一个常见定积分问题

$$I=\int_0^1 \frac{\ln(1+x)}{1+x^2}dx$$

1.

Let I denote the integral we wish to compute. The function $f(x)=\dfrac{\ln(x+1)}{x^2+1}$ does not have an elementary antiderivative. We will use Taylor series to compute I. We can find the Taylor series for the function $\dfrac{\ln(x+1)}{x^2+1}$ using the following formulas:

$$\begin{aligned}\ln(x+1)&=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots\\ \frac{1}{1+x^2}&=1-x^2+x^4-\dots\end{aligned}$$

These formulas aren't good everywhere, but they do hold in $(0,1)$. We have

$$\begin{aligned}f(x)&=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots\right)(1-x^2+x^4-x^6+\dots)\\ &=x+\left(-\frac{1}{2}\right)x^2+\left(\frac{1}{3}-1\right)x^3+\left(-\frac{1}{4}+\frac{1}{2}\right)x^4+\left(\frac{1}{5}-\frac{1}{3}+1\right)x^5+\dots\end{aligned}$$

In particular, an antiderivative is given by

$$F(x)=\frac{1}{2}x^2+\frac{1}{3}\left(-\frac{1}{2}\right)x^3+\frac{1}{4}\left(\frac{1}{3}-1\right)x^4+\frac{1}{5}\left(-\frac{1}{4}+\frac{1}{2}\right)x^5+\frac{1}{6}\left(\frac{1}{5}-\frac{1}{3}+1\right)x^6+\dots$$

The definite integral I is given by $F(1)$, i.e., the sum

$$I=\frac{1}{2}+\frac{1}{3}\left(-\frac{1}{2}\right)+\frac{1}{4}\left(\frac{1}{3}-1\right)+\frac{1}{5}\left(-\frac{1}{4}+\frac{1}{2}\right)+\frac{1}{6}\left(\frac{1}{5}-\frac{1}{3}+1\right)+\dots$$

Now we use the facts that

$$\begin{aligned}1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots&=\frac{\pi}{4}\\ 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots&=\ln(2),\end{aligned}$$

from the Taylor series for $\tan^{-1}(x)$ and $\ln(x+1)$ respectively. Notice that in the above sum, every number of the form $\dfrac{1}{r\cdot s}$, where r is even and s is odd, occurs exactly once, with a positive sign if $r+s\equiv 3 \pmod 4$ and a negative sign if $1 \pmod 4$. Therefore, it is clear that

$$\begin{aligned}I&=\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots\right)\left(\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\dots\right)\\ &=\frac{\pi}{4}\cdot\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots\right)\\ &=\frac{\pi\ln(2)}{8}.\end{aligned}$$

2.

换元 令y=tan t

$$I=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan t\right)dt=\int_0^{\frac{\pi}{4}}\ln\left(\frac{\cos t+\sin t}{\cos t}\right)dt$$

$$\int_0^{\frac{\pi}{4}}\ln\left(\frac{\sqrt{2}\sin\left(t+\frac{\pi}{4}\right)}{\cos t}\right)dt=\int_0^{\frac{\pi}{4}}\ln\sqrt{2}+\ln\left(\sin\left(t+\frac{\pi}{4}\right)\right)-\ln\left(\cos t\right)dt$$

$$=\frac{\pi}{8}\ln 2+I_1-I_2,I_1=\int_0^{\frac{\pi}{4}}\ln\left(\sin\left(t+\frac{\pi}{4}\right)\right)dt,\text{令}\frac{\pi}{4}-t=u,\text{则}I_1=I_2$$

故$I=\frac{\pi}{8}\ln 2$

3.

$$I=\int_0^1\frac{\ln\left(1+\alpha x\right)-\ln\left(1-\beta x\right)}{1+x^2}dx.\text{其中}\alpha=1,\ \beta=0$$

$$\text{则}I=\int_\beta^\alpha dy\int_0^1\frac{x}{\left(1+xy\right)\left(1+x^2\right)}dx=\int_\beta^\alpha\frac{1}{1+y^2}\left(\frac{\pi}{4}y+\frac{1}{2}\ln 2-\ln\left(1+y\right)\right)dy$$

将$\alpha=1$，$\beta=0$带入计算得I=$\frac{\pi}{8}\ln 2$

4.

换元，令$x=\frac{1-t}{1+t}$,得$I=\int_0^1\frac{\ln\left(1+\frac{1-t}{1+t}\right)}{1+t^2}dt$,

即$I=\int_0^1\frac{\ln 2}{1+t^2}dt-I$, $I=\frac{\pi}{8}\ln 2$

网名 Hilbert 整理

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