KRP	**Spring 2023**

Homework 3

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Question 1. Closure under Disjoint Union

Let C be an $\mathcal{ALC}$-Concept. If there exists a rule that can be applied, then C has a subconcept of the form $\neg D$, and D is as the form of $E \sqcap F, E \sqcup F, \neg E, \exists r.E, \forall r.E$. Let C' be the $\mathcal{ALC}$-concept after rule application.

- case 1: $\neg D = \neg(E \sqcap F)$, E, F is concept name.
  After transforming, $\neg(E \sqcap F)$ became $\neg\neg\neg E \sqcup \neg\neg\neg F$.

$$\begin{aligned}
M(\neg(E \sqcap F)) &= \{\#(E \sqcap F)\} \cup M(E) \cup M(F) \\
M(\neg\neg\neg E \sqcup \neg\neg\neg F) &= \{\#(\neg\neg E), \#(\neg E), \#E, \#(\neg\neg F), \#(\neg F), \#F\} \\
&\quad \cup M(E) \cup M(F) \\
M(C') &= (M(C) \setminus M(\neg D)) \cup M(\neg\neg\neg E \sqcup \neg\neg\neg F) \\
&= (M(C) \setminus \{\#(E \sqcap F)\}) \\
&\quad \cup \{\#(\neg\neg E), \#(\neg E), \#E, \#(\neg\neg F), \#(\neg F), \#F\}
\end{aligned}$$

  So, we can get that after transforming, $\#(E \sqcap F)$ was replaced by some smaller numbers $\#(\neg\neg E), \#(\neg E), \#E, \#(\neg\neg F), \#(\neg F), \#F$.

- case 2: $\neg D = \neg(E \sqcup F)$, E, F is concept name.
  The prove of case 2 is similar as case 1.

- case 3: $\neg D = \neg\neg E$, E is a concept name.
  After transforming, $\neg\neg E$ became E.

$$M(\neg\neg E) = \{\#\neg E, \#E\}$$
$$M(C') = (M(C) \setminus M(\neg\neg E)) \cup M(E) = M(C) \setminus \{\#\neg E, \#E\}$$

So, we can get that after transforming, $\#\neg E, \#E$ was deleted.

- case 4: $\neg D = \neg(\exists r.E)$, E is a concept name.

$$M(\neg(\exists r.E)) = \{\#(\exists r.E)\} \cup M(E)$$
$$M(\forall r.\neg E) = \{\#E\} \cup M(E)$$
$$M(C') = (M(C) \setminus M(\neg(\exists r.E))) \cup M(\forall r.\neg E) = (M(C) \setminus \{\#(\exists r.E)\}) \cup \{\#E\}$$

  So, we can get that after transforming, $\#(\exists r.E)$ was replaced by a smaller number $\#E$.

- case 5: $\neg D = \neg(\forall r.E)$, E is a concept name.
  The prove of case 5 is similar as case 4.

Finally, we can get that with transforming, the elements of $M(C)$ will became smaller over and over again, the stop condition is the elements of $M(C)$ all becomes 0. The procedure of transformations is always terminates because the numbers are limited.

Question 2. Negation Normal Norm (NNF)

(1) $\Longrightarrow$:

let $\mathcal{T}'$ is obtained from $\mathcal{T}$ by replacing each concept defnition $A \equiv C$ with the concept inclusion $A \sqsubseteq C$ and $C \sqsubseteq A$, $\mathcal{T}$ is equivalent to $\mathcal{T}'$.

Obviously, $\mathcal{T}^{\sqsubseteq} \subseteq \mathcal{T}'$, which means any model of $\subseteq \mathcal{T}'$ is also a model of $\mathcal{T}^{\sqsubseteq}$.

So, every concept name is satisfable w.r.t. $\mathcal{T}$ implies it is satisfable w.r.t. $\mathcal{T}^{\sqsubseteq}$.

$\Longleftarrow$:

If concept name C is satisfiable w.r.t. $\mathcal{T}^{\sqsubseteq}$, then there exists an interpretation $\mathcal{I}$ s.t. $C^{\mathcal{I}} \neq \emptyset$ w.r.t. $\mathcal{T}^{\sqsubseteq}$.

We expand $\mathcal{I}$ and get $\mathcal{J}$. The rule to expand is as follow: We modify recursively all $A^{\mathcal{J}}, C_1^{\mathcal{J}}, C_2^{\mathcal{I}}......, C_n^{\mathcal{J}}$ to $A^{\mathcal{I}} \cup C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}} \cup ...... \cup C_n^{\mathcal{I}}$ if GCI $A \sqsubseteq C_i$ in $\mathcal{T}^{\sqsubseteq}$ for all $1 \leq i \leq n$, until all $A^{\mathcal{J}} = C^{\mathcal{J}}$ if concept defnition $A \equiv C$ in $\mathcal{T}$.

Because the stop condition is $A^{\mathcal{J}} = C^{\mathcal{J}}$ and the expand procedure does not violate any one GCI, so $\mathcal{J}$ is a interpretation w.r.t. $\mathcal{T}$. Because we just expand some $A^{\mathcal{I}}$ and get corresponding $A^{\mathcal{J}}$, so $C^{\mathcal{J}} \neq \emptyset$ due to $C^{\mathcal{I}} \neq \emptyset$.

Therefore, C is satisfiable w.r.t. $\mathcal{T}$.

(2) Do not holds.

$$\mathcal{T} = \{A \equiv C \sqcap \neg B, B \equiv P, C \equiv P\}$$
$$\mathcal{T}^{\sqsubseteq} = \{A \sqsubseteq C \sqcap \neg B, B \sqsubseteq P, C \sqsubseteq P\}$$

Because:

$$A^{\mathcal{I}} = (C \sqcap \neg B)^{\mathcal{I}} = C^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus B^{\mathcal{I}}) = P^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus P^{\mathcal{I}}) = \emptyset$$

So concept name $\mathcal{A}$ is not satisfiable w.r.t. $\mathcal{T}$.

Let $\Delta^{\mathcal{I}} = \{a\}, A^{\mathcal{I}} = \{a\}, C^{\mathcal{I}} = \{a\}, B^{\mathcal{I}} =, P^{\mathcal{I}} = \{a\}$, obviously it satisfies $T^{\sqsubseteq}$ and $A^{\mathcal{I}} \neq \emptyset$.

So concept name A is satisfiable w.r.t. $\mathcal{T}^{\sqsubseteq}$.

Question 3. Tableau Algorithm for ABoxes with Acyclic TBoxes

- Termination.

  We had known that the tableau algorithm consistent($\mathcal{A}$) is terminate, so we only need to prove that after using $\equiv_1$-rule and $\equiv_2$-rule to unfold $\mathcal{T}$, the new Abox $\mathcal{A}'$ is finite. Obviously, we would add finite number of assertions to $\mathcal{A}$, so the new Abox $\mathcal{A}'$ is finite.

- Soundness.

  Let $\mathcal{A}' =$ consistent($\mathcal{K}$, $\mathcal{A}$).To construct a model from a complete and clash-free ABox $\mathcal{A}'$, we can use the same definitions as presented in the lecture slides to obtain an interpretation $\mathcal{J}$ of all role names and of the concept names that do not have definitions in $\mathcal{T}$. It remains to show that $\mathcal{J}$ is also a model of $\mathcal{A}'$, where the main problem is showing Property (P1) by induction. So let's prove it.

  If C doesn't have definition in $\mathcal{T}$, then we have already proved it. Otherwise C has definition. According to the $\equiv_1$-rule and $\equiv_2$-rule, if $a : A$ and $A \equiv C$, there must be $a : C$ in $\mathcal{A}'$, if $a : \neg A$ and $A \equiv C$, there must be $a : \dot{\neg} C$, by the defnition of $\mathcal{J}$, $a^{\mathcal{J}} \in A^{\mathcal{J}}$.

- Completeness.

  The $\equiv_1$-rule: if $a : A \in A$, $A \equiv C \in \mathcal{T}$, then $a^{\mathcal{I}} \in A^{\mathcal{I}}$, $A^{\mathcal{I}} = C^{\mathcal{I}}$, so $a^{\mathcal{I}} \in C^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup \{a : C\}$.

  The $\equiv_2$-rule: if $a : \neg A \in A$, $A \equiv C \in \mathcal{T}$, then $a^{\mathcal{I}} \notin A^{\mathcal{I}}$, $A^{\mathcal{I}} = C^{\mathcal{I}}$, so $a^{\mathcal{I}} \notin C^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup \{a : \dot{\neg} C\}$.

  The other rules are as the same as presented in the text book.

Therefore, consistent($\mathcal{T}, \mathcal{A}$) is a decision procedure for the consistency of $\mathcal{ALC}$-knowledge bases with acyclic TBoxes.

Question 4. Tableau Algorithm for ABoxes with Acyclic TBoxes

Doesn't hold.

If the subsumption $\neg(\forall r.A) \sqcap \forall r.C \sqsubseteq \forall r.E$ doesn't hold w.r.t. acyclic TBox $\mathcal{T}$, then $\neg(\forall r.A) \sqcap \forall r.C \sqcap \neg(\forall r.E)$ holds w.r.t. acyclic TBox $\mathcal{T}$.

We use tableau algorithm to determine it.

$$
\begin{aligned}
\mathcal{A}_0 &= \{a : \neg(\forall r.A) \sqcap \forall r.C \sqcap \neg(\forall r.E)\} \\
\mathcal{A}_1 &= \mathcal{A}_0 \cup \{a : \neg(\forall r.A), a : \forall r.C, a : \neg(\forall r.E)\} \\
&= \mathcal{A}_0 \cup \{a : \exists r.\neg A, a : \forall r.C, a : \exists r.\neg E\} \quad (\sqcap\text{-rule}) \\
\mathcal{A}_2 &= \mathcal{A}_1 \cup \{(a,b) : r, b : \neg A, (a,c) : r, c : \neg E\} \quad (\exists\text{-rule}) \\
\mathcal{A}_3 &= \mathcal{A}_2 \cup \{b : C, c : C\} \quad (\forall\text{-rule}) \\
\mathcal{A}_4 &= \mathcal{A}_3 \cup \{b : (\exists r.\neg B) \sqcap \neg A, c : (\exists r.\neg B) \sqcap \neg A\} \quad (\equiv_1\text{ -rule}) \\
\mathcal{A}_5 &= \mathcal{A}_4 \cup \{c : \exists r.A \sqcup \forall r.\neg D\} \quad (\equiv_2\text{ -rule}) \\
\mathcal{A}_6 &= \mathcal{A}_5 \cup \{b : \exists r.\neg B, c : \exists r.\neg B, c : \neg A\} \quad (\sqcap\text{-rule}) \\
\mathcal{A}_7 &= \mathcal{A}_6 \cup \{(b,d) : r, d : \neg B, (c,e) : r, e : \neg B\} \quad (\exists\text{-rule}) \\
\mathcal{A}_8 &= \mathcal{A}_7 \cup \{c : \exists r.A\} \quad (\sqcup\text{-rule}) \\
\mathcal{A}_9 &= \mathcal{A}_8 \cup \{(c,f) : r, f : A\} \quad (\exists\text{-rule})
\end{aligned}
$$

There is no rules to apply in $\mathcal{A}_9$ and it is clash free, which means $\neg(\forall r.A) \sqcap \forall r.C \sqcap \neg(\forall r.E)$ holds w.r.t. acyclic TBox $\mathcal{T}$ and $\neg(\forall r.A) \sqcap \forall r.C \sqsubseteq \forall r.E$ doesn't hold w.r.t. acyclic TBox $\mathcal{T}$

Question 5. Anywhere Blocking

- Termination.

Let $m = |\operatorname{sub}(K)|$. Termination is a consequence of the following properties of the expansion rules:

(1) There can be at most $|\operatorname{sub}(\mathcal{K})|$ rule applications of a individual name.

(2) The outdegree of each tree in the forest-shaped ABox is bounded by $|\operatorname{sub}(\mathcal{K})|$.

(3) Any path having more individual names than $2^{|\operatorname{sub}(\mathcal{K})|}$ has at least two individual names a, b such that $\operatorname{con}_{\mathcal{A}}(b) = \operatorname{con}_{\mathcal{A}}(a) \subseteq \operatorname{sub}(\mathcal{K})$. Because the ages of a individual name are strictly monotonic increasing, we assume that age $(a) <$ age(b). Therefore, there are at most 2^m individual names in a path along tree individuals that are not blocked. That means the depth of each tree in the forest-shaped *ABox* is bounded by 2^m.

- Soundness.

  Let $\mathcal{A}^{'} = \operatorname{consistent}(\mathcal{K})$. We firstly construct $\mathcal{A}^{'}$:

$$\begin{aligned}\mathcal{A}'' = &\{a : C \mid a : C \in \mathcal{A}' \text{ and } a \text{ is not blocked }\} \cup \\ &\{(a,b) : r \mid (a,b) : r \in \mathcal{A}' \text{ and } b \text{ is not blocked }\} \cup \\ &\{(a,b') : r \mid (a,b) : r \in \mathcal{A}', a \text{ is not blocked and } b \text{ is blocked by } b'\}\end{aligned}$$

  The first preconclusion we want to show is:

$$\operatorname{con}_{\mathcal{A}''}(a) = \operatorname{con}_{\mathcal{A}'}(a)$$

  - For individual assertion, a must not be blocked by its definition.
  - For role assertion, there is two case:
    case 1: $(a, b) \in r \in \mathcal{A}^{'}$, because the successor of a blocked individual is also blocked, so a must not be blocked.
    case 2: $(a, b^{'}) \notin r \in \mathcal{A}^{'}$, b' is not blocked according to the definition of anywhere blocking.

Therefore, there is no blocked individual names in $\mathcal{A}''$, which means the fir preconclusion holds.

The second preconclusion we want to show is:

$$\mathcal{A}'' \text{ is complete if } \mathcal{A}' \text{ is complete.}$$

- $\sqcap-$ rule: if $a : C \sqcap D \in \mathcal{A}''$, according to the first preconclusion $a : C \sqcap D \in \mathcal{A}'$. Completeness of $\mathcal{A}'$ implies that $\{a : C, a : D\} \subseteq \mathcal{A}'$ and then the first preconclusion implies $\{a : C, a : D\} \subseteq \mathcal{A}''$.
- $\sqcup-$ rule: if $a : C \sqcup D \in \mathcal{A}''$, according to the first preconclusion $a : C \sqcup D \in \mathcal{A}'$. Completeness of $\mathcal{A}'$ implies that at least one of $\{a : C, a : D\}$ is in $\mathcal{A}'$ and then the first preconclusion implies at least one of $\{a : C, a : D\}$ is in $\mathcal{A}''$.
- $\sqsubseteq -$ rule: if $C \sqsubseteq D \in \mathcal{T}$ and $a : C \in \mathcal{A}''$, according to the first preconclusion $a : C \sqsubseteq D \in \mathcal{A}'$. Completeness of $\mathcal{A}'$ implies that $a : D \in \mathcal{A}'$ and then the first preconclusion implies $a : D \in \mathcal{A}''$.
- $\exists-$ rule: if $a : \exists r.C \in \mathcal{A}''$, according to the first preconclusion $a : \exists r.C \in \mathcal{A}'$ and a is not blocked in $\mathcal{A}'$. Completeness of $\mathcal{A}'$ implies that $\{(a, b) : r, b : C\} \subseteq \mathcal{A}'$. If b is not blocked, $\{(a, b) : r, b : C\} \subseteq \mathcal{A}''$ according to the definition of $\mathcal{A}''$. If b is anywhere blocked by b', then b' is not blocked and $\mathrm{con}_{\mathcal{A}'}(b) \subseteq \mathrm{con}\ _{A'}(b')$ according to the definition of anywhere blocking. Therefore, $b' : C \in \mathcal{A}''$ according to the first preconclusion. By definition of $\mathcal{A}''$, we have $(a, b') : r \in \mathcal{A}''$ naturally.
- $\forall$-rule: if $\{a : \forall r.C, (a, b') : r\} \subseteq \mathcal{A}''$, then $a : \forall r.C \in \mathcal{A}'$ and neither a or b' not blocked in $\mathcal{A}'$. If $(a, b') : r \in \mathcal{A}'$, then completeness of $\mathcal{A}'$ implies $b' : C \in \mathcal{A}''$. If $(a, b') : r \notin \mathcal{A}'$, then there must be a b such that $(a, b) : r \in \mathcal{A}'$ and b is anywhere blocked by b' in $\mathcal{A}'$. Then completeness of $\mathcal{A}'$ implies $b : C \in \mathcal{A}'$. According to the definition of anywhere blocking, we

have $\text{con}_{A'}(b) \subseteq \text{con}_{A'}(b')$, and $b' : C \in \mathcal{A}'$. Then the first preconclusion implies that $b' : C \in A''$.

Now, we can construct a interpretation $\mathcal{I}$ according to $\mathcal{A}''$ just like the prove of lemma 4.5 in the text book and then we can get that $\mathcal{I}$ is a model of $\mathcal{A}''$ and $\mathcal{A}$.

For each GCI $C \sqsubseteq D \in \mathcal{T}$ and assertion $a : C \in \mathcal{A}$, we have $a : C \in \mathcal{A}''$ and $a : D \in \mathcal{A}''$. If $a^{\mathcal{I}} \in C^{\mathcal{I}}$, then $a^{\mathcal{I}} \in D^{\mathcal{I}}$ and thus $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is a model of $\mathcal{T}$.

To sum up, $\mathcal{I}$ is a model of $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, $\mathcal{K}$ is consistent if the tableau algorithm with anywhere blocking returns ”consistent”.

- Completeness.

  The prove of completeness is similar as the prove on the text book. The only difference is $\sqsubseteq$-rule: If $a : C \in \mathcal{A}$ and $C \sqsubseteq D \in \mathcal{T}$, then $a^{\mathcal{I}} \in C^{I}$ and $C^{\mathcal{I}} \sqsubseteq D^{\mathcal{I}}$ by semantics. Consequently, adding $a : D$ to $\mathcal{A}$ has no effect on the consistency of $\mathcal{K}$. Therefore, $\mathcal{I}$ is still a model of $(\mathcal{T}, \mathcal{A} \cup \{a : D\})$.

Question 6. Precompletion of Tableau Algorithm

$\Longrightarrow$:

Let $\mathcal{J}$ denote a model of $\mathcal{K}$. According to the completeness of tableau algorithm, applying any rule to $\mathcal{A}$ will not change the consistency of $\mathcal{K}$ for any model with respect to $\mathcal{K}$. Therefore, $\mathcal{J}$ is also a model with respect to $\mathcal{A}'$(precompletion of $\mathcal{K}$).

By the definition of $C^a_{\mathcal{A}}$, obviously $a^{\mathcal{J}} \in C^a_{\mathcal{A}'}$. So $C^a_{\mathcal{A}'}$ is satisfiable with respect to $\mathcal{T}$ for each individual name a that occurs in $\mathcal{A}'$.

$\Longleftarrow$:

We construct a suitable interpretation I:

- $\Delta^I = \{\text{a} \mid a : C \in \mathcal{A}'\}$
- $a^I = a$ for all individual name a in $\mathcal{A}'$
- $A^I = \{a | A \in \text{con}_{A'}(a)\}$ for each $A \in \text{sub}(\mathcal{A}')$
- $r^I = \{(a,b) | (a,b) : r \in \mathcal{A}'\}$ for each role name r in $\mathcal{A}^{'}$

If $C^a_{\mathcal{A}}$ is satisfiable for all individual names a in $\mathcal{A}'$, then there exists at least one individual name a such that $a^I \in C^a_{\mathcal{A}'} = (\sqcap_{a:C\in\mathcal{A}'} C)^I = \bigcap_{a:C\in\mathcal{A}'} C^I$, which means $a^I \in C^I$ for all individual assertion $a : C \in \mathcal{A}'$. Therefore, $\mathcal{A}'$ is consistent with the model $\mathcal{I}$.

To show that $\mathcal{T}$ is satisfied by $\mathcal{I}$, we consider each GCI $C \sqsubseteq D \in \mathcal{T}$ and each assertion $a : C \in \mathcal{A}$. Since $a : C \in \mathcal{A}$, we have $a : C \in \mathcal{A}'$, and therefore $a : D \in \mathcal{A}'$ by the completeness of $\mathcal{A}'$. Thus $a^I \in C^I$ implies $a^I \in D^I$ and thus we have $C^I \subseteq D^I$. That means $\mathcal{I}$ is a model of $\mathcal{T}$.

Hence $\mathcal{I}$ is a model of $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. So, $\mathcal{K}$ is consistent.

Question 7. Tableau Algorithm for $\mathcal{ALCN}$

- Soundness.

  For soundness, the proof is similar to the one for Lemma 4.11(in the text book), but the construction of $\mathcal{A}^{''}$ must be modified so that it leads to a complete and clash-free ABox.For example, if $\{a : (\geqslant 2r), (a,x) : r, (a,y) : r, x \neq y\} \subseteq \mathcal{A}^{'}$, and both x and y are blocked by z, then replacing $(a,x) : r$ and $(a,y) : r$ with $(a,z) : r$ in $\mathcal{A}^{''}$ would effectively merge x and y into z, resulting in a clash. So we extend $\mathcal{A}^{'}$ by adding some copies of blocking individual name for each individual name they block. For example, x is blocked by z, we add a new individual name z_1 and concept assertion $z_1 : C$ for all $z : C$ and role assertions $(z_1, y) : r$ for all $(z, y) : r$. Trivially, our new $\mathcal{A}^{''}$ is still complete and clash-free because any rule could be applied to it would have applied to its original individual name. Now we could prove the soundness using the known method, except when x is blocked by z, we treat it as it is blocked by z_1. Finally, we can copy the equality assertions from $\mathcal{A}'$ to $\mathcal{A}''$ and use these in the model construction to ensure that each individual name occurring in $\mathcal{A}$ is appropriately interpreted.

- Completeness.
  - The $\geq$-rule: If $a : (\geq n\ r) \in \mathcal{A}$, then $a^{\mathcal{J}} \in (\geq n\ r)^{\mathcal{J}}$. We can extend $\mathcal{J}$ to get a new model $\mathcal{J}'$ of $\mathcal{A}$, we just add n new individual names $d_i, 1 \leq i \leq n$ and $(a^{\mathcal{J}'}, d_i^{\mathcal{J}'}) \in (\geq nr)^{\mathcal{J}'}$, so $\mathcal{A} \cup \{(a, d_i) : r, d_i : C \mid 1 \leq i \leq n\} \cup \{d_i \neq d_j \mid 1 \leq i < j \leq n\}$ is still consistent.
  - The $\leq$-rule: If $a : (\leq n\ r) \in \mathcal{A}$, then $a^{\mathcal{J}} \in (\leq n\ r)^{\mathcal{J}}$, which means there at most n distingusih individual names d such that $(a^{\mathcal{J}}, d^{\mathcal{J}})$. Therefore, at least one $\mathcal{A}\left[b_j \mapsto b_i\right] \cup \{b_i = b_j\}$ is consistent.

Question 8. Tableau Algorithm for $\mathcal{ALCQ}$

If we use the proposed algorithm:

$$\mathcal{A}_0 = \{a :\leq 1r.(D \sqcap E), (a,b) : r, b : C \sqcap D, (a,c) : r, c : D \sqcap E, c : \neg C\}$$

$$\mathcal{A}_1 = \{a :\leq 1r.(D \sqcap E), (a,b) : r, b : C \sqcap D, (a,c) : r, c : D \sqcap E, c : \neg C\}$$
$$\cup \{b : C, b : D\} \quad (\sqcap\text{-rule})$$

$$\mathcal{A}_2 = \{a :\leq 1r.(D \sqcap E), (a,b) : r, b : C \sqcap D, (a,c) : r, c : D \sqcap E, c : \neg C, b : C, b : D\}$$
$$\cup \{c : D, c : E\} \quad (\sqcap\text{-rule})$$

$$\mathcal{A}_3 = \{a :\leq 1r.(D \sqcap E), (a,b) : r, b : C \sqcap D, (a,c) : r, c : D \sqcap E, c : \neg C, b : C, b : D,$$
$$c : D, b : E\} \cup \{b : E\} \quad (\sqsubseteq\text{-rule})$$

There is no rules can be applied and $\mathcal{A}_3$ is clash-free, the algorithm will return "consistent".

But $\mathcal{K}$ is not consistent. If there is a model I w.r.t. $\mathcal{K}$. According to $b:\ C \sqcap D$ and $C \sqsubseteq E$ we can know that $b : D \sqcap E$, we can also know that $c : D \sqcap E$, but $I \vDash_{\mathcal{K}} a :\leq 1r.(D \sqcap E)$ which means there is at most 1 element in$(D \sqcap E)^I$ so it must be $b^I = c^I$. But c : ¬C and b ∈ C ,so there is a clash.

Therefore, the knowledge base isn't consistent but the proposed algorithm can't detect this.

Question 9. A Complex in ALC Extensions

(1) Let's prove a simple property first.

For any model $\mathcal{I}$ w.r.t. $\mathcal{T}$:

$$\begin{aligned} C^{\mathcal{I}} \subseteq D^{\mathcal{I}} &\Leftrightarrow \forall a^{\mathcal{I}} \in \Delta^{\mathcal{I}}, a^{\mathcal{I}} \in C^{\mathcal{I}} \text{implies } a^{\mathcal{I}} \in D^{\mathcal{I}}) \\ &\Leftrightarrow \forall a^{\mathcal{I}} \in \Delta^{\mathcal{I}}, a^{\mathcal{I}} \notin C^{\mathcal{I}} \text{ or } a^{\mathcal{I}} \in D^{\mathcal{I}} \\ &\Leftrightarrow \forall a^{\mathcal{I}} \in \Delta^{\mathcal{I}}, a^{\mathcal{I}} \in (\neg C)^{\mathcal{I}} \text{ or } a^{\mathcal{I}} \in D^{\mathcal{I}} \\ &\Leftrightarrow \forall a^{\mathcal{I}} \in \Delta^{\mathcal{I}}, a^{\mathcal{I}} \in (\neg C \sqcup D)^{\mathcal{I}} \\ &\Leftrightarrow \Delta^{\mathcal{I}} \subseteq (\neg C \sqcup D)^{\mathcal{I}} \end{aligned}$$

So if I is a model of $\mathcal{T}$, then:

$$\begin{aligned} &\text{for each GCI } C \sqsubseteq D \in \mathcal{T}, C^I \subseteq D^I \\ \Leftrightarrow\ &\text{ for each GCI } C \sqsubseteq D \in \mathcal{T}, \Delta^I \subseteq (\neg C \sqcup D)^I \\ \Leftrightarrow\ &\Delta^I \subseteq \bigcap_{C \sqsubseteq D \in \mathcal{T}} (\neg C \cup D)^I \\ \Leftrightarrow\ &I \text{ satisfy } \top \sqsubseteq \bigcap_{C \sqsubseteq D \in \mathcal{T}} \neg C \sqcup D \\ \Leftrightarrow\ &I \text{ is a model of } \mathcal{T}' \end{aligned}$$

Therefore, $\mathcal{T}$ and $\mathcal{T}'$ have the same models.

(2) $\Longrightarrow$:

If $\mathcal{K}$ is consistent, then there exists a model I for $\mathcal{K}$ such that $I \vDash \mathcal{T}$, $I \vDash \mathcal{A}$, and $I \vDash \mathcal{R}$. So what remain to us to prove is $\mathcal{I}$ satisfies $\mathcal{T}^{+}$, i.e. $(\forall r.C)^{\mathcal{I}} \subseteq (\forall r.\forall r.C)^{I}$.

For each element a^I such that $a^I \in (\forall r.C)^I$, we consider two cases:

1. If there is not b^I such that $\left(a^I, b^I\right) \in r^I$, then $b^I \in (\forall r, \forall r.C)^I$.
2. If there exists some $b^I \in \Delta^I$ such that $\left(a^I, b^I\right) \in r^I$, then $b^I \in C^I$. If $b^I \notin (\forall r.C)^I$, then there must exist $c^I \in (\neg C)^I$ such that $\left(b^I, c^I\right) \in r^I$. However, transitivity of r implies $\left(a^I, c^I\right) \in r^I$, and hence $c^I \in C^I$, which is a contradiction. Therefore, $b^I \in (\forall r.C)^I$. Therefore,we have $a^I \in (\forall r.\forall r.C)^I$.

Therefore, $(\forall r.C)^{\mathcal{I}} \subseteq (\forall r.\forall r.C)^I$, which means $\mathcal{I} \vDash \forall r.C \sqsubseteq \forall r.\forall r.C$, $\mathcal{I}$ is a model of $\mathcal{T}^{+}$.

So we have proved that $\mathcal{K}^{+}$ is consistent.

$\Longleftarrow$:

If $\mathcal{K}^{+}$ is consistent, then there exists a model $\mathcal{I}$ for $\mathcal{K}^{+}$. Because $\mathcal{T} \subseteq \mathcal{T}^{+}$, so $\mathcal{I} \vDash (\mathcal{T}, \mathcal{A})$. So what remain to us to prove is $\mathcal{I}$ satisfies $\mathcal{R}$.

Considering the form: $\forall r.C \sqsubseteq \forall r.\forall r.C \in \mathcal{T}^{+}$ for all $\text{tran}(s) \in \mathcal{R}$.

For all $(x, z) \in r^{\mathcal{I}}$, there are $y \in \Delta^{\mathcal{T}}$ such that $(x, y) \in r^{\mathcal{I}}$ and $(y, z) \in r^{\mathcal{I}}$, which means r in transitive w.r.t. $\mathcal{I}$, so $\mathcal{I}$ is a model of $\mathcal{R}$.

So we have proved that $\mathcal{K}$ is consistent.

Question 10 (with 1 bonus mark). Pushdown automata

(1) Let $A = \{0^k 1^k 2^k | k \geq 0\}$.

We firstly show that A can be recognized by 2-PDAs.

The procedure is as follow:

- read '0' from input, and push it to stack 1 until read '1'. If read symbol that is not '0' or '1', reject.
- read '1' from input, and push it to stack 2 until read '2'. If read symbol that is not '1' or '2', reject.
- read '2' from input, pop one element from both stack 1 and stack 2. If read symbol that is not '2', reject.
- When read the whole string, it will accept it if both stack 1 and stack 2 are empty, otherwise reject.

Then we show that A could not be recognized by 1-PDAs.

We prove that A is not a CFL by contradiction.

If A is a CFL, then it satisfies the pumping lemma of CFL, which is as follow:

(1) for each $i \geq 0$, $uv^ixy^iz \in A$.

(2) $vy \neq \varepsilon$.

(3) $|vxy| \leq p$.

$s = 0^p1^p2^p$, condition (3) implies that there are at most two kinds of symbols occurs at vxy.

Therefore, there are 0s, 1s and 2s with unequal length in uv^2xy^2z, which means $uv^2xy^2z \notin B$.

So B is not a CFL, it could not be recognized by 1-PDAs. Through this example, we can see that 2-PDAs are more powerful than 1-PDAs.

(2) We can use 2 stacks to simulate 3 stacks.

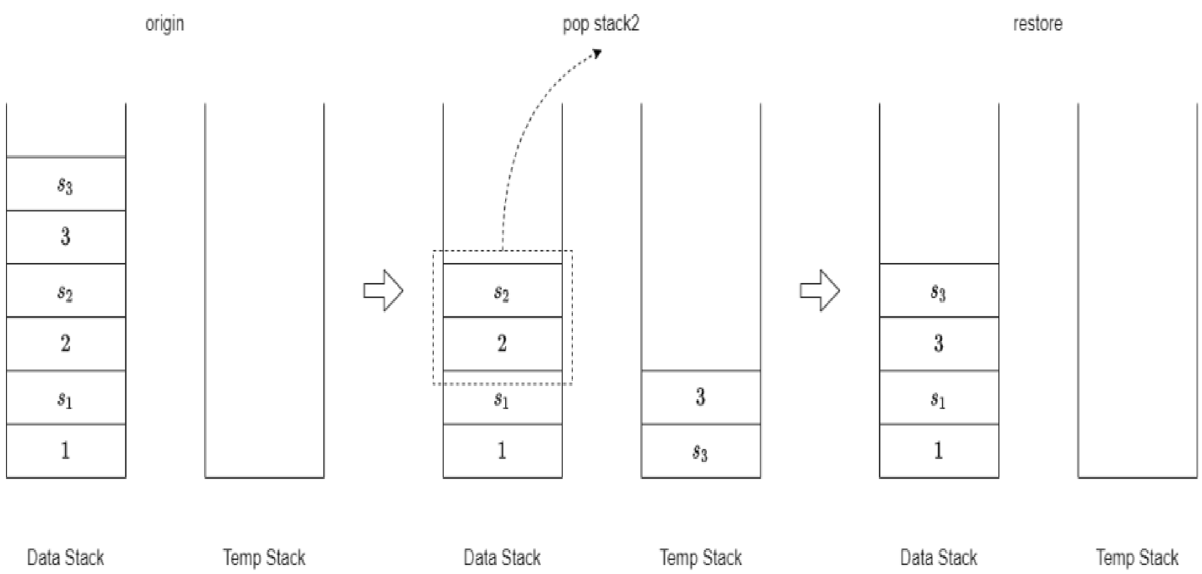
Let a 3-PDA $A = \langle Q, \Sigma, \Gamma, \delta, q_0, F\rangle$.

Let a 2-PDA $A = \langle Q, \Sigma, \Gamma', \delta, q_0, F \rangle$, $\Gamma' = \Gamma \cup \{s_1, s_2, s_3\}$, where s_1, s_2 and s_3 are label elements to identify in which stack elements of data stack are.

If we need to push a element, we firstly push the element into stack 1(we also call it data stack), then we push a label element into data stack.

If we need to pop a element with specific label, such as label s2(which means it belongs to stack 2 in 3-PDAs), we firstly pop all elements whose label is not s_2 from top of data stack, and push them into stack 2(we also call it temp stack). Once we read a label element s_2 and then pop its data element, finally restore all elements in temp stack into data stack.

See the following picture:



So 3-PDAs are not more powerful than 2-PDAs.

Question 11 (with 1 bonus mark). Turing machine

1. Scan the input string from left to right to see whether the read symbol is a

member of $\{0,1\}^*$. If there is a symbol which is not 0s or 1s,reject, otherwise go to step 2.

2. Move the head to the left-hand end, and move to right until read a 0s, write 's' and go to step 3. If there is no 0s, go to step 4.
3. Move the head to the left-hand end, and move to right until read a 1s, write 's' and go to step 2. If there is no 1s, reject.
4. Move the head to the left-hand end, and scan from left to right, if there is a symbol other than 's', reject, otherwise accept.

Question 12 (with 1 bonus mark). Decidability

A is decidable.

There are only 2 possibilities for set A : either $\{0\}$ or $\{1\}$, which are both finite sets and hence decidable.

We can define a TM to decides it: Match the input string with the only string s in language. Accept if they match, and reject otherwise.

Therefore, even though we don't know whether there will be reasonably-priced yummies in the canteens of NJU someday, the language A is decidable.