$I(\alpha)=\int_0^{\pi}\ln(1-2\alpha\cos x+\alpha^2)\,dx$

2013 高数期末

南開大學 作业纸

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一．求解下列微分方程

(1) $y'=e^{x-y}$

解：$\frac{dy}{dx}=\frac{e^x}{e^y}$

$\Rightarrow e^y dy=e^x dx$

$\Rightarrow \int e^y dy=\int e^x dx+C$

$\Rightarrow e^y=e^x+C$

$\Rightarrow e^{y-x}=C$

$\Rightarrow y=x+C$

(2) $y'+2xy=4x$

解：$y=e^{-\int 2x dx}\left(\int 4x\cdot e^{\int 2x dx}dx+C\right)$

$=e^{-x^2}\left(\int 4x\cdot e^{x^2}dx+C\right)$

$=e^{-x^2}(2e^{x^2}+C)$

$=2+Ce^{-x^2}$

~~(3) $y''+5y'+4y=0$~~

~~解：令 $y'=P$~~

~~则 $y''=P\frac{dp}{dy}$~~

~~∴原方程化为~~

~~$P\frac{dp}{dy}+5P+4y=0$~~

~~$\Rightarrow \frac{dp}{dy}+\frac{4y}{P}=5$~~

~~令 $P=uy$，则 $u+yu'+5+\frac{4}{u}=0$~~

~~$\Rightarrow yu'=-5-\frac{4}{u}-u$~~

~~$\Rightarrow u=$~~

二．判断下列广义积分敛散性

(1) $\sum_{n=1}^{\infty}\frac{n+1}{\sqrt{n^5+n^3+1}}$

解：∵ $\lim_{n\to\infty}\frac{n+1}{\sqrt{n^5+n^3+1}}\Big/\frac{1}{n^{\frac{3}{2}}}$

$=\lim_{n\to\infty}\sqrt{\frac{n^3(n+1)^2}{n^5+n^3+1}}=1$

∴原级数收敛

(2) $\sum_{n=1}^{\infty}\frac{n^4}{4^n}$

解：∵ $\lim_{n\to\infty}\sqrt[n]{\frac{n^4}{4^n}}$

$=\lim_{n\to\infty}\frac{1}{4}(\sqrt[n]{n})^4=\frac{1}{4}<1$

∴原级数收敛

(3) $\sum_{n=2}^{\infty}\frac{(-1)^n\ln^2 n}{n}$

解：记 $u_n=(-1)^n\frac{\ln^2 n}{n}$

∵ $\lim_{n\to\infty}|u_n|=\lim_{n\to\infty}\frac{\ln^2 n}{n}=\lim_{n\to\infty}\frac{2\ln n}{n}=\lim_{n\to\infty}\frac{2}{n}=0$

$|u_{n+1}|-|u_n|=\frac{\ln^2(n+1)}{n+1}-\frac{\ln^2 n}{n}=\frac{n\ln^2(n+1)-n\ln^2 n-\ln^2 n}{n(n+1)}$

$=\frac{n\ln^2(1+\frac{1}{n})-\ln^2 n}{n(n+1)}=\frac{1-\ln^2 n}{n(n+1)}<0$

∴交错级数 $\sum_{n=2}^{\infty}\frac{(-1)^n\ln^2 n}{n}$ 收敛

讨论 $\sum_{n=2}^{\infty}|u_n|$.

∵ $\lim_{n\to\infty}\frac{\ln^2 n}{n}\Big/\frac{1}{n}=\lim_{n\to\infty}\ln^2 n=+\infty$.

∴ $\sum_{n=2}^{\infty}|u_n|$ 发散

∴原级数条件收敛

(4) $\int_1^{+\infty}\sqrt{x}\sin\frac{1}{x}dx$

解：∵ $\lim_{x\to+\infty}\sqrt{x}\sin\frac{1}{x}\Big/\frac{1}{\sqrt{x}}=\lim_{x\to+\infty}\frac{\sin\frac{1}{x}}{\frac{1}{x}}=1$

∴原级数发散．

一．(3) $y''+5y'+4y=0$．

$\lambda^2+5\lambda+4=0$

$(\lambda+1)(\lambda+4)=0$

$\lambda=-1\quad \lambda=-4$

∴ $y=C_1e^{-x}+C_2e^{-4x}$．

$I(\alpha)=\int_0^{\pi} \ln(1-2\alpha\cos x+\alpha^2)\,dx$

三、将 $f(x)=\frac{1}{x-2}-\frac{1}{x-1}$ 展开成 x 的幂级数并求收敛域

解：$\because \frac{1}{x-2}=-\frac{1}{2-x}=-\frac{1}{2}\left(\frac{1}{1-\frac{x}{2}}\right)=-\frac{1}{2}\left(1+\frac{x}{2}+\frac{x^2}{4}+\frac{x^3}{8}+\cdots+\frac{x^n}{2^n}+\cdots\right)=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}$

$-\frac{1}{x-1}=\frac{1}{1-x}=1+x+x^2+\cdots+x^n=\sum_{n=0}^{\infty}x^n$

$\therefore f(x)=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}+\sum_{n=0}^{\infty}x^n=\sum_{n=0}^{\infty}x^n\left(1-\frac{1}{2^{n+1}}\right)$

又对于 $\frac{1}{x-2}=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{2^n}$，$\rho_1=\lim_{n\to\infty}\sqrt[n]{\frac{1}{2^n}}=\frac{1}{2}$，$\Rightarrow R_1=2$.

当 $x=2$ 时，$\frac{1}{x-2}=-\frac{1}{2}\sum_{n=0}^{\infty}1=\infty$，不成立，当 $n=-2$ 时，$\frac{1}{x-2}=-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n$ 不收敛

$\therefore \frac{1}{x-2}=\sum_{n=0}^{\infty}-\frac{x^n}{2^{n+1}}$ 的收敛域为 $(-2,2)$

同理，$\frac{1}{1-x}$ 的收敛域为 $(-1,1)$

综上，$f(x)$ 的收敛域为 $(-1,1)$

四、将 $f(x)=\sin x\ (0\le x\le\pi)$ 展开成正弦级数（幂级数？）

解：将 $f(x)$ 拓展成奇函数：$f(x)=\sin x$，$(-\pi\le x\le\pi)$

则有 $a_n=a_0=0$.

$b_n=\frac{2}{\pi}\int_0^{\pi}\sin x\sin nx\,dx\quad(n\ne 1)$

$=\frac{1}{\pi}\int_0^{\pi}[\cos(x-nx)-\cos(x+nx)]\,dx$

$=\frac{1}{\pi}\int_0^{\pi}\cos(n-1)x\,dx-\frac{1}{\pi}\int_0^{\pi}\cos(1+n)x\,dx$

$=\frac{1}{\pi(n-1)}\sin(n-1)x\Big|_0^{\pi}-\frac{1}{\pi(1+n)}\sin(n+1)x\Big|_0^{\pi}$

$=0$

$n=1$ 时，$b_n=\frac{2}{\pi}\int_0^{\pi}\sin^2x\,dx=\frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin^2x\,dx=\frac{4}{\pi}\cdot\frac{\pi}{2}\cdot\frac{1}{2}=1$

$\therefore f(x)=\sin x\ \cdots$

按幂级数：令 $F(x)=e^{ix}$

则 $e^{ix}=\sum_{n=0}^{\infty}\frac{(ix)^n}{n!}$

$=1+ix-\frac{x^2}{2!}-\frac{ix^3}{3!}+\frac{x^4}{4!}+\cdots+\frac{(ix)^n}{n!}\cdots$

$=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right.$

$\therefore f(x)=\sin x=\sum_{n=1}^{\infty}\frac{x^{2n-1}}{(2n-1)!}\cdot(-1)^{n-1}$

假期末

五. 计算

(1) $\int_0^{\frac{1}{e}} \frac{1}{x\ln^2 x}dx$　　(2) $\lim\limits_{x\to 0} \frac{\int_x^{x^2} e^{-xy^2}dy}{x}$

解: $\int_0^{\frac{1}{e}} \frac{1}{x\ln^2 x}dx$

$= \int_0^{\frac{1}{e}} \frac{1}{\ln^2 x} d\ln x$

$= \lim\limits_{a\to 0^+} -\frac{1}{\ln x}\Big|_a^{\frac{1}{e}}$

$= \lim\limits_{a\to 0^+} \frac{1}{\ln a} - \frac{1}{-1} = 1$

解: 原式 $= \lim\limits_{x\to 0}\left[2xe^{-x^5} - e^{-x^3} + \int_x^{x^2} -2xye^{-xy^2}dy\right]$

$= \lim\limits_{x\to 0}\left[-1 - 2x\int_x^{x^2} ye^{-xy^2}dy\right]$

$= 0$

六. 设 $a_0=4$, $a_1=1$, $n\geq 2$ 时 $a_n = \frac{a_{n-2}}{n(n-1)}$, 求幂级数 $\sum\limits_{n=0}^{\infty} a_n x^n$ 的和函数

解: ∵ $n\geq 2$ 时 $a_n = \frac{a_{n-2}}{n(n-1)}$

∴ $n\to\infty$ 时有 $a_n = \frac{a_{n-2}}{n(n-1)} = \frac{1}{n(n-1)}\cdot\frac{a_{n-4}}{(n-2)(n-3)} = \cdots = \frac{a_1}{n(n-1)\cdots 2} = \frac{1}{n!}$

∴ $\sum\limits_{n=0}^{\infty} a_n x^n = 4 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = 4 + \sum\limits_{n=1}^{\infty}\frac{x^n}{n!}$

令 $S(x) = \sum\limits_{n=1}^{\infty}\frac{x^n}{n!}$

则 $S'(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = 1 + \sum\limits_{n=1}^{\infty}\frac{x^n}{n!} = 1 + S(x)$

∴ $S(x) = e^x - 1$

∴ $\sum\limits_{n=0}^{\infty} a_n x^n = 4 + S(x) = e^x + 3$

七. 计算积分 $I(\alpha) = \int_0^{\pi} \ln(1 - 2\alpha\cos x + \alpha^2)dx$　$|\alpha|<1$

解: ∵ $I'(\alpha) = \int_0^{\pi} \frac{2\alpha - 2\cos x}{1 - 2\alpha\cos x + \alpha^2}dx = 2\alpha\int_0^{\pi}\frac{1}{1-2\alpha\cos x+\alpha^2}dx - 2\int_0^{\pi}\frac{\cos x}{1-2\alpha\cos x+\alpha^2}dx$

$\xlongequal{t=\tan\frac{x}{2}} 2\alpha\int_0^{+\infty}\frac{1}{(1+\alpha^2) - 2\alpha\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2} - \frac{(1+\alpha^2)}{\alpha}\int_0^{+\infty}\frac{1}{(1+\alpha^2) - 2\alpha\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2} + \int_0^{\pi}\frac{1}{\alpha}dx$

$\cancel{= \frac{\alpha-1}{(\alpha+1)^3}\cdot\frac{\pi}{2} - \frac{(1+\alpha^2)}{\alpha}\cdot\frac{\alpha-1}{(\alpha+1)^3}\cdot\frac{\pi}{2} + \frac{\pi}{\alpha}}$

∴ $\cancel{I(\alpha) = \int_0^{\alpha} I'(t)}dt$

=

=

$\cancel{= \frac{\pi(\alpha^3+6\alpha^2+4\alpha+3)}{2\alpha(\alpha+1)^3}}$

得 $\cancel{I'(\alpha)=\frac{\pi}{\alpha}+\int_0^{+\infty}(\alpha^2-1)}$

$$\cancel{I'(\alpha)=\frac{\pi}{\alpha}+(\alpha^2-1)\int_0^{\pi}\frac{1}{1-2\alpha\cos x+\alpha^2}}\,dx$$

$$\cancel{=\frac{\pi}{\alpha}+(\alpha^2-1)\int_0^{+\infty}\frac{2\,dt}{(1+\alpha^2)-2\alpha\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{1}{1+t^2}}$$

$$=\cdots$$

$$\cancel{=\frac{\pi}{\alpha}+\frac{\pi}{\alpha^2-1}}$$

$$\therefore\ \cancel{I(\alpha)=\int_0^{\alpha}I'(t)\,dt-I(0)=\int_0^{\alpha}I'(t)\,dt=\lim_{b\to0^+}\left(\pi\ln t+\frac{\pi}{2}\ln(t-1)-\frac{\pi}{2}\ln(t+1)\right)\Big|_b^{\alpha}}$$

$$\cancel{=\frac{\pi}{2}\lim_{b\to0}\left[(2\ln\alpha-\ln(\alpha+1)+\ln(\alpha-1))-\ln\frac{b^2(b-1)}{b+1}\right]}.$$

$$I'(\alpha)=\int_0^{\pi}\frac{2\alpha-\cos x}{1+\alpha^2-2\alpha\cos x}\,dx$$

$$=\frac{1}{\alpha}\int_0^{\pi}\frac{2\alpha^2-2\alpha\cos x}{1+\alpha^2-2\alpha\cos x}\,dx$$

$$=\frac{1}{\alpha}\int_0^{\pi}\frac{(\alpha^2-1)+(\alpha^2-2\alpha\cos x+1)}{1+\alpha^2-2\alpha\cos x}\,dx$$

$$=\frac{\alpha^2-1}{\alpha}\int_0^{\pi}\frac{1}{(1+\alpha^2)-2\alpha\cos x}\,dx+\frac{1}{\alpha}\int_0^{\pi}dx$$

$$=\frac{\pi}{\alpha}+\frac{\alpha^2-1}{\alpha}\int_0^{+\infty}\frac{1}{(1+\alpha^2)-2\alpha\cdot\frac{1-t^2}{1+t^2}}\cdot\frac{2\,dt}{1+t^2}$$

$$=\frac{\pi}{\alpha}+\frac{2(\alpha^2-1)}{\alpha}\cdot\frac{1}{(1+\alpha)^2}\int_0^{+\infty}\frac{dt}{t^2+\left(\frac{1-\alpha}{1+\alpha}\right)^2}$$

$$=\frac{\pi}{\alpha}+\frac{2(\alpha-1)}{\alpha(1+\alpha)}\cdot\frac{1+\alpha}{1-\alpha}\arctan t\cdot\frac{1+\alpha}{1-\alpha}\Big|_0^{+\infty}$$

$$=\frac{\pi}{\alpha}-\frac{2}{\alpha}\cdot\frac{\pi}{2}=0$$

$\therefore\ I(\alpha)-I(0)=\int_0^{\alpha}I'(t)\,dt=0.$ 又 $I(0)=0$ $\therefore\ I(\alpha)=0.$