计算方法 **Spring 2023**

Homework 1

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第 2 题

Solution. 由泰勒公式:$e(x^n) \approx n{x^*}^{n-1}(x^* - x)$

$\therefore e_r(x^n) \approx \frac{e(x^n)}{(x^*)^n} = n\frac{x^*-x}{x^*} = ne_r(x) = 0.02n$

□

第 5 题

Solution. $f(R) = \frac{4}{3}\pi R^3, e^*(f(R)) = 4\pi {R^*}^3(R^* - R)$

$\therefore e_r^*(f(R)) = e^*(f(R))/f(R^*) = 3(R^* - R) = 3e^*(R) \leq 1\%$

$\therefore e^*(R) \leq 1/300$

□

第 8 题

Solution. 令 $x = arctanA, y = arctanB, tan(x - y) = \frac{tanx - tany}{1+tan(x)tan(y)} = \frac{A-B}{1+AB}$

$\therefore x - y = arctan(\frac{A-B}{1+AB})$, 即 $arctanA - arctanB = arctan(\frac{A-B}{1+AB})$

$\int_N^{N+1} \frac{1}{x^2+1} = arctan(N + 1) - arctan(N) = arctan(\frac{1}{1+N(N+1)})$

□

第 13 题

Solution. 令 $y = x - \sqrt{x^2 - 1}$

$\sqrt{899} \approx 29.9833, 30 - 29.9833 \approx 0.0167$

$\therefore y^* = 0.0167, |y^* - y| \leq \frac{1}{2} \times 10^{-4}$

$\therefore \epsilon(f(x)) \approx \frac{1}{|y^*|}|y^* - y| \leq \frac{\frac{1}{2}\times 10^{-4}}{0.0167} \leq 0.3 \times 10^{-2}$

若用等价公式 $ln(x - \sqrt{x^2 - 1}) = -ln(x + \sqrt{x^2 - 1})$

则 $y^* = 59.9833$

$\therefore \epsilon(f(x)) \approx \frac{1}{|y^*|}|y^* - y| \leq \frac{\frac{1}{2}\times 10^{-4}}{59.9833} \leq 8.34 \times 10^{-7}$

□

第 1 题

Solution. 注意到右端项是 $y=x$ 在 $x_0=0,x_1=1,...,x_n=n$ 处的 Lagrange 插值多项式。

$\therefore x \approx \sum\limits_{i=0}^{n} l_i(x)x_i = \sum\limits_{i=0}^{n}(\prod\limits_{k=0,k\neq i}^{n} \frac{x-k}{i-k})i$

余项是 $R_n(x) = x - \sum\limits_{i=0}^{n} l_i(x)x_i = \frac{x^{(n+1)}|_{x=\xi}}{(n+1)!}\omega_{n+1}(x) = 0$

$\therefore x = \sum\limits_{i=0}^{n}(\prod\limits_{k=0,k\neq i}^{n} \frac{x-k}{i-k})i$

□

第 2 题

Solution. 令 $x_0=1,y_0=0;x_1=-1,y_1=-3;x_2=2,y_2=4$

使用 Lagrange 插值法,

$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x-1)(x_0-x_2)} = -\frac{1}{2}(x+1)(x-2)$

$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{1}{6}(x-1)(x-2)$

$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{3}(x-1)(x+1)$

$\therefore L_2(x) = \sum\limits_{i=0}^{2} y_i l_i(x) = \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3}$

使用 Newton 插值法，

一阶差商：$f[x_0,x_1] = \frac{3}{2}, f[x_1,x_2] = \frac{7}{3}$

二阶差商: $f[x_0,x_1,x_2] = \frac{5}{6}$

$\therefore N_2(x) = y_0 + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1) = \frac{5}{6}x^2 + \frac{3}{2}x - \frac{7}{3}$

□

第 4 题

Solution.

当 $x \in [x_k, x_{k+1}]$ 时，线性插值多项式为：

$$L_1(x) = \cos x_k \frac{x-x_{k+1}}{x_k-x_{k+1}} + \cos x_{k+1}\frac{x-x_k}{x_{k+1}-x_k}$$

其中 $x_k = \frac{k}{60} \times \frac{\pi}{180} = \frac{k\pi}{10800}$.

误差估计

$$\begin{aligned} |\cos x - L_1'(x)| &= |\cos x - L_1(x) + L_1(x) - L_1'(x)| \\ &\leq |\cos x - L_1(x)| + |L_1(x) - L_1'(x)| \end{aligned}$$

将误差估计分为两部分分别计算

$$\begin{aligned} |\cos x - L_1(x)| &= \left|\frac{1}{2}(-\cos\xi)(x-x_k)(x-x_{k+1})\right| \\ &\leq \frac{1}{2}|(x-x_k)(x-x_{k+1})| \\ &\leq \frac{1}{2} \times \left(\frac{1}{2} \times \frac{\pi}{10800}\right)^2 \\ &\approx 1.06 \times 10^{-8} \end{aligned}$$

$$\begin{aligned}|L_1(x)-L_1'(x)| &= |e(f^*(x_k))|\frac{x_{k+1}-x}{x_{k+1}-x_k}+|e(f^*(x_{k+1}))|\frac{x-x_k}{x_{k+1}-x_k}\\ &\le \max\{|e(f^*(x_k))|,|e(f^*(x_{k+1}))|\}\left(\frac{x_{k+1}-x}{x_{k+1}-x_k}+\frac{x-x_k}{x_{k+1}-x_k}\right)\\ &= \max\{|e(f^*(x_k))|,|e(f^*(x_{k+1}))|\}\end{aligned}$$

由有效数字的定义可得

$$|e(f^*(x_k))| \le \frac{1}{2}\times 10^{m_k-4}$$

所以有

$$\max\{|e(f^*(x_k))|,|e(f^*(x_{k+1}))|\} \le \max\left\{\frac{1}{2}\times 10^{m_k-4},\frac{1}{2}\times 10^{m_{k+1}-4}\right\}=\frac{1}{2}\times 10^{\max\{m_k,m_{k+1}\}-4}$$

综上所述

$$|\cos x - L_1'(x)| \le 1.06\times 10^{-8}+\frac{1}{2}\times 10^{\max\{m_k,m_{k+1}\}-4}$$

在区间 $[0,\frac{\pi}{2}]$ 上可得

$$|\cos x - L_1'(x)| \le 1.06\times 10^{-8}+\frac{1}{2}\times 10^{-5}=0.50106\times 10^{-5}$$

□

第 6 题

Solution. i). 注意到等式左边是 $y=x^k$ 在 $(x_j,x_j^k), j=0,1,...,n$ 处的 Lanrange 插值多项式

$\therefore x^k \approx \sum\limits_{j=0}^{n} x_j^k l_j(x)$

$R_n(x)=x^k-\sum\limits_{j=0}^{n} x_j^k l_j(x)=\frac{(x^k)^{(n+1)}|_{x=\xi}}{(n+1)!}\omega_{n+1}(x)=0$

$\therefore \sum\limits_{j=0}^{n} x_j^k l_j(x)\equiv x^k$

ii).

$$\begin{aligned}\sum_{j=0}^{n}(x_j-x)^k l_j(x) &= \sum_{j=0}^{n}\left[l_j(x)\sum_{i=0}^{k}\tbinom{k}{i}x_j^i(-x)^{k-i}\right]\\ &=\sum_{j=0}^{n}\sum_{i=0}^{k}\left[\tbinom{k}{i}x_j^i(-x)^{k-i}l_j(x)\right]\\ &=\sum_{i=0}^{k}\sum_{j=0}^{n}\left[\tbinom{k}{i}x_j^i(-x)^{k-i}l_j(x)\right]\\ &=\sum_{i=0}^{k}\left[\tbinom{k}{i}(-x)^{k-i}\sum_{j=0}^{n}x_j^i l_j(x)\right]\\ &=\sum_{i=0}^{k}\tbinom{k}{i}(-x)^{k-i}x^i\\ &=(x-x)^k\equiv 0\end{aligned}$$

□

第 8 题

Solution. 根据截断误差公式:$R_n(x) = f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(x)$

$R_2(x) = \frac{1}{3!}f'''(\xi)\left(x - x_{i-1}\right)\left(x - x_i\right)\left(x - x_{i+1}\right), \quad \xi \in (x_{i-1}, x_{i+1})$

其中：$x_{i-1} = x_i - h, x_{i+1} = x_i + h$

$$|R_2(x)| = \frac{1}{6}e^x\left|(x - x_{i-1})(x - x_i)(x - x_{i+1})\right| \leqslant \frac{1}{6}\mathrm{e}^4 \max_{x_{i-1} \leqslant x \leqslant x_{i+1}} \left|\left(x - x_{i-1}\right)\left(x - x_i\right)\left(x - x_{i+1}\right)\right|$$
$$\leqslant \frac{1}{6}\mathrm{e}^4\frac{2}{3}\frac{1}{\sqrt{3}}h^3 = \frac{\mathrm{e}^4}{9\sqrt{3}}h^3$$

其中第二个不等式根据求导得到。

令 $\frac{\mathrm{e}^4}{9\sqrt{3}}h^3 \leq 10^{-6}$, 得到 $h \leq 0.00658$

□

第 17 题

Solution.

由 Hermite 插值函数的条件可知

$$H_3(x_k) = f_k, H_3(x_{k+1}) = f_{k+1}, H_3'(x_k) = f_k', H_3'(x_{k+1}) = f_{k+1}'$$

由此可知: $R_3(x)$ 有二重零点 x_K, X_{k+1}, 则

$$R_3(x) = f(x) - H_3(x) = g(x)(x - x_k)^2(x - x_{k+1})^2$$

令 $h(t) = f(t) - H_3(t) - g(t)(t - x_k)^2(t - x_{k+1})^2$

则 $h(x_k) = h(x_{k+1}) = 0, h'(x_k) = h'(x_{k+1}) = 0, h(x) = 0$

在 $[x_k, x]$ 和 $[x, x_{k+1}]$ 上对 $h(x)$ 使用 Rolle 中值定理可得 $\exists\xi_1 \in [x_k, x], \quad \exists\xi_2 \in [x, x_{k+1}]$ 使得 $h'(\xi_1) = h'(\xi_2) = 0$

在 $[x_k, \xi_1], [\xi_1, \xi_2]$ 和 $[\xi_2, x_{k+1}]$ 对 $h'(x)$ 使用 Rolle 定理可得 $\exists\xi_{11} \in [x_k, \xi_1], \xi_{22} \in [\xi_1, \xi_2], \xi_{33} \in [\xi_2, x_{k+1}]$ 使得 $h''(\xi_{11}) = h''(\xi_{22}) = h''(\xi_{33} = 0)$。

同理再用两次 Rolle 定理，可得 $h^{(4)}(\xi) = f^{(4)}(t) - k(x) \times 4!$，可解得

$$k(x) = \frac{1}{4!}f^{(4)}(\xi)$$

所以有

$$R_3(x) = \frac{1}{4!}f^{(4)}(\xi)(x - x_k)^2(x - x_{k+1})^2$$

令 $x_k = a + kh, h = \frac{b-a}{n}$，在 $[x_k, x_{k+1}]$ 上有

$$|f(x) - H_3(x)| = \frac{1}{4!}|f^{(4)}(x)|(x - x_k)^2(x - x_{k+1})^2$$
$$\leq \frac{1}{4!}\max|f^{(4)}(x)|\max(x - x_k)^2(x - x_{k+1})^2$$

由于

$$\max(x - x_k)^2(x - x_{k+1})^2 = \max(s^2(s-1)^2h^4) = \frac{1}{16}h^4 \tag{1}$$

误差估计为

$$|f(x) - I_h(x)| \leq \frac{1}{384}h^4 \max_{a \leq x \leq b}|f^{(4)}(x)| \tag{2}$$

□

第 19 题

Solution. $x_0 = 0, y_0 = 0, m_0 = 0; x_1 = 1, y_1 = 1, m_1 = 1$

满足上面条件的 Hermite 插值多项式为：

$H_3(x) = \sum_{j=0}^{1} [y_j \alpha_j(x) + m_j \beta_j(x)]$

$= \left[1 - 2\frac{x-1}{1-0}\right] \left[\frac{x-0}{1-0}\right]^2 + (x-1) \left[\frac{x-0}{1-0}\right]^2 = 2x^2 - x^3$

令 $P(x) = H_3(x) + ax^2(x-1)^2$,

由 $P(2) = 1$, 解得 $a = \frac{1}{4}$

$\therefore P(x) = 2x^2 - x^3 + \frac{1}{4}x^2(x-1)^2 = \frac{1}{4}x^2(x-3)^2$

$\square$