Sensitivity Analysis

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Outline

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1. Motivation
   - Why perform SA?
   - Main contributors
2. Introduction
   - Notations
   - Uncertainty Quantification
   - Sensitivity Analysis
3. Sensitivity Analysis methods
   - Local Sensitivity methods
   - Variance based methods
   - Moment independent sensitivity measures
4. Conclusions

Motivation to perform Sensitivity Analysis

- Sensitivity Analysis (SA) identifies the most *important*[1] model inputs.
- Given more precise values of these *important* inputs, one can decrease uncertainty in the model output.
- In addition, one can exclude *unimportant* parameters from the UQ study to decrease its computational cost.
- Moreover, SA helps in
  - studying physics of the modelled process,
  - finding sources of errors,
  - etc.

[1] By *important* we mean that these parameters influence greatly the variability of the model output.

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I have always done uncertainty quantification. The difference now is that it is capitalized. Bill Browning

Ilya Meyerovich Sobol[2,3] and Andrea Saltelli[4,5]

[2] Ilya Sobol and Sergei Kucherenko. "Global sensitivity indices for nonlinear mathematical models. Review". In: *Wilmott* 2005.1 (2005), pp. 56–61. ISSN: 15406962.

[3] Ilya Sobol. "Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates". In: *Mathematics and Computers in Simulation* 55.1 (2001). The Second IMACS Seminar on Monte Carlo Methods, pp. 271 –280. ISSN: 0378-4754.

[4] Andrea Saltelli et al. "Variance based sensitivity analysis of model output. Design and estimator for the total sensitivity index". In: *Computer Physics Communications* 181.2 (2010), pp. 259–270. ISSN: 0010-4655.

[5] Andrea Saltelli et al. "Why So Many Published Sensitivity Analyses Are False. A Systematic Review of Sensitivity Analysis Practices". In: *ArXiv e-prints* 1 (2017), pp. 1–35. arXiv: 1711.11359.

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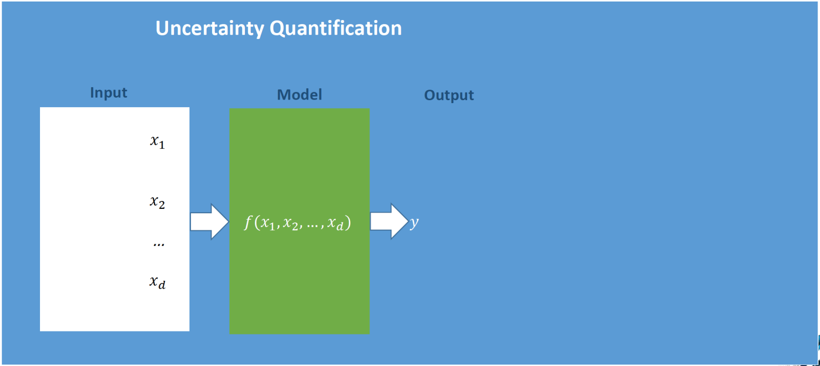
Notations

Assume that our computational model is described by a function f:

$$f(x) = y,$$

where $x = (x_1, \cdots, x_d)$ is a d-dimensional vector with model inputs, and y is a model output.



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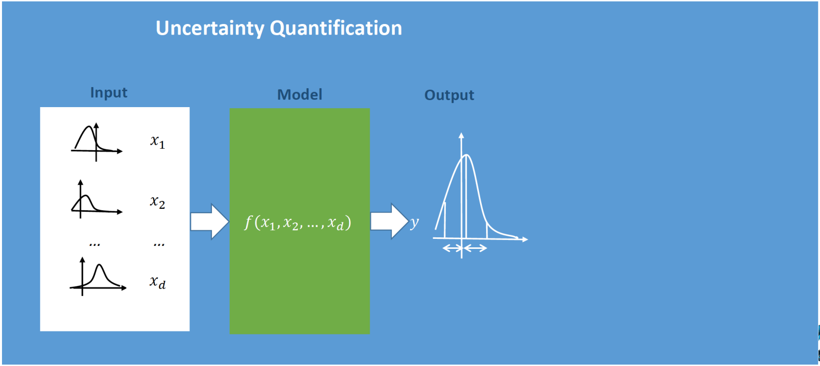
Uncertainty Quantification

However, usually we are not given with the precise values of the inputs x, but we can represent it as random values with some probability density functions $p_{x_1}, \cdots, p_{x_d}$. In this case, the model output is no longer a scalar, but a random value as well:

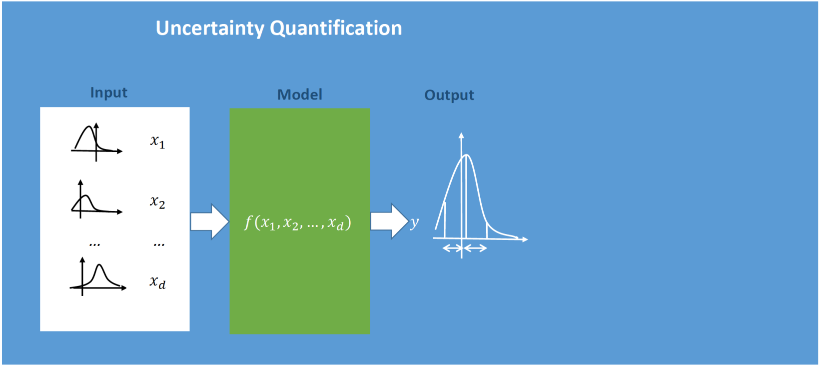
$$f(p_{x_1}, \cdots, p_{x_d}) = p_y,$$

where p_y is the probability density functions of y.



Uncertainty Quantification

Th goal of Uncertainty Quantification is to measure and analyse the influence of the uncertain inputs to the model output.



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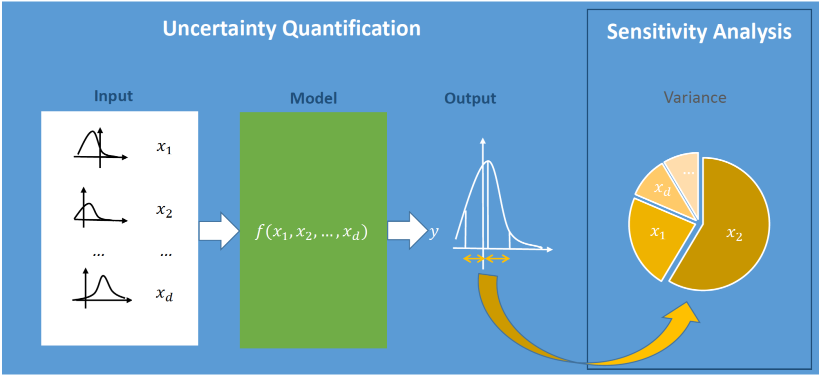
Sensitivity Analysis

"Sensitivity analysis is the study of how uncertainty in the output of a model (numerical or otherwise) can be apportioned to different sources of uncertainty in the model input factors, factors from now on."[6]



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Local Sensitivity methods (One-at-a-time)

Let $y^* = f(x^*)$ is the required solution. Then local sensitivity is measured by computing the partial derivatives with respect to an input factor x_k for $k \leq d$:

$$\left(\frac{\partial y}{\partial x_k}\right)_{x=x^*}$$

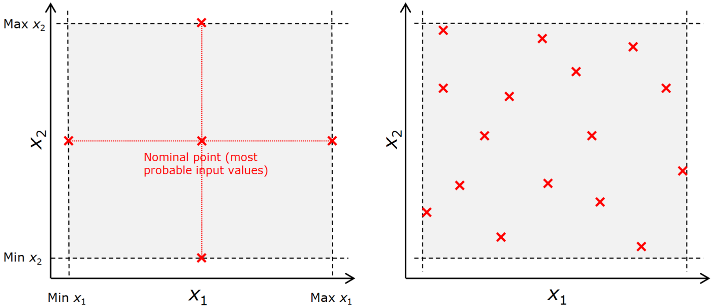

Figure: Local (left) versus Global (right) design[7]

[7] Andrea Saltelli et al. "Why So Many Published Sensitivity Analyses Are False. A Systematic Review of Sensitivity Analysis Practices". In: *ArXiv e-prints* 1 (2017), pp. 1–35. arXiv: 1711.11359.

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Requirements for SA techniques

According to Saltelli, a Sensitivity Analysis method must be[8]:

- **Global**: the entire input distribution of the model parameters is under consideration.
- **Quantitative**: sensitivity can be measured.
- **Model free**: no assumptions on the model functional relationship to its inputs is required.

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ANOVA decomposition

The ANOVA decomposition of a function $y = f(x_1, \dots, x_n) = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}$, $x \in [0, 1]^n$, and y has a finite variance, $\mathrm{Var}(y) < \infty$, i.e. $f \in L^2((0, 1)^n)$. The function f can be represented by

$$f(x) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j}^{n} f_{ij}(x_i, x_j) + \cdots + f_{12\dots n}(x_1, x_2, \cdots, x_n), \qquad (1)$$

where the elements in the decomposition satisfy

$$\int f_{i_1 \dots i_s}(x_{i_1}, \cdots, x_{i_s}) dx_{i_k} = 0 \text{ for } 1 \le k \le s, 1 \le i_1 < \cdots < i_s \le n. \qquad (2)$$

It can be proved that condition (2) implies the uniqueness of decomposition (1). As a byproduct, it is easy to see that the mean value of the function f is

$$\int f(x)dx = f_0.$$

Sobol sensitivity indices

Moreover, the decomposition in (1) allows us to decompose also the variance of the function $f(x)$:

$$\mathrm{Var}(f(x)) = \sum_{s=1}^{n} \sum_{i_1 < \cdots < i_s}^{n} \mathrm{Var}(f_{i_1 \ldots i_s}(x_{i_1}, \cdots, x_{i_s})),$$

where

$$\mathrm{Var}(f_{i_1 \ldots i_s}(x_{i_1}, \cdots, x_{i_s})) = \int f^2_{i_1 \ldots i_s}(x_{i_1}, \cdots, x_{i_s}) dx_{i_1}, \cdots, x_{i_s}$$

is a partial variance.

The Sobol sensitivity indices are defined as the ratio between a partial variance and the total variance:

$$S_{i_1 \ldots i_s} = \frac{\mathrm{Var}(f_{i_1 \ldots i_s}(x_{i_1}, \cdots, x_{i_s}))}{\mathrm{Var}(f(x))}.$$

Monte Carlo estimation of Sobol indices

An approximation of the first order Sobol sensitivity indices can be represented by

$$\mathrm{Var}(f_{X_i}(X_i)) = \mathrm{Var}_{X_i}(E_{\mathbf{X}_{-\mathbf{i}}}(Y|X_i)) \approx \frac{1}{N}\sum_{j=1}^{N} f(A)_j f(B_A^{(i)})_j - f_0,$$

and the total variance of a parameter X_i, can be computed as

$$E_{\mathbf{X}_{-\mathbf{i}}}(\mathrm{Var}_{X_i}(Y|\mathbf{X}_{-\mathbf{i}})) \approx \frac{1}{N}\sum_{j=1}^{N} f(A)_j f(A_B^{(i)})_j - f_0,$$

where $f(A)$ and $f(B)$ are the model outputs with inputs from matrices A and B, correspondingly, and $f(A_B^{(i)})_j$ is output with values of all inputs from A except the i-th factor, which is taken from matrix B.

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- **Global**: the entire input distribution of the model parameters is under consideration.
- **Quantitative**: sensitivity can be measured.
- **Model free**: no assumptions on the model functional relationship to its inputs is required.
- **Moment-independent**: no reference to any specific moment of the model output[10].

[9] Emanuele Borgonovo. "A new uncertainty importance measure". In: *Reliability Engineering & System Safety* 92.6 (2007), pp. 771–784.

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Moment-independent sensitivity indicator δ

A moment-independent sensitivity measure, introduced by Borgonovo[11] is

$$\delta_i = \frac{1}{2} E_i[|f_Y(y) - f_{Y|X_i=x_i}(y)|],$$

where $f_Y(y)$ is the unconditional distribution of the endogenous variable, $f_{Y|X_i=x_i}(y)$ is a conditional distribution given that exogenous variable X_i fixed at x_i.

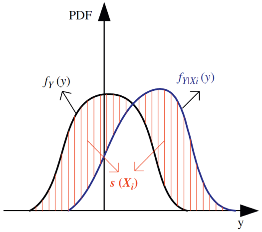

Fig. 1. The shift of $f_{Y|X_i}(y)$ from $f_Y(y)$ is evaluated by the area $s(X_i)$.

Figure: Distance metric[12]

[11] Emanuele Borgonovo. “A new uncertainty importance measure”. In: *Reliability Engineering & System Safety* 92.6 (2007), pp. 771–784.

[12] Qiao Liu and Toshimitsu Homma. “A new computational method of a moment-independent uncertainty importance measure”. In: *Reliability Engineering & System Safety* 94.7 (2009), pp. 1205–1211.

Quantile based global sensitivity measures

One moment independent measure based on quantiles was proposed by Chun et al.[13]:

$$CHT_i = \frac{\sqrt{\int (q_{Y|X_i}(\alpha) - q_Y(\alpha))^2 d\alpha}}{E[Y]}.$$

A quantile-based sensitivity measures by Kucherenko and Song[14] is

$$\bar{q}_i(\alpha) = E(|q_Y(\alpha) - q_{Y|X_i}(\alpha)|).$$

A normalized versions of quantile-based sensitivity measures[14]:

$$Q_i(\alpha) = \frac{\bar{q}_i(\alpha)}{\sum_{j=1}^{d} \bar{q}_j(\alpha)}$$

[13] Moon-Hyun Chun, Seok-Jung Han, and Nam-IL Tak. "An uncertainty importance measure using a distance metric for the change in a cumulative distribution function". In: *Reliability Engineering & System Safety* 70.3 (2000), pp. 313–321.

[14] Sergei Kucherenko and Shufang Song. "Quantile based global sensitivity measures". In: *arXiv preprint arXiv:1608.02221* (2016).

Conclusions

- One must perform Uncertainty Quantification and Sensitivity Analysis (SA) for better understanding reliability of the computational models.
- Local SA can be applied in a limited number of cases.
- Variance based SA methods are recommended to apply, however, if shape of PDF of the model output is "nice".
- In the rest of the cases moment-independent SA approaches should be used.

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