Optimization Methods **Fall 2022**

Homework 3

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Notice

- The submission email is: **optfall2022@163.com**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Negative-entropy Regularization

Please show how to compute

$$\operatorname*{argmin}_{x \in \Delta^n} \quad b^\top x + c \cdot \sum_{i=1}^{n} x_i \ln x_i$$

where $\Delta^n = \{x \mid \sum_{i=1}^n x_i = 1, x_i \geq 0, i = 1, \cdots, n\}$, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

Problem 2: One inequality constraint

With $c \neq 0$, express the dual problem of

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & f(x) \leq 0 \end{aligned}$$

in terms of the conjugate $f^\star$.

Problem 3: KKT conditions

Consider the problem

$$\begin{aligned} \min_{x \in \mathbb{R}^2} \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & (x_1 - 1)^2 + (x_2 - 1)^2 \leq 2 \\ & (x_1 - 1)^2 + (x_2 + 1)^2 \leq 2 \end{aligned}$$

where $x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^\top \in \mathbb{R}^2$.

(1) Write the Lagrangian for this problem.

(2) Does strong duality hold in this problem?

(3) Write the KKT conditions for this optimization problem.

Problem 4: Equality Constrained Least-squares

Consider the equality constrained least-squares problem

$$\begin{array}{ll} \min & \frac{1}{2}\|Ax-b\|_2^2 \\ \text{s. t.} & Gx = h \end{array}$$

where $A \in \mathbf{R}^{m\times n}$ with $\mathbf{rank}\ A = n$, and $G \in \mathbf{R}^{p\times n}$ with $\mathbf{rank}\ G = p$.

(1) Derive the Lagrange dual problem with Lagrange multiplier vector v.

(2) Derive expressions for the primal solution $x^\star$ and the dual solution $v^\star$.

Problem 5: Matrix eigenvalues

We denote by $f(A)$ the sum of the largest r eigenvalues of a symmetric matrix $A \in \mathbf{S}^n$ (with $1 \leq r \leq n$), i.e.,

$$f(A) = \sum_{k=1}^{r} \lambda_k(A),$$

where $\lambda_1(A), \cdots, \lambda_n(A)$ are the eigenvalues of A sorted in decreasing order. Show that the optimal value of the optimization problem

$$\begin{array}{ll} \max & \mathrm{tr}(AX) \\ \text{s. t.} & \mathrm{tr}(X) = r, \\ & 0 \preceq X \preceq I \end{array}$$

with variable $X \in \mathbf{S}^n$ being equal to $f(A)$.