hw06

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1 Homework 6

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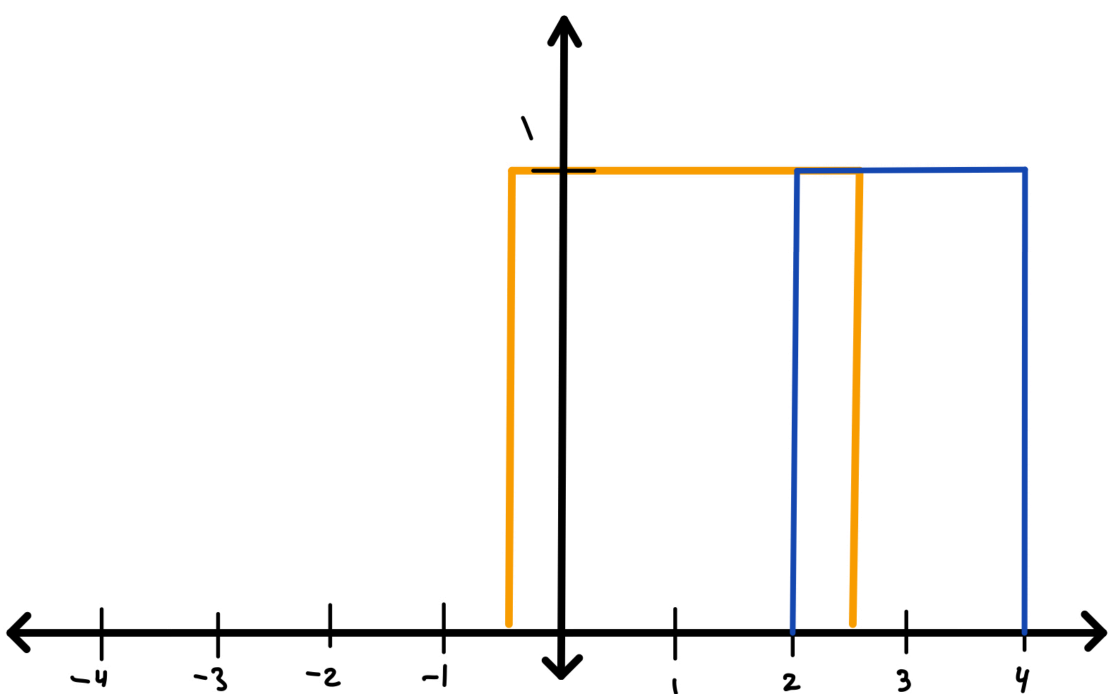
[14]:
```
# better image quality
import matplotlib as mpl
%matplotlib inline
mpl.rcParams['figure.dpi'] = 300
```

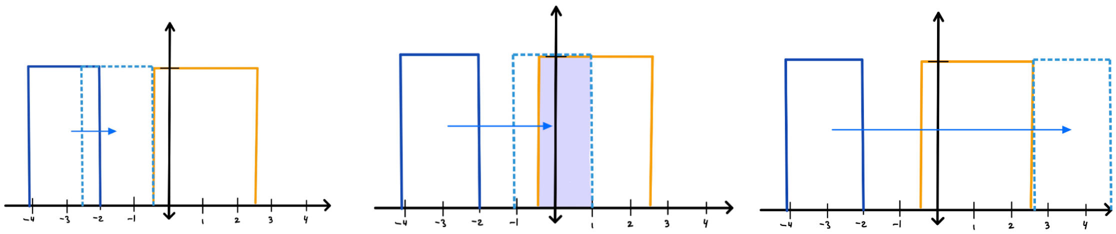
1.1 Problem 1

We can find an analytical solution to the convolution using a graphical approach. Sketching out the two functions we get:



Let $f(x) = \Pi(\frac{x-1}{3})$ $(orange)$ and $g(x) = \Pi(\frac{x-3}{2})$ $(blue)$. We can find the solution to $f(x) * g(x)$ graphically.

Following the procedure for graphically interpreting the convolution, we can fix $f(x)$ and flip $g(x)$ about the y axis. Then slide $g(x)$ over until the functions overlap, and we can infer the bounds of our convolution:



From the graphs we can see that there will be 3 integrals to evaluate:

1.1.1 Section 1

Integrate from $u = -\frac{1}{2}$ to $u = x - 2$:

$$\int_{-\frac{1}{2}}^{x-2} f(u)g(x-u)du = \int_{-\frac{1}{2}}^{x-2} (1)(1)du = u\Big|_{-\frac{1}{2}}^{x-2} = x - 2 - -\frac{1}{2} = x - 1\frac{1}{2}$$

The bounds of this equation in x space can be calculated based on our right edge:

Left bound: $u = -\frac{1}{2} = x - 1 \Rightarrow x = 1\frac{1}{2}$

Right bound: $u = 1\frac{1}{2} = x - 2 \Rightarrow x = 3\frac{1}{2}$

1.1.2 Section 2

Integrate from $u = x - 4$ to $u = x - 2$:

$$\int_{x-4}^{x-2} f(u)g(x-u)du = \int_{x-4}^{x-2} (1)(1)du = u\Big|_{x-4}^{x-2} = x - 2 - (x - 4) = 2$$

The bounds of this equation in x space can be calculated using the edges of the plot:

Left bound: $u = -\frac{1}{2} = x - 4 \Rightarrow 3\frac{1}{2}$

Right bound: $u = 2\frac{1}{2} = x - 2 \Rightarrow 4\frac{1}{2}$

1.1.3 Section 3

Integrate from $u = x - 4$ to $u = 2\frac{1}{2}$:

$$\int_{x-4}^{2\frac{1}{2}} f(u)g(x-u)du = \int_{x-4}^{2\frac{1}{2}} (1)(1)du = u\Big|_{x-4}^{2\frac{1}{2}} = 2\frac{1}{2} - (x - 4) = 6\frac{1}{2} - x$$

The bounds of this equation in x space can be calculated using the edges of the plot:

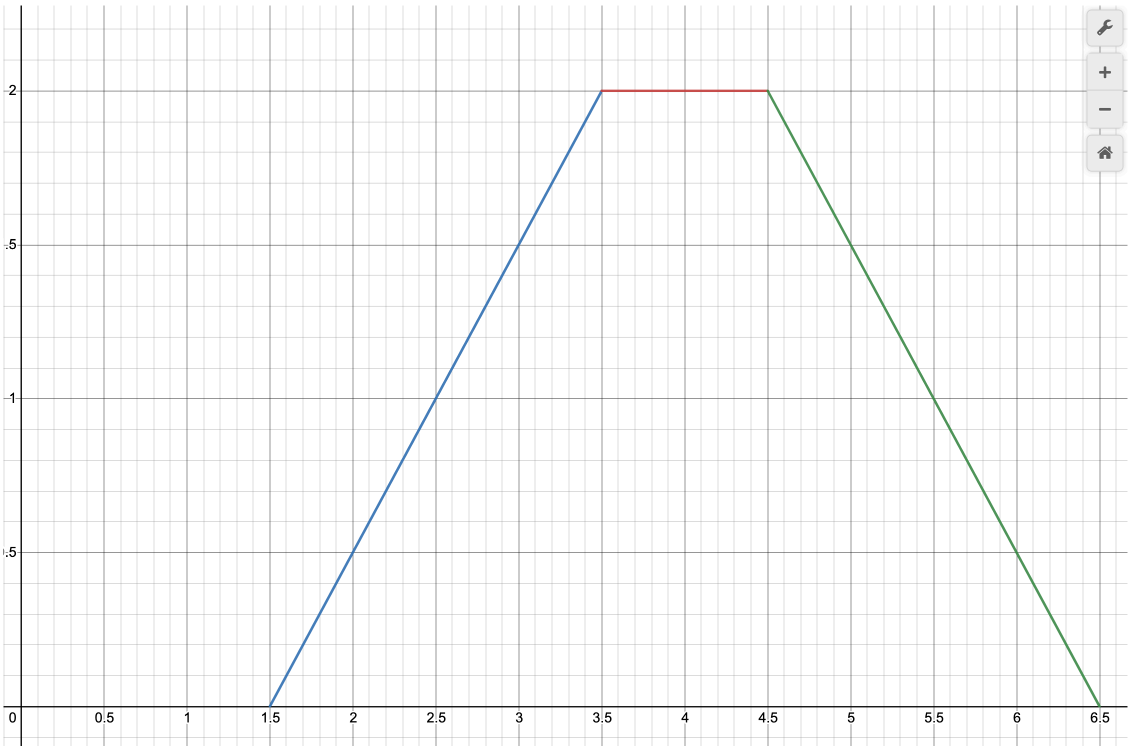
Left bound: $u = \frac{1}{2} = x - 4 \Rightarrow 4\frac{1}{2}$

Right bound: $u = 2\frac{1}{2} = x - 4 \Rightarrow 6\frac{1}{2}$

Thus, our final piecewise function looks like this:

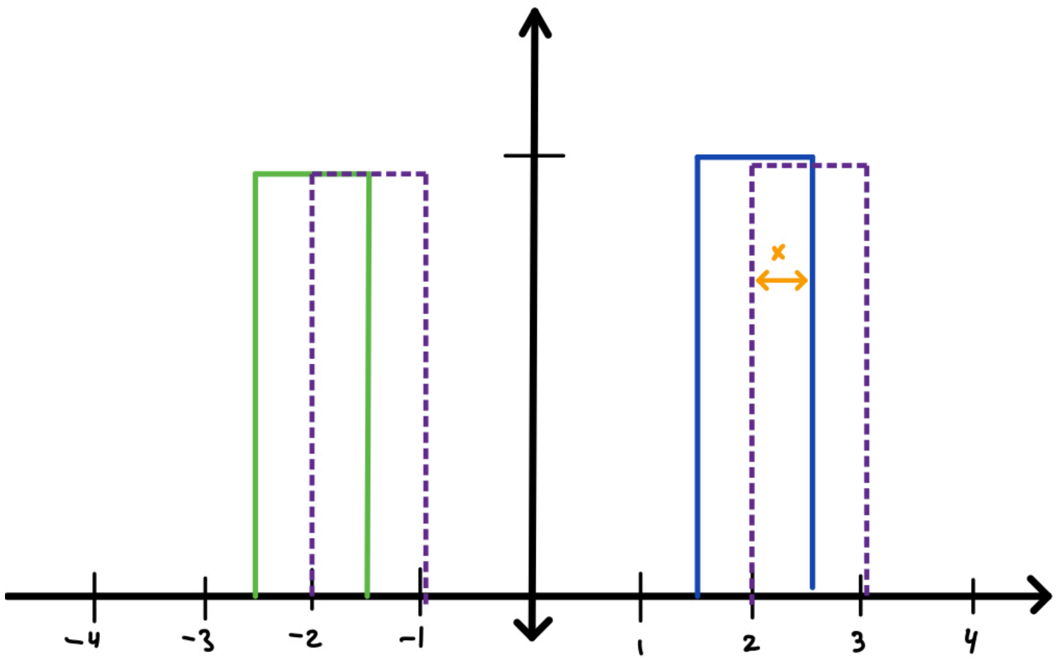
$$h(x) = \begin{cases} x - 1\frac{1}{2} & 1\frac{1}{2} \leq x \leq 3\frac{1}{2} \\ 2 & 3\frac{1}{2} \leq x \leq 4\frac{1}{2} \\ -x + 6\frac{1}{2} & 4\frac{1}{2} \leq x \leq 6\frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$$

Here is a graph of the analytical convolution:



1.2 Problem 2

Here is a plot of our system:



By intuition, we can see that there will be 6 distinct regions and subsequent integrals to calculate:

1. Initially as we shift $f_2(x)$ to the right.
2. When $\Pi_2(x+2)$ begins to overlap with $\Pi_1(x-2)$.
3. When $\Pi_2(x+2)$ begins to **stop** overlapping with $\Pi_1(x-2)$.
4. Initially as we shift $f_2(x)$ to the left.
5. When $\Pi_2(x-2)$ begins to overlap with $\Pi_1(x+2)$.
6. When $\Pi_2(x-2)$ begins to **stop** overlapping with $\Pi_1(x+2)$.

1.2.1 Section 1 (two integrals):

$$\int_{x-2\frac{1}{2}}^{-1\frac{1}{2}} f(u)g(x-u)du + \int_{x+1\frac{1}{2}}^{2\frac{1}{2}} f(u)g(x-u)du = \int_{x-2\frac{1}{2}}^{-1\frac{1}{2}} (1)(1)du + \int_{x+1\frac{1}{2}}^{2\frac{1}{2}} (1)(1)du$$

$$u\Big|_{x-2\frac{1}{2}}^{-1\frac{1}{2}} + u\Big|_{x+1\frac{1}{2}}^{2\frac{1}{2}} = -1\tfrac{1}{2} - (x - 2\tfrac{1}{2}) + 2\tfrac{1}{2} - (x + 1\tfrac{1}{2})$$

$$2 - 2x$$

1.2.2 Section 2:

$$\int_{1\frac{1}{2}}^{x-1\frac{1}{2}} f(u)g(x-u)du = \int_{1\frac{1}{2}}^{x-1\frac{1}{2}} (1)(1)du$$

$$u\Big|_{1\frac{1}{2}}^{x-1\frac{1}{2}} = 1\tfrac{1}{2} - (x - 1\tfrac{1}{2})$$

$$x - 3$$

1.2.3 Section 3:

$$\int_{x-2\frac{1}{2}}^{2\frac{1}{2}} f(u)g(x-u)du = \int_{x-2\frac{1}{2}}^{2\frac{1}{2}} f(1)(1)du$$

$$u\Big|_{x-2\frac{1}{2}}^{2\frac{1}{2}} = 2\tfrac{1}{2} - (x - 2\tfrac{1}{2})$$

$$5 - x$$

1.2.4 Section 4 (two integrals):

$$\int_{-2\frac{1}{2}}^{x-1\frac{1}{2}} f(u)g(x-u)du + \int_{1\frac{1}{2}}^{x+2\frac{1}{2}} f(u)g(x-u)du = \int_{-2\frac{1}{2}}^{x-1\frac{1}{2}} (1)(1)du + \int_{1\frac{1}{2}}^{x+2\frac{1}{2}} (1)(1)du$$

$$u\Big|_{-2\frac{1}{2}}^{x-1\frac{1}{2}} + u\Big|_{1\frac{1}{2}}^{x+2\frac{1}{2}} = (x - 1\tfrac{1}{2}) - (-2\tfrac{1}{2}) + (x + 2\tfrac{1}{2}) - 1\tfrac{1}{2}$$

$$2x + 2$$

1.2.5 Section 5:

$$\int_{x+1\frac{1}{2}}^{-1\frac{1}{2}} f(u)g(x-u)du = \int_{x+1\frac{1}{2}}^{-1\frac{1}{2}} (1)(1))du$$

$$u\Big|_{x+1\frac{1}{2}}^{-1\frac{1}{2}} = -1\tfrac{1}{2} - (x + 1\tfrac{1}{2})$$

$$-3 - x$$

1.2.6 Section 6:

$$\int_{-2\frac{1}{2}}^{x+2\frac{1}{2}} f(u)g(x-u)du = \int_{-2\frac{1}{2}}^{x+2\frac{1}{2}} (1)(1)du$$

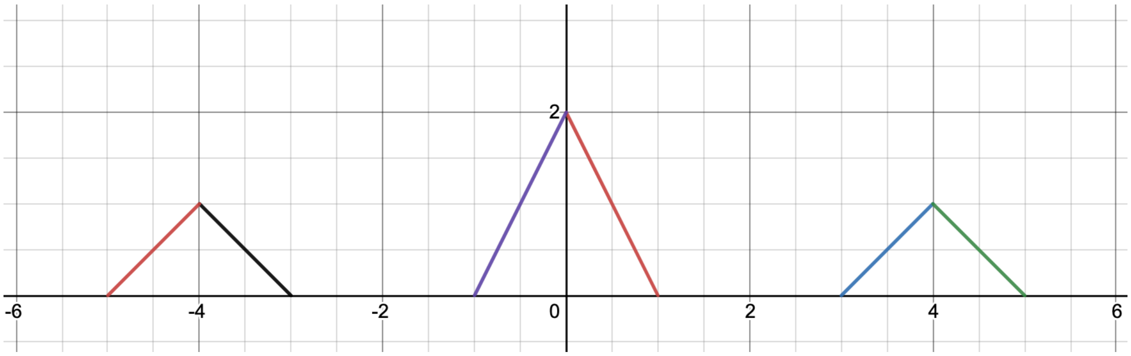
$$u\Big|_{-2\frac{1}{2}}^{x+2\frac{1}{2}} = (x + 2\tfrac{1}{2}) - (-2\tfrac{1}{2})$$

$$x + 5$$

Thus, our final piecewise function will look like this:

$$h(x) = \begin{cases} 2-2x & 0 \leq x \leq 1 \\ x-3 & 3 \leq x \leq 4 \\ 5-x & 4 \leq x \leq 5 \\ 2x+2 & -1 \leq x \leq 0 \\ -3-x & -4 \leq x \leq -3 \\ x+5 & -5 \leq x \leq -4 \\ 0 & \text{elsewhere} \end{cases}$$

This plot looks like this:



1.3 Problem 3

We can plot our system in Python:

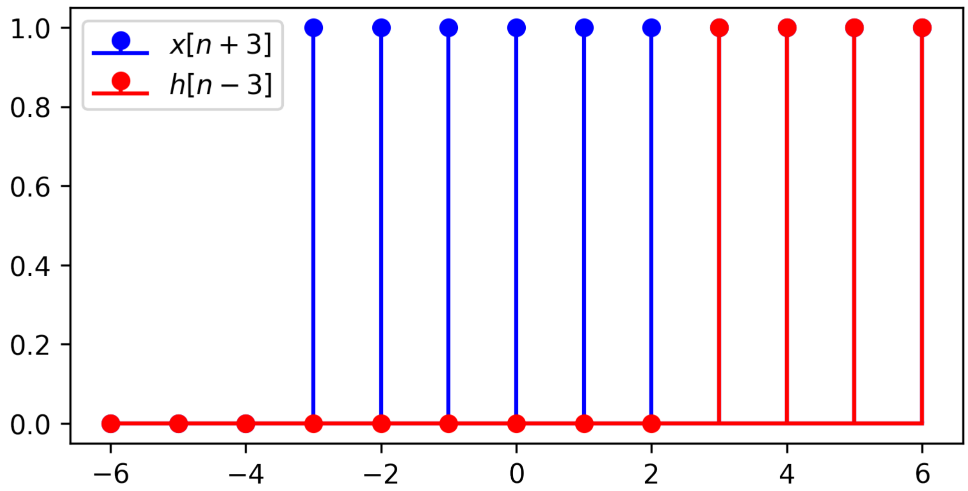
[26]:
```
import matplotlib.pyplot as plt
import numpy as np

locs = np.linspace(-6,6, num=13)
x=[0,0,0,1,1,1,1,1,1,1,1,1,1]
h=[0,0,0,0,0,0,0,0,0,1,1,1,1]

plt.rcParams["figure.figsize"] = (6,3)
plt.stem(
    locs,
    x,
    linefmt='b-',
    markerfmt="bo",
    basefmt="b-",
    label="$x[n+3]$"
)
plt.stem(
    locs,
    h,
    linefmt='r-',
    markerfmt="ro",
    basefmt="r-",
    label="$h[n-3]$"
```

```
)
plt.legend()
```

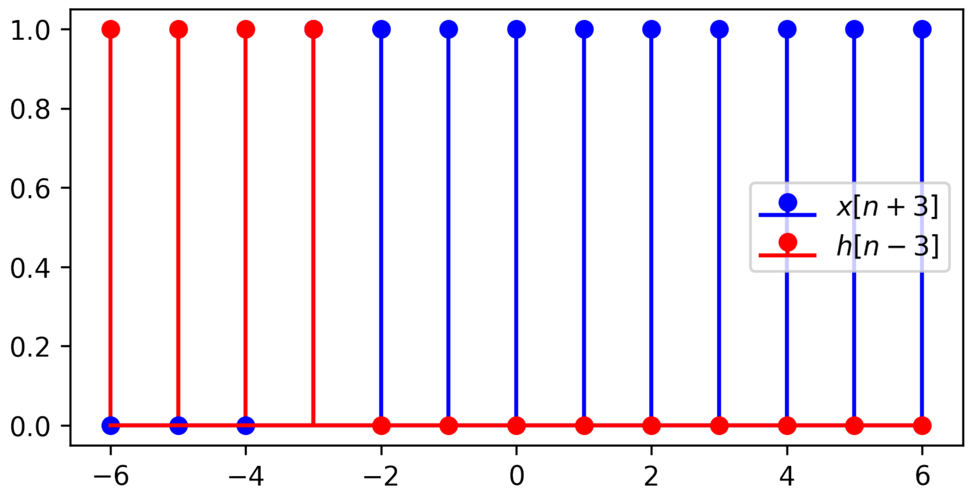
[26]: <matplotlib.legend.Legend at 0x10b3cfbe0>



We can flip $h[n=3]$ and begin shifting to the right in increments of k to intuitively build an analytical solution for our convolution:

[27]:
```
plt.rcParams["figure.figsize"] = (6,3)
plt.stem(
    locs,
    x,
    linefmt='b-',
    markerfmt="bo",
    basefmt="b-",
    label="$x[n+3]$"
)
plt.stem(
    locs,
    np.flip(h),
    linefmt='r-',
    markerfmt="ro",
    basefmt="r-",
    label="$h[n-3]$"
)
plt.legend()
```

[27]: <matplotlib.legend.Legend at 0x10b67b3d0>



Right off the bat we overlap by one discrete time unit. The total overlapping "area" is $1 \cdot 1 = 1$ Shifting over by $k = 1$ we get a total "area" of $1 \cdot 1 + 1 \cdot 1 = 2$. This idea can be extended to build a pattern. Let the convolution $x[n+3] * h[n-3] = y[n]$:

$k = 0 \rightarrow y[n] = 1$

$k = 1 \rightarrow y[n] = 2$

$k = 2 \rightarrow y[n] = 3$

$\vdots$

This yields the general solution:

$y[n] = n + 1$

1.4 Problem 4

1.4.1 a.) $m[n] = x[n] * z[n]$

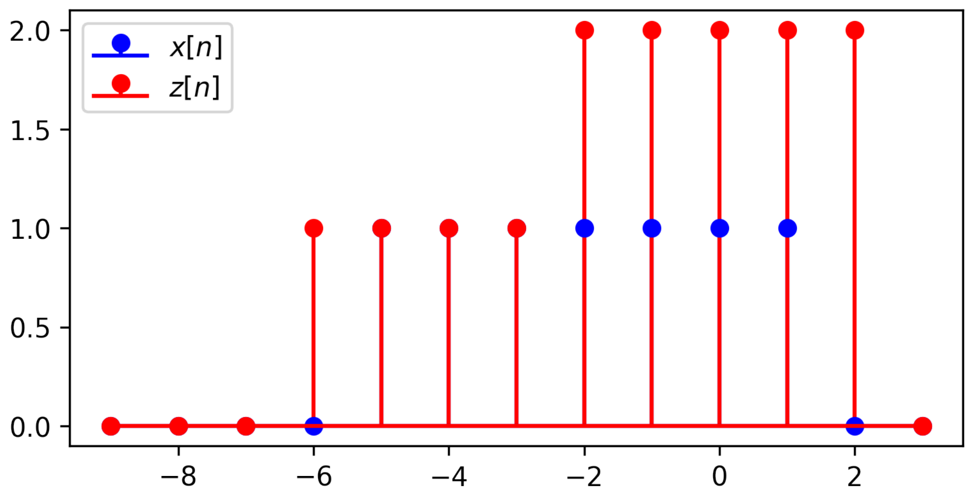
We can plot the functions in python:

[40]:
```
locs = np.linspace(-9,3, num=13)
x = [0,0,0,0,1,1,1,1,1,1,1,0,0]
z = np.flip([0,2,2,2,2,2,1,1,1,1,0,0,0])

plt.rcParams["figure.figsize"] = (6,3)
plt.stem(
    locs,
    x,
    linefmt='b-',
    markerfmt="bo",
    basefmt="b-",
```

```
    label="$x[n]$"
)
plt.stem(
    locs,
    z,
    linefmt='r-',
    markerfmt="ro",
    basefmt="r-",
    label="$z[n]$"
)
plt.legend()
```

[40]: <matplotlib.legend.Legend at 0x10bcce820>

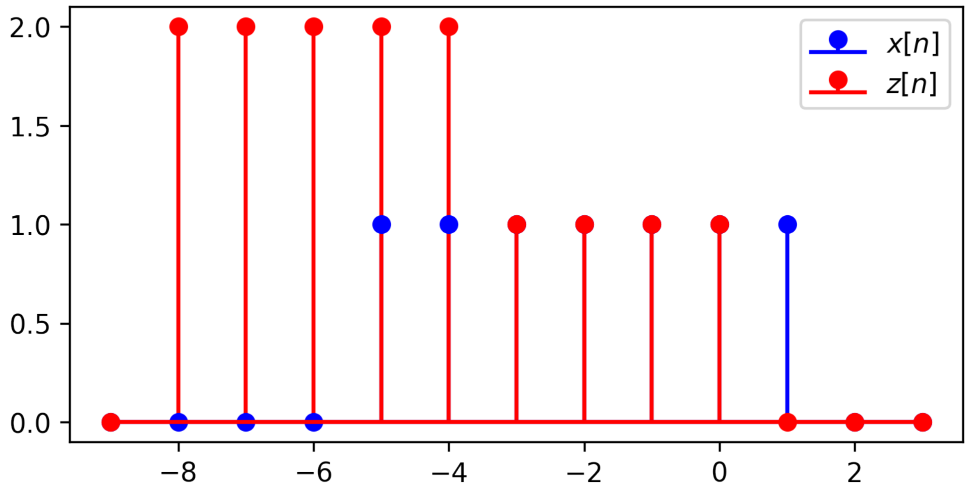


We can flip $z[n]$ about the y axis and then start shifting:

[42]:
```
plt.rcParams["figure.figsize"] = (6,3)
plt.stem(
    locs,
    x,
    linefmt='b-',
    markerfmt="bo",
    basefmt="b-",
    label="$x[n]$"
)
plt.stem(
    locs,
    np.flip(z),
    linefmt='r-',
    markerfmt="ro",
```

```
    basefmt="r-",
    label="$z[n]$"
)
plt.legend()
```

[42]: <matplotlib.legend.Legend at 0x10bd6a550>



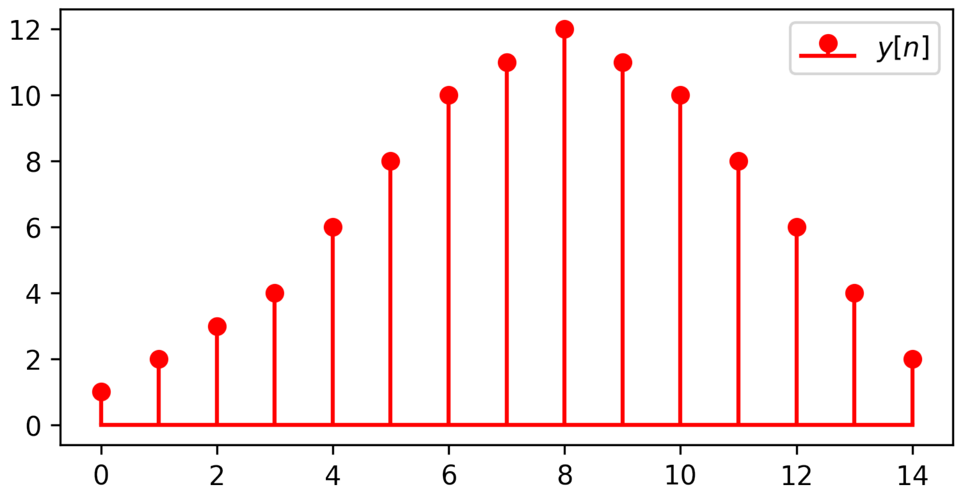
As we shift $z[n]$ for postiive and negative values of k, we finally acheive the following vector:

$$y[n] = \left[1, 2, 3, 4, 6, 8, 10, 11, 12, 11, 10, 8, 6, 4, 2\right]$$

Plotted, we achieve the following:

[44]:
```
y = [1,2,3,4,6,8,10,11,12,11,10,8,6,4,2]
plt.stem(
    y,
    linefmt='r-',
    markerfmt="ro",
    basefmt="r-",
    label="$y[n]$"
)
plt.legend()
```

[44]: <matplotlib.legend.Legend at 0x10be40910>



1.5 Problem 5

To find a matrix H that satisfies the equation $y = Hx$, we can start with the definition of a discrete-time convolution:

$$\sum_{k=0}^{\infty} x[k]h[n-k]$$

We can set our lower bound to $k = 0$ as both equtions are 0 for all values $n < 0$. From this we can start investigating the values of $y[n]$. Note that for all values of $k > n$, those summation components will be directly equal to 0. Values of $k > n$ will evaluate to $h[-1], h[-2] \ldots$ etc, which is, by our definitaion of $h[n]$ qual to zero and will evaluate to $x[n] \cdot 0 = 0$.

$n = 0 \rightarrow y[0] = x[0]h[0]$

$n = 1 \rightarrow y[0] = x[0]h[1] + x[1]h[0]$

$n = 2 \rightarrow y[0] = x[0]h[2] + x[1]h[1] + x[2]h[0] = x[2]h[0] + x[1]h[1] + x[0]h[2]$

$n = 3 \rightarrow y[0] = x[0]h[3]x[1]h[2] + x[2]h[1] + x[3]h[0] = x[3]h[0] + x[2]h[1] + x[1]h[2] + x[0]h[3]$

$\vdots$

etc

We can see a pattern emerging and build up our vectors:

$$y = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[L+M=1] \end{bmatrix} = \begin{bmatrix} h[0] & 0 & 0 & 0 & \dots & 0 \\ h[1] & h[0] & 0 & 0 & \dots & 0 \\ h[2] & h[1] & h[0] & 0 & \dots & 0 \\ \vdots & & & & & \\ h[M-1] & h[M-2] & h[M-3] & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & h[M-1] & h[0] \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[L-1] \end{bmatrix}$$

We can see now that we have a general solution to the originally proposed equation:

$$y = Hx$$