P57. **习题** 8.1 (A) 8-8, 8-10;
P71. **习题** 8.2 (A) 1-6, 1-7; 3-3; 12
P80. **习题** 8.3 (A) 1-7; 5

8. 求下列极限

(8) $\lim\limits_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)^{x^2y^2}$; (10) $\lim\limits_{\substack{x\to\infty\\ y\to\infty}}\dfrac{|x|+|y|}{x^2+y^2}$;

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1. 求下列函数一阶和二阶偏导数:

(6) $z=\tan\dfrac{x}{y}$; (7) $z=\arctan\dfrac{x}{y}$;

3. 验证下列各题中的等式成立:

(3) $u=\left(\dfrac{x-y+z}{x+y-z}\right)^n$, $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=0$.

12. 证明:函数

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}} & (x,y)\neq(0,0),\\ 0 & (x,y)=(0,0)\end{cases}$$

在(0,0)连续,但在(0,0)处不可微.

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1. 求下列复合函数的偏导数(或导数):

(7) $u=f(x,xy,xyz)$,求$\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial u}{\partial z}$;

5. 设函数 $g(r)$ 有二阶导数,$f(x,y)=g(r)$,$r=\sqrt{x^2+y^2}$,求证:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=g''(r)+\frac{1}{r}g'(r).$$

P57-习题 8.1

8. 求下列极限

(8) $\lim\limits_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)^{x^2y^2}$； (10) $\lim\limits_{\substack{x\to \infty\\ y\to \infty}}\frac{|x|+|y|}{x^2+y^2}$；

A.8-8. 解. 记 $\rho^2 = x^2 + y^2$，由

$$\left|x^2y^2 \ln\left(x^2+y^2\right)\right| \le \frac{1}{2}r^4 \ln r \to 0 \Rightarrow \lim_{\substack{x\to 0\\ y\to 0}}(x^2+y^2)^{x^2y^2} = 1$$

A.8-10. 解. 由

$$\frac{|x|+|y|}{x^2+y^2} \le \frac{1}{|x|} + \frac{1}{|y|} \to 0 \Rightarrow \lim_{\substack{x\to \infty\\ y\to \infty}}\frac{|x|+|y|}{x^2+y^2} = 0$$

在极限计算中，也可以取特定路径转化为一元函数求极限，如 $x = y = t$

P71-习题 8.2

1. 求下列函数一阶和二阶偏导数：

(6) $z=\tan\frac{x}{y}$; (7) $z=\arctan\frac{x}{y}$;

A.1-6. **解.**

$$z_x=\frac{1}{\cos^2\frac{x}{y}}\frac{1}{y}=\frac{1+z^2}{y},\qquad z_y=\frac{1}{\cos^2\frac{x}{y}}\frac{-x}{y^2}=\frac{-x(1+z^2)}{y^2}$$

$$z_{xx}=\frac{2zz_x}{y}=\frac{2z(1+z^2)}{y^2}\qquad z_{xy}=\frac{2zz_y}{y}-\frac{1+z^2}{y^2}=\frac{-2zx(1+z^2)}{y^3}-\frac{1+z^2}{y^2}$$

$$z_{yx}=\frac{-(1+z^2)}{y^2}+\frac{-2xzz_x}{y^2}=-\frac{1+z^2}{y^2}-\frac{2xz(1+z^2)}{y^3}$$

$$z_{yy}=\frac{-2xzz_y}{y^2}+\frac{2x(1+z^2)}{y^3}=\frac{2x(1+z^2)}{y^3}+\frac{2x^2z(1+z^2)}{y^4}$$

P71-习题 8.2

1. 求下列函数一阶和二阶偏导数：

(6) $z=\tan\frac{x}{y}$； (7) $z=\arctan\frac{x}{y}$；

A.1-7. **解.**

$$z_x=\frac{1}{1+\frac{x^2}{y^2}}\frac{1}{y}=\frac{y}{x^2+y^2} \qquad z_y=\frac{1}{1+\frac{x^2}{y^2}}\frac{-x}{y^2}=\frac{-x}{x^2+y^2}$$

$$z_{xx}=\frac{-2xy}{(x^2+y^2)^2} \qquad z_{xy}=\frac{1}{x^2+y^2}-\frac{2y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$z_{yy}=\frac{2xy}{(x^2+y^2)^2} \qquad z_{yx}=\frac{-1}{x^2+y^2}+\frac{2x^2}{x^2+y^2}=\frac{x^2-y^2}{(x^2+y^2)^2}$$

3. 验证下列各题中的等式成立：

(3) $u=\left(\frac{x-y+z}{x+y-z}\right)^n, x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$

A.3-3. **解.** $u=\left(\frac{2z-2y}{x+y-z}+1\right)^n=\left(\frac{2x}{x+y-z}-1\right)^n$, **依题意**

$$u_x=n\Big(\frac{2z-2y}{x+y-z}+1\Big)^{n-1}\frac{2y-2z}{(x+y-z)^2}$$

$$u_y=n\Big(\frac{2x}{x+y-z}-1\Big)^{n-1}\frac{-2x}{(x+y-z)^2}$$

$$u_z=n\Big(\frac{2x}{x+y-z}-1\Big)^{n-1}\frac{2x}{(x+y-z)^2}$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\Big(\frac{x-y+z}{x+y-z}\Big)^{n-1}\frac{x(2y-2z)-2xy+2xz}{(x+y-z)^2}=0$$

P71-习题 8.2

12. 证明：函数

$$f(x,y)=\begin{cases}\dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0)\end{cases}$$

在(0,0)连续，但在(0,0)处不可微.

A.12. 证：由 $|xy|\leq\frac{1}{2}(x^2+y^2)$，有 $(x,y)\to(0,0)$ 时

$$\left|f(x,y)\right|\leq\frac{1}{2}(x^2+y^2)^{1/2}\to 0\Rightarrow \text{函数在 }(0,0)\text{ 连续}$$

求偏导得

$$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=0\quad f_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=0$$

考虑 $r=f(x,y)-f(0,0)-f_x(0,0)x-f_y(0,0)=f(x,y)$，有 $\frac{r}{\rho}=\frac{xy}{x^2+y^2}$
$x=y=t\to 0$ 时，$\frac{r}{\rho}=\frac{1}{2}\to\frac{1}{2}$；故 $r\neq o(\rho)$，故不可微

P71-习题 8.3

1. 求下列复合函数的偏导数(或导数):

(7) $u=f(x,xy,xyz)$, 求$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$;

A.1-7. 解.

$$\begin{aligned}
\frac{\partial u}{\partial x} &= f_1(x,xy,xyz)+f_2(x,xy,xyz)y+f_3(x,xy,xyz)yz \\
&= f_1(r,s,t)+f_2(r,s,t)y+f_3(r,s,t)yz \\
&\qquad \text{其中 } r=x, s=xy, t=xyz \\
\frac{\partial u}{\partial y} &= f_2(x,xy,xyz)x+f_3(x,xy,xyz)xz \\
&= f_2(r,s,t)x+f_3(r,s,t)xz \\
\frac{\partial u}{\partial z} &= f_3(x,xy,xyz)xy = f_3(r,s,t)xy
\end{aligned}$$

P71-习题 8.3

5. 设函数 $g(r)$ 有二阶导数，$f(x,y)=g(r)$，$r=\sqrt{x^2+y^2}$，求证：

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=g''(r)+\frac{1}{r}g'(r).$$

A.5. **证. 依题意**

$$r_x=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{r}\Rightarrow f_x=g'(r)r_x=\frac{x}{r}g'(r)$$

$$\Rightarrow f_{xx}=\left(\frac{1}{r}-\frac{x}{r^2}r_x\right)g'(r)+\frac{x}{r}g''(r)r_x$$

$$=\left(\frac{1}{r}-\frac{x^2}{r^3}\right)g'(r)+\frac{x^2}{r^2}g''(r)$$

同理 $$f_{yy}=\left(\frac{1}{r}-\frac{y^2}{r^3}\right)g'(r)+\frac{y^2}{r^2}g''(r)$$

$$\Rightarrow f_{xx}+f_{yy}=\frac{1}{r}g'(r)+g''(r)$$