求极限 $\lim\limits_{x\to 0}\dfrac{\sin\left(e^x-1\right)-e^{\sin x}+1}{\sin^4 x}$.

解法一：别问，问就泰勒 ❤

由于分母是 x 的 4 阶无穷小，因此对分子使用泰勒展开方法要求整体上展开到 x 的 4 阶. 先对各层函数分别展开

$$\begin{aligned}
e^x-1&=x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+o\left(x^4\right)\\
\sin x&=x-\frac{1}{6}x^3+o\left(x^4\right)
\end{aligned}$$

然后合并，注意忽略由于内外层的较高阶无穷小复合产生的高阶无穷小

$$\begin{aligned}
\sin\left(e^x-1\right)&=\left[x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+o\left(x^4\right)\right]-\frac{1}{6}\underbrace{\left[x+\frac{1}{2}x^2+\frac{1}{6}x^3+o\left(x^3\right)\right]^3}_{=x^3+\frac{1}{2}x^4\times 3+o\left(x^4\right)}+o\left(x^4\right)\\
&=x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4-\frac{1}{6}\left[x^3+\frac{1}{2}x^4\times 3\right]+o\left(x^4\right)\\
&=x+\frac{1}{2}x^2-\frac{5}{24}x^4+o\left(x^4\right)
\end{aligned}$$

$$\begin{aligned}
e^{\sin x}&=1+\sin x+\frac{1}{2}\sin^2 x+\frac{1}{6}\sin^3 x+\frac{1}{24}\sin^4 x+o\left(x^4\right)\\
&=1+\left(x-\frac{1}{6}x^3+o\left(x^4\right)\right)+\frac{1}{2}\underbrace{\left(x-\frac{1}{6}x^3+o\left(x^4\right)\right)^2}_{=x^2-\frac{1}{6}x^4\times 2+o\left(x^4\right)}+\frac{1}{6}\underbrace{\left(x-\frac{1}{6}x^3+o\left(x^4\right)\right)^3}_{=x^3+o\left(x^4\right)}+\frac{1}{24}\underbrace{\left(x-\frac{1}{6}x^3+o\left(x^4\right)\right)^4}_{=x^4+o\left(x^4\right)}+o\left(x^4\right)\\
&=1+x+\frac{1}{2}x^2-\frac{7}{24}x^4+o\left(x^4\right)
\end{aligned}$$

最后代入原极限

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin\left(e^x-1\right)-e^{\sin x}+1}{\sin^4 x}&=\lim_{x\to 0}\frac{\left[x+\frac{1}{2}x^2-\frac{5}{24}x^4+o\left(x^4\right)\right]-\left[1+x+\frac{1}{2}x^2-\frac{7}{24}x^4+o\left(x^4\right)\right]+1}{\sin^4 x}\\
&=\lim_{x\to 0}\frac{-\frac{5}{24}x^4+\frac{7}{24}x^4+o\left(x^4\right)}{\sin^4 x}\\
&=-\frac{1}{12}
\end{aligned}$$

解法二：合理地综合使用各种方法才能提高解题速度.

注意到分子部分有两项含有指数函数，且其余部分是常数，于是可使用 L’Hospital 法则和关于 e 的等价无穷小

$$\begin{aligned}
\left[\sin\left(e^x-1\right)-e^{\sin x}+1\right]'&=e^x\cos\left(e^x-1\right)-e^{\sin x}\cos x=e^{x+\ln\cos\left(e^x-1\right)}-e^{\sin x+\ln\cos x}=e^{\sin x+\ln\cos x}\left(e^{\left[x+\ln\cos\left(e^x-1\right)\right]-\left[\sin x+\ln\cos x\right]}-1\right)\\
&\sim e^{\left[x+\ln\cos\left(e^x-1\right)\right]-\left[\sin x+\ln\cos x\right]}-1\left(\text{注意验证 } e^{\sin x+\ln\cos x}\to 1\right)\\
&\sim\left[x+\ln\cos\left(e^x-1\right)\right]-\left[\sin x+\ln\cos x\right]\left(\text{注意验证}\left[x+\ln\cos\left(e^x-1\right)\right]-\left[\sin x+\ln\cos x\right]\to 0\right)\\
&=\left[x-\sin x\right]+\left[\ln\cos\left(e^x-1\right)-\ln\cos x\right]
\end{aligned}$$

又有

$$x-\sin x=\frac{1}{6}x^3+o\left(x^3\right),$$

$$\begin{aligned}
\ln\cos\left(e^x-1\right)-\ln\cos x&=\ln\frac{\cos\left(e^x-1\right)}{\cos x}\sim\frac{\cos\left(e^x-1\right)}{\cos x}-1\\
&=\sec x\left[\cos\left(e^x-1\right)-\cos x\right]\sim\cos\left(e^x-1\right)-\cos x\left(\text{注意验证}\sec x\to 1\right)\\
&\xlongequal{\text{Lagrange中值定理}}-\sin\xi\cdot\left(e^x-1-x\right),\xi\text{介于 }e^x-1\text{ 和 }x\\
&\sim-\sin x\cdot\left(e^x-1-x\right)\sim-\frac{1}{2}x^3\left(\begin{array}{l}\text{注意由于洛必达之后分子只要求精确到三阶，}\\ \text{故}\sin\xi\text{只需要精确到 }x\text{ 即可，而 }e^x-1=x+o(x)\end{array}\right)
\end{aligned}$$

于是

$$
\begin{aligned}
\lim_{x\to0}\frac{\sin\left(e^x-1\right)-e^{\sin x}+1}{\sin^4 x}&=\lim_{x\to0}\frac{\sin\left(e^x-1\right)-e^{\sin x}+1}{x^4}\xlongequal{\text{L'Hospital法则}}\lim_{x\to0}\frac{\left[\sin\left(e^x-1\right)-e^{\sin x}+1\right]'}{\left(x^4\right)'}\\
&=\lim_{x\to0}\frac{\left[x-\sin x\right]+\left[\ln\cos\left(e^x-1\right)-\ln\cos x\right]}{4x^3}\\
&\xlongequal{\text{极限四则运算法则}}\lim_{x\to0}\frac{x-\sin x}{4x^3}+\lim_{x\to0}\frac{\ln\cos\left(e^x-1\right)-\ln\cos x}{4x^3}\\
&=\lim_{x\to0}\frac{\frac{1}{6}x^3+o\left(x^3\right)}{4x^3}+\lim_{x\to0}\frac{-\frac{1}{2}x^3}{4x^3}\\
&=\frac{1}{24}-\frac{1}{8}=-\frac{1}{12}
\end{aligned}
$$

解法三：观察结构，对症下药，合理的拆项+等价无穷小/泰勒展开往往能大幅度提高解题速度.

注意到

$$
\sin\left(e^x-1\right)-e^{\sin x}+1=\left[\sin\left(e^x-1\right)-\left(e^x-1\right)\right]+\left[e^x-e^{\sin x}\right]
$$

且

$$
\sin\left(e^x-1\right)-\left(e^x-1\right)=-\frac{1}{6}\left(e^x-1\right)^3+o\left(\left(e^x-1\right)^4\right)=-\frac{1}{6}\left(x+\frac{1}{2}x^2\right)^3+o\left(x^4\right)=-\frac{1}{6}\left(x^3+\frac{3}{2}x^4\right)+o\left(x^4\right)
$$

$$
e^x-e^{\sin x}=e^{\sin x}\left(e^{x-\sin x}-1\right)=\left(1+\sin x\right)\left(\frac{1}{6}x^3+o\left(x^4\right)\right)=\left(1+x+o(x)\right)\left(\frac{1}{6}x^3+o\left(x^4\right)\right)=\frac{1}{6}x^3+\frac{1}{6}x^4+o\left(x^4\right)
$$

于是有

$$
\begin{aligned}
\lim_{x\to0}\frac{\sin\left(e^x-1\right)-e^{\sin x}+1}{\sin^4 x}&=\lim_{x\to0}\frac{\left[\sin\left(e^x-1\right)-\left(e^x-1\right)\right]+\left[e^x-e^{\sin x}\right]}{\sin^4 x}\\
&=\lim_{x\to0}\frac{\left[-\frac{1}{6}\left(x^3+\frac{3}{2}x^4\right)+o\left(x^4\right)\right]+\left[\frac{1}{6}x^3+\frac{1}{6}x^4+o\left(x^4\right)\right]}{\sin^4 x}\\
&=\lim_{x\to0}\frac{-\frac{1}{6}\left(x^3+\frac{3}{2}x^4\right)+\frac{1}{6}x^3+\frac{1}{6}x^4+o\left(x^4\right)}{\sin^4 x}=\lim_{x\to0}\frac{-\frac{1}{4}x^4+\frac{1}{6}x^4+o\left(x^4\right)}{\sin^4 x}\\
&=-\frac{1}{12}
\end{aligned}
$$

总结：此题还能写出其他解法，但都跳不出常规方法（等价无穷小、泰勒展开、洛必达、拉格朗日中值定理等）的圈子，就不赘述了。对于这类带有较复杂复合函数的高阶无穷小作商类型的极限，观察结构、合理拆分然后等价无穷小，再使用泰勒展开等方法，才能简化解题过程，提高解题速度。