homework11

P277. 习题 12.4 (A) 1-1,1-3

P288. 习题 12.5 (A) 1-3； 2-2；

1. 求下列函数项级数的收敛域：

(1) $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{x^n}$; (3) $\sum\limits_{n=0}^{\infty} \dfrac{1}{2n+1}\left(\dfrac{1-x}{1+x}\right)^n$;

1. 求下列幂级数的收敛半径和收敛域： (2) $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{n^2}$;

2. 求下列幂级数的收敛域及其和函数： (2) $\sum\limits_{n=1}^{\infty} nx^{2n-1}$

补 1. 求幂级数 $\sum\limits_{n\geq 0}(2n+1)(3n+2)x^n$

补 2. 求级数 $\sum\limits_{n\geq 0}\dfrac{1}{4n+3}\left(\dfrac{1}{4}\right)^n, \sum\limits_{n\geq 0}\dfrac{1}{4n+3}\left(\dfrac{-1}{4}\right)^n$

补 3. 求级数 $\sum\limits_{n\geq 1}\dfrac{4n+3}{n(n+1)(n+2)}$

补 4. 求级数 $\sum\limits_{n\geq 0}\dfrac{(2n)!}{(n!)^2}x^{2n}$ 满足的一阶微分方程，并求该级数

P277 习题 12.4

1. 求下列函数项级数的收敛域：

(1) $\sum\limits_{n=1}^{\infty}\frac{n^2}{x^n}$； (3) $\sum\limits_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{1-x}{1+x}\right)^n$；

A.1-1. 解. 记 $a_n=\frac{n^2}{x^n}$

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{(n+1)^2}{n^2}\frac{1}{|x|}\to\frac{1}{|x|}\Rightarrow |x|>1\text{时，级数收敛}$$

$$x=\pm 1\text{时，}|a_n|=n^2\nrightarrow 0\Rightarrow\text{级数发散}\Rightarrow\text{收敛域为}(-\infty,-1)\cup(1,\infty)$$

A.1-3. 解. 记 $u=\frac{1-x}{1+x}$，原式 $=\sum\limits_{n\geq 0}\frac{1}{2n+1}u^n$

$$\sqrt[n]{\frac{1}{2n+1}}\to 1\Rightarrow R=1$$

$$u=1\text{时，原式}=\sum_{n\geq 0}\frac{1}{2n+1}\text{发散}\quad u=-1\text{时，原式}=\sum_{n\geq 0}\frac{(-1)^n}{2n+1}\text{收敛}$$

$$\text{由}-1\leq\frac{1-x}{1+x}<1\Rightarrow\frac{2}{1+x}\geq 0\text{且}\frac{-2x}{1+x}<0\Rightarrow\text{收敛域为}(0,\infty)$$

P288 习题 12.5

1. 求下列幂级数的收敛半径和收敛域： (2) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$;

A.1-3. **解**. 记 $a_n = \frac{1}{n^2}$， $S(x) = \sum_{n\geq 1} a_n x^n$

$$\Rightarrow \frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \to 1 \Rightarrow \text{收敛半径} R = 1$$

$$x = 1\text{时，原式} = \sum_{n\geq 1} \frac{1}{n^2}\text{收敛，} x = -1\text{时，原式} = \sum_{n\geq 1} \frac{(-1)^n}{n^2}\text{收敛}$$

$$\Rightarrow \text{收敛域为}[-1, 1]$$

P277 习题 12.4

2. 求下列幂级数的收敛域及其和函数： (2) $\sum_{n=1}^{\infty} nx^{2n-1}$

A.2-2. 解. 记 $x_n = nx^{2n-1}$

$$\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{n}x^2 \to x^2 \Rightarrow x^2 < 1\text{时，级数收敛}$$

$$x^2 = 1\text{时，原式} = x\sum_{n\geq 1} n\text{发散} \Rightarrow \text{收敛域为}(-1,1)$$

$$\text{记}S(x) = \sum_{n\geq 1} nx^{2n-1}, x \in (-1,1)$$

$$\Rightarrow \int_0^x S(t)\,dt = \sum_{n\geq 1} n\int_0^x t^{2n-1}\,dt = \sum_{n\geq 1}\frac{x^{2n}}{2} = \frac{x^2}{2(1-x^2)}$$

$$\Rightarrow S(x) = \frac{1}{2}\left(\frac{x^2}{1-x^2}\right)' = \frac{1}{2}\left(\frac{1}{1-x^2} - 1\right)' = \frac{x}{(1-x^2)^2}$$

幂级数习题补充题

求幂级数 $\sum\limits_{n\geq 0}(2n+1)(3n+2)x^n$

补充 1. 解. 依题意知，级数收敛域为 $(-1,1)$，取

$$S(x)=\sum_{n\geq 0}x^n=\frac{1}{1-x}$$

$$\Rightarrow S'(x)=\sum_{n\geq 1}nx^{n-1}=\sum_{n\geq 0}(n+1)x^n=\frac{1}{(1-x)^2},$$

$$\Rightarrow S''(x)=\sum_{n\geq 2}n(n-1)x^{n-2}=\sum_{n\geq 0}(n+1)(n+2)x^n=\frac{2}{(1-x)^3}$$

$$\Rightarrow \sum_{n\geq 0}(2n+1)(3n+2)x^n=\sum_{n\geq 0}\big(6(n+1)(n+2)-11(n+1)+1\big)x^n$$

$$=\frac{12}{(1-x)^3}-\frac{11}{(1-x)^2}+\frac{1}{(1-x)}$$

幂级数习题补充题

求幂级数 $\sum\limits_{n\geq 0}(2n+1)(3n+2)x^n$

补充 1. 解. 依题意知，级数收敛域为 $(-1,1)$，$x>0$ 时

$$\sum_{n\geq 0}x^n=\frac{1}{1-x}\Rightarrow\sum_{n\geq 0}x^{n+\frac{1}{2}}=\frac{\sqrt{x}}{1-x}\Rightarrow\sum_{n\geq 0}(n+\frac{1}{2})x^{n-\frac{1}{2}}=\frac{1}{2}\frac{x^{-1/2}}{(1-x)}+\frac{\sqrt{x}}{(1-x)^2}$$

$$\Rightarrow\sum_{n\geq 0}(n+\frac{1}{2})x^{n+\frac{2}{3}}=\frac{1}{2}\frac{x^{\frac{2}{3}}}{(1-x)}+\frac{x^{5/3}}{(1-x)^2}$$

$$\Rightarrow\sum_{n\geq 0}(n+\frac{1}{2})(n+\frac{2}{3})x^{n-\frac{1}{3}}=\frac{1}{3}\frac{x^{\frac{-1}{3}}}{(1-x)}+\frac{1}{2}\frac{x^{2/3}}{(1-x)^2}+\frac{5}{3}\frac{x^{2/3}}{(1-x)^2}+\frac{2x^{5/3}}{(1-x)^3}$$

$$\Rightarrow\sum_{n\geq 0}(2n+1)(3n+2)x^n=\frac{2}{1-x}+\frac{3x}{(1-x)^2}+\frac{10x}{(1-x)^2}+\frac{12x^2}{(1-x)^3}$$

$$=\frac{12}{(1-x)^3}-\frac{11}{(1-x)^2}+\frac{1}{1-x}$$

求级数 $\sum\limits_{n\geq 0} \dfrac{1}{4n+3}\left(\dfrac{1}{4}\right)^n, \sum\limits_{n\geq 0} \dfrac{1}{4n+3}\left(\dfrac{-1}{4}\right)^n$

补充 2-1. 解. 记 $S(x)=\sum\limits_{n\geq 0}\dfrac{x^{4n+3}}{4n+3}$，则收敛域为 $(-1,1)$，

$$\Rightarrow S'(x)=\sum_{n\geq 0}x^{4n+2}=\frac{x^2}{1-x^4}=\frac{1}{4}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)-\frac{1}{2}\frac{1}{1+x^2}$$

$$\Rightarrow S(x)=\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|-\frac{1}{2}\arctan x+c$$

$$\text{由 } S(0)=0\Rightarrow S(x)=\frac{1}{4}\ln\frac{1+x}{1-x}-\frac{1}{2}\arctan x$$

$$\Rightarrow \text{原式}=2\sqrt{2}S\left(\frac{1}{\sqrt{2}}\right)=\sqrt{2}\ln(\sqrt{2}+1)-\sqrt{2}\arctan\frac{1}{\sqrt{2}}$$

幂级数习题补充题

求级数 $\sum\limits_{n\geq 0}\frac{1}{4n+3}\left(\frac{1}{4}\right)^n, \sum\limits_{n\geq 0}\frac{1}{4n+3}\left(\frac{-1}{4}\right)^n$

补充 2-2. **解.** **记** $S(x)=\sum\limits_{n\geq 0}\frac{(-1)^n x^{4n+3}}{4n+3}$，**则收敛域为** $[-1,1]$，

$$\Rightarrow S(0)=0 \quad S'(x)=\sum_{n\geq 0}(-1)^n x^{4n+2}=\frac{x^2}{1+x^4}$$

$$=\frac{a}{1+x^2+\sqrt{2}x}+\frac{b(2x+\sqrt{2})}{1+x^2+\sqrt{2}x}+\frac{c}{1-\sqrt{2}x+x^2}+\frac{d(2x-\sqrt{2})}{1-\sqrt{2}x+x^2}$$

$$=\frac{1}{4}\left(\frac{1}{x^2+\sqrt{2}x+1}+\frac{1}{x^2-\sqrt{2}x+1}\right)+\frac{1}{4\sqrt{2}}\left(\frac{2x-\sqrt{2}}{x^2-\sqrt{2}x+1}-\frac{2x+\sqrt{2}}{x^2+\sqrt{2}x+1}\right)$$

$$\Rightarrow S(x)=\frac{\sqrt{2}}{4}\left(\arctan(\sqrt{2}x+1)+\arctan(\sqrt{2}x-1)\right)+\frac{1}{4\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}$$

$$\text{原式}=2\sqrt{2}S\left(\frac{1}{\sqrt{2}}\right)=\arctan 2-\frac{1}{2}\ln 5$$

幂级数习题补充题

求级数 $\sum\limits_{n\geq 1}\frac{4n+3}{n(n+1)(n+2)}$

补充 3. 解. 由

$$\frac{4x+3}{x(x+1)(x+2)}=\frac{a}{x}+\frac{b}{x+1}+\frac{c}{x+2}=\frac{3}{2x}+\frac{1}{x+1}-\frac{5}{2(x+2)}$$

$$\Rightarrow \text{原式}=\sum_{n\geq 1}\Big(\frac{3}{2}\frac{1}{n}+\frac{1}{n+1}-\frac{5}{2}\frac{1}{n+2}\Big)$$

$$\text{记}S(x)=\sum_{n\geq 1}\frac{x^n}{n}\Rightarrow S'(x)=\sum_{n\geq 1}x^{n-1}=\frac{1}{1-x}\Rightarrow S(x)=-\ln(1-x)\quad \forall x\in(-1,1)$$

$$\text{记}G(x)=\sum_{n\geq 1}\Big(\frac{3}{2}\frac{1}{n}+\frac{1}{n+1}-\frac{5}{2}\frac{1}{n+2}\Big)x^n\quad \forall x\in[-1,1]$$

$$\Rightarrow G(x)=\frac{3}{2}S(x)+\frac{S(x)-x}{x}-\frac{5}{2}\frac{S(x)-x-\frac{x^2}{2}}{x^2}\quad \forall x\in(-1,1)$$

$$\Rightarrow \text{原式}=G(1)=\lim_{x\to 1-}G(x)=\lim_{x\to 1-}\frac{S(x)}{2x^2}(3x^2+2x-5)+\frac{5}{2x}+\frac{1}{4}=\frac{11}{4}$$

幂级数习题补充题

求级数 $\sum_{n\geq 0}\frac{(2n)!}{(n!)^2}x^{2n}$ 满足的一阶微分方程，并求该级数

补充 4. 解. 收敛域 记 $a_n=\frac{(2n)!}{(n!)^2}, S(x)=\sum_{n\geq 0}a_nx^{2n}$

$$\frac{a_{n+1}}{a_n}=\frac{2(n+1)(2n+1)}{(n+1)^2}\to 4\Rightarrow R=\frac{1}{4}$$

$$\text{由}\frac{1}{4}\frac{a_{n+1}}{a_n}=\frac{2n+1}{2(n+1)}\geq\sqrt{\frac{n}{n+1}}$$

$$\Rightarrow\frac{a_n}{4^n}=\Big(\frac{a_n}{a_{n-1}}\frac{1}{4}\Big)\Big(\frac{a_{n-1}}{a_{n-2}}\frac{1}{4}\Big)\cdots\Big(\frac{a_2}{a_1}\frac{1}{4}\Big)\frac{a_1}{4}\geq\frac{a_1}{4}\sqrt{\frac{1}{n}}\Rightarrow\text{收敛域为}\Big(\frac{-1}{2},\frac{1}{2}\Big)$$

求和函数 $S'(x)=\sum_{n\geq 1}a_n2nx^{2n-1}\quad (n+1)a_{n+1}-4na_n-2a_n=0\quad \forall n\geq 0$

$$\sum_{n\geq 0}(n+1)a_{n+1}x^{2n}=\frac{1}{2x}S'(x)\quad \sum_{n\geq 0}na_nx^{2n}=\frac{x}{2}S'(x)\quad \sum_{n\geq 0}a_nx^{2n}=S(x)$$

$$S'(x)\Big(\frac{1}{2x}-2x\Big)-2S(x)=0\quad S(0)=1\Rightarrow S(x)=(1-4x^2)^{-1/2}$$