ABE 55700

Homework 2 | Sterilization and Finite Difference

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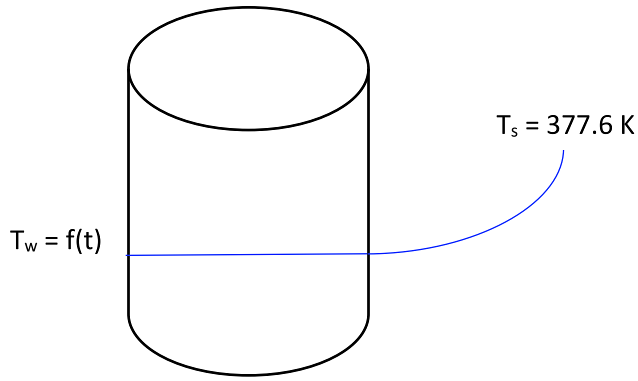
Problem 5.2-3

We have a tank that is being continuously stirred at a relatively high RPM. With this information, it is safe to make the assumption that the inside temperature of our tank is uniform. Thermodynamically, this indicates that our convective heat transfer resistance inside the tank is negligible or zero. In addition, we can assume that the walls are thin enough such that the conductivity resistance through the walls is also negligible. Thus, our temperature profile may look something like this:



We are given some information on our tank and the relevant thermodynamics of the system:

Parameter	***Value***
Volume, V [m^3]	0.0283
Initial Water Temp, T_0, [K]	288.8
Steam Bath Temp, T_s, [K]	377.6
Heat Transfer Coeff, U, [W/m^2-K]	1136
Surface Area, A, [m^2]	0.372
Desired Water Temp, T_w(t), [K]	338.7

Heat transfer is governed by the following equation:

$$\dot{q}\ [J/s] = UA\left(T_s - T_w(t)\right)$$

We can introduce some basic thermodynamic constants for our system to obtain an equation that characterizes instantaneous temperature changes in our system:

$$\frac{\dot{q}}{c_p \rho V} = \frac{dT_w}{dt} = \frac{UA}{c_p \rho V}(T_s - T_w)$$

This equation can be integrated with the given initial conditions to achieve the following:

$$ln\left(\frac{T_s - T_w}{T_s - T_0}\right) = -\frac{UA}{c_p \rho V} \cdot t$$

Using the specific heat of water, its density, and the given conditions, we can solve for the time required to reach our desired temperature. The calculations were done in excel and the calculated time was found to be 232.15 seconds.

Problem 5.4 – 5

Unsteady state heat transfer in a slab is untrivial and solving the associated differential equations is difficult since there are partial differentials involved. Numerical methods are overwhelmingly the method of choice for solving these equations and obtaining transient analysis of systems. For partial differential equations, we use a method known as the finite difference method. Beginning with the heat transfer equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

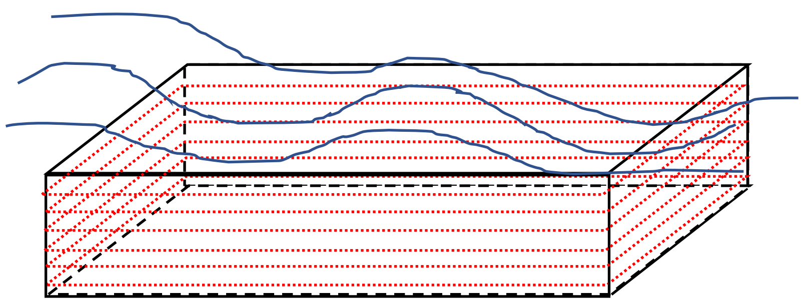
We can use the definition of derivatives to create an equation to estimate temperature changes over the course of a space and time:

$$Conduction\ inside:\quad T^{n}_{t+\Delta t} = \frac{1}{M}[T_t^{n+1} + (M-2)T_t^{n} + T_t^{n-1}]$$

$$Convection\ into\ air:\quad T^{1}_{t+\Delta t} = \frac{1}{M}[2NT_t^{0} + (M-(2N+2)T_t^{1} + 2T_2^{t}]$$

$$Insulated\ Boundary: T^{f}_{t+\Delta t} = \frac{1}{M}\left[(M-2)T_t^{f} + 2T_t^{f-1}\right]$$

Where M is $\frac{\Delta x^2}{\alpha \Delta t}$. α is the thermal diffusivity of our system. N is $\frac{h\Delta x}{k}$. We can "slice" the meat slab into sections and analyze the temperature gradient across each slice.



The system parameters are given as the following:

Thickness, L (mm)	45.7
Thickness, L (m)	0.0457
To, meat, (C)	37.78
To, meat, (K)	310.78
Tair, (C)	-1.11
Tair, (K)	271.89
h, conv. (W/m^2-K)	38
k, cond. (W/m-k)	0.498
diffusivity, α, m^2/h	0.000464

diffusivity, α, m^2/s	1.28889E-07
num slices	5
M	4
N	0.697429719
Δx	0.00914
Δt	162.0380172

Employing the finite difference method and the appropriate equations in the appropriate regions in excel, we obtain a profile that looks like this:

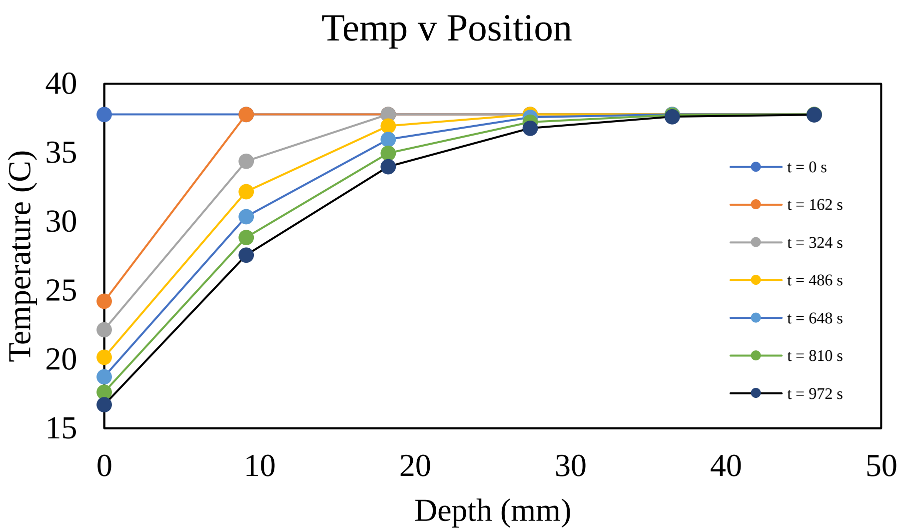


Figure 1. The above graph gives the temperature profile inside our slab of meat for various times. Initially, at time = 0, the temperature is a uniform 37.7 deg C. Over time, the temperature decreases inside our slab. This occurs much more rapidly at the face of the slab being blasted with chilled air. The steady state profile would follow a linear profile.

The exact temperature profile after 0.27 hours is:

Depth, mm [node]	Temperature, °C
0 (n = 1)	16.71
9.14 (n = 2)	27.57
18.28 (n = 3)	34.00
27.42 (n = 4)	36.78
36.56 (n = 5)	37.61
45.7 (n = 6)	37.75

Problem 9.12 – 5

In sterilization studies, the F_o value of a process is the time at 250°F that will produce the same degree of sterilization as the given process at its temperature T. More precisely, it is the equivalent-minutes of exposure to 250°C. The desired F_0 of our process is 2.60 min. This indicates that over the course of our process we want an equivalent exposure of 2.60 minutes at 250°C. Typically, for constant temperature systems and processes, we can use the following equation to calculate the F_0 value:

$$F_0 = t \cdot 10^{\frac{T-250}{z}}$$

The z-value is a system-dependent constant that is usually obtained from empirical studies. For a non-constant temperature system, our equation becomes a little more complex by summing the individual pieces of our process and their subsequent F_0 values:

$$F_0 = \sum_{i=1}^{n} (t_{i+1} - t_i) 10^{\frac{T_i - 250}{z}}$$

Using the following data…

Time (min)	T (°F)	T (°C)
0	110	43.33333
20	165	73.88889
40	205	96.11111
60	228	108.8889
80	232	111.1111
90	225	107.2222
100	160	71.11111

With z-values of 18°F and 10°C. The calculations were completed in excel, and the F_0 values for our Celsius and Fahrenheit measurements were 2.67 and 2.74 minutes respectively.

Problem 9.12 – 6

Sterility is incredibly important for food processes. Safety is the number one priority when it comes to the consumer market and understanding how this occurs and mathematically characterizing the process is vital. It is known that sterilization occurs as a first-order process:

$$\frac{dN}{dt} = -kN$$

Where k is often a function of the system parameters, namely temperature. In this problem, we are given this kinetic constant as:

$$k = 7.94 \times 10^{38} e^{-(68.7 \times 10^3)/1.987T}$$

We are given Temperature versus time data in a table. We can calculate the kinetic constant at the various temperatures and plot this data over time (Fig. 2).

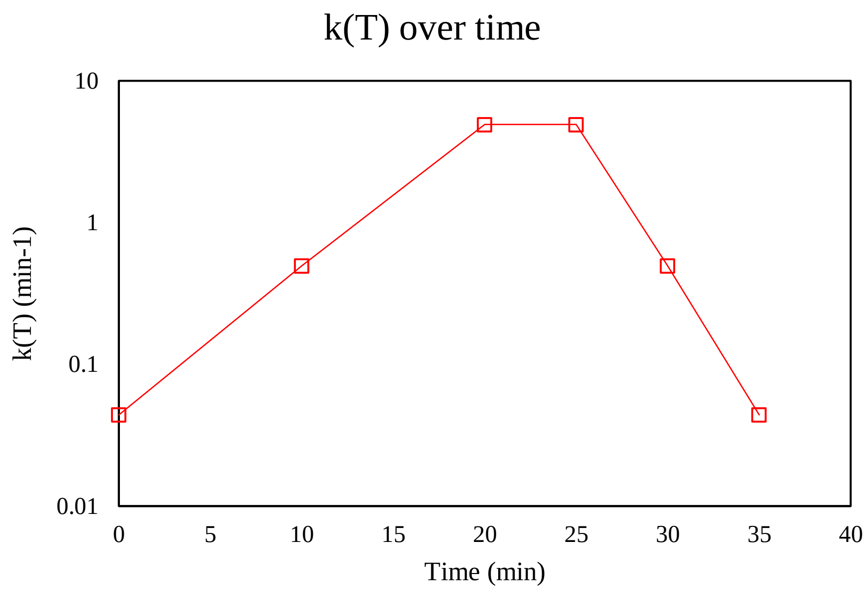


Figure 2. k(T) plotted over time. One can see that it increases in the middle of the process. This is a direct result of the increase in temperature at that time.

We can characterize the sterilization mathematically as the following equation(s):

$$\nabla = ln\frac{N_0}{N} = \int_0^t k\, dt$$

Obliviously, we do not have continuous data on the kinetic constant over time, so we can approximate the integral with the following midpoint Riemann sum formula:

$$\int_0^t k\, dt \approx \sum_{i=0}^{5} \frac{k(T)_{i+1} + k(T)_i}{2} \Delta t$$

When conducting these calculations in excel, we obtain a value, $\nabla = 67.28$. This allows us to calculate the final sterility level with the following equation:

$$N = \frac{N_0}{e^{\nabla}}$$

Again, with excel, we can obtain a value of N = 8.17e-19.

Problem 9.12 – 7

As stated before, the F_o value of a process is the time at 250°F that will produce the same degree of sterilization as the given process at its temperature T. More precisely, it is the equivalent-minutes of exposure to 250°C. However, in this case we are interested in exposure to the temperature 65.2°C. We know that an Equivalent F_0 can be calculated with the following equation:

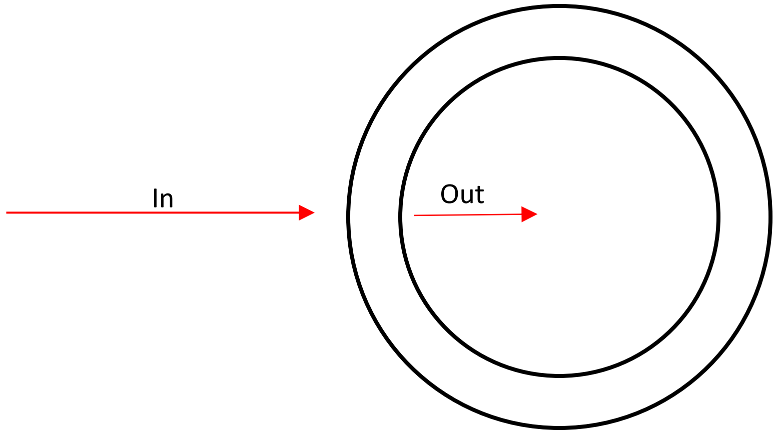
$$F_0 = t \cdot 10^{\frac{T - 65.6°C}{5°C}}$$

Rearranging, we can calculate the needed pasteurization time:

$$t = \frac{F_0}{10^{\frac{T - 65.6°C}{5°C}}}$$

Using the given values and constants, we can then calculate that the needed pasteurization time is 32.67 minutes. The calculations are relatively trivial and an excel sheet was not needed to conduct this.

Derivation of Equation 5.4 – 23



We start with ol' faithful: $Accumulation = In - Out + Generation - Consumption$

We are examining the heat transfer right at the boundary of our cylindrical system such that there is convection at the surface, and conduction underneath. Our starting equation becomes:

$$\rho \mathrm{c_p} V \frac{\partial \mathrm{T}}{\partial \mathrm{t}} = \underbrace{h\mathrm{A_{conv}} \frac{\partial \mathrm{T}}{\partial \mathrm{r}}|_{\mathrm{r + dr}}}_{\text{Convection at surface}} - \underbrace{k\mathrm{A_{cond}} \frac{\partial \mathrm{T}}{\partial \mathrm{r}}|_{\mathrm{r}}}_{\text{Conduction under surface}}$$

Dividing both sides by $\rho \mathrm{c_p}$ lets us define some constants, namely, thermal diffusivity, α, and a convective thermal diffusivity, β:

$$\alpha = \frac{\mathrm{k}}{\rho \mathrm{c_p}}, \beta = \frac{\mathrm{h}}{\rho \mathrm{c_p}}$$

Introducing the volume of our cylinder, and the two areas:

$$V = \pi((r + \Delta r)^2 - \mathrm{r}^2),\ \mathrm{A_{conv}} = 2\pi(r + \Delta r),\ \mathrm{A_{cond}} = 2\pi r$$

We can simplify the equation and cancel out many terms related to volume to obtain the following equation:

$$\frac{\partial T}{\partial t} = \frac{\beta}{r\Delta r}(r + \Delta r)\frac{\partial T}{\partial r}|_{r + dr} - \frac{\alpha}{\Delta r}\frac{\partial T}{\partial r}|_r$$

From here, we can use the definition of derivatives to obtain a numerically solvable equation too estimate the temperature profile within a cylindrical system:

$$\frac{\partial T}{\partial t} = \frac{T_{t+\Delta t}^n - T_t^n}{\Delta t}, \quad \frac{\partial T}{\partial r}|_{r + dr} = \frac{T_t^0 - T_t^1}{\Delta r}, \quad \frac{\partial T}{\partial r}|_r = \frac{T_t^1 - T_t^2}{\Delta r}$$

$$\frac{T_{t+\Delta t}^{n} - T_{t}^{n}}{\Delta t} = \frac{\beta}{r\Delta r}(r+\Delta r)\frac{T_{t}^{0} - T_{t}^{1}}{\Delta r} - \frac{\alpha}{\Delta r}\frac{T_{t}^{1} - T_{t}^{2}}{\Delta r}$$

Defining the two following constants:

$$M = \frac{\alpha \Delta \mathrm{t}}{\Delta \mathrm{r}^2}, \quad N = \frac{\beta \Delta \mathrm{t}}{\Delta \mathrm{r}^2}$$

We can then get the following equation which can be programmed to obtain our temperature profile:

$$T_{t+\Delta t}^{n} = N(r+\Delta r)T_{t}^{0} - (N(r+\Delta r) + M + 1)T_{t}^{1} + MT_{t}^{2}$$

Excel Sheets

Problem 5.2 – 3

	A	B	C	D	E	F	G
1	*Volume, m^3*	0.0283					
2	*To, K*	288.8					
3	*Ts, K*	377.6					
4	*U, W/m^2-K*	1136					
5	*A, m^2*	0.372					
6	*T desired*	338.7					
7	*Cp, J/kg-K*	4200					
8	*ρ, kg/m^3*	1000					
9	*logorithm*	-0.82539					
10	*Big Constant*	-0.00356					
11	*Time (s)*	232.1533					
12							
13							
14							

Problem 5.4 – 5

	A	B
1	*Thickness, L (mm)*	45.7
2	*Thickness, L (m)*	0.0457
3	*To, meat, (C)*	37.78
4	*To, meat, (K)*	310.78
5	*Tair, (C)*	-1.11
6	*Tair, (K)*	271.89
7	*h, conv. (W/m^2-K)*	38
8	*k, cond. (W/m-k)*	0.498
9	*diffusivity, α, m^2/h*	0.000464
10	*diffusivity, α, m^2/s*	1.28889E-07
11	*num slices*	5
12	*M*	4
13	*N*	0.697429719
14	*Δx*	0.00914
15	*Δt*	162.0380172
16	*t,final (hrs)*	0.27
17	*t,final (s)*	972
18	*num,time*	5.998592284

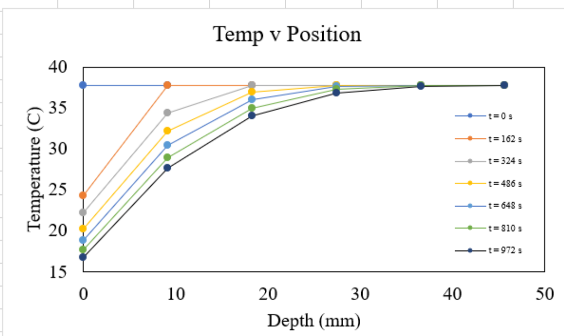
Conduction in Slab (K)

t\n	6	5	4	3	2	1
0	310.78	310.78	310.78	310.78	310.78	310.78
162.038	310.78	310.78	310.78	310.78	310.78	297.2185
324.076	310.78	310.78	310.78	310.78	307.3896	295.1668
486.1141	310.78	310.78	310.78	309.9324	305.1815	293.1612
648.1521	310.78	310.78	310.5681	308.9566	303.3642	291.7538
810.1901	310.78	310.727	310.2182	307.9614	301.8597	290.6322
972.2281	310.7535	310.6131	309.7812	307.0001	300.5782	289.7103

Conduction in Slab (C)

t\n	1	2	3	4	5	6
0	37.78	37.78	37.78	37.78	37.78	37.78
162.038	37.78	37.78	37.78	37.78	37.78	24.21848
324.076	37.78	37.78	37.78	37.78	34.38962	22.16682
486.1141	37.78	37.78	37.78	36.9324	32.18152	20.16125
648.1521	37.78	37.78	37.5681	35.95658	30.36417	18.75378
810.1901	37.78	37.72703	37.2182	34.96136	28.85968	17.63218
972.2281	37.75351	37.61306	36.78119	34.00015	27.57822	16.71025

	t = 0 s	t = 162 s	t = 324 s	t = 486 s	t = 648 s	t = 810 s	t = 972 s
45.7	37.78	37.78	37.78	37.78	37.78	37.78	37.75351
36.56	37.78	37.78	37.78	37.78	37.78	37.72703	37.61306
27.42	37.78	37.78	37.78	37.78	37.5681	37.2182	36.78119
18.28	37.78	37.78	37.78	36.9324	35.95658	34.96136	34.00015
9.14	37.78	37.78	34.38962	32.18152	30.36417	28.85968	27.57822
0	37.78	24.21848	22.16682	20.16125	18.75378	17.63218	16.71025



Problem 9.12 – 5

	A	B	C	D	E
1	Time (min)	T (°F)	T (°C)	F' (°F)	F' (°C)
2	0	110	43.33333		
3	20	165	73.88889	3.3362E-07	3.42266E-07
4	40	205	96.11111	0.00037915	0.000388972
5	60	228	108.8889	0.06324555	0.064884522
6	80	232	111.1111	1.1989685	1.230039009
7	90	225	107.2222	1	1.025914365
8	100	160	71.11111	0.40842387	0.419007911
9				**2.6710174**	**2.740235121**
10					
11	z (°F)	18			
12	z (°C)	10			
13	Fo (min)	2.6			

9.12 – 6

	A	B	C	D	E	F	H	I
1	*Time (min)*	*Temp (°C)*	*Temp (K)*	*k(T)*	*k-mid(T)*	*Area*		
2	0	100	373	4.40E-02	0.269472101	2.694721013	*N0*	1E+12
3	10	110	383	4.95E-01	2.708483114	27.08483114	*N*	8.17179E-19
4	20	120	393	4.92E+00	4.922024262	24.61012131		
5	25	120	393	4.92E+00	2.708483114	13.54241557		
6	30	110	383	4.95E-01	0.269472101	1.347360507		
7	35	100	373	4.40E-02	0.022001118			
8						69.27944954