Calculus note

微积分笔记

Victory won't come to us unless we go to it.

整理：不愿意留名 & 我也不想的

整理时间：July 13, 2017

Email: 不愿意告诉联系方式 @qq.com

Version: 1.00

目　录

第 1 章 函数与极限

1.1 等式

1. 常用级数求和

(a) $\sum_{k=0}^{n} k=\frac{1}{2}n\,(n+1)$

(b) $\sum_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1)\quad \sum_{k=1}^{n}(2k-1)^2=\frac{1}{3}n(4n^2-1)\quad \sum_{k=1}^{n}(2k)^2$

(c) $\sum_{k=1}^{n} k^3=(1+2+\cdots+n)^2=\frac{1}{4}n^2(n+1)^2$

(d) $\sum_{k=1}^{n} k^4=\frac{1}{30}n\,(n+1)\,(2n+1)\left(3n^2+3n-1\right)$

(e) $\sum_{k=0}^{n} k^5=\frac{1}{12}n^2(n+1)^2\left(2n^2+2n-1\right)$

(f) $1\cdot 2+2\cdot 3+\cdots+n\cdot(n+1)=\frac{1}{3}n(n+1)(n+2)$

(g) $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\cdots+n\cdot(n+1)\cdot(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$

2. 三角求和公式

(a) $\sum_{k=0}^{n}\cos(x+k\alpha)=\frac{1}{\sin\frac{\alpha}{2}}\sin\frac{(n+1)\alpha}{2}\cos\left(x+\frac{n\alpha}{2}\right)$

(b) $\sum_{k=0}^{n}\sin(2k-1)x=\frac{(\sin nx)^2}{\sin x}$

(c) $\sum_{k=0}^{n}\sin^2 kx=\frac{n}{2}-\frac{\cos(n+1)x\cdot\sin nx}{2\sin x}$

(d) $\sum_{k=0}^{n}\sin kx=\frac{1}{\sin\frac{x}{2}}\sin\left(\frac{nx}{2}\right)\sin\left(\frac{(n+1)x}{2}\right),\ \sum_{k=0}^{n}\cos kx=\frac{1}{\sin\frac{x}{2}}\sin\left(\frac{nx}{2}\right)\cos\left(\frac{(n+1)x}{2}\right)$

(e) $\sin^3 x=\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$

(f) $\cos^3 x=\frac{1}{4}(\cos 3x+3\cos x)$

(g) $\tan x=\cot x-2\cot 2x$

(h) $\sin^4 x - \cos^4 x = -\cos 2x$

(i) $\cos n\pi = (-1)^n$

3. (Newton 二项式) $(a+b)^n = \sum\limits_{r=0}^{n} C_n^r a^{n-r} b^r, C_k^n = \dfrac{n!}{k!(n-k)!}$

4. $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1})$

5. $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots - ab^{n-2} + b^{n-1})$

6. $a^3 + b^3 + c^3 - 3ab = (a+b+c)(a^2+b^2+c^2-ab-ac-bc)$

7. (B.Pascal 恒等式) $C_{k-1}^n + C_k^n = C_k^{n+1}$

1.2 不等式

1. 均值不等式: $H_n < G_n < A_n < Q_n$ 被称为均值不等式。简记为"调几算方"。

   其中：$H_n = \dfrac{1}{\sum\limits_{i=1}^{n} \frac{1}{x_i}} = \dfrac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$ ，被称为调和平均数

   $$G_n = \sqrt[n]{\prod_{i=1}^{n} x_i} = \sqrt[n]{x_1 x_2 \cdots x_n}$$，被称为几何平均数。

   $$A_n = \frac{\sum\limits_{i=1}^{n} x_i}{n} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$ ，被称为算术平均数。

   $$Q_n = \sqrt{\frac{\sum\limits_{i=1}^{n} x_i^2}{n}} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}$$，被称为平方平均数。

2. 柯西 (Cauchy) 不等式 $\left(\sum\limits_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum\limits_{i=1}^{n} (a_i)^2\right)\left(\sum\limits_{i=1}^{n} (b_i)^2\right)$

3. 伯努利 (Bernoulli) 不等式 对实数 $x > -1$, 在 $n > 1$ 时, 有 $(1+x)^n \geq 1 + nx$ 成立; 在 $0 < n < 1$ 时，有 $(1+x)^n \leq 1 + nx$ 成立。可以看到等号成立当且仅当 $n = 0, 1$ 或 $x = 0$ 时。

4. $n! < \left(\dfrac{n+1}{2}\right)^n, n > 1 \qquad \left(\dfrac{n+1}{e}\right)^n < n! < e\left(\dfrac{n+1}{e}\right)^{n+1}$

5. $\left(\dfrac{n}{e}\right)^n < n! < e\left(\dfrac{n}{2}\right)^n$

6. $\left(1+\frac{1}{n}\right)^{n}<\left(1+\frac{1}{n+1}\right)^{n+1} \leq e \leq\left(1+\frac{1}{n+1}\right)^{n+2}<\left(1+\frac{1}{n}\right)^{n+1}$

7. $(2n)!! > (2n+1)!!, n > 1 \quad \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}$

8. $\frac{1}{n+1}<\ln \left(1+\frac{1}{n}\right)<\frac{1}{n}$

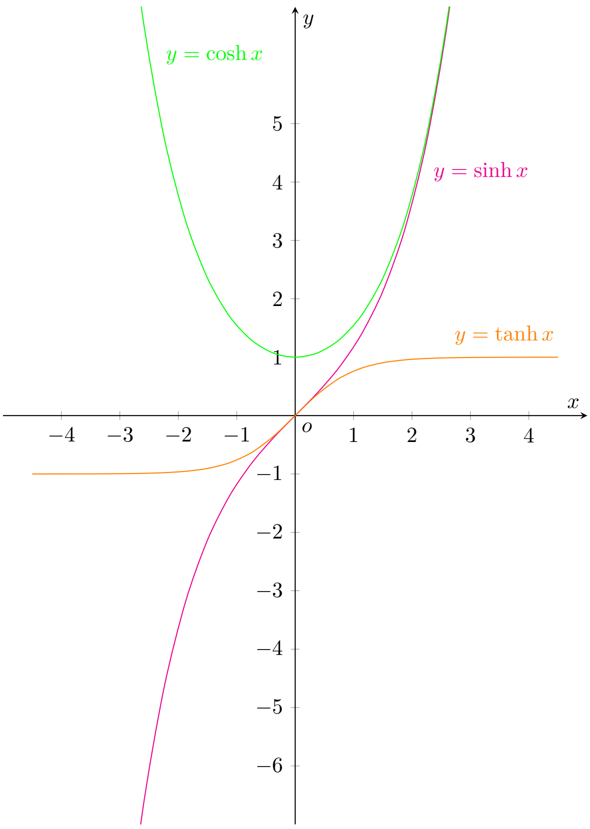
9. $\frac{k}{n+k}<\ln \left(1+\frac{k}{n}\right)<\frac{k}{n}$，其中 $k \in N_{+}$

10. 当 $0<x<\frac{\pi}{2}$ 时，$\sin x+\tan x>2x$；$\frac{2x}{\pi}<\sin x<x$；$\frac{\tan x}{x}>\frac{x}{\sin x}$

11. 当 $x>0$ 时，$\ln(1+x)>\frac{\arctan x}{1+x}$

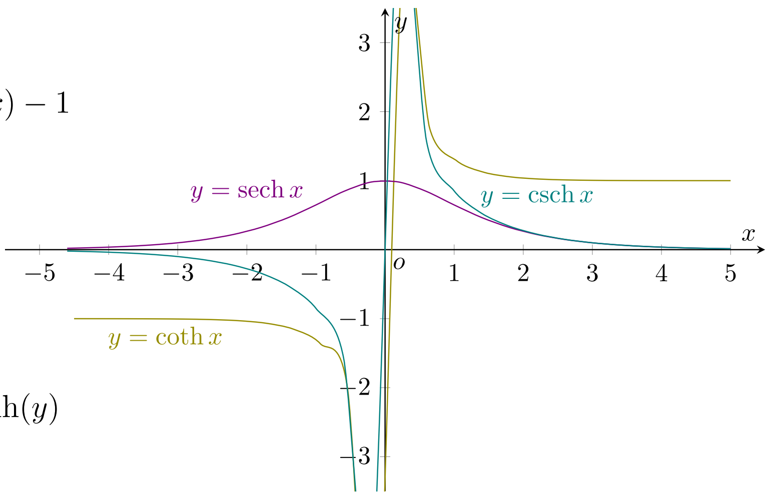
1.3 双曲函数

$$\sinh(x)=\frac{e^{x}-e^{-x}}{2}$$
$$\cosh(x)=\frac{e^{x}+e^{-x}}{2}$$
$$\tanh(x)=\frac{\sinh(x)}{\cosh(x)}=\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}=\frac{e^{2x}-1}{e^{2x}+1}$$
$$\coth(x)=\frac{\cosh(x)}{\sinh(x)}=\frac{e^{x}+e^{-x}}{e^{x}-e^{-x}}=\frac{e^{2x}+1}{e^{2x}-1}$$
$$\operatorname{sech}(x)=\frac{1}{\cosh(x)}=\frac{2}{e^{x}+e^{-x}}$$
$$\operatorname{csch}(x)=\frac{1}{\sinh(x)}=\frac{2}{e^{x}-e^{-x}}$$



$$\cosh^{2}(x)-\sinh^{2}(x)=1$$
$$1-\tanh^{2}(x)=\operatorname{sech}^{2}(x)$$
$$\coth^{2}(x)-1=\operatorname{csch}^{2}(x)$$
$$\sinh(x+y)=\sinh(x)\cosh(y)+\cosh(x)\sinh(y)$$
$$\cosh(x+y)=\cosh(x)\cosh(y)+\sinh(x)\sinh(y)$$
$$\tanh(x+y)=\frac{\tanh(x)+\tan(y)}{1+\tanh(x)\tanh(y)}$$

$$
\begin{aligned}
&\sinh(2x) = 2\sinh(x)\cosh(x) \\
&\cosh(2x) = \cosh^2(x) + \sinh^2(x) = 2\cosh^2(x) - 1 \\
&\tanh(2x) = \frac{2\tanh(x)}{1+\tanh^2(x)} \\
&\sinh^2\left(\frac{x}{2}\right) = \frac{\cosh(x)-1}{2} \\
&\cosh^2\left(\frac{x}{2}\right) = \frac{\cosh(x)+1}{2} \\
&\cosh(x+y) = \cosh(x)\cosh(y) + \sinh(x)\sinh(y)
\end{aligned}
$$



Exercise 1.1: 证明

$$3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}$$

Proof:

$$
\begin{aligned}
3 &= \sqrt{9} = \sqrt{1+8} = \sqrt{1+2\times 4} \\
&= \sqrt{1+2\sqrt{16}} = \sqrt{1+2\sqrt{1+15}} = \sqrt{1+2\sqrt{1+3\times 5}} \\
&= \sqrt{1+2\sqrt{1+3\sqrt{25}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4\times 6}}} \\
&= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{36}}}} \\
&\vdots \\
&= \sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}
\end{aligned}
$$

□

1.4 极限存在准则

Exercise 1.2: 设 $x_1 = a, x_2 = b, x_n = \frac{1}{2}(x_{n-1} - x_{n-2})\ (n \geqslant 2)$. 证明: 数列 $\{x_n\}$ 收敛, 并求 $\lim\limits_{n\to\infty} x_n$

Solution 因为

$$x_n - x_{n-1} = -\frac{1}{2}(x_{n-1} - x_{n-2}),\ (n \geqslant 3)$$

求积得

$$\prod_{i=3}^{n}(x_i - x_{i-1}) = \prod_{i=3}^{n}\left[-\frac{1}{2}(x_{i-1} - x_{i-2})\right] = \prod_{i=2}^{n-1}\left(-\frac{1}{2}\right)(x_i - x_{i-1})$$

化简得

$$x_n - x_{n-1} = \left(-\frac{1}{2}\right)^{n-2}(x_2 - x_1) = \left(-\frac{1}{2}\right)^{n-2}(b-a)$$

求和得

$$x_n - x_1 = (b-a) \times \frac{1-\left(-\frac{1}{2}\right)^{n-1}}{1-\left(-\frac{1}{2}\right)}$$

即

$$x_n = \frac{2}{3}(b-a)\left[1-\left(-\frac{1}{2}\right)^{n-1}\right] + a$$

故

$$\lim_{n\to\infty} x_n = \frac{1}{3}(a+2b)$$

Exercise 1.3: 设数列 $\{x_n\}$ 满足 $x_1 = a > 1$, 且满足递推

$$x_{n+1} = 1 + \ln\left(\frac{x_n^2}{1+\ln x_n}\right), n = 2, 3, \cdots$$

求证: $\{x_n\}$ 收敛, 并求出极限值

Proof: 先利用数学归纳法证明 $x_n > 1$, 现在假设 $x_n > 1$ 则只需要证明

$$\ln\left(\frac{x_n^2}{1+\ln x_n}\right) > 0 \Longleftrightarrow x_n^2 - 1 - \ln x_n > 0$$

考虑函数 $f(x) = x^2 - 1 - \ln x, x > 1$, 易得 $f'(x) > 0$, 所以 $f(x) > f(1) = 0$
接着证明 $x_n < x_{n+1}$, 那么只要证明

$$x_{n+1} - 1 - \ln\left(\frac{x_n^2}{1+\ln x_n}\right) > 0$$

考虑函数

$$g(x) = x - 1 - 2\ln x + \ln(1+\ln x), x > 1$$

易得

$$g'(x) = \frac{x - 1 + x\ln x - 2\ln x}{x(1+\ln x)}, x > 1$$

考虑函数 $h(x) = x - 1 + x\ln x - 2\ln x, x > 1$, 易得 $g(x) > 0$
或者考虑

$$G(x) = 1 + 2\ln x + \ln(1+\ln x) \Longrightarrow G'(x) = \frac{1+2\ln x}{x(1+\ln x)}$$

利用导数易得 $x(1+\ln x) \geqslant 1 + 2\ln x, x \geqslant 1$, 故有 $0 < G'(x) < 1$
那么有

$$0 < x_{n+1} = G(x_n) = \int_1^{x_n} G'(x)\,\mathrm{d}x + 1 \leqslant 1 + (x_n - 1) = x_n$$

综上知: 数列 $\{x_n\}$ 单调递减有下界, 故数列 $\{x_n\}$ 收敛, 设极限值为 A, 有

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = A$$

即

$$A=1+\ln\left(\frac{A^2}{1+A}\right)\Longrightarrow A=1$$

□

Exercise 1.4: $a_n>0$, 且$a_{n+1}-\dfrac{1}{a_{n+1}}=a_n+\dfrac{1}{a_n}\quad(n\geq 1)$, 求 $\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n}}\sum\limits_{j=1}^{n}\dfrac{1}{a_j}$

Proof: 假设 $0<a_n<M$

$$a_{n+1}-a_n=\frac{1}{a_n}+\frac{1}{a_{n+1}}\Rightarrow a_n-a_1=\sum_{i=1}^{n-1}\frac{1}{a_i}+\sum_{i=2}^{n}\frac{1}{a_i}\geq 2\frac{n-1}{M}$$

令 $n\to+\infty$, a_n 无界, 与假设矛盾！ 显然 a_n 严格单调递增, 故 $a_n\to+\infty(n\to+\infty)$
将 $a_{n+1}-\dfrac{1}{a_{n+1}}=a_n+\dfrac{1}{a_n}$ 两边平方得

$$a_{n+1}^2+\frac{1}{a_{n+1}^2}=a_n^2+\frac{1}{a_n^2}+4$$

从而

$$a_n+\frac{1}{a_n}=\sqrt{4n+a_1^2+\frac{1}{a_1^2}-2}$$

用 Stolz 公式, 故

$$\lim_{n\to\infty}\frac{\sum\limits_{i=1}^{n}\frac{1}{a_i}}{\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}+\sqrt{n+1}}{a_{n+1}}=\lim_{n\to\infty}\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{4n+a_1^2+\frac{1}{a_1^2}-2}+\frac{1}{a_{n+1}}}=1$$

□

Exercise 1.5: 设 $y_0\geq 2, y_n=y_{n-1}^2-2(n\in\mathbb{N}), S_n=\dfrac{1}{y_0}+\dfrac{1}{y_0y_1}+\cdots+\dfrac{1}{y_0y_1\cdots y_n}$,
证明: $\lim\limits_{n\to+\infty}S_n=\dfrac{y_0-\sqrt{y_0^2-4}}{2}$

Proof: 若 $y_0=2$, 则$y_n=2, n\in\mathbb{N}$. 此时

$$\lim_{n\to+\infty}S_n=\sum_{n=1}^{\infty}\frac{1}{2^n}=1=\frac{y_0-\sqrt{y_0^2-4}}{2}$$

若 $y_0>2$, 这时记 $\alpha=\dfrac{y_0-\sqrt{y_0^2-4}}{2}$, 此时 $y_0=\alpha+\dfrac{1}{\alpha}$. 一般地，

$$y_n=\alpha^{2^n}+\alpha^{-2^n},\quad n\in\mathbb{N}$$

因此

$$\begin{aligned}y_0y_1y_2\cdots y_n&=(\alpha+\alpha^{-1})(\alpha^2+\alpha^{-2})(\alpha^{2^2}+\alpha^{-2^2})\cdots(\alpha^{2^n}+\alpha^{-2^n})\\&=\frac{\alpha^{2^{n+1}}-\alpha^{-2^{n+1}}}{\alpha-\alpha^{-1}}\\&=\frac{\alpha}{\alpha^2-1}\cdot\frac{\alpha^{2^{n+2}}-1}{\alpha^{2^{n+1}}}\end{aligned}$$

故

$$\begin{aligned}\frac{1}{y_0y_1y_2\cdots y_n}&=\frac{\alpha^2-1}{\alpha}\cdot\frac{\alpha^{2^{n+1}}}{\alpha^{2^{n+2}}-1}\\&=\frac{\alpha^2-1}{\alpha}\cdot\frac{\alpha^{2^{n+1}}+1-1}{\alpha^{2^{n+2}}-1}\\&=\frac{\alpha^2-1}{\alpha}(\frac{1}{\alpha^{2^{n+1}}-1}-\frac{1}{\alpha^{2^{n+2}}-1})\end{aligned}$$

因此

$$\begin{aligned}S_n&=\sum_{k=0}^{n}\frac{1}{y_0y_1y_2\cdots y_k}\\&=\sum_{k=0}^{n}\frac{\alpha^2-1}{\alpha}\left(\frac{1}{\alpha^{2^{k+1}}-1}-\frac{1}{\alpha^{2^{k+2}}-1}\right)\\&=\frac{\alpha^2-1}{\alpha}\left(\frac{1}{\alpha^2-1}-\frac{1}{\alpha^{2^{n+2}}-1}\right)\end{aligned}$$

注意到 $\alpha<1$, 最终

$$\lim_{n\to\infty}S_n=\frac{\alpha^2-1}{\alpha}\left(\frac{1}{\alpha^2-1}+1\right)=\alpha=\frac{y_0-\sqrt{y_0^2-4}}{2}$$

□

Exercise 1.6: 设数列 a_n 满足级数 $|a_1|+|a_2|+\cdots+|a_n|+\cdots$ 收敛, 证明: $\lim\limits_{p\to\infty}(|a_1|^p+|a_2|^p+\cdots+|a_n|^p+\cdots)^{\frac{1}{p}}$ 的极限存在, 并求之.

Proof: 记

$$||a||_p=(|a_1|^p+|a_2|^p+\cdots+|a_n|^p+\cdots)^{\frac{1}{p}},\ (p>0)$$

由于

$$|a_1|+|a_2|+\cdots+|a_n|+\cdots$$

收敛

所以 $\lim\limits_{n\to\infty}|a_n|=0, \sup|a_n|$ 存在

易证 $|a_n|\le||a||_q\ (q>1,n=1,2,3,\cdots)$, 于是 $\sup|a_n|\le||a||_q$

对 $1<q<p$

$$\begin{aligned}||a||_p&=(|a_1|^p+|a_2|^p+\cdots+|a_n|^p+\cdots)^{\frac{1}{p}}\\&=(|a_1|^{p-q}|a_1|^q+|a_2|^{p-q}|a_2|^q+\cdots+|a_n|^{p-q}|a_n|^q+\cdots)^{\frac{1}{p}}\\&\le||a||_q^{\frac{p-q}{p}}(|a_1|^q+|a_2|^q+\cdots+|a_n|^q+\cdots)^{\frac{1}{p}}\\&\le||a||_q^{\frac{p-q}{p}}||a||_q^{\frac{q}{p}}\\&=||a||_q\end{aligned}$$

故 $||a||_p\le||a||_q$, 所以 $||a||_p$ 关于 p 单调递减且有下界. 于是有

$$a_n\le||a||_p\le(\sup a_n)^{1-\frac{q}{p}}||a||_q^{\frac{q}{p}}$$

当 $p \to +\infty$ 时, 有夹逼定理, $\lim\limits_{p\to+\infty} ||a||_p = \sup|a_n|$ □

Exercise 1.7: 设数列 $\{a_n\}$ 满足 $a_1 = 1, a_{n+1} = a_n + \dfrac{1}{a_1 + a_2 + \cdots + a_n}$，求 $\lim\limits_{n\to\infty} \dfrac{a_n}{\sqrt{2\ln n}}$

Proof: 易知 $\{a_n\}$ 单调递增，且趋于 ∞ , 所以

$$1 \leq \frac{a_{n+1}}{a_n} \leq 1 + \frac{1}{na_n}$$

$$1 \leq n+1-n\frac{a_n}{a_{n+1}} \leq \frac{1+\frac{n+1}{na_n}}{1+\frac{1}{na_n}}, \quad \lim_{n\to\infty} \frac{1+\frac{n+1}{na_n}}{1+\frac{1}{na_n}} = 1$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{(n+1)a_{n+1} - na_n}{a_{n+1}} = 1 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{na_n}{a_1 + a_2 + \cdots + a_n} = 1$$

$$\therefore \quad \lim_{n\to\infty} \frac{n}{(\sum\limits_{i=1}^{n} a_i)^2} = 0$$

$$\begin{aligned}
\therefore \quad \lim_{n\to\infty} \frac{a_n^2}{2\ln n} &= \lim_{n\to\infty} \frac{n}{2}(a_{n+1}^2 - a_n^2) \\
&= \lim_{n\to\infty} \frac{n}{2} \left(\frac{2a_n}{\sum\limits_{i=1}^{n} a_i} + \frac{1}{(\sum\limits_{i=1}^{n} a_i)^2} \right) \\
&= 1
\end{aligned}$$

□

Exercise 1.8:

Proof:

□

1.5 求极限

Exercise 1.9: 求极限

$$\lim_{n\to\infty} \left\{ \tan(\pi\sqrt{n^2 + \left[\frac{6n}{11}\right]}) + 4\sin(\pi\sqrt{4n^2 + \left[\frac{8n}{11}\right]}) \right\}$$

Solution:

$$\tan\pi(\sqrt{n^2 + \left[\frac{6n}{11}\right]}) = \tan(\pi\sqrt{n^2 + \left[\frac{6n}{11}\right]} - n\pi)$$

$$\pi\sqrt{n^2 + \left[\frac{6n}{11}\right]} - n\pi = \frac{\left[\frac{6}{11}n\right]}{\sqrt{n^2 + [\frac{6}{11}n]} + \sqrt{n^2}}\pi$$

考虑下列不等式

$$\frac{\frac{6}{11}n - 1}{\sqrt{n^2 + \frac{6}{11}n} + \sqrt{n^2}} \leq \frac{\left[\frac{6}{11}n\right]}{\sqrt{n^2 + [\frac{6}{11}n]} + \sqrt{n^2}} \leq \frac{\left[\frac{6}{11}n\right]}{2n} \leq \frac{3}{11}$$

当 $n\to\infty$, 左边等于 $\frac{3}{11}$ 故

$$\lim_{n\to\infty}\tan(\pi\sqrt{n^2+\left[\frac{6n}{11}\right]})=\tan\frac{3}{11}\pi$$

同样的方法，可以计算出

$$\lim_{n\to\infty}\sin(\pi\sqrt{4n^2+\left[\frac{8n}{11}\right]})=\sin\frac{2}{11}\pi$$

对于 $\tan\frac{3}{11}\pi+4\sin\frac{2}{11}\pi=\sqrt{11}$ 的计算，这里不再给出。 □

Exercise 1.10: 求极限

$$\lim_{n\to+\infty}(\frac{2}{3})^{\frac{1}{2^{n-1}}}(\frac{4}{7})^{\frac{1}{2^{n-2}}}\cdots(\frac{2^{n-1}}{2^n-1})^{\frac{1}{2}}$$

Solution: 令

$$x_n=(\frac{2}{3})^{\frac{1}{2^{n-1}}}(\frac{4}{7})^{\frac{1}{2^{n-2}}}\cdots(\frac{2^{n-1}}{2^n-1})^{\frac{1}{2}}$$

则

$$\begin{aligned}\ln x_n&=\frac{1}{2^{n-1}}\ln\frac{2}{3}+\frac{1}{2^{n-2}}\ln\frac{4}{7}+\cdots+\frac{1}{2}\ln\frac{2^{n-1}}{2^n-1}\\&=\frac{1}{2^{n-1}}\left(\ln\frac{2}{3}+2\ln\frac{4}{7}+\cdots+2^{n-2}\ln\frac{2^{n-1}}{2^n-1}\right)\end{aligned}$$

应用 Stolz 公式求极限

$$\begin{aligned}\lim_{n\to\infty}\ln x_n&=\lim_{n\to\infty}\frac{2^{n-2}\ln\frac{2^{n-1}}{2^n-1}}{2^{n-1}-2^{n-2}}\\&=\lim_{n\to\infty}\ln\frac{1}{2-\frac{1}{2^{n-1}}}\\&=\ln\frac{1}{2}\end{aligned}$$

故

$$\lim_{n\to\infty}x_n=\lim_{n\to\infty}(\frac{2}{3})^{\frac{1}{2^{n-1}}}(\frac{4}{7})^{\frac{1}{2^{n-2}}}\cdots(\frac{2^{n-1}}{2^n-1})^{\frac{1}{2}}=\frac{1}{2}$$

□

Exercise 1.11: 求极限

Solution:

□

1.6 闭区间上连续函数的性质

第 2 章 导数与微分

2.1 高阶导数

2.1.1 高阶导数

Exercise 2.1: $y=\sin^4 x+\cos^4 x$, 求 $y^{(n)}$

Solution

$$\begin{aligned} y &= \sin^4 x+\cos^4 x=\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x \\ &= 1-\frac{1}{2}\sin^2 2x=1-\frac{1}{4}(1-\cos 4x)=\frac{3}{4}+\frac{1}{4}\cos 4x \end{aligned}$$

Exercise 2.2: 已知 $y=x^2e^{2x}$, 求 $y^{(20)}$

Solution 设 $u=e^{2x}$, $v=x^2$, 则

$$u^{(k)}=2^k e^{2x}\ (k=1,2,\cdots,20)$$

$$v'=2x,\ \ v''=2,\ \ v^{(k)}=0\ (k=3,4,\cdots,20)$$

代入莱布尼茨公式, 得

$$\begin{aligned} y^{(20)} &= (x^2e^{2x})^{(20)} \\ &= 2^{20}e^{2x}\cdot x^2+20\cdot 2^{19}e^{2x}\cdot 2x+\frac{20\cdot 19}{2!}2^{18}e^{2x}\cdot 2 \\ &= 2^{20}e^{2x}(x^2+20x+95) \end{aligned}$$

莱布尼茨公式

$$(uv)^n=\sum_{k=0}^{n}C_n^k u^{(n-k)}v^{(k)}$$

Exercise 2.3: 求 $y=x^2e^{3x}$ 的 n 阶导数

Solution 设 $u=e^{3x}$, $v=x^2$, 则

$$u^{(k)}=2^k e^{2x}\ (k=1,2,\cdots,20)$$

$$v'=2x,\ \ v''=2,\ \ v^{(k)}=0\ (k=3,4,\cdots,20)$$

代入莱布尼茨公式, 得

$$
\begin{aligned}
y^{(n)} &= (x^2e^{3x})^{(n)} \\
&= (e^{3x})^{(n)}x^2 + n(e^{3x})^{(n-1)}(x^2)' + \frac{n(n-1)}{2}(e^{3x})^{(n-2)}(x^2)'' \\
&= 3^{n-2}e^{3x}\left[9x^2 + 6nx + n(n-1)\right]
\end{aligned}
$$

莱布尼茨公式

$$(uv)^n = \sum_{k=0}^{n} C_n^k u^{(n-k)}v^{(k)}$$

Exercise 2.4: 设 $f(x) = \dfrac{1}{1-x^2+x^4}$ 求 $f^{(100)}(0)$

Solution 因为

$$f(x) = \frac{1}{1-x^2+x^4} = \frac{1+x^2}{1+x^6}$$

由带皮亚诺余项的麦克劳林公式, 有

$$f(x) = (1+x^2)(1-x^6+\cdots+x^{96}-x^{102}+o(x^{102}))$$

所以 $f(x)$ 展开式的 100 次项为 0

即有 $\dfrac{f^{(100)}(0)}{100!} = 0$, 故 $f^{(100)}(0) = 0$

Exercise 2.5: 设 $f(x) = e^x \sin 2x$ 求 $f^{(4)}(0)$

Solution 由麦克劳林公式

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + o(x^n)$$

则

$$f(x) = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + o(x^3)\right)\left(2x - \frac{1}{3!}(2x)^3 + o(x^4)\right)$$

所以 $f(x)$ 展开式的 4 次项为

$$\frac{2}{3!}x^4 - \frac{1}{3!}(2x)^3 \cdot x = -x^4$$

即有 $\dfrac{f^{(4)}(0)}{4!} = -x^4$, 故 $f^{(4)}(0) = -24$

2.2 函数的微分

Definition 2.1

设函数 $f(x)$ 在某区间内有定义, x_0 及 $x_0+\Delta x$ 在这区间内, 如果函数的增量

$$\Delta y = f(x_0+\Delta x) - f(x_0)$$

可表示为

$$\Delta y = A\Delta x + o(\Delta x)$$

其中 A 是不依赖于 Δx 的常数, 那么称函数 $y=f(x)$ 在点 x_0 是可微的, 而 $A\Delta x$ 叫做函数 $y=f(x)$ 在点 x_0 相应与自变量增量 Δx 的微分, 记作 $\mathrm{d}y$, 即

$$\mathrm{d}y = A\Delta x$$

Note: 当 $f(x)$ 在点 x_0 可微时, 其微分一定是 $\mathrm{d}y = f'(x_0)\Delta x$

Note: 当 $f(x)$ 在任意点 x 的微分, 称为函数的微分, 记作 $\mathrm{d}y$ 或 $\mathrm{d}f(x)$, 即 $\mathrm{d}y = f'(x)\Delta x$

Note: 通常把自变量 x 的增量 Δx 称为自变量的微分, 记作 $\mathrm{d}x$, 即 $\mathrm{d}x = \Delta x$

第 3 章 微分中值定理与导数的应用

3.1 微分中值定理

Theorem 3.1 费马定理

设函数 $f(x)$ 在点 x_0 的某领域 $U(x_0)$ 内有定义, 并且在 x_0 处可导, 如果对任意的 $x \in U(x_0)$, 有

$$f(x) \leqslant f(x_0) \quad (\text{或} f(x) \geqslant f(x_0))$$

那么 $f'(x_0) = 0$

Theorem 3.2 罗尔 (Rolle) 定理

如果函数 $f(x)$ 满足

(1) 在闭区间 $[a, b]$ 上连续;

(2) 在开区间 (a, b) 内可导;

(3) 在区间端点的函数值相等，即 $f(a) = f(b)$,

那么在 (a, b) 内至少有一点 $\xi(a < \xi < b)$, 使得函数 $f(x)$ 在该点的导数等于零, 即 $f'(\xi) = 0$

Theorem 3.3 拉格朗日 (Lagrange) 中值定理

如果函数 $f(x)$ 满足

(1) 在闭区间 $[a, b]$ 上连续;

(2) 在开区间 (a, b) 内可导;

那么在 (a, b) 内至少有一点 $\xi(a < \xi < b)$, 使等式 $f(b) - f(a) = f'(\xi)(b - a)$ 成立

Corollary 3.1

如果函数 $f(x)$ 在区间 I 上的导数恒为零, 那么 $f(x)$ 在区间 I 上是一个常数

Corollary 3.2

如果函数 $f(x)$ 在区间 I 上 $f(x) = g(x)$ 恒成立，则 $f(x)$ 在区间 I 上有 $f(x) = g(x) + C$

Theorem 3.4 柯西中值定理

如果函数 $f(x)$ 及 $F(x)$ 满足

(1) 在闭区间 $[a,b]$ 上连续;

(2) 在开区间 (a,b) 内可导;

(3) 对任一 $x \in (a,b), F'(x) \neq 0$,

那么在 (a,b) 内至少有一点 $\xi(a < \xi < b)$, 使等式 $\dfrac{f(a) - f(b)}{F(a) - F(b)} = \dfrac{f'(\xi)}{F'(\xi)}$ 成立

Exercise 3.1: 设 $f(x)$ 在 $[0,1]$ 上可微, $f(0)=0, f(1)=1$. 三个正数 $\lambda_1,\lambda_2,\lambda_3$ 的和为 1, 证明: $(0,1)$ 内存在三个不同数 ξ_1,ξ_2,ξ_3, 使得

$$\frac{\lambda_1}{f'(\xi_1)}+\frac{\lambda_2}{f'(\xi_2)}+\frac{\lambda_3}{f'(\xi_3)}=1$$

Proof: 设 $0<x_1<x_2<1$, 对 $f(x)$ 在区间 $[0,x_1],[x_1,x_2],[x_2,1]$ 上分别使用拉格朗日中值定理可得

$$f'(\xi_1)=\frac{f(x_1)-f(0)}{x_1-0}=\frac{f(x_1)}{x_1},\quad \xi_1\in(0,x_1)\Longrightarrow\frac{\lambda_1}{f'(\xi_1)}=\frac{\lambda_1 x_1}{f(x_1)}$$
$$f'(\xi_2)=\frac{f(x_2)-f(x_1)}{x_2-x_1},\quad \xi_1\in(x_1,x_2)\Longrightarrow\frac{\lambda_2}{f'(\xi_2)}=\frac{\lambda_2(x_2-x_1)}{f(x_2)-f(x_1)}$$
$$f'(\xi_3)=\frac{f(1)-f(x_2)}{1-x_2}=\frac{1-f(x_2)}{1-x_2},\quad \xi_1\in(x_2,1)\Longrightarrow\frac{\lambda_3}{f'(\xi_3)}=\frac{\lambda_3(1-x_2)}{1-f(x_2)}$$

欲使

$$\frac{\lambda_1}{f'(\xi_1)}+\frac{\lambda_2}{f'(\xi_2)}+\frac{\lambda_3}{f'(\xi_3)}=1$$

只需

$$f(x_1)=\lambda_1, f(x_2)=\lambda_2-\lambda_1$$

又 $f(0)=0, f(1)=1$, 由连续函数的介值定理知,
存在 $x_1\in(0,1)$, 使得 $f(x_1)=\lambda_1$ 和 存在 $x_2\in(0,1)$, 使得 $f(x_1)=\lambda_2-\lambda_1$ □

Exercise 3.2: 设 $f(x)$ 在 $[0,1]$ 上可导且 $f(0)=0, f(1)=1$. 且 $f(x)$ 在 $[0,1]$ 上严格递增证明: $(0,1)$ 内存在 $\xi_i\in(0,1)$ $(1\leqslant i\leqslant n)$, 使得

$$\frac{1}{f'(\xi_1)}+\cdots+\frac{1}{f'(\xi_n)}=n$$

Proof: 设 $\xi_i\in(0,1)$, 对 $f(x)$ 在区间 $[0,x_1],[x_2,x_3],\cdots,[x_{n-1},1]$ 上分别使用拉格朗日中值定理可得

$$f'(\xi_1)=\frac{f(x_1)-f(0)}{x_1-0}=\frac{f(x_1)}{x_1},\quad \xi_1\in(0,x_1)\Longrightarrow\frac{1}{f'(\xi_1)}=\frac{x_1}{f(x_1)}$$
$$f'(\xi_2)=\frac{f(x_2)-f(x_1)}{x_2-x_1},\quad \xi_1\in(x_1,x_2)\Longrightarrow\frac{1}{f'(\xi_2)}=\frac{x_2-x_1}{f(x_2)-f(x_1)}$$
$$\vdots$$
$$f'(\xi_n)=\frac{f(1)-f(x_{n-1})}{1-x_{n-1}}=\frac{1-f(x_{n-1})}{1-x_{n-1}},\quad \xi_n\in(x_{n-1},1)\Longrightarrow\frac{1}{f'(\xi_n)}=\frac{1-x_{n-1}}{1-f(x_{n-1})}$$

欲使

$$\frac{1}{f'(\xi_1)}+\cdots+\frac{1}{f'(\xi_2)}=n$$

只需

$$f(x_1)=\frac{1}{n}, f(x_2)=\frac{2}{n},\cdots,f(x_{n-1})=\frac{n-1}{n}$$

又 $f(0)=0, f(1)=1$, 由连续函数的介值定理, 存在 $x_k\in(0,1)$, $k\in[1,n-1]$, 使得 $f(x_k)=\dfrac{k}{n}$
证毕 □

Exercise 3.3: 设 $f(x)$ 在 $[a,b]$ 上连续, 在 (a,b) 内可导 $(0<a<b)$, $f(a)\neq f(b)$,

证明存在 $\xi,\eta\in(a,b)$, 使得 $\dfrac{f'(\xi)}{2\xi}=\dfrac{\ln\frac{b}{a}}{b^2-a^2}\eta f'(\eta)$

✎ **Solution** 考虑

$$\frac{f'(\xi)}{2\xi}\Longrightarrow g'(x)=x^2\Longrightarrow \text{构造 } g(x)=x^2$$

令 $g(x)=x^2, g(x)$ 与 $f(x)$ 在 $[a,b]$ 上连续, 在 (a,b) 内可导, 由柯西中值定理知 $\exists\,\xi\in[a,b]$

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\xi)}{g'(\xi)}=\frac{f'(\xi)}{2\xi}\Longrightarrow f(b)-f(a)=\frac{(b^2-a^2)f'(\xi)}{2\xi}$$

考虑

$$\eta f'(\eta)=\frac{f'(\eta)}{\frac{1}{\eta}}\Longrightarrow g'(x)=\ln x\Longrightarrow \text{构造 } g(x)=\ln x$$

令 $g(x)=\ln x, g(x)$ 与 $f(x)$ 在 $[a,b]$ 上连续, 在 (a,b) 内可导, 由柯西中值定理知 $\exists\,\eta\in[a,b]$

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(\eta)}{g'(\eta)}=\eta f'(\eta)\Longrightarrow f(b)-f(a)=\ln\frac{b}{a}\eta f'(\eta)$$

故 $\exists\,\xi,\eta\in(a,b)$ 使得 $\dfrac{f'(\xi)}{2\xi}=\dfrac{\ln\frac{b}{a}}{b^2-a^2}\eta f'(\eta)$. 得证

Exercise 3.4: 设 $f(x)$ 在 $[0,1]$ 上连续，在 $(0,1)$ 上可导，且 $f(0)=0, f(1)=1$
证明：存在两个不同的常数 $\eta,\xi\in(0,1)$ 使得 $f'(\xi)f'(\eta)=1$

✎ **Solution** 构造函数令 $F(x)=f(x)+x-1$
因为 $F(0)F(1)<0$ 故由零点定理知存在 $x_0\in(0,1)$ 使得 $F(x_0)=f(x_0)+x_0-1=0$, 即 $f(x_0)=1-x_0$
在 $(0,x_0)$ 和 $(x_0,1)$ 上分别对 $f(x)$ 用拉格朗日中值定理可得

$$f(x_0)-f(0)=f'(\xi)(x_0-0)\Leftrightarrow\frac{1-x_0}{x_0}=f'(\xi),\xi\in(0,x_0)$$

$$f(1)-f(x_0)=f'(\eta)(1-x_0)\Leftrightarrow\frac{x_0}{1-x_0}=f'(\eta),\eta\in(x_0,1)$$

于是有

$$f'(\xi)f'(\eta)=\frac{1-x_0}{x_0}\times\frac{x_0}{1-x_0}=1$$

因此存在两个不同的常数 $\eta,\xi\in(0,1)$ 使得 $f'(\xi)f'(\eta)=1$

Exercise 3.5: 设 $f(x)$ 在 $[0,1]$ 上二阶可导，且 $f(0)=f(1)=0,\ f'(1)=1$
求证：存在 $\xi\in(0,1)$ 使得 $f''(\xi)=2$

✎ **Solution** 令 $F(x)=f(x)-x^2$, 则 $\quad F(0)=f(0), F(1)=f(1)-1,$
且 $F(x)$ 满足拉格朗日中值定理的条件. 由拉格朗日中值定理,
$\exists\, C\in(0,1)$, 使 $F'(C)=\dfrac{F(1)-F(0)}{1-0}=-1$
又 $F'(x)=f'(x)-2x, F'(1)=f'(1)-2=-1$

且 $F(x)$ 在 $[C,1]$ 满足罗尔定理的条件. 根据罗尔定理 $\exists\xi\in(C,1)$, 使 $F''(\xi)=0$
即 $f''(\xi)-2=0$, 也即存在 $\xi\in(0,1)$ 使得 $f''(\xi)=2$

Exercise 3.6: 设 $f(x)$ 在区间 $[a,b]$ 上连续, 开区间 (a,b) 内二阶可导,
$f(a)=f(b)=0, \int_a^b f(x)\,\mathrm{d}x=0$. 证明

(1) 至少存在一点 $\xi\in(a,b)$, 使得 $f'(\xi)=f(\xi)$;

(2) 至少存在一点 $\eta\in(a,b)$, $\eta\neq\xi$, 使得 $f''(\eta)=f(\eta)$

Solution 令 $g(x)=f(x)e^{-x}$, 由 $\int_a^b f(x)\,\mathrm{d}x=0$ 知存在 $f(\lambda)=0\,(0<\lambda<b)$
且 $g(\lambda)=g(a)=0, g(x)$ 在区间 $[a,b]$ 上连续, 开区间 (a,b) 内二阶可导,
由罗尔定理知, 至少存在一点 $\xi_1\in(a,\lambda)$, 使得 $g'(\xi_1)=0\,(a<\xi_1<\lambda)$
同理, 至少存在一点 $\xi_2\in(\lambda,b)$, 使得 $g'(\xi_2)=0\,(\lambda<\xi_2<b)$
令 $h(x)=f^2(x)-[f'(x)]^2$, 易知 $h(\xi_1)=h(\xi_2)$
$h(x)$ 在 (ξ_1,ξ_2) 上满足罗尔定理的条件, 因此至少存在一点 $\eta\in(\xi_1,\xi_2)$, $\eta\neq\xi$, 使得 $h'(\eta)=0$, 即 $f''(\eta)=f(\eta)$

3.2 洛必达法则

Exercise 3.7: 求极限

$$\lim_{x\to0}\frac{1-\cos x\sqrt{\cos x}\sqrt[3]{\cos x}\cdots\sqrt[n]{\cos x}}{(x+\sin x)^2}$$

Solution

$$\begin{aligned}
&\lim_{x\to0}\frac{1-\cos x\sqrt{\cos x}\sqrt[3]{\cos x}\cdots\sqrt[n]{\cos x}}{(x+\sin x)^2}\\
=&\lim_{x\to0}\frac{1-e^{\ln\left(\cos x\sqrt{\cos x}\sqrt[3]{\cos x}\cdots\sqrt[n]{\cos x}\right)}}{(x+\sin x)^2}\\
=&-\lim_{x\to0}\frac{\ln(\cos x)+\ln(\sqrt[2]{\cos x})+\cdots+\ln(\sqrt[n]{\cos x})}{(x+\sin x)^2}\\
=&\lim_{x\to0}\frac{\tan x+\tan 2x+\cdots+\tan nx}{2(x+\sin x)(1+\cos x)}\\
=&\frac{1}{4}\lim_{x\to0}\frac{\tan x+\tan 2x+\cdots+\tan nx}{x+\sin x}\\
=&\frac{1}{4}\lim_{x\to0}\frac{\sec^2 x+2\sec^2 2x+3\sec^2 3x+\cdots+n\sec^2 nx}{1+\cos x}\\
=&\frac{1}{8}(1+2+3+\cdots+n)=\frac{n(n+1)}{16}
\end{aligned}$$

Exercise 3.8: 求极限

$$\lim_{x\to0^+}\left(\frac{1}{x}\right)^{\tan x}$$

Solution

$$\lim_{x\to0^+}\left(\frac{1}{x}\right)^{\tan x}=\exp\lim_{x\to0^+}\tan x\ln\left(\frac{1}{x}\right) \tag{3.1}$$
$$=\exp\lim_{x\to0^+}(-x\ln x)=\exp\lim_{x\to0^+}\left(-\frac{\ln x}{\frac{1}{x}}\right) \tag{3.2}$$
$$=\exp\lim_{x\to0^+}\left(-\frac{\frac{1}{x}}{-\frac{1}{x^2}}\right) \tag{3.3}$$
$$=\exp\lim_{x\to0^+}x=1 \tag{3.4}$$

Exercise 3.9: 求极限

$$\lim_{n\to a}\frac{\sin x-\sin a}{x-a}$$

Solution

$$\lim_{n\to a}\frac{\sin x-\sin a}{x-a}\xlongequal{\text{洛必达}}\lim_{n\to a}\frac{\cos a}{1}=\cos a$$

Solution

$$\begin{aligned}\lim_{n\to a}\frac{\sin x-\sin a}{x-a}&=\lim_{n\to a}\frac{2\cos\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a}\\&=\lim_{n\to a}\frac{2\cos\frac{x+a}{2}\cdot\frac{x-a}{2}}{x-a}\\&=\lim_{n\to a}\cos\frac{x+a}{2}\\&=\cos a\end{aligned}$$

Solution

$$\lim_{n\to a}\frac{\sin x-\sin a}{x-a}=\left.\frac{d\sin x}{dx}\right|_{x=a}=\cos a$$

Exercise 3.10: 求极限

$$\lim_{x\to0^+}\frac{\ln\cot x}{\ln x}$$

Solution

$$\begin{aligned}\lim_{x\to0^+}\frac{\ln\cot x}{\ln x}&\xlongequal{\text{洛必达}}\lim_{x\to0^+}\frac{\frac{1}{\cot x}\times(-\csc^2x)}{\frac{1}{x}}\\&=-\lim_{x\to0^+}\frac{x\tan x}{\sin^2x}\\&=-\lim_{x\to0^+}\frac{x\times x}{x^2}=-1\end{aligned}$$

Exercise 3.11: 求极限

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\ln\cot x}{\tan x}$$

Solution

$$
\begin{aligned}
\lim_{x\to\frac{\pi}{2}^-}\frac{\ln\cot x}{\tan x} &\xlongequal{\text{洛必达}} \lim_{x\to\frac{\pi}{2}^-}\frac{\frac{1}{\cot x}\times(-\csc^2 x)}{\sec^2 x}\\
&\xlongequal{\text{化简}} -\lim_{x\to\frac{\pi}{2}^-}\cot x\\
&\xlongequal{\text{带值}} 0
\end{aligned}
$$

Solution

$$
\begin{aligned}
\lim_{x\to\frac{\pi}{2}^-}\frac{\ln\cot x}{\tan x} &\xlongequal{t=\frac{\pi}{2}-x} \lim_{x\to 0^+}\frac{\ln\cot\left(\frac{\pi}{2}-t\right)}{\tan\left(\frac{\pi}{2}-t\right)}\\
&= \lim_{x\to 0^+}\frac{\ln\tan t}{\cot t}
\end{aligned}
$$

Exercise 3.12: 求极限

$$
\lim_{x\to+\infty}\left[\left(x^3-x^2+\frac{x}{2}\right)e^{\frac{1}{x}}-\sqrt{x^6+1}\right]
$$

Solution

$$
\begin{aligned}
I &= \lim_{x\to+\infty}\left[\left(x^3-x^2+\frac{x}{2}\right)e^{\frac{1}{x}}-\sqrt{x^6+1}\right]\\
&\xlongequal{u=\frac{1}{x}} \lim_{u\to 0^+}\frac{\left(1-u+\frac{u^2}{2}e^u\right)-\sqrt{1+u^6}}{u^3}\\
&= \lim_{u\to 0^+}\frac{\frac{u^2e^u}{2}-\frac{3u^5}{\sqrt{1+u^6}}}{3u^2} = \lim_{u\to 0^+}\frac{\frac{e^u}{2}-\frac{3u^3}{\sqrt{1+u^6}}}{3}\\
&\xlongequal{\text{代值}} \frac{1}{6}
\end{aligned}
$$

Exercise 3.13: 求极限

$$
\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)
$$

Solution

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right) &= \lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}\\
&= \lim_{x\to 0}\frac{(\sin x-x)(\sin x+x)}{x^4}\\
&= \lim_{x\to 0}\frac{\sin x-x}{x^3}\times\lim_{x\to 0}\frac{\sin x+x}{x}\\
&= \lim_{x\to 0}\frac{\cos x-1}{3x^2}\times\lim_{x\to 0}\frac{\cos x+1}{1}\\
&= 2\lim_{x\to 0}\frac{-\frac{1}{2}x^2}{3x^2}\\
&= -\frac{1}{3}
\end{aligned}
$$

Exercise 3.14: 设 $f(x)$ 在 $x=0$ 的某领域内二阶可导, 且 $\lim\limits_{x\to 0}\left(\dfrac{\sin 3x}{x^3}+\dfrac{f(x)}{x^2}\right)=0$.

求 $f(0), f'(0), f''(0), \lim\limits_{x\to 0}\dfrac{f(x)+3}{x^2}$

Solution 由题意

$$
\begin{aligned}
\lim_{x\to 0}\left(\frac{\sin 3x}{x^3}+\frac{f(x)}{x^2}\right) &= \lim_{x\to 0}\frac{\sin 3x+xf(x)}{x^3}=\lim_{x\to 0}\frac{\sin 3x-3x+3x+xf(x)}{x^3}\\
&=\lim_{x\to 0}\frac{\sin 3x-3x}{x^3}+\lim_{x\to 0}\frac{3x+xf(x)}{x^3}\\
&=-\frac{9}{2}+\lim_{x\to 0}\frac{3+f(x)}{x^2}=0
\end{aligned}
$$

故

$$
\lim_{x\to 0}\frac{3+f(x)}{x^2}=\frac{9}{2}\Longrightarrow \lim_{x\to 0}f(x)=-3
$$

且 $f(x)$ 在 $x=0$ 的某领域内二阶可导, 故

$$
f(0)=\lim_{x\to 0}f(x)=-3
$$

以及

$$
\lim_{x\to 0}\frac{3+f(x)}{x^2}\xlongequal{\text{洛必达}}\lim_{x\to 0}\frac{f'(x)}{2x}=\frac{9}{2}\Longrightarrow f'(0)=\lim_{x\to 0}f'(x)=0
$$

由上式

$$
\lim_{x\to 0}\frac{f'(x)}{2x}\xlongequal{\text{洛必达}}\lim_{x\to 0}\frac{f''(x)}{2}=\frac{9}{2}\Longrightarrow f''(0)=\lim_{x\to 0}f''(x)=9
$$

3.3 泰勒公式

Theorem 3.5 泰勒中值定理

如果函数 $f(x)$ 在点 x_0 的某个领域 $U(x_0)$ 内有 $(n+1)$ 阶导数, 那么对任一 $x\in U(x_0)$, 有

$$
f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)
$$

其中 $R_n(x)=o[(x-x_0)^n]$. 称为皮亚诺形式的余项

Theorem 3.6 泰勒中值定理

如果函数 $f(x)$ 在 x_0 处具有 n 阶导数, 那么存在 x_0 的一个领域，对于该领域内的任一 x，有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)$$

其中 $R_n(x)=\dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1},\quad \xi$ 在 x 与 x_0 之间. $R_n(x)$ 称为拉格朗日形式的余项

Theorem 3.7 泰勒中值定理

如果函数 $f(x)$ 在 x_0 处具有 n 阶导数, 那么存在 x_0 的一个领域，对于该领域内的任一 x，有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)$$

其中 $R_n(x)=\dfrac{f^{(n+1)}(x_0+\theta(x-x_0))}{(n+1)!}(x-x_0)^{n+1},\quad \theta\in(0,1)$, $R_n(x)$ 称为柯西形式的余项

Theorem 3.8 泰勒中值定理

若函数 $f(x)$ 在点 x_0 的领域 $U(x_0)$ 内有连续的 $n+1$ 阶导数, 则 $\forall x\in U(x_0)$， 有

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+R_n(x)$$

其中 $R_n(x)=\dfrac{1}{n!}\displaystyle\int_{x_0}^{x}f^{(n+1)}(t)(x-t)^n dt$ 称为积分型余项

Exercise 3.15: 设 $f(x)$ 在 $[a,b]$ 上具有二阶导数, 且 $f'(a)=f'(b)=0$

证明: $\exists\,\xi\in(a,b)$, 使

$$\left|f''(\xi)\right|\geqslant\frac{4}{(b-a)^2}\left|f(b)-f(a)\right|$$

✎ **Solution** 将 $f\left(\dfrac{a+b}{2}\right)$ 分别在 a 和点 b 展开成泰勒公式, 并考虑到 $f'(a)=f'(b)=0$, 有

$$f\left(\frac{a+b}{2}\right)=f(a)+\frac{1}{2}f''(\xi_1)\left(\frac{b-a}{2}\right)^2,\quad a<\xi_1<\frac{a+b}{2} \tag{3.5}$$

$$f\left(\frac{a+b}{2}\right)=f(b)+\frac{1}{2}f''(\xi_2)\left(\frac{b-a}{2}\right)^2,\quad \frac{a+b}{2}<\xi_1<b \tag{3.6}$$

由 (3.6) − (3.5), 得

$$f(b)-f(a)+\frac{1}{8}\left[f''(\xi_2)-f''(\xi_1)\right](b-a)^2=0$$

故

$$\frac{4\left|f(b)-f(a)\right|}{(b-a)^2}\leqslant\frac{1}{2}\left(\left|f''(\xi_1)\right|+\left|f''(\xi_2)\right|\right)\leqslant f''(\xi)$$

其中 $f''(\xi)=\max\left\{\left|f''(\xi_1)\right|,\left|f''(\xi_2)\right|\right\}$

Exercise 3.16: 求极限

$$\lim_{x\to0}\frac{(1+\cos x)^x-2^x}{\sin^3 x}$$

✎ **Solution**

$$\begin{aligned}
\lim_{x\to0}\frac{(1+\cos x)^x-2^x}{\sin^3 x}&=\lim_{x\to0}\frac{2^x\left(\left(\frac{1+\cos x}{2}\right)^x-1\right)}{x^3}\\
&=\lim_{x\to0}2^x\cdot\lim_{x\to0}\frac{e^{x\ln\left(\frac{1+\cos x}{2}\right)}-1}{x^3}\\
&=\lim_{x\to0}\frac{\ln\left(\frac{1+\cos x}{2}\right)}{x^2}=\lim_{x\to0}\frac{\ln\left(1+\frac{\cos x-1}{2}\right)}{x^2}\\
&=\lim_{x\to0}\frac{\frac{\cos x-1}{2}}{x^2}=-\frac{1}{4}
\end{aligned}$$

Exercise 3.17: 求极限

$$\lim_{x\to+\infty}\left(\sqrt[3]{x^3+3x^2}-\sqrt[4]{x^4-2x^3}\right)$$

✎ **Solution**

$$\begin{aligned}
\lim_{x\to+\infty}\left(\sqrt[3]{x^3+3x^2}-\sqrt[4]{x^4-2x^3}\right)&=\lim_{x\to+\infty}x\left(\sqrt[3]{1+\frac{3}{x}}-\sqrt[4]{1-\frac{2}{x}}\right)\\
&=\lim_{x\to+\infty}x\left(\left(1+\frac{1}{3}\times\frac{3}{x}+o\left(\frac{1}{x}\right)\right)-\left(1-\frac{1}{4}\times\frac{2}{x}+o\left(\frac{1}{x}\right)\right)\right)\\
&=\lim_{x\to+\infty}\left(x\times\left(\frac{3}{2x}+o\left(\frac{1}{x}\right)\right)\right)\\
&=\frac{3}{2}
\end{aligned}$$

Exercise 3.18: 求极限

$$\lim_{x\to0}\frac{1}{x}\left(\frac{1}{x}-\cot x\right)$$

Solution

$$\begin{aligned}\lim_{x\to0}\frac{1}{x}\left(\frac{1}{x}-\cot x\right)&=\lim_{x\to0}\frac{\sin x-x\cos x}{x^2\sin x}\\&=\lim_{x\to0}\frac{\sin x-x\cos x}{x^3}\\&=\lim_{x\to0}\frac{\cos x-(\cos x-x\sin x)}{3x^2}\\&=\frac{1}{3}\end{aligned}$$

Solution

$$\begin{aligned}\lim_{x\to0}\frac{1}{x}\left(\frac{1}{x}-\cot x\right)&=\lim_{x\to0}\frac{\sin x-x\cos x}{x^2\sin x}\\&=\lim_{x\to0}\frac{\left(x-\frac{1}{6}x^3+o(x^3)\right)-x\left(1-\frac{1}{2}x^2+o(x^2)\right)}{x^3}\\&=\lim_{x\to0}\frac{\frac{1}{3}x^3+o(x^3)}{x^3}\\&=\frac{1}{3}\end{aligned}$$

Exercise 3.19: 求极限

$$\lim_{x\to1}\frac{x^x-x}{\ln x-x+1}$$

Solution

$$\begin{aligned}\lim_{x\to1}\frac{x^x-x}{\ln x-x+1}&=\lim_{x\to1}\frac{x\left(e^{(x-1)\ln x}-1\right)}{\ln x-x+1}\\&=\lim_{x\to1}x\times\lim_{x\to1}\frac{e^{(x-1)\ln x}-1}{\ln x-x+1}\\&=\lim_{x\to1}\frac{e^{(x-1)^2+o\left((x-1)^2\right)}-1}{-\frac{1}{2}(x-1)^2+o\left((x-1)^2\right)}\\&=\lim_{x\to1}\frac{(x-1)^2+o\left((x-1)^2\right)}{-\frac{1}{2}(x-1)^2+o\left((x-1)^2\right)}\\&=-2\end{aligned}$$

Exercise 3.20: 求极限

$$\lim_{n\to\infty}\left(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1}\right)$$

Solution

$$\begin{aligned}\lim_{n\to\infty}\left(\sqrt{n+1}-2\sqrt{n}+\sqrt{n-1}\right)&=\lim_{n\to\infty}\sqrt{n}\left(\sqrt{1+\frac{1}{n}}-2+\sqrt{1-\frac{1}{n}}\right)\\&=\lim_{n\to\infty}\sqrt{n}\left(\left(1+\frac{1}{2}\times\frac{1}{n}+o\left(\frac{1}{n}\right)\right)-2+\left(1-\frac{1}{2}\times\frac{1}{n}+o\left(\frac{1}{n}\right)\right)\right)\\&=\lim_{n\to\infty}\sqrt{n}o\left(\frac{1}{n}\right)\\&=0\end{aligned}$$

Exercise 3.21: 求极限

$$\lim_{x\to+\infty}\left((x-1)e^{\frac{x}{2}+\arctan x}-e^{\pi}x\right)$$

Solution

$$\begin{aligned}
&\lim_{x\to+\infty}\left((x-1)e^{\frac{x}{2}+\arctan x}-e^{\pi}x\right)\\
=&\lim_{x\to+\infty}\left((x-1)e^{\frac{\pi}{2}+\left(\frac{\pi}{2}-\arctan\left(\frac{1}{x}\right)\right)}-e^{\pi}x\right)\\
=&e^{\pi}\lim_{x\to+\infty}\left((x-1)e^{-\arctan\left(\frac{1}{x}\right)}-x\right)\\
=&e^{\pi}\lim_{x\to+\infty}\left((x-1)\left(1-\frac{1}{x}+o\left(\frac{1}{x}\right)\right)-x\right)\\
=&-2e^{\pi}
\end{aligned}$$

注:

$$\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2}$$

Exercise 3.22: 求极限

$$\lim_{n\to\infty}\left(n^2\sqrt{\frac{n}{n+1}}-\left(n^2+1\right)\sqrt{\frac{n+1}{n+2}}\right)$$

Solution

$$\begin{aligned}
&\lim_{n\to\infty}\left(n^2\sqrt{\frac{n}{n+1}}-\left(n^2+1\right)\sqrt{\frac{n+1}{n+2}}\right)\\
=&\lim_{n\to\infty}\frac{n^3\sqrt{n+2}-\left(n^2+1\right)(n+1)\sqrt{n}}{\sqrt{n(n+1)(n+2)}}\\
=&\lim_{n\to\infty}\frac{\left(n^{\frac{7}{2}}\sqrt{1+\frac{2}{n}}\right)-\left(n^{\frac{7}{2}}+n^{\frac{5}{2}}+n^{\frac{3}{2}}+\sqrt{n}\right)}{n^{\frac{3}{2}}}\\
=&\lim_{n\to\infty}\frac{\left(n^{\frac{7}{2}}\left(1+\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)\right)-\left(n^{\frac{7}{2}}+n^{\frac{5}{2}}+n^{\frac{3}{2}}+\sqrt{n}\right)}{n^{\frac{3}{2}}}\\
=&-\frac{3}{2}
\end{aligned}$$

Note:

$$\sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+o\left(x^2\right)$$

Exercise 3.23: 求极限

$$\lim_{x\to0}x^6\left(\frac{1}{\sin^8 x}-\frac{1}{x^8}\right)$$

Solution

$$\begin{aligned}
\lim_{x\to0} x^6\left(\frac{1}{\sin^8 x}-\frac{1}{x^8}\right) &= \lim_{x\to0}\frac{x^8-\sin^8 x}{x^2\sin^8 x}\\
&= \lim_{x\to0}\frac{(x^4-\sin^4 x)(x^4+\sin^4 x)}{x^{10}}\\
&=2\lim_{x\to0}\frac{x^4-\sin^4 x}{x^6}\\
&=2\lim_{x\to0}\frac{(x^2-\sin^2 x)(x^2+\sin^2 x)}{x^6}\\
&=4\lim_{x\to0}\frac{x^2-\sin^2 x}{x^4}\\
&=4\lim_{x\to0}\frac{(x-\sin x)(x+\sin x)}{x^4}\\
&=8\lim_{x\to0}\frac{x-\sin x}{x^3}\\
&=8\lim_{x\to0}\frac{1-\cos x}{3x^2}=\frac{8}{3}\lim_{x\to0}\frac{\frac{1}{2}x^2}{x^2}\\
&=\frac{4}{3}
\end{aligned}$$

Exercise 3.24: 求极限

$$\lim_{x\to1}\frac{\ln x-\sin(x-1)}{\sqrt[3]{2x-x^3}-1}$$

Solution

$$\begin{aligned}
\lim_{x\to1}\frac{\ln x-\sin(x-1)}{\sqrt[3]{2x-x^3}-1} &= \lim_{x\to1}\frac{\ln(1+(x-1))-\sin(x-1)}{\sqrt[3]{1+(2x-x^3-1)}-1}\\
&\xlongequal{t=x-1}\lim_{t\to0}\frac{\ln(1+t)-\sin t}{\sqrt[3]{1+(-t^3-3t^2-t)}-1}\\
&=\lim_{t\to0}\frac{\left(t-\frac{1}{2}t^2+o(t^2)\right)-\left(t-\frac{1}{6}t^3+o(t^3)\right)}{-\frac{1}{3}t+o(t)}\\
&=\lim_{t\to0}\frac{-\frac{1}{2}t^2+o(t^2)}{-\frac{1}{3}t+o(t)}=0
\end{aligned}$$

Note:

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+o(x^5),x\in(-\infty,+\infty)$$

$$\ln(1+x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}x^{n+1}=x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3),x\in(-1,1]$$

$$(1+x)^{\frac{1}{3}}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+o\left(x^4\right),x\in(-1,1)$$

Exercise 3.25: 求极限

$$\lim_{x\to0}\frac{\sin^2 x}{\sqrt{1+x\sin x}-\sqrt{\cos x}}$$

✎ **Solution**

$$\begin{aligned}
\lim_{x\to0}\frac{\sin^2x}{\sqrt{1+x\sin x}-\sqrt{\cos x}} &= \lim_{x\to0}\frac{\sin^2x\left(\sqrt{1+x\sin x}+\sqrt{\cos x}\right)}{\left(\sqrt{1+x\sin x}-\sqrt{\cos x}\right)\left(\sqrt{1+x\sin x}+\sqrt{\cos x}\right)}\\
&= 2\lim_{x\to0}\frac{\sin^2x}{1+x\sin x-\cos x}\\
&= 2\lim_{x\to0}\frac{x^2+o\left(x^2\right)}{1+x\left(x+o\left(x\right)\right)-\left(1-\frac{1}{2}x^2+o\left(x^2\right)\right)}\\
&= 2\lim_{x\to0}\frac{x^2+o\left(x^2\right)}{\frac{3}{2}x^2+o\left(x^2\right)}=\frac{4}{3}
\end{aligned}$$

Exercise 3.26: 求极限

$$\lim_{x\to0^+}\frac{x^{(\sin x)^x}-(\sin x)^{x^{\sin x}}}{x^3}$$

✎ **Solution**

$$\begin{aligned}
&\lim_{x\to0^+}\frac{x^{(\sin x)^x}-(\sin x)^{x^{\sin x}}}{x^3}\\
=&\lim_{x\to0^+}\frac{e^{(\sin x)^x\ln x}-e^{x^{\sin x}\ln\sin x}}{x^3}\\
=&\lim_{x\to0^+}\frac{e^{\left(x-\frac{1}{6}x^3+o\left(x^3\right)\right)^x\ln x}-e^{x^{\left(x-\frac{1}{6}x^3+o\left(x^3\right)\right)}\ln\left(x-\frac{1}{6}x^3+o\left(x^3\right)\right)}}{x^3}\\
=&\lim_{x\to0^+}\frac{e^{x^{\left(x-\frac{1}{6}x^3\right)}\ln\left(x-\frac{1}{6}x^3\right)}}{x}\times\lim_{x\to0^+}\frac{e^{\left(x-\frac{1}{6}x^3\right)\ln x-x^{\left(x-\frac{1}{6}x^3\right)}\ln\left(x-\frac{1}{6}x^3\right)}-1}{x^2}\\
=&1\times\lim_{x\to0^+}\frac{\left(x-\frac{1}{6}x^3\right)\ln x-x^{\left(x-\frac{1}{6}x^3\right)}\ln\left(x-\frac{1}{6}x^3\right)}{x^2}\\
=&\lim_{x\to0^+}\frac{\left[\left(x-\frac{1}{6}x^3\right)^x-x^{\left(x-\frac{1}{6}x^3\right)}\right]\ln x}{x^2}-\lim_{x\to0^+}\frac{x^{\left(x-\frac{1}{6}x^3\right)}\left(-\frac{1}{6}x^2\right)}{x^2}\\
=&0-\left(-\frac{1}{6}\right)=\frac{1}{6}
\end{aligned}$$

Exercise 3.27: 求极限

$$\lim_{x\to0}\frac{x^x-\sin^x x}{x^2\arctan x}$$

✎ **Solution**

$$\begin{aligned}
\lim_{x\to0}\frac{x^x-\sin^x x}{x^2\arctan x} &= \lim_{x\to0}\frac{x^x-\sin^x x}{x^3}\\
&= \lim_{x\to0}\frac{x^x\left(1-e^{x\ln\sin x-x\ln x}\right)}{x^3}\\
&= -\lim_{x\to0}x^x\times\lim_{x\to0}\frac{\ln\frac{\sin x}{x}}{x^2}\\
&= -\lim_{x\to0}\frac{\frac{\sin x-x}{x}}{x^2}\\
&= \frac{1}{6}
\end{aligned}$$

Exercise 3.28: 求极限

$$\lim_{x\to 0}\left(\frac{1}{\arctan^2 x}-\frac{1}{x^2}\right)$$

Solution

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\arctan^2 x}-\frac{1}{x^2}\right) &= \lim_{x\to 0}\frac{x^2-\arctan^2 x}{x^2\arctan^2 x}\\
&= \lim_{x\to 0}\frac{(x-\arctan x)(x+\arctan x)}{x^4}\\
&= \lim_{x\to 0}\frac{x-\arctan x}{x^3}\times\lim_{x\to 0}\frac{x+\arctan x}{x}\\
&= 2\lim_{x\to 0}\frac{1-\frac{1}{x^2+1}}{3x^2} = 2\lim_{x\to 0}\frac{\frac{x^2}{x^2+1}}{3x^2}\\
&= \frac{2}{3}
\end{aligned}$$

Exercise 3.29: 求极限

$$\lim_{x\to 0}\left(\frac{1}{\arctan^2 x}-\frac{1}{x^2}\right)$$

Solution

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{1}{\arctan^2 x}-\frac{1}{x^2}\right) &= \lim_{x\to 0}\frac{x^2-\arctan^2 x}{x^2\arctan^2 x}\\
&= \lim_{x\to 0}\frac{(x-\arctan x)(x+\arctan x)}{x^4}\\
&= \lim_{x\to 0}\frac{x-\arctan x}{x^3}\times\lim_{x\to 0}\frac{x+\arctan x}{x}\\
&= 2\lim_{x\to 0}\frac{x-\left(x-\frac{1}{3}x^3+o(x^3)\right)}{x^3}\\
&= \frac{2}{3}
\end{aligned}$$

Exercise 3.30: 求极限

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$$

Solution

$$\begin{aligned}
\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} &= \lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x}\\
&= e\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)-1}-1}{x}\\
&= e\lim_{x\to 0}\frac{e^{\frac{\ln(1+x)-x}{x}}-1}{x} = e\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}\\
&= e\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x} = -\frac{e}{2}\lim_{x\to 0}\frac{1}{1+x}\\
&= -\frac{e}{2}
\end{aligned}$$

Exercise 3.31: 求极限

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x}$$

Solution

$$\begin{aligned}
\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{x} &= \lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-e}{x}\\
&= e\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)-1}-1}{x}\\
&= e\lim_{x\to 0}\frac{e^{\frac{\ln(1+x)-x}{x}}-1}{x} = e\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}\\
&= e\lim_{x\to 0}\frac{(x-\frac{1}{2}x^2+o(x^2))-x}{x^2}\\
&= -\frac{e}{2}
\end{aligned}$$

Exercise 3.32: 求极限

$$\lim_{x\to\infty}\left(\frac{x^{x+1}}{(1+x)^x}-\frac{x}{e}\right)$$

Solution

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{x^{x+1}}{(1+x)^x}-\frac{x}{e}\right) &= \lim_{x\to\infty}\left(\frac{x}{\left(1+\frac{1}{x}\right)^x}-\frac{x}{e}\right)\\
&= \frac{1}{e}\lim_{x\to\infty}x\left[\frac{1}{\exp\left(x\ln\left(1+\frac{1}{x}\right)-1\right)}-1\right]\\
&= \frac{1}{e}\lim_{x\to\infty}x\left[\exp\left(1-x\left(\frac{1}{x}-\frac{1}{2x^2}+o\left(\frac{1}{x^2}\right)\right)\right)-1\right]\\
&= \frac{1}{e}\lim_{x\to\infty}x\left[\exp\left(\frac{1}{2x}+o\left(\frac{1}{x}\right)\right)-1\right]\\
&= \frac{1}{2e}
\end{aligned}$$

Exercise 3.33: 求极限

$$\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-(1+2x)^{\frac{1}{2x}}}{\sin x}$$

Solution

$$\begin{aligned}
\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-(1+2x)^{\frac{1}{2x}}}{\sin x} &= \lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}\left(1-\exp\left(\frac{1}{2x}\ln(1+2x)-\frac{1}{x}\ln(1+x)\right)\right)}{x}\\
&= -e\lim_{x\to 0}\frac{\frac{1}{2x}\ln(1+2x)-\frac{1}{x}\ln(1+x)}{x}\\
&= -e\lim_{x\to 0}\frac{\frac{1}{2x}\left(2x-2x^2+o(x^2)\right)-\frac{1}{x}\left(x-\frac{1}{2}x^2+o(x^2)\right)}{x}\\
&= -e\lim_{x\to 0}\frac{\frac{3}{2}x+o(x)}{x}\\
&= \frac{1}{2}e
\end{aligned}$$

Exercise 3.34: 求极限

$$\lim_{x\to\infty} n\left(e^2-\left(1+\frac{1}{n}\right)^{2n}\right)$$

Solution

$$\begin{aligned}
\lim_{x\to\infty} n\left(e^2-\left(1+\frac{1}{n}\right)^{2n}\right) &= \lim_{x\to\infty} n\left(e^2-e^{2n\ln(1+\frac{1}{n})}\right)\\
&= e^2\lim_{x\to\infty} n\left(1-e^{2n\ln(1+\frac{1}{n})-2}\right)\\
&= e^2\lim_{x\to\infty} n\left(2-2n\ln\left(1+\frac{1}{n}\right)\right)\\
&= e^2\lim_{x\to\infty} n\left(2-2n\left(\frac{1}{n}-\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right)\right)\\
&= e^2\lim_{x\to\infty} n\left(\frac{1}{n}+o\left(\frac{1}{n}\right)\right)\\
&= e^2
\end{aligned}$$

Exercise 3.35: 求极限

$$\lim_{x\to 0}\frac{e^{(1+x)^{\frac{1}{x}}}-(1+x)^{\frac{e}{x}}}{x^2}$$

Solution

$$\frac{1}{x}\ln(1+x)=\frac{1}{x}\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)\right)=1-\frac{1}{2}x+\frac{1}{3}x^2+o(x^2)$$

$$\frac{e}{x}\ln(1+x)=e-\frac{e}{2}x+\frac{e}{3}x^2+o(x^2)$$

$$e^{\frac{1}{x}\ln(1+x)}=e^{1-\frac{1}{2}x+\frac{1}{3}x^2+o(x^2)}=e\cdot e^{-\frac{1}{2}x+\frac{1}{3}x^2+o(x^2)}$$

$$e^{-\frac{1}{2}x+\frac{1}{3}x^2+o(x^2)}=1-\frac{1}{2}x+\frac{1}{3}x^2+o(x^2)+\frac{1}{2}\left(-\frac{1}{2}x+o(x)\right)^2=1-\frac{1}{2}x+\frac{11}{24}x^2+o(x^2)$$

$$\begin{aligned}
\lim_{x\to 0}\frac{e^{(1+x)^{\frac{1}{x}}}-(1+x)^{\frac{e}{x}}}{x^2} &= \lim_{x\to 0}\frac{(1+x)^{\frac{e}{x}}\left(e^{e^{\frac{1}{x}\ln(1+x)}-\frac{e}{x}\ln(1+x)}-1\right)}{x^2}\\
&= e^e\lim_{x\to 0}\frac{e^{\frac{1}{x}\ln(1+x)}-\frac{e}{x}\ln(1+x)}{x^2}\\
&= e^{e+1}\lim_{x\to 0}\frac{\left(1-\frac{1}{2}x+\frac{11}{24}x^2+o(x^2)\right)-\left(1-\frac{1}{2}x+\frac{1}{3}x^2+o(x^2)\right)}{x^2}\\
&= \frac{1}{8}e^{e+1}
\end{aligned}$$

Exercise 3.36: 求极限

$$\lim_{x\to 0}\frac{\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}}-e^2}{x^2}$$

Solution

$$\begin{cases}\tan x = x+\dfrac{1}{3}x^3+o(x^3)\\ \dfrac{1}{1-x}=1+x+x^2+x^3+o(x^3)\end{cases}\implies \frac{2\tan x}{1-\tan x}=2x+2x^2+\frac{8}{3}x^3+o\left(x^3\right)$$

又因为

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)$$

所以

$$\Longrightarrow \ln\left(1 + \frac{2\tan x}{1-\tan x}\right) = 2x + \frac{4}{3}x^3 + o\left(x^3\right)$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}} - e^2}{x^2} &= \lim_{x\to 0}\frac{e^{\frac{1}{x}\ln\left(\frac{1+\tan x}{1-\tan x}\right)} - e^2}{x^2} \\
&= \lim_{x\to 0}\frac{e^2\left(e^{\frac{1}{x}\ln\left(1+\frac{2\tan x}{1-\tan x}\right)-2} - 1\right)}{x^2} \\
&= e^2\lim_{x\to 0}\frac{e^{\frac{1}{x}\left(2x+\frac{4}{3}x^3+o\left(x^3\right)\right)-2} - 1}{x^2} = e^2\lim_{x\to 0}\frac{e^{\frac{4}{3}x^2+o\left(x^2\right)} - 1}{x^2} \\
&= e^2\lim_{x\to 0}\frac{\frac{4}{3}x^2 + o\left(x^2\right)}{x^2} = \frac{4e^2}{3}
\end{aligned}$$

Exercise 3.37: 求极限

$$\lim_{x\to+\infty} x^{\frac{3}{2}}\left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right)$$

Solution

$$\begin{aligned}
&\lim_{x\to+\infty} x^{\frac{3}{2}}\left(\sqrt{x+1} + \sqrt{x-1} - 2\sqrt{x}\right) \\
=& \lim_{x\to+\infty} x^2\left(\sqrt{1+\frac{1}{x}} + \sqrt{1-\frac{1}{x}} - 2\right) \\
=& \lim_{t\to 0^+}\frac{\sqrt{1+t} + \sqrt{1-t} - 2}{t^2} \\
=& \lim_{t\to 0^+}\frac{\left(1+\frac{1}{2}t-\frac{1}{8}t^2+o\left(t^2\right)\right) + \left(1-\frac{1}{2}t-\frac{1}{8}t^2+o\left(t^2\right)\right) - 2}{t^2} \\
=& \lim_{t\to 0^+}\frac{-\frac{1}{4}t^2 + o\left(t^2\right)}{t^2} = -\frac{1}{4}
\end{aligned}$$

Note:

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + o\left(x^2\right)$$

Exercise 3.38: 求极限

$$\lim_{x\to 0}\frac{\tan\tan x - \sin\sin x}{\tan x - \sin x}$$

Solution

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan(\tan x) - \sin(\sin x)}{\tan x - \sin x} &= \lim_{x\to 0}\frac{\tan(\tan x) - \sin(\tan x)}{\tan x - \sin x} + \lim_{x\to 0}\frac{\sin(\tan x) - \sin(\sin x)}{\tan x - \sin x} \\
&= \lim_{x\to 0}\frac{\frac{1}{2}\tan^3 x + o(\tan^3 x)}{\frac{1}{2}x^3 + o(x^3)} + \lim_{x\to 0}\frac{2\cos\frac{\tan x+\sin x}{2}\sin\frac{\tan x-\sin x}{2}}{\tan x - \sin x} \\
&= 1 + 1 = 2
\end{aligned}$$

Solution

$$
\begin{aligned}
\lim_{x\to0}\frac{\tan\tan x-\sin\sin x}{\tan x-\sin x} &= \lim_{x\to0}\frac{\tan\tan x-\tan\sin x+\tan\sin x-\sin\sin x}{\tan x-\sin x}\\
&= \lim_{x\to0}\frac{\tan\tan x-\tan\sin x}{\tan x-\sin x}+\lim_{x\to0}\frac{\tan\sin x-\sin\sin x}{\tan x-\sin x}\\
&= \lim_{x\to0}\frac{\tan\tan x-\tan\sin x}{\tan x-\sin x}+\lim_{x\to0}\frac{\tan\sin x\,(1-\cos\sin x)}{\tan x\,(1-\cos x)}\\
&= \underset{\varepsilon\in(\sin x,\tan x)}{(\tan\varepsilon)'}+\lim_{x\to0}\frac{x\times\frac{1}{2}x^2}{x\times\frac{1}{2}x^2}=\frac{1}{\cos^2\varepsilon}+1=1+1=2
\end{aligned}
$$

Exercise 3.39: 求极限

Solution

3.4 函数的单调性与曲线的凹凸性

3.4.1 曲线的凹凸性与拐点

Definition 3.1

设函数 f 在区间 I 上定义. 若对每一对点 $x_1,\ x_2\in I,\ x_1\neq x_2$ 和每个 $\lambda\in(0,1)$ 成立不等式

$$f(\lambda x_1+(1-\lambda)x_2)\geqslant\lambda f(x_1)+(1-\lambda)f(x_2) \tag{3.7}$$

则称 f 为区间 I 上的下凹函数

Theorem 3.9

如 f 为区间 I 上的二阶可微下凸函数, 则对任何 $x_1,x_2,\cdots,x_n\in I$ 与满足条件 $\lambda_1+\lambda_2+\cdots+\lambda_n=1$ 的 n 个正数 $\lambda_1,\lambda_2,\cdots,\lambda_n$ 成立不等式

$$\lambda_1 f(x_1)+\lambda_2 f(x_2)+\cdots+\lambda_n f(x_n)\geqslant f(\lambda_1x_1+\lambda_2x_2+\cdots+\lambda_nx_n)$$

又若 f 严格下凸, 则上述不等式成立等号的充分必要条件是

$$x_1=x_2=\cdots=x_n$$

3.5 函数的极值与最大值最小值

Exercise 3.40: 求

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{n+1}$$

Solution 设

$$f(x)=x^{n+1}-x(x-1)(x-2)\cdots(x-n)$$

由插值公式我们有

$$f(x)=\sum_{k=0}^{n}\left(\prod_{j\neq k}\frac{(x-j)}{k-j}\right)f(k)$$

比较两边 x^n 系数:

$$1+2+\cdots+n=\sum_{k=0}^{n}\left(\prod_{j\neq k}\frac{k^{n+1}}{k-j}\right)$$

化简得

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{n+1}=\frac{n(n+1)!}{2}$$

3.6 渐近线

Definition 3.2 水平渐近线

曲线 $y=f(x)$ 上点 $\big(x,f(x)\big)$ 与直线 $y=c$ 的距离为 $\big|f(x)-c\big|$, 当 $\lim\limits_{x\to+\infty}f(x)=c$, $\lim\limits_{x\to-\infty}f(x)=c$, $\lim\limits_{x\to\infty}f(x)=c$ 三种情况之一成立, 直线 $y=c$ 为曲线 $y=f(x)$ 的水平渐近线

Note: *一条曲线最多两条水平渐近线*

Definition 3.3 铅直渐近线

若 $\lim\limits_{x\to x_0}f(x)=\infty\left(\text{或}\lim\limits_{x\to x_0^-}f(x)=\infty\text{ 或 }\lim\limits_{x\to x_0^+}f(x)=\infty\right)$, 则直线 $x=x_0$ 为曲线 $y=f(x)$ 的铅直渐近线

Note: *当 $x=c$ 为函数 $f(x)$ 的无穷间断点时, $x=x_0$ 为曲线 $y=f(x)$ 的铅直渐近线*

Definition 3.4 斜渐近线

若 $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=k\neq 0$ 且 $\lim\limits_{x\to\infty}\left[f(x)-kx\right]=b$, 则直线 $y=kx+b$ 为曲线 $y=f(x)$ 的斜渐近线

Note: 有时需要分 $x\to-\infty$ 或 $x\to+\infty$ 加以讨论. 一条曲线最多两条斜渐近线

Definition 3.5 极坐标渐近线

对于以极坐标表示的曲线 $r=f(\theta)$, 其渐近线为 $r\sin(\theta_0-\theta)=p$, 其中 $\lim\limits_{\theta\to\theta_0}f(\theta)=\infty$, $\lim\limits_{\theta\to\theta_0}r(\theta_0-\theta)$.

3.7 曲率

3.8 方程的近似解

3.9 不等式

Exercise 3.41: 设 $x>0$, 证明 $\sqrt{x+1}-\sqrt{x}=\dfrac{1}{2\sqrt{x+\theta(x)}}$, 其中 $\dfrac{1}{4}<\theta(x)<\dfrac{1}{2}$

Solution(by 蓝兔兔): 由题易得

$$\begin{aligned}\theta(x)&=\frac{1}{4(\sqrt{1+x}-\sqrt{x})^2}-x\\&=\frac{1}{4}\left(2\sqrt{x^2+x}-2x+1\right)\end{aligned}$$

令 $g(x)=2\sqrt{x^2+x}-2x+1$, 则有 $g(0)=1$ 因为

$$g'(x)=\frac{2x+1}{\sqrt{x^2+x}}-2=\frac{(\sqrt{1+x}-\sqrt{x})^2}{\sqrt{x^2+x}}\geqslant 0$$

由此可知 $g(x)\uparrow$

又 $\lim\limits_{x\to 0}g(x)=g(0)=1$ 以及 $\lim\limits_{x\to+\infty}g(x)=1+2\lim\limits_{x\to+\infty}(\sqrt{x^2+x}-x)=2$

所以 $\theta(x)=\dfrac{1}{4}g(x)\in\left(\dfrac{1}{4},\dfrac{1}{2}\right)$

Solution(by Hilbert): 由题

$$\sqrt{x+1}-\sqrt{x}=\frac{1}{\sqrt{1+x}+\sqrt{x}}=\frac{1}{2\sqrt{x+\theta(x)}}\Longleftrightarrow\sqrt{1+x}+\sqrt{x}=2\sqrt{x+\theta(x)}$$

故

$$\begin{aligned}
\theta(x) &= \left(\frac{\sqrt{1+x}+\sqrt{x}}{2}\right)^2 - x = \left(\frac{\sqrt{1+x}+\sqrt{x}}{2}\right)^2 - \left(\sqrt{x}\right)^2 \\
&= \frac{3\sqrt{x}+\sqrt{1+x}}{2} \cdot \frac{\sqrt{1+x}-\sqrt{x}}{2} \\
&= \frac{\sqrt{x}+\sqrt{1+x}+2\sqrt{x}}{4\left(\sqrt{x}+\sqrt{1+x}\right)} \\
&= \frac{1}{4} + \frac{1}{2} \cdot \frac{\sqrt{x}}{\sqrt{x}+\sqrt{1+x}} \\
&= \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{1+\sqrt{1+\frac{1}{x}}}
\end{aligned}$$

显然 $\theta(x)\uparrow$, **且** $\lim\limits_{x\to 0^+}\theta(x) = \dfrac{1}{4}$ **以及** $\lim\limits_{x\to +\infty}\theta(x) = \dfrac{1}{2}$

故 $\theta(x) \in \left(\dfrac{1}{4}, \dfrac{1}{2}\right)$

第 4 章 不定积分

4.1 不定积分的概念与性质

表 4.1: 部分初等函数积分表

$$\int \sec x \, \mathrm{d}x = \ln|\sec x + \tan x| + C \qquad \int \csc x \, \mathrm{d}x = \ln|\csc x - \cot x| + C$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C \qquad \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan \frac{x}{a} + C$$

$$\int \frac{\mathrm{d}x}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C \qquad \int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln \left| x + \sqrt{x^2 \pm a^2} \right| + C$$

4.2 不定积分的计算

Exercise 4.1: 求不定积分

$$\int e^{x \sin + cosx} \left(\frac{x^4 \cos^3 x - x \sin x + \cos x}{x^2 \cos^2 x} \right) \mathrm{d}x$$

Solution 注意到

$$\frac{\mathrm{d}}{\mathrm{d}x} \left(e^{x \sin + \cos x} \right) = x \cos x e^{x \sin + \cos x}$$

以及

$$\int \frac{\cos x - x \sin x}{x^2 \cos^2 x} \mathrm{d}x = -\frac{1}{x \cos x}$$

故

$$\begin{aligned}
I &= \int e^{x \sin + \cos x} \left(\frac{x^4 \cos^3 x - x \sin x + \cos x}{x^2 \cos^2 x} \right) \mathrm{d}x \\
&= \int e^{x \sin + \cos x} x^2 \cos x \, \mathrm{d}x + \int e^{x \sin + \cos x} \left(\frac{\cos x - x \sin x}{x^2 \cos^2 x} \right) \mathrm{d}x \\
&= \int x \, \mathrm{d} \left(e^{x \sin + \cos x} \right) + \int e^{x \sin + \cos x} \mathrm{d} \left(-\frac{1}{x \cos x} \right) \\
&= x e^{x \sin + \cos x} - \int e^{x \sin + \cos x} \mathrm{d}x \\
&\quad - \frac{1}{x \cos x} e^{x \sin + \cos x} + \int \frac{1}{x \cos x} x \cos x e^{x \sin + \cos x} \mathrm{d}x \\
&= x e^{x \sin + \cos x} - \frac{e^{x \sin + \cos x}}{x \cos x} + C
\end{aligned}$$

Exercise 4.2: 求不定积分

$$I=\int\frac{f'(x)+f(x)g'(x)}{f(x)\left[c+f(x)e^{g(x)}\right]}\,\mathrm{d}x$$

Solution 注意到

$$\left(c+f(x)e^{g(x)}\right)'=e^{g(x)}\left[f'(x)+f(x)g'(x)\right]$$

故

$$\begin{aligned}
I&=\int\frac{e^{g(x)}\left[f'(x)+f(x)g'(x)\right]}{f(x)e^{g(x)}\left[c+f(x)e^{g(x)}\right]}\,\mathrm{d}x\\
&=\int\frac{\mathrm{d}\left[c+f(x)e^{g(x)}\right]}{f(x)e^{g(x)}\left[c+f(x)e^{g(x)}\right]}\\
&=\frac{1}{c}\int\left[\frac{1}{f(x)e^{g(x)}}-\frac{1}{c+f(x)e^{g(x)}}\right]\mathrm{d}\left[c+f(x)e^{g(x)}\right]\\
&=\frac{1}{c}\left\{\int\frac{\mathrm{d}\left[c+f(x)e^{g(x)}\right]}{f(x)e^{g(x)}}-\int\frac{\mathrm{d}\left[c+f(x)e^{g(x)}\right]}{c+f(x)e^{g(x)}}\right\}\\
&=\frac{1}{c}\left[\ln\left|f(x)e^{g(x)}\right|-\ln\left|c+f(x)e^{g(x)}\right|\right]+C
\end{aligned}$$

Exercise 4.3: 求不定积分

$$\int\left(\frac{\arctan x}{x-\arctan x}\right)^2\mathrm{d}x$$

Solution

$$\begin{aligned}
\int\left(\frac{\arctan x}{x-\arctan x}\right)^2\mathrm{d}x&\xlongequal{t=\arctan x}\int\frac{t^2\sec^2 t}{(\tan t-t)^2}\,\mathrm{d}t\\
&=\int\frac{t^2}{(\sin t-t\cos t)^2}\,\mathrm{d}t=\int\frac{t}{\sin t}\,\mathrm{d}\left(\frac{1}{\sin t-t\cos t}\right)\\
&=\frac{t}{\sin t(\sin t-t\cos t)}-\int\frac{1}{\sin t-t\cos t}\times\frac{\sin t-t\cos t}{\sin^2 t}\,\mathrm{d}t\\
&=\frac{t}{\sin t(\sin t-t\cos t)}+\cot t+C\\
&=\frac{x\arctan x}{x-\arctan x}+C
\end{aligned}$$

Exercise 4.4: 求不定积分

Solution

$$\begin{aligned}
\int\frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}&=2\int\frac{\mathrm{d}\sqrt{x-a}}{\sqrt{b-x}}\\
&=2\int\frac{\mathrm{d}\sqrt{x-a}}{\sqrt{(b-a)-\left(\sqrt{x-a}\right)^2}}\\
&=2\arcsin\sqrt{\frac{x-a}{b-a}}+C
\end{aligned}$$

Exercise 4.5: 求不定积分

$$\int \frac{1}{\sin^6 x + \cos^6 x} \mathrm{d}x$$

Solution 注意到

$$\begin{aligned}
\frac{1}{\sin^6 x + \cos^6 x} &= \frac{\sin^2 x + \cos^2 x}{\sin^4 x(1 - \cos^2 x) + \cos^4 x(1 - \sin^2 x)} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x - \sin^4 x \cos^2 x - \cos^4 x \sin^2 x} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x(\sin^2 x + \cos^2 x)} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x}
\end{aligned}$$

故

$$\begin{aligned}
\int \frac{1}{\sin^6 x + \cos^6 x} \mathrm{d}x &= \int \frac{\sin^2 x + \cos^2 x}{\sin^4 x + \cos^4 x - \sin^2 x \cos^2 x} \mathrm{d}x \\
&= \int \frac{\tan^2 x + 1}{\tan^4 x - \tan^2 x + 1} \mathrm{d}(\tan x) \\
&\xlongequal{t=\tan x} \int \frac{t^2 + 1}{t^4 - t^2 + 1} \mathrm{d}t \\
&= \int \frac{1}{\left(t - \frac{1}{t}\right)^2 + 1} \mathrm{d}\left(t - \frac{1}{t}\right) \\
&= \arctan\left(t - \frac{1}{t}\right) + C \\
&= -\arctan\left(2\cot x\right) + C
\end{aligned}$$

Exercise 4.6: 求不定积分

$$\int \frac{1}{(x^2 + x + 1)^2} \mathrm{d}x$$

Solution

$$\begin{aligned}
\int \frac{1}{(x^2 + x + 1)^2} \mathrm{d}x &= \frac{4}{3} \int \frac{\overbrace{3/4 + (x + 1/2)^2}^{x^2+x+1} - (x + 1/2)^2}{(x^2 + x + 1)^2} \mathrm{d}x \\
&= \frac{4}{3} \int \frac{1}{\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}} \mathrm{d}x + \frac{2}{3} \int \left(x + \frac{1}{2}\right) \mathrm{d}\left(\frac{1}{x^2 + x + 1}\right) \\
&= \frac{8}{3\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} + \frac{2}{3} \frac{x + \frac{1}{2}}{x^2 + x + 1} - \frac{2}{3} \int \frac{1}{x^2 + x + 1} \mathrm{d}x \\
&= \frac{4}{3\sqrt{3}} \arctan \frac{2x + 1}{\sqrt{3}} + \frac{1}{3} \frac{2x + 1}{x^2 + x + 1} + C
\end{aligned}$$

Exercise 4.7: 求不定积分

$$I = \int \frac{16x + 11}{(x^2 + 2x + 2)^2} dx$$

✎ **Solution**

$$
\begin{aligned}
I &= 8\int \frac{2x+2}{\left(x^2+2x+2\right)^2}dx - \int \frac{5}{\left(x^2+2x+2\right)^2}dx \\
&= 8\int \frac{1}{\left(x^2+2x+2\right)^2}d\left(x^2+2x+2\right) - 5\int \frac{1}{\left(x^2+2x+2\right)^2}dx \\
&= -\frac{8}{x^2+2x+2} - 5\int \frac{1+(x+1)^2-(x+1)^2}{\left(x^2+2x+2\right)^2}dx \\
&= -\frac{8}{x^2+2x+2} - 5\int \frac{1}{(x+1)^2+1}d(x+1) - \frac{5}{2}\int (x+1)d\left(\frac{1}{x^2+2x+2}\right) \\
&= -\frac{8}{x^2+2x+2} - 5\arctan(x+1) - \frac{5x+5}{2\left(x^2+2x+2\right)} + \frac{5}{2}\int \frac{1}{(x+1)^2+1}d(x+1) \\
&= -\frac{5x+21}{2\left(x^2+2x+2\right)} - \frac{5}{2}\arctan(x+1) + C
\end{aligned}
$$

➼ **Exercise 4.8:** 求不定积分

$$\int \frac{16x+11}{\left(x^2+2x+2\right)^2}dx$$

✎ **Solution**

$$
\begin{aligned}
\int \frac{16x+11}{\left(x^2+2x+2\right)^2}dx &= 8\int \frac{2x+2}{\left(x^2+2x+2\right)^2}dx - \int \frac{5}{\left(x^2+2x+2\right)^2}dx \\
&= 8\int \frac{1}{\left(x^2+2x+2\right)^2}d\left(x^2+2x+2\right) - 5\underbrace{\int \frac{1}{\left((x+1)^2+1\right)^2}dx}_{x+1=\tan t} \\
&= -\frac{8}{x^2+2x+2} - 5\int \frac{\sec^2 t}{\left(\tan^2 t+1\right)^2}dt \\
&= -\frac{8}{x^2+2x+2} - 5\int \cos^2 t dt \\
&= -\frac{8}{x^2+2x+2} - 5\int \frac{1+\cos 2t}{2}dt \\
&= -\frac{8}{x^2+2x+2} - \frac{5}{2}\int dt - \frac{5}{4}\int \cos 2t d(2t) \\
&= -\frac{8}{x^2+2x+2} - \frac{5}{2}t - \frac{5}{4}\sin 2t + C \\
&= -\frac{5x+21}{2\left(x^2+2x+2\right)} - \frac{5}{2}\arctan(x+1) + C
\end{aligned}
$$

➼ **Exercise 4.9:** 求不定积分

$$\int \frac{1}{x+\sqrt{1-x^2}}\mathrm{d}x$$

Solution

$$
\begin{aligned}
\int \frac{1}{x+\sqrt{1-x^2}}\mathrm{d}x &= \frac{1}{2}\int \frac{1+\frac{x}{\sqrt{1-x^2}}+1-\frac{x}{\sqrt{1-x^2}}}{x+\sqrt{1-x^2}}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x + \frac{1}{2}\int \frac{1}{x+\sqrt{1-x^2}}\mathrm{d}(x+\sqrt{1-x^2}) \\
&= \frac{1}{2}\arcsin x + \frac{1}{2}\ln|x+\sqrt{1-x^2}| + C
\end{aligned}
$$

Exercise 4.10: 求不定积分

$$\int \frac{\sin x - x\cos x}{x(x-\sin x)}\,\mathrm{d}x$$

Solution

$$
\begin{aligned}
\int \frac{\sin x - x\cos x}{x(x-\sin x)}\,\mathrm{d}x &= \int \frac{-(x-\sin x)+(x-x\cos x)}{x(x-\sin x)}\,\mathrm{d}x \\
&= \int \frac{-1}{x}\,\mathrm{d}x + \int \frac{1-\cos x}{x-\sin x}\,\mathrm{d}x \\
&= \ln\left|\frac{x-\sin x}{x}\right| + C
\end{aligned}
$$

Exercise 4.11: 求不定积分

$$\int \sqrt{\tan x}dx$$

Solution

$$
\begin{aligned}
\int \sqrt{\tan x}dx \xlongequal{\sqrt{\tan x}=t} 2\int \frac{t^2}{1+t^4}dt &= \int \frac{1+t^2}{1+t^4}dt - \int \frac{1-t^2}{1+t^4}dt \\
&= \int \frac{1}{\left(t-\frac{1}{t}\right)^2+2}d\left(t-\frac{1}{t}\right) - \int \frac{1}{\left(t+\frac{1}{t}\right)^2-2}d\left(t+\frac{1}{t}\right) \\
&= \frac{\sqrt{2}}{2}\arctan\left(\frac{t^2-1}{\sqrt{2}t}\right) + \frac{\sqrt{2}}{4}\ln\left|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right| + c \\
&= \frac{\sqrt{2}}{2}\arctan\left(\frac{\tan x-1}{2\sqrt{\tan x}}\right) + \frac{\sqrt{2}}{4}\ln\left|\frac{\tan x+\sqrt{2\tan x}+1}{\tan x-\sqrt{2\tan x}+1}\right| + c
\end{aligned}
$$

Exercise 4.12: 求不定积分

$$\int \frac{x^2}{\sqrt{1+x+x^2}}\,\mathrm{d}x$$

Solution

$$
\begin{aligned}
\int \frac{x^2}{\sqrt{1+x+x^2}}\,\mathrm{d}x &\xlongequal{x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan t} \int \left(\frac{\sqrt{3}}{2}\tan t-\frac{1}{2}\right)^2 \sec t\,\mathrm{d}t \\
&= \frac{3}{4}\int \tan^2 t\sec t\,\mathrm{d}t-\frac{\sqrt{3}}{2}\int \tan t\sec t\,\mathrm{d}t+\frac{1}{4}\int \sec t\,\mathrm{d}t \\
&= \frac{3}{4}\int \sec t(\sec^2 t-1)\,\mathrm{d}t-\frac{\sqrt{3}}{2}\sec t+\frac{1}{4}\ln|\sec t+\tan t| \\
&= \frac{3}{4}\int \sec^3 t\,\mathrm{d}t-\frac{3}{4}\int \sec t\,\mathrm{d}t-\frac{\sqrt{3}}{2}\sec t+\frac{1}{4}\ln|\sec t+\tan t| \\
&= \frac{3}{4}\sec t\tan t-\frac{3}{4}\int \tan^2 t\sec t\,\mathrm{d}t-\frac{3}{4}\int \sec t\,\mathrm{d}t-\frac{\sqrt{3}}{2}\sec t+\frac{1}{4}\ln|\sec t+\tan t| \\
&= \frac{3}{8}\sec t\tan t-\frac{\sqrt{3}}{2}\sec t-\frac{1}{8}\ln|\sec t+\tan t|+C \\
&= \frac{1}{4}(2x-3)\sqrt{x^2+x+1}-\frac{1}{8}\ln|2\sqrt{x^2+x+1}+2x+1|+C
\end{aligned}
$$

Exercise 4.13: 求不定积分

$$
\int \frac{1}{1+x^4}\,\mathrm{d}x
$$

Solution

$$
\begin{aligned}
I &= \int \frac{1}{1+x^4}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{x^2+1}{1+x^4}\mathrm{d}x-\frac{1}{2}\int \frac{x^2-1}{1+x^4}\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{x}\right)^2+2}\mathrm{d}\left(x-\frac{1}{x}\right)-\frac{1}{2}\int \frac{1}{\left(x+\frac{1}{x}\right)^2-2}\mathrm{d}\left(x+\frac{1}{x}\right) \\
&= \frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}}+\frac{1}{4\sqrt{2}}\ln\left|\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right|+C
\end{aligned}
$$

Exercise 4.14: 求不定积分

$$
\int \sin x\sin 2x\sin 3x\,\mathrm{d}x
$$

Solution

$$
\begin{aligned}
\int \sin x\sin 2x\sin 3x\,\mathrm{d}x &= \frac{1}{2}\int (\cos x-\cos 3x)\sin 3x\,\mathrm{d}x \\
&= \frac{1}{2}\int \cos x\sin 3x\,\mathrm{d}x-\frac{1}{2}\int \cos 3x\sin 3x\,\mathrm{d}x \\
&= \frac{1}{4}\int (\sin 2x+\sin 4x)\,\mathrm{d}x-\frac{1}{4}\int \sin 6x\,\mathrm{d}x \\
&= -\frac{1}{8}\cos 2x-\frac{1}{16}\cos 4x+\frac{1}{24}\cos 6x+C
\end{aligned}
$$

Note:

$$
\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]
$$

Exercise 4.15: 求不定积分

$$\int \frac{dx}{(9x+7)\sqrt{x-2}}$$

Solution

$$\begin{aligned}
\int \frac{dx}{(9x+7)\sqrt{x-2}} &= \int \frac{dx}{\left((9+\sqrt{x-2})^2\right)\sqrt{x-2}} \\
&= \int \frac{2d(\sqrt{x-2})}{9+\left(\sqrt{x-2}\right)^2} \\
&= \frac{2}{3}\arctan\frac{\sqrt{x-2}}{3}+c
\end{aligned}$$

Exercise 4.16: 求不定积分

$$\int \frac{1}{(1+x)\sqrt{x^2+x+1}}dx$$

Solution

$$\begin{aligned}
\int \frac{1}{(1+x)\sqrt{x^2+x+1}}dx &= \int \frac{1}{(1+x)\sqrt{(1+x)^2-(x+1)+1}}dx \\
&= \int \frac{dx}{(1+x)^2\sqrt{1-\frac{1}{1+x}+\frac{1}{(1+x)^2}}} \\
&= -\int \frac{d\left(\frac{1}{1+x}\right)}{\sqrt{1-\frac{1}{1+x}+\frac{1}{(1+x)^2}}} \\
&= -\int \frac{d\left(\frac{1}{1+x}-\frac{1}{2}\right)}{\sqrt{\left(\frac{1}{1+x}-\frac{1}{2}\right)^2+\frac{3}{4}}} \\
&= \ln(1+x)-\ln(2\sqrt{x^2+x+1}-x+1)+C
\end{aligned}$$

Exercise 4.17: 求不定积分

$$\int \frac{2^x\times 3^x}{9^x-4^x}dx$$

Solution

$$\begin{aligned}
\int \frac{2^x\times 3^x}{9^x-4^x}dx &= \int \frac{\frac{2^x}{3^x}}{1-\left(\frac{2^x}{3^x}\right)^2}dx \xlongequal{d\left(\frac{2^x}{3^x}\right)=\frac{2^x}{3^x}\ln\frac{2}{3}dx} \frac{1}{\ln\frac{2}{3}}\int \frac{1}{1-\left(\frac{2^x}{3^x}\right)^2}d\left(\frac{2^x}{3^x}\right) \\
&= \frac{1}{\ln\frac{2}{3}}\int \frac{1}{\left(1-\frac{2^x}{3^x}\right)\left(1+\frac{2^x}{3^x}\right)}d\left(\frac{2^x}{3^x}\right) \\
&= \frac{1}{2\ln\frac{2}{3}}\left[\int \frac{1}{1-\frac{2^x}{3^x}}d\left(\frac{2^x}{3^x}\right)-\int \frac{1}{1+\frac{2^x}{3^x}}d\left(\frac{2^x}{3^x}\right)\right] \\
&= \frac{1}{2\ln\frac{2}{3}}\left(\ln\left|1-\frac{2^x}{3^x}\right|-\ln\left|1+\frac{2^x}{3^x}\right|\right)+c \\
&= \frac{1}{2\ln\frac{2}{3}}\ln\left|\frac{1-\frac{2^x}{3^x}}{1+\frac{2^x}{3^x}}\right|+c = \frac{1}{2\ln\frac{2}{3}}\ln\left|\frac{3^x-2^x}{3^x+2^x}\right|+c
\end{aligned}$$

Exercise 4.18: 求不定积分

$$\int \frac{x^2}{(x\cos x-\sin x)(x\sin x+\cos x)}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
I &= \int \frac{x^2}{(x\cos x-\sin x)(x\sin x+\cos x)}\,\mathrm{d}x\\
&= \int \frac{x\cos x}{x\sin x+\cos x}\,\mathrm{d}x + \int \frac{x\sin x}{x\cos x-\sin x}\,\mathrm{d}x\\
&= \int \frac{1}{x\sin x+\cos x}\,\mathrm{d}(x\sin x+\cos x) - \int \frac{1}{x\cos x-\sin x}\,\mathrm{d}(x\cos x-\sin x)\\
&= \ln\left|\frac{x\sin x+\cos x}{x\cos x-\sin x}\right| + C
\end{aligned}$$

Exercise 4.19: 求不定积分

$$\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}}$$

Solution

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}} &= \int \frac{\sqrt[3]{x+1}}{\sqrt[3]{(x+1)^3(x-1)^4}}\mathrm{d}x = \int \frac{1}{x^2-1}\sqrt[3]{\frac{x+1}{x-1}}\mathrm{d}x\\
&\xlongequal[x=\frac{u^3+1}{u^3-1}]{\sqrt[3]{\frac{x+1}{x-1}}} \int \frac{u}{\left(\dfrac{u^3+1}{u^3-1}\right)^2-1}\cdot\frac{-6u^2}{(u^3-1)^2}\mathrm{d}u\\
&= -\frac{3}{2}\int \mathrm{d}u = -\frac{3}{2}u + C\\
&= -\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}} + C
\end{aligned}$$

Exercise 4.20: 求不定积分

$$\int \frac{1}{\sin x+\cos x}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int \frac{1}{\sin x+\cos x}\,\mathrm{d}x &= \int \frac{\cos x-\sin x}{\cos^2 x-\sin^2 x}\,\mathrm{d}x\\
&= \int \frac{1}{1-2\sin^2 x}\,\mathrm{d}(\sin x) + \int \frac{1}{2\cos^2 x-1}\,\mathrm{d}(\cos x)\\
&= -\frac{1}{\sqrt{2}}\int \frac{1}{2\sin^2 x-1}\,\mathrm{d}\big(\sqrt{2}\sin x\big) + \frac{1}{\sqrt{2}}\int \frac{1}{2\cos^2 x-1}\,\mathrm{d}\big(\sqrt{2}\cos x\big)\\
&= -\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right| + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}\cos x-1}{\sqrt{2}\cos x+1}\right| + C
\end{aligned}$$

✎ **Solution**

$$\begin{aligned}\int \frac{1}{\sin x+\cos x}\,\mathrm{d}x &= \int \frac{1}{\cos^2\left(\frac{1}{2}x\right)-\sin^2\left(\frac{1}{2}x\right)+2\cos\left(\frac{1}{2}x\right)\sin\left(\frac{1}{2}x\right)}\,\mathrm{d}x \\ &= 2\int \frac{1}{-\left(\tan\left(\frac{1}{2}x\right)-1\right)^2+2}\,\mathrm{d}\big(\tan\left(\tfrac{1}{2}x\right)-1\big) \\ &= -\frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{1}{2}x\right)-1-\sqrt{2}}{\tan\left(\frac{1}{2}x\right)-1+\sqrt{2}}\right|+C\end{aligned}$$

✎ **Solution**

$$\begin{aligned}\int \frac{1}{\sin x+\cos x}\,\mathrm{d}x &= \int \frac{1}{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2}}\int \frac{1}{\sin\left(x+\frac{\pi}{4}\right)}\,\mathrm{d}\left(x+\frac{\pi}{4}\right) \\ &= \frac{1}{\sqrt{2}}\ln\left|\tan\left(\frac{x+\frac{\pi}{4}}{2}\right)\right|+C \\ &= \frac{1}{\sqrt{2}}\ln\left|\csc\left(x+\frac{\pi}{4}\right)-\cot\left(x+\frac{\pi}{4}\right)\right|+C\end{aligned}$$

Exercise 4.21: 求不定积分

$$\int \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,\mathrm{d}x$$

✎ **Solution**

$$\begin{aligned}I &= \int \left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,\mathrm{d}x \\ &= \int e^{x+\frac{1}{x}}\,\mathrm{d}x+\int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,\mathrm{d}x \\ &= xe^{x+\frac{1}{x}}-\int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,\mathrm{d}x+\int x\left(1-\frac{1}{x^2}\right)e^{x+\frac{1}{x}}\,\mathrm{d}x \\ &= xe^{x+\frac{1}{x}}+C\end{aligned}$$

Exercise 4.22: 设 $y(x-y)^2=x$, 求积分

$$\int \frac{1}{x-3y}\,\mathrm{d}x$$

✎ **Solution** 令 $y=tx$ 则 $x=\dfrac{1}{\sqrt{t(1-t)^2}}, y=\dfrac{t}{\sqrt{t(1-t)^2}}$

当 $t\geqslant 1$ 时 $x=\dfrac{1}{(1-t)\sqrt{t}}, y=\dfrac{t}{(1-t)\sqrt{t}}\ \mathrm{d}x=\dfrac{3t-1}{2(t-1)^2t^{\frac{3}{2}}}\mathrm{d}t$

那么

$$\begin{aligned} I &= \int \frac{1}{x-3y}\,\mathrm{d}x \\ &= \int \frac{1}{2t(1-t)}\,\mathrm{d}t \\ &= \frac{1}{2}\left(\int \frac{1}{t}\,\mathrm{d}t + \int \frac{1}{1-t}\,\mathrm{d}t\right) \\ &= \frac{1}{2}\ln\left|\frac{y}{y-x}\right| + C \end{aligned}$$

当 $t<1$ 时 $x=\dfrac{1}{(t-1)\sqrt{t}}, y=\dfrac{t}{(t-1)\sqrt{t}}\ \mathrm{d}x=\dfrac{1-3t}{2(t-1)^2t^{\frac{3}{2}}}\mathrm{d}t$

那么

$$\begin{aligned} I &= \int \frac{1}{x-3y}\,\mathrm{d}x \\ &= \int \frac{1}{2t(t-1)}\,\mathrm{d}t \\ &= \frac{1}{2}\left(\int \frac{1}{t}\,\mathrm{d}t + \int \frac{1}{t-1}\,\mathrm{d}t\right) \\ &= \frac{1}{2}\ln\left|\frac{y}{x-y}\right| + C \end{aligned}$$

✎ **Solution** 令 $\begin{cases} x-y=u \\ \dfrac{x}{y}=v \end{cases}$ 即 $u^2=v$ 解得 $\begin{cases} x=\dfrac{uv}{v-1}=\dfrac{u^3}{u^2-1} \\ y=\dfrac{u}{v-1}=\dfrac{u}{u^2-1} \end{cases}$, $\mathrm{d}x=\dfrac{u^4-3u^2}{(u^2-1)^2}\mathrm{d}u$

所以

$$\begin{aligned} I &= \int \frac{1}{x-3y}\,\mathrm{d}x \\ &= \int \frac{u}{u^2-1}\,\mathrm{d}u \\ &= \frac{1}{2}\ln\left|u^2-1\right| + C \\ &= \frac{1}{2}\ln\left|(x-y)^2-1\right| + C \end{aligned}$$

Exercise 4.23: 求不定积分

✎ **Solution**

4.3 非初等表达

Exercise 4.24: 求不定积分

$$\int \frac{\arctan x}{x} dx$$

✎ **Solution** 设 $f\left(x\right)=\arctan x$ 则

$$f'\left(x\right)=\frac{1}{1+x^2}=\frac{1}{\left(1-ix\right)\left(1+ix\right)}=\frac{1}{2}\left(\frac{1}{1-ix}+\frac{1}{1+ix}\right)$$

利用幂级数展开 $f'(x)$, 首先我们知道 $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n, x\in\left(-1,1\right)$

因此

$$f'\left(x\right)=\frac{1}{\left(1-ix\right)\left(1+ix\right)}=\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(ix\right)^n+\sum_{n=0}^{\infty}\left(-ix\right)^n\right)$$

对两边积分有:

$$\begin{aligned}\int_0^x f'\left(x\right)dx&=\int_0^x\frac{1}{2}\left(\sum_{n=0}^{\infty}\left(ix\right)^n+\sum_{n=0}^{\infty}\left(-ix\right)^n\right)dx\\&=-\frac{1}{2}i\sum_{n=0}^{\infty}\frac{\left(ix\right)^{n+1}}{n+1}+\frac{1}{2}i\sum_{n=0}^{\infty}\frac{\left(-ix\right)^{n+1}}{n+1}\\&=-\frac{1}{2}i\sum_{n=1}^{\infty}\frac{\left(ix\right)^n}{n}+\frac{1}{2}i\sum_{n=1}^{\infty}\frac{\left(-ix\right)^n}{n}\end{aligned}$$

所以:

$$f\left(x\right)=\arctan x=\frac{1}{2}i\sum_{n=1}^{\infty}\frac{\left(-ix\right)^n}{n}-\frac{1}{2}i\sum_{n=1}^{\infty}\frac{\left(ix\right)^n}{n}$$

所以

$$\begin{aligned}\int\frac{\arctan x}{x}dx&=\frac{1}{2}\int\sum_{n=1}^{\infty}\frac{\left(-ix\right)^{n-1}}{n}dx-\frac{1}{2}i\int\sum_{n=1}^{\infty}\frac{\left(ix\right)^{n-1}}{n}dx\\&=\frac{1}{2}i\sum_{n=1}^{\infty}\frac{\left(-ix\right)^n}{n^2}-\frac{1}{2}i\sum_{n=1}^{\infty}\frac{\left(ix\right)^n}{n^2}+c\\&=\frac{1}{2}i\left(\mathrm{Li}_2\left(-ix\right)-\mathrm{Li}_2\left(ix\right)\right)+c\end{aligned}$$

Exercise 4.25: 求不定积分

$$\int x\tan x dx$$

✎ **Solution**

$$\begin{aligned}
\int x\tan x dx &= \int x\times\frac{\frac{e^{ix}-e^{-ix}}{2i}}{\frac{e^{ix}+e^{-ix}}{2}}dx=-\int ix\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}}dx\\
&=-\int ix\frac{e^{2ix}-1}{e^{2ix}+1}dx=-\int ixdx+2i\int\frac{x}{e^{2ix}+1}dx\\
&\xlongequal{e^{2ix}=t}-\frac{1}{2}ix^2+2i\int\frac{\frac{1}{2i}\ln t}{t+1}\frac{1}{2it}dt=-\frac{1}{2}ix^2-\frac{1}{2}i\int\frac{\ln t}{(t+1)\,t}dt\\
&=-\frac{1}{2}ix^2-\frac{1}{2}i\left(\int\frac{\ln t}{t}dt-\int\frac{\ln t}{t+1}dt\right)\\
&=-\frac{1}{2}ix^2-\frac{1}{2}i\left(\frac{1}{2}\ln^2 t-\ln t\ln(t+1)+\int\frac{\ln(1+t)}{t}dt\right)\\
&=-\frac{1}{2}ix^2-\frac{1}{2}i\left(\frac{1}{2}\ln^2 t-\ln t\ln(t+1)+\int\sum_{k=1}^{\infty}\frac{(-t)^{k-1}}{k}dt\right)\\
&=-\frac{1}{2}ix^2-\frac{1}{2}i\left(\frac{1}{2}\ln^2 t-\ln t\ln(t+1)-\sum_{k=1}^{\infty}\frac{(-t)^{k}}{k^2}\right)+c\\
&=-\frac{1}{2}ix^2-\frac{1}{2}i\left(\frac{1}{2}\ln^2 t-\ln t\ln(t+1)-\mathrm{Li}_2\left(-t\right)\right)+c\\
&=\frac{1}{2}ix^2+x\ln(e^{2ix}+1)+\frac{1}{2}i\mathrm{Li}_2\left(-e^{2ix}\right)+c
\end{aligned}$$

Exercise 4.26: 求不定积分

$$\int\cos\frac{1}{x}dx$$

✎ **Solution**

$$\begin{aligned}
\int\cos\frac{1}{x}dx &\xlongequal{x=\frac{1}{t}}-\int\frac{\cos t}{t^2}dt=\int\cos t d\frac{1}{t} &(4.1)\\
&=\frac{\cos t}{t}-\int\frac{\sin t}{t}dt &(4.2)\\
&=\frac{\cos t}{t}-\mathrm{Si}\left(t\right)+c &(4.3)\\
&=x\cos\frac{1}{x}-\mathrm{Si}\left(\frac{1}{x}\right)+c &(4.4)
\end{aligned}$$

Exercise 4.27: 求不定积分

$$\int\sin x\log x dx$$

✎ **Solution**

$$\begin{aligned}
\int\sin x\log x dx &=-\int\log x d\cos x &(4.5)\\
&=-\log x\cos x+\int\frac{\cos x}{x}dx &(4.6)\\
&=-\log x\cos x+\mathrm{Ci}(x)+c &(4.7)
\end{aligned}$$

Exercise 4.28: 求不定积分

$$\int \frac{x}{\tan x}dx$$

Solution

$$\begin{aligned}
\int \frac{x}{\tan x}dx &= \int \frac{x\cos x}{\sin x}dx = \int \frac{x\times\frac{e^{ix}+e^{-ix}}{2}}{\frac{e^{ix}-e^{-ix}}{2i}}dx \\
&= \int xi\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}}dx = \int xi\frac{\left(e^{ix}-e^{-ix}+2e^{-ix}\right)}{e^{ix}-e^{-ix}}dx \\
&= \int ixdx + 2\int \frac{ie^{-ix}x}{e^{ix}-e^{-ix}}dx = \frac{1}{2}ix^2 + 2\int \frac{ix}{e^{2ix}-1}dx \\
&= \frac{1}{2}ix^2 - 2\int \frac{ix}{1-e^{2ix}}dx \\
&\xlongequal{e^{2ix}=t} \frac{ix^2}{2} - 2\int \frac{i\times\frac{1}{2i}\ln t}{1-t}\times\left(\frac{1}{2it}\right)dt \\
&= \frac{ix^2}{2} + \frac{i}{2}\int \frac{\ln t}{t(1-t)}dt = \frac{1}{2}ix^2 + \frac{i}{2}\left(\int \frac{\ln t}{t}dt + \int \frac{\ln t}{1-t}dt\right) \\
&= \frac{1}{2}ix^2 + \frac{i}{2}\left(\int \ln t d\ln t - \ln t\ln(1-t) + \int \frac{\ln(1-t)}{t}dt\right) \\
&= \frac{1}{2}ix^2 + \frac{i}{2}\left(\frac{1}{2}\ln^2 t - \ln t\ln(1-t) - \frac{1}{n}\sum_{n=1}^{\infty}\int t^{n-1}dt\right) \\
&= \frac{1}{2}ix^2 + \frac{i}{2}\left(\frac{1}{2}\ln^2 t - \ln t\ln(1-t) - \sum_{n=1}^{\infty}\frac{t^n}{n^2}\right) + c \\
&= \frac{1}{2}ix^2 + \frac{i}{2}\left(\frac{1}{2}\ln^2 t - \ln t\ln(1-t) - \mathrm{Li}_2(t)\right) + c \\
&= x\ln\left(1-e^{2ix}\right) - \frac{1}{2}i\left(x^2 + \mathrm{Li}_2\left(e^{2ix}\right)\right) + c
\end{aligned}$$

Exercise 4.29: 求不定积分

$$\int \left(\frac{\sin x}{x}\right)^2 dx$$

Solution

$$\int \left(\frac{\sin x}{x}\right)^2 dx = -\int \sin^2 x d\left(\frac{1}{x}\right) \tag{4.8}$$

$$= -\frac{\sin^2 x}{x} + \int \frac{\sin 2x}{x}dx \tag{4.9}$$

$$= -\frac{\sin^2 x}{x} + \int \frac{\sin 2x}{2x}d2x \tag{4.10}$$

$$= -\frac{\sin^2 x}{x} + \mathrm{Si}(2x) + c \tag{4.11}$$

Exercise 4.30: 求不定积分

$$\int \frac{xe^x}{1+e^x}dx$$

✎ **Solution**

$$
\begin{aligned}
\int \frac{xe^x}{1+e^x}dx &\overset{t=e^x}{=\!=} \int \frac{\ln t}{1+t}dt && (4.12)\\
&= \ln t\ln(1+t) - \int \frac{\ln(1+t)}{t}dt && (4.13)\\
&= \ln t\ln(1+t) - \int \sum_{n=1}^{\infty}\frac{(-t)^{n-1}}{n}dt && (4.14)\\
&= \ln t\ln(1+t) - \sum_{n=1}^{\infty}\int \frac{(-t)^{n-1}}{n}dt && (4.15)\\
&= \ln t\ln(1+t) + \sum_{n=1}^{\infty}\frac{(-t)^{n}}{n^2}dt + c && (4.16)\\
&= \mathrm{Li}_2(-t) + \ln t\ln(t+1) + c && (4.17)\\
&= \mathrm{Li}_2(-e^x) + x\ln(e^x+1) + c && (4.18)\\
& && (4.19)
\end{aligned}
$$

Exercise 4.31: 计算不定积分

$$\int \ln\left(1+\sqrt{\frac{1+x}{x}}\right)dx(x>0).$$

✎ **Solution** 令 $t=\sqrt{\frac{1+x}{x}}$, 则 $x=\frac{1}{t^2-1}$. 从而有

$$
\begin{aligned}
\int \ln\left(1+\sqrt{\frac{1+x}{x}}\right)dx &= \int \ln(1+t)d\left(\frac{1}{t^2-1}\right)\\
&= \frac{1}{t^2-1}\ln(1+t) - \int \frac{1}{t^2-1}\cdot\frac{1}{1+t}dt
\end{aligned}
$$

而

$$
\begin{aligned}
\int \frac{1}{t^2-1}\cdot\frac{1}{1+t}dt &= \frac{1}{4}\int\left(\frac{1}{t-1}-\frac{1}{t+1}-\frac{2}{(t+1)^2}\right)dt\\
&= \frac{1}{4}\ln(t-1) - \frac{1}{4}\ln(t+1) + \frac{1}{2(t+1)} + C
\end{aligned}
$$

所以

$$
\begin{aligned}
\int \ln\left(1+\sqrt{\frac{1+x}{x}}\right)dx &= \frac{1}{t^2-1}\ln(1+t) + \frac{1}{4}\ln\frac{t+1}{t-1} - \frac{1}{2(t+1)} + C\\
&= x\ln\left(1+\sqrt{\frac{1+x}{x}}\right) + \frac{1}{2}\ln\left(\sqrt{1+x}+\sqrt{x}\right) - \frac{1}{2}\frac{\sqrt{x}}{\sqrt{1+x}+\sqrt{x}} + C\\
&= x\ln\left(1+\sqrt{\frac{1+x}{x}}\right) + \frac{1}{2}\ln\left(\sqrt{1+x}+\sqrt{x}\right) + \frac{1}{2}x - \frac{1}{2}\sqrt{x+x^2} + C.
\end{aligned}
$$

Exercise 4.32: 求不定积分

$$\int \frac{dx}{x^2\sqrt{x^2-1}}$$

Solution

$$\begin{aligned}
\int \frac{dx}{x^2\sqrt{x^2-1}} &= \int \frac{1}{x^3\sqrt{1-\frac{1}{x^2}}}dx \\
&= \int \frac{1}{\sqrt{1-\frac{1}{x^2}}}d\left(-\frac{1}{2x^2}\right) \\
&= \frac{1}{2}\int \left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}} d\left(1-\frac{1}{x^2}\right) \\
&= \frac{1}{2}\cdot\frac{\left(1-\frac{1}{x^2}\right)^{-\frac{1}{2}}}{\frac{1}{2}}+c \\
&= \frac{\sqrt{x^2-1}}{x}+c
\end{aligned}$$

Exercise 4.33: 求不定积分

$$\int \frac{x^2dx}{(x^4+1)^2},$$

Solution

$$I = \int \frac{x^2+x^4}{(x^4+1)^2}\mathrm{d}x = \int \frac{1}{((x-\frac{1}{x})^2+2)^2}\mathrm{d}\left(x-\frac{1}{x}\right)$$

$$J = \int \frac{-x^2+x^4}{(x^4+1)^2}\mathrm{d}x = \int \frac{1}{((x+\frac{1}{x})^2-2)^2}\mathrm{d}\left(x+\frac{1}{x}\right)$$

Solution

Exercise 4.34: 求不定积分

$$\int \frac{x^2dx}{(x^4+1)^2},$$

Solution

$$\begin{aligned}
\int \frac{x^2}{(x^4+1)^2}\mathrm{d}x &= \int \frac{4x^3}{4x(x^4+1)^2}\mathrm{d}x \\
&= \int \frac{1}{4x(x^4+1)^2}\mathrm{d}(x^4+1) \\
&= \int \frac{-1}{4x}\mathrm{d}\left(\frac{1}{x^4+1}\right) \\
&= \frac{-1}{4x(x^4+1)} + \int \frac{1}{4(x^4+1)}\mathrm{d}\left(\frac{1}{x}\right)
\end{aligned}$$

$$\begin{aligned}\int \frac{1}{x^4+1}\mathrm{d}\frac{1}{x} &= \int \frac{y^4}{y^4+1}\mathrm{d}y \\ &= \int 1-\frac{1}{y^4+1}\mathrm{d}y \\ &= y-\int \frac{1}{y^4+1}\mathrm{d}y\end{aligned}$$

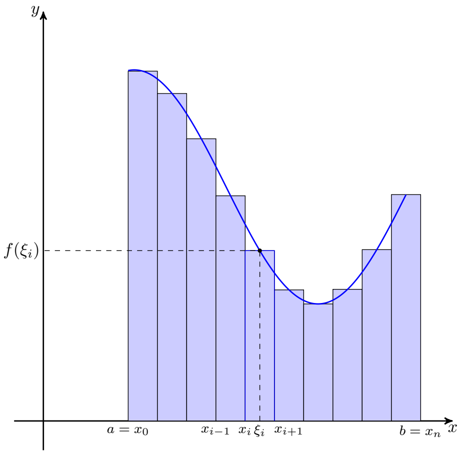
$$\begin{aligned}\int \frac{1}{y^4+1}\mathrm{d}y &= \int \frac{(y^2+1)-(y^2-1)}{2(y^4+1)}\mathrm{d}y \\ &= \int \frac{y^2+1}{2(y^4+1)}\mathrm{d}y - \int \frac{y^2-1}{2(y^4+1)}\mathrm{d}y\end{aligned}$$

$$\begin{aligned}\int \frac{y^2+1}{y^4+1}\mathrm{d}y &= \int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}\mathrm{d}y \\ &= \int \frac{1}{(y-\frac{1}{y})^2+2}\mathrm{d}\left(y-\frac{1}{y}\right)\end{aligned}$$

$$\begin{aligned}\int \frac{y^2-1}{y^4+1}\mathrm{d}y &= \int \frac{1-\frac{1}{y^2}}{y^2+\frac{1}{y^2}}\mathrm{d}y \\ &= \int \frac{1}{(y+\frac{1}{y})^2-2}\mathrm{d}\left(y+\frac{1}{y}\right)\end{aligned}$$

第 5 章 定积分

5.1 定积分的概念与性质



Definition 5.1 定积分

设函数 $f(x)$ 在 $[a,b]$ 上有界, 在 $[a,b]$ 中任意插入若干个分点

$$a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b$$

把区间 $[a,b]$ 分为若 n 个小区间

$$[x_0,x_1],[x_1,x_2],\cdots,[x_{n-1},x_n]$$

各个小区间长度依次为

$$\Delta x_1=x_1-x_0,\Delta x_2=x_2-x_1,\cdots,\Delta x_n=x_n-x_{n-1}$$

在小区间 $[x_{i-1},x_i]$ 上任取一点 $\xi_i\,(x_{i-1}\leqslant\xi\leqslant x_i)$, 作函数值 $f(\xi_i)$ 与小区间长度 Δx_i 的乘积 $f(\xi_i)\Delta x_i\,(i=1,2,\cdots,n)$, 并作出和

♡

$$S=\sum_{i=1}^{n}f(\xi_i)\Delta x_i$$

记 $\lambda=\max\{\Delta x_1,\Delta x_2,\cdots,\Delta x_n\}$, 如果当 $\lambda\to 0$ 时, 这个和的极限存在, 且与闭区间 $[a,b]$ 的分法无关及点 ξ_i 的取法无关, 那么称这个极限 I 为函数 $f(x)$ 在 $[a,b]$ 上的定积分 (简称积分), 记作 $\displaystyle\int_a^b f(x)\,\mathrm{d}x$, 即

$$\int_a^b f(x)\,\mathrm{d}x=I=\lim_{\lambda\to 0}\sum_{i=1}^{n}f(\xi_i)\Delta x_i$$

其中 $f(x)$ 叫做被积函数, $f(x)\,\mathrm{d}x$ 叫做被积表达式, x 叫做积分变量, a 叫做积分下限, b 叫做积分上限, $[a,b]$ 叫做积分区间

Definition 5.2 定积分 $\varepsilon-\delta$

设有常数 I, 如果对于任意给定的正数 ε, 总存在一个正数 δ, 使得对于区间 $[a,b]$ 的任何分法, 不论 ξ_i 在 $[x_{n-1},x_n]$ 中怎样选取, 只要 $\lambda=\max\{\Delta x_1,\Delta x_2,\cdots,\Delta x_n\}<\delta$, 总有

♡

$$\left|\sum_{i=1}^{n}f(\xi_i)\Delta x_i-I\right|<\varepsilon$$

成立, 那么称 I 是 $f(x)$ 在 $[a,b]$ 上的定积分, 记作 $\displaystyle\int_a^b f(x)\,\mathrm{d}x$

Exercise 5.1: 利用定义计算定积分

$$\int_0^1 x^2\,\mathrm{d}x$$

Solution 函数 $f(x)=x^2$ 在 $[0,1]$ 上连续, 故 $f(x)=x^2$ 在 $[0,1]$ 上可积.
将 $[0,1]$ n 等分, 其分点为 $x_i=\dfrac{i}{n}$, $(i=1,2,\cdots,n)$, 小区间 $\left[\dfrac{i-1}{n},\dfrac{i}{n}\right]$ $(i=1,2,\cdots,n)$ 长度为 $\Delta x_i=\dfrac{1}{n}$ $(i=1,2,\cdots,n)$, 取 $\xi_i=\dfrac{i}{n}$ $(i=1,2,\cdots,n)$, $\lambda=\max\{\Delta x_i\}=\dfrac{1}{n}$, 故

$$\begin{aligned}\int_0^1 x^2\,\mathrm{d}x &= \lim_{\lambda\to 0}\sum_{i=1}^n f(\xi_i)\Delta x_i=\lim_{n\to\infty}\sum_{i=1}^n \xi_i^2\Delta x_i\\ &=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\left(\frac{i}{n}\right)^2=\lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^n i^2\\ &=\lim_{n\to\infty}\frac{\frac{1}{6}n(n+1)(2n+1)}{n^3}=\frac{1}{3}\end{aligned}$$

Exercise 5.2: 利用定义计算定积分

$$\int_0^1 e^x\,\mathrm{d}x$$

Solution 函数 $f(x)=e^x$ 在 $[0,1]$ 上连续, 故 $f(x)=e^x$ 在 $[0,1]$ 上可积.
将 $[0,1]$ n 等分, 其分点为 $x_i=\dfrac{i}{n}$, $(i=1,2,\cdots,n)$, 小区间 $\left[\dfrac{i-1}{n},\dfrac{i}{n}\right]$ $(i=1,2,\cdots,n)$ 长度为 $\Delta x_i=\dfrac{1}{n}$ $(i=1,2,\cdots,n)$, 取 $\xi_i=\dfrac{i}{n}$ $(i=1,2,\cdots,n)$, $\lambda=\max\{\Delta x_i\}=\dfrac{1}{n}$, 故

$$\begin{aligned}\int_0^1 e^x\,\mathrm{d}x &= \lim_{\lambda\to 0}\sum_{i=1}^n f(\xi_i)\Delta x_i=\lim_{n\to\infty}\sum_{i=1}^n e^{\xi_i}\Delta x_i\\ &=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n e^{\frac{i}{n}}=\lim_{n\to\infty}\frac{(e-1)e^{\frac{1}{n}}}{n(e^{\frac{1}{n}}-1)}\\ &=e-1\end{aligned}$$

Exercise 5.3: 利用定义计算定积分

$$\int_a^b \frac{1}{x}\,\mathrm{d}x$$

Solution 函数 $f(x)=\dfrac{1}{x}$ 在 $[a,b]$ 上连续, 故 $f(x)=\dfrac{1}{x}$ 在 $[a,b]$ 上可积.
将 $[a,b]$ n 等分, 其分点为 $x_0=a, x_1=aq, x_2=aq^2,\cdots,x_n=aq^n=b, q=\left(\dfrac{b}{a}\right)^{\frac{1}{n}}$,
小区间 $[aq^{i-1},aq^i]$ $(i=1,2,\cdots,n)$ 长度为 $\Delta x_i=aq^{i-1}(q-1)$ $(i=1,2,\cdots,n)$,

取 $\xi_i = aq^i\ (i=1,2,\cdots,n), \lambda = \max\{\Delta x_i\} = aq^{n-1}(q-1) \sim \frac{b}{n}\ln\left(\frac{b}{a}\right)$, 故

$$\begin{aligned}\int_a^b \frac{1}{x}\,\mathrm{d}x &= \lim_{\lambda\to 0}\sum_{i=1}^n f(\xi_i)\Delta x_i = \lim_{n\to\infty}\sum_{i=1}^n \frac{1}{\xi_i}\Delta x_i\\ &= \lim_{n\to\infty}\sum_{i=1}^n \frac{aq^{i-1}(q-1)}{aq^i} = \lim_{n\to\infty} n(1-q^{-1})\\ &= \lim_{n\to\infty} n\left(1-\left(\frac{a}{b}\right)^{\frac{1}{n}}\right) = \lim_{n\to\infty} n\left(1-e^{\frac{1}{n}\ln\left(\frac{a}{b}\right)}\right)\\ &= \ln\left(\frac{b}{a}\right)\end{aligned}$$

Exercise 5.4: 求极限

$$\lim_{n\to\infty}\frac{\sqrt[n]{1+2!+3!+\cdots+n!}}{n}$$

Solution 由于

$$\frac{\sqrt[n]{n!}}{n} \leqslant \frac{\sqrt[n]{1+2!+3!+\cdots+n!}}{n} \leqslant \frac{\sqrt[n]{n\times n!}}{n}$$

而

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} == \exp\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \ln\frac{i}{n}\right\} = \exp\left(\int_0^1 \ln x dx\right) = \frac{1}{e}$$

$$\lim_{n\to\infty}\frac{\sqrt[n]{n\times n!}}{n} = \lim_{n\to\infty}\sqrt[n]{n}\times\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

所以由夹逼准则知所求极限为 $\frac{1}{e}$

Exercise 5.5: 求极限

$$\lim_{n\to\infty} n\left(\frac{\sin\frac{\pi}{n}}{n^2+1}+\frac{\sin\frac{2}{n}\pi}{n^2+2}+\cdots+\frac{\sin\pi}{n^2+n}\right)$$

Solution 由于

$$\frac{1}{n+1}\sum_{i=1}^n \sin\frac{i\pi}{n} \leqslant \sum_{i=1}^n \frac{\sin\frac{i\pi}{n}}{n+\frac{i}{n}} \leqslant \frac{1}{n}\sum_{i=1}^n \sin\frac{i\pi}{n}$$

而

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=1}^n \sin\frac{i\pi}{n} = \lim_{n\to\infty}\frac{n}{(n+1)\pi}\times\lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^n\sin\frac{i\pi}{n} = \frac{1}{\pi}\int_0^\pi \sin x\,\mathrm{d}x = \frac{2}{\pi}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \sin\frac{i\pi}{n} = \frac{1}{\pi}\lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^n\sin\frac{i\pi}{n} = \frac{1}{\pi}\int_0^\pi \sin x\,\mathrm{d}x = \frac{2}{\pi}$$

所以由夹逼准则知所求极限为 $\frac{2}{\pi}$

Exercise 5.6: 求极限

$$\lim_{n\to\infty}\left(\frac{\sqrt{1\cdot 2}}{n^2+1}+\frac{\sqrt{2\cdot 3}}{n^2+2}+\cdots+\frac{\sqrt{n\cdot(n+1)}}{n^2+n}\right)$$

✎ **Solution** 由于

$$\frac{i}{n^2+n} \leqslant \frac{\sqrt{i\cdot(i+1)}}{n^2+i} \leqslant \frac{i+1}{n^2+1}$$

而

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2+n} = \lim_{n\to\infty}\frac{n^2}{n^2+n}\times\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n} = 1\times\int_0^1 x\,\mathrm{d}x = \frac{1}{2}$$

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i+1}{n^2+1} &= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2+1} + \lim_{n\to\infty}\frac{1}{n^2+1}\\ &= \lim_{n\to\infty}\frac{n^2}{n^2+1}\times\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n} = 1\times\int_0^1 x\,\mathrm{d}x = \frac{1}{2}\end{aligned}$$

故由夹逼准则知

$$\lim_{n\to\infty}\left(\frac{\sqrt{1\cdot 2}}{n^2+1}+\frac{\sqrt{2\cdot 3}}{n^2+2}+\cdots+\frac{\sqrt{n\cdot(n+1)}}{n^2+n}\right) = \frac{1}{2}$$

Exercise 5.7: 设 $f(x)$ 在 $[1,+\infty)$ 上是减函数, 且 $f(x)\geqslant 0$. 证明

$$\int_1^{\infty} f(x)\,\mathrm{d}x \leqslant \sum_{n=1}^{\infty} f(n) \leqslant f(1)+\int_1^{\infty} f(x)\,\mathrm{d}x$$

✎ **Solution** 一方面

$$\begin{aligned}\sum_{n=1}^{\infty} f(n) &= f(1)+\sum_{n=2}^{\infty} f(n) = f(1)+\sum_{n=2}^{\infty}\int_{n-1}^{n} f(n)\,\mathrm{d}x\\ &< f(1)+\sum_{n=2}^{\infty}\int_{n-1}^{n} f(x)\,\mathrm{d}x = f(1)+\int_1^{\infty} f(x)\,\mathrm{d}x\end{aligned}$$

另一方面

$$\sum_{n=1}^{\infty} f(n) = \sum_{n=1}^{\infty}\int_n^{n+1} f(n)\,\mathrm{d}x > \sum_{n=1}^{\infty}\int_n^{n+1} f(x)\,\mathrm{d}x = \int_1^{\infty} f(x)\,\mathrm{d}x$$

因此

$$\int_1^{\infty} f(x)\,\mathrm{d}x \leqslant \sum_{n=1}^{\infty} f(n) \leqslant f(1)+\int_1^{\infty} f(x)\,\mathrm{d}x$$

Exercise 5.8: 求 $\sum\limits_{n=1}^{100} n^{-\frac{1}{2}}$ 的整数部分

✎ **Solution** 一方面

$$\begin{aligned}\sum_{n=1}^{100} n^{-\frac{1}{2}} &= 1+\sum_{n=2}^{100} n^{-\frac{1}{2}} = 1+\sum_{n=2}^{100}\int_{n-1}^{n} n^{-\frac{1}{2}}\,\mathrm{d}x\\ &< 1+\sum_{n=2}^{100}\int_{n-1}^{n} x^{-\frac{1}{2}}\,\mathrm{d}x = 1+\int_1^{100} x^{-\frac{1}{2}}\,\mathrm{d}x = 19\end{aligned}$$

或者

$$\sum_{n=1}^{100} n^{-\frac{1}{2}} < \int_{1}^{101} \frac{1}{\sqrt{x-\frac{1}{2}}}\,\mathrm{d}x = 2\sqrt{100.5} - \sqrt{2} \approx 18.636$$

另一方面

$$\sum_{n=1}^{100} n^{-\frac{1}{2}} = \sum_{n=1}^{100}\int_{n}^{n+1} n^{-\frac{1}{2}}\,\mathrm{d}x > \sum_{n=1}^{100}\int_{n}^{n+1} x^{-\frac{1}{2}}\,\mathrm{d}x = \int_{1}^{101} x^{-\frac{1}{2}}\,\mathrm{d}x = 2\left(\sqrt{101}-1\right) \approx 18.1$$

因此 $\sum\limits_{n=1}^{100} n^{-\frac{1}{2}}$ 的整数部分为 18

Exercise 5.9: 求极限

$$I = \lim_{n\to\infty}\left(\frac{1}{[n\pi]} + \frac{1}{[n\pi+1]} + \cdots + \frac{1}{4n}\right)$$

Solution1. 一方面

$$\begin{aligned}
\frac{1}{[n\pi]} + \frac{1}{[n\pi+1]} + \cdots + \frac{1}{4n} &= \sum_{k=0}^{4n-[n\pi]} \frac{1}{[n\pi]+k} \\
&< \sum_{k=0}^{4n-[n\pi]} \int_{k-1}^{k} \frac{1}{[n\pi]+x}\,\mathrm{d}x \\
&= \int_{-1}^{4n-[n\pi]} \frac{1}{[n\pi]+x}\,\mathrm{d}x \to \ln\frac{4}{\pi}
\end{aligned}$$

另一方面

$$\begin{aligned}
\frac{1}{[n\pi]} + \frac{1}{[n\pi+1]} + \cdots + \frac{1}{4n} &= \sum_{k=0}^{4n-[n\pi]} \frac{1}{[n\pi]+k} \\
&> \sum_{k=0}^{4n-[n\pi]} \int_{k}^{k+1} \frac{1}{[n\pi]+x}\,\mathrm{d}x \\
&= \int_{0}^{4n-[n\pi]+1} \frac{1}{[n\pi]+x}\,\mathrm{d}x \to \ln\frac{4}{\pi}
\end{aligned}$$

因此 $\lim\limits_{n\to\infty}\left(\frac{1}{[n\pi]} + \frac{1}{[n\pi+1]} + \cdots + \frac{1}{4n}\right) = \ln\frac{4}{\pi}$

Solution2. 考虑欧拉常数的定义

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \ln n + \gamma + \varepsilon_n$$

故有

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{[n\pi-1]} = \ln[n\pi-1] + \gamma + \varepsilon_{[n\pi]-1} \tag{5.1}$$

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{4n} = \ln(4n) + \gamma + \varepsilon_{4n} \tag{5.2}$$

由 (5.2)−(5.1) 得

$$\frac{1}{[n\pi]}+\frac{1}{[n\pi+1]}+\cdots+\frac{1}{4n}=\ln\frac{4n}{[n\pi-1]}+\varepsilon_{4n}-\varepsilon_{[n\pi]-1}$$

因此 $\lim\limits_{n\to\infty}\left(\frac{1}{[n\pi]}+\frac{1}{[n\pi+1]}+\cdots+\frac{1}{4n}\right)=\ln\frac{4}{\pi}$

✎ **Solution3.** 显然

$$n\pi-1<[n\pi]\leqslant n\pi$$

故有

$$\frac{1}{n\pi-1}+\frac{1}{n\pi}+\cdots+\frac{1}{4n-1}<I\leqslant\frac{1}{n\pi}+\frac{1}{n\pi+1}+\cdots+\frac{1}{4n}$$

其中

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n\pi}+\frac{1}{n\pi+1}+\cdots+\frac{1}{4n}\right)&=\lim_{n\to\infty}\sum_{i=1}^{4n}\frac{1}{n\pi+i}+\lim_{n\to\infty}\frac{1}{n\pi}\\&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{(4-\pi)n}\frac{1}{\pi+\frac{i}{n}}=\int_0^{4-\pi}\frac{1}{\pi+x}\,\mathrm{d}x=\ln\frac{4}{\pi}\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{n\pi-1}+\frac{1}{n\pi}+\cdots+\frac{1}{4n-1}\right)&=\lim_{n\to\infty}\sum_{i=1}^{4n}\frac{1}{n\pi+i}-\lim_{n\to\infty}\frac{1}{n\pi}+\lim_{n\to\infty}\frac{1}{n\pi}+\lim_{n\to\infty}\frac{1}{n\pi-1}\\&=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{(4-\pi)n}\frac{1}{\pi+\frac{i}{n}}=\int_0^{4-\pi}\frac{1}{\pi+x}\,\mathrm{d}x=\ln\frac{4}{\pi}\end{aligned}$$

故由夹逼准则知 $\lim\limits_{n\to\infty}\left(\frac{1}{[n\pi]}+\frac{1}{[n\pi+1]}+\cdots+\frac{1}{4n}\right)=\ln\frac{4}{\pi}$

Exercise 5.10: 求极限

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x(t-[t])^2\,\mathrm{d}t$$

✎ **Solution** 当 $n\leqslant t\leqslant n+1$ 时

$$\begin{aligned}\int_0^x(t-[t])^2\,\mathrm{d}t\int_0^n(t-[t])^2\,\mathrm{d}t+\int_n^x(t-[t])^2\,\mathrm{d}t\\&=\sum_{i=1}^{n-1}\int_i^{i+1}(t-[t])^2\,\mathrm{d}t+\int_n^x(t-n)^2\,\mathrm{d}t\\&=\sum_{i=1}^{n-1}\int_i^{i+1}(t-i)^2\,\mathrm{d}t-\frac{1}{3}(n-x)^3=\frac{1}{3}\left[n+(x-n)^3\right]\end{aligned}$$

所以

$$\frac{n}{3(n+1)}\leqslant\frac{1}{x}\int_0^x(t-[t])^2\,\mathrm{d}t\leqslant\frac{n+1}{3n},n=1,2\cdots$$

由于 $\lim\limits_{n\to\infty}\dfrac{n}{3(n+1)}=\lim\limits_{n\to\infty}\dfrac{n+1}{3n}=\dfrac{1}{3}$, 并且当 $n\to\infty$ 时有 $x\to\infty$, 所以

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x (t-[t])^2\,\mathrm{d}t=\frac{1}{3}$$

Exercise 5.11: 求积分

$$\int_0^{+\infty}\frac{\left[\frac{x}{\pi}\right]}{x^3}\,\mathrm{d}x$$

Solution 当 $n\pi\leqslant x\leqslant(n+1)\pi$ 时

$$\begin{aligned}\int_0^{+\infty}\frac{\left[\frac{x}{\pi}\right]}{x^3}\,\mathrm{d}x&=\sum_{i=0}^{\infty}\int_{i\pi}^{(i+1)\pi}\frac{\left[\frac{x}{\pi}\right]}{x^3}\,\mathrm{d}x=\sum_{i=0}^{\infty}\int_{i\pi}^{(i+1)\pi}\frac{i}{x^3}\,\mathrm{d}x\\&=\sum_{i=0}^{\infty}\left[\frac{i}{2\pi^2}\left(\frac{1}{i^2}-\frac{1}{(i+1)^2}\right)\right]\\&=\frac{1}{2\pi^2}\sum_{i=0}^{\infty}\left[\frac{1}{k}-\frac{1}{k+1}+\frac{1}{(k+1)^2}\right]=\frac{1}{12}\end{aligned}$$

其中:

$$\lim_{n\to\infty}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right)=\frac{\pi^2}{6}$$

Exercise 5.12: 求极限

$$\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{n}{n^2+k^2}$$

Solution 一方面

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{n}{n^2+k^2}&=\lim_{n\to\infty}\sum_{k=1}^{n^2}\int_{k-1}^{k}\frac{n}{n^2+k^2}\mathrm{d}x\\&\leqslant\lim_{n\to\infty}\sum_{k=1}^{n^2}\int_{k-1}^{k}\frac{n}{n^2+x^2}\mathrm{d}x=\int_0^{n^2}\frac{n}{n^2+x^2}\mathrm{d}x=\frac{\pi}{2}\end{aligned}$$

另一方面

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{n}{n^2+k^2}&=\lim_{n\to\infty}\sum_{k=1}^{n^2}\int_{k}^{k+1}\frac{n}{n^2+k^2}\mathrm{d}x\\&\geqslant\lim_{n\to\infty}\sum_{k=1}^{n^2}\int_{k}^{k+1}\frac{n}{n^2+x^2}\mathrm{d}x=\int_1^{n^2+1}\frac{n}{n^2+x^2}\mathrm{d}x=\frac{\pi}{2}\end{aligned}$$

故由夹逼准则知

$$\lim_{n\to\infty}\sum_{k=1}^{n^2}\frac{n}{n^2+k^2}=\frac{\pi}{2}$$

Exercise 5.13: 求极限

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+\frac{i^2+1}{n}}$$

✎ **Solution**

$$
\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+\frac{i^2+1}{n}} &= \lim_{n\to\infty}\frac{1}{n+\frac{n^2+1}{n}}+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+\frac{(n-1)^2+1}{n^2}}\\
&= \lim_{\lambda\to 0}\sum_{i=1}^{n}\frac{1}{1+(\xi_i)^2}\Delta x_i,\quad \xi_i=\frac{(n-1)^2+1}{n^2},\Delta x_i=\frac{1}{n}\\
&= \int_0^1\frac{1}{1+x^2}\,\mathrm{d}x=\frac{\pi}{4}
\end{aligned}
$$

Exercise 5.14: 求极限

$$
\lim_{n\to\infty}\frac{\left[1^\alpha+3^\alpha+\cdots+(2n+1)^\alpha\right]^{\beta+1}}{\left[2^\beta+4^\beta+\cdots+(2n)^\beta\right]^{\alpha+1}}\quad(\alpha,\beta\neq -1)
$$

✎ **Solution**

$$
\begin{aligned}
I &= \lim_{n\to\infty}\frac{\left[1^\alpha+3^\alpha+\cdots+(2n+1)^\alpha\right]^{\beta+1}}{\left[2^\beta+4^\beta+\cdots+(2n)^\beta\right]^{\alpha+1}}\quad(\alpha,\beta\neq -1)\\
&= 2^{\alpha-\beta}\lim_{n\to\infty}\frac{\left\{\frac{2}{n}\left[\left(\frac{1}{n}\right)^\alpha+\left(\frac{3}{n}\right)^\alpha+\cdots+\left(\frac{2n+1}{n}\right)^\alpha\right]\right\}^{\beta+1}}{\left\{\frac{2}{n}\left[\left(\frac{2}{n}\right)^\beta+\left(\frac{4}{n}\right)^\beta+\cdots+\left(\frac{2n}{n}\right)^\beta\right]\right\}^{\alpha+1}}\\
&= 2^{\alpha-\beta}\frac{\left\{\int_0^2 x^\alpha\,\mathrm{d}x\right\}^{\beta+1}}{\left\{\int_0^2 x^\beta\,\mathrm{d}x\right\}^{\alpha+1}}=2^{\alpha-\beta}\frac{\left\{\frac{1}{\alpha+1}x^{\alpha+1}\Big|_0^2\right\}^{\beta+1}}{\left\{\frac{1}{\beta+1}x^{\beta+1}\Big|_0^2\right\}^{\alpha+1}}\\
&= 2^{\alpha-\beta}\frac{(\beta+1)^{\alpha+1}}{(\alpha+1)^{\beta+1}}
\end{aligned}
$$

Exercise 5.15: 求极限

$$
\lim_{n\to\infty}\frac{1}{n^4}\prod_{i=1}^{2n}\sqrt[n]{n^2+i^2}
$$

✎ **Solution** 取对数, 我们有

$$
\begin{aligned}
\ln\left(\frac{1}{n^4}\prod_{i=1}^{2n}\sqrt[n]{n^2+i^2}\right) &= \sum_{i=1}^{2n}\frac{\ln(n^2+i^2)}{n}-\ln n^4\\
&= \sum_{i=1}^{2n}\frac{\ln n^2}{n}+\frac{1}{n}\sum_{i=1}^{2n}\ln\left(1+\left(\frac{i}{n}\right)^2\right)-\ln n^4\\
&= \frac{1}{n}\sum_{i=1}^{2n}\ln\left(1+\left(\frac{i}{n}\right)^2\right)
\end{aligned}
$$

从而可得

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n^4}\prod_{i=1}^{2n}\sqrt[n]{n^2+i^2}&=\exp\left(\lim_{n\to\infty}\sum_{i=1}^{2n}\ln\left(1+\left(\frac{i}{n}\right)^2\right)\frac{1}{n}\right)\\&=\exp\left(\int_0^2\ln(1+x^2)\,\mathrm{d}x\right)=25e^{2\arctan 2-4}\end{aligned}$$

Exercise 5.16: 求极限

$$\lim_{n\to\infty}\frac{(1^1\cdot 2^2\cdot 3^3\cdots n^n)^{\frac{1}{n^2}}}{\sqrt{n}}$$

Solution 取对数, 我们有

$$\begin{aligned}\ln\left(\frac{(1^1\cdot 2^2\cdot 3^3\cdots n^n)^{\frac{1}{n^2}}}{\sqrt{n}}\right)&=\frac{1}{n^2}\sum_{i=1}^{n}i\ln i-\frac{1}{2}\ln n\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\ln\frac{i}{n}+\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\ln n-\frac{1}{2}\ln n\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\ln\frac{i}{n}+\frac{n^2+n}{2n^2}\ln n-\frac{1}{2}\ln n\\&=\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\ln\frac{i}{n}+\frac{\ln n}{2n}\end{aligned}$$

从而可得

$$\begin{aligned}\lim_{n\to\infty}\frac{(1^1\cdot 2^2\cdot 3^3\cdots n^n)^{\frac{1}{n^2}}}{\sqrt{n}}&=\exp\left(\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{i}{n}\ln\frac{i}{n}+\frac{\ln n}{2n}\right)\right)\\&=\exp\left(\int_0^1 x\ln x\,\mathrm{d}x\right)=e^{-\frac{1}{4}}\end{aligned}$$

Exercise 5.17: 求极限

$$I=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{(n+k)(n+k+1)}$$

Solution

$$\begin{aligned}I&=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n+k}-\frac{k}{n+k+1}\right)\\&=\lim_{n\to\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}+\frac{2}{n+2}-\frac{2}{n+3}-\cdots+\frac{n}{n+n}-\frac{n}{n+n+1}\right)\\&=\lim_{n\to\infty}\left(\left(\sum_{k=1}^{n}\frac{k}{n+k}\right)-\frac{n}{n+n+1}\right)\\&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}-\frac{1}{2}=\int_0^1\frac{1}{1+x}\,\mathrm{d}x-\frac{1}{2}\\&=\ln 2-\frac{1}{2}\end{aligned}$$

Exercise 5.18: 求极限

$$\int_0^n [x]\,\mathrm{d}x$$

Solution

$$\int_0^n [x]\,\mathrm{d}x = \sum_{k=1}^{n}\int_{k-1}^{k}[x]\,\mathrm{d}x = \sum_{k=1}^{n}(k-1) = \frac{1}{2}n(n-1)$$

Exercise 5.19: 求极限

$$\int_0^1 \left(\left[\frac{2}{x}\right] - 2\left[\frac{1}{x}\right]\right)\mathrm{d}x$$

Solution 当$n \leqslant \frac{2}{x} < n+1$ 即 $\frac{1}{2(n+1)} < x \leqslant \frac{1}{2n}$ 时, $\left[\frac{2}{x}\right] = n$;

同样的, 当$n \leqslant \frac{1}{x} < n+1$ 即 $\frac{1}{n+1} < x \leqslant \frac{1}{n}$ 时, $\left[\frac{1}{x}\right] = n$;

由于

$$\left(\frac{1}{n+1}, \frac{1}{n}\right] = \left(\frac{2}{2n+2}, \frac{2}{2n}\right] = \left(\frac{2}{2n+2}, \frac{2}{2n+1}\right] \bigcup \left(\frac{2}{2n+1}, \frac{2}{2n}\right]$$

当 $\frac{2}{2n+2} < x \leqslant \frac{2}{2n+1}$ 时, $\left[\frac{2}{x}\right] = 2n+1$, $\left[\frac{1}{x}\right] = n$, 此时有

$$\left[\frac{2}{x}\right] - 2\left[\frac{1}{x}\right] = (2n+1) - 2n = 1$$

当 $\frac{2}{2n+1} < x \leqslant \frac{2}{2n}$ 时, $\left[\frac{2}{x}\right] = 2n$, $\left[\frac{1}{x}\right] = n$ 此时有

$$\left[\frac{2}{x}\right] - 2\left[\frac{1}{x}\right] = 2n - 2n = 0$$

因此,

$$\begin{aligned}
I &= \int_0^1 \left(\left[\frac{2}{x}\right] - 2\left[\frac{1}{x}\right]\right)\mathrm{d}x = \sum_{n=1}^{\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}} \left(\left[\frac{2}{x}\right] - 2\left[\frac{1}{x}\right]\right)\mathrm{d}x \\
&= \sum_{n=1}^{\infty}\int_{\frac{2}{2n+2}}^{\frac{2}{2n+1}} \left(\left[\frac{2}{x}\right] - 2\left[\frac{1}{x}\right]\right)\mathrm{d}x + \sum_{n=1}^{\infty}\int_{\frac{2}{2n+1}}^{\frac{2}{2n}} \left(\left[\frac{2}{x}\right] - 2\left[\frac{1}{x}\right]\right)\mathrm{d}x \\
&= \sum_{n=1}^{\infty}\int_{\frac{2}{2n+2}}^{\frac{2}{2n+1}} \mathrm{d}x + \sum_{n=1}^{\infty}\int_{\frac{2}{2n+1}}^{\frac{2}{2n}} 0\,\mathrm{d}x = \sum_{n=1}^{\infty}\left(\frac{2}{2n+1} - \frac{2}{2n+2}\right) \\
&= 2\left(\frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\right) \\
&= 2\left(-\ln 2 + 1 - \frac{1}{2}\right) \\
&= \ln 4 - 1 = 2\ln 2 - 1
\end{aligned}$$

Note:

$$\ln 2 = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{(-1)^n}{n+1} + \cdots$$

Exercise 5.20: 求极限

$$I=\lim_{n\to\infty}\frac{1^p+3^p+\cdots+(2n-1)^p}{n^{p+1}}$$

Solution 考虑 $f(x)=x^p\ (x\in[0,2])$. 将 $[0,2]$ n 等分, 分点为 $\frac{2i}{n}, (i=1,2,\cdots,n)$, 小区间长度为 $\Delta x_i=\frac{2}{n}\,(i=1,2,\cdots,n)$, 取 $\xi_i=\frac{2i-1}{n}\,(i=1,2,\cdots,n)$, $\lambda=\max\{\Delta x_i\}=\frac{2}{n}$, 故

$$I=\frac{1}{2}\lim_{n\to\infty}\frac{2}{n}\sum_{k=1}^{n}\left(\frac{2k-1}{n}\right)^p=\frac{1}{2}\lim_{\lambda\to 0}\sum_{i=1}^{n}(\xi_i)^p\Delta x_i=\frac{1}{2}\int_0^2 x^p\,\mathrm{d}x=\frac{2^p}{p+1}$$

Exercise 5.21: 求极限

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sin\left(\frac{i-\frac{1}{2}}{n}\pi\right)$$

Solution1. 考虑 $f(x)=\sin(\pi x)\ (x\in[0,1])$. 将 $[0,1]$ n 等分, 分点为 $\frac{i}{n}, (i=1,2,\cdots,n)$, 小区间长度为 $\Delta x_i=\frac{1}{n}\,(i=1,2,\cdots,n)$, 取 $\xi_i=\frac{i-\frac{1}{2}}{n}\,(i=1,2,\cdots,n)$, $\lambda=\max\{\Delta x_i\}=\frac{1}{n}$, 故

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sin\left(\frac{i-\frac{1}{2}}{n}\pi\right)=\lim_{\lambda\to 0}\sum_{i=1}^{n}\sin(\xi_i\pi)\Delta x_i=\int_0^1\sin(\pi x)\,\mathrm{d}x=\frac{2}{\pi}$$

Solution2. 考虑 $f(x)=\sin x\ (x\in[0,\pi])$. 将 $[0,\pi]$ n 等分, 分点为 $\frac{i\pi}{n}, (i=1,2,\cdots,n)$, 小区间长度为 $\Delta x_i=\frac{\pi}{n}\,(i=1,2,\cdots,n)$, 取 $\xi_i=\frac{i-\frac{1}{2}}{n}\pi\,(i=1,2,\cdots,n)$, $\lambda=\max\{\Delta x_i\}=\frac{\pi}{n}$, 故

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sin\left(\frac{i-\frac{1}{2}}{n}\pi\right)&=\frac{1}{\pi}\lim_{n\to\infty}\frac{\pi}{n}\sum_{i=1}^{n}\sin\left(\frac{i-\frac{1}{2}}{n}\pi\right)\\&=\frac{1}{\pi}\lim_{\lambda\to 0}\sum_{i=1}^{n}\sin(\xi_i)\Delta x_i=\frac{1}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x=\frac{2}{\pi}\end{aligned}$$

Exercise 5.22: 设 $f(x)$ 在 $[a,b]$ 可积, $F(x)$ 是 $f(x)$ 在 $[a,b]$ 上的一个原函数, 试用定积分的定义和拉格朗日中值定理证明牛顿莱布尼茨公式

$$\int_a^b f(x)\,\mathrm{d}x=F(a)-F(b)$$

Solution 用分点

$$a=x_0<x_1<x_2<\cdots<x_{n-1}<x_n=b$$

将 $[a,b]$ 分为 n 个小区间, 记 $\Delta x_i = x_i - x_{i-1}\,(i=1,2,\cdots,n)$, $\lambda = \max\limits_{1\leqslant i\leqslant n} \Delta x_i$
应用拉格朗日中值定理, 必存在 $\xi_i \in (x_{i-1}, x_i)$ 使得

$$F(x_i) - F(x_{i-1}) = F'(\xi_i)(x_i - x_{i-1})$$

于是

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \lim_{\lambda\to 0}\sum_{i=1}^n f(\xi_i)\Delta x_i \\ &= \lim_{\lambda\to 0}\sum_{i=1}^n \big(F(x_i) - F(x_{i-1})\big) \\ &= F(b) - F(a)\end{aligned}$$

Exercise 5.23: 证明

$$\lim_{n\to\infty}\int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x = 0$$

Solution $\forall\,\varepsilon > 0\,(\varepsilon < \pi)$, 因

$$\left|\int_0^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,\mathrm{d}x\right| \leqslant \frac{\pi}{2}\sin^n\left(\frac{\pi}{2} - \frac{\varepsilon}{2}\right)$$

而 $\lim\limits_{n\to\infty}\dfrac{\pi}{2}\sin^n\left(\dfrac{\pi}{2} - \dfrac{\varepsilon}{2}\right) = 0$, 所以 $\exists\, N \in \mathbb{N}$, 当 $n > N$ 时

$$0 < \frac{\pi}{2}\sin^n\left(\frac{\pi}{2} - \frac{\varepsilon}{2}\right) < \frac{\varepsilon}{2}$$

又

$$\left|\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,\mathrm{d}x\right| \leqslant \int_{\frac{\pi}{2}-\frac{\varepsilon}{2}}^{\frac{\pi}{2}} \mathrm{d}x = \frac{\varepsilon}{2}$$

故 $\forall\,\varepsilon > 0$, $\exists \in \mathbb{N}$, 当 $n > N$ 时有

$$\left|\int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x\right| \leqslant \left|\int_0^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,\mathrm{d}x\right| + \left|\int_{\frac{\pi}{2}}^{\frac{\pi}{2}-\frac{\varepsilon}{2}} \sin^n x\,\mathrm{d}x\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

由极限的定义即得原式成立

5.2 微积分基本公式

Exercise 5.24: 设 $f(x) = \displaystyle\int_0^x \cos\frac{1}{t}\,\mathrm{d}t$, 求 $f'(0)$

✎ **Solution1** 显然 $f(0)=0$, 所以

$$
\begin{aligned}
f'(0) &= \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{1}{x}\int_0^x \cos\frac{1}{t}\,\mathrm{d}t \\
&= \lim_{x\to 0}\frac{1}{x}\int_0^x t^2\,\mathrm{d}\left(\sin\frac{1}{t}\right) = \lim_{x\to 0}\frac{1}{x}\left(x^2\sin\frac{1}{x}-\int_0^x 2t\sin\frac{1}{t}\,\mathrm{d}x\right) \\
&= \lim_{x\to 0}\frac{\int_0^x 2t\sin\frac{1}{t}\,\mathrm{d}t}{x} \xlongequal{\text{洛必达}} \lim_{x\to 0}\left(2x\sin\frac{1}{x}\right) = 0
\end{aligned}
$$

✎ **Solution2** 对 $\forall\varepsilon>0$, 取 $\delta=\dfrac{\varepsilon}{2}$, 当 $0<x<\delta$ 时, 有

$$
\begin{aligned}
\left|\frac{\int_0^x \cos\frac{1}{t}\,\mathrm{d}t}{x}\right| &= \left|\frac{\int_{+\infty}^{\frac{1}{x}} -\frac{\cos u}{u^2}\,\mathrm{d}u}{x}\right| = \frac{1}{x}\left|\frac{\sin u}{u}\Big|_{\frac{1}{x}}^{+\infty} + \int_{\frac{1}{x}}^{+\infty}\frac{2\sin u}{u^3}\,\mathrm{d}u\right| \\
&\leqslant \frac{1}{x}\left[\left|-\frac{\sin\frac{1}{x}}{\frac{1}{x^2}}\right| + \frac{2}{x}\int_{\frac{1}{x}}^{+\infty}\frac{1}{u^3}\,\mathrm{d}u\right] \\
&= x\left|\sin\frac{1}{x}\right| + \frac{1}{x}\left(-\frac{1}{u^2}\right)\Big|_{\frac{1}{x}}^{+\infty} = x\left|\sin\frac{1}{x}\right| + x \leqslant 2x < 2\sigma = \varepsilon
\end{aligned}
$$

同理, 当 $-\delta<x<0$ 时, 也有 $\left|\dfrac{\int_0^x \cos\frac{1}{t}\,\mathrm{d}t}{x}\right| < \varepsilon$,

所以

$$
f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{\int_0^x \cos\frac{1}{t}\,\mathrm{d}t}{x} = 0
$$

5.3 定积分的换元法和分部积分法

Theorem 5.1

设 $f(x)$ 在 $[0,1]$ 连续, 则

(1) $\displaystyle\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x = \int_a^{\frac{\pi}{2}} f(\cos x)\,\mathrm{d}x$

(2) $\displaystyle\int_0^{\pi} xf(\sin x)\,\mathrm{d}x = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(\sin x)\,\mathrm{d}x$

Theorem 5.2

设 $f(x)$ 是连续的周期性函数, 周期为 T, 则

(1) $\displaystyle\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x$

(2) $\displaystyle\int_a^{a+nT} f(x)\,\mathrm{d}x = n\int_0^T f(x)\,\mathrm{d}x$

Theorem 5.3 华里士公式

$$\begin{aligned}
I_n &= \int_0^{\frac{\pi}{2}} \sin^n x\,\mathrm{d}x = \int_0^{\frac{\pi}{2}} \cos^n x\,\mathrm{d}x \\
&= \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\cdots\cdot\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & n\text{为正偶数},\ I_0 = \dfrac{\pi}{2} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\cdots\cdot\dfrac{4}{5}\cdot\dfrac{2}{3}, & n\text{为大于 1 的正奇数},\ I_1 = 1 \end{cases}
\end{aligned}$$

Note:

$$\int_a^b f(x)\,\mathrm{d}x = \sum_{k=1}^{n}\int_{a+\frac{k-1}{n}(b-a)}^{a+\frac{k}{n}(b-a)} f(x)\,\mathrm{d}x$$

Note:

$$\frac{1}{\binom{m+n}{m}} = m\int_0^1 (1-x)^n x^{m-1}\,\mathrm{d}x$$

Note:

$$\frac{1}{(1+x)^y} = \frac{1}{\Gamma(y)}\int_0^{+\infty} t^{y-1}e^{-t-tx}dt$$

Exercise 5.25: 设 $F(x) = \displaystyle\int_x^{x+2\pi} \frac{\sin t}{\sin^2 t+1}\,\mathrm{d}t$, 则 $F(x)$

Solution

$$\begin{aligned}
F(x) &= \int_x^{x+2\pi} \frac{\sin t}{\sin^2 t+1}\,\mathrm{d}t = \int_0^{2\pi} \frac{\sin t}{\sin^2 t+1}\,\mathrm{d}t \\
&\xlongequal{u=t-\pi} \int_{-\pi}^{2\pi} \frac{\sin u}{\sin^2 u+1}\,\mathrm{d}u = 0
\end{aligned}$$

Note: $f(x)$ 是连续的周期性函数, 周期为 T, 则

$$\int_a^{a+T} f(x)\,\mathrm{d}x = \int_0^T f(x)\,\mathrm{d}x$$

Exercise 5.26: 令 $P_n(x)=\sum_{k=0}^{n}\frac{x^k}{k!}$, 计算极限

$$\lim_{n\to\infty}\frac{1}{n!}\int_0^1\frac{2n!\sin x-n!e^{2x}+x^n}{e^{2x}+\sin x+\cos x+P_n(x)}\,\mathrm{d}x$$

Solution 令

$$f(x)=e^{2x}+\sin x+\cos x+P_n(x)=e^{2x}+\sin x+\cos x+\sum_{k=0}^{n}\frac{x^k}{k!}$$

则

$$f'(x)=2e^{2x}+\cos x-\sin x+\sum_{k=0}^{n-1}\frac{x^k}{k!}$$

那么有

$$f(x)-f'(x)=-e^{2x}+2\sin x+\frac{x^n}{n!}$$

故

$$\begin{aligned}
I&=\lim_{n\to\infty}\frac{1}{n!}\int_0^1\frac{2n!\sin x-n!e^{2x}+x^n}{e^{2x}+\sin x+\cos x+P_n(x)}\,\mathrm{d}x\\
&=\lim_{n\to\infty}\int_0^1\frac{f(x)-f'(x)}{f(x)}\,\mathrm{d}x\\
&=\lim_{n\to\infty}\left\{\int_0^1\mathrm{d}x-\int_0^1\frac{1}{f(x)}\,\mathrm{d}(f(x))\right\}\\
&=\lim_{n\to\infty}\left\{\Big[x\Big]_0^1-\Big[\ln(f(x))\Big]_0^1\right\}\\
&=1-\ln(e^2+\sin 1+\cos 1+e)
\end{aligned}$$

Exercise 5.27: 计算积分

$$\int_0^{\frac{\pi}{2}}\sqrt{\frac{1-\cos x}{1+\cos x}}\ln^2\left(\frac{1-\cos x}{1+\cos x}\right)\frac{\mathrm{d}x}{\sin x}$$

Solution 注意到

$$\frac{1-\cos x}{1+\cos x}=\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)^2}=\frac{\sin^2 x}{(1+\cos x)^2}$$

以及

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{1+\cos x}\right)=\frac{1}{1+\cos x}$$

故

$$\begin{aligned}
I&=\int_0^{\frac{\pi}{2}}\sqrt{\frac{1-\cos x}{1+\cos x}}\ln^2\left(\frac{1-\cos x}{1+\cos x}\right)\frac{\mathrm{d}x}{\sin x}\\
&=4\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos x}\ln^2\left(\frac{\sin x}{1+\cos x}\right)\mathrm{d}x\\
&\xlongequal{\frac{\sin x}{1+\cos x}=t}4\int_0^1\ln^2 t\,\mathrm{d}t\\
&=8
\end{aligned}$$

Exercise 5.28: 计算积分

$$\int_1^2 \frac{2x-3}{\sqrt{-x^2+3x-2}}\,\mathrm{d}x$$

Solution

$$\begin{aligned}
\int_1^2 \frac{2x-3}{\sqrt{-x^2+3x-2}}\,\mathrm{d}x &\xlongequal{t=x-\frac{3}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2\left(t+\frac{3}{2}\right)-3}{\sqrt{-\left(t+\frac{3}{2}\right)^2+3\left(t+\frac{3}{2}\right)-2}}\,\mathrm{d}t \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2t}{\sqrt{-t^2+\frac{11}{4}}}\,\mathrm{d}t \\
&= 0
\end{aligned}$$

Exercise 5.29: 设 $I_n = \displaystyle\int_0^1 \sqrt[n]{x^{n^2}+x^{n^2n}}\,\mathrm{d}x \quad n \geqslant 2$ 求极限 $\displaystyle\lim_{n\to\infty} n(nI_n-1)$

Solution

$$\begin{aligned}
I_n &= \int_0^1 \sqrt[n]{x^{n^2}+x^{n^2n}}\,\mathrm{d}x = \int_0^1 x^{n-1}\sqrt[n]{x^n+1}\,\mathrm{d}x \\
&\xlongequal{x^n+1=t} \frac{1}{n}\int_0^2 t^{\frac{1}{n}}\,\mathrm{d}t \\
&= \frac{2^{\frac{n+1}{n}}-1}{n+1}
\end{aligned}$$

$$\begin{aligned}
I_n &= \lim_{n\to\infty} n(nI_n-1) = \lim_{n\to\infty}\left(\frac{n^2 2^{\frac{1}{n}+1}-n(2n+1)}{n+1}\right) \\
&= \lim_{n\to\infty} n\left(\frac{2^{1+\frac{1}{n}}-1}{1+\frac{1}{n}}-1\right) \\
&= \lim_{n\to\infty} n\left(2^{1+\frac{1}{n}}-2-\frac{1}{n}\right) = \lim_{n\to\infty}\left(2\cdot\frac{2^{\frac{1}{n}}-1}{\frac{1}{n}}-1\right) \\
&= 2\ln 2-1
\end{aligned}$$

Exercise 5.30: 计算积分:

$$\int_0^\pi \left(\sin x \ln\left|\frac{x-\pi}{x}\right| + \frac{\sqrt{x}}{\sqrt{\pi-x}+\sqrt{x}}\right) dx$$

Solution

Exercise 5.31: 计算积分

$$\int_0^{\frac{\pi}{2}} \left(\frac{x}{\sin x}\right)^2 \mathrm{d}x$$

Solution

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \left(\frac{x}{\sin x}\right)^2 \mathrm{d}x = \int_0^{\frac{\pi}{2}} x^2 d(-\cot x) \\
&= \Big[-x^2\cot x\Big]_0^{\frac{\pi}{2}} + 2\int_0^{\frac{\pi}{2}} x\cot x\,\mathrm{d}x \\
&= 0 + 2\int_0^{\frac{\pi}{2}} x\,\mathrm{d}(\ln\sin x) \\
&= \Big[2x\ln\sin x\Big]_0^{\frac{\pi}{2}} - 2\int_0^{\frac{\pi}{2}} \ln\sin x\,\mathrm{d}x \\
&= 0 - 2\times\left(-\frac{\pi}{2}\ln 2\right) \\
&= \pi\ln 2
\end{aligned}
$$

Exercise 5.32: 计算积分

$$
\int_0^{\frac{\pi}{2}} \ln\sin x\,\mathrm{d}x
$$

Solution

$$
J = \int_0^{\frac{\pi}{2}} \ln\sin x\,\mathrm{d}x \xlongequal{x=\frac{\pi}{2}-u} \int_{\frac{\pi}{2}}^0 \ln\sin\left(\frac{\pi}{2}-u\right)(-\mathrm{d}u) = \int_0^{\frac{\pi}{2}} \ln\cos x\,\mathrm{d}x
$$

$$
\begin{aligned}
J &= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \ln\sin x\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \ln\cos x\,\mathrm{d}x\right) \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right)\mathrm{d}x \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln\sin 2x\,\mathrm{d}x \\
&\xlongequal{u=2x} -\frac{\pi}{4}\ln 2 + \frac{1}{4}\int_0^{\pi} \ln\sin u\,\mathrm{d}u \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{4}\int_0^{\frac{\pi}{2}} \ln\sin u\,\mathrm{d}u + \overbrace{\frac{1}{4}\int_{\frac{\pi}{2}}^{\pi} \ln\sin u\,\mathrm{d}u}^{t=u-\frac{\pi}{2}} \\
&= -\frac{\pi}{4}\ln 2 + \frac{1}{4}\int_0^{\frac{\pi}{2}} \ln\sin u\,\mathrm{d}u + \frac{1}{4}\int_0^{\frac{\pi}{2}} \ln\cos t\,\mathrm{d}t \\
&= -\frac{\pi}{2}\ln 2
\end{aligned}
$$

✎ **Solution**

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x &\xlongequal{x=2t} 2\int_0^{\frac{\pi}{4}} \ln\sin 2t\mathrm{d}t \\
&= 2\int_0^{\frac{\pi}{4}} (\ln 2 + \ln\sin x + \ln\cos x)\mathrm{d}t \\
&= \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{4}} \ln\sin t\mathrm{d}t + 2\underbrace{\int_0^{\frac{\pi}{4}} \ln\cos t\mathrm{d}t}_{u=\frac{\pi}{2}-t} \\
&= \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{4}} \ln\sin t\mathrm{d}t + 2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\sin\left(\frac{\pi}{2}-u\right)(-\mathrm{d}u) \\
&= \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{4}} \ln\sin t\mathrm{d}t + 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\sin u\mathrm{d}u \\
&= \frac{\pi}{2}\ln 2 + 2\int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x
\end{aligned}$$

$$\Longrightarrow \int_0^{\frac{\pi}{2}} \ln\sin x\mathrm{d}x = -\frac{\pi}{2}\ln 2$$

Exercise 5.33: 计算积分: $\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx$

✎ **Solution**

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx &\xlongequal{\text{令} x=\tan t} \int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt \\
&\xlongequal{\text{令} t=\frac{\pi}{4}-u} \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan u}{1+\tan u}\right)du \\
&= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan u}\right)du \\
&= \frac{\pi}{4}\ln 2 - \int_0^{\frac{\pi}{4}} \ln(1+\tan u)\,du
\end{aligned}$$

$$\Longrightarrow \int_0^{\frac{\pi}{4}} \ln(1+\tan t)dt = \frac{\pi}{8}\ln 2$$

$$\Longrightarrow \int_0^1 \frac{\ln(1+x^2)}{1+x^2}dx = \frac{\pi}{8}\ln 2$$

Exercise 5.34: 计算积分: $\int_0^{2\pi} x\sin^8 x\,\mathrm{d}x$

✎ **Solution**

$$\begin{aligned}
\int_0^{2\pi} x\sin^8 x\,\mathrm{d}x &\xlongequal{t=x-\pi} \int_{-\pi}^{\pi}(t+\pi)\sin^8(\pi+t)\,\mathrm{d}t\\
&= \pi\int_{-\pi}^{\pi}\sin^8 t\,\mathrm{d}t + \int_{-\pi}^{\pi} t\sin^8 t\,\mathrm{d}t\\
&= 2\pi\int_0^{\pi}\sin^8 t\,\mathrm{d}t + 0\\
&= 2\pi\int_0^{\pi}\sin^8 t\,\mathrm{d}t\\
&\xlongequal{u=x-\frac{\pi}{2}} 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^8 u\,\mathrm{d}u\\
&= 4\pi\int_0^{\frac{\pi}{2}}\cos^8 u\,\mathrm{d}u\\
&= 4\pi\times\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\\
&= \frac{35\pi^2}{64}
\end{aligned}$$

Exercise 5.35: 计算积分

$$\int_0^1 \frac{dx}{(x^2-x+1)^{\frac{3}{2}}}$$

✎ **Solution**

$$\begin{aligned}
\int_0^1 \frac{dx}{(x^2-x+1)^{\frac{3}{2}}} \xlongequal{t=x-\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dt}{\left(t^2+\frac{3}{4}\right)^{\frac{3}{2}}} &= \frac{4}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{t^2+\frac{3}{4}-t^2}{\left(t^2+\frac{3}{4}\right)^{\frac{3}{2}}}dt\\
&= \frac{4}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\left(t^2+\frac{3}{4}\right)^{\frac{1}{2}}}dt - \frac{4}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{t^2}{\left(t^2+\frac{3}{4}\right)^{\frac{3}{2}}}dt\\
&= \frac{4}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\left(t^2+\frac{3}{4}\right)^{\frac{1}{2}}}dt + \frac{4}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} t d\left(\frac{1}{\left(t^2+\frac{3}{4}\right)^{\frac{1}{2}}}\right)\\
&= \frac{4}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\left(t^2+\frac{3}{4}\right)^{\frac{1}{2}}}dt + \frac{3}{4}\left[\frac{t}{\left(t^2+\frac{3}{4}\right)^{\frac{1}{2}}}\right]_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{4}{3}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\left(t^2+\frac{3}{4}\right)^{\frac{1}{2}}}dt\\
&= \frac{4}{3}
\end{aligned}$$

Exercise 5.36: 计算积分

$$\int_0^{\pi} \frac{x\sin 2x\sin\left(\frac{\pi}{2}\cos t\right)}{2x-\pi}\,\mathrm{d}x$$

✎ **Solution(西西)**

$$
\begin{aligned}
I &= \int_0^{\pi} \frac{x\sin 2x \sin\left(\frac{\pi}{2}\cos t\right)}{2x-\pi}\,\mathrm{d}x \\
&\xlongequal{t=2x-\pi} \frac{1}{4}\int_{-\pi}^{\pi} \frac{(t+\pi)\sin t \sin\left(\frac{\pi}{2}\sin\left(\frac{t}{2}\right)\right)}{t}\,\mathrm{d}t \\
&= \frac{1}{4}\int_{-\pi}^{\pi} 2\sin\frac{t}{2}\cos\frac{t}{2}\sin\left(\frac{\pi}{2}\sin\frac{t}{2}\right)\,\mathrm{d}t \\
&\xlongequal{x=\sin\frac{t}{2}} \int_{-1}^{1} x\sin\left(\frac{\pi}{2}x\right)\,\mathrm{d}x \\
&= 2\int_0^1 x\sin\left(\frac{\pi}{2}x\right)\,\mathrm{d}x \\
&= 2\Big[-\frac{2x}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big]_0^1 + 2\int_0^1 \frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\,\mathrm{d}x \\
&= 2\Big[\frac{4}{\pi^2}\sin\left(\frac{\pi}{2}x\right)\Big]_0^1 \\
&= \frac{8}{\pi^2}
\end{aligned}
$$

Exercise 5.37: 计算积分

$$
\int_{-2}^{-1} \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x
$$

✎ **Solution**

$$
\begin{aligned}
\int_{-2}^{-1} \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x &= \int_{-2}^{-1} \frac{x^2-1}{x\sqrt{x^2-1}}\,\mathrm{d}x \\
&= \int_{-2}^{-1} \frac{x}{\sqrt{x^2-1}}\,\mathrm{d}x - \int_{-2}^{-1} \frac{1}{x\sqrt{x^2-1}}\,\mathrm{d}x \\
&= \int_{-2}^{-1} \frac{1}{2\sqrt{x^2-1}}\,\mathrm{d}(x^2-1) - \int_{-2}^{-1} \frac{1}{\sqrt{1-(\frac{1}{x})^2}}\,\mathrm{d}\left(\frac{1}{x}\right) \\
&= \Big[\sqrt{x^2-1}\Big]_{-2}^{-1} - \Big[\arcsin\left(\frac{1}{x}\right)\Big]_{-2}^{-1} \\
&= \frac{\pi}{3} - \sqrt{3}
\end{aligned}
$$

Exercise 5.38: 计算积分

$$
\int \frac{1}{x\sqrt{x^2-2x-3}}\,\mathrm{d}x
$$

Solution

$$
\begin{aligned}
\int \frac{1}{x\sqrt{x^2-2x-3}}\,\mathrm{d}x &= \int \frac{1}{x\sqrt{(x-1)^2-4}}\,\mathrm{d}x \\
&\xlongequal{x-1=2\sec t} \int \frac{2\tan t\sec t}{2(2\sec t+1)\tan t}\,\mathrm{d}t = \int \frac{1}{2+\cos t}\,\mathrm{d}t \\
&= \int \frac{2-\cos t}{4-\cos^2 t}\,\mathrm{d}t \\
&= 2\int \frac{1}{4\sin^2 t+3\cos^2 t}\,\mathrm{d}t - \int \frac{\cos t}{3+\sin^2 t}\,\mathrm{d}t \\
&= \int \frac{1}{(2\tan t)^2+3}\,\mathrm{d}(2\tan t) - \int \frac{1}{3+\sin^2 t}\,\mathrm{d}(\sin t) \\
&= \frac{1}{\sqrt{3}}\arctan\frac{2\tan t}{\sqrt{3}} - \frac{1}{2\sqrt{3}}\arctan\frac{\sin t}{\sqrt{3}} + C \\
&= \frac{1}{\sqrt{3}}\arctan\frac{\frac{2\tan t}{\sqrt{3}} - \frac{\sin t}{\sqrt{3}}}{1+\frac{2\tan t}{\sqrt{3}}\times\frac{\sin t}{\sqrt{3}}} + C \\
&= -\frac{1}{\sqrt{3}}\arctan\frac{x+3}{\sqrt{3}\sqrt{x^2-2x-3}} + C
\end{aligned}
$$

Exercise 5.39: 计算积分:

$$
\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\{(2x+1)\sqrt{x^2-x+1}+(2x-1)\sqrt{x^2+x+1}\}\sqrt{x^4+x^2+1}}dx
$$

Solution

$$
\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x}{\{(2x+1)\sqrt{x^2-x+1}+(2x-1)\sqrt{x^2+x+1}\}\sqrt{x^4+x^2+1}}dx \\
=&\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x[(2x+1)\sqrt{x^2-x+1}-(2x-1)\sqrt{x^2+x+1}]}{[(2x+1)^2(x^2-x+1)-(2x-1)^2(x^2+x+1)]\sqrt{x^4+x^2+1}}dx \\
=&\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x[(2x+1)\sqrt{x^2-x+1}-(2x-1)\sqrt{x^2+x+1}]}{6x\sqrt{x^4+x^2+1}}dx \\
=&\frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(2x+1)\sqrt{x^2-x+1}-(2x-1)\sqrt{x^2+x+1}}{\sqrt{x^4+x^2+1}}dx \\
=&\frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(2x+1)\sqrt{x^2-x+1}-(2x-1)\sqrt{x^2+x+1}}{\sqrt{(x^2-x+1)(x^2+x+1)}}dx \\
=&\frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2x+1}{\sqrt{x^2+x+1}}dx - \frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{2x-1}{\sqrt{x^2-x+1}}dx \\
=&\frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d(x^2+x+1)}{\sqrt{x^2+x+1}} - \frac{1}{6}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{d(x^2-x+1)}{\sqrt{x^2-x+1}} \\
=&\frac{1}{3}\left[\sqrt{x^2+x+1}\right]_{-\frac{1}{2}}^{\frac{1}{2}} - \frac{1}{3}\left[\sqrt{x^2-x+1}\right]_{-\frac{1}{2}}^{\frac{1}{2}} \\
=&\frac{\sqrt{7}-\sqrt{3}}{3}
\end{aligned}
$$

Exercise 5.40: 计算积分:

$$\int_0^1 \frac{x}{\{(2x-1)\sqrt{x^2+x+1}+(2x+1)\sqrt{x^2-x+1}\}\sqrt{x^4+x^2+1}}dx$$

Solution

$$a(x)=\sqrt{x^2+x+1}=\sqrt{(x+\frac{1}{2})^2+\frac{3}{4}},\Rightarrow a'(x)=\frac{x+\frac{1}{2}}{a(x)}$$

$$\begin{aligned}
\text{原式} &= \frac{1}{2}\int_0^1 \frac{x}{[(x-\frac{1}{2})a(x)+(x+\frac{1}{2})a(-x)]a(x)a(-x)}dx \\
&= \frac{1}{2}\int_0^1 \frac{x}{a^2(x)(a^2-x)[a'(x)-a'(-x)]}dx = \frac{1}{2}\int_0^1 \frac{x[a'(x)+a'(-x)]dx}{a^2(x)a^2(-x)\{[a'(x)]^2-[a'(-x)]^2\}} \\
&= \frac{1}{2}\int_0^1 \frac{x[a'(x)+a'(-x)]dx}{(x+\frac{1}{2})^2a^2(-x)^2-(x-\frac{1}{2})^2a^2(x)} = \frac{2}{3}\int_0^1 \frac{x[a'(x)+a'(-x)]dx}{2x} \\
&= \frac{1}{3}\int_0^1 [a'(x)+a'(-x)] = \frac{a(1)-a(-1)}{3} = \frac{\sqrt{3}-1}{3}
\end{aligned}$$

Exercise 5.41: 计算积分

$$\int_{\frac{1}{e}}^{e} \frac{\ln^2 x}{1+x}\mathrm{d}x$$

Solution

$$\begin{aligned}
\int_{\frac{1}{e}}^{e} \frac{\ln^2 x}{1+x}\mathrm{d}x &= \int_e^{\frac{1}{e}} \frac{\ln^2\left(\frac{1}{t}\right)}{1+\frac{1}{t}} \times \frac{-1}{t^2}\mathrm{d}t = \int_{\frac{1}{e}}^{e} \frac{\ln^2 t}{t(t+1)}\mathrm{d}t \\
&= \int_{\frac{1}{e}}^{e} \frac{\ln^2 t}{t}\mathrm{d}t - \int_{\frac{1}{e}}^{e} \frac{\ln^2 t}{t+1}\mathrm{d}t \\
&= \frac{1}{2}\int_{\frac{1}{e}}^{e} \frac{\ln^2 t}{t}\mathrm{d}t \\
&= \frac{1}{2}\left[\frac{1}{3}\ln^3 t\right]_{\frac{1}{e}}^{e} \\
&= \frac{1}{3}
\end{aligned}$$

Exercise 5.42: 计算积分

$$\int_{-\infty}^{+\infty} \frac{\sin^3 x}{x^3}dx$$

Solution 注意到

$$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$$

所以

$$\begin{aligned}
\int_{-\infty}^{+\infty}\frac{\sin^3 x}{x^3}dx &= \int_{-\infty}^{+\infty}\frac{\frac{3}{4}\sin x-\frac{1}{4}\sin 3x}{x^3}dx\\
&=\frac{3}{4}\int_{-\infty}^{+\infty}\frac{\sin x}{x^3}dx-\frac{1}{4}\int_{-\infty}^{+\infty}\frac{\sin 3x}{x^3}dx\\
&=\frac{3}{4}\int_{-\infty}^{+\infty}\sin xd\left(\frac{-1}{2x^2}\right)-\frac{1}{4}\int_{-\infty}^{+\infty}\sin 3xd\left(\frac{-1}{2x^2}\right)\\
&=\left[\frac{-3\sin x}{8x^2}\right]_{-\infty}^{+\infty}+\frac{3}{8}\int_{-\infty}^{+\infty}\frac{\cos x}{x^2}dx+\left[\frac{\sin 3x}{8x^2}\right]_{-\infty}^{+\infty}-\frac{3}{8}\int_{-\infty}^{+\infty}\frac{\cos 3x}{x^2}dx\\
&=\frac{3}{8}\int_{-\infty}^{+\infty}\cos xd\left(\frac{-1}{x}\right)-\frac{3}{8}\int_{-\infty}^{+\infty}\cos 3xd\left(\frac{-1}{x}\right)\\
&=\left[\frac{-3\cos x}{8x^2}\right]_{-\infty}^{+\infty}-\frac{3}{8}\int_{-\infty}^{+\infty}\frac{\sin x}{x}dx+\left[\frac{-3\cos 3x}{8x^2}\right]_{-\infty}^{+\infty}+\frac{9}{8}\int_{-\infty}^{+\infty}\frac{\sin 3x}{3x}d(3x)\\
&=-\frac{3\pi}{8}+\frac{9\pi}{8}=\frac{3\pi}{4}
\end{aligned}$$

Exercise 5.43: 计算积分

$$\int_0^1 x\arcsin\left(2\sqrt{x(1-x)}\right)\,dx$$

Solution

$$\begin{aligned}
\int_0^1 x\arcsin\left(2\sqrt{x(1-x)}\right)\,dx &= -\int_1^0(1-t)\arcsin\left(2\sqrt{(1-t)t}\right)\,dt\\
&=\int_0^1(1-t)\arcsin\left(2\sqrt{(1-t)t}\right)\,dt\\
&=\frac{1}{2}\int_0^1\arcsin\left(2\sqrt{x(1-x)}\right)\,dx\\
&=\left[\frac{1}{2}x\arcsin\left(2\sqrt{x(1-x)}\right)\right]_0^1-\frac{1}{2}\int_0^1\frac{x(2x-1)}{|2x-1|\sqrt{x-x^2}}\,dx\\
&=\frac{1}{2}\int_0^1 x\,d\left(\frac{1}{x^2+1}\right)-\frac{1}{2}\int_1^{+\infty}x\,d\left(\frac{1}{x^2+1}\right)\\
&=\frac{1}{2}
\end{aligned}$$

Exercise 5.44: 计算积分: $\int_0^\infty \frac{\ln x}{x^2+3x+2}\mathrm{d}x$

Solution 在 $\int_0^\infty \frac{\ln x}{x^2+3x+2}dx$ 中作代换 $x=\sqrt{2}u$ 得:

$$\sqrt{2}\int_0^\infty\frac{\ln(\sqrt{2}u)}{2u^2+3\sqrt{2}u+2}dx=\frac{\sqrt{2}\ln 2}{2}\int_0^\infty\frac{1}{2u^2+3\sqrt{2}u+2}dx+\sqrt{2}\int_0^\infty\frac{\ln u}{2u^2+3\sqrt{2}u+2}dx$$

其中后者积分为 0

所以

$$\begin{aligned}\int_0^{\infty}\frac{\ln x}{x^2+3x+2}dx &= \frac{\sqrt{2}\ln 2}{4}\int_0^{\infty}\frac{1}{u^2+\frac{3\sqrt{2}}{2}u+1}dx \\ &= \frac{\sqrt{2}\ln 2}{4}\int_0^{\infty}\frac{1}{(u^2+\frac{3\sqrt{2}}{4})^2-\frac{1}{8}}dx = \frac{\sqrt{2}\ln 2}{4}\int_{\frac{3\sqrt{2}}{4}}^{\infty}\frac{1}{x^2-\frac{1}{8}}dx \\ &= \frac{\sqrt{2}\ln 2}{4}\int_{\frac{3\sqrt{2}}{4}}^{\infty}\frac{1}{x^2-\frac{1}{8}}dx = \frac{\sqrt{2}\ln 2}{4}\left(\sqrt{2}\ln 2\right) = \frac{\ln^2 2}{2}\end{aligned}$$

Exercise 5.45: 计算积分: $\int_0^1 \ln(1-x)\ln x\ln(1+x)\mathrm{d}x$

Solution

$$\begin{aligned}I &= \int_0^1 \ln(1-x)\ln x\ln(1+x)dx \\ &= \int_0^1 \ln(1-x)\ln(1+x)d(x\ln x-x+1) \\ &= \int_0^1 (x\ln x-x+1)\left[\frac{\ln(1+x)}{1-x}-\frac{\ln(1-x)}{1+x}\right]dx \\ &=2\int_0^1 (x\ln x-x+1)\left[\sum_{n=0}^{\infty}(H_{2n+1}-H_n)x^{2n+1}\right]dx \qquad (H_0=0) \\ &=2\sum_{n=0}^{\infty}(H_{2n+1}-H_n)\int_0^1 (x\ln x-x+1)x^{2n+1}dx \\ &=2\sum_{n=0}^{\infty}\frac{H_{2n+1}-H_n}{(2n+3)(2n+2)}-2\sum_{n=0}^{\infty}\frac{H_{2n+1}-H_n}{(2n+3)^2} \\ &=\frac{\pi^2}{6}-\ln^2 2-2+2\ln 2-2\left[\frac{7\zeta(3)}{16}+2-\ln 2-\frac{\pi^2}{8}-\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)^2}\right] \\ &=\frac{5\pi^2}{12}-\ln^2 2-6+4\ln 2+\frac{21\zeta(3)}{8}-\frac{\pi^2\ln 2}{2}\end{aligned}$$

Exercise 5.46: 计算积分:

$$\int_0^1 \frac{\ln(1+x)}{x(1+x^2)}\mathrm{d}x$$

Solution 此题需要用到

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$$

记 $F(a)=\int_0^1 \frac{\ln(1+ax)}{x(1+x^2)}dx$, 其中 $a>0$, 则 $F'(a)=\int_0^1 \frac{1}{(1+ax)(1+x^2)}dx$

采用部分分式

$$\frac{1+a^2}{(1+ax)(1+x^2)}=\frac{a^2}{1+ax}+\frac{1-ax}{1+x^2}$$

有

$$F'(a)=\frac{a}{1+a^2}\ln(1+a)+\frac{1}{1+a^2}\frac{\pi}{4}-\frac{a}{2(1+a^2)}\ln 2,$$

又因为 $F(0)=0$
所以

$$F(1)=\int_0^1\frac{\ln(1+x)}{x(1+x^2)}dx=\int_0^1\left[\frac{x}{1+x^2}\ln(1+x)+\frac{1}{1+x^2}\frac{\pi}{4}-\frac{x}{2(1+x^2)}\ln 2\right]dx$$
$$=\int_0^1\frac{x\ln(1+x)}{1+x^2}dx+\frac{\pi^2}{16}-\frac{\ln^2 2}{4}=\int_0^1\frac{\ln(1+x)}{x}-\frac{\ln(1+x)}{x(1+x^2)}dx$$

移项即有

$$\int_0^1\frac{\ln(1+x)}{x(1+x^2)}dx=\frac{1}{2}\left(\frac{\pi^2}{16}-\frac{\ln^2 2}{4}\right)+\frac{1}{2}\int_0^1\frac{\ln(1+x)}{x}dx$$

对于 $\int_0^1\frac{\ln(1+x)}{x}dx$ 利用 $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots$
便有

$$\int_0^1\frac{\ln(1+x)}{x}dx=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}\ldots=\frac{\pi^2}{12}$$

所以

$$\int_0^1\frac{\ln(1+x)}{x(1+x^2)}dx=\frac{1}{2}\left(\frac{\pi^2}{16}-\frac{\ln^2 2}{4}\right)+\frac{1}{2}\frac{\pi^2}{12}=\frac{7\pi^2}{96}-\frac{\ln^2 2}{8}$$

Exercise 5.47: 计算积分

$$I=\int_0^1\ln(1+x)\ln(1-x)\mathrm{d}x$$

Solution 因为

$$\ln(1+x)\ln(1-x)=\sum_{n=1}^{\infty}\frac{H_n-H_{2n}-\frac{1}{2n}}{n}x^{2n}$$

所以

$$\int_0^1\ln(1+x)\ln(1-x)dx=\sum_{n=1}^{\infty}\frac{H_n-H_{2n}}{n(2n+1)}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2(2n+1)}$$

Since

$$
\begin{aligned}
I &= \int_0^1 \ln(2-x)\ln x dx \\
&= -\int_0^1 x[\frac{\ln(2-x)}{x} - \frac{\ln x}{2-x}]dx \\
&= 1 - 2\ln 2 + \int_0^1 \frac{x\ln x}{2-x}dx \\
&= 1 - 2\ln 2 + 2\int_0^{\frac{1}{2}} \frac{(2x)\ln(2x)}{2-2x}dx \\
&= 1 - 3\ln 2 + 2\ln^2 2 + 2\int_0^{\frac{1}{2}} \frac{x\ln x}{1-x}dx \\
&= 1 - 3\ln 2 + 2\ln^2 2 + 2\sum_{k=0}^{\infty}\int_0^{\frac{1}{2}} x^{k+1}\ln x dx \\
&= 1 - 3\ln 2 + 2\ln^2 2 - \sum_{k=0}^{\infty} \frac{\ln 2}{(k+2)2^{k+1}} + \frac{1}{(k+2)^2 2^{k+1}}dx \\
&= 1 - 3\ln 2 + 2\ln^2 2 - \ln 2[2\ln 2 - 1] - \frac{\pi^2}{6} + \ln^2 2 + 1 \qquad \text{The value of } \mathrm{Li}_2(\frac{1}{2}) \\
&= 2 - \frac{\pi^2}{6} - 2\ln 2 + \ln^2 2
\end{aligned}
$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)} = \frac{\pi^2}{6} + 4\ln 2 - 4$$

所以

$$\sum_{n=1}^{\infty} \frac{H_{2n} - H_n}{n(2n+1)} = \frac{\pi^2}{12} - \ln^2 2$$

Exercise 5.48: 计算积分: $\int_0^{\infty} \frac{\ln x}{x^2+3x+9}\mathrm{d}x$

Solution 在 $\int_0^{\infty} \frac{\ln x}{x^2+3x+9}dx$ 中做代换 $x = 3u$ 有

$$3\int_0^{\infty} \frac{\ln(3u)}{9u^2+9u+9}du = \frac{\ln 3}{3}\int_0^{\infty} \frac{1}{u^2+u+1}du + \frac{1}{3}\int_0^{\infty} \frac{\ln u}{u^2+u+1}du$$

如果在 $\int_0^{\infty} \frac{\ln u}{u^2+u+1}du$ 做代换 $u = \frac{1}{t}$ 即得

$$\int_0^{\infty} \frac{\ln u}{u^2+u+1}du = \int_0^{\infty} \frac{\ln \frac{1}{t}}{t^2+t+1}dt \Rightarrow \int_0^{\infty} \frac{\ln u}{u^2+u+1}du = 0$$

所以

$$\int_0^{\infty}\frac{\ln x}{x^2+3x+9}dx=\frac{\ln 3}{3}\int_0^{\infty}\frac{1}{u^2+u+1}du=\frac{\ln 3}{3}\int_0^{\infty}\frac{1}{(u+\frac{1}{2})^2+\frac{3}{4}}du$$
$$=\frac{\ln 3}{3}\left[\frac{2}{\sqrt{3}}\arctan\frac{2(u+\frac{1}{2})}{\sqrt{3}}\right]_0^{\infty}=\frac{2\ln 3}{3\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$$
$$=\frac{2\pi\ln 3}{9\sqrt{3}}$$

Exercise 5.49: 计算积分: $\displaystyle\int_0^{\infty}\frac{\arctan x}{x^2+x+1}\mathrm{d}x$

Solution 在 $\displaystyle\int_0^{\infty}\frac{\arctan x}{x^2+x+1}dx$ 中做倒代换即有

$$\int_0^{\infty}\frac{\arctan x}{x^2+x+1}dx=\int_0^{\infty}\frac{\arctan\frac{1}{x}}{\frac{1}{x^2}+\frac{1}{x}+1}\frac{1}{x^2}dx=\int_0^{\infty}\frac{\frac{\pi}{2}-\arctan x}{x^2+x+1}dx$$

其中利用了恒等式 $\arctan\dfrac{1}{x}+\arctan x=\dfrac{\pi}{2}$

所以

$$\int_0^{\infty}\frac{\arctan x}{x^2+x+1}dx=\frac{\pi}{4}\int_0^{\infty}\frac{1}{x^2+x+1}dx=\frac{\pi}{4}\int_0^{\infty}\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}dx$$
$$=\frac{\pi}{4}\left[\frac{2}{\sqrt{3}}\arctan\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right]_0^{\infty}=\frac{\pi}{2\sqrt{3}}\left[\frac{\pi}{2}-\frac{\pi}{6}\right]=\frac{\pi^2}{6\sqrt{3}}$$

Exercise 5.50: 计算积分:

$$\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{(1+\sin 2\theta)^2}d\theta$$

Solution

$$\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{(1+\sin 2\theta)^2}d\theta=\int_0^{\frac{\pi}{2}}\frac{2\sin\theta\cos\theta}{(\sin\theta+\cos\theta)^2}d\theta=\int_0^{\frac{\pi}{2}}\frac{2\tan\theta\sec^2x}{(\tan\theta+1)^4}d\theta$$
$$=\int_0^{\frac{\pi}{2}}\frac{2\left(\tan\theta+1\right)-2}{(\tan\theta+1)^4}d\left(\tan\theta+1\right)$$
$$=2\int_0^{\frac{\pi}{2}}\frac{1}{(\tan\theta+1)^3}d\left(\tan\theta+1\right)-2\int_0^{\frac{\pi}{2}}\frac{1}{(\tan\theta+1)^4}d\left(\tan\theta+1\right)$$
$$=\left[\frac{-1}{(\tan\theta+1)^2}\right]_0^{\frac{\pi}{2}}-\left[\frac{-2}{3(\tan\theta+1)^3}\right]_0^{\frac{\pi}{2}}=\frac{1}{3}$$

Exercise 5.51: 计算积分:

$$\int_0^1\frac{\ln^2 x}{1+x^2}dx$$

Solution 注意到

$$\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^n x^{2n}$$

所以

$$\begin{aligned}\int_0^1 \ln^2 x \sum_{n=0}^{\infty}(-1)^n x^{2n}dx &= \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{2n}\ln^2 x dx \left(\frac{x^{2n+1}}{2n+1}\right)\\ &= \sum_{n=0}^{\infty}(-1)^n \int_0^1 \ln^2 x d\left(\frac{x^{2n+1}}{2n+1}\right)\\ &= \sum_{n=0}^{\infty}(-1)^n \left(\left[\frac{x^{2n+1}\ln^2 x}{2n+1}\right]_0^1 - 2\int_0^1 \frac{x^{2n}\ln x}{(2n+1)}dx\right)\\ &= \sum_{n=0}^{\infty}(-1)^n \int_0^1 \ln x d\left(\frac{-x^{2n+1}}{(2n+1)^2}\right)\\ &= \sum_{n=0}^{\infty}(-1)^n \left(\left[-\frac{x^{2n+1}\ln x}{(2n+1)^2}\right]_0^1 + 2\int_0^1 \frac{x^{2n}}{(2n+1)^2}dx\right)\\ &= 2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\\ &= \frac{\pi^3}{16} \approx 1.93789229\end{aligned}$$

✎ **Solution 注意到**

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

令

$$f(a) = \int_0^1 \frac{x^a}{1+x^2}dx$$

所以

$$\begin{aligned}f(a) &= \int_0^1 x^a \sum_{n=0}^{\infty}(-1)^n x^{2n}dx = \sum_{n=0}^{\infty}(-1)^n \int_0^1 x^{2n+a}dx\\ &= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1+a}\end{aligned}$$

所以

$$f''(a) = \int_0^1 \frac{x^a \ln^2 x}{1+x^2}dx = \sum_{n=0}^{\infty}(-1)^n \frac{2}{(2n+1+a)^3}$$

因此

$$f''(0) = \int_0^1 \frac{\ln^2 x}{1+x^2}dx = 2\sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n+1)^3} = \frac{\pi^3}{16} \approx 1.93789229$$

Exercise 5.52: 计算积分:

$$\int_0^{\frac{\pi}{2}} \tan x \cdot \frac{a+\cos x \cdot \ln(\tan x)}{1+\tan x}\,\mathrm{d}x$$

Solution

$$\int_0^{\frac{\pi}{2}} \tan x \cdot \frac{a+\cos x\cdot\ln(\tan x)}{1+\tan x}\,\mathrm{d}x = a\int_0^{\frac{\pi}{2}} \frac{\tan x}{1+\tan x}\,\mathrm{d}x + \int_0^{\frac{\pi}{2}} \frac{\sin x\cdot\ln(\tan x)}{1+\tan x}\,\mathrm{d}x$$

其中

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\tan x}{1+\tan x}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}}\mathrm{d}x - \int_0^{\frac{\pi}{2}}\frac{1}{1+\tan x}\,\mathrm{d}x\\ &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}}\frac{\cos x}{\cos x+\sin x}\,\mathrm{d}x\\ &\xlongequal{t=\frac{\pi}{2}-x} \frac{\pi}{2} - \int_0^{\frac{\pi}{2}}\frac{\sin t}{\cos t+\sin t}\,\mathrm{d}t\\ &= \frac{\pi}{4}\end{aligned}$$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin x\cdot\ln(\tan x)}{1+\tan x}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}}\frac{\sin x\cos x\cdot\ln(\tan x)}{\cos x+\sin x}\,\mathrm{d}x\\ &\xlongequal{t=\frac{\pi}{2}-x} -\int_0^{\frac{\pi}{2}}\frac{\cos t\sin t\cdot\ln(\tan t)}{\cos t+\sin t}\,\mathrm{d}t\\ &= 0\end{aligned}$$

故

$$\int_0^{\frac{\pi}{2}} \tan x \cdot \frac{a+\cos x\cdot\ln(\tan x)}{1+\tan x}\,\mathrm{d}x = \frac{a\pi}{4}$$

Exercise 5.53: 计算积分:

$$\int_0^1 \frac{\ln^2(1-x)\ln x}{x}dx$$

Solution 方法 1

$$\begin{aligned}\int_0^1 \frac{\ln^2(1-x)\ln x}{x}dx &= \int_0^1 \ln(1-x)\ln x d\mathrm{Li}_2(x)\\ &= \int_0^1 \mathrm{Li}_2(x)\frac{\ln(1-x)}{x}dx - \int_0^1 \mathrm{Li}_2(x)\frac{\ln x}{1-x}dx\\ &= -\frac{1}{2}\mathrm{Li}_2^2(1) - \int_0^1 \frac{\ln x}{1-x}\sum_{n=1}^{\infty}\frac{x^n}{n^2}dx\\ &= -\frac{1}{2}\mathrm{Li}_2^2(1) + \sum_{n=1}^{\infty}\frac{1}{n^2}\sum_{k=n+1}^{\infty}\frac{1}{k^2}\\ &= -\frac{\pi^4}{72} + \frac{\pi^4}{120} = -\frac{\pi^4}{180}\end{aligned}$$

✎ **Solution 方法 2**

$$\begin{aligned}
\int_0^1 \frac{\ln^2(1-x)\ln x}{x}dx &= \frac{1}{2}\ln^2 x\ln^2(1-x)\Big|_0^1 + \int_0^1 \frac{\ln^2 x\ln(1-x)}{1-x}dx \\
&= \int_0^1 \sum_{k=1}^{\infty}(-1)^{2k-1}H_k x^k \ln^2 x dx = \sum_{k=1}^{\infty}(-1)^{2k-1}H_k \int_0^1 x^k \ln^2 x dx \\
&= 2\sum_{k=1}^{\infty}(-1)^{2k-1}\frac{H_k}{(k+1)^3} \\
&= 2\sum_{k=1}^{\infty}(-1)^{2k-1}\left[\frac{H_{k+1}}{(k+1)^3} - \frac{1}{(k+1)^4}\right] \\
&= 2\sum_{k=1}^{\infty}(-1)^{2k-1}\left[\frac{H_k}{k^3} - \frac{1}{k^4}\right] \\
&= -\frac{\pi^4}{36} + \frac{\pi^4}{45} = -\frac{\pi^4}{180}
\end{aligned}$$

Exercise 5.54: 计算积分:

$$\int_0^{\frac{\pi}{4}} \ln\sin x dx$$

✎ **Solution 我们知道卡特兰常数 G 有一个定义**

$$\mathrm{G} = \int_0^{\frac{\pi}{4}} \ln\cot x dx = \int_0^{\frac{\pi}{4}} \ln\cos x dx - \int_0^{\frac{\pi}{4}} \ln\sin x dx \quad (\mathrm{a})$$

$$\begin{aligned}
令 \int_0^{\frac{\pi}{2}} \ln\sin x dx &\overset{x=2t}{=\!=\!=} 2\int_0^{\frac{\pi}{4}} \ln\sin t dt \\
&= 2[\int_0^{\frac{\pi}{4}} \ln\cos x dx + \int_0^{\frac{\pi}{4}} \ln\sin x dx + \frac{\pi}{4}\ln 2] \\
&= 2(\int_0^{\frac{\pi}{4}} \ln\cos x dx + \int_0^{\frac{\pi}{4}} \ln\sin x dx) + \frac{\pi}{2}\ln 2 = -\frac{\pi}{2}\ln 2
\end{aligned}$$

得到 $\int_0^{\frac{\pi}{4}} \ln\cos x dx + \int_0^{\frac{\pi}{4}} \ln\sin x dx = -\frac{\pi}{2}\ln 2 \quad (\mathrm{b})$

结合 a, b 得到

$$\begin{aligned}
\int_0^{\frac{\pi}{4}} \ln\sin x dx &= -\frac{1}{2}\left(\frac{\pi}{2}\ln 2 + G\right) \\
\int_0^{\frac{\pi}{4}} \ln\cos x dx &= \frac{1}{2}\left(-\frac{\pi}{2}\ln 2 + G\right)
\end{aligned}$$

Exercise 5.55: 计算积分:

$$\int_0^1 \frac{1}{2-x}\ln\frac{1}{x}dx$$

✎ **Solution 此题需用到**

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$$

$$
\begin{aligned}
\int_0^1 \frac{1}{2-x}\ln\frac{1}{x}dx &= -\int_0^1 \frac{\ln x}{2-x}dx \\
&\xlongequal{2-x=t} \int_2^1 \frac{1}{t}\ln(2-t)dt \\
&= -\int_1^2 \frac{\ln 2}{t}dt - \int_1^2 \frac{\ln(1-\frac{t}{2})}{t}dt \\
&= -(\ln 2)^2 + \int_1^2 2\sum_{n=1}^{\infty}\frac{(-1)^n(-\frac{t}{2})^n}{n}\cdot\frac{1}{t}dt \\
&= -(\ln 2)^2 + \int_1^2 \frac{1}{t}\sum_{n=1}^{\infty}\frac{t^n}{2^t\cdot n}dt \\
&= -(\ln 2)^2 + \sum_{n=1}^{\infty}\frac{t^n}{2^n n^2}\Big|_1^2 = -(\ln 2)^2 + \sum_{n=1}^{\infty}\frac{1}{n^2} - \sum_{n=1}^{\infty}\frac{1}{2^n n^2} \\
&= -(\ln 2)^2 + \frac{\pi^2}{6} - \sum_{n=1}^{\infty}\frac{1}{2^n n^2} \\
&= -(\ln 2)^2 + \frac{\pi^2}{6} - \left(\frac{\pi^2}{12} - \frac{(\ln 2)^2}{2}\right) \\
&= \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2}
\end{aligned}
$$

Note: 设 $f(x) = \sum_{n=1}^{\infty}\frac{x^n}{n^2}$, 当 $x \in (0,1)$ 时, 有:$f(x) + f(1-x) + \ln x\ln(1-x) = \sum_{n=1}^{\infty}\frac{1}{n^2}$

Exercise 5.56: 计算积分:

$$\int_0^1 \left(\frac{\arcsin x}{x}\right)^3 dx$$

Solution 这里需要一些公式

$$\cot x = \frac{\cos x}{\sin x},\ \csc x = \frac{1}{\sin x}$$

$$\frac{d}{dx}\cot x = -\csc^2 x,\ \frac{d}{dx}\csc x = -\cot x\csc x,\ \csc^2 x = \cot^2 x + 1$$

做代换 $x = \sin u$, 有

$$\int_0^1 \left(\frac{\arcsin x}{x}\right)^3 dx = \int_0^{\frac{\pi}{2}} u^3\frac{\cos u}{\sin^3 u}du = \int_0^{\frac{\pi}{2}} u^3\cot u\csc^2 u du$$

利用分布积分

$$= -\int_0^{\frac{\pi}{2}} u^3\cot u d(\cot u) = \int_0^{\frac{\pi}{2}} (3u^2\cot u - u^3\csc^2 u)\cot u du$$

其中利用了 $\cot\frac{\pi}{2} = 0$ 这个值所以

$$\int_0^{\frac{\pi}{2}} u^3\cot u\csc^2 u du = 3\int_0^{\frac{\pi}{2}} u^2\cot^2 u du - \int_0^{\frac{\pi}{2}} u^3\cot u\csc^2 u du$$

移项:

$$\int_0^{\frac{\pi}{2}} u^3 \cot u \csc^2 u du = \frac{3}{2}\int_0^{\frac{\pi}{2}} u^2 \cot^2 u du = \frac{3}{2}\int_0^{\frac{\pi}{2}} u^2(\csc^2 u - 1) du$$
$$= -\frac{3}{2}\int_0^{\frac{\pi}{2}} u^2 d(\cot u) - \frac{3}{2}\int_0^{\frac{\pi}{2}} u^2 du = 3\int_0^{\frac{\pi}{2}} u\cot u du - \frac{\pi^3}{16}$$

留意到

$$\int_0^{\frac{\pi}{2}} u\cot u du = \int_0^{\frac{\pi}{2}} u d(\ln(\sin u)) = -\int_0^{\frac{\pi}{2}} \ln(\sin x) dx$$

做代换 $x = \frac{\pi}{2} - u$ 有 $\int_0^{\frac{\pi}{2}} \ln(\sin x)dx = \int_0^{\frac{\pi}{2}} \ln(\cos x)dx$ 所以

$$2\int_0^{\frac{\pi}{2}} \ln(\sin x)dx = \int_0^{\frac{\pi}{2}} \ln(\frac{\sin 2x}{2})dx = \int_0^{\frac{\pi}{2}} \ln(\sin 2x)dx - \frac{\pi}{2}\ln 2$$
$$= \frac{1}{2}\int_0^{\pi} \ln(\sin x)dx - \frac{\pi}{2}\ln 2 = \int_0^{\frac{\pi}{2}} \ln(\sin x)dx - \frac{\pi}{2}\ln 2$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \ln(\sin x)dx = -\frac{\pi}{2}\ln 2 \Rightarrow \int_0^{\frac{\pi}{2}} u\cot u du = \frac{\pi}{2}\ln 2$$

最后就有

$$\int_0^1 \left(\frac{\arcsin x}{x}\right)^3 dx = \frac{3\pi}{2}\ln 2 - \frac{\pi^3}{16}$$

Exercise 5.57: 计算积分:

$$\int_0^{\infty} \frac{dx}{\sqrt{x}[x^2 + (1+2\sqrt{2})x + 1][1 - x + x^2 + \cdots + x^{50}]}$$

Solution 先计算积分

$$I = \int_0^{\infty} \frac{dx}{\sqrt{x}[x^2 + ax + 1]\sum\limits_{k=0}^{n}(-x)^k} \tag{5.3}$$

$$\begin{aligned} I &= \int_0^{\infty} \frac{(-1)^n x^{n+1} dx}{\sqrt{x}[x^2 + ax + 1]\sum\limits_{k=0}^{n}(-x)^k} \\ &= \frac{1}{2}\int_0^{\infty} \frac{1 - (-x)^{n+1}}{\sqrt{x}[x^2 + ax + 1]\sum\limits_{k=0}^{n}(-x)^k} dx \\ &= \frac{1}{2}\int_0^{\infty} \frac{1 + x}{\sqrt{x}[x^2 + ax + 1]} dx \\ &= \int_0^{\infty} \frac{1 + x^2}{x^4 + ax^2 + 1} dx \\ &= \int_0^{\infty} \frac{1}{(x - \frac{1}{x})^2 + 2 + a} d(x - \frac{1}{x}) \\ &= \frac{1}{\sqrt{2+a}}\arctan \frac{x - \frac{1}{x}}{\sqrt{2+a}}\Big|_0^{\infty} = \frac{\pi}{\sqrt{2+a}} \end{aligned}$$

所以

$$\int_0^\infty \frac{dx}{\sqrt{x}[x^2+(1+2\sqrt{2})x+1][1-x+x^2+\cdots+x^{50}]} = \frac{\pi}{\sqrt{2+a}}$$

Exercise 5.58: 计算积分:

$$\int_0^\infty \frac{e^{-x}(1-e^{-6x})}{x(1+e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x})}dx$$

Solution

$$\begin{aligned}
&\int_0^\infty \frac{e^{-x}(1-e^{-6x})}{x(1+e^{-2x}+e^{-4x}+e^{-6x}+e^{-8x})}dx \\
&=\int_0^\infty \frac{e^{-x}(1-e^{-6x})(1-e^{-2x})}{x(1-e^{-10x})}dx \\
&=\int_0^\infty \frac{1}{x}\sum_{k=0}^\infty e^{-10kx}\cdot e^{-x}(1-e^{-6x})(1-e^{-2x})dx \\
&=\sum_{k=0}^\infty \int_0^\infty \frac{e^{-(10k+1)x}-e^{-(10k+3)x}-e^{-(10k+7)x}+e^{-(10k+9)x}}{x}dx \\
&=\sum_{k=0}^\infty \int_0^\infty \left[\frac{e^{-(10k+1)x}-e^{-(10k+3)x}}{x}+\frac{-e^{e^{-(10k+9)x}-(10k+7)x}}{x}\right]dx \\
&=\sum_{k=0}^\infty \left(f(0)\ln\left[\frac{-(10k+3)}{-(10k+1)}\right]+f(0)\ln\left[\frac{-(10k+7)}{-(10k+9)}\right]\right)=\sum_{k=0}^\infty\left(\ln\left(\frac{10k+3}{10k+1}\right)+\ln\left(\frac{10k+7}{10k+9}\right)\right) \\
&=\sum_{k=0}^\infty \ln\frac{(10k+3)(10k+7)}{(10k+1)(10k+9)} \\
&=\ln\prod_{k=0}^\infty \frac{(10k+3)(10k+7)}{(10k+1)(10k+9)}=\ln\prod_{k=0}^\infty\frac{(k+\frac{3}{10})(k+\frac{7}{10})}{(k+\frac{1}{10})(k+\frac{9}{10})} \\
&=\ln\frac{\Gamma(\frac{1}{10})\Gamma(\frac{9}{10})}{\Gamma(\frac{3}{10})\Gamma(\frac{7}{10})}=\ln\frac{\sin\frac{3\pi}{10}}{\sin\frac{\pi}{10}}\approx 0.962424
\end{aligned}$$

Theorem 5.4 Froullani 积分公式

设 $f(x)$ 在 $(0,+\infty)$ 上连续,$a>0, b>0$, 有

1. 若 $f(0), f(+\infty)$ 存在，则 $\displaystyle\int_0^{+\infty} \frac{f(ax)-f(bx)}{x} dx = [f(0)-f(+\infty)] \ln \frac{b}{a}$;
2. 若 $f(0)$ 存在，且 $\forall>0$, $\displaystyle\int_A^{+\infty} \frac{f(x)}{x} dx$ 存在，
   则 $\displaystyle\int_0^{+\infty} \frac{f(ax)-f(bx)}{x} dx = f(0) \ln \frac{b}{a}$;
3. 若 $f(+\infty)$ 存在，且 $\forall>0$, $\displaystyle\int_0^{A} \frac{f(x)}{x} dx$ 存在，
   则 $\displaystyle\int_0^{+\infty} \frac{f(ax)-f(bx)}{x} dx = -f(+\infty) \ln \frac{b}{a}$;

Exercise 5.59: 计算积分:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\ln(\tan x)\, \mathrm{d}x$$

Solution 令 $t=\ln(\tan x)$ 则: $\mathrm{d}t = \left(\frac{1}{\tan x}+\tan x\right)$, $\mathrm{d}x = \left(\mathrm{e}^{-t}+\mathrm{e}^{t}\right) \mathrm{d}x$ 原积分

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\ln(\tan x)\, \mathrm{d}x = \int_0^{\infty} \frac{\ln t}{\mathrm{e}^t+\mathrm{e}^{-t}} \mathrm{d}t = \int_0^{\infty} \frac{\mathrm{e}^{-t}\ln t}{1+\mathrm{e}^{-2t}} \mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-t} \ln t \sum_{k=0}^{\infty} (-1)^k \mathrm{e}^{-2kt} \mathrm{d}t$$

所以有

$$\begin{aligned}
\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\ln(\tan x) \mathrm{d}x &= \sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} \mathrm{e}^{-(2k+1)t} \ln t \mathrm{d}t \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \int_0^{\infty} \mathrm{e}^{-t} \ln t \mathrm{d}t - \sum_{k=0}^{\infty} (-1)^k \frac{\ln(2k+1)}{2k+1} \int_0^{\infty} \mathrm{e}^{-t} \mathrm{d}t \\
&= -\frac{\pi}{4}\gamma + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\ln(2k+1)}{2k+1} = -\frac{\pi}{4}\gamma + \left[\frac{\pi}{4}\gamma + \frac{\pi}{4} \ln \frac{\Gamma^4\left(\frac{3}{4}\right)}{\pi}\right]
\end{aligned}$$

再由公式

$$\Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right) = \sqrt{2}\pi$$

故

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\ln(\tan x)\, \mathrm{d}x = \frac{\pi}{2} \ln\left[\frac{\sqrt{2\pi}\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\right]$$

Exercise 5.60: 证明:

$$\int_0^1 \left(\sqrt[a]{1-x^b} - \sqrt[b]{1-x^a}\right) \mathrm{d}x = 0,\ \text{其中 } a, b>0$$

因为

✎ **Solution**

$$\begin{aligned}\int_0^1 \sqrt[a]{1-x^b}\,\mathrm{d}x &\xlongequal[x=\sqrt[b]{1-t^a}]{t=\sqrt[a]{1-x^b}} \int_1^0 t\,\mathrm{d}\left(\sqrt[b]{1-t^a}\right)\\ &= t\sqrt[b]{1-t^a}\Big|_1^0 - \int_1^0 \sqrt[b]{1-t^a}\,\mathrm{d}t\\ &= \int_0^1 \sqrt[b]{1-t^a}\,\mathrm{d}t\end{aligned}$$

因此

$$\int_0^1 \left(\sqrt[a]{1-x^b}-\sqrt[b]{1-x^a}\right)\mathrm{d}x = 0$$

Exercise 5.61: 证明:

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sqrt{\sin 2x}}\,\mathrm{d}x$$

✎ **Solution**

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sqrt{\sin 2x}}\,\mathrm{d}x &\xlongequal{x=\frac{\pi}{2}-x} \int_0^{\frac{\pi}{2}} \frac{\cos t}{1+\sqrt{\sin 2t}}\,\mathrm{d}t\\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{1+\sqrt{\sin 2x}}\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{1+\sqrt{(\sin x-\cos x)^2-1}}\,\mathrm{d}x\\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\left(\sin x-\cos x\right)}{1+\sqrt{(\sin x-\cos x)^2-1}}\\ &= \frac{1}{2}\int_{-1}^{1} \frac{\mathrm{d}u}{1+\sqrt{u^2-1}}\\ &\xlongequal{x=\sin t} \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos u}{1+\cos t}\,\mathrm{d}t = \frac{\pi}{2}-1\end{aligned}$$

Exercise 5.62: 设 $f(x)$ 在 $(0,+\infty)$ 内为单调可导函数, 它的反函数为 $f^{-1}(x)$, 且 $f(x)$ 满足等式 $\displaystyle\int_2^{f(x)} f^{-1}(t)\,\mathrm{d}t = \frac{1}{3}x^{\frac{3}{2}}-9$, 则 $f(x)=(\qquad)$

(A) $\sqrt{x}-1$ (B) $\sqrt{x}+1$ (C) $2\sqrt{x}-1$ (D) $2\sqrt{x}+1$

✎ **Solution** 令 $\frac{1}{3}x^{\frac{3}{2}}-9=0 \Longrightarrow x=9$, 又 $f(x)$ 在 $(0,+\infty)$ 内为单调可导函数故 $f(9)=2$ 代入选项可知 A 正确

令 $f^{-1}(t)=u \Longrightarrow t=f(u) \Longrightarrow \mathrm{d}t=f'(u)\mathrm{d}u$

$$\begin{aligned}\int_2^{f(x)} f^{-1}(t)\,\mathrm{d}t &\xlongequal{f^{-1}(t)=u} \int_9^x uf'(u)\,\mathrm{d}u \\ &= uf(u)\Big|_9^x - \int_9^x f(u)\,\mathrm{d}u \\ &= xf(x)-9f(9)-\int_9^x f(u)\,\mathrm{d}u = \frac{1}{3}x^{\frac{3}{2}}-9 \end{aligned} \tag{5.4}$$

对 (5.4) 求导

$$xf'(x)=\frac{1}{2}x^{\frac{1}{2}} \Longleftrightarrow f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$$

对上式积分可得

$$f(x)=\sqrt{x}+C$$

代入 $f(9)=2$ 得 $f(x)=\sqrt{x}-1$

Exercise 5.63: 证明:

Solution

5.4 无初等解析

Exercise 5.64: 计算积分: $\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{x}\sin x\mathrm{d}x$

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sqrt{x}\sin x dx &\xlongequal[dx=\pi tdt]{\sqrt{x}=\sqrt{\frac{\pi}{2}}t} \pi\sqrt{\frac{\pi}{2}}\int_0^1 t^2\sin\left(\frac{1}{2}\pi t^2\right)dt \\ &= \sqrt{\frac{\pi}{2}}\int_0^1 td\left(-\cos\left(\frac{\pi}{2}t^2\right)\right) \\ &= \left[-\sqrt{\frac{\pi}{2}}t\cos\left(\frac{\pi}{2}t^2\right)\right]_0^1+\sqrt{\frac{\pi}{2}}\int_0^1\cos\left(\frac{\pi}{2}t^2\right)dt \\ &= \sqrt{\frac{\pi}{2}}C(1)\approx 0.977451\end{aligned}$$

Note: *Fresnel Integrals*

$$C(x)=\int_0^x\cos\left(\frac{1}{2}\pi t^2\right)dt$$

$$S(x)=\int_0^x\sin\left(\frac{1}{2}\pi t^2\right)dt$$

Exercise 5.65: 证明:

Solution

5.5 反常积分的审敛法　Γ 函数

5.5.1 反常积分的审敛法

Theorem 5.5 无穷积分的 Abel 判别法

若 $\int_a^{+\infty} f(x)\,\mathrm{d}x$ 收敛; $g(x)$ 在 $[a,+\infty)$ 上单调有界, 则 $\int_a^{+\infty} f(x)g(x)\,\mathrm{d}x$ 收敛

Theorem 5.6 无穷积分的 Dirichlet 判别法

若 $g(x)$ 在 $[a,+\infty)$ 上单调有界, 且 $\lim\limits_{x\to+\infty} g(x)=0$; $F(u)=\int_a^u f(x)\,\mathrm{d}x$ 在 $[a,+\infty)$ 上有界, 则 $\int_a^{+\infty} f(x)g(x)\,\mathrm{d}x$ 收敛

Theorem 5.7 有界瑕积分的 A–D 判别法

若 $f(x)$ 在 $[a,b]$ 上只有一个奇点 b

1. (Abel 判别法) 若 $\int_a^b f(x)\,\mathrm{d}x$ 收敛; $g(x)$ 在 $[a,b)$ 上单调有界,
   则 $\int_a^b f(x)g(x)\,\mathrm{d}x$ 收敛
2. (Dirichlet 判别法) 若 $g(x)$ 在 $[a,b)$ 上单调有界, 且 $\lim\limits_{x\to b^-} g(x)=0$;
   $F(\eta)=\int_a^{b-\eta} f(x)\,\mathrm{d}x$ 在 $[0,b-a)$ 上有界, 则 $\int_a^b f(x)g(x)\,\mathrm{d}x$ 收敛

Exercise 5.66: 证明 $\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$ 条件收敛

Solution

$$\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x = \int_0^1 \frac{\sin x}{x}\,\mathrm{d}x + \int_1^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$$

令 $g(x) = \begin{cases} \frac{\sin x}{x} & ,0 < x \leqslant 1 \\ 1 & ,x = 0 \end{cases}$, 因为 $\lim\limits_{x\to 0} g(x) = \frac{\sin x}{x} = 1$ 故 $g(x)$ 在 $[0,1]$ 上连续,
所以 $g(x)$ 在 $[0,1]$ 上可积, 且

$$\int_0^1 \frac{\sin x}{x}\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x$$

即 $\int_0^1 \frac{\sin x}{x}\,\mathrm{d}x$ 存在. 下面往证 $\int_1^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$ 收敛
$f(x) = \sin x$ 在 $[1,+\infty)$ 连续, 且对 $\forall x \in [1,+\infty)$, 有

$$F(u) = \int_1^u \sin x\,\mathrm{d}x = \cos 1 - \cos u$$

$$|F(u)| = |\cos 1 - \cos u| \leqslant 2$$

而 $\frac{1}{x}$ 在 $[1,+\infty)$ 单调递减并趋向于 0, 故由 Dirichlet 判别法可知 $\int_1^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$ 收敛
在证无穷积分 $\int_1^{+\infty} \left|\frac{\sin x}{x}\right|\mathrm{d}x$ 发散
已知 $\forall x \in [1,+\infty)$, 有 $|\sin x| \geqslant \sin^2 x$, 从而

$$\left|\frac{\sin x}{x}\right| \geqslant \frac{\sin^2 x}{x} = \frac{1-\cos 2x}{2x} = \frac{1}{2x} - \frac{\cos 2x}{2x}$$

同理可证明无穷积分 $\int_1^{+\infty} \frac{\cos 2x}{2x}\,\mathrm{d}x$ 收敛, 而 $\int_1^{+\infty} \frac{1}{2x}\,\mathrm{d}x$ 发散
由于 $\int_1^{+\infty} \left(\frac{1}{2x} - \frac{\cos 2x}{2x}\right)\mathrm{d}x$ 发散, 由比较判别法可知 $\int_1^{+\infty} \left|\frac{\sin x}{x}\right|\mathrm{d}x$ 发散
综上所述, 无穷积分 $\int_0^{+\infty} \frac{\sin x}{x}\,\mathrm{d}x$ 条件收敛

5.5.2 Γ 函数

Exercise 5.67: f $\alpha_1 + \alpha_2 + \cdots + \alpha_n = \beta_1 + \beta_2 + \cdots + \beta_n$ then

$$\frac{\Gamma(\beta_1)\cdot\cdots\cdot\Gamma(\beta_n)}{\Gamma(\alpha_1)\cdot\cdots\cdot\Gamma(\alpha_n)} = \prod_{k\geqslant 0} \frac{(k+\alpha_1)\cdot\cdots\cdot(k+\alpha_n)}{(k+\beta_1)\cdot\cdots\cdot(k+\beta_n)}$$

Proof: according to Euler's definition for the gamma function

$$\Gamma(z) = \frac{m^z m!}{z(z+1)\cdot\cdots\cdot(z+m)} \tag{5.5}$$

therefore we have

$$\frac{\Gamma(\beta_1)\cdot\cdots\cdot\Gamma(\beta_n)}{\Gamma(\alpha_1)\cdot\cdots\cdot\Gamma(\alpha_n)}=\prod_{j=1}^{n}\frac{\Gamma(\beta_j)}{\Gamma(\alpha_j)}=\lim_{m\to\infty}\prod_{j=1}^{n}\frac{\frac{m^{\beta_j}m!}{\beta_j(\beta_j+1)\cdot\cdots\cdot(\beta_j+m)}}{\frac{m^{\alpha_j}m!}{\alpha_j(\alpha_j+1)\cdot\cdots\cdot(\alpha_j+m)}}$$
$$=\lim_{m\to\infty}\prod_{j=1}^{n}m^{\beta_j-\alpha_j}\prod_{k=0}^{m}\frac{\alpha_j+k}{\beta_j+k}$$
$$=\lim_{m\to\infty}\prod_{k=0}^{m}\prod_{j=1}^{n}\frac{\alpha_j+k}{\beta_j+k}$$
$$=\lim_{m\to\infty}\prod_{k=0}^{m}\frac{(k+\alpha_1)\cdot\cdots\cdot(k+\alpha_n)}{(k+\beta_1)\cdot\cdots\cdot(k+\beta_n)}$$

□

Exercise 5.68: 计算

$$\int_0^{+\infty}\frac{x^3}{e^x-1}\,\mathrm{d}x$$

Solution

$$\int_0^{+\infty}\frac{x^3}{e^x-1}\,\mathrm{d}x=\int_0^{+\infty}x^3\left(\sum_{n=1}^{\infty}e^{-nx}\right)\mathrm{d}x$$
$$=\sum_{n=1}^{\infty}\int_0^{+\infty}x^3e^{-nx}\,\mathrm{d}x$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^{+\infty}t^3e^{-t}\,\mathrm{d}t\ ,\ t=nx$$
$$=\sum_{n=1}^{\infty}\frac{1}{n^4}\Gamma(4)=16\sum_{n=1}^{\infty}\frac{1}{n^4}$$
$$=16\times\frac{\pi^4}{90}=\frac{\pi^4}{15}$$

Exercise 5.69: 计算积分:

$$\int_0^1\frac{1}{\sqrt{1+x^4}}dx$$

Solution 我们有

$$J=\int_0^{+\infty}\frac{1}{\sqrt{1+x^4}}dx\xlongequal{x^4=t}\frac{1}{4}\int_0^{+\infty}\frac{t^{-\frac{3}{4}}}{(1+t)^{\frac{1}{2}}}dt=\frac{1}{4}B(\frac{1}{4},\frac{1}{4})=\frac{1}{4}\frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{4})}{\Gamma(\frac{1}{2})}=\frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}$$

对积分 $\int_1^{+\infty}\frac{1}{\sqrt{1+x^4}}dx$ 做变量替换, 令 $t=\frac{1}{x}$, 可得

$$\int_1^{+\infty}\frac{1}{\sqrt{1+x^4}}dx=\int_0^1\frac{1}{\sqrt{1+t^4}}dt$$

由此知

$$J=\int_0^1\frac{1}{\sqrt{1+x^4}}dx+\int_1^{+\infty}\frac{1}{\sqrt{1+x^4}}dx=2\int_0^1\frac{1}{\sqrt{1+t^4}}dt=2I$$

所以

$$I=\int_0^1\frac{1}{\sqrt{1+x^4}}dx=\frac{J}{2}=\frac{\Gamma^2(\frac{1}{4})}{8\sqrt{\pi}}$$

Exercise 5.70: 计算积分:

$$\int_{-\infty}^{+\infty}\frac{x^2e^x}{(e^x+1)^2}\,\mathrm{d}x$$

Solution

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{x^2e^x}{(e^x+1)^2}\,\mathrm{d}x&=2\int_0^{+\infty}\frac{x^2e^x}{(e^x+1)^2}\,\mathrm{d}x=2\int_0^{+\infty}x^2\,\mathrm{d}\left(\frac{1}{e^x+1}\right)\\&=\frac{2x^2}{e^x+1}\bigg|_0^{+\infty}-4\int_0^{+\infty}\frac{x}{e^x+1}\,\mathrm{d}x\\&=4\int_0^{+\infty}\frac{xe^{-x}}{1+e^{-x}}\,\mathrm{d}x=4\int_0^{+\infty}xe^{-x}\sum_{n=0}^{\infty}(-1)^ne^{-nx}\,\mathrm{d}x\\&=4\sum_{n=0}^{\infty}(-1)^n\int_0^{+\infty}xe^{-(n+1)x}\,\mathrm{d}x=4\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}\int_0^{+\infty}te^{-t}\,\mathrm{d}t\\&=4\sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^2}=\frac{\pi^2}{3}\end{aligned}$$

Exercise 5.71: 计算积分:

$$\lim_{n\to 0}\sqrt[n]{n!}$$

Solution

$$\begin{aligned}\lim_{n\to 0}\sqrt[n]{n!}&=\lim_{n\to 0}\exp\left\{\frac{\ln(n!)}{n}\right\}\\&=\exp\left\{\lim_{n\to 0}\frac{\ln\Gamma(n+1)}{n}\right\}\\&=\exp\left\{\lim_{x\to 0^+}\frac{\ln\Gamma(x+1)}{x}\right\}\\&=\exp\left\{\lim_{x\to 0^+}\frac{\Gamma'(x+1)}{\Gamma(x+1)}\right\}\\&=e^{\psi(1)}=e^{-\gamma}\end{aligned}$$

Exercise 5.72: 计算积分:

$$\int_0^1\ln\Gamma(x)dx$$

Solution 本题需用到的公式

$$\Gamma(x)\Gamma(1-x)=\frac{\pi}{\sin\pi x},x\in(0,1)\qquad \text{——余元公式}$$

$$\begin{aligned}I&=\int_0^1\ln\Gamma(x)dx\xlongequal{t=1-x}-\int_1^0\ln\Gamma(1-t)dt=\int_0^1\ln\Gamma(1-t)dt\\&=\int_0^1\ln\Gamma(1-x)dx\end{aligned}$$

$$
\begin{aligned}
2I &= \int_0^1 \ln\Gamma(x)dx + \int_0^1 \ln\Gamma(1-x)dx \\
&= \int_0^1 \left(\ln\Gamma(x) + \ln\Gamma(1-x)\right)dx = \int_0^1 \ln\frac{\pi}{\sin\pi x}dx \\
&= \int_0^1 \ln\pi dx - \int_0^1 \ln\sin\pi x dx = \ln\pi - \int_0^1 \ln\sin\pi x dx \\
&\xlongequal{\pi x=t} \ln\pi - \frac{1}{\pi}\int_0^{\pi}\ln\sin t dt \\
&= \ln\pi - \frac{1}{\pi}\underbrace{\int_0^{\frac{\pi}{2}}\ln\sin t dt}_{=-\frac{\pi}{2}\ln 2} - \frac{1}{\pi}\underbrace{\int_{\frac{\pi}{2}}^{\pi}\ln\sin t dt}_{u=\pi-t} \\
&= \ln\pi + \frac{1}{2}\ln 2 + \frac{1}{\pi}\int_{\frac{\pi}{2}}^{0}\ln\sin u du \\
&= \ln\pi + \frac{1}{2}\ln 2 - \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln\sin u du \\
&= \ln(2\pi)
\end{aligned}
$$

$$\Longrightarrow I = \int_0^1 \ln\Gamma(x)dx = \frac{1}{2}\ln(2\pi)$$

Exercise 5.73: 计算积分:

$$\int_0^1 \frac{x^2-1}{(x^2+1)\ln x}\mathrm{d}x$$

Solution

$$
\begin{aligned}
\int_0^1 \frac{x^2-1}{(x^2+1)\ln x}\mathrm{d}x &= \int_0^1 \frac{x+1}{x^2+1}\int_0^1 x^t\mathrm{d}t\mathrm{d}x = \int_0^1\int_0^1 \frac{x^{t+1}+x^t}{x^2+1}\mathrm{d}x\mathrm{d}t \\
&= \int_0^1\left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{1}{x+3} - \frac{1}{x+4} + \cdots\right)\mathrm{d}x \\
&= \left(\ln\frac{2}{1} + \ln\frac{3}{2}\right) - \left(\ln\frac{4}{3} + \ln\frac{5}{4}\right) + \cdots \\
&= \ln\frac{3}{1} - \ln\frac{5}{3} + \ln\frac{7}{5} - \ln\frac{9}{7} + \ln\frac{11}{9} - \cdots \\
&= \ln\left(\frac{3}{1}\cdot\frac{3}{5}\cdot\frac{7}{5}\cdot\frac{7}{9}\cdot\frac{11}{9}\cdots\right) \\
&= \lim_{n\to\infty}\ln\left\{\frac{\Gamma^2\left(\frac{5}{4}\right)\Gamma^2\left(\frac{4n+3}{4}\right)}{\Gamma^2\left(\frac{3}{4}\right)\Gamma^2\left(\frac{4n+5}{4}\right)}(4n+3)\right\} \\
&= 2\ln\left\{\frac{2\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right\}
\end{aligned}
$$

最后用了 Gautschi′s inequality.

Exercise 5.74: 计算积分:

$$\int_0^1 \frac{5x^4(1+x^{10075})}{(1+x^5)^{2017}}\mathrm{d}x$$

Solution 因为

$$\mathrm{Beta}(x,y)=\int_0^1\frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\mathrm{d}t\ \ (Re\,x>0,Re\,y>0)$$

$$\begin{aligned}\int_0^1\frac{5x^4(1+x^{10075})}{(1+x^5)^{2017}}\mathrm{d}x &\xlongequal{x^5=t}\int_0^1\frac{1+t^{2015}}{(1+t)^{2017}}\mathrm{d}t\\ &=\int_0^1\frac{x^{1-1}+t^{2016-1}}{(1+t)^{2017}}\mathrm{d}t\\ &=\mathrm{B}(1,2016)\\ &=\frac{0!2015!}{2016!}=\frac{1}{2016}\end{aligned}$$

Note: 用到的公式

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\ (x>0,y>0)$$

Exercise 5.75: 计算积分:

$$\int_0^{\frac{\pi}{2}}\sin^{\frac{5}{2}}xdx$$

Solution 因为

$$\mathrm{Beta}(x,y)=2\int_0^{\frac{\pi}{2}}\sin^{2x-1}\theta\cos^{2y-1}\theta d\theta\ \ (Re\,x>0,Re\,y>0)$$

所以

$$\int_0^{\frac{\pi}{2}}\sin^{\frac{5}{2}}xdx=\frac{1}{2}B\left(\frac{7}{4},\frac{1}{2}\right) \tag{5.6}$$

$$=\frac{1}{2}\frac{\Gamma\left(\frac{7}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{9}{4}\right)} \tag{5.7}$$

$$=\frac{6\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{5\Gamma\left(\frac{1}{4}\right)}\approx 0.718884 \tag{5.8}$$

Note: 用到的公式

$$Beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\ (x>0,y>0) \tag{5.9}$$

$$\Gamma\left(n+\frac{1}{4}\right)=\frac{\prod\limits_{i=1}^{n}(4i-3)}{4^n}\Gamma\left(\frac{1}{4}\right)=\frac{1\cdot5\cdot9\cdot13\cdot\cdots\cdot(4n-3)}{4^n}\Gamma\left(\frac{1}{4}\right)\ \ (n=1,2,3,\cdots) \tag{5.10}$$

$$\Gamma\left(n+\frac{3}{4}\right)=\frac{\prod\limits_{i=1}^{n}(4i-1)}{4^n}\Gamma\left(\frac{3}{4}\right)=\frac{3\cdot7\cdot11\cdot15\cdot\cdots\cdot(4n-1)}{4^n}\Gamma\left(\frac{3}{4}\right)\ \ (n=1,2,3,\cdots) \tag{5.11}$$

特殊值

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}=\left(-\frac{1}{2}\right)! \tag{5.12}$$

$$\Gamma\left(\frac{1}{4}\right)\approx 3.6256099082\cdots \tag{5.13}$$

$$\Gamma\left(\frac{3}{4}\right)\approx 1.2254167024\cdots \tag{5.14}$$

Exercise 5.76: 求极限:

$$\lim_{x\to 1^-}\sqrt{1-x}\int_0^{+\infty}x^{t^2}\,\mathrm{d}t$$

Solution 由于

$$\begin{aligned}\int_0^{+\infty}x^{t^2}\,\mathrm{d}t&=\int_0^{+\infty}e^{-t^2\ln(\frac{1}{x})}\,\mathrm{d}t\\&\xlongequal{u=t\sqrt{\ln(\frac{1}{x})}}\frac{1}{\sqrt{\ln(\frac{1}{x})}}\int_0^{+\infty}e^{-u^2}\,\mathrm{d}u\\&=\frac{1}{2}\sqrt{\frac{\pi}{\ln(\frac{1}{x})}}\\&\sim\frac{1}{2}\sqrt{\frac{\pi}{1-x}}\end{aligned}$$

故

$$\begin{aligned}\lim_{x\to 1^-}\sqrt{1-x}\int_0^{+\infty}x^{t^2}\,\mathrm{d}t&=\lim_{x\to 1^-}\left(\sqrt{1-x}\times\frac{1}{2}\sqrt{\frac{\pi}{1-x}}\right)\\&=\frac{\sqrt{\pi}}{2}\end{aligned}$$

Exercise 5.77: 计算积分:

$$\int_0^{+\infty}e^{-(ax^2+bx)}\mathrm{d}x$$

Solution

$$\begin{aligned}I&=\int_0^{+\infty}e^{-(ax^2+bx)}\mathrm{d}x\\&=\int_0^{+\infty}e^{-a\left[\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)-\left(\frac{b}{2a}\right)^2\right]}\mathrm{d}x\\&=e^{\frac{b^2}{4a}}\int_0^{+\infty}e^{-a\left(x+\frac{b}{2a}\right)^2}\mathrm{d}x\\&=e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\int_0^{+\infty}e^{-\left(\sqrt{a}\left(x+\frac{b}{2a}\right)\right)^2}\mathrm{d}\left(\sqrt{a}\left(x+\frac{b}{2a}\right)\right)\\&=e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\int_0^{+\infty}e^{-x^2}\mathrm{d}x=e^{\frac{b^2}{4a}}\frac{1}{2\sqrt{a}}\int_0^{+\infty}t^{-\frac{1}{2}}e^{-t}dt=\frac{1}{2}\Gamma(\tfrac{1}{2})\\&=e^{\frac{b^2}{4a}}\frac{1}{\sqrt{a}}\cdot\frac{\sqrt{\pi}}{2}=\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}\end{aligned}$$

Exercise 5.78: 计算积分:

$$\int_0^1 \sqrt{(1-x^2)^3}\mathrm{d}x$$

Solution

$$\begin{aligned}\int_0^1 \sqrt{(1-x^2)^3}dx &\xlongequal{x^2=t} \frac{1}{2}\int_0^1 t^{-\frac{1}{2}}(1-t)^{\frac{3}{2}}dt \\ &= \frac{1}{2}B\left(\frac{1}{2},\frac{5}{2}\right) \\ &= \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{2\Gamma(3)} \\ &= \frac{3\pi}{16} \approx 0.58905\end{aligned}$$

Exercise 5.79: 计算积分:

$$\int_0^1 \sin(\pi x)\log\Gamma(x)\mathrm{d}x$$

Solution

$$\int_0^1 \sin(\pi x)\log\Gamma(x)dx = \frac{1}{\pi}\left(1+\log\left(\frac{\pi}{2}\right)\right)$$

$$\begin{aligned}I &= \int_0^1 \sin(\pi x)\log\Gamma(x)dx \xlongequal{t=1-x} -\int_1^0 \sin(t\pi)\log\Gamma(1-t)dt \\ &= \int_0^1 \sin(t\pi)\log\Gamma(1-t)dt \\ I &= \frac{1}{2}\left(\int_0^1 \sin(\pi x)\log\Gamma(x)dx + \int_0^1 \sin(x\pi)\log\Gamma(1-x)dx\right) \\ &= \frac{1}{2}\int_0^1 \sin(\pi x)\log\left(\Gamma(x)+\Gamma(1-x)\right)dx \\ &= \frac{1}{2}\int_0^1 \sin(\pi x)\log\left(\frac{\pi}{\sin \pi x}\right)dx \\ &= \frac{1}{\pi}\left(1+\ln\frac{\pi}{2}\right)\end{aligned}$$

5.6 特殊函数

Exercise 5.80: 计算积分:

$$\int_0^1 \frac{e^{-x^2}}{\sqrt{1-x^2}}dx$$

Solution 因为

$$I_n(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos(n\theta)\,d\theta$$

$$\int_0^1 \frac{e^{-x^2}}{\sqrt{1-x^2}}dx \xlongequal{令x=\cos t} -\int_{\frac{\pi}{2}}^{0} e^{-\cos^2 t}dt \tag{5.15}$$

$$\xlongequal{令u=\frac{\pi}{2}-t} \int_0^{\frac{\pi}{2}} e^{-\sin^2 u}du = \int_0^{\frac{\pi}{2}} e^{-\frac{1-\cos 2u}{2}}du = \frac{1}{\sqrt{e}}\int_0^{\frac{\pi}{2}} e^{\frac{1}{2}\cos 2u}du \tag{5.16}$$

$$\xlongequal{令\theta=2u} \frac{1}{2\sqrt{e}}\int_0^{\pi} e^{\frac{1}{2}\cos\theta}d\theta \tag{5.17}$$

$$= \frac{\pi I_0\left(\frac{1}{2}\right)}{2\sqrt{e}} \tag{5.18}$$

Exercise 5.81: 证明:

$$\int_0^{2\pi} e^{\sin x}\sin x\,\mathrm{d}x = \int_0^{2\pi} e^{\sin x}\cos^2 x\,\mathrm{d}x = 2\pi I_1(1)$$

Solution

$$\begin{aligned}
\int_0^{2\pi} e^{\sin x}\sin x\,\mathrm{d}x &= \int_0^{2\pi} e^{\sin x}\,\mathrm{d}(-\cos x) \\
&= \Big[-e^{\sin x}\cos x\Big]_0^{2\pi} + \int_0^{2\pi} e^{\sin x}\cos^2 x\,\mathrm{d}x \\
&= 0 + \int_0^{2\pi} e^{\sin x}\cos^2 x\,\mathrm{d}x \\
&= \int_0^{2\pi} e^{\sin x}\cos^2 x\,\mathrm{d}x
\end{aligned}$$

又因为

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\,d\theta$$

$$\begin{aligned}
\int_0^{2\pi} e^{\sin x}\sin x\,\mathrm{d}x &= \underbrace{\int_0^{\frac{\pi}{2}} e^{\sin x}\sin x\,\mathrm{d}x}_{u=\frac{\pi}{2}+x} + \underbrace{\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} e^{\sin x}\sin x\,\mathrm{d}x}_{t=x-\frac{\pi}{2}} + \underbrace{\int_{\frac{3\pi}{2}}^{2\pi} e^{\sin x}\sin x\,\mathrm{d}x}_{t=x-\frac{\pi}{2}} \\
&= -\int_{\frac{\pi}{2}}^{\pi} e^{-\cos u}\cos u\,\mathrm{d}u + \int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t + \underbrace{\int_{\pi}^{\frac{3\pi}{2}} e^{\cos t}\cos t\,\mathrm{d}t}_{v=t-\pi} \\
&= -\int_{\frac{\pi}{2}}^{\pi} e^{-\cos t}\cos t\,\mathrm{d}t + \int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t - \int_0^{\frac{\pi}{2}} e^{-\cos v}\cos v\,\mathrm{d}v \\
&= \int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t - \underbrace{\int_0^{\pi} e^{-\cos t}\cos t\,\mathrm{d}t}_{x=\pi-t} \\
&= \int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t - \int_{\pi}^{0} e^{\cos x}\cos x\,\mathrm{d}x \\
&= 2\int_0^{\pi} e^{\cos t}\cos t\,\mathrm{d}t \\
&= 2\pi I_1(1) \approx 3.551
\end{aligned}$$

Exercise 5.82: 证明:

$$\int_0^{2\pi} e^{\cos x}\cos x\,\mathrm{d}x > 0$$

Solution

$$\begin{aligned}
\int_0^{2\pi} e^{\cos x}\cos x\,\mathrm{d}x &= \int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x + \underbrace{\int_{\pi}^{2\pi} e^{\cos x}\cos x\,\mathrm{d}x}_{t=2\pi-x}\\
&= \int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x - \int_{\pi}^{0} e^{\cos t}\cos t\,\mathrm{d}t\\
&= 2\int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x = 2\int_0^{\pi} e^{\cos x}\,\mathrm{d}(\sin x)\\
&= 2\sin x e^{\cos x}\Big|_0^{\pi} + 2\int_0^{\pi} e^{\cos x}\sin x\,\mathrm{d}x\\
&> 0
\end{aligned}$$

又因为

$$I_n(z) = \frac{1}{\pi}\int_0^{\pi} e^{z\cos\theta}\cos(n\theta)\,\mathrm{d}\theta$$

$$\begin{aligned}
\int_0^{2\pi} e^{\cos x}\cos x\,\mathrm{d}x &= \int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x + \underbrace{\int_{\pi}^{2\pi} e^{\cos x}\cos x\,\mathrm{d}x}_{t=2\pi-x}\\
&= \int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x - \int_{\pi}^{0} e^{\cos t}\cos t\,\mathrm{d}t\\
&= 2\int_0^{\pi} e^{\cos x}\cos x\,\mathrm{d}x\\
&= 2\pi I_1(1) \approx 3.551
\end{aligned}$$

$$\begin{aligned}
&C(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right)dt \xlongequal[du=\sqrt{\frac{\pi}{2}}dt]{\frac{\pi t^2}{2}=u^2} \sqrt{\frac{2}{\pi}}\int_0^{\sqrt{\frac{\pi}{2}}x}\cos u^2 du\\
&\Longrightarrow \int_x^0 \cos x^2 dx = -\int_0^x \cos x^2 dx = -\sqrt{\frac{\pi}{2}}C\left(\sqrt{\frac{2}{\pi}}x\right)
\end{aligned}$$

5.7 积分不等式

Exercise 5.83: 设 $f(x)$ 在 $[0,1]$ 上有连续导数, 且 $f(0)=0$

求证:

$$\int_0^1 f^2(x)\,\mathrm{d}x \leqslant \frac{1}{2}\int_0^1 f'^2(x)\,\mathrm{d}x$$

Proof: 因为

$$f(x) = f(x) - f(0) = \int_0^x f'(x)\,\mathrm{d}x$$

由柯西积分不等式有

$$\begin{aligned} f^2(x) &= \left(\int_0^x f'(x)\,\mathrm{d}x\right)^2 \leqslant \int_0^x 1^2\,\mathrm{d}x \cdot \int_0^x f'^2(x)\,\mathrm{d}x \\ &= x\int_0^x f'^2(x)\,\mathrm{d}x \leqslant x\int_0^1 f'^2(x)\,\mathrm{d}x \end{aligned}$$

所以

$$\int_0^1 f^2(x)\,\mathrm{d}x \leqslant \int_0^1 x\,\mathrm{d}x \cdot \int_0^1 f'^2(x)\,\mathrm{d}x = \frac{1}{2}\int_0^1 f'^2(x)\,\mathrm{d}x$$

□

Exercise 5.84: 设 $f(x)$ 在 $[0,1]$ 上有连续导数, 且 $f(0)=0, f(1)=0$

求证:

$$\int_0^1 f^2(x)\,\mathrm{d}x \leqslant \frac{1}{8}\int_0^1 f'^2(x)\,\mathrm{d}x$$

Proof: 因为

$$f(x) = f(x) - f(0) = \int_0^x f'(x)\,\mathrm{d}x$$

由柯西积分不等式有

$$\begin{aligned} f^2(x) &= \left(\int_0^x f'(x)\,\mathrm{d}x\right)^2 \leqslant \int_0^x 1^2\,\mathrm{d}x \cdot \int_0^x f'^2(x)\,\mathrm{d}x \\ &= x\int_0^x f'^2(x)\,\mathrm{d}x \leqslant x\int_0^1 f'^2(x)\,\mathrm{d}x \end{aligned}$$

所以

$$\int_0^{\frac{1}{2}} f^2(x)\,\mathrm{d}x \leqslant \int_0^{\frac{1}{2}} x\,\mathrm{d}x \cdot \int_0^{\frac{1}{2}} f'^2(x)\,\mathrm{d}x = \frac{1}{8}\int_0^{\frac{1}{2}} f'^2(x)\,\mathrm{d}x$$

又

$$f(x) = f(x) - f(1) = -\int_x^1 f'(x)\,\mathrm{d}x$$

同理可得

$$\int_{\frac{1}{2}}^1 f^2(x)\,\mathrm{d}x \leqslant \frac{1}{8}\int_{\frac{1}{2}}^1 f'^2(x)\,\mathrm{d}x$$

因此

$$\int_0^1 f^2(x)\,\mathrm{d}x \leqslant \frac{1}{8}\int_0^1 f'^2(x)\,\mathrm{d}x$$

□

Exercise 5.85: 试证:

$$\frac{16}{9} < \int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\,\mathrm{d}x < \frac{418}{225}$$

Proof:

□

Exercise 5.86: $f_0(x)$ 在 $[0,1]$ 上可积, $f_0(x)>0$; $f_n(x) = \sqrt{\int_0^x f_{n-1}(t)\mathrm{d}t}, \quad (n=1,2,\ldots)$, 求 $\lim\limits_{n\to\infty} f_n(x)$.

☞ **Proof:** 设 $0<\delta<1$. 因为 $f_0(x)$ 在 $[0,1]$ 上可积且 $f_0(x)>0$,
所以 $f_1(x)=\sqrt{\int_0^x f_0(t)dt}$ 是区间 $[0,1]$ 上的连续函数，故存在正数 m,M, 使得

$$f_1(x)\le M \qquad (x\in[0,1])$$
$$f_1(x)\ge m \qquad (x\in[\delta,1])$$

对任一自然数 n, 用数学归纳法可以证明如下不等式

$$m^{\frac{1}{2^n}}a_n(x-\delta)^{1-\frac{1}{2^n}}\le f_{n+1}(x)\le M^{\frac{1}{2^n}}a_n x^{1-\frac{1}{2^n}} \tag{5.19}$$

其中

$$a_n=\left(\frac{2}{2^2-1}\right)^{\frac{1}{2^{n-1}}}\left(\frac{2^2}{2^3-1}\right)^{\frac{1}{2^{n-2}}}\cdots\left(\frac{2^{n-1}}{2^n-1}\right)^{\frac{1}{2}}$$

当 $n=1$ 时, 有

$$f_2(x)=\sqrt{\int_0^x f_1(t)dt}\le M^{\frac{1}{2}}x^{1-\frac{1}{2}}=M^{\frac{1}{2}}a_1x^{1-\frac{1}{2}}$$

设 $n-1$ 时结论成立, 则对 n 有

$$\begin{aligned}f_{n+1}(x)&=\sqrt{\int_0^x f_n(t)dt}\le M^{\frac{1}{2^n}}a_{n-1}^{\frac{1}{2}}\sqrt{\int_0^x t^{1-\frac{1}{2^{n-1}}}dt}\\&=M^{\frac{1}{2^n}}a_{n-1}^{\frac{1}{2}}\frac{2^{\frac{n-1}{2}}}{(2^n-1)^{\frac{1}{2}}}=M^{\frac{1}{2^n}}a_nx^{1-\frac{1}{2^n}}\end{aligned}$$

故 (5.19) 式右边的不等式对一切自然数 n 都成立, 同理可证左边的不等式亦真.
因为

$$\ln a_n=\frac{1}{2^{n-1}}\ln\frac{2}{2^2-1}+\frac{1}{2^{n-2}}\ln\frac{2^2}{2^3-1}+\cdots+\frac{1}{2}\ln\frac{2^{n-1}}{2^n-1}\qquad(n=1,2,\cdots)$$

所以根据特普利茨定理（容易验证此时条件全部满足）有

$$\lim_{n\to+\infty}\ln a_n=\lim_{n\to+\infty}\ln\frac{1}{2-\dfrac{1}{2^{n-1}}}=\ln\frac{1}{2}$$

于是

$$\lim_{n\to+\infty}M^{\frac{1}{2^n}}a_nx^{1-\frac{1}{2^n}}=\frac{x}{2}\qquad\lim_{n\to+\infty}m^{\frac{1}{2^n}}a_n(x-\delta)^{1-\frac{1}{2^n}}=\frac{x-\delta}{2}$$

由 δ 的任意性即知对任一切 $x\in(0,1]$ 有

$$\lim_{n\to+\infty}f_{n+1}(x)=\frac{x}{2}$$

又因 $f_{n+1}(0)=0\quad(n=1,2,\cdots)$ 所以对一切 $x\in[0,1]$ 有

$$\lim_{n\to+\infty}f_{n+1}(x)=\frac{x}{2}$$

□

Exercise 5.87:

☞ **Proof:**

□

第 6 章 定积分的应用

6.1 定积分在几何学上的应用

6.1.1 平面图形的面积

6.1.2 直角坐标类型

(1) 若 $D=\{(x,y)|\varphi_1(x)\leqslant y\leqslant\varphi_2(x),a\leqslant x\leqslant b\}, \varphi_1(x),\varphi_2(x)$ 连续, 则 D 的面积为

$$S_D=\int_a^b[\varphi_2(x)-\varphi_1(x)]\,\mathrm{d}x$$

(2) 若 $D=\{(x,y)|\psi_1(y)\leqslant x\leqslant\psi_2(y),c\leqslant y\leqslant d\}, \psi_1(y),\psi_2(y)$ 连续, 则 D 的面积为

$$S_D=\int_c^d[\psi_2(y)-\psi_1(y)]\,\mathrm{d}y$$

Exercise 6.1: 求抛物线 $y^2=4ax$ 与过焦点的弦所围成的平面图形面积 A 的最小值

Solution 易知抛物线 $y^2=4ax$ 的焦点坐标为 $(a,0)$

故设过焦点的弦方程为

$$y=k(x-a)$$

联立 $\begin{cases}y=k(x-a)\\ y^2=4ax\end{cases}$ 解得

焦点坐标为 $\left(\dfrac{ak^2+2a-2\sqrt{a^2(k^2+1)}}{k^2},\dfrac{2a(\sqrt{(k^2+1)}+1)}{k}\right)$ 和 $\left(\dfrac{ak^2+2a+2\sqrt{a^2(k^2+1)}}{k^2},\dfrac{2a(1-\sqrt{\cdots}}{\cdots}\right.$

记 $y_1=\dfrac{2a(\sqrt{(k^2+1)}+1)}{k}\ y_2=\dfrac{2a(1-\sqrt{(k^2+1)})}{k}$

则面积 A 为

$$\begin{aligned}A&=\int_{y_1}^{y_2}\left(a+\frac{y}{k}-\frac{y^2}{4a}\right)\mathrm{d}y\\&=a(y_2-y_1)+\frac{y_2^2-y_1^2}{2k}-\frac{y_2^3-y_1^3}{12a}\\&=\frac{8a^2(1+k^2)^{\frac{3}{2}}}{3k^3}\\&=\frac{8a^2}{3}\left(1+\frac{1}{k^2}\right)^{\frac{3}{2}}\end{aligned}$$

显然 $A\uparrow$, 故 $A_{\min}=\lim\limits_{k\to\infty}\dfrac{8a^2}{3}\left(1+\dfrac{1}{k^2}\right)^{\frac{3}{2}}=\dfrac{8a^2}{3}$

6.1.3 极坐标类型

若 $D=\{(\rho,\theta)|\rho_1(\theta)\leqslant\rho\leqslant\rho_2(\theta),\alpha\leqslant\theta\leqslant\beta\}$, $\rho_1(\theta),\rho_2(\theta)$ 连续, 则 D 的面积为

$$S_D=\frac{1}{2}\int_\alpha^\beta\left[\rho_2^2(\theta)-\rho_1^2(\theta)\right]\,\mathrm{d}\theta$$

这里 (ρ,θ) 为极坐标。

设平面图形由曲线 $\rho=\rho(\theta)$ 及射线 $\theta=\alpha$, $\theta=\beta$ 所围成, 求其面积 S。

$$S=\int_\alpha^\beta\frac{1}{2}\left[\rho(\theta)\right]^2\mathrm{d}\theta$$

6.1.4 体积

(1) 设立体 Ω 介于平面 $x=a$ 与 $x=b$ 之间, $\forall x\in(a,b)$, 过点 x 且与 x 轴垂直的平面截立体 Ω 的截面面积为连续函数 $A(x)$, 则立体的体积为 $V_\Omega=\int_a^b A(x)\,\mathrm{d}x$

(2.1) 将由 x 轴, 直线 $x=a,x=b\ (a<b)$, 及连续曲线 $y=f(x)\,(f(x)\geqslant0)$ 所围成的曲边梯形绕 x 轴旋转一周所得到的旋转体的体积 $V_x=\pi\int_a^b f^2(x)\,\mathrm{d}x$

(2.2) 将由 y 轴, 直线 $y=c,y=d\ (c<d)$, 及连续曲线 $x=\varphi(y)\,(\varphi(y)\geqslant0)$ 所围成的曲边梯形绕 y 轴旋转一周所得到的旋转体的体积 $V_y=\pi\int_c^d\varphi^2(y)\,\mathrm{d}y$

(3) 将由 x 轴, 直线 $x=a,x=b\ (a<b)$, 及连续曲线 $y=f(x)\,(f(x)\geqslant0)$ 所围成的曲边梯形绕 y 轴旋转一周所得到的旋转体的体积 $V_y=2\pi\int_a^b xf(x)\,\mathrm{d}x$

Exercise 6.2: 底面由圆 $x^2+y^2=4$ 围成，且垂直与 x 轴的所有截面都是正方形的立体体积为 (　)

✎ **Solution** $x>0$ 时, 对于任一 x 的取值
正方形边长 $=2\sqrt{4-x^2}$, 正方形面积 $=\left(2\sqrt{4-x^2}\right)^2$
所求体积

$$\begin{aligned}V&=2\int_0^2\left(2\sqrt{4-x^2}\right)^2\,\mathrm{d}x\\&=8\int_0^2\left(4-x^2\right)\,\mathrm{d}x\\&=8\left[4x-\frac{x^3}{3}\right]_0^2\\&=42\frac{2}{3}\end{aligned}$$

✎ **Solution** 所求体积

$$
\begin{aligned}
V &= 2\int_0^2 \left(2\sqrt{4-x^2}\right)^2 \,\mathrm{d}x \\
&= 8\int_0^2 \left(4-x^2\right) \,\mathrm{d}x \\
&= 8\left[4x-\frac{x^3}{3}\right]_0^2 \\
&= 42\frac{2}{3}
\end{aligned}
$$

6.1.5 平面曲线的弧长

第 7 章 微分方程

7.1 微分方程的基本概念

7.2 可分离变量的微分方程

Exercise 7.1: 设 $f(x)$ 在 $(-\infty, +\infty)$ 上有定义, 对任何 x, y 恒有

$$f(x+y) = f(x) + f(y) + 2xy$$

又 $f(x)$ 在点 $x = 0$ 处可导, 且 $f'(0) = 1$, 求 $f(x)$ 的表达式

Solution 首先在等式

$$f(x+y) = f(x) + f(y) + 2xy$$

令 $x = y = 0$ 得到 $f(0) = 0$.
对固定的 x 以及任意的 $y \neq 0$ 都有

$$\frac{f(x+y) - f(x)}{y} = \frac{f(y)}{y} + 2x$$

即

$$\frac{f(x+y) - f(x)}{y} = \frac{f(y) - f(0)}{y - 0} + 2x$$

令 $y \to 0$, 由 $f'(0) = 1$ 则得到 $f'(x) = 2x + 1$ 解这个微分方程并注意到 $f'(0) = 1$
就有 $f(x) = x^2 + x$

7.3 齐次方程

7.4 一阶线性微分方程

Exercise 7.2: 求微分方程 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2x+1}{2xy}\cos^2(xy^2) - \dfrac{y}{2x}$ 的通解

Solution 法 1 两边同乘 $2xy$ 得

$$2xyy' = (2x+1)\cos^2(xy^2) - y^2$$

移项

$$2xyy' + y^2 = (2x+1)\cos^2(xy^2)$$

注意到

$$(xy^2)' = 2xyy' + y^2$$

故

$$(xy^2)' \sec^2(xy^2) = 2x + 1$$

即

$$\left[\tan(x^2y)\right]' = 2x + 1$$

上式两边对 x 积分可得

$$\tan(x^2y) = x^2 + x + C$$

Solution 法 2 令 $xy^2 = u$, 则 $2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} + y^2 = \dfrac{\mathrm{d}u}{\mathrm{d}x}$

Note:

$$\text{分离变量} = \begin{cases} 1 \quad \text{能分} \longrightarrow \text{分} \\ 2 \quad \text{不能分} \longrightarrow \text{代} \end{cases}$$

Exercise 7.3: 求微分方程 $(x - e^y)y' = 1$ 的通解

Solution

$$(x - e^y)y' = 1 \Longrightarrow \frac{1}{x - e^y}\frac{\mathrm{d}x}{\mathrm{d}y} = 1 \Longrightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = x - e^y$$

故

$$x = e^{\int \mathrm{d}y}\left[-\int e^y \cdot e^{-\int \mathrm{d}y}\,\mathrm{d}y + C\right] = e^y(c - y)$$

7.5 恰当方程与积分因子

7.5.1 恰当方程

Definition 7.1 恰当方程

假设 $M(x,y), N(x,y)$ 在某矩形内是 x,y 的连续函数, 且具有连续的一阶偏导数, 有

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{7.1}$$

如果方程 (7.1) 的左端恰好是某个二元函数 $u(x,y)$ 的全微分, 即

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y \equiv \mathrm{d}u(x,y) \equiv \frac{\partial u}{\partial x}\mathrm{d}x + \frac{\partial u}{\partial y}\mathrm{d}y$$

称 (7.1) 为恰当方程.

容易验证 (7.1) 的通解为 $u(x,y) = C$, 这里 C 为任意常数

Theorem 7.1

$\dfrac{\partial M}{\partial y}=\dfrac{\partial N}{\partial x}$ 是 (7.1) 为恰当方程的充分必要条件

Note: 一些简单二元函数的全微分, 如

$$\begin{aligned}
y\,\mathrm{d}x+x\,\mathrm{d}y&=\mathrm{d}(xy)\\
\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{y^2}&=\mathrm{d}\left(\frac{x}{y}\right)\\
\frac{-y\,\mathrm{d}x+x\,\mathrm{d}y}{x^2}&=\mathrm{d}\left(\frac{y}{x}\right)\\
\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{xy}&=\mathrm{d}\left(\ln\left|\frac{x}{y}\right|\right)\\
\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{x^2+y^2}&=\mathrm{d}\left(\arctan\frac{x}{y}\right)\\
\frac{y\,\mathrm{d}x-x\,\mathrm{d}y}{x^2-y^2}&=\frac{1}{2}\mathrm{d}\left(\ln\left|\frac{x-y}{x+y}\right|\right)
\end{aligned}$$

Exercise 7.4: 求微分方程: $(3x^2+6xy^2)\,\mathrm{d}x+(6x^2y+4y^3)\,\mathrm{d}y=0$ 的通解

Solution 这里 $M(x,y)=3x^2+6xy^2, N(x,y)=6x^2y+4y^3$, 这时

$$\frac{\partial M}{\partial y}=12xy,\qquad \frac{\partial N}{\partial x}=12xy$$

因此方程是恰当方程

现在求 $u(x,y)$ 使它同时满足如下两个方程

$$\frac{\partial u}{\partial x}=3x^2+6xy^2 \tag{7.2}$$

$$\frac{\partial u}{\partial y}=6x^2y+4y^3 \tag{7.3}$$

由 (7.2) 对 x 积分, 得到

$$u=x^3+3x^2y^2+\varphi(y) \tag{7.4}$$

为了确定 $\varphi(y)$, 将 (7.4) 对 y 求导数, 并使它满足 (7.5.1), 即得

$$\frac{\partial u}{\partial y}=6x^2y+\frac{\mathrm{d}\varphi(y)}{\mathrm{d}y}=6x^2y+4y^3$$

于是

$$\frac{\mathrm{d}\varphi(y)}{\mathrm{d}y}=4y^3$$

积分后可得

$$\varphi(y)=y^4$$

将 $\varphi(y)$ 代入 (7.4) 得到

$$u(x,y)=x^3+3x^2y^2+y^4$$

因此, 方程的通解为

$$x^3+3x^2y^2+y^4=C$$

这里 C 为任意常数

Solution2 这里 $M(x,y)=3x^2+6xy^2, N(x,y)=6x^2y+4y^3$, 这时

$$\frac{\partial M}{\partial y}=12xy,\qquad \frac{\partial N}{\partial x}=12xy$$

因此方程是恰当方程

并由题得

$$3x^2\mathrm{d}x+4y^3\mathrm{d}y+6xy^2\mathrm{d}x+6x^2y\mathrm{d}y=0$$

即

$$\mathrm{d}x^3+dy^4+3y^2\mathrm{d}x^2+3x^2\mathrm{d}y^2=0$$

或者写成

$$\mathrm{d}(x^3+y^4+3x^2y^2)=0$$

于是, 方程的通解为

$$x^3+3x^2y^2+y^4=C$$

这里 C 为任意常数

7.5.2 积分因子法

Exercise 7.5: 求微分方程: $y'+P(x)y=Q(x)$ 的通解

Solution 两边同乘 $u(x)$, 原方程变为

$$u(x)y'+u(x)P(x)y=u(x)Q(x)$$

使得

$$[u(x)y]'=u(x)y'+u'(x)y=u(x)y'+u(x)P(x)y$$

于是

$$u'(x)=u(x)P(x)\Longrightarrow u(x)=e^{\int P(x)\,\mathrm{d}x}$$

于是, 我们得到如下积分因子法

方程 $y'+P(x)y=Q(x)$ 两端同乘以积分因子 $u(x)=e^{\int P(x)\,\mathrm{d}x}$, 得

$$e^{\int P(x)\,\mathrm{d}x}y'+P(x)ye^{\int P(x)\,\mathrm{d}x}=Q(x)e^{\int P(x)\,\mathrm{d}x}$$

$$\Longrightarrow\left(ye^{\int P(x)\,\mathrm{d}x}\right)'=Q(x)e^{\int P(x)\,\mathrm{d}x}$$

上式两端同时积分可得

$$ye^{\int P(x)\,\mathrm{d}x} = \int Q(x)e^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C$$

即

$$y = e^{-\int P(x)\,\mathrm{d}x}\left(\int Q(x)e^{\int P(x)\,\mathrm{d}x}\,\mathrm{d}x + C\right)$$

7.6 可降阶的高阶微分方程

7.7 高阶线性微分方程

Exercise 7.6: 设 f 是二次可微函数, 对于任何实数 x, y 都满足函数方程

$$f^2(x) - f^2(y) = f(x+y)f(x-y)$$

试求 f 的表达式

Solution 首先在等式

$$f^2(x) - f^2(y) = f(x+y)f(x-y)$$

令 $x = y = 0$ 得到 $f(0) = 0$.
又对其两边关于 x, y 先后求两次偏导数得

$$2f(x)f'(x) = f'(x+y)f(x-y) + f(x+y)f'(x-y)$$

$$0 = f''(x+y)f(x-y) - f(x+y)f''(x-y)$$

作变量代换 $x + y = u, x - y = v$ 则对于任何实数 u, v 都有

$$f''(u)f(v) = f(u)f''(v)$$

如果 $f(v) \equiv 0$, 则该函数方程的解为 $f(x) \equiv 0$.
若 $f(v) \not\equiv 0$, 存在一点 v_0 使得 $f(v_0) \neq 0$, 则可令 $c = \dfrac{f''(v_0)}{f(v_0)}$, 即化为 $f''(u) = cf(u)$. 根据初始条件 $f(0) = 0$ 即可求得解为

$$f(u) = \begin{cases} A\sinh\sqrt{c}x, & c > 0 \\ Au, & c = 0,\ \text{其中}A\text{是任意常数} \\ A\sin\sqrt{-c}x, c < 0 \end{cases}$$

Exercise 7.7: 求微分方程: $y'' - (y')^2 + y' = 0$ 的通解

Solution 变形得:

$$y'' - (y')^2 + y' = 0 \Longleftrightarrow \frac{y'' - (y')^2}{y^2} = -\frac{y'}{y^2}$$

对上式积分得:

$$\Longrightarrow \frac{y'}{y} = \frac{1}{y} + C_1$$

整理得:

$$\Longrightarrow y' - C_1 y = 1$$

左右同乘 $e^{-C_1 x}$

$$\Longleftrightarrow e^{-C_1 x} y' - C_1 e^{-C_1 x} y = e^{-C_1 x}$$

对上式积分得:

$$e^{-C_1 x} y = -\frac{1}{C_1} e^{-C_1 x} + C_2$$

通解为:

$$y = C_2 e^{C_1 x} - \frac{1}{C}$$

Exercise 7.8: 求微分方程: $y'' = 1 + y'^2$ 的通解

Solution 移项得

$$\frac{y''}{1 + y'^2} = 1$$

对上式积分得

$$\arctan y' = x + c_1$$

所以

$$y' = \tan(x + c_1)$$

对上式积分得

$$y = -\ln|\cos(x + c_1)| + c_2$$

7.8 常系数齐次线性微分方程

7.9 常系数非齐次线性微分方程

7.9.1 非齐次线性微分方程的解的叠加原理

Definition 7.2 解的叠加原理

设 y_1^* 和 y_2^* 分别是非齐次线性方程

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f_1(x) \tag{7.5}$$

和

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f_2(x) \tag{7.6}$$

的特解, 则 $y_1^* + y_2^*$ 是非齐次线性方程

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f_1(x) + f_2(x) \tag{7.7}$$

的特解

Definition 7.3 复数解的叠加原理

设线性方程

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f(x) + ig(x) \tag{7.8}$$

(其中 $a_i(x)$ $(i = 1, 2, 3, \cdots, n)$, $f(x)$ 和 $g(x)$ 均为实函数) 有复数解 $y = u^* + iv^*$, 则这个解的实部 u^* 和虚部 v^* 分别是线性方程

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = f(x) \tag{7.9}$$

和

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = g(x) \tag{7.10}$$

的解

7.9.2 $f(x) = e^{\lambda x} P_m(x)$ 型

$y'' + py' + qy = e^{\lambda x} P_m(x)$ **的特解形式为:**

$$y^* = x^k Q_m(x) e^{\lambda x} \qquad k = \begin{cases} 0 & \text{当 } \lambda \text{ 不是特征根} \\ 1 & \text{当 } \lambda \text{ 是特征单根} \\ 2 & \text{当 } \lambda \text{ 是特征重根} \end{cases}$$

7.9.3 $f(x) = e^{\lambda x} [P_l(x) \cos \omega x + P_n(x) \sin \omega x]$ 型

$y'' + py' + qy = e^{\lambda x} [P_l(x) \cos \omega x + P_n(x) \sin \omega x]$ **特解的设法:**

设 $m = \max\{l, n\}$

$$y^* = x^k e^{\lambda x} [Q_m(x) \cos \omega x + R_m(x) \sin \omega x] \qquad k = \begin{cases} 0 & \text{当 } \lambda + \omega i \text{ 不是特征根} \\ 1 & \text{当 } \lambda + \omega i \text{ 是特征根} \end{cases}$$

Theorem 7.2 刘维尔公式

若 y_1 是二阶线性微分方程 $y'' + p(x)y' + q(x)y = 0$ 的一个解, 则该方程与 y_1 线性无关的另一个解为

$$y_2(x) = y_1(x) \int \frac{1}{y_1^2(x)} e^{-\int p(x)\,\mathrm{d}x} \,\mathrm{d}x$$

Exercise 7.9: 求通解 $y'' + 3y' + 2y = 3xe^{-x}$

Solution 对两边同时积分

$$\int (y'' + 3y' + 2y) \,\mathrm{d}x = \int 3xe^{-x} \,\mathrm{d}x$$

左边

$$\begin{aligned} \int (y'' + 3y' + 2y) \,\mathrm{d}x &= \int y'' \,\mathrm{d}x + 3 \int y' \,\mathrm{d}x + 2 \int y \,\mathrm{d}x \\ &= y' + 3y + 2 \int y \,\mathrm{d}x \end{aligned}$$

Exercise 7.10: 求通解 $y'' + y = x\cos 2x$

Solution 特征方程

$$r^2 + 1 = 0$$

特征根

$$r = \pm i$$

对应齐次方程的通解

$$Y = C_1 \cos x + C_2 \sin x$$

$$l = 1, n = 0, m = \max\{1, 0\} = 1, \lambda = 0, \omega = 2, \lambda + \omega i = 2i$$

$\lambda + \omega i = 2i$ 不是特征根，故设特解为

$$y^* = (ax + b)\cos 2x + (cx + d)\sin 2x$$

求导为:

$$\begin{aligned} y^{*\prime} &= a\cos 2x - 2(ax + b)\sin 2x + c\sin 2x + 2(cx + d)\cos 2x \\ &= (a + 2cx + 2d)\cos 2x + (c - 2ax - 2b)\sin 2x \end{aligned}$$

再次求导

$$\begin{aligned} y^{*\prime\prime} &= 2c\cos 2x - 2(a + 2cx + 2d)\sin 2x - 2a\sin 2x + 2(c - 2ax - 2b)\cos 2x \\ &= 4(c - ax - b)\cos 2x - 4(a + cx + d)\sin 2x \end{aligned}$$

带入原方程, 得

$$(-3ax - 3b + 4c)\cos 2x - (3cx + 3d + 4a)\sin 2x = x\cos 2x$$

比较 $\cos 2x$, $\sin 2x$ 的系数, 得

$$-3ax - 3b + 4c = x, -(3cx + 3d + 4a) = 0$$

$$-3a = 1, -3b + 4c = 0, c = 0, 3d + 4a = 0$$

解得

$$a = -\frac{1}{3}, b = c = 0, d = \frac{4}{9}$$

特解为

$$\begin{aligned} y^* &= (ax + b)\cos 2x + (cx +)\sin 2x \\ &= -\frac{1}{3}x\cos 2x + \frac{4}{9}\sin 2x \end{aligned}$$

故所求通解为

$$y = C_1 \cos x + C_2 \sin x - \frac{1}{3} x \cos 2x + \frac{4}{9} \sin 2x$$

Exercise 7.11: 求通解 $y'' + 2y' + 5y = \sin 2x$

Solution 特征方程

$$r^2 + 2r + 5 = 0$$

特征根

$$r_1 = -1 - 2i, r_2 = -1 + 2i$$

对应齐次方程的通解

$$Y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x)$$

$$l = 0, n = 0, m = \max\{0, 0\} = 0, \lambda = 0, \omega = 2, \lambda + \omega i = 2i$$

$\lambda + \omega i = 2i$ 不是特征根，故设特解为

$$y^* = a \cos 2x + b \sin 2x$$

求导为:

$$y^{*\prime} = -2a \sin 2x + 2b \cos 2x$$

再次求导

$$y^{*\prime\prime} = -4a \cos 2x - 4b \sin 2x$$

带入原方程, 得

$$(-4a + 4b + 5a) \cos 2x + (-4b - 4a + 5b) \sin 2x = \sin 2x$$

比较 $\cos 2x$, $\sin 2x$ 的系数, 得

$$a + 4b = 0, b - 4a = 1$$

解得

$$b = \frac{1}{17}, a = -\frac{4}{17}$$

特解为

$$\begin{aligned} y^* &= a \cos 2x + b \sin 2x \\ &= -\frac{4}{17} \cos 2x + \frac{1}{17} \sin 2x \end{aligned}$$

故所求通解为

$$y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x) - \frac{4}{17} \cos 2x + \frac{1}{17} \sin 2x$$

Exercise 7.12: 求通解 $y'' + 2y' + 10y = xe^{-x}\cos 3x$

Solution

$$l = 1, n = 0, m = \max\{1, 0\} = 1, \lambda = -1, \omega = 3, \lambda + \omega i = -1 + 3i$$

$\lambda + \omega i = -1 + 3i$ 是特征根，故设特解为

$$y^* = xe^{-x}((ax + b)\cos 3x + (cx + d)\sin 3x)$$

求导为：

$$\begin{aligned} y^{*\prime} =& e^{-x}\Big(\big((-a + 3c)x^2 + (3d + 2a - b)x + b\big)\cos 3x \\ & + \big(-(3a + c)x^2 + (2c - 3b - d)x + d\big)\sin 3x\Big) \end{aligned}$$

再次求导

$$\begin{aligned} y^{*\prime\prime} =& e^{-x}\Big(\big((-8a - 6c)x^2 + (-4a - 8b + 12c - 6d)x + (2a - 2b + 6d)\big)\cos 3x \\ & + \big((6a - 8c)x^2 + (-12a + 6b - 4c - 8d)x + (-6b + 2c - 2d)\big)\sin 3x\Big) \end{aligned}$$

带入原方程, 得

$$\big(12cx + (2a + 6d)\big)\cos 3x + \big(-6b + 2c\big)\sin 3x = x\cos 3x$$

比较 $\cos 3x$, $\sin 3x$ 的系数, 得

$$\begin{cases} 12c = 1 \\ 2a + 6d = 0 \\ -6b + 2c = 0 \end{cases} \quad \text{解得} \begin{cases} a = -3d \\ b = \dfrac{1}{36} \\ c = \dfrac{1}{12} \end{cases}$$

特解为

$$\begin{aligned} y^* &= xe^{-x}((ax + b)\cos 3x + (cx + d)\sin 3x) \\ &= xe^{-x}\left(\frac{1}{36}\cos 3x + \frac{1}{12}x\sin 3x\right) \end{aligned}$$

故所求通解为

$$y = e^{-x}(C_1\cos 3x + C_2\sin 3x) + xe^{-x}\left(\frac{1}{36}\cos 3x + \frac{1}{12}x\sin 3x\right)$$

Solution2 特征方程

$$r^2 + 2r + 10 = 0$$

特征根

$$r_1 = -1 - 3i, r_2 = -1 + 3i$$

对应齐次方程的通解

$$Y = e^{-x}(C_1 \cos 3x + C_2 \sin 3x)$$

Exercise 7.13: 求通解 $y'' + y = \sec x$

Solution 特征方程

$$r^2 + 1 = 0$$

特征根

$$r = \pm i$$

对应齐次方程的通解

$$Y = C_1 \cos x + C_2 \sin x$$

朗斯基行列式为

$$W(x) = \begin{vmatrix} \cos x & \sin x \\ (\cos x)' & (\sin x)' \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

其中

$$v_1(x) = -\int \frac{\sec x \sin x}{1} \mathrm{d}x = \ln|\cos x|$$

$$v_2(x) = -\int \frac{\sec x \cos x}{1} \mathrm{d}x = x$$

故特解为

$$y^* = \cos x \ln(\cos x) + x \sin x$$

那么所求通解为

$$y = C_1 \cos x + C_2 \sin x + \cos x \ln(\cos x) + x \sin x$$

7.10 欧拉方程

Definition 7.4 欧拉方程

形如

$$x^n y^{(n)} + p_1 x^{n-1} y^{(n-1)} + \cdots + p_{n-1} x y' + p_n y = f(x) \tag{7.11}$$

的方程（其中 $p_1, p_2, \cdots p_n$ 为常数），叫做欧拉方程

作变换令 $x=e^t$ 或 $t=\ln x$，将自变量 x 换成 t，我们有

$$x=\ln t \Longrightarrow \mathrm{d}t=\frac{1}{x}\mathrm{d}x \Leftrightarrow \frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{x}, \frac{\mathrm{d}x}{\mathrm{d}t}=x$$

$$\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t}\\
&\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t}\right)\frac{\mathrm{d}t}{\mathrm{d}x}=\left(-\frac{\frac{\mathrm{d}x}{\mathrm{d}t}}{x^2}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{1}{x}\frac{\mathrm{d}^2y}{\mathrm{d}t^2}\right)\frac{1}{x}=\frac{1}{x^2}\left(\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-\frac{\mathrm{d}y}{\mathrm{d}t}\right)\\
&\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=\frac{1}{x^3}\left(\frac{\mathrm{d}^3y}{\mathrm{d}t^3}-3\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2\frac{\mathrm{d}y}{\mathrm{d}t}\right)
\end{aligned}$$

采用记号 D 表示对 t 求导的运算 $\dfrac{\mathrm{d}}{\mathrm{d}t}$, 那么上述计算结果可以写成

$$\begin{aligned}
&xy'=Dy\\
&x^2y''=\frac{\mathrm{d}^2y}{\mathrm{d}t^2}-\frac{\mathrm{d}y}{\mathrm{d}t}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}-\frac{\mathrm{d}}{\mathrm{d}t}\right)y=(D^2-D)y=D(D-1)y\\
&x^3y'''=\frac{\mathrm{d}^3y}{\mathrm{d}t^3}-3\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+2\frac{\mathrm{d}y}{\mathrm{d}t}=(D^3-3D^2+2D)y=D(D-1)(D-2)y
\end{aligned}$$

一般地, 有

$$x^ky^{(k)}=D(D-1)\cdots(D-k+1)y$$

将它带入欧拉方程 (7.11) 便得到一个以 t 为自变量的常系数线性微分方程. 在求出这个解后, 把 t 换成 $\ln x$，即得原方程的解.

第 8 章 差分方程

8.1 差分方程概述

Definition 8.1

设自变量 t 取离散的整数值 $t=0,1,2,\cdots$，而 y 是 t 的函数，记为 $y_t=f(t)$。当自变量从 t 变到 $t+1$ 时，相应的函数值的改变量称为函数 $y(t)$ 在 t 处的一阶差分，记为

$$\Delta y_t=y(t+1)-y(t)$$

或

$$\Delta y_t=y_{t+1}-y_t$$

函数 $y(t)$ 在 t 处的二阶差分记为

$$\Delta^2 y_t=\Delta(\Delta y_t)=y_{t+2}-2y_{t+1}+y_t$$

函数 $y(t)$ 在 t 处的 n 阶差分记为

$$\Delta^n y_t=\Delta(\Delta^{n-1}y_t)=\sum_{i=0}^{n}C_n^i(-1)^i y_{t+n-i}$$

Example 8.1: 求 $y_t=C$ 的各阶差分

Solution: $\Delta y_t=y_{t+1}-y_t=0$，且其各阶差分都为 0 □

Properties: 当 a,b,C 为常数，u_t 和 v_t 为 t 的函数时，有以下结论成立

(1) $\Delta(C)=0$;

(2) $\Delta(Cy_t)=C\Delta y_t$;

(3) $\Delta(au_t+bv_t)=a\Delta u_t+b\Delta v_t$;

(4) $\Delta(u_tv_t)=u_t\Delta v_{t+1}+v_{t+1}\Delta u_t$;

(5) $\Delta\left(\dfrac{u_t}{v_t}\right)=\dfrac{v_t\Delta u_t-u_t\Delta v_t}{v_tv_{t+1}}$;

Definition 8.2 差分方程

一般地，含未知函数有和未知函数差分的方程称为差分方程
差分方程的一般形式为

$$F\left(t, y_t, y_{t+1}, \cdots, y_{t+n}\right)=0$$

或

$$G\left(t, y_t, \Delta y_t, \cdots, \Delta^n y_t\right)=0$$

其中 F，G 为表达式，t 是自变量

差分方程中含有未知的最高阶数称为差分方程的阶

满足差分方程的函数称为差分方程的解

一般地，不含有任意常数的解称为特解，n 阶差分方程的含有 n 个彼此独立的任意常数的解称为差分方程的通解

Definition 8.3

n 阶非齐次线性差分方程形如

$$a_0(t) y_{t+n}+a_1(t) y_{t+n-1}+\cdots+a_n(t) y_t=f(t)$$

其中右端项 $f(t)$ 和各项系数 $a_0(t), a_1(t), \cdots, a_n(t)$ 为已知函数。相应的齐次线性差分方程为

$$a_0(t) y_{t+n}+a_1(t) y_{t+n-1}+\cdots+a_n(t) y_t=0$$

Definition 8.4

设有二阶非齐次线性差分方程

$$y_{t+2}+a(t) y_{t+1}+b(t) y_t=f(t) \tag{8.1}$$

相应的齐次线性差分方程为

$$y_{t+2}+a(t) y_{t+1}+b(t) y_t=0 \tag{8.2}$$

其中系数 $b(t) \neq 0$

Theorem 8.1

若 $y_t^{(1)}$ 和 $y_t^{(2)}$ 都是方程 (8.2) 的解，则对任意常数 C_1，C_2，$C_1y_t^{(1)}+C_2y_t^{(2)}$ 也是方程 (8.2) 的解。

Theorem 8.2

若 $y_t^{(1)}$ 和 $y_t^{(2)}$ 是 (8.2) 的线性无关的特解，则对任意常数 C_1，C_2，$C_1y_t^{(1)}+C_2y_t^{(2)}$ 是它的通解

Theorem 8.3

若 $y_t^{(1)}$ 和 $y_t^{(2)}$ 都是非齐次方程 (8.1) 的解，则 $y_t^{(1)}-y_t^{(2)}$ 是齐次方程 (8.2) 的解

Theorem 8.4

若 $y^{(c)}$ 是齐次方程 (8.2) 的通解，$\bar{y}$ 是非齐次方程 (8.1) 特解，则 $y=y^{(c)}+\bar{y}$ 是非齐次方程 (8.1) 的通解

8.2 一阶常系数线性差分方程

8.2.1 迭代法

一阶常系数非齐次线性差分方程的一般形式为

$$y_{t+1}-py_t=f\left(t\right) \tag{8.3}$$

其中常数系数 $p\neq 0$，未知函数项 y_{t+1} 和 y_t 为一次的，右端项 $f\left(t\right)$ 为已知函数。与其相应的齐次方程为

$$y_{t+1}-py_t=0 \tag{8.4}$$

齐次差分方程 (8.4) 的通解为

$$y_t=Cp^t,t=0,1,2,\cdots \tag{8.5}$$

当 $f(t)=b$ 为常数，非齐次差分方程 (8.3) 的通解为

$$y_t=\begin{cases} Cp^t+\dfrac{b}{1-p}, & p=1, \\ C+bt, & p=1. \end{cases} \tag{8.6}$$

8.2.2 待定系数法

1. 设非齐次差分方程 (8.3) 的右端为 $f(t)=P_n(t)$

   (1) 当 $p=1$ 时，设其为

   $$y_t=t\left(b_0+b_1t+b_2t^2+\cdots+b_nt^n\right)$$

   (2) 当 $p\neq 1$ 时，设其为

   $$y_t=b_0+b_1t+b_2t^2+\cdots+b_nt^n$$

2. 设非齐次差分方程 (8.3) 的右端为 $f(t)=\lambda^tP_n(t)$
   其中：λ 为已知常数，$P_n(t)$ 为的 n 次多项式
   设所求特解为

   $$y_t=t^k\lambda^t\left(b_0+b_1t+b_2t^2+\cdots+b_nt^n\right)$$

   其中当 $p=\lambda$ 时 $k=1$，当 $p\neq\lambda$ 时 $k=0$

Example 8.2: 求 $y_{t+1}-5y_t=3$ 的通解和满足 $y\big|_{t=0}=\dfrac{7}{3}$ 的特解

✎ **Solution:** 该差分方程中 $p=5$，$b=3$，由式 (8.6) 得到方程通解

$$y_t=C\cdot 5^t+\frac{3}{1-5}=C\cdot 5^t-\frac{3}{4}$$

将 $y_0=\dfrac{7}{3}$ 带入上式得到 $C=\dfrac{37}{12}$，故所求特解为

$$y_t=\frac{37}{12}\cdot 5^t-\frac{3}{4}$$

□

Example 8.3: 求 $y_{t+1}-y_t=3+2t$ 的通解

✎ **Solution:** 由式 (8.5) 得到齐次方程的通解为 $y_t=C$

$$y_t=C\cdot 5^t+\frac{3}{1-5}=C\cdot 5^t-\frac{3}{4}$$

因为 $p=1$ 故设所求方程的特解为 $\bar{y}_t=t(b_0+b_1t)$
代入方程得

$$(t+1)\big(b_0+b_1(t+1)\big)-t(b_0+b_1t)=3+2t$$

所以

$$\begin{cases}2b_1=2\\ b_0+b_1=3\end{cases}\Longrightarrow\begin{cases}b_1=1\\ b_0=2\end{cases}$$

故所求通解为

$$y_t=C+2t+t^2$$

□

Example 8.4: 求 $y_{t+1}-3y_t=7\cdot 2^t$ 的通解

✎ **Solution:** 由式 (8.5) 得到齐次方程的通解为 $y_t=C\cdot 3^t$，C 为常数

$$y_t=C\cdot 5^t+\frac{3}{1-5}=C\cdot 5^t-\frac{3}{4}$$

因为 $3=p\neq\lambda=2$ 故设所求方程的特解为 $y_t^*=b\cdot 2^t$
代入方程得

$$b\cdot 2^{t+1}-3b\cdot 2^t=7\cdot 2^t$$

解得

$$b=-7$$

故所求方程特解为

$$\bar{y}_t=-7\cdot 2^t$$

通解为

$$y_t=C\cdot 3^t-7\cdot 2^t$$

□

8.3 二阶常系数线性差分方程

二阶常系数非齐次线性差分方程的一般形式为

$$y_{t+2}+py_{t+1}+qy_t=f(t) \tag{8.7}$$

其中 p,q 为常数系数 $(q\neq 0)$，未知函数项 y_{t+2},y_{t+1} 和 y_t 为一次的，右端项 $f(t)$ 为已知函数。与其相应的齐次方程为

$$y_{t+2}+py_{t+1}+qy_t=0 \tag{8.8}$$

将 $y_t = \lambda^t$ 代入 (8.8) 得到

$$\lambda^2 + p\lambda + q = 0 \tag{8.9}$$

容易证明 $y_t = \lambda^t$ 为 (8.8) 的解，当且仅当 λ 为 (8.9) 的解，因此称二次代数方程 (8.9) 为 (8.7) 和 (8.8) 的特征方程，其根为特征根。特征根有两个

$$\lambda_{1,2} = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4q}\right)$$

1. 当 $p^2 > 4q$ 时，特征方程有一对互异实根
   $\lambda_1 = \frac{1}{2}\left(-p + \sqrt{p^2 - 4q}\right), \lambda_2 = \frac{1}{2}\left(-p - \sqrt{p^2 - 4q}\right)$
   (8.8) 通解为 $y_t = C_1\lambda_1^t + C_2\lambda_2^t$，其中 C_1,C_2 为任意常数

2. 当 $p^2 = 4q$ 时，特征方程有二重实根 $\lambda_1 = \lambda_2 = -\frac{p}{2}$，
   (8.8) 通解为 $y_t = (C_1 + C_2 t)\left(-\frac{p}{2}\right)^t$，其中 C_1，C_2 为任意常数

3. 当 $p^2 < 4q$ 时，特征方程有共轭复根 $\lambda_{1,2} = \alpha \pm i\beta$
   特征根的实部 $\alpha = -\frac{p}{2}$，
   特征根的虚部 $\beta = \frac{1}{2}\sqrt{4q - p^2}$
   $r = \sqrt{\alpha^2 + \beta^2}$ 其中 $\cos\theta = \frac{\alpha}{r}, \sin\theta = \frac{\beta}{r}, \theta\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
   (8.8) 通解为 $y_t = r^t\left(C_1\cos(\theta t) + C_2\sin(\theta t)\right)$，其中 C_1，C_2 为任意常数

8.3.1 求二阶常系数非齐次线性差分方程的特解

1. 设 $f(t) = P_n(t)$，即 (8.7) 右端为一个已知的 n 次多项式

   $$y_{t+2} + py_{t+1} + qy_t = P_n(t)$$

   方程可改写为

   $$\Delta^2 y_t + (p+2)\Delta y_t + (1+p+q)y_t = P_n(t)$$

   (a) 当 $1 + p + q \neq 0$ 时，
   设

   $$y_t = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$$

   (b) 当 $1 + p + q = 0, p + 2 \neq 0$ 时，
   设

   $$y_t = t\left(b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n\right)$$

(c) 当 $1+p+q=0, p+2=0$ 时，
设

$$y_t = t^2\left(b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n\right)$$

2. 设 $f(t) = \lambda^t P_n(t)$，此时有

$$y_{t+2} + p y_{t+1} + q y_t = \lambda^t P_n(t)$$

设 (8.7) 有特解

$$y_t = \lambda^t t^k\left(b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n\right)$$

其中 k 等于 λ(作为特征根) 的重数

Example 8.5: 求 $y_{t+2} + 5y_{t+1} + 4y_t = 0$ 的通解

✎ **Solution:** 其特征方程为

$$\lambda^2 + 5\lambda + 4 = 0$$

有特征根 $\lambda_1 = -1, \lambda_2 = -4$
所求通解为

$$y_t = C_1(-1)^t + C_2(-4)^2$$

其中 C_1，C_2 为任意常数 □

Example 8.6: 求 $y_{t+2} - 6y_{t+1} + 9y_t = 0$ 的通解

✎ **Solution:** 其特征方程为

$$\lambda^2 - 6\lambda + 9 = 0$$

有特征根

$$\lambda_1 = \lambda_2 = 3$$

所求通解为

$$y_t = (C_1 + tC_2)3^t$$

其中 C_1，C_2 为任意常数 □

Example 8.7: 求 $y_{t+2} + 4y_t = 0$ 的通解解

✎ **Solution:** 其特征方程 $\lambda^2 + 4 = 0$，特征根 $\lambda = \pm 2i$
实部

$$\alpha = -\frac{p}{2} = 0$$

虚部

$$\beta = \frac{1}{2}\sqrt{4q - p^2} = 2$$

$$r = \sqrt{\alpha^2 + \beta^2} = 2, \sin\beta = \frac{\beta}{r} = 1$$

故所求通解为

$$y_t = 2^t\left(C_1 \sin\frac{\pi}{2}t + C_2 \sin\frac{\pi}{2}t\right)$$

其中 C_1，C_2 为任意常数 □

Example 8.8: 求 $y_{t+2} - 3y_{t+1} + 2y_4 = 4$ 的通解

✎ **Solution:** 特征方程为 $\lambda^2 - 3\lambda + 2 = 0$，特征根 $\lambda_1 = 1, \lambda_2 = 2$

对应的齐次方程的通解

$$y_t = C_1 + C_2 2^t$$

因 $1 + p + q = 1 - 3 + 2 = 0, p + 2 = -3 + 2 = -1 \neq 0$

故设非齐次方程的特解为 $\bar{y_t} = bt$

将其代入差分方程得

$$b(t+2) - 3b(t+1) + 2bt = 4$$

解得 $b = -4$，所求通解为

$$y_t = C_1 + C_2 2^t - 4t$$

其中 C_1，C_2 为任意常数 □

Example 8.9: 求 $y_{t+2} + y_{t+1} - 2y_4 = 12t$ 的通解

✎ **Solution:** 其特征方程为

$$\lambda^2 + \lambda - 2 = 0$$

特征根

$$\lambda_1 = 1, \lambda_2 = -2$$

对应的齐次方程的通解

$$y_t = C_1 + C_2(-2)^t$$

因为

$$1 + p + q = 1 + 1 - 2 = 0, p + 2 = 1 + 2 = 3 \neq 0$$

故设非齐次方程的一个特解为

$$\bar{y_t} = t(b_0 + b_1 t)$$

将其代入差分方程得

$$(t+2)(b_0 + b_1(t+2)) + (t+1)(b_0 + b_1(t+1)) - 2t(b_0 + b_1 t) = 12t$$

整理得

$$6b_1 t + 3b_0 + 5b_1 = 12x$$

比较系数，得

$$\begin{cases} 6b_1 = 12, \\ 3b_0 + 5b_1 = 0, \end{cases}$$

解得 $b_0 = -\dfrac{10}{3}, b_1 = 2$
故所求通解为

$$y_t = C_1 + C_2(-2)^t - \frac{10}{3}t + 2x^2$$

其中 C_1，C_2 为任意常数 □

Example 8.10: 求 $y_{t+2} - 6y_{t+1} + 9y_t = 3^t$ 的通解

✎ **Solution:** 特征方程为

$$\lambda^2 - 6\lambda + 9 = 0$$

有特征根

$$\lambda_1 = \lambda_2 = 3$$

$f(t) = 3^t P_0(t)$，因 $\lambda = 3$ 为二重根，故设特解为 $\bar{y}_t = bt^2 3^t$
将其代入差分方程得

$$b(t+2)^2 3^{t+2} - 6b(t+1)^2 3^{t+1} + 9b^2 3^t = 3^t$$

解得 $b = \dfrac{1}{18}$，特解为 $\bar{y}_t = \dfrac{1}{18}t^2 3^t$ 所求通解为

$$y_t = (C_1 + C_2 t)3^t + \frac{1}{18}t^2 3^t$$

□

Example 8.11: 求 $y_{t+2} - 4y_{t+1} + 4y_t = 5^t$ 的通解

✎ **Solution:** 特征方程为

$$\lambda^2 - 4\lambda + 4 = 0$$

有特征根

$$\lambda_1 = \lambda_2 = 2$$

$f(t) = 5^t P_0(t)$，因 $\lambda = 5$ 不是特征根，故设特解为 $\bar{y}_t = b3^t$
将其代入差分方程得

$$b3^{t+2} - 4b3^{t+1} + 4b3^t = 5^t$$

解得 $b = \dfrac{1}{9}$，非齐次方程的特解为 $\bar{y}_t = \dfrac{1}{9}5^t$
所求通解为

$$y_t = (C_1 + C_2 t)2^t + \frac{1}{9}5^t$$

其中 C_1，C_2 为任意常数 □

Example 8.12: 求 $y_{t+2} - 3y_{t+1} + 2y_t = 2^t$ 的通解

✎ **Solution:** 特征方程为

$$\lambda^2 - 3\lambda + 2 = 0$$

有特征根

$$\lambda_1 = 1, \lambda_2 = 2$$

$f(t) = 2^t P_0(t)$，因 $\lambda = 2$ 是单特征根，故设特解为 $\bar{y}_t = bt2^t$
将其代入差分方程得

$$b(t+2)2^{t+2} - 3(t+1)b3^{t+1} + 2bt3^t = 2^t$$

解得 $b = \dfrac{1}{2}$，非齐次方程的特解为 $\bar{y}_t = \dfrac{1}{2}2^t = 2^{t-1}$
所求通解为

$$y_t = C_1 + \left(C_2 + \frac{1}{2}\right)2^t$$

其中 C_1，C_2 为任意常数 □

第 9 章 向量代数与空间解析几何

9.1 向量及其线性运算

9.2 数量积向量积混合积

9.3 平面及其方程

9.4 空间直线及其方程

9.5 曲面及其方程

9.5.1 旋转曲面

1. 曲线 C $f(y,z)=0$ 绕 z 轴旋转一周得旋转曲面 $f(\pm\sqrt{x^2+y^2},z)=0$
2. 曲线 C $f(y,z)=0$ 绕 y 轴旋转一周得旋转曲面 $f(y,\pm\sqrt{x^2+z^2})=0$

9.5.2 柱面

9.5.3 二次曲面

9.6 空间曲线及其方程

第 10 章 多元函数微分法及其应用

10.1 多元函数的基本概念

Exercise 10.1: 设实数 x, y, z 满足

$$e^x + e^y + e^z = 2 + e^{x+y+z}$$

求极限

$$\lim_{(x,y,z)\to(0,0,0)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{x+y+z}{12} \right)$$

Solution 注意

$$\frac{1}{e^x - 1} + \frac{1}{e^y - 1} + \frac{1}{e^z - 1} = -1$$

且由泰勒或者伯努利函数得

$$\frac{1}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^{k-1}$$

其中 B_k 表示第 k 个伯努利数. 即有

$$\frac{1}{e^x - 1} + \frac{1}{e^y - 1} + \frac{1}{e^z - 1} = \frac{1}{x} - \frac{1}{2} + \frac{x}{12} + \frac{1}{y} - \frac{1}{2} + \frac{y}{12} + \frac{1}{z} - \frac{1}{2} + \frac{z}{12}$$

即

$$\lim_{(x,y,z)\to(0,0,0)} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{x+y+z}{12} \right) = \frac{1}{2}$$

Exercise 10.2: 求极限

$$\lim_{\substack{x\to 0 \\ y\to 0}} \frac{\sin(x^2 y + y^4)}{x^2 + y^2}$$

Solution 因为 $|\sin x| \leq |x|$, 因而有

$$0 \leq \left| \frac{\sin(x^2 y + y^4)}{x^2 + y^2} \right| \leq \left| \frac{x^2 y + y^4}{x^2 + y^2} \right|$$

又

$$\begin{aligned} \left| \frac{x^2 y + y^4}{x^2 + y^2} \right| &\leq \frac{x^2}{x^2 + y^2} \times |y| + \frac{y^2}{x^2 + y^2} \times y^2 \\ &\leq |y| + y^2 \to 0 \end{aligned}$$

由夹逼准则知道极限为 0

Exercise 10.3: 求极限

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\sin(x^3+y^3)}{x^2+y^2}$$

Solution 由于

$$\begin{aligned}|\sin(x^3+y^3)| &\le |x^3|+|y^3|\\ &\le (|x|+|y|)(x^2+y^2)\end{aligned}$$

从而

$$\lim_{\substack{x\to 0\\ y\to 0}} \left|\frac{\sin(x^3+y^3)}{x^2+y^2}\right| \le \lim_{\substack{x\to 0\\ y\to 0}}(|x|+|y|)=0$$

由夹逼准则知

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\sin(x^3+y^3)}{x^2+y^2}=0$$

Exercise 10.4: 求极限 $\lim\limits_{\substack{x\to\infty\\ y\to\infty}} \dfrac{x+y}{x^2-xy+y^2}$ ——16 中科院数分

Solution 由于

$$\left|\frac{x+y}{x^2-xy+y^2}\right| \le \frac{\left|\frac{1}{y}+\frac{1}{x}\right|}{\left|\frac{x}{y}-1+\frac{y}{x}\right|} \le \frac{\left|\frac{1}{y}+\frac{1}{x}\right|}{\left|\frac{x}{y}+\frac{y}{x}\right|-1} \le \left|\frac{1}{y}+\frac{1}{x}\right|$$

显然

$$\lim_{\substack{x\to\infty\\ y\to\infty}} \left|\frac{1}{y}+\frac{1}{x}\right|=0$$

故由夹逼准则知

$$\lim_{\substack{x\to\infty\\ y\to\infty}} \frac{x+y}{x^2-xy+y^2}=0$$

Solution 由于

$$\left|\frac{x+y}{x^2-xy+y^2}\right| \le \frac{2\,|x+y|}{x^2+y^2} \le 2\frac{|x|+|y|}{x^2+y^2} \le 2\left(\frac{1}{|x|}+\frac{1}{|y|}\right)$$

显然

$$\lim_{\substack{x\to\infty\\ y\to\infty}} 2\left(\frac{1}{|x|}+\frac{1}{|y|}\right)=0$$

故由夹逼准则知

$$\lim_{\substack{x\to\infty\\ y\to\infty}} \frac{x+y}{x^2-xy+y^2}=0$$

Solution 注意到

$$x^2+y^2-xy \ge 2xy-xy=xy$$

由于

$$\left|\frac{x+y}{x^2-xy+y^2}\right| \le \left|\frac{x+y}{xy}\right| \le \left(\frac{1}{|y|}+\frac{1}{|x|}\right)$$

显然

$$\lim_{\substack{x\to\infty\\y\to\infty}}\left(\frac{1}{|y|}+\frac{1}{|x|}\right)=0$$

故由夹逼准则知

$$\lim_{\substack{x\to\infty\\y\to\infty}}\frac{x+y}{x^2-xy+y^2}=0$$

Exercise 10.5: 求极限

$$\lim_{\substack{x\to\infty\\y\to\infty}}\frac{x^2+y^2}{x^4+y^4}$$

Solution 由于

$$\begin{aligned}\frac{x^2+y^2}{x^4+y^4}&=\frac{x^4}{x^4+y^4}\times\frac{1}{x^2}+\frac{y^4}{x^4+y^4}\times\frac{1}{y^2}\\&\le\frac{1}{x^2}+\frac{1}{y^2}\end{aligned}$$

显然

$$\lim_{\substack{x\to\infty\\y\to\infty}}\left(\frac{1}{|y|}+\frac{1}{|x|}\right)=0$$

故由夹逼准则知

$$\lim_{\substack{x\to\infty\\y\to\infty}}\frac{x+y}{x^2-xy+y^2}=0$$

Exercise 10.6: 求极限

$$\lim_{\substack{x\to+\infty\\y\to+\infty}}(x^2+y^2)e^{-(x+y)}$$

Solution 由于

$$\begin{aligned}\frac{(x^2+y^2)}{e^{x+y}}&=\frac{x^2}{e^{x+y}}+\frac{y^2}{e^{x+y}}\\&\le\frac{x^2}{e^x}+\frac{y^2}{e^y}\end{aligned}$$

而

$$\lim_{x\to+\infty}\frac{x^2}{e^x}=\lim_{x\to+\infty}\frac{2x}{e^x}=\lim_{x\to+\infty}\frac{2}{e^x}=0$$

以及

$$\lim_{y\to+\infty}\frac{y^2}{e^y}=\lim_{y\to+\infty}\frac{2y}{e^y}=\lim_{y\to+\infty}\frac{2}{e^y}=0$$

故由夹逼准则知

$$\lim_{\substack{x\to+\infty\\y\to+\infty}}(x^2+y^2)e^{-(x+y)}=0$$

Exercise 10.7: 求极限

$$\lim_{\substack{x\to+\infty\\y\to+\infty}}\left(\frac{xy}{x^2+y^2}\right)^{x^2}$$

Solution 注意到

$$0\le\frac{xy}{x^2+y^2}\le\frac{\frac{1}{2}(x^2+y^2)}{x^2+y^2}=\frac{1}{2}$$

所以

$$0 \le \left(\frac{xy}{x^2+y^2}\right)^{x^2} \le \left(\frac{1}{2}\right)^{x^2}$$

由于

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \left(\frac{1}{2}\right)^{x^2} = 0$$

从而

$$\lim_{\substack{x\to+\infty\\ y\to+\infty}} \left(\frac{xy}{x^2+y^2}\right)^{x^2} = 0$$

Exercise 10.8: 求极限

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2)$$

Solution 令 $x^2+y^2=t$ 则 $t\to 0^+$ 所以有

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{t\to 0^+} t\ln t$$

又

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} = \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = 0$$

所以

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = 0$$

Exercise 10.9: 求极限

$$\lim_{(x,y)\to(0,0)} x^2\ln(x^2+y^2)$$

Solution 因为

$$\lim_{(x,y)\to(0,0)} x^2\ln(x^2+y^2) = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}(x^2+y^2)\ln(x^2+y^2)$$

接着我们令 $x^2+y^2=t$ 则 $t\to 0^+$ 那么

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim_{t\to 0^+} t\ln t$$

又

$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+} \frac{\ln t}{1/t} = \lim_{t\to 0^+} \frac{1/t}{-1/t^2} = 0$$

所以

$$\lim_{(x,y)\to(0,0)} x^2\ln(x^2+y^2) = 0$$

Exercise 10.10: 求极限

$$\lim_{(x,y)\to(0,0)} x\ln(x^2+y^2)$$

Solution 因为

$$\lim_{\substack{x\to 0\\ y\to 0}} x\ln(x^2+y^2) = 2\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2}\ln\sqrt{x^2+y^2}$$

接着我们令 $\sqrt{x^2+y^2}=t$ 则 $t\to 0^+$ 那么

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\sqrt{x^2+y^2}\ln\sqrt{x^2+y^2} &= \lim_{t\to 0^+} t\ln t\\ &= \lim_{t\to 0^+}\frac{\ln t}{1/t} = \lim_{t\to 0^+}\frac{1/t}{-1/t^2} = 0\end{aligned}$$

所以

$$\lim_{(x,y)\to(0,0)} x\ln(x^2+y^2) = 0$$

Exercise 10.11: 求极限

$$\lim_{(x,y)\to(0,0)}\frac{x-y}{x+y}$$

Solution 当 (x,y) 沿着$y=kx$趋向于 $(0,0)$ 点时, 有

$$\lim_{\substack{x\to 0\\ y\to 0}}\frac{x-y}{x+y} = \lim_{\substack{x\to 0\\ y=kx}}\frac{x-y}{x+y} = \lim_{x\to 0}\frac{x-kx}{x+kx} = \frac{1-k}{1+k}$$

显然它的值随着 k 值的变化而变化，故极限不存在 (不满足极限的唯一性)

Exercise 10.12: 求极限

$$\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}$$

Solution 当 (x,y) 沿着$y=x$趋向于 $(0,0)$ 点时, 有

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2} &= \lim_{\substack{x\to 0\\ y=x}}\frac{x^2y^2}{x^2y^2+(x-y)^2}\\ &= \lim_{x\to 0}\frac{x^2x^2}{x^2x^2+(x-x)^2} = 1\end{aligned}$$

当 (x,y) 沿着$y=0$趋向于 $(0,0)$ 点时, 有

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2} &= \lim_{\substack{x\to 0\\ y=0}}\frac{x^2y^2}{x^2y^2+(x-y)^2}\\ &= \lim_{x\to 0}\frac{x^2 0^2}{x^2 0^2+(x-0)^2} = 0\end{aligned}$$

因此极限不存在

Exercise 10.13: 求极限

$$\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y}$$

✎ **Solution** 当 (x,y) 沿着$y=kx^3-x^2$趋向于 $(0,0)$ 点时, 有

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x^3+y^3}{x^2+y} &= \lim_{\substack{x\to 0\\ y=kx^3-x^2}}\frac{x^3+y^3}{x^2+y}\\ &= \lim_{x\to 0}\frac{x^3+(kx^3-x^2)^3}{x^2+kx^3-x^2}\\ &= \frac{1}{k}\end{aligned}$$

显然它的值随着 k 值的变化而变化，故极限不存在

Exercise 10.14: 求极限

$$\lim_{(x,y)\to(0,0)} x\frac{\ln(1+xy)}{x+y}$$

✎ **Solution** 当 (x,y) 沿着$y=x^\alpha-x$趋向于 $(0,0)$ 点时, 有

$$\begin{aligned}\lim_{(x,y)\to(0,0)} x\frac{\ln(1+xy)}{x+y} &= \lim_{(x,y)\to(0,0)}\frac{x^2y}{x+y}\\ &= \lim_{\substack{x\to 0\\ x^\alpha-x}}\frac{x^2y}{x+y} = \lim_{x\to 0}\frac{x^{\alpha+2}-x^3}{x^\alpha}\\ &= \lim_{x\to 0}(x^2-x^{3-\alpha}) = \begin{cases}-1, & \alpha=3\\ 0, & \alpha<3\\ 0, & \alpha>3\end{cases}\end{aligned}$$

故极限不存在

Exercise 10.15: 求极限

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x+y}$$

✎ **Solution** 当 (x,y) 沿着$y=x^2-x$趋向于 $(0,0)$ 点时, 有

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{xy}{x+y} &= \lim_{\substack{x\to 0\\ y=x^2-x}}\frac{xy}{x+y}\\ &= \lim_{x\to 0}\frac{x(x^2-x)}{x+x^2-x}\\ &= \lim_{x\to 0}(x-1) = -1\end{aligned}$$

当 (x,y) 沿着$y=x$趋向于 $(0,0)$ 点时, 有

$$\lim_{(x,y)\to(0,0)}\frac{xy}{x+y} = \lim_{\substack{x\to 0\\ y=x}}\frac{xy}{x+y} = \lim_{x\to 0}\frac{x^2}{2x} = 0$$

故极限不存在

Exercise 10.16: 求极限

✎ **Solution**

10.2 偏导数

10.3 全微分

Exercise 10.17: 证明：函数 $f(x,y)=\sqrt[3]{x^2y}$ 在 (0,0) 点的偏导数存在且在 (0,0) 处不可微

Solution 显然有 $f(x,0)=0, f(0,y)=0$, 由偏导数的定义知道

$$f'_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{\sqrt[3]{0^2y}-0}{x}=0$$

以及

$$f'_y(0,0)=\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y}=\lim_{y\to 0}\frac{\sqrt[3]{x^20}-0}{y}=0$$

即偏导数 $f(x,y)$ 在 $(0,0)$ 处偏导数存在并且

$$f'_x(0,0)=f'_y(0,0)=0$$

又因为 $f(x,y)$ 在 $(0,0)$ 的全增量

$$\Delta f(x,y)=f(\Delta x,\Delta y)-f(0,0)=\sqrt[3]{(\Delta x)^2\Delta y}$$

记

$$\Delta f(x,y)=f'_x(0,0)\Delta x+f'_y(0,0)\Delta y+\omega=\omega$$

则有

$$\omega=\sqrt[3]{(\Delta x)^2\Delta y}$$

由微分的定义可知道, 如果 $f(x,y)$在$(0,0)$可微, 那么必然有 ω 是 $\sqrt{(\Delta x)^2+(\Delta y)^2}$ 的高阶无穷小量

下面证明极限 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{\sqrt[3]{(\Delta x)^2\Delta y}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$ 不存在, 这一结果也就说明 $f(x,y)$在$(0,0)$不可微

考虑 $\Delta y=k\Delta x$ 则

$$\lim_{\substack{\Delta x\to 0\\ \Delta y=k\Delta x}}\frac{\sqrt[3]{(\Delta x)^2\Delta y}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\frac{\sqrt[3]{k}}{\sqrt{1+k^2}} \tag{10.1}$$

显然他的值是随着 k 值的改变而改变, 故上式极限不存在, 即 $f(x,y)$ 在 $(0,0)$ 处不可微

Exercise 10.18: 证明：函数 $f(x,y)=\dfrac{xy}{x^2+y^2}$ 在 (0,0) 点的偏导数存在且在 (0,0) 处不可微

Solution 显然有 $f(\Delta x,0)=0, f(0,\Delta y)=0$, 由偏导数的定义知道

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{\frac{\Delta x\times 0}{\Delta x^2+0^2}-0}{\Delta x}=0$$

以及

$$f_y'(0,0)=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{y}=\lim_{y\to 0}\frac{\frac{0\times\Delta y}{0^2+\Delta y^2}-0}{\Delta y}=0$$

即偏导数 $f(x,y)$ 在 $(0,0)$ 处偏导数存在并且

$$f_x'(0,0)=f_y'(0,0)=0$$

又因为 $f(x,y)$ 在 $(0,0)$ 的全增量

$$\Delta f(x,y)=f(\Delta x,\Delta y)-f(0,0)=\frac{\Delta x\Delta y}{\Delta x^2+\Delta y^2}$$

记

$$\Delta f(x,y)=f_x'(0,0)\Delta x+f_y'(0,0)\Delta y+\omega=\omega$$

则有

$$\frac{\Delta x\Delta y}{\Delta x^2+\Delta y^2}$$

由微分的定义可知道, 如果 $f(x,y)$在$(0,0)$可微, 那么必然有 ω 是 $\dfrac{\Delta x\Delta y}{\Delta x^2+\Delta y^2}$ 的高阶无穷小量

下面证明极限 $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\dfrac{\frac{\Delta x\Delta y}{\Delta x^2+\Delta y^2}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$ 不存在, 这一结果也就说明 $f(x,y)$在$(0,0)$不可微

考虑 $\Delta y=k\Delta x$ 则

$$\lim_{\substack{\Delta x\to 0\\ \Delta y=k\Delta x}}\frac{\sqrt[3]{(\Delta x)^2\Delta y}}{\sqrt{(\Delta x)^2+(\Delta y)^2}}=\frac{\sqrt[3]{k}}{\sqrt{1+k^2}}\tag{10.2}$$

显然他的值是随着 k 值的改变而改变, 故上式极限不存在, 即 $f(x,y)$ 在 $(0,0)$ 处不可微

Exercise 10.19: 设 $f(x,y)$ 可微, 且 $f(x,2x)=x$, $f_1'(x,2x)=x^2$, 求 $f_2'(x,2x)$

Solution 对 $f(x,2x)=x$ 两边对 x 求导

$$f_1'(x,2x)+2f_2'(x,2x)=1$$

由 $f_1'(x,2x)=x^2$ 可得

$$f_2'(x,2x)=\frac{1}{2}(1-x^2)$$

Exercise 10.20: 设 $u(x,y)$ 的所有二阶偏导数都连续, 并且 $\dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}=0$.
已知 $u(x,2x)=x, u_x(x,2x)=x^2$ 试求: $u_{xx}(x,2x), u_{xy}(x,2x), u_{yy}(x,2x)$

Solution 对 $u(x,2x)=x$ 两边对 x 求导

$$u_x'(x,2x)+2u_y'(x,2x)=1$$

由 $u_x(x,2x)=x^2$ 可得

$$u_y'(x,2x)=\frac{1}{2}(1-x^2)$$

上式两边对 x 求导

$$u'_{xy}(x,2x)+2u''_{yy}(x,2x)=-x \tag{10.3}$$

对 $u'_x(x,2x)=x^2$ 两边对 x 求导

$$u''_{xx}(x,2x)+2u''_{xy}(x,2x)=2x \tag{10.4}$$

利用 $u_{xx}=u_{yy}, u_{xy}=u_{yx}$, 联立式 (10.3) 和 (10.4) 求解可得

$$u_{xx}(x,2x)=u_{yy}(x,2x)=-\frac{4}{3}x \qquad u_{xy}(x,2x)=\frac{5}{3}x$$

Exercise 10.21: 设 $a,b\neq 0$, f 具有二阶连续偏导数, 且

$$a^2 f_{xx}+b^2 f_{yy}=0 \qquad f(ax,bx)=ax \qquad f_x(ax,bx)=bx^2$$

试求 $f_{xx}(ax,bx)$, $f_{xy}(ax,bx)$, $f_{yy}(ax,bx)$

Solution

Exercise 10.22: 设 $f(x,y)$ 在 $\mathbb{R}$ 上具有连续偏导数, 且 $f(x,x^2)=1$

1. 若 $f_x(x,x^2)=x$, 求 $f_y(x,x^2)$
2. 若 $f_y(x,y)=x^2+2y$, 求 $f(x,y)$

Solution

Exercise 10.23: 设 $z=f(x,y)$ 有连续二阶偏导数, 且

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial y^2} \qquad f(x,2x)=5x^2 \qquad f'_x(x,2x)=2x$$

求 $f(2,1)=$ ________

Solution 对 $f(x,2x)=5x^2$ 两边对 x 求导

$$f'_x(x,2x)+2f'_y(x,2x)=10x$$

由 $f'_x(x,2x)=2x$ 可得

$$f'_y(x,2x)=4x \tag{10.5}$$

上式两边对 x 求导

$$f'_{xy}(x,2x)+2f''_{yy}(x,2x)=4 \tag{10.6}$$

对 $f'_x(x,2x)=2x$ 两边对 x 求导

$$f''_{xx}(x,2x)+2f''_{xy}(x,2x)=2 \tag{10.7}$$

且 $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\partial^2 z}{\partial y^2}$ 联立 (10.6) 与 (10.7) 解得

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial y^2}=2,\ f''_{xy}(x,2x)=0$$

故

$$\frac{\partial^2 z}{\partial y^2}=2\Longrightarrow\frac{\partial z}{\partial y}=2y+h(x)\Longrightarrow f(x,y)=y^2+h(x)y+g(x)$$

再结合条件 $f(x,2x)=5x^2$ 以及式 (10.5) 可得

$$h(x)=0\qquad g(x)=x^2$$

因此

$$f(x,y)=x^2+y^2$$

故

$$f(2,1)=5$$

Exercise 10.24: 求极限

Solution

10.4 隐函数的求导公式

10.5 多元函数微分学的几何应用

10.6 方向导数与梯度

10.7 多元函数的极值及其求法

10.8 二元函数的泰勒公式

Exercise 10.25: 设 $f(x,y)$ 在 $x^2+y^2\leqslant 1$ 上有连续的二阶偏导数, $f_{xx}^2+2f_{xy}^2+f_{yy}^2\leqslant M$. 若 $f(0,0)=0$, $f_x(0,0)=f_y(0,0)=0$, 证明

$$\left|\iint\limits_{x^2+y^2\leqslant 1} f(x,y)\,\mathrm{d}x\,\mathrm{d}y\right|\leqslant\frac{\pi\sqrt{M}}{4}$$

Proof:在点 $(0,0)$ 展开 $f(x,y)$ 得

$$\begin{aligned}f(x,y)&=\frac{1}{2}\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 f(\theta x,\theta y)\\&=\frac{1}{2}\left(x^2\frac{\partial^2}{\partial x^2}+2xy\frac{\partial^2}{\partial x\partial y}+y^2\frac{\partial^2}{\partial y^2}\right)^2 f(\theta x,\theta y)\end{aligned}$$

其中 $\theta\in(0,1)$

记 $(u,v,w)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\right)f(\theta x,\theta y)$, 则

$$f(x,y)=\frac{1}{2}\left(ux^2+2vxy+w^2y\right)$$

由于 $\| (u, \sqrt{2}v, w) \| = \sqrt{u^2 + 2v^2 + w^2} \leqslant \sqrt{M}$ 以及 $\| (x^2, \sqrt{2}xy, y^2) \| = x^2 + y^2$, 我们有

$$\left|(u, \sqrt{2}v, w) \cdot (x^2, \sqrt{2}xy, y^2)\right| \leqslant \sqrt{M}(x^2 + y^2)$$

即

$$|f(x, y)| \leqslant \frac{1}{2}\sqrt{M}(x^2 + y^2)$$

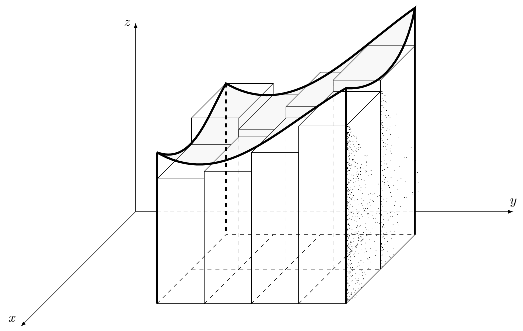
从而

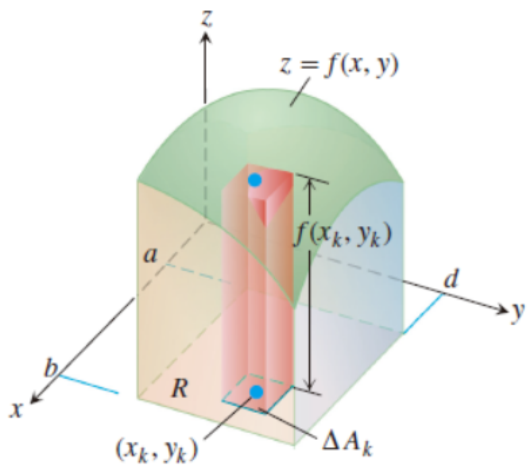
$$\left|\iint\limits_{x^2+y^2\leqslant 1} f(x, y)\,\mathrm{d}x\,\mathrm{d}y\right| \leqslant \frac{\sqrt{M}}{2}\iint\limits_{x^2+y^2\leqslant 1}(x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y = \frac{\pi\sqrt{M}}{4}$$

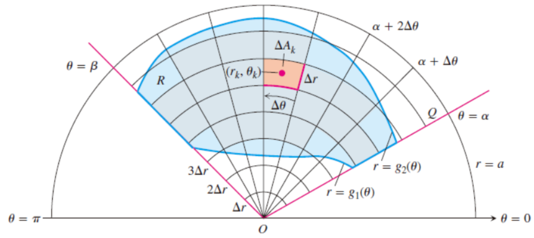
□

10.9 最小二乘法

第 11 章 重积分







11.1 二重积分的概念与性质

Theorem 11.1 二重积分的中值定理

设函数 $f(x,y)$ 在闭区间 D 上连续, σ 是 D 的面积, 则在 D 上至少存在一点 (ξ,η), 使得

$$\iint\limits_D f(x,y)\,\mathrm{d}\sigma = f(\xi,\eta)\sigma$$

Exercise 11.1: 求极限

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{(n+i+1)^2}+\frac{1}{(n+i+2)^2}+\cdots+\frac{1}{(n+i+i)^2}\right)$$

Solution

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{(n+i+1)^2}+\frac{1}{(n+i+2)^2}+\cdots+\frac{1}{(n+i+i)^2}\right)\\
&=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{(n+i+j)^2}=\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{n^2}\cdot\frac{1}{(1+\frac{i}{n}+\frac{j}{n})^2}\\
&=\int_0^1\int_0^x\frac{1}{(1+x+y)^2}dydx=\int_0^1\left[-\frac{1}{1+x+y}\right]_{y=0}^{y=x}dx=\int_0^1\left(\frac{1}{x+1}-\frac{1}{2x+1}\right)dx\\
&=\ln 2-\frac{1}{2}\ln 3=\ln\left(\frac{2}{\sqrt{3}}\right)
\end{aligned}$$

Exercise 11.2: 求极限

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}$$

Solution

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{i^2+j^2}&=\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\frac{i}{n}+\frac{j}{n}}{\left(\frac{i}{n}\right)^2+\left(\frac{j}{n}\right)^2}\\
&=\int_0^1\int_0^1\frac{x+y}{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\\
&=\frac{1}{2}\int_0^1\left(\ln(1+y^2)-2\ln y\right)\,\mathrm{d}y+\int_0^1\arctan\frac{1}{y}\,\mathrm{d}y\\
&=\frac{1}{2}\ln 2-\int_0^1\frac{y^2}{1+y^2}\,\mathrm{d}y+\int_0^1\mathrm{d}y+\frac{\pi}{2}-\int_0^1\arctan y\mathrm{d}y\\
&=\frac{\pi}{2}+\ln 2
\end{aligned}$$

Exercise 11.3: 求极限

$$\lim_{n\to\infty}\frac{\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{j}i}{n^3}$$

Solution1

$$\begin{aligned}
\lim_{n\to\infty}\frac{\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{j}i}{n^3}&=\lim_{n\to\infty}\left(\frac{1}{n^3}+\frac{1+2}{n^3}+\cdots+\frac{1+2+\cdots+n}{n^3}\right)\\
&=\lim_{n\to\infty}\frac{1\times 2+2\times 3+\cdots+n(n+1)}{2n^3}\\
&=\lim_{n\to\infty}\frac{(1^2+1)+(2^2+2)+\cdots+(n^2+n)}{2n^3}\\
&=\lim_{n\to\infty}\frac{(1+2+\cdots+n)+(1^2+2^2+\cdots+n^2)}{2n^3}\\
&=\lim_{n\to\infty}\frac{\frac{1}{2}n(n+1)+\frac{1}{6n(n+1)(2n+1)}}{2n^3}\\
&=\frac{1}{6}
\end{aligned}$$

✎ **Solution2**

$$\lim_{n\to\infty}\frac{\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{j}i}{n^3}=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{1}{n}\sum_{i=1}^{j}\frac{i}{n}=\int_0^1\mathrm{d}y\int_0^y x\,\mathrm{d}x=\frac{1}{2}\int_0^1 y^2\,\mathrm{d}y=\frac{1}{6}$$

✎ **Solution3**

$$\begin{aligned}\lim_{n\to\infty}\frac{\sum\limits_{j=1}^{n}\sum\limits_{i=1}^{j}i}{n^3}&=\lim_{n\to\infty}\frac{1+(1+2)+\cdots+(1+2+\cdots+n)}{n^3}\\&\xlongequal{Stolz}\lim_{n\to\infty}\frac{1+2+\cdots+n}{n^3-(n-1)^3}\\&=\lim_{n\to\infty}\frac{\frac{1}{2}n(n+1)}{3n^2-3n+1}\\&=\frac{1}{6}\end{aligned}$$

Exercise 11.4: 设区域 $D{:}x^2+y^2\le r^2$, 求 $\lim\limits_{r\to0}\dfrac{1}{\pi r^2}\iint\limits_D e^{x^2-y^2}\cos(x+y)dxdy$

✎ **Solution**

$$\begin{aligned}&\lim_{r\to0}\frac{1}{\pi r^2}\iint\limits_D e^{x^2-y^2}\cos(x+y)dxdy\\=&\lim_{r\to0}\frac{1}{\pi r^2}e^{\xi^2-\eta^2}\cos(\xi+\eta)\cdot\pi r^2\\=&\lim_{\substack{r\to0\\(\xi,\eta)\to(0,0)}}e^{\xi^2-\eta^2}\cos(\xi+\eta)=1\end{aligned}$$

Note: 设函数 $f(x,y)$ 在闭区间 D 上连续, σ 是 D 的面积, 则在 D 上至少存在一点 (ξ,η) , 使得

$$\iint\limits_D f(x,y)\,\mathrm{d}\sigma=f(\xi,\eta)\sigma$$

11.2 二重积分的计算法

Exercise 11.5: 计算积分

$$\int_0^2\int_0^4(6-x-y)dxdy$$

✎ **Solution**

$$\begin{aligned}
&\int_0^2\int_0^4 (6-x-y)dxdy \\
&=\int_0^2 \left[6x-\frac{1}{2}x^2-xy\right]_0^4 dy \\
&=\int_0^2 (16-4y)dy \\
&=\left[16y-2y^2\right]_0^2=24
\end{aligned}$$

Exercise 11.6: 证明

$$I=\int_0^1 dx\int_0^1 (xy)^{xy}dy=\int_0^1 y^y dy=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}$$

✎ **Solution** 令 $xy=t$，我们有 (注意 $xy=0$ 时给定 $xy^{xy}=1$)

$$\begin{aligned}
I&=\int_0^1 dy\int_0^1 (xy)^{xy}dx=\int_0^1\frac{dy}{y}\int_0^1 (xy)^{xy}d(xy) \\
&=\int_0^1\frac{dy}{y}\int_0^y t^t dt=\int_0^1\left(\int_0^y t^t dt\right)d\ln y \\
&=\ln y\cdot\int_0^y t^t dt\Big|_0^1-\int_0^1 y^y\ln y dy=-\int_0^1 y^y\ln y dy
\end{aligned}$$

注意到

$$\int_0^1 y^y(1+\ln y)dy=\int_0^1 d(y^y)=\left[y^y\right]_0^1=\lim_{x\to 1^-}y^y-\lim_{x\to 0^+}y^y=1-1=0$$

故

$$I=\int_0^1 dx\int_0^1 (xy)^{xy}dy=\int_0^1 y^y(1+\ln y)dy-\int_0^1 y^y\ln y dy=\int_0^1 y^y dy$$

进一步

$$\int_0^1 y^y dy=\int_0^1 e^{y\ln y}dx=\int_0^1\sum_{n=0}^{\infty}\frac{(y\ln y)^n}{n!}dy=\sum_{n=0}^{\infty}\int_0^1\frac{(y\ln y)^n}{n!}dy$$

因为

$$\begin{aligned}
\int_0^1 (y\ln y)^n dy&=\int_0^1\frac{\ln^n y}{n+1}dy^{n+1} \\
&=\left[\frac{y^{n+1}}{n+1}\ln^n y\right]_0^1-\int_0^1\frac{n}{n+1}y^n\ln^{n-1}ydy \\
&=-\frac{n}{(n+1)^2}\int_0^1 \ln^{n-1}ydy^{n+1} \\
&=\left[-\frac{n}{(n+1)^2}y^{n+1}\ln^{n-1}y\right]_0^1+\int_0^1\frac{n\,(n-1)}{(n+1)^2}y^n\ln^{n-2}ydy \\
&=\cdots=\frac{(-1)^n n!}{(n+1)^{n+1}}
\end{aligned}$$

所以

$$\int_0^1 y^y dy = \sum_{n=0}^{\infty}\int_0^1 \frac{(y\ln y)^n}{n!}dy = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n} \approx 0.783430\cdots$$

所以

$$I = \int_0^1 dx\int_0^1 (xy)^{xy}dy = \int_0^1 y^y dy = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}$$

Exercise 11.7: 计算积分 $\displaystyle\iint\limits_D (x+y)\mathrm{d}x\mathrm{d}y$ 其中 D 是由 $x^2+y^2\leqslant 2$ 和 $x^2+y^2\geqslant 2x$ 所围成的区域

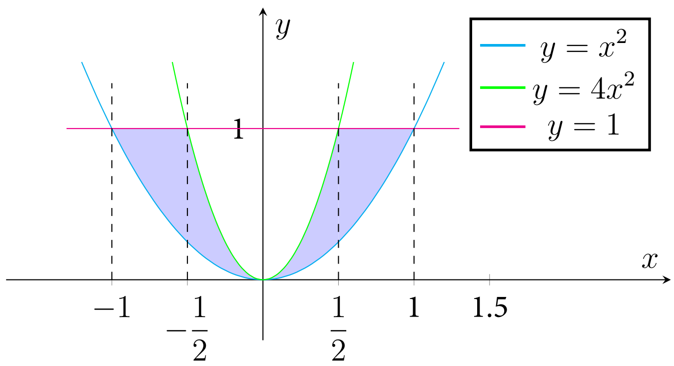
Solution

$$\begin{aligned}
\iint\limits_D (x+y)\mathrm{d}x\mathrm{d}y &= \iint\limits_D x\mathrm{d}x\mathrm{d}y + \iint\limits_D y\mathrm{d}x\mathrm{d}y = 2\iint\limits_{D_1} x\mathrm{d}x\mathrm{d}y + 0\\
&= 2\iint\limits_{D_{11}} x\mathrm{d}x\mathrm{d}y + 2\iint\limits_{D_{12}} x\mathrm{d}x\mathrm{d}y\\
&= 2\int_{\frac{\pi}{2}}^{\pi}\mathrm{d}\theta\int_0^{\sqrt{2}}\rho^2\cos\theta\mathrm{d}\rho + 2\int_0^1 dx\int_{\sqrt{2x-x^2}}^{\sqrt{2-x^2}} xdy\\
&= 2\int_{\frac{\pi}{2}}^{\pi}\left[\frac{1}{3}\rho^3\cos\theta\right]_0^{\sqrt{2}}\mathrm{d}\theta + 2\int_0^1 x\sqrt{2-x^2}\mathrm{d}x - 2\int_0^1 x\sqrt{2x-x^2}\mathrm{d}x\\
&= \frac{4\sqrt{2}}{3}\int_{\frac{\pi}{2}}^{\pi}\cos\theta\mathrm{d}\theta + \left[-\frac{2}{3}\sqrt{(2-x^2)^3}\right]_0^1\\
&\quad + \int_0^1 (2-2x)\sqrt{2x-x^2}\mathrm{d}x - 2\int_0^1 \sqrt{1-(x-1)^2}\mathrm{d}x\\
&= \frac{4\sqrt{2}}{3}\left[\sin\theta\right]_{\frac{\pi}{2}}^{\pi} - \frac{2}{3} + \frac{4\sqrt{2}}{3} + \left[\frac{2}{3}\sqrt{(2x-x^2)^3}\right]_0^1 - 2\times\frac{\pi}{4}\\
&= -\frac{\pi}{2}
\end{aligned}$$

Exercise 11.8: 计算积分 $\displaystyle\iint\limits_D (x+y)\mathrm{d}\sigma$ 其中 D 是由 $y=x^2$，$y=4x^2$，$y=1$ 所围成

Solution 区域 D 如图

$$\begin{aligned}
\iint\limits_D (x+y)\,\mathrm{d}\sigma &= \iint\limits_D x\,\mathrm{d}\sigma + \iint\limits_D y\,\mathrm{d}\sigma\\
&= 0 + 2\int_0^1 \mathrm{d}y\int_{\frac{\sqrt{y}}{2}}^{\sqrt{y}} y\,\mathrm{d}x\\
&= 2\int_0^1 \left[xy\right]_{\frac{\sqrt{y}}{2}}^{\sqrt{y}}\mathrm{d}y\\
&= \int_0^1 y^{\frac{3}{2}}\mathrm{d}y = \left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^1 = \frac{2}{5}
\end{aligned}$$



Solution

$$
\begin{aligned}
\iint_D (x+y)\,\mathrm{d}\sigma &= \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{-\frac{\sqrt{y}}{2}} (x+y)\,\mathrm{d}x + \int_0^1 \mathrm{d}y \int_{\frac{\sqrt{y}}{2}}^{\sqrt{y}} (x+y)\,\mathrm{d}x \\
&= \int_0^1 \left[\frac{1}{2}x^2 + xy\right]_{-\sqrt{y}}^{-\frac{\sqrt{y}}{2}} \mathrm{d}y + \int_0^1 \left[\frac{1}{2}x^2 + xy\right]_{\frac{\sqrt{y}}{2}}^{\sqrt{y}} \mathrm{d}y \\
&= \int_0^1 \left(\frac{1}{2}y^{\frac{3}{2}} - \frac{3}{8}y\right) \mathrm{d}y + \int_0^1 \left(\frac{1}{2}y^{\frac{3}{2}} + \frac{3}{8}y\right) \mathrm{d}y \\
&= \int_0^1 y^{\frac{3}{2}} \mathrm{d}y = \left[\frac{2}{5}y^{\frac{5}{2}}\right]_0^1 = \frac{2}{5}
\end{aligned}
$$

Exercise 11.9: 计算积分

$$
\int_0^1 \mathrm{d}y \int_y^1 \left(\frac{e^{x^2}}{x} - e^{y^2}\right) \mathrm{d}x
$$

Solution

$$
\begin{aligned}
I &= \int_0^1 \mathrm{d}y \int_y^1 \left(\frac{e^{x^2}}{x} - e^{y^2}\right) \mathrm{d}x \\
&= \int_0^1 \mathrm{d}y \int_y^1 \frac{e^{x^2}}{x} \mathrm{d}x - \int_0^1 \mathrm{d}y \int_y^1 e^{y^2} \mathrm{d}x \\
&= \int_0^1 \mathrm{d}x \int_0^x \frac{e^{x^2}}{x} \mathrm{d}y - \int_0^1 \mathrm{d}y \int_y^1 e^{y^2} \mathrm{d}x \\
&= \int_0^1 e^{x^2}\,\mathrm{d}x - \int_0^1 (1-y)e^{y^2}\,\mathrm{d}y \\
&= \int_0^1 ye^{y^2}\,\mathrm{d}y \\
&= \left[\frac{1}{2}e^{y^2}\right]_0^1 = \frac{e-1}{2}
\end{aligned}
$$

Exercise 11.10: 设平面区域 $D = \{(x,y) | 1 \leqslant x^2 + y^2 \leqslant 4, x \geqslant 0, y \geqslant 0\}$, **设** $f(x,y)$ **为** D

上的连续函数，且有

$$f(x,y)=\sin(\pi\sqrt{x^2+y^2})-\frac{1}{\pi}\iint\limits_D \frac{xf(x,y)}{x+y}\,\mathrm{d}x\mathrm{d}y$$

求 $f(x,y)$

✎ **Solution** 由

$$f(x,y)=\sin(\pi\sqrt{x^2+y^2})-\frac{1}{\pi}\iint\limits_D \frac{xf(x,y)}{x+y}\,\mathrm{d}x\mathrm{d}y$$

得

$$\frac{xf(x,y)}{x+y}=\frac{x\sin(\pi\sqrt{x^2+y^2})}{x+y}-\frac{1}{\pi}\frac{x}{x+y}\iint\limits_D \frac{xf(x,y)}{x+y}\,\mathrm{d}x\mathrm{d}y$$

注意到 $\iint\limits_D \frac{xf(x,y)}{x+y}\,\mathrm{d}x\mathrm{d}y$ 是个常数，故令 $C=\iint\limits_D \frac{xf(x,y)}{x+y}\,\mathrm{d}x\mathrm{d}y$

则

$$C=\iint\limits_D \frac{xf(x,y)}{x+y}\,\mathrm{d}x\mathrm{d}y=\iint\limits_D \frac{x\sin(\pi\sqrt{x^2+y^2})}{x+y}\,\mathrm{d}x\mathrm{d}y-\frac{C}{\pi}\iint\limits_D \frac{x}{x+y}\,\mathrm{d}x\mathrm{d}y$$

其中

$$\begin{aligned}\iint\limits_D \frac{x\sin(\pi\sqrt{x^2+y^2})}{x+y}\,\mathrm{d}x\mathrm{d}y&=\iint\limits_D \frac{y\sin(\pi\sqrt{x^2+y^2})}{x+y}\,\mathrm{d}x\mathrm{d}y\ (\text{轮换对称性})\\&=\frac{1}{2}\iint\limits_D \sin(\pi\sqrt{x^2+y^2})\,\mathrm{d}x\mathrm{d}y\\&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_1^2\rho\sin(\pi\rho)\mathrm{d}\rho\\&=-\frac{3}{4}\end{aligned}$$

$$\begin{aligned}\iint\limits_D \frac{x}{x+y}\,\mathrm{d}x\mathrm{d}y&=\iint\limits_D \frac{y}{x+y}\,\mathrm{d}x\mathrm{d}y\ (\text{轮换对称性})\\&=\frac{1}{2}\iint\limits_D \mathrm{d}x\mathrm{d}y\\&=\frac{15\pi}{8}\end{aligned}$$

由此可知 $C=-\frac{23}{6}$

故

$$f(x,y)=\sin(\pi\sqrt{x^2+y^2})+\frac{23}{6\pi}$$

Exercise 11.11: 求极限

$$\lim_{y\to+\infty}\left(\frac{\sqrt{\pi}}{2}y-\int_0^y dx\int_0^x e^{-y^2}dy\right)$$

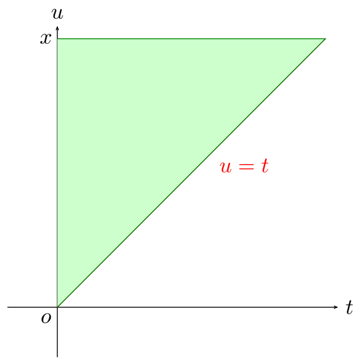
Solution

Exercise 11.12: 求极限

$$\lim_{x\to+\infty} x\left(\frac{\sqrt{\pi}}{2}-\int_0^x e^{-t^2}\,\mathrm{d}t\right)$$

Solution

$$\lim_{x\to+\infty} x\left(\frac{\sqrt{\pi}}{2}-\int_0^x e^{-t^2}\,\mathrm{d}t\right)\lim_{x\to+\infty}\int_0^{+\infty}\left(\frac{\sqrt{\pi}}{2}-\int_0^x e^{-t^2}\,\mathrm{d}t\right)\mathrm{d}u$$



$$\begin{aligned}
&=\lim_{x\to+\infty}\int_0^{+\infty}\left(\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t-\int_0^x e^{-t^2}\,\mathrm{d}t\right)\mathrm{d}u\\
&=\lim_{x\to+\infty}\int_0^{+\infty}\left(\int_x^{+\infty} e^{-t^2}\,\mathrm{d}t\right)\mathrm{d}u\\
&=\lim_{x\to+\infty}\int_0^{+\infty} e^{-t^2}\,\mathrm{d}t\int_0^t \mathrm{d}u\\
&=\frac{1}{2}
\end{aligned}$$

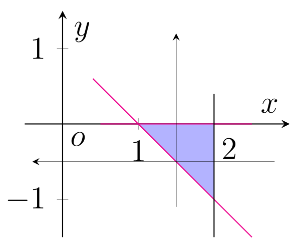
11.2.1 交换积分次序

Exercise 11.13: 交换二重积分的积分次序

$$\int_{-1}^0 \mathrm{d}y\int_2^{1-y} f(x,y)\mathrm{d}x$$

Solution

$$\begin{aligned}
\int_{-1}^0 \mathrm{d}y\int_2^{1-y} f(x,y)\mathrm{d}x &= -\int_{-1}^0 \mathrm{d}y\int_{1-y}^2 f(x,y)\mathrm{d}x\\
&= -\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y\\
&= -\int_1^2 \mathrm{d}x\int_{1-x}^0 f(x,y)\mathrm{d}y\\
&= \int_1^2 \mathrm{d}x\int_0^{1-x} f(x,y)\mathrm{d}y
\end{aligned}$$



Note: 注意积分上下限次序

Exercise 11.14: 交换二重积分的积分次序

$$\int_0^{2\pi} \mathrm{d}x\int_0^{\sin x} f(x,y)\mathrm{d}y$$

✎ **Solution**

$$\int_0^{2\pi} \mathrm{d}x \int_0^{\sin x} f(x,y)\mathrm{d}y = \int_0^1 \mathrm{d}y \int_{\arcsin y}^{\pi-\arcsin y} f(x,y)\mathrm{d}x - \int_{-1}^0 \mathrm{d}y \int_{\pi-\arcsin y}^{2\pi+\arcsin y} f(x,y)\mathrm{d}x$$

Exercise 11.15: 交换二重积分的积分次序

$$I = \int_0^1 \mathrm{d}x \int_0^1 f(x,y)\mathrm{d}y$$

✎ **Solution**

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \mathrm{d}\theta \int_0^{\sec\theta} f(\rho\cos\theta,\rho\sin\theta)\rho\mathrm{d}\rho + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{\csc\theta} f(\rho\cos\theta,\rho\sin\theta)\rho\mathrm{d}\rho \\ &= \int_0^1 \rho\mathrm{d}\rho \int_0^{\frac{\pi}{2}} f(\rho\cos\theta,\rho\sin\theta)\mathrm{d}\theta + \int_1^{\sqrt{2}} \rho\mathrm{d}\rho \int_{\arccos\frac{1}{\rho}}^{\arcsin\frac{1}{\rho}} f(\rho\cos\theta,\rho\sin\theta)\mathrm{d}\theta \end{aligned}$$

11.2.2 二重积分的换元法

Theorem 11.2 二重积分的换元公式

设 $f(x,y)$ 在 xOy 平面上的闭区域 D 上连续, 若变换

$$T:\ x = x(u,v),\ y = y(u,v)$$

将 uOv 平面上的闭区域 D' 变为 xOy 平面上的 D, 且满足

(1) $x(u,v), y(u,v)$ 在 D' 上具有一阶连续偏导数

(2) 在 D' 上雅可比式

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \neq 0$$

(3) 变换 $T:\ D' \rightrightarrows D$ 是一对一的

则有

$$\iint\limits_D f(x,y)\,\mathrm{d}x\mathrm{d}y = \iint\limits_{D'} f[x(u,v), y(u,v)]\big|J\big|\,\mathrm{d}u\mathrm{d}v$$

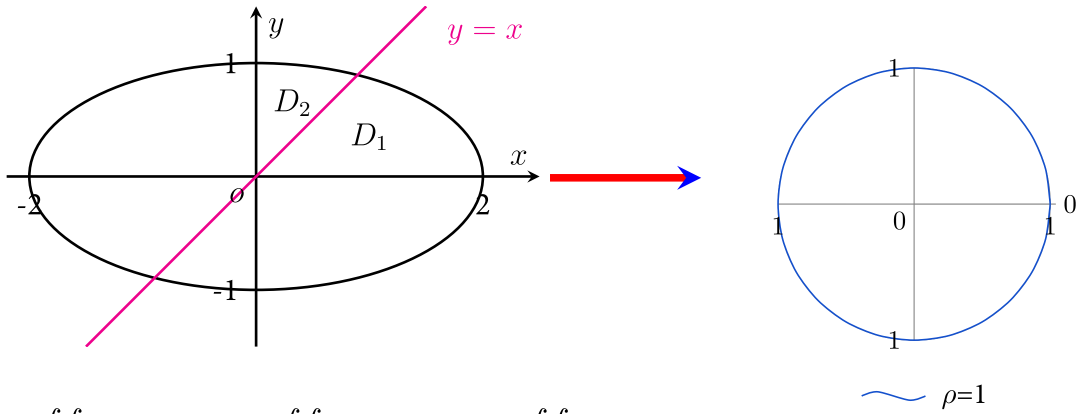
Exercise 11.16: 设平面区域 $D = \left\{(x,y)\middle|\dfrac{x^2}{4} + y^2 \leqslant 1, x \geqslant 0, y \geqslant 0\right\}$,

计算二重积分 $\iint_D |x-y|\,\mathrm{d}\sigma$

Solution 作代换

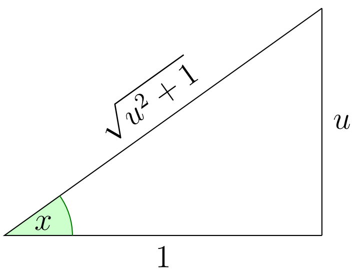
$$\begin{cases} x = 2\rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Longrightarrow J(\rho,\theta) = \frac{\partial(x,y)}{\partial(\rho,\theta)} = \begin{vmatrix} 2\cos\theta & -2\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = 2\rho$$

$x = y \Longrightarrow \theta = \arctan 2$



$$
\begin{aligned}
\iint_D |x-y|\,\mathrm{d}\sigma &= \iint_{D_1}(x-y)\,\mathrm{d}\sigma + \iint_{D_2}(y-x)\,\mathrm{d}\sigma \\
&= \int_0^{\arctan 2}\mathrm{d}\theta\int_0^1 2\rho(2\rho\cos\theta-\rho\sin\theta)\,\mathrm{d}\rho \\
&\quad + \int_{\arctan 2}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 2\rho(\rho\sin\theta-2\rho\cos\theta)\,\mathrm{d}\rho \\
&= \int_0^{\arctan 2}\frac{2}{3}(2\cos\theta-\sin\theta)\mathrm{d}\theta + \int_{\arctan 2}^{\frac{\pi}{2}}\frac{2}{3}(\sin\theta-2\cos\theta)\mathrm{d}\theta \\
&= \frac{2}{3}(2\sin\theta+\cos\theta)\Big|_0^{\arctan 2} + \frac{2}{3}(-\cos\theta-2\sin\theta)\Big|_{\arctan 2}^{\frac{\pi}{2}} \\
&= \frac{2}{3}(4\sin\arctan 2+2\cos\arctan 2-3) \\
&= \frac{4}{3}\sqrt{5}-2
\end{aligned}
$$

其中



$$\tan x = u \Longrightarrow x = \arctan u$$

$$
\begin{cases}
\sin x = \sin\arctan u = \dfrac{u}{\sqrt{u+1}} \\
\cos x = \cos\arctan u = \dfrac{1}{\sqrt{u+1}}
\end{cases}
$$

Exercise 11.17: **计算积分** $\displaystyle\iint\limits_{\mathrm{D}} \frac{(x+y)\ln\left(1+\frac{y}{x}\right)}{\sqrt{1-x-y}}\,\mathrm{d}x\,\mathrm{d}y$ **其中区域** D **是由直线** $x+y=1$ **与两坐标轴所围成的三角形区域**

✎ **Solution 作代换**

$$
\begin{cases}
u = x+y \\
v = \dfrac{y}{x}
\end{cases}
\Longrightarrow
\begin{cases}
x = \dfrac{u}{1+v} \\
y = \dfrac{uv}{1+v}
\end{cases}
$$

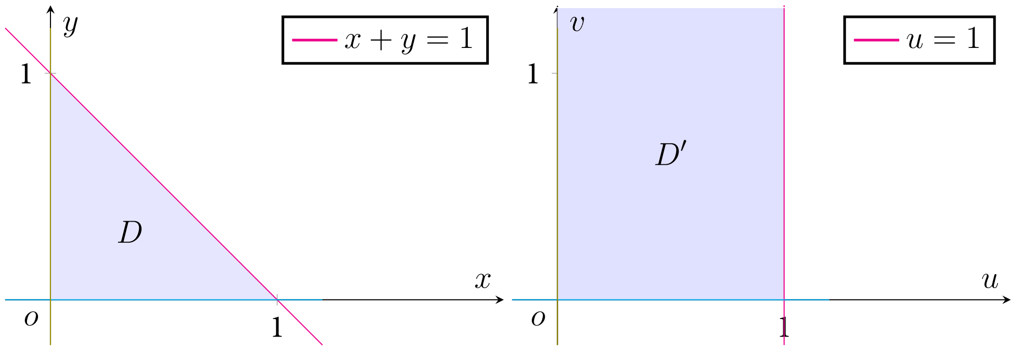
其雅可比行列式为

$$
J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{1}{1+v} & -\dfrac{u}{(1+v)^2} \\ \dfrac{v}{1+v} & \dfrac{u}{(1+v)^2} \end{vmatrix} = \frac{u}{(1+v)^2}
$$

区域 D 变为 D' , 即

$$\begin{cases} x=0 \Longrightarrow \dfrac{u}{1+v}=0 \Longrightarrow u=0 \\ y=0 \Longrightarrow \dfrac{uv}{1+v}=0 \Longrightarrow uv=0 \\ x+y=1 \Longrightarrow \dfrac{u}{1+v}+\dfrac{uv}{1+v}=1 \Leftrightarrow u=1 \end{cases}$$

区域 D 与区域 D' 如图所示



那么有

$$\begin{aligned} I &= \iint\limits_{D'} \frac{u\ln(1+v)}{\sqrt{1-u}} \cdot \frac{|u|}{(1+v)^2}\,\mathrm{d}u\mathrm{d}v \\ &= \int_0^{+\infty} \mathrm{d}v \int_0^1 \frac{u^2\ln(1+v)}{(1+v)^2\sqrt{1-u}}\mathrm{d}u \\ &= \int_0^{+\infty} \frac{\ln(1+v)}{(1+v)^2}\mathrm{d}v \int_0^1 \frac{u^2}{\sqrt{1-u}}\mathrm{d}u \end{aligned}$$

其中

$$\begin{aligned} J &= \int_0^{+\infty} \frac{\ln(1+v)}{(1+v)^2}\mathrm{d}v \\ &= \left[-\frac{\ln(1+v)}{1+v}\right]_0^{+\infty} + \int_0^{+\infty} \frac{1}{(1+v)^2}\mathrm{d}v \\ &= 0 - \left[\frac{1}{1+v}\right]_0^{+\infty} = 1 \end{aligned}$$

$$\begin{aligned} K &= \int_0^1 \frac{u^2}{\sqrt{1-u}}\mathrm{d}u \\ &= \mathrm{B}\left(3, \frac{1}{2}\right) \\ &= \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{2\sqrt{\pi}}{\frac{15\sqrt{\pi}}{8}} = \frac{16}{15} \end{aligned}$$

故

$$I = \iint\limits_{D} \frac{(x+y)\ln\left(1+\frac{y}{x}\right)}{\sqrt{1-x-y}}\,\mathrm{d}x\,\mathrm{d}y = 1 \cdot \frac{16}{15} = \frac{16}{15}$$

Exercise 11.18: 计算

$$\iint\limits_{\sqrt{x}+\sqrt{y}\leqslant 1} \sqrt[3]{\sqrt{x}+\sqrt{y}}\,\mathrm{d}x\mathrm{d}y$$

Solution 作变换

$$\begin{cases} x = \rho^4 \cos^4\theta \\ y = \rho^4 \sin^4\theta \end{cases} \implies J = 16\rho^7 \cos^3\theta \sin^3\theta$$

在这变换下,区域 $D = \left\{(x, y) \middle| \sqrt{x} + \sqrt{y} \leqslant 1\right\}$ 对应区域 $D' = \left\{(\rho, \theta) \middle| 0 \leqslant \theta \leqslant \dfrac{\pi}{2}, 0 \leqslant \rho \leqslant 1\right\}$ 因此有

$$\iint\limits_{\sqrt{x}+\sqrt{y}\leqslant 1} \sqrt[3]{\sqrt{x} + \sqrt{y}}\,\mathrm{d}x\mathrm{d}y = 16\int_0^{\frac{\pi}{2}} \cos^3\theta \sin^3\theta \mathrm{d}\theta \int_0^1 \rho^{\frac{23}{3}}\mathrm{d}\rho = \frac{2}{13}$$

Exercise 11.19: 证明

$$\iint\limits_{S} f(ax + by + c)\mathrm{d}x\mathrm{d}y = 2\int_{-1}^{1} \sqrt{1 - u^2} f\left(u\sqrt{a^2 + b^2} + c\right)\mathrm{d}u$$

其中 $\mathrm{S} : x^2 + y^2 \leqslant 1, a^2 + b^2 \neq 0$

Solution 作正交变换:

$$u = \frac{1}{\sqrt{a^2 + b^2}}(ax, by), v = \frac{1}{\sqrt{a^2 + b^2}}(ay, bx)$$

则 $x^2 + y^2 = u^2 + v^2$, 因此 $x^2 + y^2 \leqslant 1$ 变成 $u^2 + v^2 \leqslant 1$ 且

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{a^2 + b^2}\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = 1$$

所以

$$\iint\limits_{S} f(ax + by + c)\mathrm{d}x\mathrm{d}y = \iint\limits_{u^2+v^2\leqslant 1} f\left(\sqrt{a^2 + b^2}u + c\right)\mathrm{d}x\mathrm{d}v$$

而

$$\left\{u^2 + v^2 \leqslant 0\right\} = \left\{(u, v)| - 1 \leqslant u \leqslant 1, -\sqrt{1 - u^2} \leqslant v \leqslant \sqrt{1 - u^2}\right\}$$

所以

$$\begin{aligned} &\iint\limits_{S} f(ax + by + c)\mathrm{d}x\mathrm{d}y \\ &= \int_{-1}^{1}\mathrm{d}u \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} f\left(u\sqrt{a^2 + b^2} + c\right)\mathrm{d}v \\ &= \int_{-1}^{1} f\left(u\sqrt{a^2 + b^2} + c\right)\mathrm{d}u \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \mathrm{d}v \\ &= 2\int_{-1}^{1} \sqrt{1 - u^2} f\left(u\sqrt{a^2 + b^2} + c\right)\mathrm{d}u \end{aligned}$$

Exercise 11.20: 证明

$$I = \iint\limits_{\Sigma} f(ax + by + cz)\mathrm{d}s\mathrm{d}y = 2\pi\int_{-1}^{1} f\left(\sqrt{a^2 + b^2 + c^2}u\right)\mathrm{d}u$$

其中, Σ 为球面单位 $x^2+y^2+z^2=1$

✎ **Solution**

Exercise 11.21: 证明

$$\int_0^{2\pi} dx \int_0^{\pi} \sin y e^{\sin y(\cos x-\sin x)} dy = \sqrt{2}(e^{\sqrt{2}}-e^{-\sqrt{2}})\pi$$

✎ **Solution**

$$\begin{aligned}
I &= \int_0^{2\pi} \mathrm{d}x \int_0^{\pi} \sin y e^{\sin y(\cos x-\sin x)} \,\mathrm{d}y \\
&= \int_0^{2\pi} \mathrm{d}x \int_0^{\pi} \sin y e^{\sqrt{2}\sin y\cos x} \,\mathrm{d}y \\
&= \oint_{|r|=1} e^{\sqrt{2}x} \,\mathrm{d}S \\
&= 2\int_{-1}^{1} e^{\sqrt{2}x} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\sqrt{1-x^2-y^2}} \,\mathrm{d}y \right) \mathrm{d}x \\
&= 2\int_{-1}^{1} e^{\sqrt{2}x} \left(\arctan\left(\frac{y}{\sqrt{1-x^2-y^2}} \right) \Bigg|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \right) \mathrm{d}x \\
&= 2\int_{-1}^{1} e^{\sqrt{2}x} (\pi) \,\mathrm{d}x
\end{aligned}$$

11.3 三重积分

11.3.1 利用直角坐标系计算三重积分

将三重积分化为三次积分

$$\iiint_{\Omega} f(x,y,z)\,\mathrm{d}v = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} \mathrm{d}y \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,\mathrm{d}z$$

先计算一个二重积分、再计算一个定积分

$$\iiint_{\Omega} f(x,y,z)\,\mathrm{d}v = \int_{c_1}^{c_2} \mathrm{d}z \iint_{D_z} f(x,y,z)\,\mathrm{d}x\mathrm{d}y$$

11.3.2 利用柱面坐标计算三重积分

柱面坐标＝极坐标＋竖坐标

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases} \qquad \begin{cases} 0 \leqslant \rho < +\infty \\ 0 \leqslant \theta \leqslant 2\pi \\ -\infty < z < +\infty \end{cases}$$

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}v = \iiint\limits_{\Omega} f(\rho,\theta,z)\rho\,\mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z$$

11.3.3 利用球面坐标计算三重积分

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = z\cos\varphi \end{cases} \qquad \begin{cases} 0 \leqslant r < +\infty \\ 0 \leqslant \varphi \leqslant \pi \\ 0 \leqslant \theta \leqslant 2\pi \end{cases}$$

$$\iiint\limits_{\Omega} f(x,y,z)\,\mathrm{d}v = \iiint\limits_{\Omega} F(r,\varphi,\theta)r^2\sin\varphi\,\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta$$

11.4 重积分的应用

对于平面薄片，面密度 $\rho(x,y)$ 连续，D 是薄片所占的平面区域，则计算重心 $\overline{x},\overline{y}$ 的公式为

$$\overline{x} = \frac{\iint\limits_{D} x\rho(x,y)\,\mathrm{d}\sigma}{\iint\limits_{D} \rho(x,y)\,\mathrm{d}\sigma}, \qquad \overline{y} = \frac{\iint\limits_{D} y\rho(x,y)\,\mathrm{d}\sigma}{\iint\limits_{D} \rho(x,y)\,\mathrm{d}\sigma}$$

Exercise 11.22: 计算 $\iint\limits_{D} (x+y)\mathrm{d}x\mathrm{d}y$**，其中** $D:\ x^2+y^2 \leqslant x+y+1$

Solution 区域 D

$$D = \left\{ \{x,y\} \middle| \left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 \leqslant \frac{3}{2} \right\}$$

而

$$\overline{x} = \frac{\iint\limits_{D} x\rho(x,y)\,\mathrm{d}\sigma}{\iint\limits_{D} \rho(x,y)\,\mathrm{d}\sigma} = \frac{\iint\limits_{D} x\mathrm{d}x\mathrm{d}y}{\iint\limits_{D} \mathrm{d}x\mathrm{d}y} = \frac{1}{2}$$

$$\overline{y} = \frac{\iint\limits_{D} y\rho(x,y)\,\mathrm{d}\sigma}{\iint\limits_{D} \rho(x,y)\,\mathrm{d}\sigma} = \frac{\iint\limits_{D} x\mathrm{d}x\mathrm{d}y}{\iint\limits_{D} \mathrm{d}x\mathrm{d}y} = \frac{1}{2}$$

因此

$$\iint\limits_{D} (x+y)\mathrm{d}x\mathrm{d}y = \frac{3}{2}\pi$$

Note:

$$
\text{形心}:(\overline{x},\overline{y}) \quad \iint\limits_D x\mathrm{d}\sigma = \overline{x}\iint\limits_D \mathrm{d}\sigma \qquad \iint\limits_D y\mathrm{d}\sigma = \overline{y}\iint\limits_D \mathrm{d}\sigma
$$

11.5 含参变量的积分

Exercise 11.23: Evaluate

$$
\int_0^{+\infty} \frac{\sin x}{x}\mathrm{d}x
$$

Solution

$$
\begin{aligned}
\int_0^{+\infty} \frac{\sin x}{x}\mathrm{d}x &= \int_0^{+\infty} \sin x\left(\int_0^{+\infty} e^{-xy}\mathrm{d}y\right)\mathrm{d}x \\
&= \int_0^{+\infty}\left(\int_0^{+\infty} e^{-xy}\sin x\mathrm{d}x\right)\mathrm{d}y \\
&= \int_0^{+\infty}\left[-\frac{y\sin x+\cos x}{e^{xy}(y^2+1)}\right]_0^{+\infty}\mathrm{d}y \\
&= \int_0^{+\infty}\frac{1}{y^2+1}\mathrm{d}y \\
&= \Big[\arctan x\Big]_0^{+\infty} \\
&= \frac{\pi}{2}
\end{aligned}
$$

Note:

$$
\frac{1}{x^p} = \frac{1}{\Gamma(p)}\int_0^{+\infty} t^{p-1}e^{-xt}\mathrm{d}t \quad (x>0)
$$

Exercise 11.24: Evaluate

$$
\int_0^{+\infty} \cos x^2\,\mathrm{d}x
$$

Solution

$$\begin{aligned}
\int_0^{+\infty}\cos x^2\mathrm{d}x &\xlongequal[\mathrm{d}x=\frac{1}{2}u^{-\frac{1}{2}}\mathrm{d}u]{u=x^2}\int_0^{+\infty}\frac{\cos u}{2\sqrt{u}}\mathrm{d}u\\
&=\int_0^{+\infty}\frac{1}{2\sqrt{u}}\mathrm{d}(\sin u)\\
&=\lim_{u\to+\infty}\frac{\sin u}{2\sqrt{u}}-\lim_{u\to0^+}\frac{\sin u}{2\sqrt{u}}+\frac{1}{4}\int_0^{+\infty}\frac{\sin u}{u^{\frac{3}{2}}}\mathrm{d}u\\
&=\frac{1}{4}\int_0^{+\infty}\left(\frac{2}{\sqrt{\pi}}\int_0^{+\infty}\sqrt{v}e^{-uv}\mathrm{d}v\right)\sin u\mathrm{d}u\\
&=\frac{1}{2\sqrt{\pi}}\int_0^{+\infty}\sqrt{v}\left[\int_0^{+\infty}e^{-uv}\sin u\mathrm{d}u\right]\mathrm{d}v\\
&=\frac{1}{2\sqrt{\pi}}\int_0^{+\infty}\sqrt{v}\left[-\frac{\cos u+v\sin u}{e^{uv}(v^2+1)}\right]_0^{+\infty}\mathrm{d}v\\
&=\frac{1}{2\sqrt{\pi}}\int_0^{+\infty}\frac{\sqrt{v}}{1+v^2}\mathrm{d}v\\
&\xlongequal{t=v^2}\frac{1}{4\sqrt{\pi}}\int_0^{+\infty}\frac{t^{-\frac{1}{4}}}{1+t}\mathrm{d}t\\
&=\frac{1}{4\sqrt{\pi}}\mathrm{B}\left(\frac{3}{4},\frac{1}{4}\right)=\frac{1}{4\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}\\
&\xlongequal{\text{余元公式}}\frac{1}{4\sqrt{\pi}}\cdot\frac{\pi}{\sin\frac{\pi}{4}}=\frac{\sqrt{2\pi}}{4}
\end{aligned}$$

Note: *Equation*

$$\frac{1}{x^p}=\frac{1}{\Gamma(p)}\int_0^{+\infty}t^{p-1}e^{-xt}\mathrm{d}t\quad(x>0)$$

Beta function

$$\mathrm{B}(x,y)=\int_0^{+\infty}\frac{t^{x-1}}{(1+t)^{x+y}}\mathrm{d}t\quad(\mathrm{Re}\,x>0,\mathrm{Re}\,y>0)$$

Relationship between gamma function and beta function

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}\quad(\mathrm{Re}\,x>0,\mathrm{Re}\,y>0)$$

Euler's reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}\quad(0<z<1)$$

Exercise 11.25: 计算积分

$$\int_0^1\sin(\log(x))\frac{x^b-x^a}{\log(x)}dx$$

Solution 换元令 $x=e^t$ 则: $dx=e^tdt$ 那么

$$I=\int_0^1\sin(\log(x))\frac{x^b-x^a}{\log(x)}dx=-\int_{-\infty}^0\sin(t)e^t\frac{e^{bt}-e^{at}}{t}dt=-\int_{-\infty}^0\sin(x)e^x\frac{e^{bx}-e^{ax}}{x}dx$$

又因为

$$\frac{e^{bx}-e^{ax}}{x}=\left.\frac{e^{tx}}{x}\right|_a^b=\int_a^b e^{tx}dt$$

所以

$$\begin{aligned}
I&=\int_0^1\sin(\log(x))\frac{x^b-x^a}{\log(x)}dx=-\int_{-\infty}^0\int_a^b\sin(x)e^xe^{tx}dtdx\\
&=-\int_a^b dt\int_{-\infty}^0\sin(x)e^{(t+1)x}dx\\
&=-\int_a^b\left[\frac{1}{t^2+2t+2}e^{(t+1)x}((t+1)\sin x-\cos x)\right]_{-\infty}^0dt\\
&=\int_a^b\frac{1}{t^2+2t+2}dt\\
&=\int_a^b\frac{1}{(t+1)^2+1}dt=\int_a^b\frac{1}{(t+1)^2+1}d(t+1)\\
&=\left[\arctan(t+1)\right]_a^b=\arctan(b+1)-\arctan(a+1)
\end{aligned}$$

✎ **Solution** 换元令 $x=e^t$ 则: $dx=e^tdt$ 那么

$$I=\int_0^1\sin(\log(x))\frac{x^b-x^a}{\log(x)}dx=-\int_{-\infty}^0\sin(t)e^t\frac{e^{bt}-e^{at}}{t}dt=-\int_{-\infty}^0\sin(x)e^x\frac{e^{bx}-e^{ax}}{x}dx$$

因为

$$\frac{\partial I}{\partial b}=-\int_{-\infty}^0\sin(t)e^{(b+1)t}dt=\frac{1}{b^2+2b+2}$$

所以

$$I(a,b)=I(0,b)-I(0,a)=\arctan(b+1)-\arctan(a+1)$$

Exercise 11.26: 计算积分

$$\int_0^{+\infty}\frac{\sin x}{xe^x}\,\mathrm{d}x$$

✎ **Solution**

$$\begin{aligned}
I(\alpha)&=\int_0^{+\infty}\frac{\sin x}{xe^{\alpha x}}\mathrm{d}x\\
I(0)&=\int_0^{+\infty}\frac{\sin x}{xe^{0x}}\mathrm{d}x=\int_0^{+\infty}\frac{\sin x}{x}\mathrm{d}x=\frac{\pi}{2}\\
I'(\alpha)&=-\int_0^{+\infty}\frac{\sin x}{e^{\alpha x}}\mathrm{d}x=\left[\frac{\alpha\sin x+\cos x}{(\alpha^2+1)e^{\alpha x}}\right]_0^{+\infty}=-\frac{1}{\alpha^2+1}\\
I(1)-I(0)&=-\int_0^1\frac{1}{\alpha^2+1}\mathrm{d}\,\alpha=-\arctan 1=-\frac{\pi}{4}\\
\int_0^{+\infty}\frac{\sin x}{xe^x}\mathrm{d}x&=I(1)=-\frac{\pi}{4}+I(0)=-\frac{\pi}{4}+\frac{\pi}{2}=\frac{\pi}{4}
\end{aligned}$$

✎ **Solution 注意到**

$$\sin x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}$$

以及

$$\arctan x=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

故

$$\begin{aligned}
I&=\int_0^{+\infty}\frac{\sin x}{xe^x}\,\mathrm{d}x\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\int_0^{+\infty}x^{2n}e^{-x}\,\mathrm{d}x\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\Gamma(2n+1)\\
&=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}=\arctan 1\\
&=\frac{\pi}{4}
\end{aligned}$$

Exercise 11.27: 计算积分

$$\int_0^{+\infty}\frac{\sqrt{x}\ln x}{1+x^2}\,\mathrm{d}x$$

✎ **Solution(西西) 注意到**

$$\int_0^{+\infty}\frac{x^{-a}}{1+x}\,\mathrm{d}x=\mathrm{B}(1-a,a)=\pi\csc(a\pi)$$

上式两边对 a 求导得:

$$\int_0^{+\infty}\frac{x^{-a}\ln x}{1+x}\,\mathrm{d}x=\pi^2\csc(a\pi)\cot(a\pi)$$

令 $a=\dfrac{1}{4}$. 再换 $x\to x^2$ 即有

$$\int_0^{+\infty}\frac{\sqrt{x}\ln x}{1+x^2}\,\mathrm{d}x=\frac{\sqrt{2}\pi^2}{2}$$

Exercise 11.28: 计算积分

$$\int_0^{\pi}\ln(2+\cos x)\mathrm{d}x$$

✎ **Solution** 令 $I(\alpha)=\int_0^\pi \ln(\alpha+\cos x)\mathrm{d}x,\alpha>1$, 易知 $I(\alpha,x)$ 可导

$$\begin{aligned}I'(\alpha)&=\int_0^\pi \frac{\mathrm{d}x}{\alpha+\cos x}\\&=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\alpha+\cos x}+\int_{\frac{\pi}{2}}^{\pi}\frac{\mathrm{d}x}{\alpha+\cos x}\\&=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\alpha+\cos x}+\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\alpha-\sin x}\\&=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\alpha+\sin x}+\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}x}{\alpha-\sin x}\\&=\int_0^{\frac{\pi}{2}}\frac{2\alpha}{\alpha^2-\sin^2 x}\mathrm{d}x\\&=-\int_0^{\frac{\pi}{2}}\frac{2\alpha d(\cot x)}{(\alpha\cot x)^2+\alpha^2-1}\\&=-\frac{2}{\sqrt{\alpha^2-1}}\arctan\frac{\alpha\cot x}{\sqrt{\alpha^2-1}}\Big|_0^{\frac{\pi}{2}}\\&=\frac{\pi}{\sqrt{\alpha^2-1}}\end{aligned}$$

所以

$$I(\alpha)=\pi\ln(\alpha+\sqrt{\alpha^2-1})+C\Rightarrow I(1)=\pi\ln(1+0)+C=C$$

因为

$$I(1)=\int_0^\pi \ln(1+\cos x)\mathrm{d}x=\pi\ln 2+4\int_0^{\frac{\pi}{2}}\ln\cos t\mathrm{d}t=-\pi\ln 2$$

所以

$$I(\alpha)=\pi\ln\frac{\alpha+\sqrt{\alpha^2-1}}{2}$$

令 $\alpha=2$, 可得

$$\therefore\qquad \int_0^\pi \ln(2+\cos x)\mathrm{d}x=\pi\ln\frac{\sqrt{3}+2}{2}$$

Exercise 11.29: 计算积分

$$I=\int_0^1\frac{1-x}{\ln x}\left(x+x^2+x^{2^2}+x^{2^3}+\cdots\right)\mathrm{d}x$$

✎ **Solution** 考虑含参变量 a 的积分所确定的函数

$$I(a)=\int_0^1\frac{x^a-1}{\ln x}\mathrm{d}x$$

易得 $I(0)=0$ 以及

$$\frac{\partial I(a)}{\partial a}=\int_0^1 x^a\,\mathrm{d}x=\frac{1}{a+1}\tag{11.1}$$

式 (11.1) 在 $[0,1]$ 对 a 积分得

$$I(a)-I(0)=\int_0^1\frac{1}{a+1}\,\mathrm{d}x\Longrightarrow I(a)=\ln(a+1)$$

因此有

$$\int_0^1\frac{1-x}{\ln x}x^k\,\mathrm{d}x=\int_0^1\frac{(x^k-1)-(x^{k+1}-1)}{\ln x}\,\mathrm{d}x=\ln\frac{k+1}{k+2}$$

故

$$I=\int_0^1\frac{1-x}{\ln x}\sum_{k=0}^{\infty}x^{2^k}\,\mathrm{d}x=\ln\prod_{k=0}^{\infty}\frac{2^k+1}{2^k+2}=\ln\left(\frac{1}{2}\prod_{k=0}^{\infty}\frac{2^k+1}{2^{k-1}+1}\right)=-\ln 3$$

Exercise 11.30: 计算积分: $\displaystyle\int_0^{+\infty}\frac{\cos ax-\cos bx}{x}\mathrm{d}x$, 其中 $a,b>0$

Solution 方法 1:

$$\begin{aligned}\int_0^{+\infty}\frac{\cos ax-\cos bx}{x}dx&=\lim_{\substack{\varepsilon\to0^+\\\delta\to+\infty}}\int_\varepsilon^\delta\frac{\cos ax-\cos bx}{x}dx\\&=\lim_{\substack{\varepsilon\to0^+\\\delta\to+\infty}}\left[\int_\varepsilon^\delta\frac{\cos ax}{x}dx-\int_\varepsilon^\delta\frac{\cos bx}{x}dx\right]\end{aligned}$$

分别作变量代换 $ax=u,bx=u$, 得

$$\begin{aligned}&=\lim_{\substack{\varepsilon\to0^+\\\delta\to+\infty}}\left[\int_{a\varepsilon}^{a\delta}\frac{\cos x}{x}dx-\int_{b\varepsilon}^{b\delta}\frac{\cos x}{x}dx\right]=\lim_{\substack{\varepsilon\to0^+\\\delta\to+\infty}}\left[\int_{a\varepsilon}^{b\varepsilon}\frac{\cos x}{x}dx-\int_{a\delta}^{b\delta}\frac{\cos x}{x}dx\right]\\&=\lim_{\varepsilon\to0^+}\int_{a\varepsilon}^{b\varepsilon}\frac{\cos x}{x}dx-\lim_{\delta\to0^+}\int_{a\delta}^{b\delta}\frac{\cos x}{x}dx\end{aligned}$$

因为 $\displaystyle\int_1^{+\infty}\frac{\cos x}{x}dx$ 收敛 (可由 Dirichlet 判别法得到)

所以

$$\lim_{\delta\to0^+}\int_{a\delta}^{b\delta}\frac{\cos x}{x}dx=\lim_{\delta\to0^+}\left[\int_1^{b\delta}\frac{\cos x}{x}dx-\int_1^{a\delta}\frac{\cos x}{x}dx\right]=0$$

对于前面那个极限

$$\lim_{\varepsilon\to0^+}\int_{a\varepsilon}^{b\varepsilon}\frac{\cos x}{x}dx=\lim_{\varepsilon\to0^+}\left[\int_{a\varepsilon}^{b\varepsilon}\frac{\cos x-1}{x}dx+\int_{a\varepsilon}^{b\varepsilon}\frac{1}{x}dx\right]=\ln\frac{b}{a}+\lim_{\varepsilon\to0^+}\int_{a\varepsilon}^{b\varepsilon}\frac{\cos x-1}{x}dx$$

由于 $\displaystyle\int_0^1\frac{\cos x-1}{x}dx$ 收敛, 同理有 $\displaystyle\lim_{\varepsilon\to0^+}\int_{a\varepsilon}^{b\varepsilon}\frac{\cos x-1}{x}dx=0$

因此

$$\int_0^{+\infty}\frac{\cos ax-\cos bx}{x}dx=\ln\frac{b}{a}$$

注: 这个方法可以计算此题的一般形式 $\displaystyle\int_0^{+\infty}\frac{f(ax)-f(bx)}{x}dx$ 称为 Froullani 积分, 其中 $f(x)$ 需要满足适当条件

✎ **Solution** **方法 2** 记 $F(t)=\int_0^{+\infty}\frac{e^{-tx}(\cos ax-\cos bx)}{x}dx$, 则易验证 $F(x)$ 在 $[0,+\infty]$ 上一致收敛

而

$$F'(t)=-\int_0^{+\infty}e^{-tx}(\cos ax-\cos bx)dx=\frac{t}{b^2+t^2}-\frac{t}{a^2+t^2}$$

$\Rightarrow F(t)=\frac{1}{2}\ln\left(\frac{b^2+t^2}{a^2+t^2}\right)+C$, 其中 C 为积分常数

留意到 $F(+\infty)=0$

所以

$$0=\frac{1}{2}\lim_{t\to+\infty}\ln\left(\frac{b^2+t^2}{a^2+t^2}\right)+C\Rightarrow C=0$$

所以

$$F(t)=\frac{1}{2}\ln\left(\frac{b^2+t^2}{a^2+t^2}\right)$$

令 $t\to0^+$ 即有 $\int_0^{+\infty}\frac{\cos ax-\cos bx}{x}dx=\ln\frac{b}{a}$

Exercise 11.31: 计算积分:

$$\int_0^1\frac{\arctan\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}\mathrm{d}x$$

✎ **Solution**

$$\begin{aligned}
\frac{\pi^2}{16}&=\int_0^1\int_0^1\frac{\mathrm{d}x\mathrm{d}y}{(1+x^2)(1+y^2)}\\
&=\int_0^1\int_0^1\left(\frac{1}{(1+x^2)(2+x^2+y^2)}+\frac{1}{(1+y^2)(2+x^2+y^2)}\right)\mathrm{d}x\mathrm{d}y\\
&=2\int_0^1\int_0^1\frac{1}{(1+x^2)(2+x^2+y^2)}\mathrm{d}y\mathrm{d}x\\
&=2\int_0^1\frac{1}{(1+x^2)\sqrt{2+x^2}}\arctan\frac{1}{\sqrt{2+x^2}}\mathrm{d}x\\
&=2\int_0^1\left(\frac{\pi}{2(1+x^2)\sqrt{2+x^2}}-\frac{\arctan\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}\right)\mathrm{d}x\\
&=\frac{\pi^2}{6}-2\int_0^1\frac{\arctan\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}\mathrm{d}x
\end{aligned}$$

$$\Rightarrow\int_0^1\frac{\arctan\sqrt{2+x^2}}{(1+x^2)\sqrt{2+x^2}}dx=\frac{5}{96}\pi^2$$

第 12 章 曲线积分与曲面积分

12.1 对弧长的曲线积分

$$\int_L f(x,y)\,\mathrm{d}s=\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i)\Delta s_i$$

其中 $f(x,y)$ 叫做被积函数, L 叫做积分弧段.

函数 $f(x,y)$ 在闭曲线 L 上对弧长的曲线积分记为 $\oint_L f(x,y)\,\mathrm{d}s$

函数 $f(x,y,z)$ 在空间曲线弧 Γ 上对弧长的曲线积分为

$$\int_\Gamma f(x,y,z)\,\mathrm{d}s=\lim_{\lambda\to 0}\sum_{i=1}^{n} f(\xi_i,\eta_i,\zeta_i)\Delta s_i$$

Theorem 12.1 几何意义

在三维空间中画出 xOy 平面内曲线 L 为准线, 母线平行于 z 轴的柱面, $\int_L f(x,y)\,\mathrm{d}s$ 表示柱面上以 L 为底以 $f(x,y)$ 为的部分柱面面积的代数和，对应 $f(x,y)\geqslant 0$ 的部分面积为正，对应 $f(x,y)\leqslant 0$ 的部分面积为负.

特别的 $\int_L \mathrm{d}s=s$. 即被积函数为 1 时，对弧长的曲线积分等于积分曲线 L 的弧长.

12.1.1 计算

第 13 章 无穷级数

13.1 常数项级数

Note:

$$2\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^m}=m\zeta(m+1)-\sum_{k=1}^{m-2}\zeta(m-k)\zeta(k+1)$$

Note:

$$\ln(\sin x)=-\ln 2-\sum_{k=1}^{\infty}\frac{\cos(2kx)}{k}$$

13.1.1 反三角函数

Note:

$$\begin{aligned}
&\arctan\frac{1}{n^2+n+1}=\arctan(n+1)-\arctan(n)\\
&\arctan\frac{1}{2n^2}=\arctan\frac{1}{2n-1}-\arctan\frac{1}{2n+1}\\
&\arctan\frac{2}{n^2}=\arctan\frac{1}{n-1}-\arctan\frac{1}{n+1}\\
&\arctan\frac{2n}{n^4+n^2+2}=\arctan(n^2+n+1)-\arctan(n^2-n+1)
\end{aligned}$$

Theorem 13.1 比较审敛法的极限形式

设 $\sum\limits_{n=1}^{\infty}u_n$ 和 $\sum\limits_{n=1}^{\infty}v_n$ 都是正项级数,

1. 如果 $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l\,(0\leqslant l<+\infty)$, 且级数 $\sum\limits_{n=1}^{\infty}v_n$ 收敛, 那么级数 $\sum\limits_{n=1}^{\infty}u_n$ 收敛
2. 如果 $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=l>0$ 或 $\lim\limits_{n\to\infty}\frac{u_n}{v_n}=+\infty$, 且级数 $\sum\limits_{n=1}^{\infty}v_n$ 发散, 那么级数 $\sum\limits_{n=1}^{\infty}u_n$ 发散

Exercise 13.1: 证明

$$\sum_{n=1}^{\infty}\frac{(n-1)!}{n(x+1)(x+2)\cdots(x+n)}=\sum_{k=1}^{\infty}\frac{1}{(x+k)^2}$$

Proof: 我们知道 Gamma 函数有

$$\Gamma(x+1)=x\Gamma(x)$$

$$\Longrightarrow \Gamma(x+n+1)=(x+n)(x+n-1)\cdots(x+1)\Gamma(x+1)$$

这样

$$\frac{(n-1)!}{n(x+1)(x+2)\cdots(x+n)}=\frac{\Gamma(x+1)\Gamma(n)}{n\Gamma(x+n+1)}=\frac{B(x+1,n)}{n}$$

于是

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{B(x+1,n)}{n}&=\sum_{k=1}^{\infty}\frac{1}{n}\int_0^1 t^{n-1}(1-t)^x dt\\
&=\int_0^1\left(\sum_{n=1}^{\infty}\frac{t^{n-1}}{n}\right)(1-t)^x dt\\
&=\int_0^1\left[-\frac{\ln(1-t)}{t}\right](1-t)^x dt\\
&\xlongequal{z=1-t}\int_0^1\left[-\frac{\ln z}{1-z}\right]z^x dz\\
&=\int_0^1(-1)\sum_{k=1}^{\infty}z^{x+k-1}\ln z dz\\
&=\sum_{k=1}^{\infty}(-1)\int_0^1 z^{x+k-1}\ln z dz\\
&\xlongequal{z=e^{-u}}\sum_{k=1}^{\infty}\int_0^{\infty}ue^{-u(x+k)}du\\
&\xlongequal{y=u(x+k)}\sum_{k=1}^{\infty}\int_0^{\infty}\frac{1}{(x+k)^2}ye^{-y}dy\\
&=\sum_{k=1}^{\infty}\frac{1}{(x+k)^2}
\end{aligned}$$

□

13.1.2 调和级数

Note:

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=\ln n+\gamma+\frac{1}{2n}-\sum_{k=1}^{p}\frac{B_{2k}}{2kp^{2k}}+R(n,p)$$

$$\gamma=\sum_{k=1}^{n-1}\frac{1}{k}-\ln n+\frac{1}{2n}+\frac{1}{12n^2}+\cdots+\frac{B_{2r}}{2r}\frac{1}{n^{2r}}+\frac{B_{2r+2}}{2(r+1)}\frac{\theta}{n^{2r+2}},\theta\in(0,1)$$

13.2 幂级数

Theorem 13.2

设 $\sum_{n=0}^{\infty} a_n x^n$，$\sum_{n=0}^{\infty} b_n x^n$ 的收敛半径各为 R_a，R_b 则对 $|x| < R = \min\{R_a, R_b\}$ 有

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n$$

Theorem 13.3 欧拉 (Euler) 公式

$$e^{xi} = \cos x + i \sin x \Longleftrightarrow \begin{cases} \cos x = \dfrac{e^{xi} + e^{-xi}}{2} \\ \sin x = \dfrac{e^{xi} - e^{-xi}}{2i} \end{cases}$$

13.3 傅里叶级数

Definition 13.1 三角级数

形如

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi t}{l} + b_n \sin\frac{n\pi t}{l}\right) \tag{13.1}$$

的级数叫三角级数，其中 a_0,a_n,$b_n$$(n = 1, 2, 3, \dots)$ 都是常数

令 $\frac{\pi t}{l} = x$,(13.1) 式成为

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{13.2}$$

这就把以周期为 $2l$ 的三角级数转换成以 2π 为周期的三角级数

Theorem 13.4

组成三角函数系

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots, \cos nx, \sin nx, \cdots \tag{13.3}$$

在区间 $[-\pi, \pi]$ 上正交, 即在三角函数系 (13.3) 中任何不同的两个函数的乘积在区间 $[-\pi, \pi]$ 上的积分等于 0, 即

$$\int_{-\pi}^{\pi} \cos nx \mathrm{d}x = 0 \quad (n = 1, 2, 3, \cdots)$$
$$\int_{-\pi}^{\pi} \sin nx \mathrm{d}x = 0 \quad (n = 1, 2, 3, \cdots)$$
$$\int_{-\pi}^{\pi} \sin kx \cos nx \mathrm{d}x = 0 \quad (k, n = 1, 2, 3, \cdots)$$
$$\int_{-\pi}^{\pi} \cos kx \cos nx \mathrm{d}x = 0 \quad (k, n = 1, 2, 3, \cdots, k \neq n)$$
$$\int_{-\pi}^{\pi} \sin kx \sin nx \mathrm{d}x = 0 \quad (k, n = 1, 2, 3, \cdots, k \neq n)$$

Theorem 13.5

设设 $f(x)$ 是周期为 2π 的周期函数, 且能展开成三角级数

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \tag{13.4}$$

右端级数可逐项积分, 则有

$$\begin{cases} a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos nx \mathrm{d}x \quad (n = 0, 1, 2, 3, \cdots) \\ b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx \mathrm{d}x \quad (n = 1, 2, 3, \cdots) \end{cases} \tag{13.5}$$

如果公式 (13.5) 中的积分都存在, 这时他们定出的系数 a_0,a_1,b_1,$\cdots$ 叫做函数 $f(x)$ 的傅里叶 *(Fourier)* 系数, 将这些系数带入到 (13.4) 式的右端, 所得到的三角级数

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{13.6}$$

叫做函数 $f(x)$ 的傅里叶级数

Theorem 13.6 收敛定理, 狄利克雷 (Dirichlet) 充分条件

设 $f(x)$ 是周期为 2π 的周期函数, 如果它满足:

1. 在一个周期内连续或者只有有限个第一类间断点,
2. 在一个周期内至多只有有限个极值点.

那么 $f(x)$ 的傅里叶级数收敛, 并且

当 x 是 $f(x)$ 的连续点时, 级数收敛于 $f(x)$

当 x 是 $f(x)$ 的间断点时, 级数收敛于 $\frac{1}{2}[f(x^-)+f(x^+)]$.

♣

Definition 13.2

对周期为 2π 的奇函数 $f(x)$, 其傅里叶级数为正弦级数, 它的傅里叶系数为

$$\begin{cases} a_n = 0 & (n=0,1,2,3,\cdots) \\ b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin nx\mathrm{d}x & (n=1,2,3,\cdots) \end{cases} \tag{13.7}$$

即知奇函数的傅里叶级数只是含有正弦项的正弦级数

$$\sum_{n=1}^{\infty} b_n \sin nx \tag{13.8}$$

♡

对周期为 2π 的偶函数 $f(x)$, 其傅里叶级数为余弦级数, 它的傅里叶系数为

$$\begin{cases} a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx\mathrm{d}x & (n=0,1,2,3,\cdots) \\ b_n = 0 & (n=1,2,3,\cdots) \end{cases} \tag{13.9}$$

即知偶函数的傅里叶级数是只含有常数项和余弦项的余弦级数

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{13.10}$$

Theorem 13.7

设周期为 $2l$ 的周期函数 $f(x)$ 满足收敛定理的条件，则它的傅里叶级数展开式为

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right)\quad(x\in C)\tag{13.11}$$

其中

$$\begin{cases}a_n=\dfrac{1}{l}\displaystyle\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\mathrm{d}x\quad(n=0,1,2,3,\cdots)\\ b_n=\dfrac{1}{l}\displaystyle\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\mathrm{d}x\quad(n=1,2,3,\cdots)\end{cases}\tag{13.12}$$

$$C=\left\{x\middle|f(x)=\frac{1}{2}[f(x^-)+f(x^+)]\right\}$$

当 $f(x)$ 是奇函数时

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{l}\quad(x\in C)\tag{13.13}$$

其中

$$b_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\mathrm{d}x\quad(n=1,2,3,\cdots)\tag{13.14}$$

当 $f(x)$ 是偶函数时

$$f(x)=\frac{a_0}{2}\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}\quad(x\in C)\tag{13.15}$$

其中

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x\quad(n=0,1,2,3,\cdots)\tag{13.16}$$

Theorem 13.8 狄利克雷 (Dirichlet) 收敛定理

设 $f(x)$ 是以 $2l$ 为周期的可积函数, 如果在 $[-l,l]$ 上 $f(x)$ 满足:

1. 连续或只有有限个第一类间断点;
2. 只有有限个极值点;

则 $f(x)$ 的傅里叶级数处处收敛, 记其和函数为 $S(x)$, 则

$$S(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right) \tag{13.17}$$

且 $S(x)=\begin{cases} f(x) & x \text{ 为连续点} \\ \dfrac{f(x-0)+f(x+0)}{2} & x \text{ 为第一类间断点} \\ \dfrac{f(-l+0)+f(l-0)}{2} & x \text{ 为端点} \end{cases}$

Theorem 13.9 Euler-Fourier 公式

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx dx,\ n=0,1,2,3,\cdots. \tag{13.18}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx dx,\ n=1,2,3,\cdots. \tag{13.19}$$

上面两式称为 *Euler-Fourier* 公式

设周期为 π 的函数 $f(x)$ 在 $[-\pi,\pi]$ 上可积或绝对可积 $f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ 等式右端的三角级数称为 $f(x)$ 的 *Fourier* 级数，相应的 a_n 和 b_n 称为 $f(x)$ 的 *Fourier* 系数

Theorem 13.10 Parseval 等式

设 $f(x)$ 是 $[-\pi,\pi]$ 上的可积和平方可积函数，且有 $f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$

则

$$\frac{a_n^2}{2}=\sum_{n=1}^{\infty}\left(a_n^2+b_n^2\right)=\frac{1}{\pi}\int_{-\pi}^{\pi}f^2(x)dx$$

13.4 级数求和

Exercise 13.2: 求

$$\sum_{n=0}^{+\infty}\frac{m!n!}{(m+n)!}$$

✎ **Solution** 注意到

$$\frac{1}{\binom{m+n}{m}}=m\int_0^1(1-x)^n x^{m-1}\,\mathrm{d}x$$

那么有

$$\sum_{n=0}^{+\infty}\frac{m!n!}{(m+n)!}=m\int_0^1\sum_{n=0}^{+\infty}(1-x)^n x^{m-1}\,\mathrm{d}x=m\int_0^1 x^{m-2}\,\mathrm{d}x=\frac{m}{m-1}$$

第 14 章 综合题

14.1 积分级数极限

Exercise 14.1: 求极限

$$\lim_{n\to\infty}\frac{1}{(2n-1)^{2011}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3 x\cos^2 x\mathrm{d}x$$

Solution: 根据推广的积分第一中值定理，对每个正整数 $n\ \exists\ \theta_n\in(0,1)$ 使得

$$\int_{2n\pi}^{(2n+1)\pi}x^{2010}\sin^3 x\cos^2 x\mathrm{d}x=((2n+\theta_n)\pi)^{2010}\int_{2n\pi}^{(2n+1)\pi}\sin^3 x\cos^2 x\mathrm{d}x$$

由此得

$$\begin{aligned}&\int_{2n\pi}^{(2n+1)\pi}x^{2010}\sin^3 x\cos^2 x\mathrm{d}x\\=&\left((2n\pi)^{2010}+o(n^{2010})\right)\int_{2n\pi}^{2n\pi+\pi}\sin^3 x\cos^2 x\mathrm{d}x\\=&\left((2n\pi)^{2010}+o(n^{2010})\right)\left(\frac{\cos 5x}{80}-\frac{\cos 3x}{48}-\frac{\cos x}{8}\right)\Bigg|_{2n\pi}^{(2n+1)\pi}\\=&\frac{4}{15}((2n\pi)^{2010}+o(n^{2010}))\ \ (n\to\infty)\end{aligned}$$

另外

$$(2n+1)^{2011}-(2n-1)^{2011}=4022(2n)^{2010}+o(n^{2010})\ \ (n\to\infty)$$

根据 Stolz 定理

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{(2n-1)^{2011}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3 x\cos^2 x\mathrm{d}x\\=&\lim_{n\to\infty}\frac{\displaystyle\int_{2n\pi}^{(2n+1)\pi}x^{2010}\sin^3 x\cos^2 x\mathrm{d}x}{(2n+1)^{2011}-(2n-1)^{2011}}\\=&\frac{2}{30165}\lim_{n\to\infty}\frac{(2n\pi)^{2010}+o(n^{2010})}{(2n)^{2010}+o(n^{2010})}\\=&\frac{2\pi^{2010}}{30165}\end{aligned}$$

此题的更一般结果为

$$\lim_{n\to\infty}\frac{1}{(2n-1)^{p+1}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^p\sin^3 x\cos^2 x\mathrm{d}x=\frac{2\pi^p}{15(p+1)}(p>0)$$

□

Exercise 14.2: 计算极限

$$\lim_{n\to\infty}\frac{1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}}{e^n}$$

Solution 解法 1

$$e^n=1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}+\frac{1}{n!}\int_0^n e^x(n-x)^n\mathrm{d}x$$

原命题等价于

$$\lim_{n\to\infty}\frac{e^{-n}}{n!}\int_0^n e^x(n-x)^n\mathrm{d}x=\frac{1}{2}\quad \text{而} n!=\sqrt{2n\pi}(\frac{n}{e})^n e^{\frac{\theta}{12n}},\theta\in(0,1)$$

$$\Leftrightarrow \lim_{n\to\infty}\sqrt{n}\int_0^1[e^x(1-x)]^n\mathrm{d}x=\sqrt{\frac{\pi}{2}}$$

注意到 $e^{-\frac{x^2}{2}}\geq(1-x)e^x(x\geq 0)$

$$\therefore\qquad \overline{\lim_{n\to\infty}}\sqrt{n}\int_0^1[e^x(1-x)]^n\mathrm{d}x\leq\overline{\lim_{n\to\infty}}\int_0^1\sqrt{n}e^{-\frac{nx^2}{2}}\mathrm{d}x=\sqrt{\frac{\pi}{2}}$$

考虑

$$f(x)=(1-x)e^x-e^{-\frac{ax^2}{2}}(x\geq 0,a\geq 1),f'(x)=xe^x(ae^{-\frac{ax^2}{2}-x}-1)$$

$\because\quad \lim\limits_{x\to 0^+}(ae^{-\frac{ax^2}{2}-x}-1)=a-1>0$，故存在 $x_a\in(0,1)$，使得 $ae^{-\frac{ax^2}{2}-x}-1>0$

$$\begin{aligned}(1-x)e^x\geq e^{-\frac{ax^2}{2}}(x\in[0,x_a])\Rightarrow \underline{\lim_{n\to\infty}}\sqrt{n}\int_0^1[e^x(1-x)]^n\mathrm{d}x\\ \geq\underline{\lim_{n\to\infty}}\int_0^{x_a}\sqrt{n}e^{-\frac{nax^2}{2}}\mathrm{d}x\\ =\sqrt{\frac{\pi}{2a}}\end{aligned}$$

因为 a 是任意的，所以

$$\underline{\lim_{n\to\infty}}\sqrt{n}\int_0^1[e^x(1-x)]^n\mathrm{d}x\geq\sqrt{\frac{\pi}{2}}$$

综上得

$$\lim_{n\to\infty}\sqrt{n}\int_0^1[e^x(1-x)]^n\mathrm{d}x=\sqrt{\frac{\pi}{2}}$$

Solution 解法 2

$$\because\quad (1+n+\frac{n^2}{n!}+\cdots+\frac{n^n}{n!})=e^n-\int_0^n e^t\frac{(n-t)^n}{n!}dt\xlongequal{n-t=x}e^n-e^n\int_0^n\frac{x^ne^{-x}}{n!}\mathrm{d}x$$

两边除以 e^n

$$\therefore\ a_n=1-\int_0^n\frac{x^ne^{-x}}{n!}\mathrm{d}x$$

下面求 $\lim\limits_{n\to\infty}\int_0^n \frac{x^n e^{-x}}{n!}\mathrm{d}x$

令 $\eta = n^{-\frac{1}{2}+z}, 0<\varepsilon<\frac{1}{6}$

$$
\begin{aligned}
\because \int_0^n \frac{x^n e^{-x}}{n!}\mathrm{d}x &\xlongequal{x=n(z+1)} \int_{-1}^0 \frac{e^{-n(z+1)}(z+1)^n n^{n+1}}{n!}dz \\
&= n\frac{n^n}{n!e^n}\int_{-1}^0 e^{-nz}(z+1)^n dz \\
&= \sqrt{\frac{n}{2\pi}}[1+o(\frac{1}{n})]\int_{-1}^0 [e^{-z}(z+1)]^n dz \\
&= \sqrt{\frac{n}{2\pi}}[1+o(\frac{1}{n})][\int_{-1}^{\eta} [e^{-z}(z+1)]^n dz + \int_{-\eta}^0 [e^{-z}(1+z)]^n dz] \\
&= I_1 + I_2
\end{aligned}
$$

设 $f(z)=e^{-z}(1+z), (z\le 0), f'(z) = -e^{-z}\cdot z \ge 0$

$$
\therefore \int_{-1}^{\eta}[e^{-z}(1+z)]^n dz < (1-\eta)[e^{-\eta}(1-\eta)]^n < [e^{-\eta}(1-\eta)]^n
$$

$$
\therefore\ I_1 = o(\sqrt{n}e^{-\frac{1}{2}n^{2z}})
$$

下面考虑 I_2

$$
\because\ e^{-z}(1+z) = e^{-z+\ln(z+1)} = e^{-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4(1+\theta(z))^4}} \quad (0<\theta(z)<1)
$$

$$
\begin{aligned}
I_2 &= \sqrt{\frac{n}{2\pi}}[1+o(n^{-1+4z})]\int_{-\eta}^0 e^{-n(\frac{x^2}{2}-\frac{z^2}{3})}dz \\
&= \sqrt{\frac{n}{2\pi}}[1+o(n^{-1+4z})]\int_{-\eta}^0 e^{-n\frac{z^2}{2}}(1+n\frac{z^3}{3})dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-n^z}^0 e^{-\frac{y^2}{2}}dy(1+\frac{y^3}{3\sqrt{n}})dy
\end{aligned}
$$

$$
\begin{aligned}
\therefore \lim_{n\to\infty} a_n &= 1 - \lim_{n\to\infty}\int_0^n \frac{x^n e^{-x}}{n!}\mathrm{d}x \\
&= 1 - (\lim_{n\to\infty}(I_1+I_2)) \\
&= 1 - \lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-n^z}^0 e^{-\frac{y^2}{2}}dy(1+\frac{y^3}{3\sqrt{n}})dy \\
&= 1 - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^0 e^{-\frac{y^2}{2}}dy - \lim_{n\to\infty}\frac{1}{2\sqrt{\pi n}}\int_{-\infty}^0 \frac{y^3}{3}e^{-\frac{y^2}{2}}dy \\
&= 1 - \frac{1}{2} - \lim_{n\to\infty}\frac{1}{2\sqrt{n\pi}}(-\frac{2}{3}) \\
&= \frac{1}{2}
\end{aligned}
$$

从这个解答也可以看出

$$\begin{aligned}&(1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!})\\&=e^n-e^n\int_0^n\frac{x^n}{n!}e^{-x}\mathrm{d}x\\&=\frac{1}{n!}\int_0^n(x+n)^ne^{-x}\mathrm{d}x\\&=\frac{n^n}{n!}\int_0^n(1+\frac{x}{n})^ne^{-x}\mathrm{d}x\sim\frac{n^n}{n!}\sqrt{\frac{n\pi}{2}}\end{aligned}$$

✎ **Solution** 解法 3 考虑 Taylor 公式的积分形式, 有

$$e^n=1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}+\frac{1}{n!}\int_0^n e^x(n-x)^n\mathrm{d}x$$

$$\begin{aligned}1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}&=e^n-\int_0^n e^t(n-t)^n dt\\ \text{令}(n-t=x)\qquad&=e^n-e^n\int_0^n\frac{x^n}{n!}e^{-x}\mathrm{d}x\\ \text{注意到}(\int_0^{+\infty}\frac{x^n}{n!}e^{-x}\mathrm{d}x=1)\qquad&=e^n(\int_0^{+\infty}\frac{x^n}{n!}e^{-x}\mathrm{d}x-\int_0^n\frac{x^n}{n!}e^{-x}\mathrm{d}x)\\&=e^n\int_n^{+\infty}\frac{x^n}{n!}e^{-x}\mathrm{d}x\\&=\frac{1}{n!}\int_n^{+\infty}x^ne^{n-x}\mathrm{d}x\\ \text{令}(n-x=-t)\qquad&=\frac{1}{n!}\int_0^{+\infty}(n+t)^ne^{-t}dt\\&=\frac{n^n}{n!}\int_0^{+\infty}(1+\frac{x}{n})^ne^{-x}\mathrm{d}x\end{aligned}$$

由 Striling 公式得

$$\begin{gathered}\frac{n^n}{n!}\int_0^{+\infty}(1+\frac{x}{n})^ne^{-x}\mathrm{d}x\sim\frac{n^n}{n!}\sqrt{\frac{n\pi}{2}}\\ n!\sim n^ne^{-n}\sqrt{2n\pi}\end{gathered}$$

所以

$$\begin{aligned}\lim_{n\to+\infty}\frac{1+n+\frac{n^2}{2!}+\cdots+\frac{n^n}{n!}}{e^n}&=\lim_{n\to+\infty}\frac{\frac{n^n}{n!}\int_0^{+\infty}(1+\frac{x}{n})^ne^{-x}\mathrm{d}x}{e^n}\\&=\lim_{n\to+\infty}\frac{\frac{n^n}{n!}\sqrt{\frac{n\pi}{2}}}{e^n}\\&=\lim_{n\to+\infty}\frac{\frac{n^n}{n^ne^{-n}\sqrt{2n\pi}}\sqrt{\frac{n\pi}{2}}}{e^n}\\&=\frac{1}{2}\end{aligned}$$

证明:

$$\frac{n^n}{n!}\int_0^{\infty}\left(1+\frac{x}{n}\right)^n e^{-x}dx \sim \frac{n^n}{n!}\sqrt{\frac{n\pi}{2}}$$

因为

$$\left(1+\frac{x}{n}\right)^n e^{-x} = e^{-\frac{x^2}{2n}+\frac{x^3}{3n^2}+o(\frac{x^3}{n^3})}$$

所以

$$\int_0^1 (1+\frac{x}{n})^n e^{-x}dx = \int_0^1 e^{-\frac{x^2}{2n}+\frac{x^3}{3n^2}+o(\frac{x^3}{n^3})}dx = \sqrt{2n}\int_0^n e^{-t^2}e^{o(\frac{1}{\sqrt{n}})}dt \sim \sqrt{2n}\frac{\sqrt{\pi}}{2}$$

下面考察

$$\frac{n^n}{n!}\int_1^{\infty}\left(1+\frac{x}{n}\right)^n e^{-x}dx = \frac{n^{n+1}}{n!}\int_n^{\infty}(1+x)^n e^{-nx}dx < \frac{n^{n+1}}{n!}e^{-\frac{n^2}{2}}\int_n^{\infty}(1+x)^n e^{-nx/2}dx$$

因为

$$\ln(n^{n+1}e^{-\frac{n^2}{2}}) = (n+1)\ln n - \frac{n^2}{2} = n^2\left[(1+\frac{1}{n})\frac{\ln n}{n} - \frac{1}{2}\right]$$

所以 $\lim\limits_{n\to\infty} n^{n+1}e^{-\frac{n^2}{2}} = 0$, **且由** $e^{\frac{nx}{2}} > \dfrac{\left(\frac{nx}{2}\right)^{n+2}}{(n+2)!} \Rightarrow e^{-\frac{nx}{2}} < \dfrac{(n+2)!}{(\frac{nx}{2})^{n+2}}$

$$\Rightarrow \frac{n^{n+1}}{n!}e^{-\frac{n^2}{2}}\int_n^{\infty}(1+x)^n e^{-nx/2}dx < \frac{n^{n+1}e^{-\frac{n^2}{2}}(n+1)(n+2)}{(\frac{n}{2})^{n+2}}\int_n^{\infty}\left(1+\frac{1}{x}\right)^n\frac{1}{x^2}dx$$

所以

$$\lim_{n\to\infty}\frac{(n+1)(n+2)}{(\frac{n}{2})^{n+2}} = 0. \lim_{n\to\infty}\int_n^{\infty}\left(1+\frac{1}{x}\right)^n\frac{1}{x^2}dx = 0.$$

所以

$$\lim_{n\to\infty}\frac{n^{n+1}}{n!}e^{-\frac{n^2}{2}}\int_n^{\infty}(1+x)^n e^{-nx/2}dx = 0$$

所以

$$\frac{n^n}{n!}\int_0^{\infty}\left(1+\frac{x}{n}\right)^n e^{-x}dx \sim \frac{n^n}{n!}\sqrt{\frac{n\pi}{2}}$$

✎ **Solution** 解法 4 考虑中心极限定理

我们来证一个一般性的结论设 $x_1, x_2, \cdots$ **为相互独立且服从参数** λ **的普阿松分布** $P(x_i = k) = \dfrac{1}{k!}e^{-\lambda}$

$\therefore \sum\limits_{i=1}^{n} x_i$ **服从参数** $n\lambda$ **的普阿松分布, 即** $P\left(\sum\limits_{i=1}^{n} x_i = k\right) = \dfrac{(n\lambda)}{k!}e^{-n\lambda}$ **因为**

$$E(\sum_{i=1}^{n} x_i) = n\lambda, \mathrm{var}(\sum_{i=1}^{n} x_i) = n\lambda$$

由中心极限定理对任意的 x 有

$$\lim_{n\to\infty}\left(\frac{\sum\limits_{i=1}^{n}x_i-n\lambda}{\sqrt{n\lambda}}<x\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$$

因为

$$P\left(\frac{\sum\limits_{i=1}^{n}x_i-n\lambda}{\sqrt{n\lambda}}<x\right)=P\left(\sum_{i=1}^{n}x_i<n\lambda+x\sqrt{n\lambda}\right)=\sum_{k=0}^{\left[n\lambda+x\sqrt{n\lambda}\right]}\frac{(n\lambda)^k}{k!}e^{-n\lambda}$$

所以

$$\lim_{n\to\infty}e^{-n\lambda}\sum_{k=0}^{\left[n\lambda+x\sqrt{n\lambda}\right]}\frac{(n\lambda)^k}{k!}e^{-n\lambda}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{t^2}{2}}dt$$

所以取 $x=0,\lambda=1$ 即得到: $\lim\limits_{n\to\infty}e^{-n}\sum\limits_{k=0}^{n}\dfrac{n^k}{k!}=\dfrac{1}{2}$

Exercise 14.3: 设 $a_n=\dfrac{\sum\limits_{i=0}^{n}\frac{n^i}{i!}}{e^n}$, 我们来计算 $\lim\limits_{n\to\infty}a_n$

Solution 因为

$$\left(1+n+\frac{n^2}{2!}+\cdots\cdots+\frac{n^n}{n!}\right)=e^n-\int_0^n\frac{(n-t)^n}{n!}e^tdt\overset{n-t=x}{=}e^n-e^n\int_0^n\frac{x^ne^{-x}}{n!}dx$$

两边除以 e^n

$$\Rightarrow a_n=1-\int_0^n\frac{x^ne^{-x}}{n!}dx,\text{ 即求 }\lim_{n\to\infty}\int_0^n\frac{x^ne^{-x}}{n!}dx\text{ 就好}$$

先令 $\eta=n^{-\frac{1}{2}+\varepsilon},0<\varepsilon<\dfrac{1}{6}$.

因为

$$\begin{aligned}\int_0^n\frac{x^ne^{-x}}{n!}dx&\xlongequal{x=n(z+1)}\int_{-1}^{0}\frac{e^{-n(z+1)}(z+1)^nn^{n+1}}{n!}dz\\&=n\frac{n^n}{n!e^n}\int_{-1}^{0}e^{-nz}(z+1)^ndz\\&=\sqrt{\frac{n}{2\pi}}\left[1+o(\frac{1}{n})\right]\int_{-1}^{0}\left[e^{-z}(1+z)\right]^ndz\\&=\sqrt{\frac{n}{2\pi}}\left[1+o(\frac{1}{n})\right]\left[\int_{-1}^{\eta}\left[e^{-z}(1+z)\right]^ndz+\int_{-\eta}^{0}\left[e^{-z}(1+z)\right]^ndz\right]\\&=I_1+I_2\end{aligned}$$

设 $f(z)=e^{-z}(1+z),(z\le 0).f'(z)=-e^{-z}z\ge 0.$

所以

$$\int_{-1}^{\eta}\left[e^{-z}(1+\mathrm{z})\right]^n dz<(1-\eta)[e^{-\eta}(1-\eta)]^n<\left[e^{-\eta}(1-\eta)\right]^n$$

所以 $I_1=o(\sqrt{n}e^{-\frac{1}{2}n^{2\varepsilon}})$

再来考虑 I_2, $e^{-z}(1+z)=e^{-z+\ln(z+1)}=e^{-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4(1+\theta(z))^4}},0<\theta(z)<1$

所以

$$\begin{aligned}
I_2&=\sqrt{\frac{n}{2\pi}}\left[1+o(n^{-1+4\varepsilon})\right]\int_{-\eta}^{0}e^{-n(\frac{z^2}{2}-\frac{z^3}{3})}dz\\
&=\sqrt{\frac{n}{2\pi}}\left[1+o(n^{-1+4\varepsilon})\right]\int_{-\eta}^{0}e^{-n\frac{z^2}{2}}(1+n\frac{z^3}{3})dz\\
&=\frac{1}{\sqrt{2\pi}}\int_{-n^{\varepsilon}}^{0}e^{-\frac{y^2}{2}}dy\left(1+\frac{y^3}{3\sqrt{n}}\right)dy
\end{aligned}$$

所以

$$\begin{aligned}
\lim_{n\to\infty}a_n&=1-\lim_{n\to\infty}\int_0^n\frac{x^ne^{-x}}{n!}dx=1-\left(\lim_{n\to\infty}(I_1+I_2)\right)\\
&=1-\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-n^{\varepsilon}}^{0}e^{-\frac{y^2}{2}}dy\left(1+\frac{y^3}{3\sqrt{n}}\right)dy
\end{aligned}$$

所以

$$\begin{aligned}
\lim_{n\to\infty}a_n&=1-\lim_{n\to\infty}\int_0^n\frac{x^ne^{-x}}{n!}dx=1-\left(\lim_{n\to\infty}(I_1+I_2)\right)\\
&=1-\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-n^{\varepsilon}}^{0}e^{-\frac{y^2}{2}}dy\left(1+\frac{y^3}{3\sqrt{n}}\right)dy\\
&=1-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{-\frac{y^2}{2}}dy-\lim_{n\to\infty}\frac{1}{2\sqrt{\pi n}}\int_{-\infty}^{0}\frac{y^3}{3}e^{-\frac{y^2}{2}}dy\\
&=1-\frac{1}{2}-\lim_{n\to\infty}\frac{1}{2\sqrt{\pi n}}\cdot\left(-\frac{2}{3}\right)\\
&=\frac{1}{2}
\end{aligned}$$

得证, 从这个解答也可以看出

$$\begin{aligned}
\left(1+n+\frac{n^2}{2!}+\cdots\cdots+\frac{n^n}{n!}\right)&=\mathrm{e}^n-e^n\int_0^n\frac{x^n}{n!}e^{-x}=\frac{1}{n!}\int_0^{\infty}(x+n)^ne^{-x}dx\\
&=\frac{n^n}{n!}\int_0^{\infty}\left(1+\frac{x}{n}\right)^ne^{-x}dx\\
&\sim\frac{n^n}{n!}\sqrt{\frac{n\pi}{2}}
\end{aligned}$$

Exercise 14.4: 求极限 (西西 2017 年新年祝福)

$$\lim_{x\to+\infty}\sqrt{x}e^{-x}\left(\sum_{k=1}^{+\infty}\frac{x^k}{k!\sqrt{k}}\right)$$

✎ **Solution(西西) 注意**

$$\sqrt{x}-\sqrt{k}=\frac{x-k}{\sqrt{x}+\sqrt{k}}$$

则有

$$\left|\frac{e^x}{\sqrt{x}}-\sum_{k=1}^{+\infty}\frac{x^k}{k!\sqrt{k}}\right|\le\frac{1}{\sqrt{x}}+\left|\sum_{k=1}^{+\infty}\frac{x^k}{k!}\frac{\sqrt{x}-\sqrt{k}}{\sqrt{kx}}\right|\le\frac{1}{\sqrt{x}}+\sum_{k=1}^{+\infty}\frac{x^k}{k!}\frac{|x-k|}{\sqrt{kx}}$$

由柯西不等式

$$\sum_{k=1}^{+\infty}\frac{x^k}{k!}\frac{|x-k|}{\sqrt{k}}\le\left(\sum_{k=1}^{+\infty}\frac{x^k}{k!}\frac{1}{k}\right)^{\frac{1}{2}}\left(\sum_{k=1}^{+\infty}\frac{x^k}{k!}(x-k)^2\right)^{\frac{1}{2}}$$

且

$$\sum_{k=1}^{+\infty}\frac{x^k}{k!}\frac{1}{k}\le 2\sum_{k=1}^{+\infty}\frac{x^k}{k!}\frac{1}{k+1}=\frac{2}{x}\sum_{k=1}^{+\infty}\frac{x^{k+1}}{(k+1)!}\le\frac{2}{x}e^x$$

且

$$\sum_{k=0}^{+\infty}\frac{x^k}{k!}(x-k)^2\le\sum_{k=0}^{+\infty}\frac{x^k}{k!}(x-k)^2=xe^x$$

所以

$$\left|\frac{e^x}{\sqrt{x}}-\sum_{k=1}^{+\infty}\frac{x^k}{k!\sqrt{k}}\right|\le\frac{1}{\sqrt{x}}+\sqrt{2}\frac{e^x}{x}$$

所以

$$\lim_{x\to+\infty}\sqrt{x}e^{-x}\left(\sum_{k=1}^{+\infty}\frac{x^k}{k!\sqrt{k}}\right)=1$$

✎ **Solution(那日蓝天) 引理: 设** $\sum_{k=1}^{+\infty}\varphi(k)$ **和** $\sum_{k=1}^{+\infty}\psi(k)$ **收敛, 且** $\lim_{k\to+\infty}\frac{\varphi(k)}{\psi(k)}=1$ **则** $\lim_{k\to+\infty}\frac{\sum_{k=1}^{+\infty}\varphi(k)x^k}{\sum_{k=1}^{+\infty}\psi(k)x^k}=1$

因为

$$\lim_{n\to\infty}\frac{n!\sqrt{n}}{\Gamma\left(n+\frac{3}{2}\right)}\xlongequal{Stiring}\lim_{n\to\infty}\frac{\sqrt{n}n^{n+\frac{1}{2}}e^{-n}}{\left(n+\frac{1}{2}\right)^{n+1}e^{-n-\frac{1}{2}}}=1$$

所以

$$\lim_{x\to+\infty}\sqrt{x}e^{-x}\left(\sum_{k=1}^{+\infty}\frac{x^k}{k!\sqrt{k}}\right)=\lim_{x\to+\infty}e^{-x}f(x)$$

其中

$$f(x)=\sum_{n=1}^{\infty}\frac{x^{n+\frac{1}{2}}}{\Gamma\left(n+\frac{3}{2}\right)}$$

$f(x)$ 满足方程

$$f'(x)=f(x)+\frac{2\sqrt{x}}{\sqrt{\pi}}\qquad(f(0)=0)$$

解之得

$$f(x)=\frac{2}{\sqrt{\pi}}e^{x}\int_{0}^{x}\sqrt{x}e^{-x}\,\mathrm{d}x$$

从而

$$\lim_{x\to+\infty}\sqrt{x}e^{-x}\left(\sum_{k=1}^{+\infty}\frac{x^k}{k!\sqrt{k}}\right)=\frac{2}{\sqrt{\pi}}e^{x}\int_{0}^{+\infty}x^{\frac{1}{2}}e^{-x}\,\mathrm{d}x=1$$

Exercise 14.5: 令 $P_n(x)=\sum_{k=0}^{n}\frac{x^k}{k!}$, **计算极限**

$$\lim_{n\to\infty}\frac{1}{n!}\int_0^1\frac{2n!\sin x-n!e^{2x}+x^n}{e^{2x}+\sin x+\cos x+P_n(x)}\,\mathrm{d}x$$

Solution 令

$$f(x)=e^{2x}+\sin x+\cos x+P_n(x)=e^{2x}+\sin x+\cos x+\sum_{k=0}^{n}\frac{x^k}{k!}$$

则

$$f'(x)=2e^{2x}+\cos x-\sin x+\sum_{k=0}^{n-1}\frac{x^k}{k!}$$

那么有

$$f(x)-f'(x)=-e^{2x}+2\sin x+\frac{x^n}{n!}$$

故

$$\begin{aligned}
I&=\lim_{n\to\infty}\frac{1}{n!}\int_0^1\frac{2n!\sin x-n!e^{2x}+x^n}{e^{2x}+\sin x+\cos x+P_n(x)}\,\mathrm{d}x\\
&=\lim_{n\to\infty}\int_0^1\frac{f(x)-f'(x)}{f(x)}\,\mathrm{d}x\\
&=\lim_{n\to\infty}\left\{\int_0^1\mathrm{d}x-\int_0^1\frac{1}{f(x)}\,\mathrm{d}(f(x))\right\}\\
&=\lim_{n\to\infty}\left\{\Big[x\Big]_0^1-\Big[\ln(f(x))\Big]_0^1\right\}\\
&=1-\ln(e^2+\sin 1+\cos 1+e)
\end{aligned}$$

Exercise 14.6: **求极限**

$$\lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^n+\left(\frac{2}{n}\right)^n+\cdots+\left(\frac{n}{n}\right)^n\right)$$

Solution 利用不等式

$$\left(\frac{n-i}{n}\right)^n\le e^{-i}$$

可得

$$\sum_{i=1}^{n}\left(\frac{i}{n}\right)^n=\sum_{k=0}^{n-1}\left(\frac{n-k}{n}\right)^n\le\sum_{k=0}^{n-1}e^{-k}\le\sum_{k=0}^{\infty}e^{-k}=\frac{e}{e-1}$$

另一方面，对于固定的正整数 k ，截取题目数列的后 $k+1$ 项，由于是有限项，所以可以逐项求极限，可得原极限大于等于

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(\frac{n-i}{n}\right)^n &= \sum_{i=0}^{k}\lim_{n\to\infty}\left(\frac{n-i}{n}\right)^n\\ &= \sum_{i=0}^{k}e^{-i}\\ &= \frac{1-e^{-k-1}}{1-e^{-1}}\end{aligned}$$

再令 $k\to\infty$ 即得

$$\lim_{n\to\infty}\left(\left(\frac{1}{n}\right)^n+\left(\frac{2}{n}\right)^n+\cdots+\left(\frac{n}{n}\right)^n\right)=\frac{e}{e-1}$$

Exercise 14.7: 求极限

$$\lim_{n\to\infty} n\left[\frac{e}{e-1}-\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}\right]$$

Solution(小灰灰)

$$\begin{aligned}
I &= \lim_{n\to\infty} n\left[\frac{e}{e-1}-\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}\right] = \lim_{n\to\infty} n\left[\sum_{k=0}^{\infty}e^{-k}-\sum_{k=0}^{n-1}\left(1-\frac{k}{n}\right)^{n}\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}-\left(1-\frac{k}{n}\right)^{n}+\sum_{k=n}^{\infty}e^{-k}\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}\left(1-\left(1-\frac{k}{n}\right)^{n}e^{k}\right)\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}\left(-\ln\left(1-\frac{k}{n}\right)^{n}-k\right)+O\left(\left(-\ln\left(1-\frac{k}{n}\right)^{n}-k\right)^{2}\right)\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}\left(n\left(\frac{k}{n}+\frac{k^2}{2n^2}+o\left(\frac{k^3}{3n^3}\right)\right)-k\right)+o\left(\left(-\ln\left(1-\frac{k}{n}\right)^{n}-k\right)^{2}\right)\right]\\
&= \lim_{n\to\infty} n\left[\sum_{k=0}^{n-1}e^{-k}\left(\frac{k^2}{2n}+O\left(\frac{k^3}{n^2}\right)\right)+O\left(\left(\frac{k^2}{2n}\right)^{2}\right)\right]\\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1}e^{-k}\frac{k^2}{2}+\frac{1}{n}o\left(\sum_{k=0}^{n-1}e^{-k}k^3\right)+\frac{1}{4n}o\left(\sum_{k=0}^{n-1}e^{-k}k^4\right)\\
&= \lim_{n\to\infty}\sum_{k=0}^{n-1}e^{-k}\frac{k^2}{2}=\sum_{k=0}^{\infty}e^{-k}\frac{k^2}{2}=S=\sum_{k=1}^{\infty}e^{-k+1}\frac{(k-1)^2}{2}\\
&= eS-\sum_{k=1}^{\infty}e^{-k+1}\frac{2k-1}{2}=\frac{1}{2e-2}\sum_{k=1}^{\infty}e^{-k+1}(2k-1)\\
&= \frac{1}{2e-2}\sum_{k=0}^{\infty}e^{-k}(2k+1)=\frac{1}{2e-2}+e^{-1}S+\frac{1}{2e-2}\sum_{k=1}^{\infty}2e^{-k}\\
&= \frac{1}{1-e^{-1}}\frac{1}{2e-2}\left(1+\sum_{k=1}^{\infty}2e^{-k}\right)=\frac{e^{-1}(e^{-1}+1)}{2(1-e^{-1})^3}\\
&= \frac{e(e^2+1)}{2(e-1)^3}
\end{aligned}$$

Exercise 14.8: 求极限

$$\lim_{n\to\infty} n\left[\frac{e(-e^2+2e+11)(5e+1)}{24(e-1)^5}-n\left(n\left(\frac{e}{e-1}-\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}\right)-\frac{e(e+1)}{2(e-1)^3}\right)\right]$$

✎ **Solution(西西) 我们如果利用泰勒公式就可以达到很好的结果**

$$\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}=\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)^{n}=\sum_{k=1}^{n}e^{n\ln\left(1-\frac{k}{n}\right)}$$

注意到

$$e^{n\ln\left(1-\frac{k}{n}\right)}=e^{-k}\left(1-\frac{k^2}{2n}+\frac{k^3(3k-8)}{24n^2}-\frac{k^4(k^2-8k+12)}{48n^3}\right)+o\left(\frac{1}{n^4}\right)$$

且注意到

$$\sum_{k=0}^{n}e^{-k}=\frac{e}{e-1}$$

$$\sum_{k=0}^{n}k^2e^{-k}=\frac{e(e+1)}{(e-1)^3}$$

和

$$\sum_{k=0}^{n}k^3(3k-8)e^{-k}=\frac{e(-e^2+2e+11)(5e+1)}{(e-1)^5}$$

$$\sum_{k=0}^{n}k^4(k^2-8k+12)e^{-k}=\frac{e(21+365e+502e^2-138e^3-35e^4+5e^5)}{(e-1)^7}$$

带入即可得到

$$\begin{aligned}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}=&\frac{e}{e-1}-\frac{1}{2n}\cdot\frac{e(e+1)}{(e-1)^3}+\frac{1}{24n^2}\cdot\frac{e(-e^2+2e+11)(5e+1)}{(e-1)^5}\\&-\frac{1}{48n^3}\cdot\frac{e(21+365e+502e^2-138e^3-35e^4+5e^5)}{(e-1)^7}+o\left(\frac{1}{n^4}\right)\end{aligned}$$

那么我们可以达到

$$\begin{aligned}&\lim_{n\to\infty}n\left[\frac{e(-e^2+2e+11)(5e+1)}{24(e-1)^5}-n\left(n\left(\frac{e}{e-1}-\sum_{k=1}^{n}\left(\frac{k}{n}\right)^{n}\right)-\frac{e(e+1)}{2(e-1)^3}\right)\right]\\&=\frac{e(21+365e+502e^2-138e^3-35e^4+5e^5)}{48(e-1)^7}\end{aligned}$$

Exercise 14.9: 证明

$$\lim_{n\to\infty}\frac{n!}{n^n}\left(\sum_{k=0}^{n}\frac{n^k}{k!}-\sum_{k=n+1}^{\infty}\frac{n^k}{k!}\right)=\frac{4}{3}$$

✎ **Solution(西西): 我们有**

$$e^n=\sum_{k=0}^{n}\frac{n^k}{k!}+\sum_{k=n+1}^{\infty}\frac{n^k}{k!}=\sum_{k=0}^{n}\frac{n^k}{k!}+\frac{1}{n!}\int_0^n e^t(n-t)^n dt$$

所以

$$\sum_{k=n+1}^{\infty}\frac{n^k}{k!}=\frac{1}{n!}\int_0^n e^t(n-t)^n dt$$

$$\sum_{k=0}^{n}\frac{n^k}{k!}=e^n-\frac{1}{n!}\int_0^n e^t(n-t)^n dt$$

因此，只要计算

$$\lim_{n\to\infty}\frac{n!}{n^n}\left(e^n-\frac{2}{n!}\int_0^n e^t(n-t)^n dt\right)$$

下面来估计

$$\int_0^n e^t(n-t)^n dt$$

我们有

$$\begin{aligned}\int_0^n e^t(n-t)^n dt&=n^{n+1}\int_0^1 e^{nz}(1-z)^n dz\\&=n^{n+1}\int_0^1 e^{n(z+\ln(1-z))}dz\\&=n^{n+1}\int_0^1 e^{-\frac{1}{2}nz^2-\frac{1}{3}nz^3+o(nz^3)}dz\\&=n^{n+1}\int_0^1 e^{-\frac{1}{2}nz^2}\left(1-\frac{1}{3}nz^3+o(nz^3)\right)dz\end{aligned}$$

$$\begin{aligned}&\frac{n!}{n^n}\left(e^n-\frac{2}{n!}\int_0^n e^t(n-t)^n dt\right)\\=&\frac{n!e^n}{n^n}-2n\left[\int_0^1 e^{-\frac{1}{2}nz^2}\left(1-\frac{1}{3}nz^3+o(nz^3)\right)dz\right]\\=&\left(\sqrt{2\pi n}e^{\frac{\theta_n}{12n}}-2n\int_0^1 e^{-\frac{1}{2}nz^2}dz\right)+2n\int_0^1 e^{-\frac{1}{2}nz^2}\left(\frac{1}{3}nz^3+o(nz^3)\right)dz\end{aligned}$$

其中 $\theta_n\in(0,1)$

显然有

$$\lim_{n\to\infty}\left(\sqrt{2\pi n}e^{\frac{\theta_n}{12n}}-2n\int_0^1 e^{-\frac{1}{2}nz^2}dz\right)=0$$

$$\lim_{n\to\infty}2n\int_0^1 e^{-\frac{1}{2}nz^2}\left(\frac{1}{3}nz^3+o(nz^3)\right)dz=\lim_{n\to\infty}\frac{4}{3}\left(\int_0^{\frac{n}{2}}e^{-z}zdz+o\left(\frac{1}{n}\right)\right)=\frac{4}{3}$$

所以

$$\lim_{n\to\infty}\frac{n!}{n^n}\left(\sum_{k=0}^{n}\frac{n^k}{k!}-\sum_{k=n+1}^{\infty}\frac{n^k}{k!}\right)=\frac{4}{3}$$

Exercise 14.10: 设 $A_n=\dfrac{n}{n^2+1}+\dfrac{n}{n^2+2^2}+\cdots+\dfrac{n}{n^2+n^2}$ 求极限

$$\lim_{n\to+\infty}n^4\left(\frac{1}{24}-n\left(n\left(\frac{\pi}{4}-A_n\right)-\frac{1}{4}\right)\right)$$

✎ **Solution** 这里提供一个一般的方法.

Definition 14.1 Euler-maclaurin 求和公式

设函数 $f \in C^{(2m+2)}\big[a,b\big], h=\dfrac{b-a}{n}, x_i=a+ih, i=0,1,\cdots,n$, 则

$$\begin{aligned}&\frac{b-a}{n}\sum_{i=1}^{n}\frac{1}{2}[f(x_{i-1})+f(x_i)]-\int_a^b f(x)dx\\ =&\sum_{k=1}^{m}\frac{B_{2k}}{(2k)!}h^{2k}\left[f^{(2k-1)}(b)-f^{(2k-1)}(a)\right]+\frac{B_{2m+2}}{(2m+2)!}h^{2m+2}f^{(2m+2)}(\xi)(b-a)\end{aligned}$$

其中 $\xi\in[a,b]$, $B_{2k}(k=1,2,\cdots,m+1)$ 是 Bernoulli 数且 $B_2=\dfrac{1}{6}, B_4=-\dfrac{1}{30}, B_6=\dfrac{1}{42}$

取 $a=0, b=1, f(x)=\dfrac{1}{1+x^2}$,

则 $h=\dfrac{1}{n}, x_i=\dfrac{i}{n}, A_n=\dfrac{1}{n}\sum\limits_{i=1}^{n}f(x_i)$， 则

$$\begin{aligned}A_n+\frac{1}{4n}-\frac{\pi}{4}&=\frac{1}{2}\left[\left(A_n-\frac{1}{2n}+\frac{1}{n}\right)+A_n\right]-\frac{\pi}{4}\\&=\frac{B_2}{2!}\cdot\frac{1}{n^2}[f'(1)-f'(0)]+\frac{B_4}{4!}\cdot\frac{1}{n^4}[f'''(1)-f'''(0)]\\&\quad+\frac{B_6}{6!}\cdot\frac{1}{n^6}\left[f^{(5)}(1)-f^{(5)}(0)\right]+\frac{B_8}{8!}\cdot\frac{1}{n^8}f^{(8)}(\xi)\end{aligned}$$

其中, $\xi\in[0,1]$ 也即

$$n^4\left(\frac{1}{24}-n\left(n\left(\frac{\pi}{4}-A_n\right)-\frac{1}{4}\right)\right)=\frac{1}{2016}+\frac{B_8}{8!}\cdot\frac{1}{n^8}f^{(8)}(\xi)$$

注意到 $f^{(8)}(\xi)$ 有界, 因此 $n\to+\infty$ 时所求极限为 $\dfrac{1}{2016}$

Exercise 14.11: 设 $n\in N^+, I_n=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin^2 nt}{\sin t}dt$, 计算极限 $\displaystyle\lim_{n\to\infty}\frac{I_n}{\ln n}$

✎ **Solution** 利用 $\sin^2 nt=\dfrac{1-\cos 2nt}{2}$,可得 $I_n=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sin^2 nt}{\sin t}dt=\int_0^{\frac{\pi}{2}}\frac{1-\cos 2nt}{2\sin t}dt$

所以

$$\begin{aligned}I_{n+1}-I_n&=\int_0^{\frac{\pi}{2}}\frac{1-\cos 2(n+1)t}{2\sin t}dt-\int_0^{\frac{\pi}{2}}\frac{1-\cos 2nt}{2\sin t}dt\\&=\int_0^{\frac{\pi}{2}}\frac{\cos 2nt-\cos 2(n+1)t}{2\sin t}dt\end{aligned}$$

利用和差化积公式

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

有:

$$I_{n+1}-I_n=\int_0^{\frac{\pi}{2}}\frac{2\sin 2(n+1)t\sin t}{2\sin t}dt=\int_0^{\frac{\pi}{2}}\sin(2n+1)tdt$$
$$=\left[-\frac{\cos(2n+1)t}{2n+1}\right]_0^{\frac{\pi}{2}}=\frac{1}{2n+1}$$

所以 $I_n=1+\frac{1}{3}+\cdots+\frac{1}{2n+1}$, 显然当 $n\to+\infty$ 时, $\lim\limits_{n\to\infty}\frac{n}{2n+1}$=$\frac{1}{2}$

应用 $Stolz$ 定理有:

$$\lim_{n\to\infty}\frac{I_n}{\ln n}=\lim_{n\to\infty}\frac{I_{n+1}-I_n}{\ln(n+1)-\ln n}=\lim_{n\to\infty}\frac{\frac{1}{2n+1}}{\ln(1+\frac{1}{n})}=\lim_{n\to\infty}\frac{n}{2n+1}=\frac{1}{2}$$

我以前从未写过类文件，所以，写这个模板的过程必然是折腾的过程，在写模板的过程中，最主要参考了《写给LaTeX 2ε 类与宏包的作者》[1]、moderncv.cls 文件、武汉大学黄正华老师的论文模板、《LaTeX 2ε 完全学习手册》[2]、*The Not So Short Introduction to LaTeX 2ε*[3] 以及各大 LaTeX 疑问解答网站，在此为无私奉献的组织和个人表示感谢！

忍不住插个图！

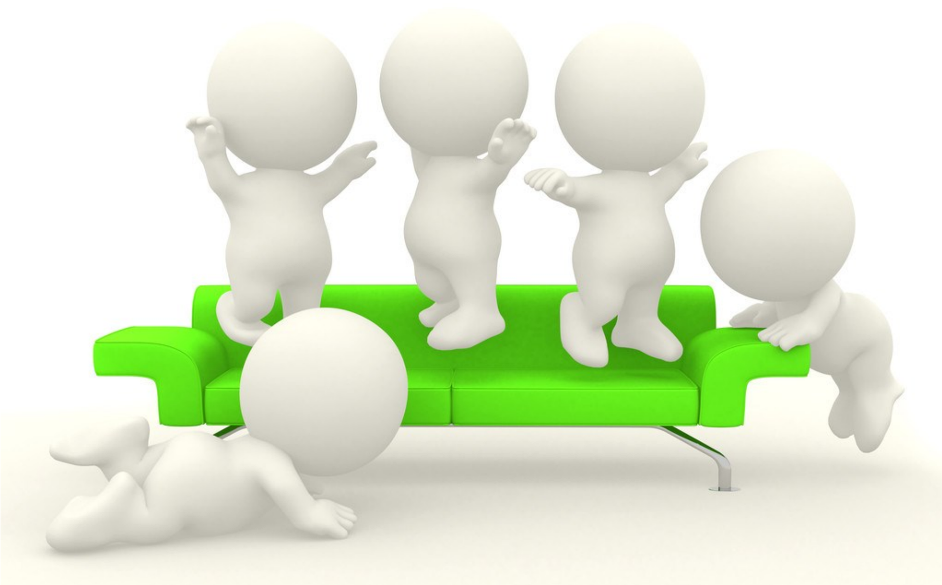


图 **14.1:** Happiness,We have it!

第 15 章 ElegantBook 开服说明

15.1 关于我们

我们目前都是学生，接触 LaTeX 的时间也不是很长，因此，对于此模板的错误还请多多包涵！目前，模板的拓展性或者可移植性有待完善，所以，我们强烈建议用户不要大幅修改模板文件，我们的初衷是提供一套模板，让初学者能够使用一些比较美观，优雅的模板。如果在使用过程中，想修改一些简单的东西需要帮忙，请联系我们，我们的邮箱是：elegantlatex2e@gmail.com。我们将竭尽全力提供帮助！

值此版本发行之际，我们 ElegantLaTeX 项目组向大家重新介绍一下我们的工作，我们的主页是 http://elegantlatex.tk，我们这个项目致力于打造一系列美观、优雅、简便的模板方便使用者记录学习历史。其中目前在实施或者在规划中的子项目有书籍模板 ElegantBook、笔记模板 ElegantNote、幻灯片模板 ElegantSlide。这些子项目的名词是一体的，请在使用这些名词的时候不要将其断开（如 Elegant Note 是不正确的写法）。并且，ElegantLaTeX Book 指的即是 ElegantBook。

15.2 文档说明

15.2.1 编译方式

本模板基于 book 文类，所以 book 的选项对于本模板也是有效的。但是，只支持 XeLaTeX，编码为 UTF-8，推荐使用 TeXlive 编译。作者编写环境为 Win8.1(64bit)+TeXlive 2013，由于使用了参考文献，所以，编译顺序为 XeLaTeX->BibTeX->XeLaTeX->XeLaTeX。

本文特殊选项设置共有 3 类，分为颜色、数学字体以及章标题显示风格。

15.2.2 选项设置

第一类为颜色主题设置，内置 3 组颜色主题，分别为 green（默认），cyan，blue，另外还有一个自定义的选项 nocolor，用户必须在使用模板的时候选择某个颜色主题或选择 nocolor 选项。调用颜色主题 green 的方法为`\documentclass[green]{elegantbook}`或者使用`\documentclass[color=green]{elegantbook}`。需要改变颜色的话请选择 nocolor 选项或者使用 color=none，然后在导言区定义 main、seco、thid 颜色，具体的方法如下：

```
\definecolor{main}{RGB}{70,70,70}     %定义main颜色值
```

```
\definecolor{seco}{RGB}{115,45,2}    %定义seco颜色值
\definecolor{thid}{RGB}{0,80,80}     %定义thid颜色值
\base{blackbase.pdf}                 %可以改为自己想要的图案
```

第二类为数学字体设置，有两个可选项，分别是 mathpazo（默认）和 mtpro2 字体，调用 mathpazo 字体使用 `\documentclass[mathpazo]{elegantbook}`，调用 mtpro2 字体时需要把 `mathpazo` 换成 `mtpro`，mathpazo 不需要用户自己安装字体，mtpro2 的字体需要自己安装。

	green	cyan	blue	主要使用的环境
main				newthem newlemma newcorol
seco				newdef
thid				newprop

表 15.1: ElegantBook 模板中的三套颜色主题

第三类为章标题显示风格，包含 `hang`（默认）与 `display` 两种风格，区别在于章标题单行显示（`hang`）与双行显示（`display`），本说明使用了 `hang`。调用方式为 `\documentclass[hang]{elegantbook}` 或者 `\documentclass[titlestyle=hang]{elegantbook}`。

综合起来，同时调用三个选项使用 `\documentclass[color=X,Y,titlestyle=Z]{elegantbook}`。其中 `X` 可以选择 `green,cyan,blue,none`；`Y` 可以选择 `mathpazo` 或者 `mtpro`；`Z` 可以选择 `hang` 或者 `display`。

15.2.3 数学环境简介

在我们这个模板中，定义了三大类环境

1. 定理类环境，包含标题和内容两部分。根据格式的不同分为 3 种
    - newthem、newlemma、newcorol 环境，颜色为主颜色 main，三者编号均以章节为单位；
    - newdef 环境，含有一个可选项，编号以章节为单位，颜色为 seco；
    - newprop 环境，含有一个可选项，编号以章节为单位，颜色为 thid。
2. 证明类环境，有newproof、note、remark、solution 环境，特点是，有引导符和引导词，并且 newproof、solution 环境有结束标志。

3. 结论类环境，有conclusion、assumption、property 环境，三者均以粗体的引导词为开头，和普通段落格式一致。
4. 示例类环境— example、exercise环境，编号以章节为单位，其中 exercise 环境有引导符。
5. 自定义环境— custom，带一个必选参数，格式与 conclusion 环境很类似。

15.2.4 可编辑的字段

在模板中，可以编辑的字段分别为作者`\author`、`\email`、`\zhtitle`、`\zhend`、`\entitle`、`\enend`、`\version`。并且，可以根据自己的喜好把封面水印效果的`cover.pdf` 替换掉，以及封面中用到的`logo.pdf`。

第 16 章　ElegantBook 写作示例

16.1　灵魂不随便出卖，代码也不随便瞎写

Definition 16.1 Wiener Process

If z is wiener process, then for any partition $t_0, t_1, t_2, \ldots$ of time interval, the random variables $z(t_1) - z(t_0), z(t_2) - z(t_1), \ldots$ are independently and normally distributed with zero means and variance $t_1 - t_0, t_2 - t_1, \ldots$

Example 16.1: E and F be two events such that $\mathbf{P}(E) = \mathbf{P}(F) = 1/2$, and $\mathbf{P}(E \cap F) = 1/3$, let $\mathscr{F} = \sigma(Y)$, X and Y be the indicate function of E and F respectively. How to compute $\mathbb{E}[X \mid \mathscr{F}]$?

Exercise 16.1: let $S = l^\infty = \{(x_n) \mid \exists M \text{ such that } \forall n, |x_n| \leq M, x_n \in \mathbb{R}\}$, $\rho_\infty(x, y) = \sup\limits_{n \geq 1} |x_n - y_n|$, show that (l^∞, ρ_∞) is complete.

Theorem 16.1 勾股定理

勾股定理的数学表达（*Expression*）为

$$a^2 + b^2 = c^2$$

其中 a, b 为直角三角形的两条直角边长，c 为直角三角形斜边长。

Note: 在本模板中，引理（*lemma*），推论（*corollary*）的样式和定理的样式一致，包括颜色，仅仅只有计数器的设置不一样。在这个例稿中，我们将不给出引理推论的例子。

Quisque ullamcorper placerat ipsum. Cras nibh. Morbi vel justo vitae lacus tincidunt ultrices. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. In hac habitasse platea dictumst. Integer tempus convallis augue. Etiam facilisis. Nunc elementum fermentum wisi. Aenean placerat. Ut imperdiet, enim sed gravida sollicitudin, felis odio placerat quam, ac pulvinar elit purus eget enim. Nunc vitae tortor. Proin tempus nibh sit amet nisl. Vivamus quis tortor vitae risus porta vehicula.

Proposition 16.1 最优性原理

如果 u^* 在 $[s,T]$ 上为最优解，则 u^* 在 $[s,T]$ 任意子区间都是最优解，假设区间为 $[t_0,t_1]$ 的最优解为 u^*，则 $u(t_0)=u^*(t_0)$，即初始条件必须还是在 u^* 上。

Fusce mauris. Vestibulum luctus nibh at lectus. Sed bibendum, nulla a faucibus semper, leo velit ultricies tellus, ac venenatis arcu wisi vel nisl. Vestibulum diam. Aliquam pellentesque, augue quis sagittis posuere, turpis lacus congue quam, in hendrerit risus eros eget felis. Maecenas eget erat in sapien mattis porttitor. Vestibulum porttitor. Nulla facilisi. Sed a turpis eu lacus commodo facilisis. Morbi fringilla, wisi in dignissim interdum, justo lectus sagittis dui, et vehicula libero dui cursus dui. Mauris tempor ligula sed lacus. Duis cursus enim ut augue. Cras ac magna. Cras nulla. Nulla egestas. Curabitur a leo. Quisque egestas wisi eget nunc. Nam feugiat lacus vel est. Curabitur consectetuer.

Suspendisse vel felis. Ut lorem lorem, interdum eu, tincidunt sit amet, laoreet vitae, arcu. Aenean faucibus pede eu ante. Praesent enim elit, rutrum at, molestie non, nonummy vel, nisl. Ut lectus eros, malesuada sit amet, fermentum eu, sodales cursus, magna. Donec eu purus. Quisque vehicula, urna sed ultricies auctor, pede lorem egestas dui, et convallis elit erat sed nulla. Donec luctus. Curabitur et nunc. Aliquam dolor odio, commodo pretium, ultricies non, pharetra in, velit. Integer arcu est, nonummy in, fermentum faucibus, egestas vel, odio.

Corollary 16.1

假设 $V(\cdot,\cdot)$ 为值函数，则跟据最大值原理，有如下推论

$$V(k,z)=\max\left\{u\big(zf(k)-y\big)+\beta\mathbb{E}V(y,z')\right\}$$

Proof: 因为 $y^*=\alpha\beta zk^\alpha$，$V(k,z)=\alpha/1-\alpha\beta\ln k_0+1/1-\alpha\beta\ln z_0+\Delta$。所以 左边 = 右边，证毕。 □

Properties: Properties of Cauchy Sequence

1. $\{x_k\}$ is cauchy sequence then $\{x_k^i\}$ is cauchy sequence.
2. $x_k\in\mathbb{R}^n$, $\rho(x,y)$ is Euclidean, then cauchy is equivalent to convergent, $(\mathbb{R}^n,\rho)$ metric space is complete.

Application: This is one example of the custom environment, the key word is given by the option of custom environment.

Definition 16.2 Contraction mapping

(S, ρ) is the metric space, $T : S \to S$, If there exists $\alpha \in (0, 1)$ such that for any x and $y \in S$, the distance

$$\rho(Tx, Ty) \leq \alpha\rho(x, y) \tag{16.1}$$

Then T is a contraction mapping.

Remarks:

1. *$T : S \to S$, where S is a metric space, if for any $x, y \in S$, $\rho(Tx, Ty) < \rho(x, y)$ is not contraction mapping.*
2. *Contraction mapping is continuous map.*

Conclusions: 看到一则小幽默，是这样说的：别人都关心你飞的有多高，只有我关心你的翅膀好不好吃！说多了都是泪啊！

参考文献

[1] T. LaTeX. . Project, “LaTeX 2$_\varepsilon$ for class and package writers,” 1999.

[2] **胡伟**, “LaTeX 2$_\varepsilon$ **完全学习手册**,” 2011.

[3] T. Oetiker, H. Partl, I. Hyna, and E. Schlegl, “The not so short introduction to LaTeX 2$_\varepsilon$,” 2010.