Optimization Methods	**Fall 2022**

Homework 1

Instructor: Lijun Zhang — *Name:* Student name*, StudentId:* Student id

Notice

- The submission email is: **optfall2022@163.com**.
- Please use the provided LaTeX file as a template.
- If you are not familiar with LaTeX, you can also use Word to generate a **PDF** file.

Problem 1: Inequalities

Let $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, where n is a positive integer. Let $\|\cdot\|$ denote the Euclidean norm.

a) Prove the triangle inequality $\|x+y\| \leq \|x\| + \|y\|$.

b) Prove $\|x+y\|^2 \leq (1+\epsilon)\|x\|^2 + (1+\frac{1}{\epsilon})\|y\|^2$ for any $\epsilon > 0$.

Hint: You may need the Young's inequality for products, i.e. if a and b are nonnegative real numbers and p and q are real numbers greater than 1 such that $1/p + 1/q = 1$, then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$.

Problem 2: Definition of convexity

Convex C_c sets are the sets satisfying the constraints below:

$$\theta x_1 + (1-\theta)x_2 \in C_c$$
$$\text{for all, } x_1, x_2 \in C_c, 0 \leq \theta \leq 1$$

Determine if each set below is convex.

a) $\{(x,y) \in \mathbb{R}^2_{++} | x/y \leq 1\}$

b) $\{(x,y) \in \mathbb{R}^2_{++} | x/y \geq 1\}$

c) $\{(x,y) \in \mathbb{R}^2_{++} | xy \leq 1\}$

d) $\{(x,y) \in \mathbb{R}^2_{++} | xy \geq 1\}$

e) $\{(x,y) \in \mathbb{R}^2_{++} | y = \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}\}$

Problem 3: Convex sets

a) Show that a polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m\}$ is convex.

b) Show that if $S \subseteq \mathbb{R}^n$ is convex, and $A \in \mathbb{R}^{m \times n}$, then $A(S) = \{Ax : x \in S\}$, is convex.

c) Show that if $S \subseteq \mathbb{R}^m$ is convex, and $A \in \mathbb{R}^{m \times n}$, then $A^{-1}(S) = \{x : Ax \in S\}$, is convex.

Problem 4: Convex cone

Let K be a convex cone. The set $K^* = \{y | x^\top y \geq 0, \forall x \in K\}$ is called the dual cone of K.

a) Show that K^* is a convex cone (even K is not convex).

b) Show that a dual cone of a subspace $V \subset \mathbb{R}^n$ (which is a cone) is its orthogonal complement $V^+ = \{y | y^\top v = 0, \forall v \in V\}$.

c) What is the dual cone of the nonnegative orthant ($\mathbb{R}^n_+$)?

Problem 5: Generalized Inequalities

Let K^* be the dual cone of a convex cone K. Prove the following,

a) K^* is indeed a convex cone.

b) $K_1 \subseteq K_2$ implies $K_2^* \subseteq K_1^*$.