Parallel-in-time methods

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Overview

Alternative: skip my talk and read "50 Years of Time Parallel Time Integration" by Martin J. Gander

- Small scale methods
- Parareal
- MultiGrid
- PFASST

ODE Solvers

(or PDE through method of lines)

Initial value problem:

$$y(t_0) = y_0$$

$$\dot{y} = f(t, y)$$

(forward) Euler

$$y_{i+1} = y_i + (t_{i-1} - t_i) f(t_i, y_i)$$

- Each y_i depends on y_{i-1}.

Runge-Kutta

$$
\begin{aligned}
k_1 &= h\ f(t_i, y_i)\\
k_2 &= h\ f(t_i + h/2, y_i + k_1/2)\\
k_3 &= h\ f(t_i + h/2, y_i + k_2/2)\\
k_4 &= h\ f(t_i + h, y_i + k_3)\\
y_{i+1} &= y_i + (k_1 + 2k_2 + 2k_3 + k_4)/6
\end{aligned}
$$

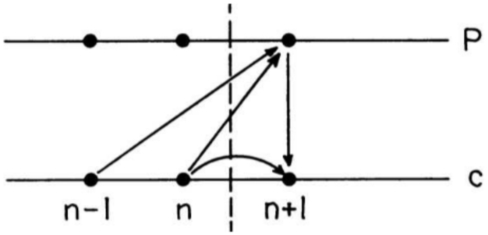
- Same problem, even within the method!

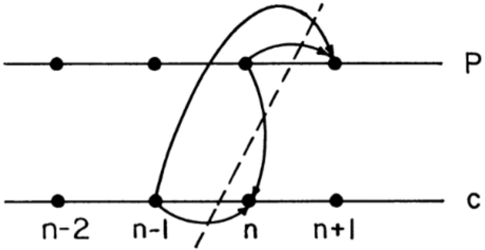
Example: Miranker and Liniger (1967)

Predictor corrector method

$$y_{n+1}^{p} = y_n^c + \frac{h}{2}(f(t, y_n^c) - f(t, y_{n-1}^c))$$
$$y_{n+1}^{c} = y_n^c + \frac{h}{2}(f(t, y_{n+1}^p) + f(t, y_n^c))$$

Example: Miranker and Liniger (1967)





Example: Miranker and Liniger (1967)

Predictor corrector method

$$\begin{aligned} y^p_{n+1} &= y^c_n + 2hf(t, y^c_n) \\ y^c_{n+1} &= y^c_n + \frac{h}{2}(f(t, y^p_n) + f(t, y^c_{n-1})) \end{aligned}$$

Parareal

- Large scale method
- Iterative
- Easy to implement
- Coarse (and cheap!) method $y_{i+1} = \mathcal{G}(y_i, t_i, t_{i+1})$
- Fine method $y_{i+1} = \mathcal{F}(y_i, t_i, t_{i+1})$

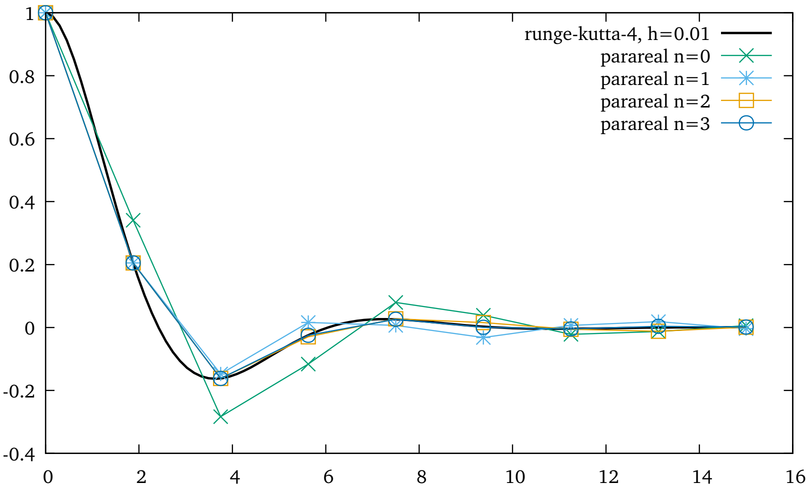
Parareal

$$y_{j+1}^{k+1} = \mathcal{G}(y_j^{k+1}, t_j, t_{j+1}) + \mathcal{F}(y_j^k, t_j, t_{j+1}) - \mathcal{G}(y_j^k, t_j, t_{j+1})$$

Example: damped harmonic oscillator

$$y'' + 2\zeta\omega_0 y' + \omega_0^2 y = 0$$

$$\begin{aligned} q' &= p \\ p' &= -2\zeta\omega_0 p - \omega_0^2 q \end{aligned}$$


runge-kutta-4, h=0.01
parareal n=0
parareal n=1
parareal n=2
parareal n=3
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
0
2
4
6
8
10
12
14
16

Instability

For most (interesting) problems (hyperbolic equations, Navier-Stokes with large Reynolds number)

- Convergence is too slow

or

- Parareal is unstable

Modifications and enhancements exist.

- Generalisation of Parareal
- Extend (spatial) MultiGrid method to time domain
- Non-intrusive
- Implemented in C++ $\rightarrow$ XBraid

- Parallel Full Approximation Scheme in Space-Time
- deferred correction method (See Gander 50 years paper)
- Shows promise in practice, but convergence behaviour is not fully understood.
- Implemented in C++ $\rightarrow$ PFASST++

- Active community: parallel-in-time.org (software directory)