Assignment 4

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⋆ **This assignment, which is due on May 31st, contributes to 10% of the final marks for this course. Moreover, the last three questions present an opportunity for students to secure up to three additional bonus marks. The bonus marks can potentially increase a student's overall marks but are subject to a maximum total of 100 for the course.**

Question 1. $\mathcal{ALC}$-Worlds Algorithm

Use the $\mathcal{ALC}$-Worlds algorithm to decide the satisfiability of the concept name B_0 w.r.t. the simple TBox:

$$\mathcal{T} := \left\{ \begin{array}{l} B_0 \equiv B_1 \sqcap B_2 \\ B_1 \equiv \exists r.B_3 \\ B_2 \equiv B_4 \sqcap B_5 \\ B_3 \equiv P \\ B_4 \equiv \exists r.B_6 \\ B_5 \equiv B_7 \sqcap B_8 \\ B_6 \equiv Q \\ B_7 \equiv \forall r.B_4 \\ B_8 \equiv \forall r.B_9 \\ B_9 \equiv \forall r.B_{10} \\ B_{10} \equiv \neg P \end{array} \right\},$$

Draw the recursion tree of a successful run and of an unsuccessful run. Does the algorithm return a positive or negative result on this input?

Question 2. Entailment Checking

Fix two concept names A and B as well as some role name r. Does the entailment

$$\{\forall r.A \sqsubseteq \exists r.A\} \models \forall r.B \sqsubseteq \exists r.B$$

hold true?

Question 3. Finite Boolean Games

Determine whether Player 1 has a winning strategy in the following finite Boolean games, where in both cases $\Gamma_1 := \{x_1, x_3\}$ and $\Gamma_2 := \{x_2, x_4\}$.

- $\psi := ((x_1 \wedge x_3) \rightarrow \neg x_2) \wedge (\neg x_1 \rightarrow x_1) \wedge (\neg x_2 \rightarrow (x_3 \vee x_4))$
- $\psi := (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (\neg x_3 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_2 \vee x_3 \vee x_4)$

Question 4. Infinite Boolean Games

Determine whether Player 2 has a winning strategy in the following infinite Boolean games where the initial configuration t_0 assigns *false* to all variables.

- $\psi := (x_1 \wedge x_2 \wedge \neg y_1) \vee (x_3 \wedge x_4 \wedge \neg y_2) \vee (\neg(x_1 \vee x_4) \wedge y_1 \wedge y_2)$

  provided that: $\Gamma_1 := \{x_1, x_2, x_3, x_4\}$ and $\Gamma_2 := \{y_1, y_2\}$
- $\psi := ((x_1 \leftrightarrow \neg y_1) \wedge (x_2 \leftrightarrow \neg y_2) \wedge (x_1 \leftrightarrow x_2)) \vee ((x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \wedge (x_1 \leftrightarrow \neg x_2))$

  provided that: $\Gamma_1 := \{x_1, x_2\}$ and $\Gamma_2 := \{y_1, y_2\}$

Question 5. Complexity of Concept Satisfiability in $\mathcal{ALC}$ Extensions

The universal role is a role u such that its extension is fixed as $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ in any interpretation $\mathcal{I}$. Let $\mathcal{ALC}^u$ be a DL extending $\mathcal{ALC}$ with the universal role.

- Show that concept satisfiability in $\mathcal{ALC}^u$ without TBoxes is EXPTIME-complete.

Question 6. Conservative Extension

Let $\mathcal{T}_1$ be an $\mathcal{EL}$ TBox, with C and D as $\mathcal{EL}$ concepts. Let us further consider $\mathcal{T}_2 := \mathcal{T}_1 \cup \{A \sqsubseteq C, D \sqsubseteq B\}$, wherein A and B are new concept names (as in Lemma 6.1).

- Show that $\mathcal{T}_2$ is a conservative extension of $\mathcal{T}_1$.
- Is this still the case after adding $A \sqsubseteq B$ to $\mathcal{T}_2$?
- What about adding $B \sqsubseteq A$?

Question 7. Subsumption in $\mathcal{EL}$

Consider the following $\mathcal{EL}$ TBox:

$$\mathcal{T} := \left\{ \begin{array}{r} A \sqsubseteq B \sqcap \exists r.C \\ B \sqcap \exists r.B \sqsubseteq C \sqcap D \\ C \sqsubseteq (\exists r.A) \sqcap B \\ (\exists r.\exists r.B) \sqcap D \sqsubseteq \exists r.(A \sqcap B) \end{array} \right\},$$

where A, B, C, D are concept names.

Use the classification procedure for $\mathcal{EL}$ to check whether the following subsumptions hold w.r.t. $\mathcal{T}$.

- $A \sqsubseteq B$
- $A \sqsubseteq \exists r.\exists r.A$
- $B \sqcap \exists r.A \sqsubseteq \exists r.C$

Question 8. Simulation

We consider simulations, which are "one-sided" variants of bisimulations. Given interpretations $\mathcal{I}$ and $\mathcal{J}$, the relation $\sigma \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a simulation from $\mathcal{I}$ to $\mathcal{J}$ if

- whenever $\mathrm{d}\ \sigma\ \mathrm{d}'$ and $\mathrm{d} \in A^{\mathcal{I}}$, then $\mathrm{d}' \in A^{\mathcal{J}}$, for all $\mathrm{d} \in \Delta^{\mathcal{I}}$, $\mathrm{d}' \in \Delta^{\mathcal{J}}$, and $A \in \mathbb{C}$;
- whenever $\mathrm{d}\ \sigma\ \mathrm{d}'$ and $(\mathrm{d}, \mathrm{e}) \in r^{\mathcal{I}}$, then there exists an $\mathrm{e}' \in \Delta^{\mathcal{J}}$ such that $\mathrm{e}\ \sigma\ \mathrm{e}'$ and $(\mathrm{d}', \mathrm{e}') \in r^{\mathcal{J}}$, for all $\mathrm{d}, \mathrm{e} \in \Delta^{\mathcal{I}}$, $\mathrm{d}' \in \Delta^{\mathcal{J}}$, and $r \in \mathbb{R}$.

We write $(\mathcal{I}, \mathsf{d}) \precsim (\mathcal{J}, \mathsf{d}')$ if there is a simulation σ from $\mathcal{I}$ to $\mathcal{J}$ such that $\mathsf{d}\ \sigma\ \mathsf{d}'$.

(a) Show that $(\mathcal{I}, \mathsf{d}) \sim (\mathcal{J}, \mathsf{d}')$ implies $(\mathcal{I}, \mathsf{d}) \precsim (\mathcal{J}, \mathsf{d}')$ and $(\mathcal{J}, \mathsf{d}') \precsim (\mathcal{I}, \mathsf{d})$.

(b) Is the converse of the implication above also true?

(c) Show that, if $(\mathcal{I}, \mathsf{d}) \precsim (\mathcal{J}, \mathsf{d}')$, then $\mathsf{d} \in C^{\mathcal{I}}$ implies $\mathsf{d}' \in C^{\mathcal{J}}$ for all $\mathcal{EL}$ concept descriptions C.

(d) Which of the constructors disjunction, negation, or value restriction (aka universal restriction) can be added to $\mathcal{EL}$ without losing the property in (c)?

(e) Show that $\mathcal{ALC}$ is more expressive than $\mathcal{EL}$.

Question 9. $\mathcal{EL}$ Extension

We consider the DL $\mathcal{EL}_{\mathsf{si}}$ extending $\mathcal{EL}$ by concept descriptions of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \delta)$ where $\mathcal{I}$ is a finite interpretation and $\mathsf{d} \in \Delta^{\mathcal{I}}$. Their semantics is defined as follows.

$$(\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d}))^{\mathcal{J}} := \{\mathsf{d}' \mid \mathsf{d}' \in \Delta^{\mathcal{J}} \text{ and } (\mathcal{I}, \mathsf{d}) \precsim (\mathcal{I}, \mathsf{d}')\}$$

Concept inclusions are then defined as usual.

- Show that each $\mathcal{EL}_{\mathsf{si}}$ concept description is equivalent to some concept descriptions of the form $\exists^{\mathsf{sim}}(\mathcal{I}, \mathsf{d})$.
- Show that $\mathcal{EL}_{\mathsf{si}}$ is more expressive than $\mathcal{EL}$.
- Show that checking subsumption in $\mathcal{EL}_{\mathsf{si}}$ without any TBox can be done in polynomial time.

Question 10 (with 1 bonus mark). $\mathcal{ALC}$-Elim Algorithm

Use the $\mathcal{ALC}$-Elim algorithm to decide satisfiability of:

- the concept name A w.r.t. $\mathcal{T} := \{A \sqsubseteq \exists r.A, \top \sqsubseteq A, \forall r.A \sqsubseteq \exists r.A\}$
- the concept description $\forall r.\forall r.\neg B$ w.r.t. $\mathcal{T} := \{\neg A \sqsubseteq B, A \sqsubseteq \neg B, \top \sqsubseteq \neg\forall r.A\}$

Give the constructed type sequence $\Gamma_0, \Gamma_1, \ldots$. In the case of satisfiability, also give the satisfying model constructed in the proof of Lemma 5.10.

Question 11 (with 1 bonus mark). $\mathcal{ALCI}$-Elim algorithm

Extend the $\mathcal{ALC}$-Elim algorithm to $\mathcal{ALCI}$. Prove the correctness (soundness) of the extended algorithm.

Question 12 (with 1 bonus mark). Subsumption in $\mathcal{ELI}$

Consider the following $\mathcal{ELI}$ TBox:

$$\mathcal{T} := \left\{ \begin{array}{r} A_1 \sqcap A_2 \sqsubseteq \exists r.B \\ \exists r^-.A_2 \sqsubseteq C \\ A \sqsubseteq A_1 \sqcap A_2 \\ \exists r.(B \sqcap C) \sqsubseteq D \end{array} \right\},$$

where A, A_1, A_2, B, C, D are concept names.

Use the classification procedure for $\mathcal{ELI}$ to check whether the following subsumptions hold w.r.t. $\mathcal{T}$.

- $A \sqsubseteq D$
- $\exists r.A \sqsubseteq \exists r.D$
- $A \sqsubseteq \exists r.A$