Final Project of Non-Linear Vibrations

Linear and Non-Linear Responses using Harmonic Balance Method

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INTRODUCTION

Newton Harmonic Balance Method (NHBM) is applied to investigate frequency and response of the system with periodic behavior. Non-Linear oscillation problems are important issues in physical science, mechanical structures & engineering researches because most of the real problems are modelled by using Non-Linear differential equations. The vibration response, stability and frequencies are the basic terms in oscillatory systems. So, we carried out a small study on different Linear and Non-Linear systems. The main challenges associated with non-linear response are unpredictability of the amplitude, resonance (we can observe when driving frequency is almost equal to natural frequency), bifurcations of responses and chaotic responses. These are the erratic responses that are observed. So, the study of these type of responses in different ways like transforming into frequency domain (i.e., by using Harmonic Balance Method) or transforming into different co-ordinates may approximate the impossible calculations either analytically and computationally and resulting with converged solutions.

DESCRIPTION

Liner Equation without Damping

Initially, Harmonic Balance method programming has done on the linear oscillating member to compare the results with the Analytical solutions. The equation that we have taken is,

$$\ddot{x} + x = \sin(\omega t)$$

The above equation is linear with a forcing function in sine terms. Our study is in discrete interval of times, so we considered some equally spaced points along time period by creating a sampled frequency (f). For this analysis, we have to consider the initial displacement/response function. Therefore, we considered

Initial response function, $x=X \sin(\omega t)$

Where, X = Amplitude of the Linear Differential Equation

ω = Driving frequency

t = Equal time space interval corresponding to the frequency

Initially, the entire variables have been declared globally for utilizing the variables in multiple functions without defining repeatedly. And continued with allocation of variables with numerals according to the requirement.

```
global N T w t F W X          %Global declaration of variables
N=19;                               %Assumed number of points
T=100;
w=2*pi/T;
t=linspace(0,T,N+1);          %Equally divided points in continuous system
t=t(1:end-1);                 %N time interval in the current system
F=sin(w*t);                   %forcing term
iw=(0:ceil(N-1)/2)*1i*(w);
miw=(-1i)*(floor (N/2):-1:1)*(w);
W= [iw, miw];                 %Appending the frequency terms
X=fft (F);                    %Forcing term in frequency domain
```

Secondly, based on the equation we found the acceleration, velocity in frequency domain by transferring the response function from time domain to frequency domain using Fast Fourier Transforms (FFT) and that is converted into time domain with the help of Inverse Fast Fourier Transforms (IFFT). Finally, the residue of the linear equation has been calculated.

```
Function residue=error ()
dotX=W.*X;
dotx=ifft(dotX);
ddotX=(W.^2).*X;
ddotx=ifft(ddotX);
x0=ddotx/W.^2;
residue=@(X) sum (abs (ddotx+x0-F))
```

The residue is to be minimized or optimized for converging the FFT solution w.r.t Analytical solution. Thus the minimized results are the responses of the Harmonic Balance Method. The below function is used in MATLAB for converging the equation. For this, initial known response value to be given for optimizing the response according to the requirement.

```
x=fminsearch (error (), X);
```

"fminsearch" is the term used for optimizing the response function. After optimization, the results are plotted with Analytical solution responses by multiplying with Amplitude, $\frac{1}{1-\omega^2}$

$$x = (\frac{1}{1-\omega^2}) * \sin(\omega t)$$

The optimized value of response using fminsearch is <u>-582.8640 e-018</u>

The below plots are taken in between Analytical response vs FFT response over a period of time and ode45 vs FFT response over a period of time.

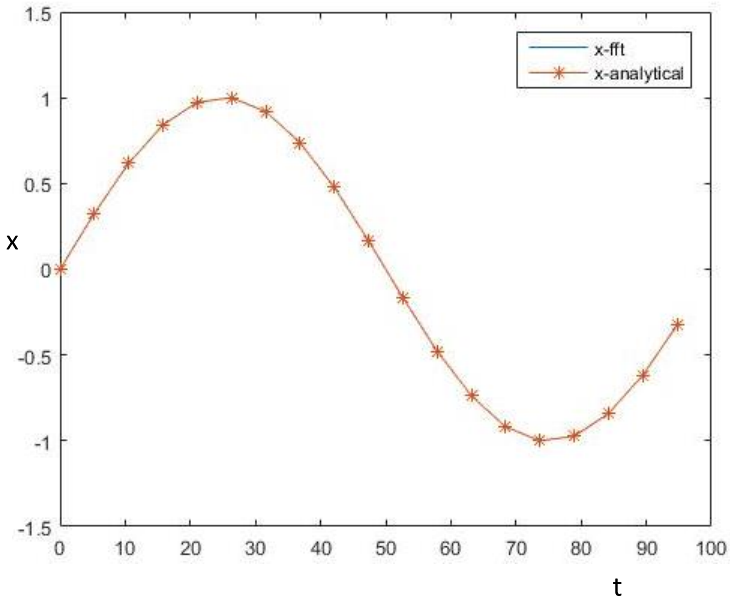

Fig1. X_fft vs X_analytical

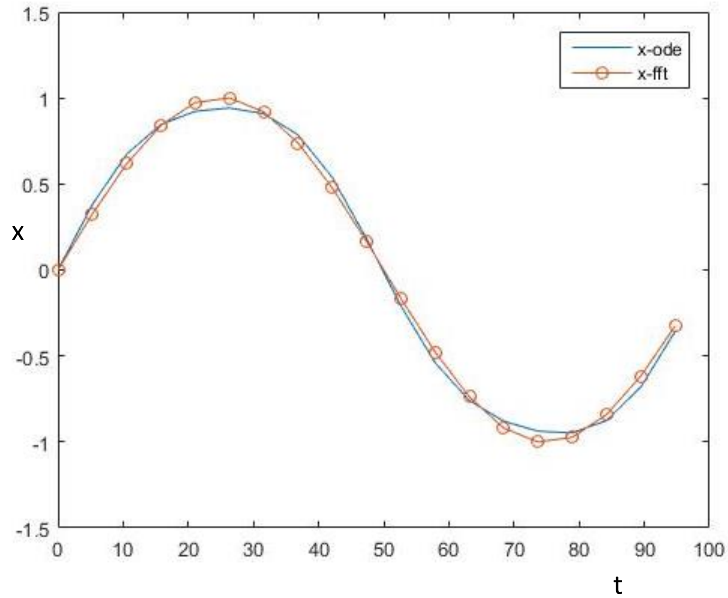

Fig2. X_ode45 vs X_fft

Liner Equation with Damping

Similar to the above procedure, the response curves have been verified by adding damping term to the linear equation. The equation is,

$$\ddot{x} + \dot{x} + x = \sin(\omega t)$$

The steady state solution of the above linear damped equation is,

$$x = \frac{\omega \cos(\omega t) - (1 - \omega^2) * \sin(\omega t)}{\omega^2 - \omega^2 - 1}$$

A graphical comparison of Analytical and FFT solution after the approximation can found below. The Harmonic Balance approximation is built by using 11 sample points and discretized by using circles on the plots.

"fminsearch" is the term used for optimizing the response function. After optimization, the results are plotted with Analytical solution responses by multiplying with Amplitude,

The optimized value of response using fminsearch is <u>-426.3639e-018.</u>

<u>Response of a Linear Differential Equation with Damping and Forcing Function using Harmonic Balance Method</u>

```
%equation x"+x'+x=sinwt
function hb_with_damping()
clc
clear all
close all
format longEng
global N T w t F W X
N=11; %No. of points assumed
T=65; %Total time
w=2*pi/T;
t=linspace(0,T,N+1); %actually they will be N equally divided points in
continuous system
t=t(1:end-1);%we required N-1 points in the current system
F=sin(w*t);%forcing term
iw=(0:ceil(N-1)/2)*1i*(w); miw=(-1i)*(floor(N/2):-1:1)*(w); %- & + values of
w
W=[iw,miw];
X=fft(F);
x=fminsearch(error(), X)
function residue=error()
dotX=W.*X;
dotx=ifft(dotX);
ddotX=(W.^2).*X;
ddotx=ifft(ddotX);
x0=ddotx/W.^2; %displacement in time domain

residue=@(X) sum(abs(ddotx+dotx+x0-F).^2);

end
 figure(1)
 a=w*sqrt(1-(ifft(X)).^2)-(ifft(X)).*(1-w^2);
 b=w*sqrt(1-F.^2)-F.*(1-w^2);
 plot(t,a./(w^2-w^4-1),t,b./(w^2-w^4-1),'*-')
 legend('x-fft','x-analytical')
 x_0=-0.1;v0=0;
 fnc = @(t,x)[x(2);sin(w*t)-x(2)-x(1)];
 [tspan,x_ode] = ode45(fnc,t,[x_0 v0]);
 figure(2)
 plot(t,x_ode(:,1),t,a./(w^2-w^4-1),'o-')
 legend('x-ode','x-fft')
end
```

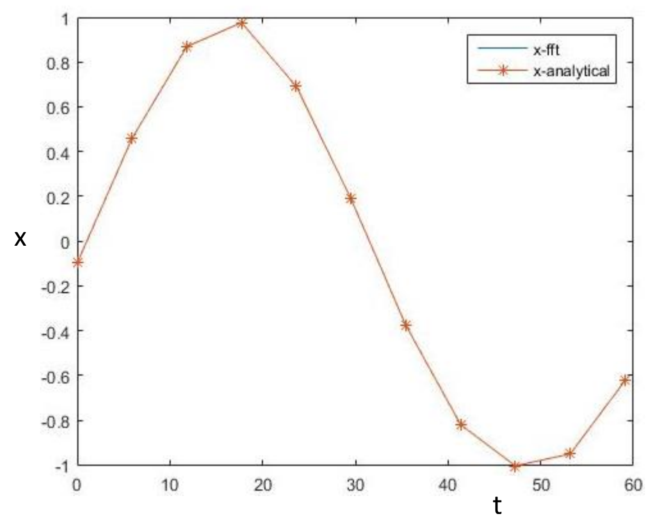

Fig3. X_fft vs X_analytical

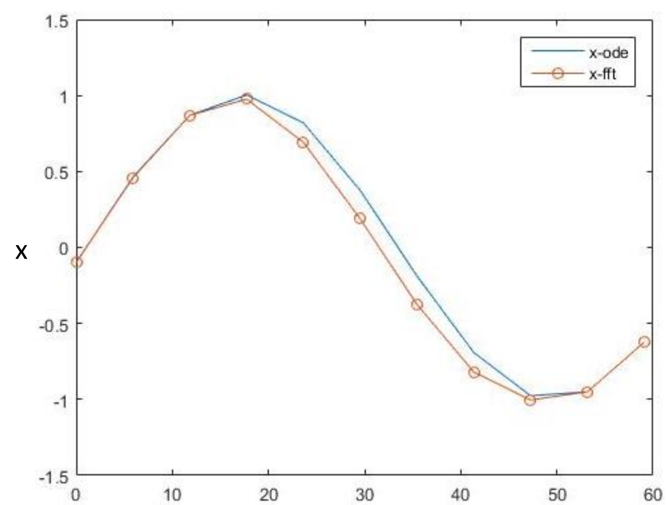

Fig4. X_ode45 vs X_fft

Response of a Non-Linear Differential Equation using Harmonic Balance Method

In this project, we have done different cases of Non-Linear Equations for comparing the analytical solutions and Harmonic Balance Method solutions and converging them with a minimum error. They are,

Duffing Oscillator

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma \cos(\omega t)$$

Duffing Oscillator, when driven by small oscillating force of γ cos(ωt) and with F_D <<1, DO will relax either into well & oscillate periodically at the driving frequency, ω and its harmonics. With periodic drive, the shape of attractors at Θ_+ & Θ_- change from a point to a closed loop in the phase space.

For larger forces, F_D>1, the response changes and generates qualitatively new types of behavior, including complicated orbits with periods greater than the drive period, 2π/3 & orbits that are not periodic at all chaos, which appears to be broad band noise. The time response of this function was found by using both Harmonic Balance method & integration method by using ode45. The signals which are resulted are compared w.r.t qualitative and quantitative difference. In integration method, we have shifted a phase of 0.9 to converge the response because initially homogenous is included with forcing function and deviates which can fit by shifting the response to the required value. The program below is for ,

$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma \cos(\omega t)$$

```
function trial2()
clc
close all
clear all
%x''+x = sin(0.9t)
w=0.08; f = w/(2*pi); n=17;
T=inv(f);
t=linspace(0,T,n)
i = sqrt(-1);
wi= [0,(1:1:((n-1)/2)),-((n-1)/2:-1:1)]*((2*pi/T))*i;
F= cos(w.*t) % forcing function
f_n = 0.1*(cos(w*t)).^3+(0.2*w*sin(w.*t)) % non linear force
f = (F-f_n)/(1-w^2)+eps;
xw = fft(f);
x=fminsearch(logic(),xw);
global wi, f, n, T, i,wi, F,xw;
```

In the above m-script, we can observe the variables globalization to work in all the functions and converting the initial response into frequency domain to work in Harmonic Balance Method.

```
function res = logic()
aw = wi.^2.*xw;
vw = wi.*xw;
at = ifft(aw);
vt = ifft(vw);
xt = at./wi.^2;
res = @(xw)(sum(abs((at+xt-F+f_n))));
end

% analytical solution
%x¨(t) + ?x?(t) + ?x(t) + ?x(t)^3  = ? cos(?t)
function ydiff=duffing(t,y)
alpha=0.1;
beta=1;
gamma=1;
delta=0.2;
dydt=[y(2); -delta*y(2)-beta*y(1)-alpha*(y(1)^3)+gamma*cos(w*t)];
ydiff=dydt;
end

function ydiff=duff(t,y)
alpha=0.1;
beta=1;
gamma=1;
delta=0.2;
dydt=[y(2); -delta*y(2)-beta*y(1)-
alpha*(y(1)^3)];
ydiff=dydt;
end

s1 = ifft(x)

figure(1)
[t,y2]=ode45(@duffing,t,[0.9;0])
[t,y1]=ode45(@duff,t,[0;0])
 y = y2-y1
plot(t,y(:,1),'*-',t,s1);

end
```

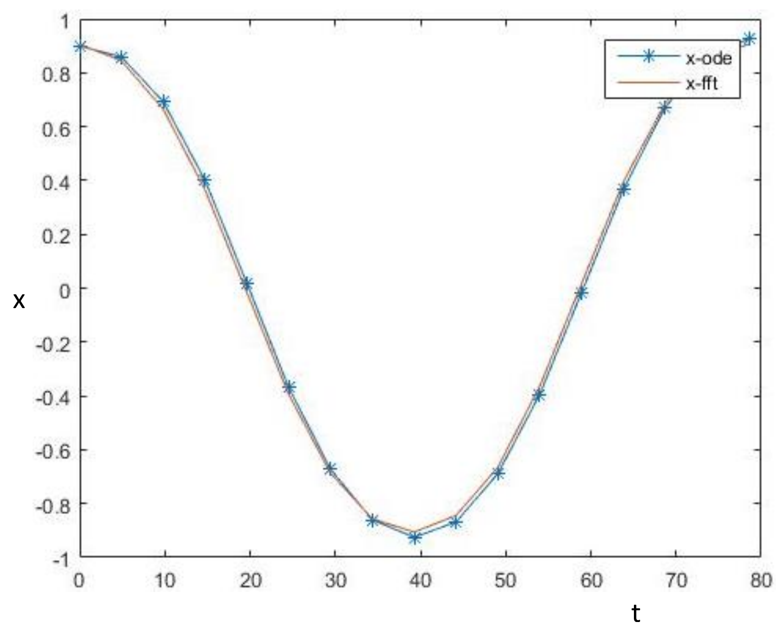

Fig5. X_ode45 vs X_fft

In the above functions, the differential equation with homogenous solution is included with forcing term in “duffing ( )” but the homogenous solution has been negotiated in “duff ( )”. Finally the plot has been drawn in between integration method and Harmonic Balance Method as shown above. The harmonic balance method have shown the acceptable approximation response for the behavior of the system.

Non-Linear Damped Pendulum

$$x + \delta\dot{x} + \beta\ddot{x} = \gamma\cos(\omega t)$$

In comparison with Duffing Oscillator, the transition shift of FFT solution at the initial time period is higher, but after some period of time due to elapse of homogeneous solution, the entire curve has converged to integration method. So, the initial displacement has taken based on the converged solution.

```
function damping_nonlinear()
clc
close all
clear all
w=0.08;
f = w/(2*pi);
n=17;
T=inv(f);
t=linspace(0,T,n)
i = sqrt(-1);
wi= [0,(1:1:((n-1)/2)),-((n-1)/2:-1:1)]*((2*pi/T))*i;
f = (cos(w.*t)-delta*(-w.*sin(w*t)))/(1-w^2)+eps;
xw = fft(f);
x=fminsearch(logic(),xw);
global wi, f, n, T, i,xw;
beta=1;
gamma=1;
delta=0.08;
function res = logic()
aw = wi.^2.*xw;
vw = wi.*xw;
at = ifft(aw);
vt = ifft(vw);
xt = at./wi.^2;
res = @(xw)(sum(abs((at+xt-(f*(1-w.^2))))));
end
% analytical solution
function ydiff=damping(t,y)
dydt=[y(2);-beta*y(1)-delta*w*sin(w*t)+gamma*cos(w*t)];
ydiff=dydt;
end
function ydiff=damp(t,y)
dydt=[y(2); -beta*y(1)-delta*w*sin(w*t)];
ydiff=dydt;
end
s1 = ifft(x)
figure(1)
[t,y2]=ode45(@damping,t,[1;0])
[t,y1]=ode45(@damp,t,[0;0])
 y = y2-y1
plot(t,y(:,1),'*-',t,s1);
legend('x-ode','x-fft')
end
```

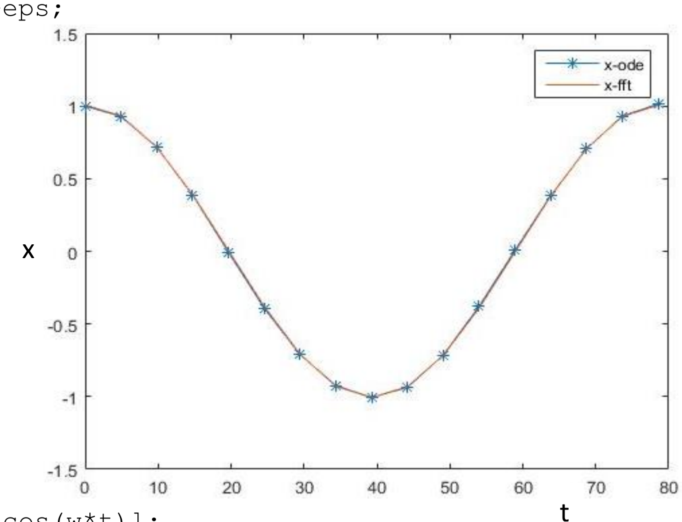

Fig6. X_ode45 vs X_fft

For the time sample at frequency ‘0.8’, the percent error is 64.286% but as the frequency decreased by 10times, the percent error has decreased to 44.44%. As the value has decreased by 20% comparatively and it is significant in the real time applications but Harmonic Balance Method solution is not satisfactory because of higher error.

One must know that the harmonic balance method captures the system response harmonically. But we have seen the transition shift in between integration method and harmonic balance method. In addition to the above, the pendulum rotating the complete cycle cannot be captured. Therefore, the user must set the magnitude of the forcing function low enough for getting the behavior appropriately.

CONCLUSION

The harmonic balance method gave an error percent higher in the non-linear damped system but on the other side it has shown almost near values in the linear systems and also in the duffing oscillator system. So, we can say that harmonic balance method has some deficiencies which should be well known by the person who are working for decreasing the errors and increasing the performance characteristics of the system.