第~~十~~三次作业

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1. 解：设 $X_1, X_2, \cdots X_m$ 和 $X_1', X_2', \cdots, X_n'$ 分别来自总体 $X \sim N(\mu, a^2)$ 的两个独立样本.

则 $\overline{X_1} \sim N(\mu, \frac{a^2}{m})$，$\overline{X_2} \sim N(\mu, \frac{a^2}{n})$

$Y = \overline{X_1} - \overline{X_2} \sim N(0, \frac{a^2}{m} + \frac{a^2}{n})$

$$\therefore P(|Y| < \varepsilon) = P(-\varepsilon < Y < \varepsilon) = P\left(-\frac{\varepsilon}{\sqrt{\frac{a^2}{m}+\frac{a^2}{n}}} < \frac{Y}{\sqrt{\frac{a^2}{m}+\frac{a^2}{n}}} < \frac{\varepsilon}{\sqrt{\frac{a^2}{m}+\frac{a^2}{n}}}\right)$$

$$= 1 - 2\Phi\left(-\frac{\varepsilon}{\sqrt{\frac{a^2}{m}+\frac{a^2}{n}}}\right)$$

2. 解：设 $Z = X+Y$，根据独立同分布和函数分布，有

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(x) f_Y(z-x)\,dx = \int_0^z \frac{\lambda^{\alpha_1}}{\Gamma(\alpha_1)} x^{\alpha_1-1} e^{-\lambda x} \frac{\lambda^{\alpha_2}}{\Gamma(\alpha_2)} (z-x)^{\alpha_2-1} e^{-\lambda(z-x)}\,dx$$

$$= \frac{\lambda^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} e^{-\lambda z} \int_0^z x^{\alpha_1-1}(z-x)^{\alpha_2-1}\,dx$$

令 $x = zt$，得

$$\int_0^z x^{\alpha_1-1}(z-x)^{\alpha_2-1}\,dx = z^{\alpha_1+\alpha_2-1}\int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\,dt = z^{\alpha_1+\alpha_2-1} B(\alpha_1, \alpha_2)$$

又 $\because B(\alpha_1, \alpha_2) = \int_0^1 t^{\alpha_1-1}(1-t)^{\alpha_2-1}\,dt = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$

代入完成证明.

3. 解：$y\le 0$时，$F_Y(y)=0$

$y>0$时，$F_Y(y)=P(X^2\le y)=P(-\sqrt{y}\le x\le\sqrt{y})$

$$=\int_{-\sqrt{y}}^{\sqrt{y}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$$

求导，得 $f_Y(y)=\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{y}}e^{-\frac{y}{2}}$

$\therefore X^2\sim\Gamma(\frac{1}{2},\frac{1}{2})$

当k为奇数时，$E(X^k)=\int_{-\infty}^{+\infty}\frac{x^k}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$，$\because \frac{x^k}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$是奇函数

$\therefore E(X^k)=0$

当k为偶数时，$E(X^k)=\int_{-\infty}^{+\infty}\frac{x^k}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

$$=\sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{x^ky^k}{2\pi}e^{-\frac{x^2+y^2}{2}}dx\,dy}$$

$$=\sqrt{\int_0^{2\pi}d\theta\int_0^{+\infty}\frac{(\rho\cos\theta)^k(\rho\sin\theta)^k}{2\pi}e^{-\frac{\rho^2}{2}}\rho\,d\rho}$$

$$=\sqrt{\frac{1}{2\pi}\frac{1}{2^{k+2}}\int_0^{4\pi}\sin^k t\,dt\int_0^{+\infty}t^ke^{-\frac{t}{2}}dt}$$

$$=\sqrt{\frac{1}{2\pi}\frac{1}{2^{k-1}}\frac{(k-1)!!}{k!!}\frac{\pi}{2}\int_0^{+\infty}t^ke^{-\frac{t}{2}}dt}$$

$$=(k-1)!!$$

4. 解：由题意 当$Z_1=\sqrt{a}(X_1+2X_2+\cdots+nX_n)\sim N(0,1)$，$Z_2=\sqrt{b}(Y_1+2Y_2+\cdots+mY_m)\sim N(0,1)$

$D(Z_n)=D(X_1+\cdots+nX_n)=(1^2+2^2+\cdots+n^2)\sigma^2=\frac{1}{6}n(n+1)(2n+1)\sigma^2$

同理可知 $D(Z_m)=\frac{1}{6}m(m+1)(2m+1)\sigma^2$

$\therefore b=\frac{6}{m(m+1)(2m+1)\sigma^2}$，$a=\frac{6}{n(n+1)(2n+1)\sigma^2}$

$\therefore Z\sim\chi^2(2)$

5、解：$Z=\dfrac{\sum_{i=1}^{n}X_i}{\sqrt{\sum_{i=1}^{n}Y_i^2}}=\dfrac{\frac{1}{n}\sum_{i=1}^{n}X_i}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}Y_i^2/n}}$，$\frac{1}{n}\sum_{i=1}^{n}X_i\sim N(0,1)$，$\frac{1}{\sqrt{n}}Y_i\sim N(0,1)$

$\therefore Z\sim t(n)$

6、解：$Y=\dfrac{(n-1)S^2}{\sigma^2}=4\sum_{i=1}^{n}(X_i-\bar{X})^2\sim\chi^2(n-1)$

$P(\sum_{i=1}^{n}(X_i-\bar{X})^2\geqslant\varepsilon)=P(4\sum_{i=1}^{n}(X_i-\bar{X})^2\geqslant4\varepsilon)=P(Y\geqslant4\varepsilon)$

7、解：$\dfrac{(n-1)S^2}{\sigma^2}\sim\chi^2(n-1)$

$\therefore T=\dfrac{\frac{1}{\sigma^2}\cdot\frac{1}{n}\cdot\sum_{i=1}^{n}(X_i-12)}{\sqrt{\frac{1}{n-1}\frac{(n-1)S^2}{\sigma^2}}}\sim t(n-1)$

$\therefore P(\frac{1}{n}\sum_{i=1}^{n}X_i\geqslant\varepsilon)=P(\dfrac{\frac{1}{\sigma^2}\cdot\frac{1}{n}\sum_{i=1}^{n}(X_i-12)}{\sqrt{\frac{1}{n-1}\frac{(n-1)S^2}{\sigma^2}}}\geqslant\dfrac{\varepsilon-12}{\sqrt{S^2}})=P(T\geqslant\dfrac{\varepsilon-12}{\sqrt{S^2}})$

7. 解：$E(X_i)=1\times0.3+1.2\times0.2+1.5\times0.5=1.29$

$E(X_i^2)=1^2\times0.3+1.2^2\times0.2+1.5^2\times0.5=1.713$

$D(X_i)=E(X_i^2)-[E(X_i)]^2=0.0489$

(1) 设X表示总收入，则 $X=\sum_{i=1}^{300}X_i$，

由于 $X_1,\cdots,X_{300}$ 独立同分布，

所以 $P(X\geq400)=P(400\leq X\leq\infty)$

$$=P\left(\frac{400-1.29\times300}{\sqrt{300}\sqrt{0.0489}}\leq\frac{\sum_{i=1}^{300}X_i-300\times1.29}{\sqrt{300}\sqrt{0.0489}}<\frac{\infty-300\times1.29}{\sqrt{300}\sqrt{0.0489}}\right)$$

$$\approx1-\Phi(3.39)=0.0003$$

(2) 设Y为售出价格为1.2元的蛋糕的数量，

则 $Y\sim B(300,0.2)$

$E(Y)=300\times0.2=60$

$D(Y)=300\times0.2\times(1-0.2)=48$

根据棣莫弗-拉普拉斯中心极限定理，

$Y\xrightarrow{d}N(60,48)$

$\therefore P(Y>60)=1-P(Y\leq60)$

$$=1-P\left(\frac{Y-60}{\sqrt{48}}\leq\frac{60-60}{\sqrt{48}}\right)$$

$$\approx1-\Phi(0)=0.5$$

9、解：(1) 由中心极限定理 $\bar{X} \sim N(2.2, 1.4^2/52)$

$$P(\bar{X}<2) = P\left(\frac{\bar{X}-2.2}{1.4/\sqrt{52}} < \frac{2-2.2}{1.4/\sqrt{52}}\right) = \Phi\left(-\frac{\sqrt{52}}{7}\right) = 1-\Phi\left(\frac{\sqrt{52}}{7}\right) = 1-\Phi(1.03)$$

$$= 0.1515$$

(2) $$P\left(\sum_{i=1}^{52} X_i \geq 100\right) = P\left(\frac{\frac{1}{52}\sum_{i=1}^{52} X_i - 2.2}{1.4/\sqrt{52}} \geq \frac{100/52-2.2}{1.4/\sqrt{52}}\right)$$

$$= 1-\Phi\left(\frac{100/52-2.2}{1.4/\sqrt{52}}\right) = 0.0764$$

11、解：(1) $\bar{X} \sim N(5, 0.3/80)$

$$\therefore P(4.9<\bar{X}<5.1) = P\left(\frac{-0.1}{\sqrt{\frac{3}{800}}} < \frac{\bar{X}-5}{\sqrt{\frac{3}{800}}} < \frac{0.1}{\sqrt{\frac{3}{800}}}\right)$$

$$= 2\Phi\left(\frac{0.1}{\sqrt{\frac{3}{800}}}\right)-1 = 2\Phi(1.633)-1 = 0.8968$$

(2) $E(\bar{X}-\bar{Y}) = 0$

$D(\bar{X}-\bar{Y}) = D(\bar{X})+D(\bar{Y}) = \frac{3}{400}$

$$P(-0.1<\bar{X}-\bar{Y}<0.1) = P\left(\frac{-0.1}{\sqrt{\frac{3}{400}}} < \frac{\bar{X}-\bar{Y}}{\sqrt{\frac{3}{400}}} < \frac{0.1}{\sqrt{\frac{3}{400}}}\right)$$

$$= 2\Phi\left(\frac{0.1}{\sqrt{\frac{3}{400}}}\right)-1 = 0.7498$$

12. 解：设 $X_1, X_2, \cdots, X_{200}$ 分别为住户 1，…，住户 200 拥有车辆数.

$E(X_i) = 0\times 0.1 + 1\times 0.6 + 2\times 0.3 = 1.2$

$E(X_i^2) = 0^2\times 0.1 + 1^2\times 0.6 + 2^2\times 0.3 = 1.8$

$D(X_i) = E(X_i^2) - [E(X_i)]^2 = 0.36$

由中心极限定理，$\sum_{i=1}^{200} X_i \sim N(200\times 1.2,\ 200\times 0.36)$

$P(\sum_{i=1}^{200} X_i \le n) \geq 0.95$

即 $0.95 \le \Phi\left(\frac{n-200\times 1.2}{\sqrt{200\times 0.36}}\right) = \Phi\left(\frac{n-240}{\sqrt{72}}\right)$

查表知，$\frac{n-240}{\sqrt{72}} \geq 1.645$

$\therefore n \geq 240 + 1.645\times\sqrt{72} = 253.96$

$\therefore$ 需要有 254 个车位.

13. 解：由中心极限定理：$\bar{X} \xrightarrow{d} N(\mu, 400/n)$

$\therefore P(|\bar{X}-\mu|<1) = P(-1 < \bar{X}-\mu < 1)$

$= P\left(\frac{-1}{20/\sqrt{n}} < \frac{\bar{X}-\mu}{20/\sqrt{n}} < \frac{1}{20/\sqrt{n}}\right)$

$\approx 2\Phi\left(\frac{1}{20/\sqrt{n}}\right) - 1 \geq 0.95$

$\therefore \Phi\left(\frac{1}{20/\sqrt{n}}\right) \geq \Phi(1.96)$

$\therefore 20/\sqrt{n} \le \frac{1}{1.96}$

解得：$n \geq (20\times 1.96)^2 = 1536.64$

$\therefore n$ 至少为 1537

14. 解：(1) 设X表示被治愈的病人个数

$X \sim B(100, 0.8)$

由棣莫弗中心极限定理：

$X \xrightarrow{P} N(80, 16)$

$\therefore P(X>75) \approx 1-\Phi\left(\frac{75-80}{4}\right) = \Phi(1.25) = 0.8944$

(2) $X \sim B(100, 0.7)$

$X \xrightarrow{P} N(70, 21)$

$\therefore P(X>75) \approx 1-\Phi\left(\frac{75-70}{\sqrt{21}}\right) = 1-\Phi(1.09) = 0.1379$