2005-2006 学年第二学期期中考试参考答案及评分标准

一、1 $\overrightarrow{AB}=\{1,2,0\}$，$\overrightarrow{AC}=\{0,1,2\}$ **3 分**

$$\cos\angle A=\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}=\frac{2}{5},\quad \therefore\angle A=\arccos\frac{2}{5}.$$ **7 分**

2 $\dfrac{\partial z}{\partial x}=f_1'\tan y+\dfrac{1}{y}f_2'$ **3 分**

$$\frac{\partial^2 z}{\partial x\partial y}=\sec^2 y\ f_1'+\tan y(x\sec^2 y\ f_{11}''-\frac{x}{y^2}\ f_{12}'')-\frac{1}{y^2}f_2'$$

$$+\frac{1}{y}(x\sec^2 y\ f_{21}''-\frac{x}{y^2}\ f_{22}'')$$ **6 分**

$$\frac{\partial^2 z}{\partial x\partial y}=\sec^2 y\ f_1'-\frac{1}{y^2}f_2'+x\tan y\sec^2 y\ f_{11}''+\frac{x}{y}\ f_{12}''(\sec^2 y-\frac{\tan y}{y})$$

$$-\frac{x}{y^3}f_{22}''.$$ **7 分**

3 $\dfrac{\partial u}{\partial x}=y^x\ln y+\dfrac{z}{x^2+z^2},\quad \dfrac{\partial u}{\partial y}=xy^{x-1},\quad \dfrac{\partial u}{\partial z}=\dfrac{-x}{x^2+z^2},$ **2 分**

$$\left.\frac{\partial u}{\partial x}\right|_A=-\frac{1}{4},\quad \left.\frac{\partial u}{\partial y}\right|_A=2,\quad \left.\frac{\partial u}{\partial z}\right|_A=-\frac{1}{4},$$ **3 分**

$gradu\big|_A=\{-\dfrac{1}{4},2,-\dfrac{1}{4}\}$，又 $\overrightarrow{AB}=\{0,4,3\},\ \vec{l}=\overrightarrow{AB}^0=\{0,\dfrac{4}{5},\dfrac{3}{5}\}.$ **5 分**

$$\left.\frac{\partial u}{\partial l}\right|_A=-\frac{1}{4}\times 0+2\times\frac{4}{5}-\frac{1}{4}\times\frac{3}{5}=\frac{29}{20}.$$ **7 分**

4 $\vec{s}_1=\{2,3,-1\},\vec{s}_2=\{-1,1,-1\}$，$M_1(1,1,0)\in L_1,M_2(-1,-2,1)\in L_2$，

$$\overrightarrow{M_1M_2}=\{-2,-3,1\},\ (\vec{s}_1,\vec{s}_2,\overrightarrow{M_1M_2})=\begin{vmatrix}2&3&-1\\-1&1&-1\\-2&-3&1\end{vmatrix}=0$$

$\therefore L_1,L_2$相交。 **3 分**

平面的法向量为：$\vec{n}=\vec{s}_1\times\vec{s}_2=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & 3 & -1\\ -1 & 1 & -1\end{vmatrix}=\{-2,3,5\}$ **5 分**

所以平面方程为：$2x-3y-5z+1=0.$ **7 分**

5 $$I=\int_0^{\frac{\pi}{2}}dx\int_0^{\cos x}f(x,y)dy+\int_{\frac{\pi}{2}}^{\pi}dx\int_0^{\cos x}f(x,y)dy$$

$$=\int_0^{\frac{\pi}{2}}dx\int_0^{\cos x}f(x,y)dy+\int_{\frac{\pi}{2}}^{\pi}dx\int_{\cos x}^{0}-f(x,y)dy$$ **3 分**

$$=\int_0^{1}dy\int_0^{\arccos y}f(x,y)dx-\int_{-1}^{0}dy\int_{\arccos y}^{\pi}f(x,y)dx$$ **7 分**

二、1 $$I=\iiint\limits_V(x+y+z)dxdydz$$

$$=\int_0^1dx\int_0^xdy\int_0^{xy}(x+y+z)dz$$ **3 分**

$$=\int_0^1dx\int_0^x[(x+y)xy+\frac{1}{2}x^2y^2]dy$$

$$=\int_0^1[\frac{1}{2}x^4+\frac{1}{3}x^4+\frac{1}{36}x^5]dx=\frac{7}{36}.$$ **7 分**

2 $$\frac{du}{dx}=f_1'+f_2'\frac{dy}{dx}+f_3'\frac{dz}{dx},$$ **2 分**

$$e^{xy}(y+x\frac{dy}{dx})-1=0\quad\Rightarrow\quad\frac{dy}{dx}=\frac{1-ye^{xy}}{xe^{xy}},$$ **4 分**

$$e^z\frac{dz}{dx}-\sin z-x\cos z\frac{dz}{dx}=0\quad\Rightarrow\quad\frac{dz}{dx}=\frac{\sin z}{e^z-x\cos z}.$$ **6 分**

$$du=[f_1'+f_2'\frac{1-ye^{xy}}{xe^{xy}}+f_3'\frac{\sin z}{e^z-x\cos z}]dx$$ **7 分**

3 V 关于 xoy 面对称，

$$I=\iiint\limits_V z^2dxdydz+\iiint\limits_V z\cos(x+y^2)dxdydz$$

$$=2\iiint_V z^2\,dxdydz+0$$ **2 分**

$$=2\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^R r^4\cos^2\varphi\sin\varphi dr$$ **5 分**

$$=\frac{4\pi R^5(2\sqrt{2}-1)}{15}.$$ **7 分**

4 $\frac{\partial f}{\partial x}=2x+4y-2=0$ 解得驻点 $x=2$

$\frac{\partial f}{\partial y}=4x+18y+1=0$ $y=-\frac{1}{2}$ **2 分**

$A=\frac{\partial^2 f}{\partial x^2}=2>0,\quad B=\frac{\partial^2 f}{\partial x\partial y}=4,\quad C=\frac{\partial^2 f}{\partial y^2}=18.$ **4 分**

$B^2-AC=16-36=-20<0,$

所以 $f(x,y)$ 在驻点 $(2,-\frac{1}{2})$ 处取得极值，且为极小值。

极小值点为 $(2,-\frac{1}{2})$, 极小值为 $f(2,-\frac{1}{2})=-\frac{9}{4}.$ **7 分**

三、设切点为 (x_0,y_0,z_0). 则切平面的法向量为 $\vec{n}=\{2x_0,2y_0,2z_0\}$,

直线的方向向量为 $\vec{s}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -1 & 1\\ 2 & -1 & 3\end{vmatrix}=\{-2,-1,1\}$, **2 分**

直线垂直于平面，故有

$\vec{n}\,//\,\vec{s}$ ，$\frac{2x_0}{-2}=\frac{2y_0}{-1}=\frac{2z_0}{1}$，又切点在球面上，有 $x_0^2+y_0^2+z_0^2=4$

解得 $(2\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{3}}),\quad(-2\sqrt{\frac{2}{3}},-\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}),$ **5 分**

切平面方程为 $2x+y-z-6\sqrt{\frac{2}{3}}=0,\quad 2x+y-z+6\sqrt{\frac{2}{3}}=0,$ **7 分**

四、$\frac{\partial z}{\partial x}=f'\frac{\partial u}{\partial x},\qquad \frac{\partial z}{\partial y}=f'\frac{\partial u}{\partial y},$ **2 分**

$$g'(u)\frac{\partial u}{\partial x}-2x\varphi(x^2)=0\quad\Rightarrow\quad\frac{\partial u}{\partial x}=\frac{2x\varphi(x^2)}{g'(u)},$$ **4 分**

$$g'(u)\frac{\partial u}{\partial y}+2y\varphi(y^2)=0\quad\Rightarrow\quad\frac{\partial u}{\partial y}=\frac{-2y\varphi(y^2)}{g'(u)},$$ **6 分**

$$y\varphi(y^2)\frac{\partial z}{\partial x}+x\varphi(x^2)\frac{\partial z}{\partial y}$$

$$=y\varphi(y^2)f'\frac{2x\varphi(x^2)}{g'(u)}+x\varphi(x^2)f'\frac{(-2y\varphi(y^2))}{g'(u)}=0$$ **7 分**

五、求交线：$\begin{cases}z=x^2+y^2\\ z=2x\end{cases}$ 得 $x^2+y^2=2x$

区域Ω在 xoy 面上的投影域为：$x^2+y^2\le 2x.$ **2 分**

$$J_z=\iiint_{\Omega}y^2(x^2+y^2)dxdydz$$ **4 分**

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}\rho d\rho\int_{\rho^2}^{2\rho\cos\theta}\rho^4\sin^2\theta dz$$ **6 分**

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}\rho^5\sin^2\theta(2\rho\cos\theta-\rho^2)d\rho$$

$$=2^9\left(\frac{1}{7}-\frac{1}{8}\right)\int_0^{\frac{\pi}{2}}(\cos^8\theta-\cos^{10}\theta)d\theta=\frac{\pi}{8}.$$ **8 分**

六、$V=\iint_D(x+y)dxdy$ **2 分**

$$=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}d\theta\int_0^{\cos\theta+\sin\theta}\rho^2(\cos\theta+\sin\theta)d\rho$$ **4 分**

$$=\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\frac{1}{3}(\cos\theta+\sin\theta)^4d\theta=\frac{4}{3}\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}\sin^4\left(\theta+\frac{\pi}{4}\right)d\theta$$

$$=\frac{4}{3}\int_0^{\pi}\sin^4 t\,dt\quad(\text{令 }t=\theta+\frac{\pi}{4})$$

$$=\frac{4}{3}[\int_0^{\frac{\pi}{2}}\sin^4 tdt+\int_{\frac{\pi}{2}}^{\pi}\sin^4 tdt] \ =\frac{\pi}{2}$$ **6 分**

$$S=\iint_D\sqrt{1+z_x'^2+z_y'^2}dxdy=\sqrt{3}\iint_D dxdy=\frac{\sqrt{3}}{2}\pi.$$ **9 分**

七、目标函数：$u=xyz$

约束条件：$ax+by+cz=2S$ **2 分**

令 $F(x,y,z)=xyz+\lambda(ax+by+cz-2S)$

$$\begin{cases}F_x'=yz+a\lambda=0\\F_y'=xz+b\lambda=0\\F_z'=xy+c\lambda=0\\ax+by+cz=2S\end{cases}$$ 得唯一驻点 $$\begin{cases}x=\dfrac{2S}{9a}\\y=\dfrac{2S}{9b}\\z=\dfrac{2S}{9c}\end{cases}$$ **4 分**

所以当点 P 位于三角形内部且距三边长度分别为$\dfrac{2S}{9a},\dfrac{2S}{9b},\dfrac{2S}{9c}$时，$xyz$ 最大，最大

值为$\dfrac{8S^3}{729abc}$. **6 分**