

一. 1. $-\frac{2}{\pi}$

2. $\frac{f'(x)}{1+f^2(x)} + g'(\sqrt{x^2+1}) \frac{x}{\sqrt{x^2+1}}$

3. $-\frac{1}{1+\tan x}$

4. $\frac{dx}{dt} = kx(N-x)$

5. $\frac{e^4+1}{4}$

二. $\lim_{x \rightarrow 0} \frac{x - \arcsin x}{e^{x^3} - 1} = \lim_{x \rightarrow 0} \frac{x - \arcsin x}{x^3} \dots\dots\dots(3 \text{ 分})$

$$= \lim_{x \rightarrow 0} \frac{1 - \frac{1}{\sqrt{1-x^2}}}{3x^2} \dots\dots\dots(6 \text{ 分})$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{1-x^2} - 1}{3x^2 \sqrt{1-x^2}} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}(-x^2)}{3x^2 \sqrt{1-x^2}} \\ = -\frac{1}{6} \dots\dots\dots(9 \text{ 分})$$

三. $\frac{1}{\cos^2(x+y)}(1 + \frac{dy}{dx}) = y^2 + 2xy \frac{dy}{dx} \dots\dots\dots(6 \text{ 分})$ (左右侧各 3 分)

解得 $\frac{dy}{dx} = \frac{1 - y^2 \cos^2(x+y)}{2xy \cos^2(x+y) - 1} \dots\dots\dots(7 \text{ 分})$

在已知方程中令 $x=0$, 得 $\tan y=1$, $y=\frac{\pi}{4} \dots\dots\dots(8 \text{ 分})$

$$\frac{dy}{dx} \Big|_{x=0} = \frac{1 - (\frac{\pi}{4})^2 \cos^2 \frac{\pi}{4}}{-1} = \frac{1}{32} \pi^2 - 1 \dots\dots\dots(9 \text{ 分})$$

四. 令 $\frac{y}{x} = u$, 即 $y = xu$, $\frac{dy}{dx} = u + x \frac{du}{dx}$ (2 分)

原方程化成 $x \frac{du}{dx} = \tan u$ (4 分)

$$\frac{\cos u}{\sin u} du = \frac{dx}{x} \quad \text{.....(5 分)}$$

$$\ln|\sin u| = \ln|x| + C_1 \quad \text{.....(7 分)}$$

$$\sin u = Cx \quad \text{.....(8 分)}$$

原方程通解为 $\sin \frac{y}{x} = Cx$ (9 分)

五.
$$f(x) = \begin{cases} \frac{1}{x^2} & x > 1 \\ 0 & x = 1 \\ \frac{-2x+1}{x^2+1} & 0 \leq x < 1 \end{cases} \quad \text{.....(3 分)}$$

$$\int_0^{+\infty} f(x) dx = \int_0^1 \frac{-2x+1}{x^2+1} dx + \int_1^{+\infty} \frac{1}{x^2} dx \quad \text{.....(5 分)}$$

$$= (-\ln(x^2+1) + \arctan x) \Big|_0^1 - \frac{1}{x} \Big|_1^{+\infty} \quad \text{.....(8 分)}$$

$$= -\ln 2 + \frac{\pi}{4} + 1 \quad \text{.....(9 分)}$$

六. 设 $f(x) = \sin^3 x \cos x - a$ (1 分)

$$f'(x) = 3\sin^2 x \cos^2 x - \sin^4 x \quad \text{.....(2 分)}$$

令 $f'(x) = 0$ 得 $x = \frac{\pi}{3}$ $x = \frac{2\pi}{3}$ (3 分)

$f(x)$ 在 $(0, \frac{\pi}{3})$, $(\frac{\pi}{3}, \frac{2\pi}{3})$, $(\frac{2\pi}{3}, \pi)$ 内单调

$$f(0) = -a < 0 \quad f(\pi) = -a < 0 \quad f(\frac{2\pi}{3}) = -\frac{3\sqrt{3}}{16} - a < 0$$

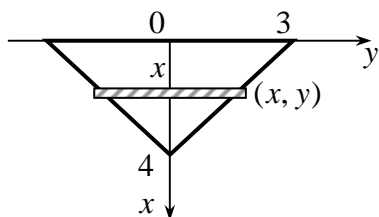
$$f(\frac{\pi}{3}) = \frac{3\sqrt{3}}{16} - a \quad \text{.....(6 分)}$$

当 $a < \frac{3\sqrt{3}}{16}$, $f(\frac{\pi}{3}) > 0$, 方程有两个不同实根.

当 $a = \frac{3\sqrt{3}}{16}$, $f(\frac{\pi}{3}) = 0$, 方程有一个实根.

当 $a > \frac{3\sqrt{3}}{16}$, $f(\frac{\pi}{3}) < 0$, 方程没有实根.(9 分)

七.



$$y = 3 - \frac{3}{4}x \quad \dots\dots\dots(1 \text{ 分})$$

$$dW = x\mu g \pi y^2 dx = \pi\mu g x(3 - \frac{3}{4}x)^2 dx \quad \dots\dots\dots(4 \text{ 分})$$

$$W = \int_0^4 \pi\mu g x(3 - \frac{3}{4}x)^2 dx \quad \dots\dots\dots(6 \text{ 分})$$

$$= \int_0^4 \frac{9}{16} \pi\mu g (16x - 8x^2 + x^3) dx$$

$$= 12\pi\mu g = 12000\pi g \text{ (J)} \quad \dots\dots\dots(9 \text{ 分})$$

八.

$$r^2 - \frac{1}{2}r - \frac{1}{2} = 0 \quad \dots\dots\dots(1 \text{ 分})$$

$$r_1 = 1 \quad r_2 = -\frac{1}{2} \quad \dots\dots\dots(2 \text{ 分})$$

$$\bar{y} = C_1 e^x + C_2 e^{-\frac{x}{2}} \quad \dots\dots\dots(4 \text{ 分})$$

设

$$y^* = x(Ax + B)e^x \quad \dots\dots\dots(6 \text{ 分})$$

代入方程得

$$A = \frac{2}{3} \quad B = -\frac{8}{9}$$

$$y^* = (\frac{2}{3}x^2 - \frac{8}{9}x)e^x \quad \dots\dots\dots(8 \text{ 分})$$

通解

$$y = C_1 e^x + C_2 e^{-\frac{x}{2}} + (\frac{2}{3}x^2 - \frac{8}{9}x)e^x \quad \dots\dots\dots(9 \text{ 分})$$

九.

由二曲线相切得

$$ax^2 = \ln x \quad 2ax = \frac{1}{x}$$

解得

$$a = \frac{1}{2e} \quad \dots\dots\dots(3 \text{ 分})$$

$$A = \int_0^{\frac{1}{2}} (e^y - \sqrt{2ey}) dy \quad \dots\dots\dots(2 \text{ 分})$$

$$= (e^y - \sqrt{2e} \frac{2}{3} y^{\frac{3}{2}}) \Big|_0^{\frac{1}{2}} = \frac{2}{3} \sqrt{e} - 1 \quad \dots\dots\dots(7 \text{ 分})$$

$$V = \int_0^{\frac{1}{2}} 2\pi y (e^y - \sqrt{2ey}) dy \quad \dots\dots\dots(9 \text{ 分})$$

$$= 2\pi (ye^y - e^y - \sqrt{2e} \frac{2}{5} e^{\frac{5}{2}}) \Big|_0^{\frac{1}{2}}$$

$$= 2\pi (1 - \frac{3}{5} \sqrt{e}) \quad \dots\dots\dots(11 \text{ 分})$$

十. 令 $x-t=u$ $\int_0^x g(x-t)dt = \int_0^x g(u)du$ (2 分)

$$f(x) = -2x^2 + \int_0^x g(u)du$$

$$f'(x) = -4x + g(x) \quad \text{.....(4 分)}$$

由题设及 $\lim_{x \rightarrow 0} \frac{g(x)}{x} = 0$, 得 $g(0) = \lim_{x \rightarrow 0} g(x) = 0$ (5 分)

$$g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x} = 0 \quad \text{.....(6 分)}$$

$$f'(0) = 0 \quad \text{故 } x=0 \text{ 是驻点} \quad \text{.....(7 分)}$$

$$f''(x) = -4 + g'(x) \quad \text{.....(8 分)}$$

$$f''(0) = -4 < 0$$

故 $x=0$ 是极值点, 且 $f(0)$ 是极大值(9 分)

十一. 令 $F(x) = f(x)\cos x$ (1 分)

$$F\left(\frac{\pi}{2}\right) = 0 \quad \text{.....(2 分)}$$

由 $\int_0^{\frac{\pi}{4}} f(x)\cos^2 x dx = 0$, 及积分中值定理, $\exists c \in [0, \frac{\pi}{4}]$, 使

$$f(c)\cos^2 c \cdot \frac{\pi}{4} = 0 \quad \text{.....(4 分)}$$

因为 $\cos c \cdot \frac{\pi}{4} \neq 0$, 故有 $F(c) = f(c)\cos c = 0$ (5 分)

根据罗尔中值定理, $\exists \xi \in (c, \frac{\pi}{2}) \subset (0, \frac{\pi}{2})$, 使

$$F'(\xi) = 0$$

即 $f'(\xi)\cos \xi + f(\xi)(-\sin \xi) = 0$

$$f'(\xi) = f(\xi)\tan \xi \quad \text{.....(7 分)}$$