Optimization Methods **Fall 2022**

Homework 1

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Notice

- The submission email is: **optfall2022@163.com**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Inequalities

Let $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, where n is a positive integer. Let $\|\cdot\|$ denote the Euclidean norm.

a) Prove the triangle inequality $\|x + y\| \le \|x\| + \|y\|$.

b) Prove $\|x + y\|^2 \le (1 + \epsilon)\|x\|^2 + (1 + \frac{1}{\epsilon})\|y\|^2$ for any $\epsilon > 0$.

Hint: You may need the Young's inequality for products, i.e. if a and b are nonnegative real numbers and p and q are real numbers greater than 1 such that $1/p + 1/q = 1$, then $ab \le \frac{a^p}{p} + \frac{b^q}{q}$.

解：

a) Prove the triangle inequality $\|x + y\| \le \|x\| + \|y\|$.

b) Prove $\|x + y\|^2 \le (1 + \epsilon)\|x\|^2 + (1 + \frac{1}{\epsilon})\|y\|^2$ for any $\epsilon > 0$.

Problem 2: Definition of convexity

Convex C_c sets are the sets satisfying the constraints below:

$$\theta x_1 + (1 - \theta)x_2 \in C_c$$

$$\text{for all, } x_1, x_2 \in C_c, 0 \le \theta \le 1$$

Determine if each set below is convex.

a) $\{(x, y) \in \mathbb{R}^2_{++} | x/y \le 1\}$

b) $\{(x, y) \in \mathbb{R}^2_{++} | x/y \ge 1\}$

c) $\{(x, y) \in \mathbb{R}^2_{++} | xy \le 1\}$

d) $\{(x,y) \in \mathbb{R}^2_{++} | xy \geq 1\}$

e) $\{(x,y) \in \mathbb{R}^2_{++} | y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\}$

Problem 3: Convex sets

a) Show that a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m\}$ is convex.

b) Show that if $S \subseteq \mathbb{R}^n$ is convex, and $A \in \mathbb{R}^{m \times n}$, then $A(S) = \{Ax : x \in S\}$, is convex.

c) Show that if $S \subseteq \mathbb{R}^m$ is convex, and $A \in \mathbb{R}^{m \times n}$, then $A^{-1}(S) = \{x : Ax \in S\}$, is convex.

Problem 4: Convex cone

Let K be a convex cone. The set $K^* = \left\{y | x^\top y \geq 0, \forall x \in K\right\}$ is called the dual cone of K.

a) Show that K^* is a convex cone (even K is not convex).

b) Show that a dual cone of a subspace $V \subset \mathbb{R}^n$ (which is a cone) is its orthogonal complement $V^+ = \{y | y^\top v = 0, \forall v \in V\}$.

c) What is the dual cone of the nonnegative orthant ($\mathbb{R}^+_n$)?

$$K^* = \{(u,v) \in \mathbb{R}^{n+1} | \|u\|_* \leq v\}$$

where the dual norm is given by $\|u\|_* = \sup\{u^\top x | \|x\| \leq 1\}$

Problem 5: Generalized Inequalities

Let K^* be the dual cone of a convex cone K. Prove the following,

a) K^* is indeed a convex cone.

b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.