left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{5.5 + 0.5 \times \left(\frac{1}{\sqrt{3}}\right)} \right] \\ &= 0.1823204 \end{aligned} \quad (22)$$

故:

$$I = I_1 + I_2 + I_3 + I_4 \approx 1.0985375 \quad (23)$$

比较

积分真值:

$$I = \int_1^3 \frac{dy}{y} = \ln 3 = 1.098612288 \dots \quad (24)$$

说明龙贝格算法比高斯求积算法更准确, 但龙贝格算法运算量更大。