§3 欧拉积分

1．计算$\Gamma(\frac{5}{2}),\ \Gamma(-\frac{5}{2}),\ \Gamma(\frac{1}{2}+n),\ \Gamma(\frac{1}{2}-n).$

解：$\Gamma(\frac{5}{2})=\frac{3}{2}\Gamma(\frac{3}{2})=\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})=\frac{3}{4}\sqrt{\pi},$

$$\Gamma(-\frac{5}{2})=\Gamma(-\frac{3}{2})/(-\frac{5}{2})=-\frac{2}{5}\Gamma(-\frac{1}{2})/(-\frac{3}{2})=\frac{4}{15}\Gamma(\frac{1}{2})/(-\frac{1}{2})$$

$$\Gamma(\frac{1}{2}+n)=\frac{2n-1}{2}\Gamma(\frac{1}{2}+(n-1))=\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{1}{2}\cdots\frac{1}{2}\Gamma(\frac{1}{2})=\frac{(2n-1)!!}{2^n}\sqrt{\pi},$$

$$\Gamma(\frac{1}{2}-n)=-\frac{2}{2n-1}\Gamma(\frac{1}{2}-(n-1))=(-\frac{2}{2n-1})(\frac{2}{2n-3})\cdots(-\frac{2}{1})\Gamma(\frac{1}{2})=\frac{(-1)^n2^n}{(2n-1)!!}\sqrt{n}.$$

2．计算$\int_0^{\frac{\pi}{2}}\sin^{2n}udu,\int_0^{\frac{\pi}{2}}\sin^{2n+1}udu.$

解：$\int_0^{\frac{\pi}{2}}\sin^{2n}udu=\int_0^{\frac{\pi}{2}}\cos^{2\cdot\frac{1}{2}-1}u\sin^{2(n+\frac{1}{2})-1}udu=\frac{1}{2}B(\frac{1}{2},n+\frac{1}{2})$

$$=\frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}=\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}.$$

$$\int_0^{\frac{\pi}{2}}\sin^{2n+1}udu=\int_0^{\frac{\pi}{2}}\cos^{2\cdot\frac{1}{2}-1}u\sin^{(2n+1)-1}udu=\frac{1}{2}B(\frac{1}{2},n+1)$$

$$=\frac{1}{2}\frac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+\frac{3}{2})}=\frac{(2n)!!}{(2n+1)!!}.$$

3．证明下列各式:

(1) $\Gamma(a)=\int_0^1(\ln\frac{1}{x})^{a-1}dx,a>0;$

(2) $\int_0^{+\infty}\frac{x^{a-1}}{1+x}dx=\Gamma(a)\Gamma(1-a),0<a<1;$

(3) $\int_0^1x^{p-1}(1-x^r)^{q-1}dx=\frac{1}{r}B(\frac{p}{r},q),p>0,q>0,r>0;$

(4) $\int_0^{+\infty}\frac{dx}{1+x^4}=\frac{\pi}{2\sqrt{2}}.$

证：(1) 由定义知: $\Gamma(a)=\int_0^{+\infty}t^{a-1}e^{-t}dt.$ 令$t=\ln\frac{1}{x}$, 则$x=e^{-t}(x\in(0,1]).$从而

$$\Gamma(a)=\int_1^0(\ln\frac{1}{x})^{a-1}\cdot x\cdot\frac{1}{\frac{1}{x}}\cdot(-\frac{1}{x^2})dx=\int_0^1(\ln\frac{1}{x})^{a-1}dx.$$

(2) $\Gamma(a)\Gamma(1-a)=B(a,1-a)\cdot\Gamma(a+1-a)=B(a,1-a)\cdot\Gamma(1)=B(a,1-a)$

$$=\int_0^{+\infty}\frac{x^{a-1}}{(1+x)^{(a+1-a)}}dx=\int_0^{+\infty}\frac{x^{a-1}}{1+x}dx.$$

(3) 令 $y=x^r$，则

$$\int_0^1 x^{p-1}(1-x^r)^{q-1}dx=\int_0^1 y^{\frac{1}{r}(p-1)}(1-y)^{q-1}dy^{\frac{1}{r}}=\int_0^1 y^{\frac{p}{r}-\frac{1}{r}}(1-y)^{q-1}\cdot\frac{1}{r}y^{\frac{1}{r}-1}dy$$

$$=\frac{1}{r}\int_0^1 y^{\frac{p}{r}-1}(1-y)^{q-1}dy=\frac{1}{r}B(\frac{p}{r},q).$$

(4) 设 $t=\frac{1}{1+x^4}$，则 $dx=-\frac{1}{4}t^{-\frac{5}{4}}(1-t)^{-\frac{3}{4}}dt$，有

$$\int_0^{+\infty}\frac{1}{1+x^4}dx=\frac{1}{4}\int_0^1 t^{-\frac{1}{4}}(1-t)^{-\frac{3}{4}}dt=\frac{1}{4}B(1-\frac{1}{4},1-\frac{3}{4})=\frac{1}{4}B(\frac{3}{4},\frac{1}{4})$$

$$=\frac{1}{4}B(\frac{3}{4},1-\frac{3}{4})=\frac{1}{4}\cdot\frac{\pi}{\sin\frac{3}{4}\pi}=\frac{\pi}{2\sqrt{2}}\text{(由余元公式)}$$

4．证明公式：$B(p,q)=B(p+1,q)+B(p,q+1).$

证明：$B(p+1,q)+B(p,q+1)=\frac{(p+1)-1}{(p+1+q)-1}B(p,q+1)+\frac{(q+1)\_1}{p+(q+1)-1}B(p,q+1-1)$

$$=\frac{p}{p+q}B(p,q)+\frac{q}{p+q}B(p,q)=B(p,q)$$

5．已知 $\Gamma(\frac{1}{2})=\sqrt{\pi}$，试证：$\int_{-\infty}^{+\infty}x^2e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$

证：$\int_{-\infty}^{+\infty}x^2e^{-x^2}dx=\int_{-\infty}^0 x^2e^{-x^2}dx+\int_0^{+\infty}x^2e^{-x^2}dx=\int_0^{+\infty}y^2e^{-y^2}dy+\int_0^{+\infty}x^2e^{-x^2}dx$

$=2\int_0^{+\infty}x^2e^{-x^2}dx$

令 $t=x^2$，则上式$=2\int_0^{+\infty}te^{-t}dt^{\frac{1}{2}}=2\int_0^{+\infty}te^{-t}\cdot\frac{1}{2}t^{-\frac{1}{2}}dt=2\int_0^{+\infty}t^{-\frac{1}{2}}e^{-t}dt=\Gamma(\frac{3}{2})=\frac{1}{2}\Gamma(\frac{1}{2})=\frac{\sqrt{\pi}}{2}.$

6．试将下列积分用欧拉积分表示,并指出参量的取值范围:

(1) $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n xdx$; (2) $\int_0^1 (\ln\frac{1}{x})^p dx$.

解：(1) $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n xdx = \frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \sin^{2\frac{m+1}{2}-1} x \cos^{2\frac{n+1}{2}-1} xdx = \frac{1}{2}B(\frac{n+1}{2},\frac{m+1}{2})$，

其中 $\frac{n+1}{2} > 0, \frac{m+1}{2} > 0$，即 $n > -1, m > -1$.

(2)由习题(3)易知

$\int_0^1 (\ln\frac{1}{x})^p dx = \int_0^1 (\ln\frac{1}{x})^{(p+1)-1} dx = \Gamma(p+1)$，其中 $p+1>0$，即 $p > -1$.