计算方法 **Spring 2023**

Homework 1

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第三章

第 6 题

$\because f^{''}(x) = -\sin x \leq 0, \forall x \in [0, \frac{\pi}{2}]$
$\therefore P_1(x) = \frac{1}{2}\left[f(a) + f(x_2)\right] + a_1\left(x - \frac{a+x_2}{2}\right)$
$a_1 = \frac{f(\frac{\pi}{2} - f(0))}{\frac{\pi}{2} - 0} = \frac{2}{\pi}$
$f^{'}(x_2) = \cos x_2 = \frac{2}{\pi}$, 解之得 $x_2 = \arccos\frac{2}{\pi} \approx 0.8807$
$f(x_2) = sin(x_2) \approx 0.7712$

$$\begin{aligned} P_1(x) &= \frac{1}{2}\left[f(a) + f(x_2)\right] + a_1\left(x - \frac{a+x_2}{2}\right) \\ &= \frac{1}{2}(0 + 0.7712) + \frac{2}{\pi}(x - \frac{0.8807}{2}) \\ &= 0.1053 + \frac{2}{\pi}x \end{aligned} \tag{1}$$

误差 $\|sin(x) - P_1(x)\|_\infty = \max\limits_{0 \leq x \leq \frac{\pi}{2}} |sin(x) - P_1(x)| = |sin(0) - P_1(0)| = 0.1053$

第 17 题

对于 $\varphi_1 = span\{1, x\}$
$(\varphi_0, \varphi_0) = \int_0^1 dx = 1, (\varphi_0, \varphi_1) = \int_0^1 xdx = \frac{1}{2}$
$(\varphi_1, \varphi_0) = \int_0^1 xdx = \frac{1}{2}, (\varphi_1, \varphi_1) = \int_0^1 x^2dx = \frac{1}{3}$
$(\varphi_0, f) = \int_0^1 x^2dx = \frac{1}{3}, (\varphi_1, f) = \int_0^1 x^3dx = \frac{1}{4}$
法方程为：

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{4} \end{bmatrix} \tag{2}$$

解得：$a_0 = -\frac{1}{6}, a_1 = 1$
最佳平方逼近:$P_1^1(x) = -\frac{1}{6} + x$
误差: $\|\delta_1^1(x)\|_\infty = \max\limits_{0 \leq x \leq 1} |x^2 - (-\frac{1}{6} + x)| = \frac{1}{6}$
对于 $\varphi_2 = span\{x^{100}, x^{101}\}$

$(\varphi_0,\varphi_0)=\int_0^1 x^{200}dx=\frac{1}{201},(\varphi_0,\varphi_1)=\int_0^1 x^{201}dx=\frac{1}{202}$
$(\varphi_1,\varphi_0)=\int_0^1 x^{201}dx=\frac{1}{202},(\varphi_1,\varphi_1)=\int_0^1 x^{202}dx=\frac{1}{203}$
$(\varphi_0,f)=\int_0^1 x^{102}dx=\frac{1}{103},(\varphi_1,f)=\int_0^1 x^{103}dx=\frac{1}{104}$
法方程为：

$$\begin{bmatrix} \frac{1}{201} & \frac{1}{202} \\ \frac{1}{202} & \frac{1}{203} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}=\begin{bmatrix} \frac{1}{103} \\ \frac{1}{104} \end{bmatrix} \tag{3}$$

解得：$a_0=\frac{2009799}{5356},a_1=-\frac{1004647}{2648}$
最佳平方逼近:$P_1^2(x)=\frac{2009799}{5356}x^{100}-\frac{1004647}{2648}x^{101}$
误差: $\|\delta_1^2(x)\|_\infty=\max\limits_{0\le x\le 1}|x^2-\frac{2009799}{5356}x^{100}-\frac{1004647}{2648}x^{101}|=\frac{4851}{5356}$
$\therefore \|\delta_1^1(x)\|_\infty<\|\delta_1^2(x)\|_\infty$

第 20 题

根据勒让德多项式的性质:

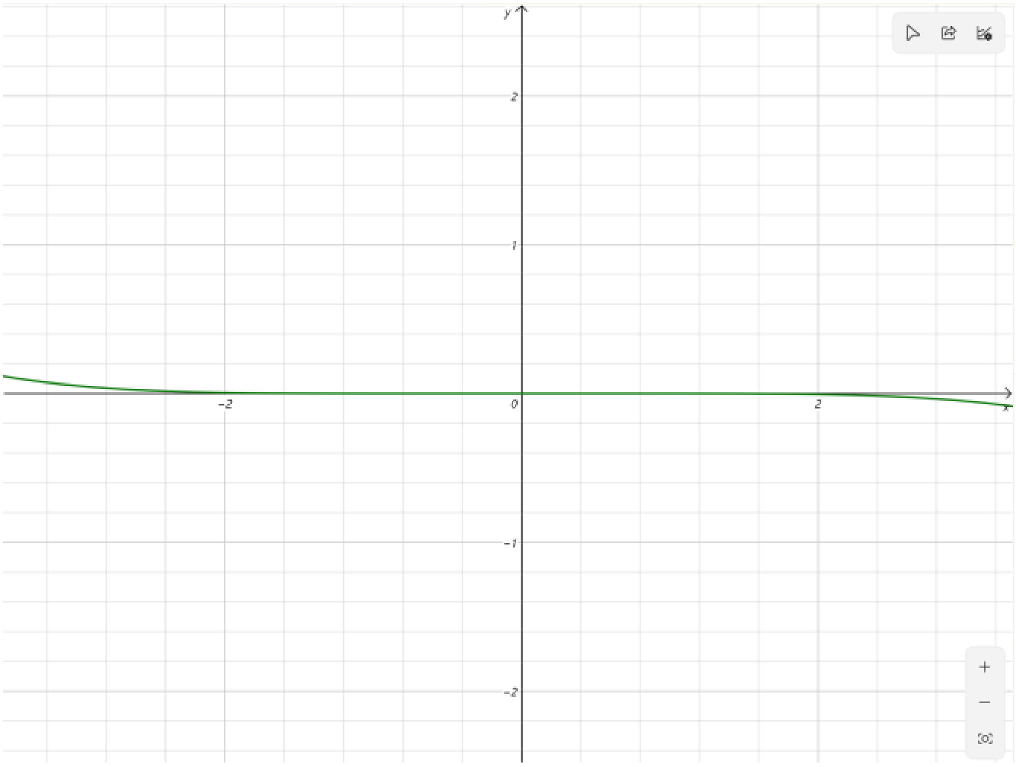
$$\begin{aligned}
&a_k=\tfrac{(f,P_k)}{(P_k,P_k)}=\tfrac{2k+1}{2}\int_{-1}^1 \sin\tfrac{x}{2}P_k(x)\mathrm{d}x\\
&a_0=\tfrac{1}{2}\int_{-1}^1 \sin\tfrac{x}{2}\cdot 1\ \mathrm{d}x=0\\
&a_1=\tfrac{3}{2}\int_{-1}^1 \sin\tfrac{x}{2}\cdot x\ \mathrm{d}x=12\sin\tfrac{1}{2}-6\cos\tfrac{1}{2}\\
&a_2=\tfrac{5}{2}\int_{-1}^1 \sin\tfrac{x}{2}\cdot \tfrac{1}{2}\left(3x^2-1\right)\mathrm{d}x=0\\
&a_3=\tfrac{7}{2}\int_{-1}^1 \sin\tfrac{x}{2}\cdot \tfrac{1}{2}\left(5x^3-3x\right)\mathrm{d}x=826\cos\tfrac{1}{2}-1512\sin\tfrac{1}{2}
\end{aligned}$$

所以:

$$\begin{aligned}
P(x)&=\left(12\sin\frac{1}{2}-6\cos\frac{1}{2}\right)x+\left(826\cos\frac{1}{2}-1512\sin\frac{1}{2}\right)\cdot\frac{1}{2}\left(5x^3-3x\right)\\
&=\left(2065\cos\frac{1}{2}-3780\sin\frac{1}{2}\right)x^3-\left(12\sin\frac{1}{2}-6\cos\frac{1}{2}\right)x\\
&\approx -0.02055x^3+0.49994x
\end{aligned}$$

误差 $E(x)=P(x)-f(x)\approx -0.02055x^3+0.49994x-\sin\frac{x}{2}$

其图形:



均方误差:
$\|\delta\|_2^2=\int_{-1}^{1}(f(x)-P(x))^2\,\mathrm{d}x\approx 1.96709\times 10^{-10}$

第 22 题

$\Phi=\operatorname{span}\{1,x^2\}$

$$\left(\varphi_0,\varphi_0\right)=\sum_{i=1}^{5}\varphi_0\left(x_i\right)^2=5,\quad\left(\varphi_0,\varphi_1\right)=\sum_{i=1}^{5}\varphi_0\left(x_i\right)\varphi_1\left(x_i\right)=5327$$
$$\left(\varphi_1,\varphi_1\right)=7277699,\quad\left(\varphi_0,y\right)=271.4,\quad\left(\varphi_1,y\right)=369321.5$$

法方程为:

$$\begin{bmatrix}5&5327\\5327&7277699\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}271.4\\369321.5\end{bmatrix}\tag{4}$$

解得: $a\approx 0.972748,b=0.050035$
所以经验公式:$y=0.972748+0.050035x^2$
均方误差:$\|\delta\|_2^2=\sum_{i=1}^{5}\left(a+bx_i^2-y_i\right)^2\approx 0.015023$

第 24 题

观察源数据的图像，可以发现 y 随着 t 单调递增，但是随着 t 越大，y 的增长率越小，可以得到 $y^{'}>0, y^{''}<0$, 可以建立拟合模型:$y=a\mathrm{e}^{-\frac{b}{t}}, a, b>0$

等式两边取对数:

$$\ln y=\ln a-\frac{b}{t}$$

令 $\Phi=\operatorname{span}\left\{1,-\frac{1}{t}\right\}$

得到:

$$\begin{gathered}(1,1)=\sum_{i=1}^{11} 1=11, \quad\left(1,-\frac{1}{t}\right)=-0.603975 \\ \left(-\frac{1}{t},-\frac{1}{t}\right)=0.062321, \quad(1, \ln y)=-87.674095, \quad\left(-\frac{1}{t}, \ln y\right)=5.032489\end{gathered} \tag{5}$$

法方程:

$$\left[\begin{array}{cc}11 & -0.603975 \\ -0.603975 & 0.062321\end{array}\right]\left[\begin{array}{c}\ln a \\ b\end{array}\right]=\left[\begin{array}{c}-87.674095 \\ 5.032489\end{array}\right] \tag{6}$$

解得: $\ln a=-7.558781, b=7.4961692, a=5.215148 \times 10^{-4}$

$\therefore y=5.215148 \times 10^{-4} \times e^{-\frac{7.4961692}{t}}$

第四章

第 1 题

(3) 为了使求积公式

$$\int_{-1}^{1} f(x) \mathrm{d} x \approx \frac{f(-1)+2 f\left(x_{1}\right)+3 f\left(x_{2}\right)}{3} \tag{7}$$

具有尽可能高的代数精度，只需取

$$f(x)=x^{m}, m=0,1,2, \ldots \tag{8}$$

对7式均准确成立即可

当 $f(x)=1$ 时，$2=\frac{1}{3}(1+2+3)$ 成立，所以7准确成立.

当 $f(x)=x, f(x)=x^{2}$ 时，代入7可得:

$$\left\{\begin{array}{l}-1+2 x_{1}+3 x_{2}=0 \\ 1+2 x_{1}^{2}+3 x_{2}^{2}=2\end{array}\right. \tag{9}$$

解得：

$$\left\{\begin{array}{l}x_{1}=-0.2898979 \\ x_{2}=0.6265986\end{array} \quad \text{或} \quad\left\{\begin{array}{l}x_{1}=0.6898979 \\ x_{2}=-0.1265986\end{array}\right.\right. \tag{10}$$

若再将 $f(x)=x^3$ 带入已经确定的求积公式，则:

$$\int_{-1}^{1} f(x)\mathrm{d}x \neq \frac{f(-1)+2f\left(x_1\right)+3f\left(x_2\right)}{3} \tag{11}$$

因此求积公式具有 2 次代数精度，求值节点：

$$\left\{\begin{array}{l} x_1=-0.2898979 \\ x_2=0.6265986 \end{array}\right. \quad \text{或} \quad \left\{\begin{array}{l} x_1=0.6898979 \\ x_2=-0.1265986 \end{array}\right. \tag{12}$$

第 2 题

(1) 使用复化梯形公式 ($h=\frac{1}{8}$):

$$T_8=\frac{h}{2}\left[f(0)+2\sum_{k=1}^{7} f\left(x_k\right)+f(1)\right]=0.1114024 \tag{13}$$

使用复化 Simpson 公式 ($h=\frac{1}{8}$):

$$S_8=\frac{h}{6}\left[f(0)+4\sum_{k=0}^{7} f\left(x_{k+\frac{1}{2}}\right)+2\sum_{k=1}^{7} f\left(x_k\right)+f(1)\right]=0.1115718 \tag{14}$$

第 7 题

设将区间分成 n 等份，则 $h=\frac{b-a}{n}$ 由误差公式:

$$|R|=\left|\frac{b-a}{12}h^2 f''(\eta)\right|=\left|\frac{b-a}{12}\left(\frac{b-a}{n}\right)^2 f''(\eta)\right| \leq \frac{(b-a)^3}{12n^2}M<\varepsilon \tag{15}$$

其中, $M=\max\limits_{a\leq x\leq b}|f''(x)|$

解得:

$$n>\sqrt{\frac{(b-a)^3 M}{12\varepsilon}}$$

第 11 题

(1) 如下表，其中 $T_0^{(k)}$ 中下标代表加速次数，上标代表二分次数。

表 1:

k	$T_0^{(k)}$	$T_1^{(k-1)}$	$T_2^{(k-2)}$	$T_3^{(k-3)}$	$T_4^{(k-4)}$
0	1.3333333				
1	1.1666667	1.1111111			
2	1.1166667	1.1000000	1.0992593		
3	1.1032107	1.0987253	1.0986403	1.0986305	
4	1.0997677	1.0986200	1.0986130	1.0986126	1.0986125

取 $I = 1.0986125$。

(2) 做变换:$y = t + 2$，则当 $y \in [1,3]$ 时，$t \in [-1,1]$. 则：

$$\int_1^3 \frac{\mathrm{d}y}{y} = \int_{-1}^1 \frac{\mathrm{d}t}{t+2} \tag{16}$$

三点高斯公式:

$$\begin{aligned}\int_1^3 \frac{\mathrm{d}y}{y} &= \int_{-1}^1 \frac{\mathrm{d}t}{t+2}\\ &\approx \frac{5}{9} f\left(-\frac{\sqrt{15}}{5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\frac{\sqrt{15}}{5}\right)\\ &\approx 0.5555556\left(\frac{1}{2-0.7745967} + \frac{1}{2+0.7745967}\right)\\ &\quad + 0.8888889 \times \frac{1}{2+0}\\ &= 1.0980393\end{aligned} \tag{17}$$

五点高斯公式：

$$\begin{aligned}\int_1^3 \frac{\mathrm{d}y}{y} = \int_{-1}^1 \frac{\mathrm{d}t}{t+2} \approx 0.2369269&\\ \left(\frac{1}{2-0.9061798} + \frac{1}{2+0.9061798}\right)&\\ + 0.4786289\left(\frac{1}{2-0.5384693} + \frac{1}{2+0.5384693}\right)&\\ + 0.5688889 \times \frac{1}{2+0} = 1.0986093&\end{aligned} \tag{18}$$

(3) 将区间 $[1,3]$ 分成四个小区间 $[1,1.5], [1.5,2], [2,2.5], [2.5,3]$

在第一个区间上, 做变换 $y = \frac{1}{4}t + \frac{5}{4}$.

$$\begin{aligned} I_1 &= \int_1^{1.5} \frac{\mathrm{d}y}{y} = \int_{-1}^{1} \frac{1\ \mathrm{d}t}{5+t} \\ &\approx 0.5 \times \left[\frac{1}{2.5 + 0.5 \times \left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{2.5 + 0.5 \times \left(\frac{1}{\sqrt{3}}\right)} \right] \\ &= 0.4054054 \end{aligned} \tag{19}$$

在第二个区间上, 做变换 $y = \frac{1}{4}t + \frac{7}{4}$.

$$\begin{aligned} I_2 &= \int_{1.5}^{2} \frac{\mathrm{d}y}{y} = \int_{-1}^{1} \frac{1\ \mathrm{d}t}{7+t} \\ &\approx 0.5 \times \left[\frac{1}{3.5 + 0.5 \times \left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{3.5 + 0.5 \times \left(\frac{1}{\sqrt{3}}\right)} \right] \\ &= 0.2876712 \end{aligned} \tag{20}$$

在第三个区间上, 做变换 $y = \frac{1}{4}t + \frac{9}{4}$.

$$\begin{aligned} I_3 &= \int_2^{2.5} \frac{\mathrm{d}x}{y} = \int_{-1}^{1} \frac{1\ \mathrm{d}t}{9+t} \\ &\approx 0.5 \times \left[\frac{1}{4.5 + 0.5 \times \left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{4.5 + 0.5 \times \left(\frac{1}{\sqrt{3}}\right)} \right] \\ &= 0.2231405 \end{aligned} \tag{21}$$

在第四个区间上, 做变换 $y = \frac{1}{4}t + \frac{11}{4}$.

$$\begin{aligned} I_4 &= \int_{2.5}^{3} \frac{\mathrm{d}y}{y} = \int_{-1}^{1} \frac{1\ \mathrm{d}t}{11+t} \\ &\approx 0.5 \times \left[\frac{1}{5.5 + 0.5 \times \left(-\frac{1}{\sqrt{3}}\right)} + \frac{1}{5.5 + 0.5 \times \left(\frac{1}{\sqrt{3}}\right)} \right] \\ &= 0.1823204 \end{aligned} \tag{22}$$

故:

$$I = I_1 + I_2 + I_3 + I_4 \approx 1.0985375 \tag{23}$$

比较

积分真值:

$$I = \int_1^3 \frac{\mathrm{d}y}{y} = \ln 3 = 1.098612288\cdots \tag{24}$$

说明龙贝格算法比高斯求积算法更准确，但龙贝格算法运算量更大。