

一. 1. $2xf'_1 + yf'_2 + zg'$

2. $\{\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\}$

3. $3x + y + 2z - 6 = 0$

4. $\frac{14}{3}\pi a^3$

5. $-\frac{2}{9\pi}$

二. 两方程两端分别对 x 求导, 得

$$\begin{cases} 1 = e^u \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \\ y = e^u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} \end{cases} \dots\dots\dots(4 \text{ 分})$$

解得 $\frac{\partial u}{\partial x} = \frac{y}{e^u + 1} \dots\dots\dots(6 \text{ 分})$

$$\frac{\partial v}{\partial x} = 1 - e^u \frac{\partial u}{\partial x} \dots\dots\dots(8 \text{ 分})$$

$$= \frac{(1-y)e^u + 1}{e^u + 1} \dots\dots\dots(9 \text{ 分})$$

三. $I = \int_{-1}^2 dx \int_{x^2}^{x+2} xy dy \dots\dots\dots(4 \text{ 分})$

$$= \frac{1}{2} \int_{-1}^2 x((x+2)^2 - x^4) dx \dots\dots\dots(7 \text{ 分})$$

$$= \frac{45}{8} \dots\dots\dots(9 \text{ 分})$$

四. $\frac{\partial z}{\partial x} = x^2 - y \quad \frac{\partial z}{\partial y} = -x + y - 2 \dots\dots\dots(2 \text{ 分})$

令 $\frac{\partial z}{\partial x} = 0 \quad \frac{\partial z}{\partial y} = 0$ 解得 $x = -1, y = 1$ 或 $x = 2, y = 4 \dots\dots\dots(4 \text{ 分})$

$$\frac{\partial^2 z}{\partial x^2} = 2x \quad \frac{\partial^2 z}{\partial x \partial y} = -1 \quad \frac{\partial^2 z}{\partial y^2} = 1 \dots\dots\dots(6 \text{ 分})$$

在点 $P_1(-1,1)$, $A = -2$ $B = -1$ $C = 1$

$$AC - B^2 = -3 < 0 \quad \text{故 } P_1(-1,1) \text{ 不是极值点} \quad \dots\dots\dots(8 \text{ 分})$$

在点 $P_2(2,4)$ $A = 4$ $B = -1$ $C = 1$

$$AC - B^2 = 3 > 0 \quad \text{且 } A > 0$$

$$\text{故 } P_2(2,4) \text{ 是极小值点, 极小值 } z|_{(2,4)} = -\frac{16}{3} \quad \dots\dots\dots(10 \text{ 分})$$

五. \quad 由题设, 有 $\frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y}$, $\dots\dots\dots(1 \text{ 分})$

$$\varphi'(x) - \frac{2x}{y^2} = \frac{1}{x} - \frac{2x}{y^2} \quad \varphi'(x) = \frac{1}{x} \quad \dots\dots\dots(2 \text{ 分})$$

$$\varphi(x) = \ln x + C \quad \dots\dots\dots(3 \text{ 分})$$

$$\text{由 } \varphi(1) = 0 \quad \text{得 } C = 0 \quad \varphi(x) = \ln x \quad \dots\dots\dots(4 \text{ 分})$$

$$u(x, y) = \int_{(1,1)}^{(x,y)} \left(\frac{y}{x} + \frac{2x}{y} \right) dx + \left(\ln x - \frac{x^2}{y^2} \right) dy \quad \dots\dots\dots(6 \text{ 分})$$

$$= \int_1^y y - \frac{1}{y^2} dy + \int_1^x \left(\frac{y}{x} + \frac{2x}{y} \right) dx \quad \dots\dots\dots(8 \text{ 分})$$

$$= y \ln x + \frac{x^2}{y} - 1 \quad \dots\dots\dots(9 \text{ 分})$$

$$\text{通解为 } y \ln x + \frac{x^2}{y} = C \quad \dots\dots\dots(10 \text{ 分})$$

六. $\quad \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \quad R = 1 \quad \dots\dots\dots(1 \text{ 分})$

$$x=1 \text{ 时, 级数为 } \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} \text{ 收敛, } \quad x=-1 \text{ 时, 级数为 } \sum_{n=0}^{\infty} \frac{1}{n+1} \text{ 发散,}$$

$$\text{收敛域为 } (-1,1] \quad \dots\dots\dots(3 \text{ 分})$$

$$\text{令 } S(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \quad \dots\dots\dots(4 \text{ 分})$$

$$S'(x) = \sum_{n=0}^{\infty} (-1)^n x^n \dots\dots\dots(5 \text{ 分})$$

$$= \frac{1}{1+x} \dots\dots\dots(6 \text{ 分})$$

$$S(x) = \int_0^x \frac{1}{1+x} dx = \ln(1+x) \dots\dots\dots(7 \text{ 分})$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1} = \begin{cases} \frac{1}{x} \ln(1+x) & x \neq 0 \\ 1 & x = 0 \end{cases} \dots\dots\dots(9 \text{ 分})$$

七. $|\vec{F}| = \frac{k}{\sqrt{x^2 + y^2 + z^2}} \quad \vec{F}^0 = \frac{\{-x, -y, -z\}}{\sqrt{x^2 + y^2 + z^2}} \dots\dots\dots(2 \text{ 分})$

$$\vec{F} = -k \frac{\{x, y, z\}}{x^2 + y^2 + z^2} \dots\dots\dots(3 \text{ 分})$$

$$W = -k \int_{AB} \frac{xdx + ydy + zdz}{x^2 + y^2 + z^2} \dots\dots\dots(6 \text{ 分})$$

$$= -k \int_0^{\frac{\pi}{2}} \frac{t}{1+t^2} dt \dots\dots\dots(8 \text{ 分})$$

$$= -\frac{k}{2} \ln(1 + \frac{\pi^2}{4}) \dots\dots\dots(9 \text{ 分})$$

八. $\mu = \sqrt{x^2 + y^2} \dots\dots\dots(1 \text{ 分})$

$$I_z = \iiint_V (x^2 + y^2) \sqrt{x^2 + y^2} dV \dots\dots\dots(2 \text{ 分})$$

$$= \int_0^{2\pi} d\theta \int_0^1 \rho^4 d\rho \int_{\rho^{-1}}^{\sqrt{1-\rho^2}} dz \dots\dots\dots(4 \text{ 分})$$

$$= 2\pi \int_0^1 \rho^4 (\sqrt{1-\rho^2} - \rho + 1) d\rho \dots\dots\dots(6 \text{ 分})$$

$$= 2\pi [\int_0^1 \rho^4 \sqrt{1-\rho^2} d\rho + \int_0^1 (\rho^4 - \rho^5) d\rho] \\ = 2\pi (\int_0^{\frac{\pi}{2}} \sin^4 t \cos^2 t dt + \frac{1}{5} - \frac{1}{6}) \dots\dots\dots(8 \text{ 分})$$

$$= \frac{\pi^2}{16} + \frac{\pi}{15} \dots\dots\dots(9 \text{ 分})$$

九. $f(x) = \frac{1}{2-(x-1)} + \ln(1+(x-1))$ (1 分)

$$= \frac{1}{2} \frac{1}{1 - \frac{x-1}{2}} + \ln(1+(x-1)) \quad \dots\dots\dots(3 \text{ 分})$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n \quad \dots\dots\dots(5 \text{ 分})$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{2^{n+1}} + \frac{(-1)^{n-1}}{n}\right) (x-1)^n \quad \dots\dots\dots(7 \text{ 分})$$

收敛域为 $(-1,3) \cap (0,2] = (0,2]$ (9 分)

十. $I = \oiint_S \left(\frac{\partial u}{\partial x} \cos \alpha + \frac{\partial u}{\partial y} \cos \beta + \frac{\partial u}{\partial z} \cos \gamma \right) dS$ (1 分)

$$= \oiint_S (4x^3 \cos \alpha + 4y^3 \cos \beta + 4z^3 \cos \gamma) dS \quad \dots\dots\dots(2 \text{ 分})$$

$$= \oiint_S 4x^3 dydz + 4y^3 dzdx + 4z^3 dxdy \quad \dots\dots\dots(3 \text{ 分})$$

$$= \iiint_V 12(x^2 + y^2 + z^2) dV \quad \dots\dots\dots(5 \text{ 分})$$

$$= 12 \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^a r^4 \sin \varphi dr \quad \dots\dots\dots(7 \text{ 分})$$

$$= 24\pi \int_0^{\pi} \sin \varphi d\varphi \cdot \int_0^a r^4 dr$$

$$= \frac{48}{5} \pi a^5 \quad \dots\dots\dots(9 \text{ 分})$$

十一. 当 $p \leq 1$, 有 $\frac{\ln n}{n^p} \geq \frac{1}{n^p}$ (2 分)

由于 $\sum_{n=1}^{\infty} \frac{1}{n^p}$ 发散, 故 $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ 发散(3 分)

当 $p > 1$, 令 $\varepsilon = p - 1$, 则 $\varepsilon > 0$, $p - \frac{\varepsilon}{2} > 1$

$$\lim_{n \rightarrow \infty} \frac{\frac{\ln n}{n^p}}{\frac{1}{n^{p-\frac{\varepsilon}{2}}}} = \lim_{n \rightarrow \infty} \frac{\ln n}{n^{\frac{\varepsilon}{2}}} = 0 \quad \dots\dots\dots(6 \text{ 分})$$

由于 $\sum_{n=1}^{\infty} \frac{1}{n^{p-\frac{\varepsilon}{2}}}$ 收敛, 故 $\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$ 收敛(7 分)