# The effect of simplex and network degree distribution on simplicial contagion models

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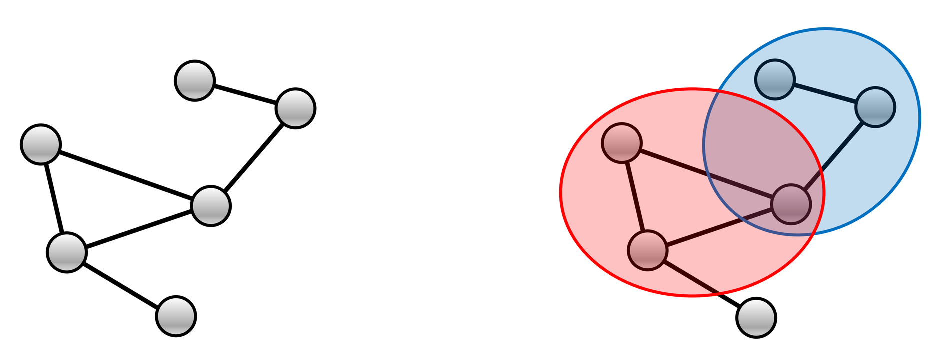
## Background

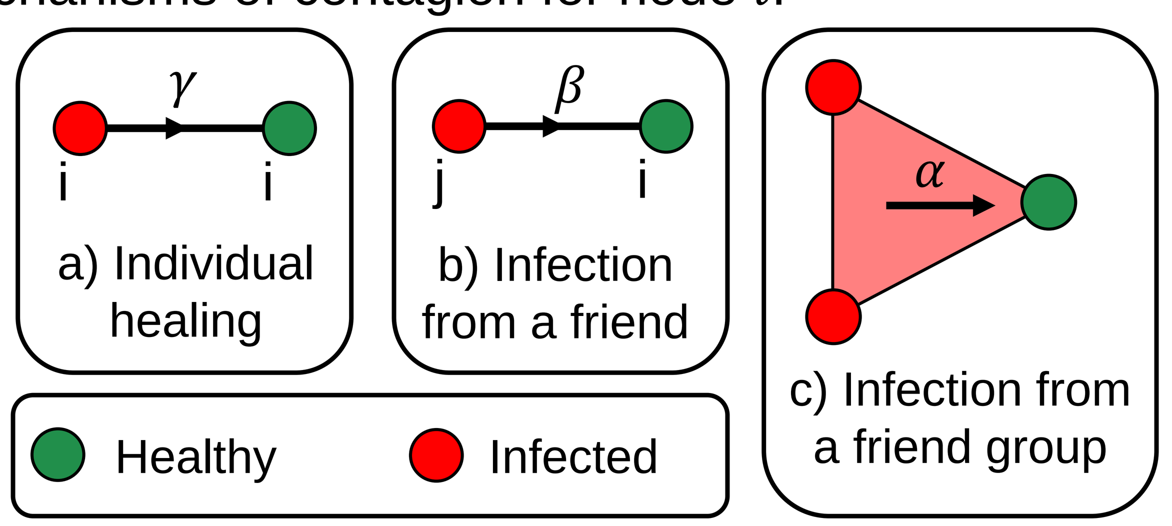
GOAL: Modeling explosive transitions present in epidemics and opinion formation (Ex. 2016 US election, memes, viral videos, flu) that are infeasible to capture using traditional network contagion models.

Hypergraphs capture higher-order interactions which more closely mimic real-world social dynamics, because groups to which we belong influence our opinions.



Network          Hypergraph

Mechanisms of contagion for node $i$:




Note that the standard model is simply a) and b)

Erdős-Rényi network with uniformly distributed simplices analyzed by Iacopini et al. in 2019

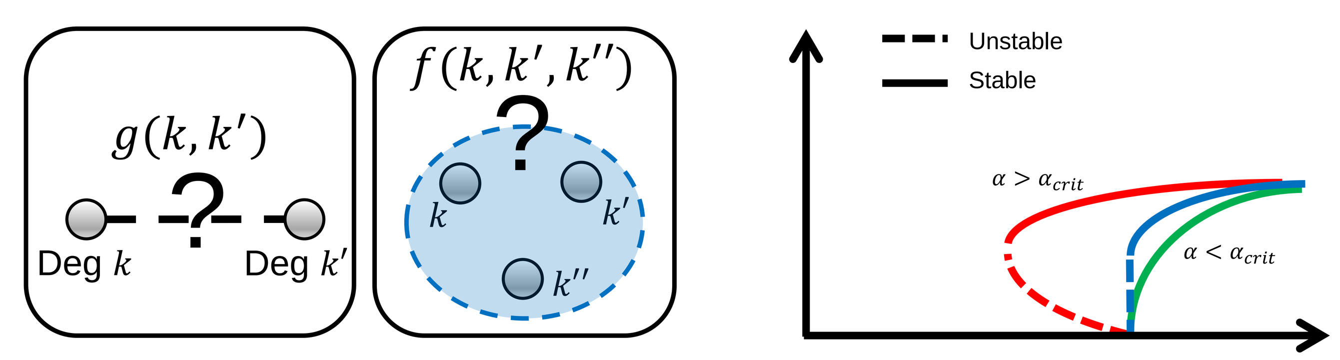
We varied the degree distribution of the underlying network and how the distribution of simplices on the nodes is related to the degree of the node and observe how the epidemic behavior is affected.

## Theory

We used a modified SIS model to include simplicial interactions.

Mean Field Theory

$$\underbrace{\frac{dx_k}{dt}}_{\substack{Rate\ of\\ Infection\\ of\ a\ node\\ of\ degree\\ k}} = \underbrace{-\gamma x_k}_{\substack{Rate\ of\\ healing}} + \underbrace{\beta(1-x_k)\sum_{k'} P_{k'} g(k,k') x_{k'}}_{\substack{Rate\ of\ infection\\ by\ infected\ neighbors}} + \underbrace{\alpha(1-x_k)\sum_{k',k''} P_{k'} P_{k''} f(k,k',k'') x_{k'} x_{k''}}_{\substack{Rate\ of\ infection\\ by\ infected\ simplices\ of\\ which\ the\ node\ is\ a\ member}}$$




We used linearization to solve for the critical value of $\beta$ at which infection occurs and found the inflection point at the critical $\beta$ to solve for the onset of hysteresis, also known as a cusp bifurcation.

We used the undirected configuration model for the network, so $g(k,k') = \frac{kk'}{n\langle k\rangle}$.

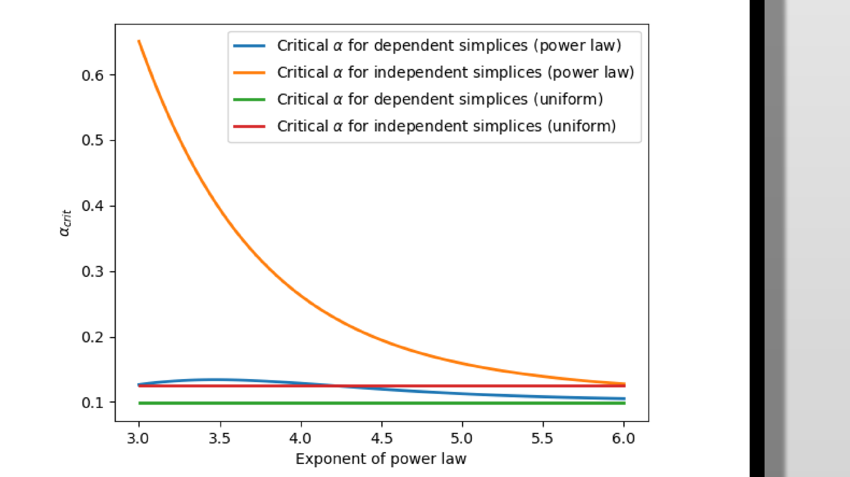
We considered two possibilities for $f(k,k',k'')$:

- The likelihood of a simplex being connected to a node depends on that node's degree
  - $f(k,k',k'') = \frac{kk'k''}{(n\langle k\rangle)^2}$
  - The critical value above which hysteresis occurs is $\alpha_{crit} = \frac{\langle k\rangle^2\langle k^3\rangle}{\langle k^2\rangle^3}$
- The likelihood of a simplex being connected to a node is identical for all nodes:
  - $f(k,k',k'') = \frac{\langle k\rangle}{n^2}$
  - The critical value above which hysteresis occurs is $\alpha_{crit} = \frac{\langle k^3\rangle}{\langle k\rangle^4}$

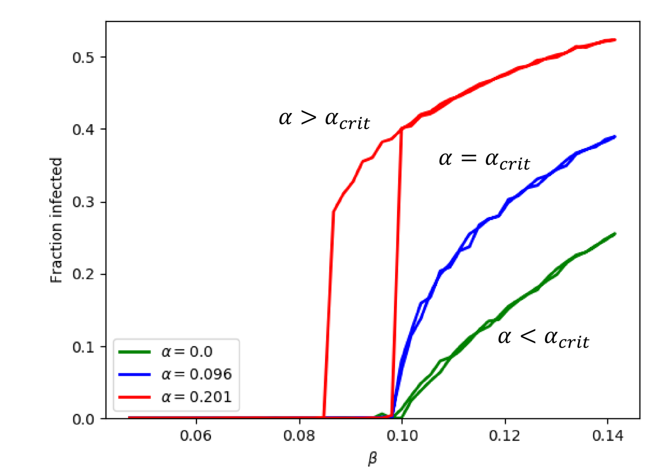
## Simulation

Degree distributions used:

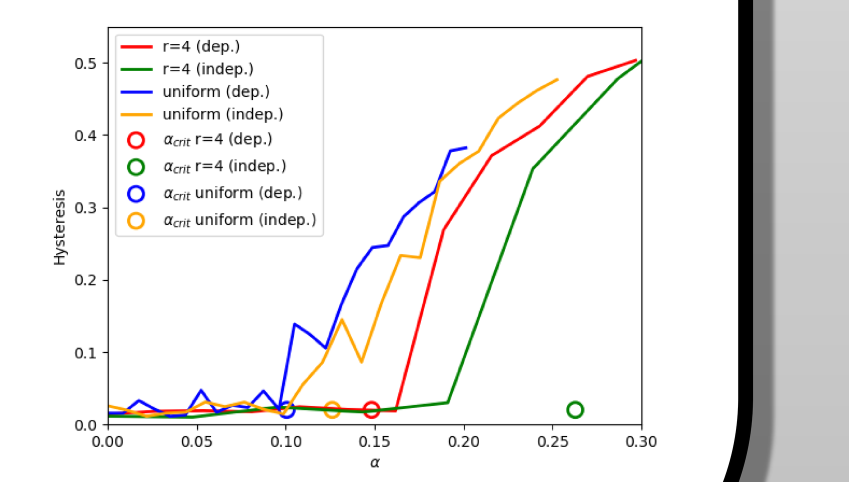
- Uniform degree distribution on $[10, 30]$
- Truncated power law degree distribution with $r = 4$ on $[13, 10000]$
- $\langle k\rangle = 20$ for both




*Onset of hysteresis as a function of the power in power law distribution*




*Simulation Infection Curves for uniform degree distribution and dependent simplices*




*Plot of the maximum distance between fixed points for different values of $\alpha$*

## Conclusions

- The network degree distribution strongly affects onset of hysteresis
- Simplices distributed according to node degree dramatically lowers the onset of hysteresis when compared with uniformly distributed simplices
- For networks with a highly heterogeneous degree distribution, explosive transitions can be suppressed by distributing the simplices independent of degree.

### References

- Iacopini, I., Petri, G., Barrat, A. *et al*. Simplicial models of social contagion. *Nat Commun* 10, 2485 (2019). doi:10.1038/s41467-019-10431-6
- de Arruda, Guilherme Ferraz, Giovanni Petri, and Yamir Moreno. Social contagion models on hypergraphs. *arXiv preprint arXiv:1909.11154* (2019).
- Soriano-Paños, David, *et al.* Explosive transitions induced by interdependent contagion-consensus dynamics in multiplex networks. *Physical Review E* 99.6 (2019): 062311.

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