KRP **Spring 2023**

Homework 4

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Question 1. $\mathcal{ALC}$-Worlds Algorithm

The role depth of all defined concept name are as follow:

$$\mathrm{rd}(B_0) = 2, \mathrm{rd}(B_1) = 1, \mathrm{rd}(B_2) = 2, \mathrm{rd}(B_3) = 0$$
$$\mathrm{rd}(B_4) = 1, \mathrm{rd}(B_5) = 2, \mathrm{rd}(B_6) = 0, \mathrm{rd}(B_7) = 2$$
$$\mathrm{rd}(B_8) = 2, \mathrm{rd}(B_9) = 1, \mathrm{rd}(B_{10}) = 0$$

Therefore, $i = \mathrm{rd}(B_0) = \max(\mathrm{rd}(B_1), \mathrm{rd}(B_2)) = 2$.

$$\mathrm{Def}_0(\mathcal{T}) = \{B_3, B_6, B_{10}\}$$
$$\mathrm{Def}_1(\mathcal{T}) = \{B_1, B_3, B_4, B_6, B_9, B_{10}\}$$
$$\mathrm{Def}_2(\mathcal{T}) = \{B_0, B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_{10}\}$$

- Successful run.

  We guess a set $\tau = \{B_0, B_1, B_2, B_3, B_4, B_5, B_6, B_7\} \subseteq \mathrm{Def}_2$ with $B_0 \in \tau$.
  recurse$(\tau, 2, \mathcal{T})$.
  recurse$(\tau, 2, \mathcal{T})$:
  τ is a type for $\mathcal{T}$ and $i \neq 0$.
  (i). for $B_1 \in \tau$ with $B_1 \equiv \exists r.B_3$: $S = B_3 \cup B_4 = \{B_3, B_4\}$, we guess $\tau_1 = \{B_3, B_4\} \subseteq \mathrm{Def}_1$ with $S \in \tau_1$.
  recurse$(\tau_1, 1, \mathcal{T})$:
  for $B_4 \in \tau_1$ with $B_4 \equiv \exists r.B_6$:$S = \{B_6\}$ We guess $\tau_1' = \{B_6\}$.

recurse($\tau_1', 0, \mathcal{T}$), because $i == 0$ so return true.

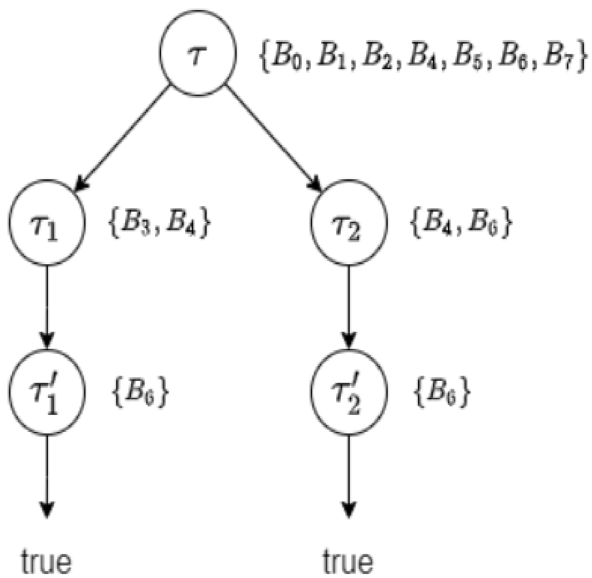
(ii). for $B_4 \in \tau$ with $B_4 \equiv \exists r.B_6$: $S = \{B_6\} \cup \{B_4\} = \{B_4, B_6\}$we guess $\tau_2 = \{B_4, B_6\}$.

recurse($\tau_2, 1, \mathcal{T}$):

$S = \{B_6\} \cup \emptyset$ we guess $\tau_2' = \{B_6\}$

recurse($\tau_2', 0, \mathcal{T}$), because $i == 0$ so return true.

Because (i) and (ii) all return true, so the algorithm finally return true.



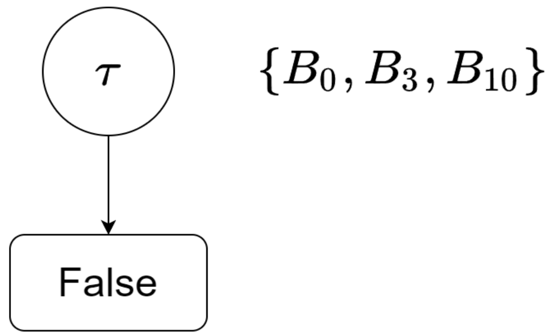
- Unsuccessful run.

  We guess a set $\tau = \{B_0, B_3, B_{10}\} \subseteq \mathrm{Def}_2$ with $B_0 \in \tau$.

  recurse($\tau, 2, \mathcal{T}$):

  τ is not a type for $\mathcal{T}$ because $B_3 \in \tau, B_{10} \in \tau$ but $B_3 \equiv P$ and $B_{10} \equiv \neg P$.

  Therefore, the algorithm return false.



Because there is a successful run, so the algorithm return a positive result.

Question 2. Entailment Checking

It holds true.

For any model $\mathcal{I} \models \{\forall r.A \sqsubseteq \exists r.A\}$, we'll prove that $\mathcal{I} \models \{\forall r.B \sqsubseteq \exists r.B\}$ for all concept B.

According to the semantics of $\mathcal{ALC}$:

$$\begin{aligned} &\mathcal{I} \models \{\forall r.A \sqsubseteq \exists r.A\} \\ &\quad \Rightarrow (\forall r.A)^{\mathcal{I}} \subseteq (\exists r.A)^{\mathcal{I}} \end{aligned}$$

For a element $a \in \Delta^{\mathcal{I}}$, if there is no element b such that $(a, b) \in r^{\mathcal{I}}$, then $a \in (\forall r.A)^{\mathcal{I}}$, so $a \in (\exists r.A)^{\mathcal{I}}$ due to $(\forall r.A)^{\mathcal{I}} \subseteq (\exists r.A)^{\mathcal{I}}$, obviously there is a contradiction.

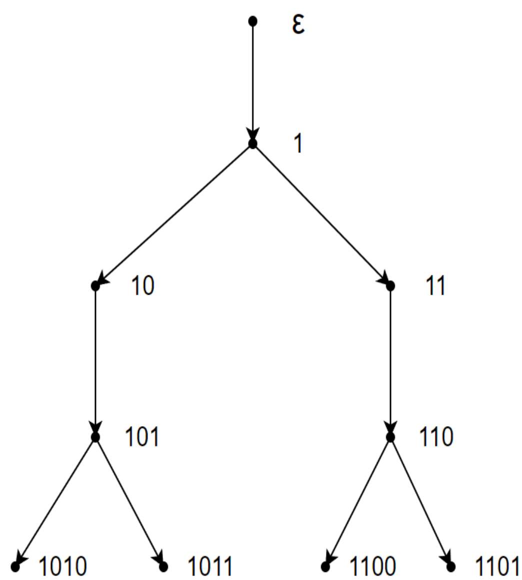
Therefore, $\forall a \in \Delta^{\mathcal{I}}$, there exist element $b \in \Delta^{\mathcal{I}}$ such that $(a, b) \in r^{\mathcal{I}}$. $\cdots$ (1)

For a element a, if $a \in (\forall r.B)^{\mathcal{I}}$, then there at least exists a element b, such that $(a, b) \in r^{\mathcal{I}}$ and $b \in B^{\mathcal{I}}$, otherwise it contradicts conclusion (1), thus $a \in (\exists r.B)^{\mathcal{I}}$.

Therefore, $(\forall r.B)^{\mathcal{I}} \subseteq (\exists r.B)^{\mathcal{I}}$, thus $\mathcal{I} \models \{\forall r.B \sqsubseteq \exists r.B\}$.

Question 3. Finite Boolean Games

(1) The figure following shows a winning strategy for Player 1 in G.



(2) If Player 2 assign $x_2 = 0 \ and \ x_4 = 1$, whatever Player1 do, there is no word t can satisfies φ.

Therefore, Player 1 doesn't have a winning strategy.

Question 4. Infnite Boolean Games

(1) Player 2 doesn't have a winning strategy.

We can show that by showing Player 1 has a winning strategy.

Player 1 assign $x_2 = 1$ and $x_3 = 1$ in the previous two turns.

After assigning, $y_1 = 0$ or $y_2 = 0$, otherwise $(\neg(x_1 \vee x_4) \wedge y_1 \wedge y_2) = 1$ and Player 1 wins.

If $y_1 =$ False, Player 1 can assign $x_1 = 1$ and then $(x_1 \wedge x_2 \wedge \neg y_1) = 1$, so Player 1 wins.

If $y_2 =$ False, Player 1 can assign $x_4 = 1$ and then $(x_3 \wedge x_4 \wedge \neg y_2) = 1$, so Player 1 wins.

So Player 2 doesn't have a winning strategy.

(2) Player 2 has a winning strategy.

If $y_1 = 0, y_2 = 0$, the formule φ is false, so Player 2 just need to assign 0 to y_1 and y_2 and he can win this game.

Question 5. Complexity of Concept Satisfability in ALC Extensions

Firstly we prove that concept satisfiability in $\mathcal{ALC}^u$ without TBoxes is EXPTIME-hard. $\cdots(1)$

We have known in class that $\mathcal{ALC}$-concept satisfiability w.r.t. general TBox is EXPTIME-hard. So we need to prove that $\mathcal{ALC}$-concept satisfiability w.r.t. general TBox can be reduced to concept satisfiability in $\mathcal{ALC}^u$.

Construct an $\mathcal{ALC}^u$-concept:

$$D_0 = C_0 \sqcap \forall u.(\bigsqcap_{C \sqsubseteq D \in \mathcal{T}} \neg C \sqcup D)$$

C_0 is satisfiable with respect to $\mathcal{T}$ iff D_0 is satisfiable with respect to $\mathcal{T}$. Now we prove it.

$\Longleftarrow$:

Let $\mathcal{I}$ be a model of D_0, $d_0 \in D_0^{\mathcal{I}}$.

Due to the universal rule, $\forall d \in D_0^{\mathcal{I}}$ we have $(d_0, d) \in u^{\mathcal{I}}$and therefore $d \in (\bigsqcap_{C \sqsubseteq D \in \mathcal{T}} \neg C \sqcup D)^{\mathcal{I}}$,which means $\forall d \in C^{\mathcal{I}}$ and $C \sqsubseteq D \in \mathcal{T}$, we have $d \in D^{\mathcal{I}}$ because $d \in (\neg C \sqcup D)^{\mathcal{I}}$.

So $\mathcal{I}$ is also a model of C_0 .

$\Longrightarrow$:

Let $\mathcal{I}$ be a model of C_0 w.r.t. $\mathcal{T}$, $d_0 \in C_0^{\mathcal{I}}$.

Modify $\mathcal{I}$ by setting $u^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

Since $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ for all $C \sqsubseteq D \in \mathcal{T}$, we have $\Delta^{\mathcal{I}} \subseteq (\neg C \sqcup D)^{\mathcal{I}}$ and therefore $d_0 \in (\forall u.(\bigsqcap_{C \sqsubseteq D \in \mathcal{T}} \neg C \sqcup D))^{\mathcal{I}} = (\forall u.\top)^{\mathcal{I}} = \Delta^{\mathcal{I}}$.

So $\mathcal{I}$ is also a model of D_0 .

And now we have prove that the satisfiability of $\mathcal{ALC}$-concept C_0 can be reduced to the satisfiability of $\mathcal{ALC}^u$-concept D_0, thus concept satisfiability in $\mathcal{ALC}^u$ without TBoxes is EXPTIME-hard.

Secondly, we prove that concept satisfiability in $\mathcal{ALC}^u$ without TBoxes has a

EXPTIME upper bound. $\cdots(2)$

We will modify $\mathcal{ALC}$-Elim algorithm to get $\mathcal{ALC}^u$-Elim algorithm, which is a EXPTIME algorithm.The only difference is the defnition of bad type:

- $\exists r.C \in \tau$ such that the set $S = \{C\} \cup \{D | \forall r.D \in \tau\}$ is no subset of any type in Γ
- $\exists u.C \in \tau$ such that the set $S' = \{C\} \cup \{D | \forall u.D \in \tau\}$ is no subset of any type in Γ

Now we prove that $\mathcal{ALC}^u$-Elim$(A_0, \mathcal{T})$ returns 'true' iff A_0 is satisfifiable w.r.t. $\mathcal{T}$.

$\Longrightarrow$:

Construct a model $\mathcal{I}$ use the result of $\mathcal{ALC}^u$-Elim$(A_0, \mathcal{T})$:

$$\Delta^{\mathcal{I}} = \Gamma_i$$
$$A^{\mathcal{I}} = \{\tau \in \Gamma_i | A \in \tau\}$$
$$r^{\mathcal{I}} = \{(\tau, \tau') \in \Gamma_i \times \Gamma_i | \forall r.C \in \tau \text{ implies } C \in \tau'\}$$

- Let $\exists u.D \in \tau$. Since τ has not been eliminated from Γ_i, it is not bad. Thus, there is a $\tau' \in \Gamma$, such that $\{D\} \subseteq \tau'$. Because we have $(\tau'', \tau') \in u^{\mathcal{I}}$ for any type τ'', we obtain $\tau'' \in (\exists r.D)^{\mathcal{I}}$ by the semantics, and it also includes $\tau \in (\exists r.D)^{\mathcal{I}}$.
- Let $\forall u.D \in \tau$. Since τ has not been eliminated from Γ_i, it is not bad. Thus, there is $D \in \tau'$ for all type τ', we obtain $\tau' \in D^{\mathcal{I}}$ and $\tau' \in (\forall u.D)^{\mathcal{I}}$ by the semantics, and it also include $\tau \in D$ and $\tau \in (\forall u.D)^{\mathcal{I}}$.

$\Longleftarrow$:

If A_0 is satisfiable with respect to $\mathcal{T}$, then there is a model $\mathcal{I}$ of A_0 and $\mathcal{T}$. Let $d_0 \in A_0^{\mathcal{I}}$. For all $d \in \Delta^{\mathcal{I}}$,

$$\text{tp}(d) = \{C \in \text{sub}(\mathcal{T}) | d \in C^{\mathcal{I}}\}$$

Define $\Psi = \{\mathrm{tp}(d) | d \in \Delta^{\mathcal{I}}\}$ and let $\Gamma_0, \Gamma_1, \cdots, \Gamma_k$ be the sequence of type sets computed by $\mathcal{ALC}^u$-Elim$(A_0, \mathcal{T})$. It is possible to prove by induction on i that no type from Ψ is ever eliminated from any set Γ_i, for $i \leq k$. So the algorithm return ”true”.

According to the conclusion (1) and (2), we can get that concept satisfability in $\mathcal{ALC}^u$ without TBoxes is EXPTIME-complete.

Question 6. Conservative Extension

(1) $\because sig(\mathcal{T}_2) = sig(\mathcal{T}_1) \cup \{A, B\} \quad \therefore sig(\mathcal{T}_1) \subseteq sig(\mathcal{T}_2)$

$\because \mathcal{T}_1 \subseteq \mathcal{T}_2 \quad \therefore$ every model of $\mathcal{T}_2$ is a model of $\mathcal{T}_1$

For every model $\mathcal{I}_1$ of $\mathcal{T}_1$, we can construct a model $\mathcal{I}_2$ as follow:

- $\Delta^{\mathcal{I}_2} = \Delta^{\mathcal{I}_1}$
- $E^{\mathcal{I}_2} = E^{\mathcal{I}_1}$ for all concept names E in $\mathcal{T}_1$, $A^{\mathcal{I}_2} = C^{\mathcal{I}_1}, B^{\mathcal{I}_2} = D^{\mathcal{I}_1}$
- $r^{\mathcal{I}_2} = r^{\mathcal{I}_1}$ for all roles in $\mathcal{T}_1$

Obviously, $\mathcal{I}_2$ is a model of $\mathcal{T}_2$ and the extensions of concept and role names from $sig(\mathcal{T}_1)$ coincide in $\mathcal{I}_1$ and $\mathcal{I}_2$.

Therefore, $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$.

(2) After adding $A \sqsubseteq B$, it still holds.

The only difference of the model $\mathcal{I}_2$ we construct with (1) is: $A^{\mathcal{I}} = \emptyset$.

We can get that $A^{\mathcal{I}} \subseteq B^{\mathcal{I}}$ so $\mathcal{I}_2$ is still a model of $\mathcal{T}_2$ and the extensions of concept and role names from $sig(\mathcal{T}_1)$ coincide in $\mathcal{I}_1$ and $\mathcal{I}_2$.

Therefore, $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$.

(3) It dose not hold.

After adding $B \sqsubseteq A$, we can get that $D \sqsubseteq C$ in $\mathcal{T}_2$.

For some model $\mathcal{I}_1$ w.r.t. $\mathcal{T}_1$, if there exists element $a \in \Delta^{\mathcal{I}_1}$ such that $a \in D^{\mathcal{I}_1}$ but $a \notin C^{\mathcal{I}_1}$, then it is impossible to find a model $\mathcal{I}_2$ w.r.t. $\mathcal{T}_2$ and the extensions of concept and role names from $sig(\mathcal{T}_1)$ coincide in $\mathcal{I}_1$ and $\mathcal{I}_2$. (because there exists $D^{\mathcal{I}_1} \neq D^{\mathcal{I}_2}$ or $C^{\mathcal{I}_1} \neq C^{\mathcal{I}_2}$)

Question 7. Subsumption in $\mathcal{EL}$

Normalization TBox $\mathcal{T}$:

$$A \sqsubseteq B \sqcap \exists r.C \rightarrow_{NF4} A \sqsubseteq B, A \sqsubseteq \exists r.C$$

$$B \sqcap \exists r.B \sqsubseteq C \sqcap D \rightarrow_{NF0} \underline{B \sqcap \exists r.B \sqsubseteq E_0}, \underline{E_0 \sqsubseteq C \sqcap D}$$

$$B \sqcap \exists r.B \sqsubseteq E_0 \rightarrow_{NF1_r} \exists r.B \sqsubseteq E_1, B \sqcap E_1 \sqsubseteq E_0$$

$$E_0 \sqsubseteq C \sqcap D \rightarrow_{NF4} E_0 \sqsubseteq C, E_0 \sqsubseteq D$$

$$C \sqsubseteq (\exists r.A) \sqcap B \rightarrow_{NF4} C \sqsubseteq \exists r.A, C \sqsubseteq B$$

$$(\exists r.\exists r.B) \sqcap D \sqsubseteq \exists r.(A \sqcap B) \rightarrow_{NF0} \underline{(\exists r.\exists r.B) \sqcap D \sqsubseteq E_2}, \underline{E_2 \sqsubseteq \exists r.(A \sqcap B)}$$

$$(\exists r.\exists r.B) \sqcap D \sqsubseteq E_2 \rightarrow_{NF1_l} \underline{(\exists r.\exists r.B) \sqsubseteq E_3}, E_3 \sqcap D \sqsubseteq E_2$$

$$(\exists r.\exists r.B) \sqsubseteq E_3 \rightarrow_{NF2} \exists r.B \sqsubseteq E_4, \exists r.E_4 \sqsubseteq E_3$$

$$E_2 \sqsubseteq \exists r.(A \sqcap B) \rightarrow_{NF3} \underline{E_5 \sqsubseteq A \sqcap B}, E_2 \sqsubseteq \exists r.E_5$$

$$E_5 \sqsubseteq A \sqcap B \rightarrow_{NF4} E_5 \sqsubseteq A, E_5 \sqsubseteq B$$

We get the normalised TBox $\mathcal{T}' = \{A \sqsubseteq B, A \sqsubseteq \exists r.C, \exists r.B \sqsubseteq E_1, B \sqcap E_1 \sqsubseteq E_0, E_0 \sqsubseteq C, E_0 \sqsubseteq D, C \sqsubseteq \exists r.A, C \sqsubseteq B, E_3 \sqcap D \sqsubseteq E_2, \exists r.B \sqsubseteq E_4, \exists r.E_4 \sqsubseteq E_3, E_2 \sqsubseteq \exists r.E_5, E_5 \sqsubseteq A, E_5 \sqsubseteq B\}$

(1) $A \sqsubseteq B$ already exists in $\mathcal{T}'$, so it holds w.r.t. to $\mathcal{T}'$.

(2) According to lemma 6.1, $\mathcal{T} \models A \sqsubseteq \exists r.\exists r.A$ iff $\mathcal{T} \cup \{F \sqsubseteq A, \exists r.\exists r.A \sqsubseteq G\} \models F \sqsubseteq G$. Normalization $\mathcal{T} \cup \{F \sqsubseteq A, \exists r.\exists r.A \sqsubseteq G\}$, we get $\mathcal{T}'' = \mathcal{T}' \cup \{F \sqsubseteq A, \exists r.A \sqsubseteq H, \exists r.H \sqsubseteq G\}$

Apply CR3 to $C \sqsubseteq \exists r.A, \exists r.A \sqsubseteq H \rightarrow C \sqsubseteq H$

Apply CR3 to $F \sqsubseteq A, A \sqsubseteq \exists r.C \rightarrow F \sqsubseteq \exists r.C$

Apply CR5 to $F \sqsubseteq \exists r.C, C \sqsubseteq H, \exists r.H \sqsubseteq G \rightarrow F \sqsubseteq G$

we have $F \sqsubseteq G$, so $A \sqsubseteq \exists r.\exists r.A$ holds.

(3) We can get $\mathcal{T}'' = \mathcal{T}' \cup \{F \sqsubseteq B, F \sqsubseteq \exists r.A, \exists r.C \sqsubseteq G\}$ just like (2).

$$\text{Apply CR5 to } A \sqsubseteq \exists r.C, C \sqsubseteq B, \exists r.B \sqsubseteq E_1 \to A \sqsubseteq E_1$$

$$\text{Apply CR3 to } B \sqcap E_1 \sqsubseteq E_0, E_0 \sqsubseteq C \to B \sqcap E_1 \sqsubseteq C$$

$$\text{Apply CR4 to } A \sqsubseteq B, A \sqsubseteq E_1, B \sqcap E_1 \sqsubseteq C \to A \sqsubseteq C$$

$$\text{Apply CR5 to } F \sqsubseteq \exists r.A, A \sqsubseteq C, \exists r.C \sqsubseteq G \to F \sqsubseteq G$$

we have $F \sqsubseteq G$, so $B \sqcap \exists r.A \sqsubseteq \exists r.C$ holds.

Question 8. Simulation

(a) According to the defnition of bisimulation $(\mathcal{I}, d) \sim (\mathcal{J}, d')$:

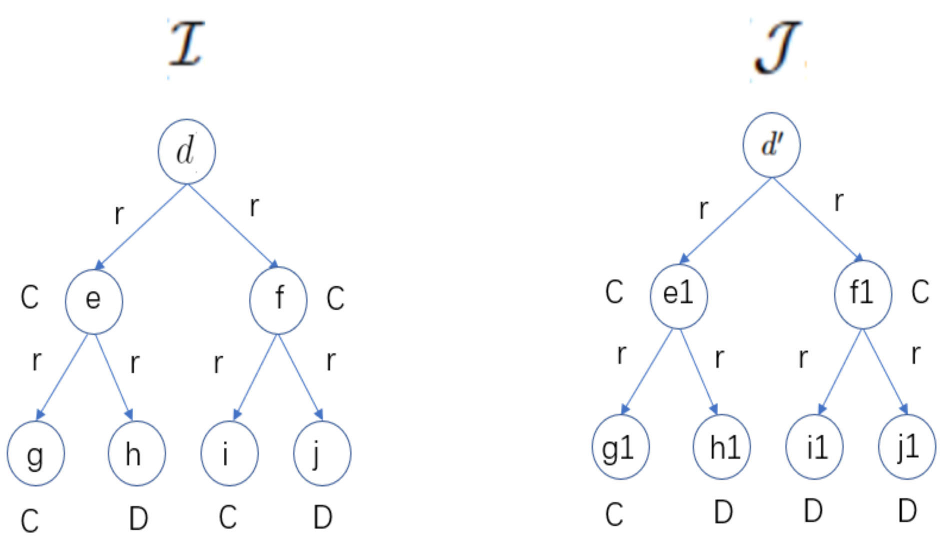
(i) $d\rho d'$ implies $d \in A^{\mathcal{I}} \Leftrightarrow d' \in A^{\mathcal{J}}$ for all $A \in \mathbb{C}$.

(ii) $d\rho d'$ and $(d, e) \in r^{\mathcal{I}}$ implies there exists $e' \in \Delta^{\mathcal{J}}$ so that $e\rho e'$ and $(d', e') \in r^{\mathcal{J}}$.

(iii) $d\rho d'$ and $(d', e') \in r^{\mathcal{J}}$ implies there exists $e \in \Delta^{\mathcal{I}}$ so that $e\rho e'$ and $(d, e) \in r^{\mathcal{I}}$.

We can get $(\mathcal{I}, d) \precsim (\mathcal{J}, d')$ because of (i) and (ii), $(\mathcal{J}, d') \precsim (\mathcal{I}, d)$ because of (i) and (iii).

(b) Here is the counterexample:



We can get $(\mathcal{I}, e) \precsim (\mathcal{J}, e1)$ and $(\mathcal{I}, f) \precsim (\mathcal{J}, f1)$ so $(\mathcal{I}, d) \precsim (\mathcal{J}, d')$.

We can get $(\mathcal{J}, e1) \precsim (\mathcal{I}, f)$ and $(\mathcal{J}, f1) \precsim (\mathcal{I}, f)$ so $(\mathcal{J}, d') \precsim (\mathcal{I}, d)$.

But obviously $(\mathcal{I}, d) \not\sim (\mathcal{J}, d')$

(c)
- Assume $C = A \in \mathbf{C}$(Concept name).

  $$d \in C^{\mathcal{I}} \text{ implies } d' \in C^{J}$$

  is an immediate consequence due to the defnition of $(\mathcal{I}, d) \sim (\mathcal{J}, d')$

- Assume $C = \top$.

  $$d \in C^{\mathcal{I}} \text{ implies } d' \in C^{J}$$

  is an immediate consequence due to the defnition of $(\mathcal{I}, d) \sim (\mathcal{J}, d')$

- Assume $C = D \sqcap E$.
  If $d \in C^{\mathcal{I}}$ then $d \in D^{\mathcal{I}} \cap E^{\mathcal{I}}$ implies $d \in D^{\mathcal{I}}, d \in E^{\mathcal{I}}$, which implies $d' \in D^{\mathcal{J}}, d' \in E^{\mathcal{J}}, d' \in (D \sqcap E)^{\mathcal{J}} = C^{\mathcal{J}}$.

- Assume $C = \exists r.D$.
  If $d \in C^{\mathcal{I}}$ then there exists an $e \in D^{\mathcal{I}}$ and $(d, e) \in r^{\mathcal{I}}$, which implies there exists an $e' \in D^{\mathcal{J}}, (d', e') \in r^{\mathcal{J}}$. So $d' \in C^{\mathcal{J}}$.

(d) For disjunction:

Assume $C = D \sqcup E$. If $d \in C^{\mathcal{I}}$, then we have $d \in D^{\mathcal{I}}$ or $d \in E^{\mathcal{I}}$. We have $d' \in D^{\mathcal{J}}$ or $d' \in E^{\mathcal{J}}$. So we have $d' \in (D \sqcup E)^{\mathcal{J}} = C^{\mathcal{J}}$

Therefore, disjunction can be added to $\mathcal{EL}$ without losing the property in (c).
For negation:

Let $A^{\mathcal{I}} = \{e\}, \Delta^{\mathcal{I}} = \{d, e\}$ and $A^{\mathcal{J}} = \{d'\}, \Delta^{\mathcal{J}} = \{d', e'\}$. We have $(\mathcal{I}, d) \eqsim (\mathcal{J}, d')$ and $d \in (\neg A)^{\mathcal{I}}$. But $d' \notin (\neg A)^{\mathcal{J}}$.

Therefore, negation can not be added without lose the property.

For value restriction:

Let $A^{\mathcal{I}} = \{a\}, r^{\mathcal{I}} = \emptyset, \Delta^{\mathcal{I}} = \{a, b\}$ and $A^{\mathcal{J}} = \{a'\}, r^{\mathcal{J}} = \{(a', b')\}, \Delta^{\mathcal{J}} = \{a', b'\}$. We have $(\mathcal{I}, a) \eqsim (\mathcal{J}, a')$ and $a \in (\forall r.A)^{\mathcal{I}}$. But $a' \notin (\forall r.A)^{\mathcal{J}}$.
Therefore, value restriction can not be added without lose the property.

(e) The above consequence in (c) states that $\mathcal{EL}$ cannot distinguish between simulate elements. But $\mathcal{ALC}$ can. Look at the example as follow:

$\Delta^{\mathcal{I}} = \{a, b, c\}, A^{\mathcal{I}} = \{a, b\}, B^{\mathcal{I}} = \{c\}$

$\Delta^{\mathcal{J}} = \{a', b', c'\}, A^{\mathcal{I}} = \{a', b', c'\}, B^{\mathcal{I}} = \{c'\}$

Obviously, $(\mathcal{I}, c) \precsim (\mathcal{J}, c')$. But $c \in (\neg A)^{\mathcal{I}}$ while $c' \notin (\neg A)^{\mathcal{J}}$

So $\mathcal{ALC}$ is more expressive than $\mathcal{EL}$

Question 9. $\mathcal{EL}$ Extension

(1) To show that each $\mathcal{EL}$si concept description is equivalent to some concept descriptions of the form $\exists\text{sim}(I, d)$, we need to demonstrate that any $\mathcal{EL}$si concept description can be represented using $\exists\text{sim}(I, d)$ and vice versa.

Let's start with an $\mathcal{EL}$si concept description of the form $\exists\text{sim}(I, \delta)$, where I is a finite interpretation and $\delta \in \Delta^I$. We want to show its equivalence to a concept description of the form $\exists\text{sim}(I, d)$.

To do this, we'll represent δ as a concept description using $\exists\text{sim}(I, d)$. Consider the concept description $\delta' = \{x \mid \exists\text{sim}(I, \delta)(x)\}$.

Now, let's analyze the semantics of both descriptions:

- $(\exists\text{sim}(I, \delta))J$: This represents the set of individuals in the interpretation J that satisfy the concept description $\exists\text{sim}(I, \delta)$. In other words, it includes individuals in J for which there exists an individual in I that is similar to them according to δ.
- $(\exists\text{sim}(I, d))J$: This represents the set of individuals in the interpretation J that satisfy the concept description $\exists\text{sim}(I, d)$. Similarly, it includes individuals in J for which there exists an individual in I that is similar to them according to d.

We need to show that $(\exists\text{sim}(I, \delta))J = (\exists\text{sim}(I, d))J$. To prove this, we'll demonstrate that $(\exists\text{sim}(I, \delta))J \subseteq (\exists\text{sim}(I, d))J$ and $(\exists\text{sim}(I, d))J \subseteq (\exists\text{sim}(I, \delta))J$.

(a) $(\exists\text{sim}(I, \delta))J \subseteq (\exists\text{sim}(I, d))J$: Let's assume an individual $a \in (\exists\text{sim}(I, \delta))J$. It means that there exists an individual b in I such that (I, δ) is similar to (J, a). Since δ' represents $\exists\text{sim}(I, \delta)$, we can say that $b \in (\exists\text{sim}(I, d))J$, as (I, d) is similar to (J, a). Therefore, $(\exists\text{sim}(I, \delta))J \subseteq (\exists\text{sim}(I, d))J$.

(b) $(\exists \mathrm{sim}(I,d))J \subseteq (\exists \mathrm{sim}(I,\delta))J$: Assume an individual $c \in (\exists \mathrm{sim}(I,d))J$. It implies that there exists an individual d in I such that (I,d) is similar to (J,c). Since δ represents $\exists \mathrm{sim}(I,\delta)$, we can say that $d \in (\exists \mathrm{sim}(I,\delta))J$, as (I,δ) is similar to (J,c). Hence, $(\exists \mathrm{sim}(I,d))J \subseteq (\exists \mathrm{sim}(I,\delta))J$.

Therefore, we have shown that $(\exists \mathrm{sim}(I,\delta))J = (\exists \mathrm{sim}(I,d))J$. This demonstrates the equivalence between the $\mathcal{EL}$si concept description $\exists \mathrm{sim}(I,\delta)$ and the concept description $\exists \mathrm{sim}(I,d)$.

By extension, we can conclude that any ELsi concept description can be represented by a concept description of the form $\exists \mathrm{sim}(I,d)$, and vice versa.

(2) Construct interpretation $\mathcal{I}$:

$$\Delta^{\mathcal{I}} = \{d\}$$
$$A^{\mathcal{I}} = \{d\} \text{ for each } A \in \mathbb{C}$$

However, there are no roles in $\mathcal{I}$. Consequently, any element simulated by d must belong to the extensions of all concepts $A \in \mathbb{C}$. The concept $\exists^{\mathrm{sim}}(\mathcal{I},d)$ is equivalent to $\prod\limits_{A\in\mathbb{C}} A$.

Now let's prove that no $\mathcal{EL}$ concept is equivalent to $\exists^{\mathrm{sim}}(\mathcal{I},d)$. Consider any concept C that is not the intersection of all concept names. There must exist a concept name $A \notin \mathrm{sub}(C)$.

We can construct a model $\mathcal{J}$ as follows:

$r^{\mathcal{J}}$ is empty for all role names in C.

$A^{\mathcal{J}} = \begin{cases} \{a,b\} & A \in \mathrm{sub}(C) \\ \{a\} & A \notin \mathrm{sub}(C) \end{cases}$ for all concept names $A \in \mathbb{C}$.

$\Delta^{\mathcal{J}} = \{a,b\}$.

In this model, any concepts of the form $\exists r.E$ will be interpreted as $\emptyset$ by $\mathcal{J}$. Consequently, any concepts in $\mathrm{sub}(C)$ will be interpreted as $\emptyset$ or a,b by $\mathcal{J}$.

However, $(\exists^{\text{sim}}(\mathcal{I}, d))^{\mathcal{J}} = a$. Thus, no $\mathcal{EL}$ concept is equivalent to the $\mathcal{EL}_{\text{si}}$ concept.

Therefore, $\mathcal{EL}_{\text{si}}$ is more expressive than $\mathcal{EL}$.

(3) To show that checking subsumption in $\mathcal{EL}$si without any TBox can be done in polynomial time, we need to demonstrate that there exists a polynomial-time algorithm that can determine whether one $\mathcal{EL}$si concept is a subsumed by another $\mathcal{EL}$si concept.

Given two $\mathcal{EL}$si concepts, C_1 and C_2, the algorithm for checking subsumption can proceed as follows:

1. If C_1 is equivalent to C_2, return true.
2. If C_1 is of the form $\exists^{\text{sim}}(I_1, d_1)$ and C_2 is of the form $\exists^{\text{sim}}(I_2, d_2)$, check if I_1 and I_2 have a non-empty intersection. If they do and $d_1 = d_2$, return true. Otherwise, return false.
3. If C_1 is of the form $\exists^{\text{sim}}(I_1, d_1)$ and C_2 is of the form D_2, recursively check if D_2 subsumes $\exists^{\text{sim}}(I_1, d_1)$. If it does, return true. Otherwise, return false.
4. If C_1 is of the form D_1 and C_2 is of the form $\exists^{\text{sim}}(I_2, d_2)$, return false.
5. If C_1 is of the form D_1 and C_2 is of the form D_2, recursively check if D_1 subsumes D_2. If it does, return true. Otherwise, return false.

This algorithm checks each possible case and terminates in a finite number of steps. The size of the input concepts and interpretations can be represented using polynomially bounded space. Thus, the algorithm runs in polynomial time.

Therefore, checking subsumption in $\mathcal{EL}$si without any TBox can be done in polynomial time.

Question 10 (with 1 bonus mark). $\mathcal{ALC}$-Elim Algorithm

(1) We firstly caculate $C_{\mathcal{T}}$ and $sub(\mathcal{T})$:

$$C_{\mathcal{T}} = A \sqcap (\neg A \sqcup \exists r.A) \sqcap (\exists r.\neg A \sqcup \exists r.A)$$
$$sub(\mathcal{T}) = \{\exists r.\neg A, \neg A, \exists r.A, \exists r.\neg A \sqcup \exists r.A, \neg A \sqcup \exists r.A, A, A \sqcap (\exists r.\neg A \sqcup \exists r.A), C_{\mathcal{T}}\}$$

Then we run $\mathcal{ALC} - Elim$ algorithm.

$\mathcal{ALC} - Elim(A, \mathcal{T})$:

loop:

$i = 0$

$\mathit{\Gamma}_0 = \{\tau_1, \tau_2\}$

$\tau_1 = \{\exists r.A, \exists r.\neg A \sqcup \exists r.A, \neg A \sqcup \exists r.A, A, A \sqcap (\exists r.\neg A \sqcup \exists r.A), C_{\mathcal{T}}\}$

i=1:

$S = \{A\} \subseteq \tau_1$, τ_1 is not bad

$S = \{\neg A\} \not\subseteq \tau_1$ or τ_2, τ_2 is bad!

$\mathit{\Gamma}_1 = \{\tau_1\}$

$i = 2, \mathit{\Gamma}_2 = \mathit{\Gamma}_1 = \{\tau_1\}$, break the loop!

$A \in \tau_1$, return true

The satisfying model $\mathcal{I}$:

$$\Delta^{\mathcal{I}} = \{\tau_1\}$$
$$A^{\mathcal{I}} = \{\tau_1\}$$
$$r^{\mathcal{I}} = \{(\tau_1, \tau_1)\}$$

(2) Add $D \sqsubseteq \forall r.\forall r.\neg B$ to $\mathcal{T}$, where D is a fresh concept name.

We firstly caculate $C_{\mathcal{T}}$, $sub(\mathcal{T})$ and τ_i:

$$C_{\mathcal{T}} = (\neg A \sqcup \neg B) \sqcap (A \sqcup B) \sqcap \exists r.\neg A \sqcap (\neg D \sqcup \forall r.\forall r.\neg B)$$

$$sub(\mathcal{T}) = \{\forall r.\neg B, \forall r.\forall r.\neg B, D, \neg D, A, \neg A, B, \neg B, \neg C \sqcup \forall r.\forall r.\neg B, \exists r.\neg A, \neg A \sqcup \neg B, A \sqcup B, (A \sqcup B) \sqcap (\neg A \sqcup \neg B), (A \sqcup B) \sqcap (\neg A \sqcup \neg B) \sqcap \exists r.\neg A, C_{\mathcal{T}}\}$$

$$\tau_0 = \{\neg C \sqcup \forall r.\forall r.\neg B, \exists r.\neg A, \neg A \sqcup \neg B, A \sqcup B, (A \sqcup B) \sqcap (\neg A \sqcup \neg B), (A \sqcup B) \sqcap (\neg A \sqcup \neg B) \sqcap \exists r.\neg A, C_{\mathcal{T}}\}$$

$\tau_1 = \tau_0 \cup \{A, \neg B, \neg D\}$ $\quad$ $\tau_2 = \tau_1 \cup \{\forall r.\neg B\}$ $\quad$ $\tau_3 = \tau_0 \cup \{A, \neg B, \forall r.\forall r.\neg B\}$

$\tau_4 = \tau_3 \cup \{D\}$ $\quad$ $\tau_5 = \tau_3 \cup \{\forall r.\neg B\}$ $\quad$ $\tau_6 = \tau_3 \cup \{\forall r.\neg B, D\}$

$\tau_7 = \tau_3 \cup \{\neg D\}$ $\quad$ $\tau_8 = \tau_3 \cup \{\neg D, \forall r.\neg B\}$ $\quad$ $\tau_9 = \tau_0 \cup \{\neg A, B, \neg D\}$

$\tau_{10} = \tau_9 \cup \{\forall r.\neg B\}$ $\quad$ $\tau_{11} = \tau_0 \cup \{\neg A, B, \forall r.\forall r.\neg B\}$ $\quad$ $\tau_{12} = \tau_{11} \cup \{D\}$

$\tau_{13} = \tau_{11} \cup \{\forall r.\neg B\}$ $\quad$ $\tau_{14} = \tau_{11} \cup \{D, \forall r.\neg B\}$ $\quad$ $\tau_{15} = \tau_{11} \cup \{\neg D\}$

$\tau_{16} = \tau_{11} \cup \{\neg D, \forall r.\neg B\}$

Then we run $\mathcal{ALC} - Elim$ algorithm.

$\mathcal{ALC} - Elim(A, \mathcal{T})$:

loop:

$i = 0$, $\mathit{\Gamma}_0 = \{\tau_i | i \in [16]\}$

$i = 1$, $\tau_2, \tau_5, \tau_6, \tau_8, \tau_{10}, \tau_{13}, \tau_{14}, \tau_{16}$ are bad.

$\mathit{\Gamma}_1 = \{\tau_1, \tau_3, \tau_4, \tau_7, \tau_9, \tau_{11}, \tau_{12}, \tau_{15}\}$

$i = 2 :$, $\tau_3, \tau_4, \tau_7, \tau_{11}, \tau_{12}, \tau_{15}$ are bad.

$\mathit{\Gamma}_2 = \{\tau_1, \tau_9\}$

$i = 3 :$

$\mathit{\Gamma}_3 = \mathit{\Gamma}_2$, break the loop!

$D \notin \tau_1$ or τ_9, $\quad$ return false

Therefore, $\forall r.\forall r.\neg B$ is not satisfiable w.r.t $\mathcal{T}$.

Question 11 (with 1 bonus mark). $\mathcal{ALCI}$-Elim algorithm

Extend definition 5.9 in text book:

Let Γ be a set of types and $\tau \in \Gamma$. Then τ is bad in Γ if:

1. there exists an $\exists r.C \in \tau$ such that the set

$$S = C \cup \{D | \forall r.D \in \tau\}$$

is no subset of any type in Γ.

or

2. there exists an $\exists r^{-}.C \in \tau$ such that the set

$$S = C \cup \{D | \forall r^{-}.D \in \tau\}$$

is no subset of any type in Γ.

The rest of the process is the same as in the textbook.

Prove of correctness(based on lemma 5.10 in text book):

Assume that $\mathcal{ALC}$-Elim$(\mathcal{A}_0, \mathcal{T})$ returns true, and let Γ_i be the set of remaining types. Then there is a $\tau_o \in \Gamma_i$ such that $\mathcal{A}_0 \in \tau_0$.

Define an interpretation I as follows:

$$\Delta^{\mathcal{I}} = \Gamma_i$$
$$A^{\mathcal{I}} = \{\tau \in \Gamma_i | A \in \tau\}$$
$$r^{\mathcal{I}} = \{(\tau, \tau') \in \Gamma_i \times \Gamma_i | \forall r.C \in \tau \quad \text{implies} \quad C \in \tau'\}$$

By induction on the structure of C, we can prove, for all $C \in sub(\mathcal{T})$ and all $\tau \in \Gamma_i$, that $C \in \tau$ implies $\tau \in C^{\mathcal{I}}$. Most cases are straightforward, using the definition of $\mathcal{I}$ and the induction hypothesis. We only do the case $C = \exists r.D$, $C = \exists r^{-}.D$ and $C = \forall r^{-}.D$ explicitly:

- Let $\exists r.D \in \tau$. Since τ has not been eliminated from Γ_i, it is not bad. Thus, there is a $\tau' \in \Gamma_i$ such that

$$\{C\} \cup \{D \mid \forall r.D \in \tau\} \subseteq \tau'.$$

By definition of $\mathcal{I}$, we have $(\tau, \tau') \in r^{\mathcal{I}}$. Since $\tau' \in C^{\mathcal{I}}$ by induction hypothesis, we obtain $\tau \in (\exists r.C)^{\mathcal{I}}$ by the semantics.

- let $\exists r^{-}.D \in \tau$. Since τ has not been eliminated from Γ_i, it is not bad. Thus, there is a $\tau' \in \Gamma_i$ such that

  $$\{D\} \cup \{E | \forall r^{-}.E \in \tau\} \subseteq \tau'$$

  If there exists $\forall r.E \in \tau'$, then $E \in \tau$ because τ' is not bad, and thus $(\tau', \tau) \in r^{\mathcal{I}}$.We obtain $\tau \in (\exists r^{-}.D)^{\mathcal{I}}$ by the semantics.

- Let $\forall r^{-}.D \in \tau$. Since τ has not been eliminated from Γ_i, it is not bad. If there is a $\tau' \in \Gamma_i$ and $(\tau', \tau) \in r^{\mathcal{I}}$, then $D \in \tau'$.
  So $\tau' \in D^{\mathcal{I}}$, we obtain $\tau \in (\forall r^{-}.D)^{\mathcal{I}}$ by semantics.

  Hence, $\mathcal{I}$ is a model of $\mathcal{T}$. Since $\mathcal{A}_0 \in \tau_0$, it is also a model of $\mathcal{A}_0$.

Question 12 (with 1 bonus mark). Subsumption in $\mathcal{ELI}$

Normalization the TBox and get

$$\mathcal{T}' = \{\{A_1, A_2\} \sqsubseteq \exists r.\{B\}, \{A_2\} \sqsubseteq \forall r.\{C\}, \{A\} \sqsubseteq \{A_1, A_2\}, \{B, C\} \sqsubseteq \forall r^-.\{D\}\}$$

(1)

Apply CR1:$\{A_1, A_2\} \sqsubseteq \{A_2\}$

Apply CR2:$\{A\} \sqsubseteq \{A_1, A_2\}, \{A_1, A_2\} \sqsubseteq \{A_2\} \rightarrow \{A\} \sqsubseteq \{A_2\}$

Apply CR2:$\{A\} \sqsubseteq \{A_1, A_2\}, \{A_1, A_2\} \sqsubseteq \exists r.\{B\} \rightarrow \{A\} \sqsubseteq \exists r.\{B\}$

Apply CR2:$\{A\} \sqsubseteq \{A_2\}, \{A_2\} \sqsubseteq \forall r.\{C\} \rightarrow \{A\} \sqsubseteq \forall r.\{C\}$

Apply CR4:$\{A\} \sqsubseteq \forall r.\{C\}, \{A\} \sqsubseteq \exists r.\{B\} \rightarrow \{A\} \sqsubseteq \exists r.\{B, C\}$

Apply CR3:$\{A\} \sqsubseteq \exists r.\{B, C\}, \{B, C\} \sqsubseteq \forall r^-.\{D\} \rightarrow \{A\} \sqsubseteq \{D\}$

We have $\{A\} \sqsubseteq \{D\}$, so $A \sqsubseteq D$ holds.

(2) According to lemma 6.1(just like Question 8(2)), we can get

$$\mathcal{T}'' = \mathcal{T}' \cup \{\{E\} \sqsubseteq \exists r.\{A\}, \{D\} \sqsubseteq \forall r^-.\{F\}\}$$

Apply CR2:$\{A\} \sqsubseteq \{D\}, \{D\} \sqsubseteq \forall r^-.\{F\} \rightarrow \{A\} \sqsubseteq \forall r^-.\{F\}$

Apply CR3:$\{E\} \sqsubseteq \exists r.\{A\}, \{A\} \sqsubseteq \forall r^-.\{F\} \rightarrow \{E\} \sqsubseteq \{F\}$

We have $\{E\} \sqsubseteq \{F\}$, so $\exists r.A \sqsubseteq \exists r.D$ holds.

(3) If we want to get $\{A\} \sqsubseteq \exists r.\{A\}$, we must apply CR4 to some $M_1 \sqsubseteq \exists r.M_2, M_1 \sqsubseteq \forall r.\{A\}$, there is no CR4-rule could apply to get $\{A\} \sqsubseteq \exists r.\{A\}$ and there isn't $\{A\} \sqsubseteq \exists r.\{A\}$, so we can conclude that $A \sqsubseteq \exists r.A$ doesn't hold.