

(05 数学分析 B 第一学期期末试题(A 卷)) 参考答案 (2006.1)

一. 1. $f(0) = 2$, 得 $d = 2$, -----(1 分)

$$f'(0) = (3ax^2 + 2bx + c)|_{x=0} = 0, \text{ 得 } c = 0, \text{ -----(2 分)}$$

$$f''(-1) = (6ax + 2b)|_{x=-1} = -6a + 2b = 0 \text{ -----(3 分)}$$

$$f(-1) = -a + b - c + d = -a + b + 2 = 4, \text{ -----(4 分)}$$

解得 $a = 1, \quad b = 3, \text{ -----(5 分)}$

因为 $f''(0) = 2b = 6 > 0$, 故 $f(0)$ 是极小值. -----(6 分)

$$2. \lim_{x \rightarrow 0} \frac{x^2 - \int_0^{x^2} \cos t^2 dt}{\int_0^{x^5} (e^x - 1) dx} = \lim_{x \rightarrow 0} \frac{2x - \cos x^4 \cdot 2x}{(e^{x^5} - 1)5x^4} \text{ -----(2 分)}$$

$$= \lim_{x \rightarrow 0} \frac{2(1 - \cos x^4)}{x^5 \cdot 5x^3} \text{ -----(4 分)}$$

$$= \lim_{x \rightarrow 0} \frac{2 \cdot \frac{1}{2} x^8}{5x^8} = \frac{1}{5}. \text{ -----(6 分)}$$

3. 令 $u = \tan x$, $u|_{x=0} = 0$,

$$\frac{dy}{dx} = f'(u) \frac{du}{dx} = e^{u^2 - 2u + 2} \frac{1}{\cos^2 x}, \text{ -----(5 分)}$$

$$\frac{dy}{dx}|_{x=0} = e^2. \text{ -----(6 分)}$$

4. 令 $t = \sqrt{1-x^2}$, 即 $x^2 = 1-t^2$, -----(1 分)

$$\int_0^1 \frac{x dx}{(3+x^2)\sqrt{1-x^2}} = \int_0^1 \frac{dt}{4-t^2} \text{ -----(3 分)}$$

$$= \frac{1}{4} \int_0^1 \left(\frac{1}{t+2} - \frac{1}{t-2} \right) dt \text{ -----(4 分)}$$

$$= \frac{1}{4} \ln \left| \frac{t+2}{t-2} \right| \Big|_0^1 = \frac{1}{4} \ln 3 \text{ -----(6 分)}$$

二. 1. $x = \theta \cos \theta, y = \theta \sin \theta,$ -----(1 分)

$$\frac{dy}{dx} = \frac{\sin \theta + \theta \cos \theta}{\cos \theta - \theta \sin \theta} \quad \text{-----}(3 \text{ 分})$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi} = \pi, \quad x|_{\theta=\pi} = -\pi, \quad y|_{\theta=\pi} = 0 \quad \text{-----}(5 \text{ 分})$$

法线方程为 $y = -\frac{1}{\pi}(x + \pi) = -\frac{1}{\pi}x - 1.$ -----(7 分)

2. $\int_0^{\frac{\pi}{2}} \frac{x - \sin x}{1 + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{x}{1 + \cos x} dx - \int_0^{\frac{\pi}{2}} \frac{\sin x}{1 + \cos x} dx$ -----(1 分)

$$= \int_0^{\frac{\pi}{2}} \frac{x}{2 \cos^2 \frac{x}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx \quad \text{-----}(2 \text{ 分})$$

$$= x \tan \frac{x}{2} \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx - \int_0^{\frac{\pi}{2}} \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} dx$$

$$= \frac{\pi}{2} + 4 \ln \left| \cos \frac{x}{2} \right| \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - \ln 4. \quad \text{-----}(7 \text{ 分})$$

3. $f(x) + \cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + 1 - \frac{x^2}{2!} + o(x^2)$ -----(2 分)

由题设, 得 $f(0) + 1 = 0, \quad f'(0) = 0, \quad \frac{f''(0)}{2} - \frac{1}{2} = 1,$ -----(5 分)

解得 $f(0) = -1, \quad f'(0) = 0, \quad f''(0) = 3.$ -----(7 分)

4. 方程变成 $\frac{dy}{dx} = \frac{x^2 y}{x^3 + y^3} = \frac{\frac{y}{x}}{1 + (\frac{y}{x})^3},$ -----(1 分)

令 $\frac{y}{x} = u,$ 即 $y = xu, \quad \frac{dy}{dx} = u + x \frac{du}{dx},$ -----(3 分)

方程变成 $u + x \frac{du}{dx} = \frac{u}{1 + u^3}, \quad \frac{1 + u^3}{u^4} du = -\frac{dx}{x},$ -----(4 分)

积分得 $-\frac{1}{3u^3} + \ln|u| = -\ln|x| + C_1,$ -----(6 分)

即 $\ln|y| = \frac{x^3}{3y^3} + C_1, \quad y = C e^{\frac{x^3}{3y^3}}.$ -----(7 分)

$$\text{三. } \int_0^{\pi} [f''(x) + f(x)] \sin x dx = \int_0^{\pi} f''(x) \sin x dx - \int_0^{\pi} f(x) d \cos x \quad \text{-----}(2 \text{ 分})$$

$$= \int_0^{\pi} f''(x) \sin x dx - f(x) \cos x \Big|_0^{\pi} + \int_0^{\pi} f'(x) \cos x dx \quad \text{-----}(4 \text{ 分})$$

$$= \int_0^{\pi} f''(x) \sin x dx + f(\pi) + f(0) + \int_0^{\pi} f'(x) d \sin x$$

$$= \int_0^{\pi} f''(x) \sin x dx + 2 + f(0) + f'(x) \sin x \Big|_0^{\pi} - \int_0^{\pi} f''(x) \sin x dx$$

$$= 2 + f(0) = 5 \quad \text{-----}(7 \text{ 分})$$

$$\therefore f(0) = 3. \quad \text{-----}(8 \text{ 分})$$

四. (图 1) 过 $M(x, y)$ 的切线 $Y - y = y'(X - x)$,

$$\text{令 } X = 0, \quad \text{得 } Y = y - xy', \quad \text{-----}(2 \text{ 分})$$

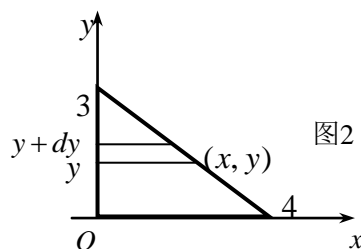
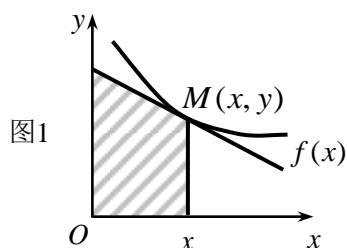
$$\text{梯形面积 } A = \frac{1}{2}(y + y - xy')x = \frac{1}{2}x(2y - xy') = 3,$$

$$\text{即 } y' - \frac{2}{x}y = -\frac{6}{x^2}, \quad y(1) = 1, \quad \text{-----}(5 \text{ 分})$$

$$\text{解得 } y = e^{\int \frac{2}{x} dx} (C + \int -\frac{6}{x^2} e^{\int \frac{2}{x} dx} dx) = Cx^2 + \frac{2}{x}, \quad \text{-----}(7 \text{ 分})$$

$$\text{由初始条件, 得 } C + 2 = 1, \quad C = -1,$$

$$\text{所求曲线为 } y = -x^2 + \frac{2}{x}. \quad \text{-----}(8 \text{ 分})$$



$$\text{五. (如图 2)} \quad x = 4 - \frac{4}{3}y, \quad \text{-----}(1 \text{ 分})$$

$$\rho = \frac{500}{\frac{1}{2} \times 3 \times 4} = \frac{250}{3} (\text{kg/m}^2), \quad \text{-----}(2 \text{ 分})$$

$$dW = \rho g \cdot y \cdot x dy = \rho g \cdot y (4 - \frac{4}{3}y) dy, \quad \text{-----}(5 \text{ 分})$$

$$W = \int_0^3 \rho g (4y - \frac{4}{3}y^2) dy \quad \text{-----}(7 \text{ 分})$$

$$= 6\rho g = 500g (J). \quad \text{-----}(8 \text{ 分})$$

六. 方程两边对 x 求导, 得

$$y'' + 3y' + 2y = 2\sin x,$$

$$y(0) = -1, \quad y'(0) = 1, \quad \text{-----}(2 \text{ 分})$$

$$r^2 + 3r + 2 = 0, \quad r_1 = -1, \quad r_2 = -2, \quad \text{-----}(4 \text{ 分})$$

$$\bar{y} = C_1 e^{-x} + C_2 e^{-2x}, \quad \text{-----}(5 \text{ 分})$$

设 $y^* = A \cos x + B \sin x, \quad \text{-----}(6 \text{ 分})$

代入方程解得 $A = -\frac{3}{5}, \quad B = \frac{1}{5},$

$$y^* = -\frac{3}{5} \cos x + \frac{1}{5} \sin x, \quad \text{-----}(7 \text{ 分})$$

通解为 $y = C_1 e^{-x} + C_2 e^{-2x} - \frac{3}{5} \cos x + \frac{1}{5} \sin x, \quad \text{-----}(8 \text{ 分})$

由初始条件得 $C_1 = 0, \quad C_2 = -\frac{2}{5},$

$$y = -\frac{2}{5} e^{-2x} - \frac{3}{5} \cos x + \frac{1}{5} \sin x. \quad \text{-----}(10 \text{ 分})$$

七. 证 1 由题设, 得 $f(0) = \lim_{x \rightarrow 0} f(x) = 0, \quad \text{-----}(1 \text{ 分})$

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x)}{x} = 1, \quad \text{-----}(2 \text{ 分})$$

当 $x \neq 0$, $f(x) = f(0) + f'(0)x + \frac{f''(\xi)}{2}x^2 = x + \frac{f''(\xi)}{2}x^2 > x. \quad \text{-----}(7 \text{ 分})$

证 2 令 $F(x) = f(x) - x, \quad \text{-----}(1 \text{ 分})$

$$F'(x) = f'(x) - 1,$$

由题设, 得 $f(0) = \lim_{x \rightarrow 0} f(x) = 0, \quad \text{-----}(3 \text{ 分})$

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x)}{x} = 1, \quad F'(0) = 0, \quad \text{-----}(4 \text{ 分})$$

$$F''(x) = f''(x) > 0,$$

$F(0) = 0$ 是极小值也是最小值, 故当 $x \neq 0$, 有 $F(x) > 0$, 即

$$f(x) > x. \quad \text{-----}(7 \text{ 分})$$

八. (1) 由 $\lim_{x \rightarrow +\infty} [f(x) - f(\frac{1}{x})] = 1$, 得 $\lim_{x \rightarrow 0^+} [f(\frac{1}{x}) - f(x)] = 1$,

$$\lim_{x \rightarrow 0^+} f(x) = \frac{1}{2} [\lim_{x \rightarrow 0^+} [f(x) + f(\frac{1}{x}) + f(x) - f(\frac{1}{x})] = -\frac{1}{2} < 0,$$

$$\lim_{x \rightarrow +\infty} f(x) = -\lim_{x \rightarrow 0^+} f(\frac{1}{x}) = \frac{1}{2} > 0,$$

故 $\exists \xi \in (0, +\infty)$, 使 $f(\xi) = 0$; -----(4 分)

(2) 令 $F(x) = f(x)e^x$,

若 $\exists \xi_1, \xi_2 \in (0, +\infty)$, $\xi_1 < \xi_2$, 使 $f(\xi_1) = f(\xi_2) = 0$,

则 $F(\xi_1) = F(\xi_2) = 0$,

因而 $\exists c \in (\xi_1, \xi_2)$, 使 $F'(c) = 0$,

但由题设, $F'(x) = [f'(x) + f(x)]e^x > 0$,

矛盾, 故 $y = f(x)$ 在 $(0, +\infty)$ 内只有一个零点. -----(7 分)