一. 1. $\dfrac{1}{3}$

2. $y''' + y'' - y' - y = 0$

3. $\dfrac{1}{2} f'(2)$

4. $\dfrac{\pi}{4}$

5. $-\dfrac{1+x}{x^3 e^{2x}}$

二. $a + b = 3$ ………………(1 分)

        $y' = 3ax^2 + 2bx$ ………………(3 分)

        $y'' = 6ax + 2b$ ………………(5 分)

        $6a + 2b = 0$ ………………(6 分)

  解得 $a = -\dfrac{3}{2}$, $b = \dfrac{9}{2}$ ………………(8 分)

三. 由题意 $\displaystyle\int f(x)dx = \dfrac{\sin x}{x} + C_1$ ……………..(2 分)

       $f(x) = \left(\dfrac{\sin x}{x} + C_1\right)' = \dfrac{x\cos x - \sin x}{x^2}$ ……………..(4 分)

       $\displaystyle\int xf'(x)dx = \int x\, df(x)$ ……………..(5 分)

       $= xf(x) - \displaystyle\int f(x)dx$ ……………..(7 分)

       $= \dfrac{x\cos x - \sin x}{x} - \dfrac{\sin x}{x} + C = \cos x - \dfrac{2\sin x}{x} + C$ ………….(8 分)

四. $1 - \dfrac{dy}{dx} - \sin y \cdot \dfrac{dy}{dx} = 0$ ……………..(3 分)

      $\dfrac{dy}{dx} = \dfrac{1}{1 + \sin y}$ ……………..(4 分)

      $\dfrac{d^2 y}{dx^2} = \dfrac{-\cos y \cdot \dfrac{dy}{dx}}{(1 + \sin y)^2}$ ……………..(6 分)

      $= \dfrac{-\cos y \cdot \dfrac{1}{1 + \sin y}}{(1 + \sin y)^2} = \dfrac{-\cos y}{(1 + \sin y)^3}$ ……………..(8 分)

五. $\displaystyle\int_1^{+\infty}\frac{1}{x^2}\arctan x\,dx = -\int_1^{+\infty}\arctan x\,d\frac{1}{x}$ ·················(1 分)

$\displaystyle = \frac{1}{x}\arctan x\Big|_1^{+\infty} + \int_1^{+\infty}\frac{1}{x(1+x^2)}dx$ ·················(3 分)

$\displaystyle = \frac{\pi}{4} + \int_1^{+\infty}(\frac{1}{x}-\frac{x}{1+x^2})dx$ ·················(5 分)

$\displaystyle = \frac{\pi}{4} + \frac{1}{2}\ln\frac{x^2}{1+x^2}\Big|_1^{+\infty}$ ·················(7 分)

$\displaystyle = \frac{\pi}{4} + \frac{1}{2}\ln 2$ ·················(9 分)

六. 方程化为 $\displaystyle\frac{dy}{dx}-\frac{2}{x}y=1$ ·················(1 分)

$\displaystyle y = e^{-\int-\frac{2}{x}dx}(C+\int e^{\int-\frac{2}{x}dx}dx)$ ·················(3 分)

$\displaystyle = x^2(C+\int\frac{1}{x^2}dx)$

$\displaystyle = x^2(C-\frac{1}{x}) = Cx^2 - x$ ·················(5 分)

$\displaystyle V(C) = \int_0^1 \pi(Cx^2-x)^2 dx$ ·················(7 分)

$\displaystyle = \pi(\frac{1}{5}C^2 - \frac{1}{2}C + \frac{1}{3})$ ·················(8 分)

$\displaystyle V'(C) = \pi(\frac{2}{5}C - \frac{1}{2})$ ·················(9 分)

令 $V'(C)=0$，得 $C=\dfrac{5}{4}$ ·················(10 分)

由于 $V''(C)=\dfrac{2}{5}\pi > 0$，故 $C=\dfrac{5}{4}$ 是极小值点也是最小值点，

所求解为 $y=\dfrac{5}{4}x^2-x$ ·················(11 分)

七.
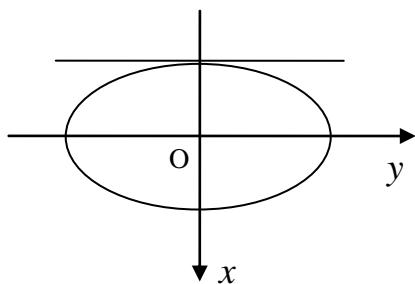
$dP = \mu g(1+x)2y\,dx$ ········(2 分)

$= 4\mu g(1+x)\sqrt{1-x^2}\,dx$ ·····(3 分)

$P = \displaystyle\int_{-1}^1 4\mu g(1+x)\sqrt{1-x^2}\,dx$ ··(5 分)

$= 8\mu g\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$ ········(6 分)

$= 2\mu g\pi = 2000\pi g\ (N)$ ······(8 分)

八.
$$r^2 + r - 2 = 0$$ ....(1 分)

$$r_1 = 1, \quad r_2 = -2$$ ....(3 分)

$$\bar{y} = C_1 e^x + C_2 e^{-2x}$$ ....(5 分)

设 $$y^* = x(Ax + B)e^x$$ ....(7 分)

$$y^{*\prime} = (Ax^2 + 2Ax + Bx + B)e^x$$

$$y^{*\prime\prime} = (Ax^2 + 4Ax + Bx + 2A + 2B)e^x$$

代入方程得 $$6A = 1, \quad 2A + 3B = -1$$ ....(9 分)

解得 $$A = \frac{1}{6}, \quad B = -\frac{4}{9}$$ ....(10 分)

$$y^* = (\frac{x^2}{6} - \frac{4}{9}x)e^x$$

通解为 $$y = C_1 e^x + C_2 e^{-2x} + (\frac{x^2}{6} - \frac{4}{9}x)e^x$$ ....(11 分)

九.
$$f = ma$$ ....(1 分)

$$a = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$ ....(3 分)

得 $$v\frac{dv}{dx} = -\sin x$$ ....(4 分)

$$v\big|_{x=0} = 2$$ ....(5 分)

$$vdv = -\sin x dx$$ ....(6 分)

$$\frac{1}{2}v^2 = \cos x + C$$ ....(7 分)

由初值得 $$C = 1$$

$$v^2 = 2(\cos x + 1)$$ ....(8 分)

十. 设 $\quad f(x) = \ln x - \dfrac{x}{e} + \int_0^1 e^{x^2} dx$ ……………………(1 分)

$$f'(x) = \dfrac{1}{x} - \dfrac{1}{e}$$ ……………………(2 分)

令 $\quad f'(x) = 0$，得 $\quad x = e$ ……………………(3 分)

$f(x)$ 在 $(0, e)$ 和 $(e, +\infty)$ 单调 ……………………(4 分)

$$\lim_{x \to 0^-} f(x) = -\infty, \quad \lim_{x \to +\infty} f(x) = -\infty$$ ……………………(6 分)

$$f(e) = \int_0^1 e^{x^2} dx > 0$$ ……………………(7 分)

故 $f(x)$ 在 $(0, e)$ 和 $(e, +\infty)$ 内各有一不同实根，

所以方程在 $(0, +\infty)$ 内有两个不同实根。 ……………………(9 分)

十一. 令 $\quad x - t = u$，得

$$F(x) = \int_0^x (x - 2u) f(u) du$$ ……………………(1 分)

$$= x \int_0^x f(u) du - 2 \int_0^x u f(u) du$$ ……………………(2 分)

$$F'(x) = \int_0^x f(u) du - x f(x)$$ …………(4 分)

$$= \int_0^x (f(u) - f(x)) du$$

因为 $f(x)$ 单调增加，故 $F'(x) < 0$，所以 $F(x)$ 单调减少 ……(6 分)

$$F(-x) = \int_0^{-x} (-x - 2u) f(u) du$$ ……………………(7 分)

令 $\quad t = -u$，得

$$F(-x) = -\int_0^x (-x + 2t) f(-t) dt$$ ……………………(8 分)

$$= -\int_0^x (x - 2t) f(t) dt$$

$$= -\int_0^x (x - 2u) f(u) du = -F(x)$$

故 $F(x)$ 是奇函数 …………………(10 分)