Assignment 2

Lecturer: Yizheng Zhao

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⋆ **This assignment, due on 23rd April, contributes to 10% of the total mark of the course.**

Question 1. Closure under Disjoint Union

Recall Theorem 3.8 from the lecture, which says that the disjoint union of a family of models of an $\mathcal{ALC}$-TBox $\mathcal{T}$ is again a model of $\mathcal{T}$. Note that the disjoint union is only defined for concept and role names.

- Extend the notion of disjoint union to individual names such that the following holds: for any family $(\mathcal{I}_\nu)_{\nu\in\Omega}$ of models of an $\mathcal{ALC}$-knowledge base $\mathcal{K}$, the disjoint union $\biguplus_{\nu\in\Omega}\mathcal{I}_\nu$ is also a model of $\mathcal{K}$.

Question 2. Closure under Disjoint Union

Let $\mathcal{K} = \{\mathcal{T}, \mathcal{A}\}$ be a consistent $\mathcal{ALC}$-KB. We write $C \sqsubseteq_\mathcal{K} D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds for every model $\mathcal{I}$ of $\mathcal{K}$.

- Prove that for all $\mathcal{ALC}$-concepts C and D we have $C \sqsubseteq_\mathcal{K} D$ iff $C \sqsubseteq_\mathcal{T} D$.

Hint: Use the modified definition of disjoint union from the previous question.

Question 3. Finite Model Property (fmp)

Let C be an $\mathcal{ALC}$-concept that is satisfiable w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$. Show truth or falsity of the following statement:

- for all $m \geq 1$ there is a finite model $\mathcal{I}_m$ of $\mathcal{T}$ such that $|C^{\mathcal{I}_m}| \geq m$.
- Does it hold if the condition "$|C^{\mathcal{I}_m}| \geq m$" is replaced by "$|C^{\mathcal{I}_m}| = m$"?

Question 4. Bisimulation over Filtration

Let C be an $\mathcal{ALC}$-concept, $\mathcal{T}$ an $\mathcal{ALC}$-TBox, $\mathcal{I}$ an interpretation and $\mathcal{J}$ its filtration w.r.t. $\mathsf{sub}(C) \cup \mathsf{sub}(\mathcal{T})$ (see Definition 3.14 for the definition of filtration). Show truth or falsity of the following statement:

- the relation $\rho = \{(\mathsf{d}, [\mathsf{d}]) \mid \mathsf{d} \in \Delta^\mathcal{I}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

Hint: If the above relation ρ were a bisimulation, why do we have to explicitly prove Lemma 3.15 in the lecture? Wouldn't Lemma 3.15 then be a consequence of Theorem 3.2?

Question 5. Bisimulation within the Same Interpretation

We define "*bisimulations on* $\mathcal{I}$" as bisimulations between an interpretation $\mathcal{I}$ and itself. Let $\mathsf{d}, \mathsf{e} \in \Delta^\mathcal{I}$ be two elements. We write $\mathsf{d} \approx_\mathcal{I} \mathsf{e}$ if they are bisimilar, i.e., if there is a bisimulation ρ on $\mathcal{I}$ such that $\mathsf{d}\ \rho\ \mathsf{e}$.

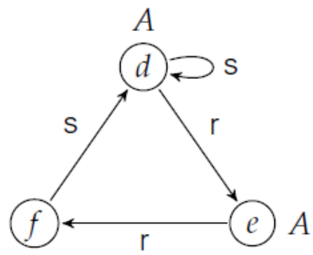
- Show that $\approx_\mathcal{I}$ is a bisimulation on $\mathcal{I}$.

Consider the interpretation $\mathcal{J}$ defined like the filtration, but with $\approx_\mathcal{I}$ instead of $\simeq$.

- Show that $\rho = \{(\mathsf{d}, [\mathsf{d}]_{\approx_{\mathcal{I}}}) \mid \mathsf{d} \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.
- Show that, if $\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$, then so is $\mathcal{J}$.
- Why can't we use the previous result to show the finite model property for $\mathcal{ALC}$?

Question 6. Unravelling

Draw the unravelling of the following interpretation $\mathcal{I}$ at d up to depth 5, i.e., restricted to d-paths of length at most 5 (see Definition 3.21):



Question 7. Tree Model Property (tmp)

- Show the truth or falsity of the following statement: if $\mathcal{K}$ is an $\mathcal{ALC}$-KB and C an $\mathcal{ALC}$-concept such that C is satisfiable w.r.t. $\mathcal{K}$, then C has a tree model w.r.t. $\mathcal{K}$.

Question 8. Tableau Algorithm

- Apply the Tableau algorithm consistent$(\mathcal{A})$ to the following ABox:

  $$\mathcal{A} = \{(b, a) : r, (a, b) : r, (a, c) : s, (c, b) : s, a : \exists s.A, b : \forall r.((\forall s.\neg A) \sqcup (\exists r.B)), c : \forall s.(B \sqcap (\forall s.\bot))\}.$$

  If $\mathcal{A}$ is consistent, draw the model generated by the algorithm.

Question 9. Extension of Tableau Algorithm

We consider the concept constructor $\rightarrow$ (implication) with the following semantics:

$$(C \rightarrow D)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid x \in C^{\mathcal{I}} \text{ implies } x \in D^{\mathcal{I}}\}.$$

To extend consistent$(\mathcal{A})$ to this constructor, we propose two alternative new expansion rules:

The deterministic $\rightarrow$-rule	
Condition:	$\mathcal{A}$ contains $a : C \rightarrow D$ and $a : C$, but not $a : D$
Action:	$\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : D\}$

The nondeterministic $\rightarrow$-rule	
Condition:	$\mathcal{A}$ contains $a : C \rightarrow D$, but neither $a : \dot{\neg} C$ nor $a : D$
Action:	$\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : X\}$ for some $X \in \{\dot{\neg} C, D\}$

For each rule, determine whether the extended algorithm remains terminating, sound, and complete.

Question 10. Modification of Tableau Algorithm

We consider an $\mathcal{ALC}$ TBox $\mathcal{T}$ consisting only of the following two kinds of axioms:

- role inclusions of the form $r \sqsubseteq s$, and
- role disjointness constraints of the form $\mathsf{disjoint}(r, s)$.

where r and s are role names. An interpretation $\mathcal{I}$ satisfies these axioms if

- $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$, and
- $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$, respectively.

Modify the Tableau algorithm $\mathsf{consistent}(\mathcal{A})$ to decide the consistency of $(\mathcal{T}, \mathcal{A})$, where $\mathcal{A}$ is an ABox and $\mathcal{T}$ a TBox containing only role inclusions and role disjointness constraints. Show that the algorithm remains terminating, sound, and complete.