KRP	**Spring 2023**

Homework 2

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Question 1. Closure under Disjoint Union

Let $\mathcal{K} =< \mathcal{T}, \mathcal{A} >$ be a $\mathcal{ALC}$ knowledge base, $(\mathcal{I}_v)_{v \in \Omega}$ a family of models of $\mathcal{K}$.

The extend the notion of disjoint union to individual names is as follow:

- $\Delta^{\mathcal{J}} = \{(d, v) | v \in \Omega \text{ and } d \in \Delta^{\mathcal{I}_v}\}$
- $A^{\mathcal{J}} = \{(d, v) | v \in \Omega \text{ and } d \in A^{\mathcal{I}_v}\}$ for all $A \in \mathbf{C}$
- $r^{\mathcal{J}} = \{((d, v), (e, v)) | v \in \Omega \text{ and } (d, e) \in r^{\mathcal{I}_v}\}$ for all $r \in \mathbf{R}$
- $a^{\mathcal{J}} = (a^{\mathcal{I}_v}, v)$

Now we prove that : for any family $(\mathcal{I}_v)_{v \in \Omega}$ of models of an ALC-knowledge base $\mathcal{K}$, the disjoint union $\mathcal{J} = \biguplus_{v \in \Omega}$ is also a model of $\mathcal{K}$. Use proof by contradiction:

If $\mathcal{J}$ is not a model of $\mathcal{T}$, then there exists a GCI: $C \sqsubseteq D$, such that $C^{\mathcal{J}} \not\subseteq D^{\mathcal{J}}$, which means there exists a element (d, v) in $C^{\mathcal{J}}$, but not in $D^{\mathcal{J}}$, so there exist a $\mathcal{I}_v$ such that $d \in C^{\mathcal{I}_v}, d \notin D^{\mathcal{I}_v}$. And what is contradictory to $\mathcal{I}_v$ is a model of $\mathcal{T}$. So $\mathcal{J}$ is a model of $\mathcal{T}$.

If $\mathcal{J}$ is not a model of $\mathcal{K}$

case 1: there exists a Assertion: $a : A$, such that $a^{\mathcal{J}} \notin A^{\mathcal{J}}$, which means there exist a $\mathcal{I}_v$ such that $a^{\mathcal{I}_v} \notin A^{\mathcal{I}_v}$. And what is contradictory to $\mathcal{I}_v$ is a model of $\mathcal{K}$.

case 2: there exists a Assertion: $(a, b) : r$, such that $((a^{\mathcal{J}}, v), (b^{\mathcal{J}}, v)) \notin r^{\mathcal{J}}$, which means there exist a $\mathcal{I}_v$ such that $(a^{\mathcal{I}_v}, a^{\mathcal{I}_v}) \notin r^{\mathcal{I}_v}$. And what is contradictory to $\mathcal{I}_v$ is a model of $\mathcal{K}$.

So $\mathcal{J}$ is a model of $\mathcal{K}$.

To sum up, $\mathcal{J} = \biguplus_{v \in \Omega}$ is also a model of $\mathcal{K}$.

Question 2. Closure under Disjoint Union

$C \sqsubseteq_{\mathcal{T}} D \Longrightarrow C \sqsubseteq_{\mathcal{K}} D$:

If $C \sqsubseteq_{\mathcal{T}} D$, then $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ hold for all models $\mathcal{I}$ of $\mathcal{T}$. For any model $\mathcal{J}$ of $\mathcal{K}$, it must a modle of $\mathcal{T}$, so we know $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$ holds for all model $\mathcal{J}$ of $\mathcal{K}$, so $C \sqsubseteq_{\mathcal{K}} D$

$C \sqsubseteq_{\mathcal{K}} D \Longrightarrow C \sqsubseteq_{\mathcal{T}} D$:

If $C \not\sqsubseteq_{\mathcal{T}} D$, then there exists a model $\mathcal{I}_0$ of $\mathcal{T}$, but $C^{\mathcal{I}_0} \not\subseteq D^{\mathcal{I}_0}$, because $\mathcal{K}$ is consistent, we can extend $\mathcal{I}_0$ such that $\mathcal{I}_0$ became a model of $\mathcal{K} =< \mathcal{T}, \mathcal{A} >$.

Then we construct a disjoint union $\mathcal{J} = \mathcal{I}_0 \cup \mathcal{I}_1$. ($\mathcal{I}_1$ is model of $\mathcal{K}$ and $C^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1}$). According to the conclusion of Q1, $\mathcal{J}$ is also a model of $\mathcal{K}$, but $C^{\mathcal{J}} \not\subseteq D^{\mathcal{J}}$. And what is contradictory to $C \sqsubseteq_{\mathcal{K}} D$. So we get $C \sqsubseteq_{\mathcal{T}} D$.

Question 3. Finite Model Property (fmp)

(1) True.

According to Finite Model Property, C has a finite model, which means there exists model $\mathcal{I}$ of $\mathcal{T}$ s.t. $|C^{\mathcal{I}}| \geq 1$.

Let $\mathcal{I}_m = \biguplus_{v \in \{1, \cdots, m\}} \mathcal{I}$, so $|C^{\mathcal{I}_m}| \geq m$.

(2) False.

Here's a counter-example: Let $C = \top$, $\mathcal{T} = \{A \sqsubseteq \exists r.\neg A, \neg A \sqsubseteq \exists r.A\}$ and $m = 1$.

If the conclusion holds, then there exists only one element in $\Delta^{\mathcal{I}}$, because: $\Delta^{\mathcal{I}} = A^{\mathcal{I}} \cup \neg A^{\mathcal{I}}$.

case 1: $A^{\mathcal{I}} = \emptyset$, so $\exists r.A = \emptyset$, according to the second GCI in $\mathcal{T}$, $\neg A = \emptyset$, and then $\Delta^{\mathcal{I}} = A^{\mathcal{I}} \cup \neg A^{\mathcal{I}} = \emptyset$.And what is contradictory.

case 2: $\neg A^{\mathcal{I}} = \emptyset$, so $\exists r.\neg A = \emptyset$, according to the first GCI in $\mathcal{T}$, $A = \emptyset$, and then $\Delta^{\mathcal{I}} = A^{\mathcal{I}} \cup \neg A^{\mathcal{I}} = \emptyset$.And what is contradictory.

So : $A^{\mathcal{I}} \neq \emptyset$ and $\neg A^{\mathcal{I}} \neq \emptyset$, which means $|\Delta^{\mathcal{I}} = A^{\mathcal{I}} \cup \neg A^{\mathcal{I}}| \geq 2$.

It doesn't hold if the condition $|C^{\mathcal{I}_m}| \geq m$ is replaced by $|C^{\mathcal{I}_m}| = m$.

Question 4. Bisimulation over Filtration

False.

Let $C = A$ and $\mathcal{T} = \{\exists r.\top \sqsubseteq \top\}$, so $S = \text{sub}(C) \cup \text{sub}(\mathcal{T}) = \{\top, A, \exists r.\top\}$.

3.4 Finite model property 61

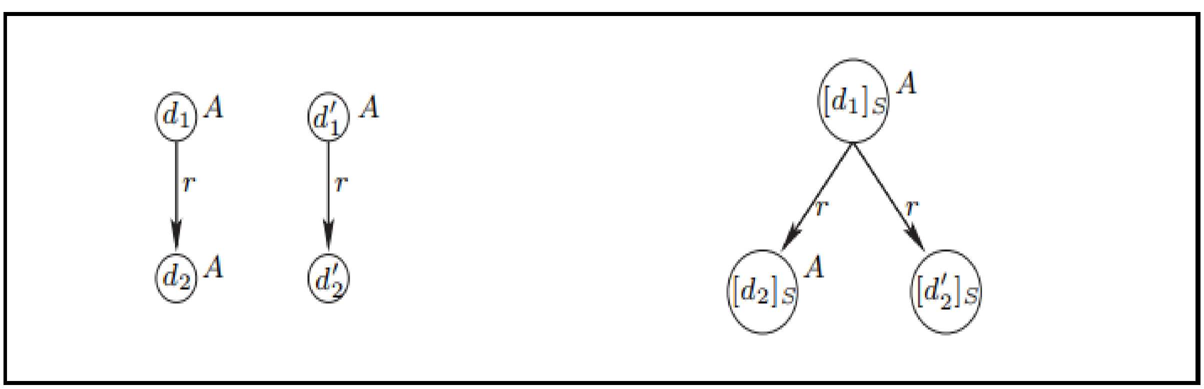


Fig. 3.4. An interpretation $\mathcal{I}$ and its S-filtration $\mathcal{J}$ for $S = \{\top, A, \exists r.\top\}$.

$[d_1]_S = \{d_1, d_1'\}, [d_2]_S = \{d_2\}, [d_2']_S = \{d_2'\}$

we can see that the relation $\rho = \{(d, [d])|d \in \Delta^{\mathcal{I}}\}$ is not a bisimmulation between $\mathcal{I}$ and $\mathcal{J}$.

Question 5. Bisimulation within the Same Interpretation

(1)
- $d_1 \approx_{\mathcal{I}} d_2$ implies there is a bisimulation ρ on $\mathcal{I}$ such that $d\rho e$,which means

$$d_1 \in A^{\mathcal{I}} \text{ iff } d_2 \in A^{\mathcal{I}} \tag{1}$$

  for all $d_1 \in \Delta^{\mathcal{I}}$, $d_2 \in \Delta^{\mathcal{I}}$ and $A \in C$.

- $d_1 \approx_{\mathcal{I}} d_2$ implies there is a bisimulation $d_1 \rho d_2$, due to the bisimualtion, $(d_1, d_1^{'}) \in r^{\mathcal{I}}$ implies there exists $d_2^{'} \in \Delta^{\mathcal{I}}$ such that

$$d_1^{'}\, \rho\, d_2^{'} \text{ and } (d_2, d_2^{'}) \in r^{\mathcal{I}} \tag{2}$$

  for all $d_1, d_1^{'}, d_2 \in \Delta^{\mathcal{I}}$ and $r \in \mathbf{R}$.

- $d_1 \approx_{\mathcal{I}} d_2$ implies there is a bisimulation $d_1 \rho d_2$, due to the bisimualtion, $(d_2, d_2^{'}) \in r^{\mathcal{I}}$ implies there exists $d_1^{'} \in \Delta^{\mathcal{I}}$ such that

$$d_1^{'}\, \rho\, d_2^{'} \text{ and } (d_1, d_1^{'}) \in r^{\mathcal{I}} \tag{3}$$

  for all $d_1, d_2^{'}, d_2 \in \Delta^{\mathcal{I}}$ and $r \in \mathbf{R}$.

So $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

(2)
- $d\, \rho\, [d]_{\approx_{\mathcal{I}}}$ implies $d \in A^{\mathcal{I}}$ iff $[d]_{\approx_{\mathcal{I}}} \in A^{\mathcal{J}}$

  If $d \in A^{\mathcal{I}}$, there must be a $[d]_{\approx_{\mathcal{I}}} \in A^{\mathcal{J}}$ by the definition of filtration. On the contrary, if $[d]_{\approx_{\mathcal{I}}} \in A^{\mathcal{J}}$, according to the definition, there exists $d' \in [d]_{\approx_{\mathcal{I}}}$ such that $d \approx_{\mathcal{I}} d'$ and $d' \in A^{\mathcal{I}}$, because $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$, so $d \in A^{\mathcal{I}}$. Therefore, $d \in A^{\mathcal{I}}$ iff $[d]_{\approx_{\mathcal{I}}} \in A^{\mathcal{J}}$.

- $d\,\rho\,[d]_{\approx_\mathcal{I}}$ and $(d,d') \in r^\mathcal{I}$ implies there exists $[d']_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}$, $d'\,\rho\,[d']_{\approx_\mathcal{I}}$ and $([d]_{\approx_\mathcal{I}}, [d']_{\approx_\mathcal{I}}) \in r^\mathcal{J}$.

  If $(d,d') \in r^\mathcal{I}$, there must be $[d']_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}$. Because of the third property of the definition of a filtration, there is $([d]_{\approx_\mathcal{I}}, [d']_{\approx_\mathcal{I}}) \in r^\mathcal{J}$, $d \in [d]_{\approx_\mathcal{I}}$, $d' \in [d']_{\approx_\mathcal{I}}$ $(d,d') \in r^\mathcal{I}$.

- $(d, [d]_{\approx_\mathcal{I}}) \in \rho$ and $([d]_{\approx_\mathcal{I}}, [e]_{\approx_\mathcal{I}}) \in r^\mathcal{J}$ implies there is $d' \in [d]_{\approx_\mathcal{I}}$, $e' \in [e]_{\approx_\mathcal{I}}$ with $(d', e') \in r^\mathcal{I}$.

  Because $d \in [d]_{\approx_\mathcal{I}}$, we can know $d \approx_\mathcal{I} d'$. And $\approx_\mathcal{I}$ is a bisimulation on $\mathcal{I}$, which implies the existence of $e \in \Delta^\mathcal{I}$ such that $e \approx_\mathcal{I} e'$ and $(d,e) \in r^\mathcal{I}$ So we can know $(e, [e]_{\approx_\mathcal{I}}) \in \rho$ and $(d,e) \in r^\mathcal{I}$ for all $d \in \Delta^\mathcal{I}$, $[d]_{\approx_\mathcal{I}}, [e]_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}$, and $r \in \mathbf{R}$.

So we show that $\rho = \{(d, [d]_{\approx_\mathcal{I}}) | d \in \Delta^\mathcal{I}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

(3) If $\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C w.r.t an $\mathcal{ALC}$-TBox $\mathcal{T}$, then $C^\mathcal{I} \neq \emptyset$.

If $d \in C^\mathcal{I}$, because there is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$, so $[d]_{\approx_\mathcal{I}} \in C^\mathcal{J}$ according to bisimulation invariance of $\mathcal{ALC}$.

It is easy to see that $\mathcal{J}$ is a model of $\mathcal{T}$. Let $D \sqsubseteq E$ be a GCI in $\mathcal{T}$ and $[d]_{\approx_\mathcal{I}} \in D^\mathcal{J}$. By bisimulation invariance, $d \in D^\mathcal{I}$ and $d \in E^\mathcal{I}$ because $\mathcal{I}$ is a model of $\mathcal{T}$.Therefore $[d]_{\approx_\mathcal{I}} \in E^\mathcal{J}$.

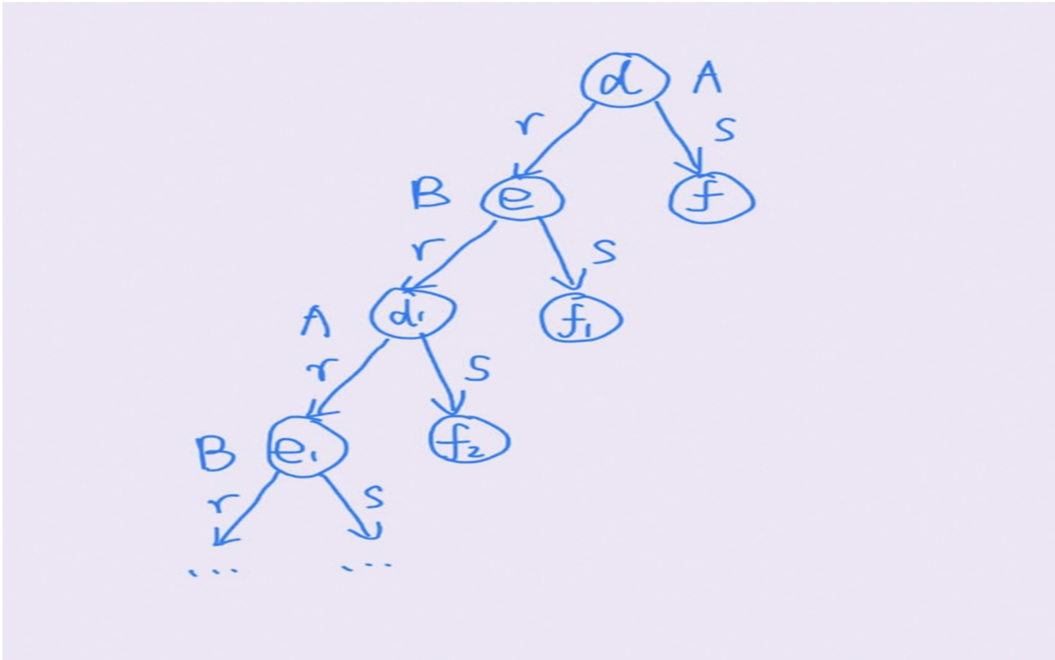
So $\mathcal{J}$ is a model of an $\mathcal{ALC}$-concept C w.r.t an $\mathcal{ALC}$-TBox $\mathcal{T}$.

(4) Because we can't get the bound of the $|\Delta^\mathcal{J}|$.

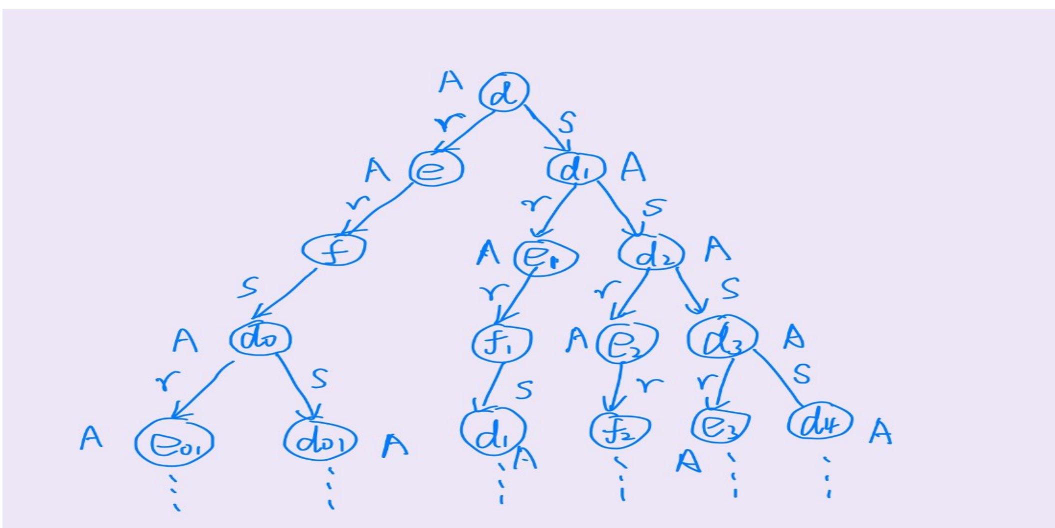
For example, $\mathcal{T} = \{A \sqsubseteq \exists r.B, B \sqsubseteq \exists r.A, A \sqcup B \sqsubseteq \exists s.\top\}$.

According the graph, we can find a Interpretation $\mathcal{J}$, but it appearently is not finite.

Question 6. Unravelling



See the following figure:



Question 7. Tree Model Property (tmp)

False.

For example, if $\mathcal{K} =< \mathcal{T}, \mathcal{A} >$ and $\mathcal{T} = \emptyset, \mathcal{A} = \{a : A, b : B, (a, b) : r, (b, a) : r\}$. For every model $\mathcal{I}$ of $\mathcal{K}$, $a^{\mathcal{I}}$ and $b^{\mathcal{I}}$ are two distinguish elements, and $(a^{\mathcal{I}}, b^{\mathcal{I}}), (b^{\mathcal{I}}, a^{\mathcal{I}}) \in r^{\mathcal{I}}$. Therefore, the model always have a cycle $a \to b \to a$, which means it is not a tree model.

Question 8. Tableau Algorithm

$\mathcal{A}_0 = \{(b,a) : r, (a,b) : r, (a,c) : s, (c,b) : s, a : \exists s.A, b : \forall r.((\forall s.\neg A) \sqcup (\exists r.B)),$
$c : \forall s.(B \sqcap (\forall s.\bot))\}$
for $a : \exists s.A$, apply the $\exists$-rule:

$$\mathcal{A}_1 = \mathcal{A}_0 \cup \{(a,d) : s, d : A\}$$

for $b : \forall r.((\forall s.\neg A) \sqcup (\exists r.B))$, apply the $\forall$-rule:

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{a : (\forall s.\neg A) \sqcup (\exists r.B)\}$$

for $c : \forall s.(B \sqcap (\forall s.\bot))$, apply the $\forall$-rule:

$$\mathcal{A}_3 = \mathcal{A}_2 \cup \{b : B \sqcap (\forall s.\bot)\}$$

for $b : B \sqcap (\forall s.\bot)$, apply the $\cup$-rule:

$$\mathcal{A}_4 = \mathcal{A}_3 \cup \{b : B, b : \forall s.\bot\}$$

for $a : (\forall s.\neg A) \sqcup (\exists r.B)$, apply $\sqcup$-rule:
case 1:

$$\mathcal{A}_5 = \mathcal{A}_4 \cup \{a : \forall s.\neg A\}$$

for $a : \forall s.\neg A$, apply the $\forall$-rule:

$$\mathcal{A}_{51} = \mathcal{A}_5 \cup \{c : \neg A, d : \neg A\}$$

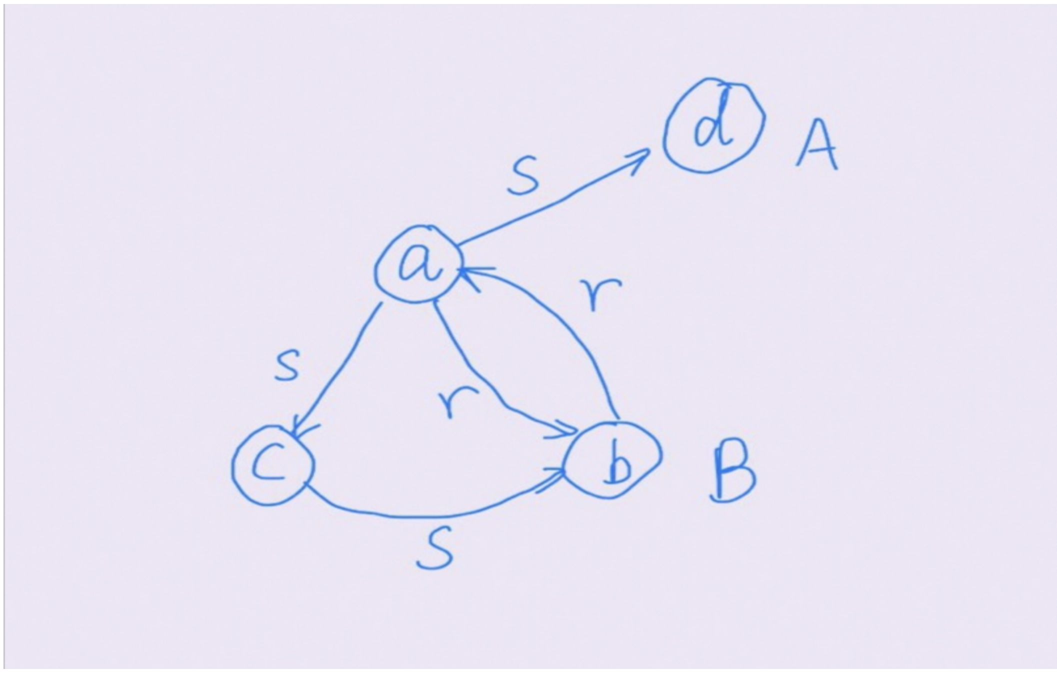
because $\mathcal{A}_{51}$ contains a clash $d : A$ and $d : \neg A$, so fail.

case 2:

$$\mathcal{A}_5 = \mathcal{A}_4 \cup \{a : \exists r.B\}$$

there is no rule is applicable and $\mathcal{A}_5$ does not contains clash.

So, we have done. $\mathcal{A}$ is consistent.



Question 9. Extension of Tableau Algorithm

Define the NNF of $\rightarrow$-constructer:$\neg(C \rightarrow D) \equiv C \sqcap \neg D$,they are semantically equivalent because $(C \rightarrow D)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} | x \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \text{ or } x \in D^{\mathcal{I}}\} = (\neg C \sqcup D)^{\mathcal{I}}$.

(1) The deterministic $\rightarrow$-rule:

Termination:

The property termination still holds.

Let $m = |\,\mathrm{sub}(\mathcal{A})|$.

a.After applying application, it will add a new assertion of the form $a : C$ and $C \in \mathrm{sub}(\mathcal{A})$. So for any individual name a, there can be at most m rule applications adding a concept assertion of the form $a : C$ and $\text{con}_{\mathcal{A}}(a) \leq m$.

b. A new individual name is added to A only when the ∃-rule is applied to an assertion of the form $a : C$ with C an existential restriction (a concept of the form ∃r.D), and for any individual name each such assertion can trigger the addition of at most one new individual name. As there can be no more than m different existential restrictions in A, a given individual name can cause the addition of at most m new individual names, and the outdegree of each tree in the forest-shaped ABox is thus bounded by m.

c. With the same original proof, the depth of each tree in the forest-shaped ABox is bounded by m.

There properties ensure that there is a bound on the size of the ABox that can be constructed via rule applications, and thus a bound on the number of recursive applications of expand.

Soundness:

The property soundness does not holds.

For example, $\mathcal{A} = \{a : (C \sqcup D) \rightarrow E, a : C, a : \neg E\}$, because $a : C \sqcup D \notin \mathcal{A}$, we could not use the deterministic rule, so there is no rule could be applicated to it and there is no clash, therefore the algorithm would return $\mathcal{A}$ is consistent. But actually the ABox is conflicting semantically.

Completeness:

The property completeness still holds.

Let $\mathcal{A}$ be consistent, and consider a model $\mathcal{I} =< \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} >$ of A. Since $\mathcal{A}$ is consistent, it cannot contain a clash. If $\mathcal{A}$ is complete - since it does not contain a clash - expand simply returns $\mathcal{A}$ and consistent returns “consistent” . If $\mathcal{A}$ is not complete, then expand calls itself recursively until $\mathcal{A}$ is complete; each call selects a rule and applies it. We will show that rule application preserves consistency by a case analysis according to the type of rule:

- The $\sqcup$-rule: If $a : C \sqcup D \in \mathcal{A}$, then $a^{\mathcal{I}} \in (C \sqcup D)^{\mathcal{I}}$ and Definition 2.2 implies that either $a^{\mathcal{I}} \in C^{\mathcal{I}}$ or $a^{\mathcal{I}} \in D^{\mathcal{I}}$. Therefore, at least one of the ABoxes $\mathcal{A}' \in \mathsf{exp}(\mathcal{A}, \sqcup\text{-rule}, a : C \sqcup D)$ is consistent. Thus, one of the calls of expand is applied to a consistent ABox.
- The $\sqcap$-rule: If $a : C \sqcap D \in \mathcal{A}$, then $a^{\mathcal{I}} \in (C \sqcap D)^{\mathcal{I}}$ and Definition 2.2 implies that both $a^{\mathcal{I}} \in C^{\mathcal{I}}$ and $a^{\mathcal{I}} \in D^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup \{a : C, a : D\}$, so $\mathcal{A}$ is still consistent after the rule is applied.
- The $\exists$-rule: If $a : \exists r.C \in \mathcal{A}$, then $a^{\mathcal{I}} \in (\exists r.C)^{\mathcal{I}}$ and Definition 2.2 implies that there is some $x \in \Delta^{\mathcal{I}}$ such that $(a^{\mathcal{I}}, x) \in r^{\mathcal{I}}$ and $x \in C^{\mathcal{I}}$. Therefore, there is a model $\mathcal{I}'$ of $\mathcal{A}$ such that, for some new individual name d, $d^{\mathcal{I}'} = x$, and that is otherwise identical to $\mathcal{I}$. This model $\mathcal{I}'$ is still a model of $\mathcal{A} \cup \{(a, d) : r, d : C\}$, so $\mathcal{A}$ is still consistent after the rule is applied.
- The $\forall$-rule: If $\{a : \forall r.C, (a, b) : r\} \subseteq \mathcal{A}$, then $a^{\mathcal{I}} \in (\forall r.C)^{\mathcal{I}}$, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$, and Definition 2.2 implies that $b^{\mathcal{I}} \in C^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup \{b : C\}$, so $\mathcal{A}$ is still consistent after the rule is applied.

We also need to prove $\rightarrow$-rule: If $a : C \rightarrow D \in \mathcal{A}$ and $a : C \in \mathcal{A}$, then $a^{\mathcal{I}} \in (C \rightarrow D)^{\mathcal{I}}$. So there is $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ or $a^{\mathcal{I}} \in D^{\mathcal{I}}$ according to the semantics of $\rightarrow$. Because we have $a^{\mathcal{I}} \in C^{\mathcal{I}}$, so there is $a^{\mathcal{I}} \in D^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup \{a : D\}$, so $\mathcal{A}$ is still consistent after applying the rule.

(2) The nondeterministic $\rightarrow$-rule:

Termination:

The property termination still holds. The proof is as the same as deterministic case.

Soundness:

The property soundness still holds.

The construction of $\mathcal{I}$ means that it trivially satisfies all role assertions in $\mathcal{A}'$.

we will show the following property by induction of the structure of concept:

if $a : C \in \mathcal{A}'$, then $a^{\mathcal{I}} \in C^{\mathcal{I}}$

Proof. Let $\mathcal{A}'$ be the set returned by $\mathsf{expand}(\mathcal{A})$. Since the algorithm returns "consistent", $\mathcal{A}'$ is a complete and clash-free ABox.

The proof then follows rather easily from the very close correspondence between $\mathcal{A}'$ and an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ that is a model of $\mathcal{A}'$, i.e., that satisfies each assertion in $\mathcal{A}'$. Given that the expansion rules never delete assertions, we have that $\mathcal{A} \subseteq \mathcal{A}'$, so $\mathcal{I}$ is also a model of $\mathcal{A}$, and is a witness to the consistency of $\mathcal{A}$. We use $\mathcal{A}'$ to construct a suitable interpretation $\mathcal{I}$ as follows:

$$
\begin{aligned}
\Delta^{\mathcal{I}} &= \{a \mid a : C \in \mathcal{A}'\}, \\
a^{\mathcal{I}} &= a \text{ for each individual name } a \text{ occurring in } \mathcal{A}', \\
A^{\mathcal{I}} &= \{a \mid A \in \mathsf{con}_{\mathcal{A}'}(a)\} \text{ for each concept name } A \text{ in } \mathsf{sub}(\mathcal{A}'), \\
r^{\mathcal{I}} &= \{(a,b) \mid (a,b) : r \in \mathcal{A}'\} \text{ for each role } r \text{ occurring in } \mathcal{A}'.
\end{aligned}
$$

we also need to prove the case when $C = D \to E$: if $a : D \to E \in \mathcal{A}'$, then completeness of $\mathcal{A}'$ implies that $\{a : E\} \subseteq \mathcal{A}'$ or $\{a : \dot{\neg} D\} \subseteq \mathcal{A}'$ (otherwise the nondeterministic $\to$-rule would be applicable). Thus $a^{\mathcal{I}} \in E^{\mathcal{I}}$ or $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus D^{\mathcal{I}}$ by induction, and hence $a^{\mathcal{I}} \in (\Delta^{\mathcal{I}} \setminus D^{\mathcal{I}}) \cup E^{\mathcal{I}} = (\neg D \sqcup E)^{\mathcal{I}} = (D \to E)^{\mathcal{I}}$ by the semantics of $\to$.

As a consequence, $\mathcal{I}$ satisfies all concept assertions in $\mathcal{A}'$ and thus in $\mathcal{A}$, and it satisfies all role assertions in $\mathcal{A}'$ and thus in $\mathcal{A}$ by definition. Hence $\mathcal{A}$ has a model and thus is consistent.

Completeness:

The property completeness still holds.

The body of the proof is the same as above, we just need to modify it a bit.

The nondeterministic $\to$-rule: If $a : C \to D \in \mathcal{A}$, then $a^{\mathcal{I}} \in (C \to D)^{\mathcal{I}}$. Thus $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ or $a^{\mathcal{I}} \in D^{\mathcal{I}}$ by the semantics of $\to$. Therefore, at least one of the ABoxes $\mathcal{A}' \in \exp(\mathcal{A}, \text{nondeterministic } \to\text{-rule}, a : C \to D)$ is consistent. Thus, one of the calls of expand is applied to a consistent ABox.

Induction Basis C is a concept name: by definition of $\mathcal{I}$, if $a : C \in \mathcal{A}'$, then $a^{\mathcal{I}} \in C^{\mathcal{I}}$ as required.

Induction Steps

- $C = \neg D$: since $\mathcal{A}'$ is clash-free, $a : \neg D \in \mathcal{A}'$ implies that $a : D \notin \mathcal{A}'$. Since all concepts in $\mathcal{A}$ are in NNF, D is a concept name. By definition of $\mathcal{I}$, $a^{\mathcal{I}} \notin D^{\mathcal{I}}$, which implies $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus D^{\mathcal{I}} = C^{\mathcal{I}}$ as required.
- $C = D \sqcup E$: if $a : D \sqcup E \in \mathcal{A}'$, then completeness of $\mathcal{A}'$ implies that $\{a : D, a : E\} \cap \mathcal{A}' \neq \emptyset$ (otherwise the $\sqcup$-rule would be applicable). Thus $a^{\mathcal{I}} \in D^{\mathcal{I}}$ or $a^{\mathcal{I}} \in E^{\mathcal{I}}$ by induction, and hence $a^{\mathcal{I}} \in D^{\mathcal{I}} \cup E^{\mathcal{I}} = (D \sqcup E)^{\mathcal{I}}$ by the semantics of $\sqcup$.
- $C = D \sqcap E$: this case is analogous to but easier than the previous one and is left to the reader as a useful exercise.
- $C = \forall r.D$: let $a : \forall r.D \in \mathcal{A}'$ and consider b with $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$. For $a^{\mathcal{I}}$ to be in $(\forall r.D)^{\mathcal{I}}$, we need to ensure that $b^{\mathcal{I}} \in D^{\mathcal{I}}$. By definition of $\mathcal{I}$, $(a, b) : r \in \mathcal{A}'$. Since $\mathcal{A}'$ is complete and $a : \forall r.D \in \mathcal{A}'$, we have that $b : D \in \mathcal{A}'$ (otherwise the $\forall$-rule would be applicable). By induction, $b^{\mathcal{I}} \in D^{\mathcal{I}}$, and since the above holds for all b with $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$, we have that $a^{\mathcal{I}} \in (\forall r.D)^{\mathcal{I}}$ by the semantics of $\forall$.
- $C = \exists r.D$: again, this case is analogous to and easier than the previous one and is left to the reader as a useful exercise.

Question 10. Modification of Tableau Algorithm

We firstly extend the definition of a clash: for some individual name a, and for some concept C, $\{a : C, a : \neg C\} \subseteq \mathcal{A}$, or for some individual names a and b, and for some role names r and s, $\{(a, b) : r, (a, b) : s\} \subseteq \mathcal{A}$ and $\{\text{disjoint}(r, s)\} \subseteq \mathcal{T}$.

Define $\sqsubseteq$-rule:

- Condition: $(a, b) : r \in \mathcal{A}, r \sqsubseteq s \in \mathcal{T}$ and $(a, b) : s \notin \mathcal{A}$.
- Action: $\mathcal{A} \to \mathcal{A} \cup \{(a, b) : s\}$.

Now, we show that the algorithm remains terminating, sound, and complete. For the sake of simplicity, we will follow the proof in Question 9 and modify it if necessary.

- Termination.

  Because the number of individual names is bounded, so the number of new role assertions added by $\sqsubseteq$-rule is bounded.

- Soundness.

  Let $\mathcal{A}'$ be the set return by expand($\mathcal{A}$). Since the algorithm returns "consistent", $\mathcal{A}'$ is a complete and clash-free ABox.

  If disjoint$(r, s) \in \mathcal{T}$, $r^{\mathcal{I}} \cap s^{\mathcal{I}} \neq \emptyset$, which means there exists $(a, b) \in r^{\mathcal{I}}$ and $(a, b) \in s^{\mathcal{I}}$. And then we can conclude $(a, b) : r, (a, b) : s \in \mathcal{A}'$ which is a clash, therefore $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$.

  If $r \sqsubseteq s \in \mathcal{T}$ but there is $(a, b) \in r^{\mathcal{I}}, (a, b) \notin s^{\mathcal{I}}$. Then by the definition of $\mathcal{I}$, $(a, b) : r \in \mathcal{A}'$ but $(a, b) : s \notin \mathcal{A}'$, which means $\mathcal{A}'$ is not a complete ABox.

  Therefore, if the consistent($\mathcal{T}$,$\mathcal{A}$) returns "consistent", then ($\mathcal{T}$,$\mathcal{A}$) is consistent.

- Completeness.

  Let $\mathcal{I}$ be a model of ($\mathcal{T}$,$\mathcal{A}$). if $(a, b) : r \in \mathcal{A}$, $r \sqsubseteq s \in \mathcal{T}$ and $(a, b) \in r^{\mathcal{I}}$. Because $\mathcal{I}$ is a model of ($\mathcal{T}$, $\mathcal{A}$), so there must be $(a, b) \in s^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is also a model of ($\mathcal{T}$, $\mathcal{A} \cup \{(a, b) : s\}$), so $\mathcal{A}$ is still consistent after the rule is applied.