Natural Frequencies and Mode Shapes of Timoshenko Beams with Attachments

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Abstract: The Laplace transform is used to obtain a solution for a Timoshenko beam on an elastic foundation with several combinations of discrete in-span attachments and with several combinations of attachments at the boundaries. These attachments include translation and torsion springs, masses, and undamped single degree-of-freedom systems. The Laplace transform technique, which apparently has not been used previously to solve for the eigenvalues of coupled systems of equations, produces a solution in which the boundary conditions can be considered independently of the number and type of in-span attachments. Taking advantage of this independence, many specific combinations of in-span attachments and boundary attachments are examined. It is shown that the Laplace transform method removes some of the drawbacks of the most frequently used methods used to solve Timoshenko beams with in-span attachments: beam partitioning, Green's functions, and Lagrange multipliers. Excellent agreement is found upon comparing previously reported results for a wide range of boundary conditions and attachments. In addition, several sets of new results are given.

Key words: Timoshenko, beams, attachments, natural frequencies, Laplace transform method

1. INTRODUCTION

Beams play an important role in the creation of mechanical, electromechanical, and civil systems. Many of these systems are subjected to dynamic excitation. As a consequence, the determination of the natural frequencies and mode shapes of linear elastic beams of various configurations and with various constraints and attachments have been studied for the last sixty years. The vibratory motion of beams are described by either the Euler (thin) beam theory or the Timoshenko beam theory, which includes the effects of the rotary inertia and the shear of the beam's cross-section. The solutions that have been obtained using these theories considered beams of constant cross-section and beam properties, and beams whose geometry and properties varied along the length of the beam. Typical attachments, both on the interior and on the boundaries, included various combinations of translation springs, torsion springs, masses, single degree-of-freedom systems, and multi-degree-of-freedom systems.

Expressions for the natural frequencies and modes shapes for Timoshenko beams with constant cross-section and constant properties and attachments only on the boundaries have been obtained by several investigators. A general solution for the frequency equation and mode shapes for a system with translation and torsion springs attached at each end and for a mass attached at one end was obtained by the author (Magrab, 1979). This result was then reduced to cases corresponding to the classical boundary conditions. No numerical results were presented. A slightly more general set of boundary conditions in which an additional mass is attached at the other end of the beam has been obtained (White and Heppler, 1995) and some numerical results presented. The general result was also reduced to cases corresponding to the classical boundary conditions. Other investigators have examined the vibrations of cantilever beams with only an end mass (Brunch and Mitchell, 1987; Abramovich and Hamburger, 1991). In all these cases, the standard solution method was used (Huang, 1961).

Timoshenko beams with attachments at their boundaries and whose cross-section varies along the length of the beam have also been examined. Several authors have investigated the case of a cantilever beam with an end mass in which the geometry and the properties vary along the length of the beam. A linearly varying tapered cantilever beam was considered and the method of Frobenius was used to arrive at the frequency equation (Lee and Lin, 1992). A cantilever beam that is composed of two contiguous beams of different cross-sections has also been investigated (Rossi et al., 1990), as was a cantilever beam with an end mass and with discrete passive piezoelectric elements distributed at various sections along the length (Maxwell and Asokanthan, 2004). In both studies, a partitioning method was used in which each section of the original beam was considered separately and the frequency equation was obtained by satisfying the boundary conditions of the original beam and the continuity conditions where the various sections of the beam meet. The solutions for each section of the beam were obtained using the standard solution method cited above. The partitioning method was used to obtain the natural frequencies for several sets of boundary conditions and for beams with different constant cross-sections along various portions of the beam and with in-span translation springs and masses. The beam sections were taken so that the ends of each section coincided with either a point of attachment or where the cross-section changed (Farghaly, 1994).

The partitioning method also has been used to consider beams with n in-span attachments by breaking up the beam into $n + 1$ beams and then satisfying at the n attachment points the appropriate continuity conditions. Using this method, a beam carrying an undamped single degree-of-freedom system for three sets of classical boundary conditions was studied (Rossi et al., 1993). The partitioning method has also been used to examine the case of a cantilever beam with a mass at its free end that has an in-span translation spring and an in-span torsion spring (Abramovich and Hamburger, 1992) and to examine a cantilever beam and beam clamped at both ends in which there are m in-span translation spring located at m distinct locations (Lin and Chang, 2005).

Other solutions have been obtained for systems having in-span attachments by using Green's functions and Lagrange multipliers. Both of these methods require the mode shapes and natural frequencies for the beam without in-span attachments. Using Green's functions, the solution for a beam carrying an n-degree of freedom system at a point for classical boundary conditions has been obtained (Kukla, 1997). In another investigation, Green's functions were used to obtain a formal solution for multiple damped single degree-of-freedom systems in translation and torsion attached to arbitrary locations along the beam (Cabanski, 2002). The Lagrange multiplier method has been used for beams with any combination of torsion and translation springs, masses, and undamped single degree-of-freedom systems for simply supported boundary conditions and for a cantilever beam with a mass at its free end (Posiadala, 1997). A different approach, in which the governing equations were combined to obtain a first order differential equation for the shearing force and then integrated numerically over m points, has been used (Matsuda et al., 1992). This study considered a cantilever Timoshenko beam with a mass at its free end and restrained with a torsion spring and translation spring and with in-span translation and torsion springs. The size of the characteristic equation depended on m.

The techniques used to date to solve for the natural frequencies and mode shapes of Timoshenko beams with in-span attachments have drawbacks. The Green's function method requires one to determine first the Green's function for each set of boundary conditions considered. The Lagrange multiplier method requires that one first determine the mode shapes and natural frequencies for the beam without any in-span attachments. The first n modes are then used to obtain numerical results after the value of n has been determined based on some convergence criterion. The beam partitioning method for n beam sections for $n - 1$ attachment locations requires an n by 4 frequency determinant and that the system of equations comprising the frequency equation has to be modified each time the boundary conditions at either end of the beam changes and, in some cases, when the relative position of each attachment of one type with respect to another attachment of another type changes.

A powerful method for solving for the natural frequencies of beams with constant cross-section and with various types of in-span and boundary attachments is the Laplace transform. The Laplace transform method is in many ways an easier method to use for systems with discrete discontinuities because the method can handle them directly; that is, without any additional considerations being required. This method apparently has not been used for the Timoshenko beam. It has, however, been used to solve for the natural frequencies of thin plates and Euler beam systems with various types of attachments (Magrab, 1968; Magrab, 1979; Jen and Magrab, 1993; Balachandran and Magrab, 2004). We shall use the Laplace transform with respect to the spatial variable to obtain a general expression from which the natural frequency coefficients and mode shapes for a Timoshenko beam of constant cross-section for a wide variety of boundary conditions and for any combination of four different types of in-span attachments can be determined. It will be seen that this method: (i) eliminates the drawbacks of the previous methods; (ii) very easily handles systems that have several different types of discrete in-span attachments applied simultaneously and that have various types of attachments at their boundaries; and (iii) the final solution is such that one can take advantage of symbolic computer problem solving environments, such as MATLAB, to examine numerous special cases of the general solution. Numerical results are compared to those obtained by other methods and several sets of new results are given.

2. FORMULATION AND SOLUTION

Consider a Timoshenko beam on an elastic foundation of stiffness k_w that has attached at different in-span locations a translational spring with stiffness k_1, a torsion spring with stiffness k_{t2}, a mass m having a rotational inertia j_m, and a single degree-of-freedom system having a mass m_{sdof} and a spring with stiffness k_{sdof}. We shall consider the general boundary conditions shown in Figure 1. Each end of the beam is restrained by a translation spring and a torsion spring. At the right end of the beam there are attached a mass m_R with a rotational inertia j_R and an undamped single degree-of-freedom system with a mass $m_{BR,sodf}$ and a

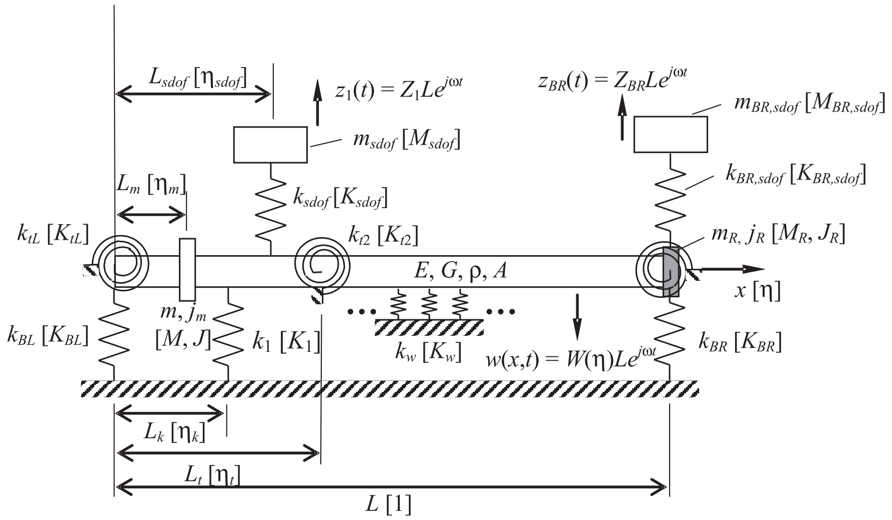

Figure 1. Notation and locations of beam attachments. Quantities in brackets are the non-dimensional equivalents.

spring stiffness $k_{BR,sodf}$. Using the definitions in Tables 1 and 2 and referring to Figure 1, the governing equations of motion for the free harmonic oscillations of a Timoshenko beam on an elastic foundation and with constant cross-section and constant properties are given by (Magrab, 1979; Cabański, 2002)

$$\begin{aligned} W''(\eta) + \gamma_{bs} R_o^2 k_\Omega W(\eta) - \Psi'(\eta) + \gamma_{bs} R_o^2 \sum_{p=1}^{3} A_p W(\eta) \delta\left(\eta - \eta_p\right) &= 0 \\ \gamma_{bs} R_o^2 \Psi''(\eta) + b_\Omega \gamma_{bs} R_o^4 \Psi(\eta) + W'(\eta) + \gamma_{bs} R_o^2 \sum_{q=1}^{2} B_p \Psi(\eta) \delta\left(\eta - \hat{\eta}_q\right) &= 0 \end{aligned} \tag{1}$$

where the prime denotes the derivative with respect to η,

$$k_\Omega = \Omega^4 - K_w \quad b_\Omega = \Omega^4 - 1/\left(\gamma_{bs} R_o^4\right)$$

and

$$\begin{array}{lll} A_1 = M\Omega^4 & \eta_1 = \eta_m & (0 < \eta_m < 1) \\ A_2 = M_{sdof} \Omega^4 A_{M_s}(\Omega) & \eta_2 = \eta_{sdof} & (0 < \eta_{sdof} < 1) \\ A_3 = -K_1 & \eta_3 = \eta_k & (0 < \eta_k < 1) \\ B_1 = J\Omega^4 & \hat{\eta}_1 = \eta_m & (0 < \eta_m < 1) \\ B_2 = -K_{t2} & \hat{\eta}_2 = \eta_t & (0 < \eta_t < 1) \end{array} \tag{2}$$

Table 1. Nomenclature for dimensional quantities.

Quantity	Units	Description
General		
x	m	x-axis
$w(x,t) = W(\eta)Le^{j\omega t}$	m	Transverse displacement of Timoshenko beam
L	m	Length of beam
k_w	N/m^2	Modulus of elastic foundation; runs continuously along the length of beam
ρ	kg/m^3	Beam density
A	m^2	Beam cross-sectional area
E	N/m^2	Young's modulus
$G = E/(2(1+\nu))$	N/m^2	Shear modulus
ν		Poisson's ratio
ω	rad/s	Radian frequency
I	m^4	Moment of inertia of beam cross-section
$r_o = \sqrt{I/A}$	m	Radius of gyration of beam cross-section
t_o §	s	Characteristic time of beam (see footnote)
$m_b = \rho AL$	kg	Mass of beam
$\psi(x,t) = \Psi(\eta)e^{j\omega t}$	rad	Angle of rotation of cross-section of Timoshenko beam due to bending only
κ		Shear correction actor, which is a constant relating A to an effective area over which the shear stress is constant; it is a function of the cross-section shape
$\gamma_{bs} = 2(1+\nu)/\kappa$		A constant that relates the shear correction factor and the wave propagation speed at very high frequencies
Boundary attachments		
k_{tL}, k_{tR}	Nm	Boundary torsion spring constants
k_{BL}, k_{BR}	N/m	Boundary transverse spring constants
j_R	kg · m^2	Rotational inertia of concentrated mass attached at $x = L$
m_R	kg	Concentrated mass attached to the beam at $x = L$
$m_{BR,sdof}$	kg	Mass of single degree-of-freedom system at $x = L$
$k_{BR,sdof}$	N/m	Spring constant of single degree-of-freedom system at $x = L$
$z_{BR}(t) = Z_{BR}Le^{j\omega t}$	m	Displacement of mass of single degree-of-freedom system attached at $x = L$
In-span attachments		
m	kg	Mass attached to the beam at $x = L_m$ $(0 < L_m < L)$
j_m	kg · m^2	Rotational inertia of mass attached at $x = L_m$ $(0 < L_m < L)$
L_α	m	Locations of various in-span attachments shown in Figure 1
k_1	N/m	Spring constant of transverse spring located at $x = L_k$ $(0 < L_k < L)$
k_{t2}	N m/rad	Spring constant of torsion spring attached at $x = L_t$ $(0 < L_t < L)$
$z_1(t) = Z_1 Le^{j\omega t}$	m	Displacement of mass of single degree-of-freedom system attached at an interior location
m_{sdof}	kg	Mass of single degree-of-freedom system at $x = L_{sdof}$ $(0 < L_{sdof} < L)$
k_{sdof}	N/m	Spring constant of single degree-of-freedom system at $x = L_{sdof}$ $(0 < L_{sdof} < L)$

§ $t_o = \sqrt{\dfrac{\rho AL^4}{EI}} = \sqrt{\dfrac{\rho L^2}{ER_o^2}} = \sqrt{\dfrac{m_b L}{AER_o^2}}$, where R_o is defined in Table II.

Table 2. Nomenclature for non-dimensional quantities.

General	Boundary attachments		In-span attachments	
$\eta = x/L$	$M_R = \dfrac{m_R}{m_b}$	$J_R = \dfrac{j_R}{m_b L^2}$	$M = \dfrac{m}{m_b}$	$J = \dfrac{j_m}{m_b L^2}$
$\eta_\alpha = L_\alpha/L$	$K_{BL} = \dfrac{k_{BL} L^3}{EI}$	$K_{BR} = \dfrac{k_{BR} L^3}{EI}$	$K_1 = \dfrac{k_1 L^3}{EI}$	$K_{t2} = \dfrac{k_{t2} L}{EI}$
$\Omega^4 = \omega^2 t_o^2$	$K_{tL} = \dfrac{k_{tL} L}{EI}$	$K_{tR} = \dfrac{k_{tR} L}{EI}$	$K_{sdof} = \dfrac{k_{sdof} L^3}{EI}$	$M_{sdof} = \dfrac{m_{sdof}}{m_b}$
$R_o = r_o/L$	$M_{BR,sdof} = \dfrac{m_{BR,sdof}}{m_b}$			
$K_w = \dfrac{k_w L^3}{EI}$	$K_{BR,sdof} = \dfrac{k_{BR,sdof} L^3}{EI}$			

$$A_{M_s}(\Omega) = \left(1 - \frac{M_{sdof}}{K_{sdof}}\Omega^4\right)^{-1}.$$

The boundary conditions at $\eta = 0$ are (Magrab, 1979)

$$\begin{aligned} W'(0) - c_1 W(0) - \Psi(0) &= 0 \\ \Psi'(0) - c_2 \Psi(0) &= 0 \end{aligned} \tag{3a}$$

and those at $\eta = 1$ are

$$\begin{aligned} W'(1) - c_3 W(1) - \Psi(1) &= 0 \\ \Psi'(1) - c_4 \Psi(1) &= 0 \end{aligned} \tag{3b}$$

where

$$\begin{aligned} c_1 &= \gamma_{bs} R_o^2 K_L \quad c_2 = K_{tL} \\ c_3 &= \gamma_{bs} R_o^2 \left(-K_R + \Omega^4 M_R + \Omega^4 M_{BR,sdof} B_{BR}(\Omega)\right) \\ c_4 &= \left(J_R \Omega^4 - K_{tR}\right) \\ B_{BR}(\Omega) &= \left(1 - \frac{M_{BR,sdof}}{K_{BR,sdof}}\Omega^4\right)^{-1}. \end{aligned} \tag{4}$$

It is noted that (1) and (3) represent the spatial eigenvalue problem.

The general representation for the boundary conditions given by (3) represent 93 different combinations of boundary conditions. They can be reduced to the classical boundary conditions by talking the appropriate limits. For example, if the end at $\eta = 1$ is simply supported, then $K_R \to \infty$ and $J_R = K_{tR} = 0$. This is equivalent to letting $c_3 \to \infty$ and setting $c_4 = 0$. Hence, we divide the first of (3b) by c_3 and take the limit as $c_3 \to \infty$ and we set $c_4 = 0$ in the second of (3b). For the details of this procedure for Euler beams see Balachandran and

Magrab (2004) and for the Euler and Timoshenko beams see Magrab (1979). We shall make use of several limiting cases when we examine special cases of the final results.

We now take the Laplace transform of (1) with respect to η and obtain

$$\begin{aligned} \left[s^2 + \gamma_{bs} R_o^2 k_\Omega\right] \bar{W}(s) - s\bar{\Psi}(s) &= \bar{G}_1(s) \\ s\bar{W}(s) + \left[\gamma_{bs} R_o^2 \left(s^2 + R_o^2 b_\Omega\right)\right] \bar{\Psi}(s) &= \bar{G}_2(s) \end{aligned} \tag{5}$$

where s is the Laplace transform parameter,

$$\begin{aligned} \bar{G}_1(s) &= sW(0) + W'(0) - \Psi(0) - \gamma_{bs} R_o^2 \sum_{p=1}^{3} A_p W(\eta_p) e^{-s\eta_p} \\ \bar{G}_2(s) &= s\gamma_{bs} R_o^2 \Psi(0) + \gamma_{bs} R_o^2 \Psi'(0) + W(0) - \gamma_{bs} R_o^2 \sum_{q=1}^{2} B_q \Psi(\hat{\eta}_q) e^{-s\hat{\eta}_q} \end{aligned} \tag{6}$$

and the over bar denotes the Laplace transform of the function. In (6), $W(0)$ is the displacement at $\eta = 0$, $W'(0)$ is the slope of the neutral axis at $\eta = 0$, $\Psi(0)$ is the rotation of the cross-section due to bending at $\eta = 0$, and $\Psi'(0)$ is a quantity that is proportional to the bending moment at $\eta = 0$. These quantities are determined from the boundary conditions at $\eta = 0$ and $\eta = 1$. The quantity $W(\eta_p)$ is the displacement at $\eta = \eta_p$ and the quantity $\Psi(\hat{\eta}_q)$ is the rotation of the cross-section at $\eta = \hat{\eta}_q$.

We see from (3a) that any two of the four unknown quantities at $\eta = 0$ can be expressed in terms of the remaining two. We choose to express $W'(0)$ in terms of $W(0)$ and $\Psi(0)$ and $\Psi'(0)$ in terms of $\Psi(0)$; that is,

$$\begin{aligned} W'(0) &= c_1 W(0) + \Psi(0) \\ \Psi'(0) &= c_2 \Psi(0). \end{aligned} \tag{7}$$

Upon substituting (7) into (6), we find that

$$\begin{aligned} \bar{G}_1(s) &= (s + c_1) W(0) - \gamma_{bs} R_o^2 \sum_{p=1}^{3} A_p W(\eta_p) e^{-s\eta_p} \\ \bar{G}_2(s) &= \gamma_{bs} R_o^2 (s + c_2) \Psi(0) + W(0) - \gamma_{bs} R_o^2 \sum_{q=1}^{2} B_q \Psi(\hat{\eta}_q) e^{-s\hat{\eta}_q}. \end{aligned} \tag{8}$$

Notice that this operation has left only two unknowns, $W(0)$ and $\Psi(0)$, which will be determined from the boundary conditions at $\eta = 1$. This reduction is possible because the 'initial conditions' resulting from the Laplace transform have system-specific interpretations.

Solving (5) for the transformed quantities $\bar{W}(s)$ and $\bar{\Psi}(s)$, we obtain

$$
\begin{aligned}
\bar{W}(s) &= \frac{1}{\bar{D}_o(s)}\left\{\left[s^3 + c_1 s^2 + R_o^2\Omega^4 s + c_1 R_o^2 b_\Omega\right] W(0) + \left[s^2 + c_2 s\right]\Psi(0)\right.\\
&\left. - \gamma_{bs} R_o^2 \sum_{p=1}^{3} A_p W(\eta_p)\left(s^2 + R_o^2 b_\Omega\right) e^{-s\eta_p} - \sum_{q=1}^{2} B_q \Psi(\hat{\eta}_q) s e^{-s\hat{\eta}_q}\right\}\\
\bar{\Psi}(s) &= \frac{1}{\bar{D}_o(s)}\left\{\left[s^3 + c_2 s^2 + \gamma_{bs} R_o^2 k_\Omega s + c_2 \gamma_{bs} R_o^2 k_\Omega\right]\Psi(0)\right.\\
&+ \left[-c_1 s/\left(\gamma_{bs} R_o^2\right) + k_\Omega\right] W(0) + \sum_{p=1}^{3} A_p W(\eta_p) s e^{-s\eta_p}\\
&\left. - \sum_{q=1}^{2} B_q \Psi(\hat{\eta}_q)\left(s^2 + \gamma_{bs} R_o^2 k_\Omega\right) e^{-s\hat{\eta}_q}\right\} \qquad (9)
\end{aligned}
$$

where

$$
\bar{D}_o(s) = \left(s^2 - \alpha^2\right)\left(s^2 + \beta^2\right) \qquad (10)
$$

and

$$
\begin{aligned}
\alpha^2, \beta^2 &= \frac{R_o^2}{2}\left\{\mp F_1 + \sqrt{F_2}\right\}\\
F_1 &= \Omega^4 + \gamma_{bs} k_\Omega\\
F_2 &= \left[\Omega^4 - \gamma_{bs} k_\Omega\right]^2 + 4k_\Omega/R_o^4 \qquad (11)
\end{aligned}
$$

provided that when $K_w = 0$

$$
\Omega^4 < \frac{1}{\gamma_{bs} R_o^4} \qquad (12a)
$$

and that when $K_w > 0$

$$
\begin{aligned}
F_2 &> 0\\
\sqrt{F_2} - F_1 &> 0. \qquad (12b)
\end{aligned}
$$

We note that

$$
\begin{aligned}
\beta^2 - \alpha^2 &= R_o^2 F_1\\
\beta^2 + \alpha^2 &= R_o^2 \sqrt{F_2}\\
\alpha^2\beta^2 &= -\gamma_{bs} R_o^4 b_\Omega k_\Omega. \qquad (13)
\end{aligned}
$$

To obtain the inverse Laplace transforms of (9), we use the transform pairs given in Appendix A to obtain

$$\begin{aligned} W(\eta) &= f_1(\eta)W(0) + f_2(\eta)\Psi(0) - H_1(\eta) \\ \Psi(\eta) &= g_1(\eta)W(0) + g_2(\eta)\Psi(0) + H_2(\eta) \end{aligned} \tag{14}$$

where

$$\begin{aligned} H_1(\eta) &= \gamma_{bs} R_o^2 \sum_{p=1}^{3} A_p f_{3p}(\eta, \eta_p) W(\eta_p) + \sum_{q=1}^{2} B_q f_{4q}(\eta, \hat{\eta}_q)\Psi(\hat{\eta}_q) \\ H_2(\eta) &= \sum_{p=1}^{3} A_p g_{3p}(\eta, \eta_p) W(\eta_p) - \sum_{q=1}^{2} B_q g_{4q}(\eta, \hat{\eta}_q)\Psi(\hat{\eta}_q) \end{aligned} \tag{15}$$

and the definitions of f_j and g_j are given in Appendix B.

We now determine the two unknown constants $W(0)$ and $\Psi(0)$ from the boundary conditions at $\eta = 1$, which are given by (3b). Upon substituting (14) into (3b), we find that

$$\begin{aligned} W(0) &= \frac{1}{D^{bc}} [d_{22} P_1 - d_{12} P_2] \\ \Psi(0) &= \frac{1}{D^{bc}} [d_{11} P_2 - d_{21} P_1] \end{aligned} \tag{16}$$

where

$$\begin{aligned} D^{bc} &= D^{bc}(c_j) = d_{11}d_{22} - d_{12}d_{21} \\ d_{11} &= c_3 f_1(1) - f_1'(1) + g_1(1) \\ d_{12} &= c_3 f_2(1) - f_2'(1) + g_2(1) \\ d_{21} &= c_4 g_1(1) - g_1'(1) \\ d_{22} &= c_4 g_2(1) - g_2'(1) \end{aligned} \tag{17}$$

and

$$\begin{aligned} P_1 &= c_3 H_1(1) - H_1'(1) - H_2(1) \\ P_2 &= -c_4 H_2(1) + H_2'(1). \end{aligned} \tag{18}$$

The definitions of f'_j and g'_j are given in Appendix B and the prime denotes the derivative with respect to η.

In anticipation of what is to follow, we have explicitly denoted in (17) the dependence of D^{bc} on the constants $c_1 \ldots c_4$, which appear in the general boundary conditions. The values of each c_j depend on the boundary conditions at each end of the beam. We shall be considering the following four sets of boundary conditions: (i) hinged at both ends, which we indicate by setting $bc = hh$; (ii) clamped at both ends, which we indicate by setting $bc = cc$; (iii) cantilever beam, which we indicate by setting $bc = cf$; and (iv) cantilever beam with attachments at its free end, which we indicate by setting $bc = cc_3$. How the various expressions are obtained for each set of boundary conditions is given subsequently.

Upon substituting (16) into (14) and collecting terms, we arrive at

$$\begin{aligned} W^{bc}(\eta) &= \frac{1}{D^{bc}}\left[\sum_{p=1}^{3} A_p C_{1p}^{bc}(\eta,\eta_p)W^{bc}(\eta_p) + \sum_{q=1}^{2} B_q C_{2q}^{bc}(\eta,\hat{\eta}_q)\Psi^{bc}(\hat{\eta}_q)\right] \\ \Psi^{bc}(\eta) &= \frac{1}{D^{bc}}\left[\sum_{p=1}^{3} A_p C_{3p}^{bc}(\eta,\eta_p)W^{bc}(\eta_p) + \sum_{q=1}^{2} B_q C_{4q}^{bc}(\eta,\hat{\eta}_q)\Psi^{bc}(\hat{\eta}_q)\right] \end{aligned} \tag{19}$$

where

$$\begin{aligned} C_{1p}^{bc}(\eta,\eta_p) &= C_{1p}^{bc}(\eta,\eta_p,c_j) = d_1(\eta)\gamma_{bs}R_o^2\left(c_3 f_{3p}(1,\eta_p) - f_{3p}'(1,\eta_p)\right) \\ &- D^{bc}\gamma_{bs}R_o^2 f_{3p}(\eta,\eta_p) - (d_1(\eta)+c_4 d_2(\eta))\, g_{3p}(1,\eta_p) + d_2(\eta)g_{3p}'(1,\eta_p) \\ C_{2q}^{bc}(\eta,\hat{\eta}_q) &= C_{2q}^{bc}(\eta,\hat{\eta}_q,c_j) = d_1(\eta)\left(c_3 f_{4q}(1,\hat{\eta}_q) - f_{4q}'(1,\hat{\eta}_q)\right) - D^{bc} f_{4q}(\eta,\hat{\eta}_q) \\ &+ (d_1(\eta)+c_4 d_2(\eta))\, g_{4q}(1,\hat{\eta}_q) - d_2(\eta)g_{4q}'(1,\hat{\eta}_q) \\ C_{3p}^{bc}(\eta,\eta_p) &= C_{3p}^{bc}(\eta,\eta_p,c_j) = d_3(\eta)\gamma_{bs}R_o^2\left(c_3 f_{3p}(1,\eta_p) - f_{3p}'(1,\eta_p)\right) + D^{bc} g_{3p}(\eta,\eta_p) \\ &- (d_3(\eta)+c_4 d_4(\eta))\, g_{3p}(1,\eta_p) + d_4(\eta)g_{3p}'(1,\eta_p) \\ C_{4q}^{bc}(\eta,\hat{\eta}_q) &= C_{4q}^{bc}(\eta,\hat{\eta}_q,c_j) = d_3(\eta)\left(c_3 f_{4q}(1,\hat{\eta}_q) - f_{4q}'(1,\hat{\eta}_q)\right) - D^{bc} g_{4q}(\eta,\hat{\eta}_q) \\ &+ (d_3(\eta)+c_4 d_4(\eta))\, g_{4q}(1,\hat{\eta}_q) - d_4(\eta)g_{4q}'(1,\hat{\eta}_q) \end{aligned} \tag{20}$$

and

$$\begin{aligned} d_1(\eta) &= d_{22} f_1(\eta) - d_{21} f_2(\eta) \\ d_2(\eta) &= -d_{12} f_1(\eta) + d_{11} f_2(\eta) \\ d_3(\eta) &= d_{22} g_1(\eta) - d_{21} g_2(\eta) \\ d_4(\eta) &= -d_{12} g_1(\eta) + d_{11} g_2(\eta). \end{aligned} \tag{21}$$

Equations (19) are in terms of five unknown constants $W^{bc}(\eta_p)$, p = 1,2,3 and $\Psi^{bc}(\hat{\eta}_q)$, q = 1,2. To obtain the characteristic equation for this system, we note that (19) must be valid at each η_p and $\hat{\eta}_q$. Thus, we evaluate (19) for $W^{bc}(\eta)$ at $\eta = \eta_p$, p = 1,2,3 and for $\Psi^{bc}(\eta)$ at $\eta = \hat{\eta}_q$, q = 1,2. This results in the following system of equations presented in matrix form

$$[a]\{Y\} = 0 \tag{22}$$

where the elements of $[a]$ and $\{Y\}$ are, respectively,

$$\begin{aligned}
a_{ij} &= A_j C^{bc}_{1j}(\eta_i, \eta_j) - \delta_{ij} D^{bc} \quad i, j = 1, 2, 3 \\
&= B_{j-3} C^{bc}_{3j-3}(\eta_i, \hat{\eta}_{j-3}) \quad i = 1, 2, 3 \;\; j = 4, 5 \\
&= A_j C^{bc}_{3j}(\hat{\eta}_{i-3}, \eta_j) \quad i = 4, 5 \;\; j = 1, 2, 3 \\
&= B_{j-3} C^{bc}_{4j-3}(\hat{\eta}_{i-3}, \hat{\eta}_{j-3}) - \delta_{ij} D^{bc} \quad i, j = 4, 5 \\
y_i &= W^{bc}(\eta_i) \quad i = 1, 2, 3 \\
&= \Psi^{bc}(\hat{\eta}_{i-3}) \quad i = 4, 5
\end{aligned} \tag{23}$$

and δ_{ij} is the Kronecker delta.

It is seen that the size of the characteristic determinant is directly proportional to the number of in-span attachments. This was made possible by (7), which is a direct result of the Laplace transform method. By being able to directly ascribe a physical interpretation to two of the four unknown constants, a system with four unknown constants is immediately reduced to a system with two unknown constants. This reduced system permitted a relatively simple analytical solution to be obtained. Since the boundary conditions were expressed in a very general form, we have, because of the analytical form of the results, a means to straightforwardly reduce the general result its numerous special cases. Thus, in a sense, we have uncoupled the boundary conditions from the number and type of in-span attachments since the form of the solution does not change as the boundary conditions change. In the next section, we shall show how this specialization is done.

The characteristic equation for a constant-cross-section Timoshenko beam on an elastic foundation with general boundary conditions and having four different types of in-span attachments applied simultaneously, each at a different location, is given by

$$\det[a] = 0 \tag{24}$$

The values of Ω that satisfy (24) are the natural frequency coefficients Ω_n for the system.

It should be realized that it is not necessary to consider that we have four different types of attachments applied at four locations. If, for example, we want to consider three single degree-of-freedom systems attached at three different locations, then we simply change the definition of each A_p accordingly. Consequently, (24) is a very general result.

The normalized mode shapes corresponding to each Ω_n are

$$\begin{aligned}
W^{bc}_n(\eta) &= \frac{1}{W_{\max}} \left(\sum_{p=1}^{3} A_p C^{bc}_{1np}(\eta, \eta_p) \widehat{W}(\eta_p) + \sum_{q=1}^{2} B_q C^{bc}_{2nq}(\eta, \hat{\eta}_q) \widehat{\Psi}(\hat{\eta}_q) \right) \\
\Psi^{bc}_n(\eta) &= \frac{1}{\Psi_{\max}} \left(\sum_{p=1}^{3} A_p C^{bc}_{3np}(\eta, \eta_p) \widehat{W}(\eta_p) + \sum_{q=1}^{2} B_q C^{bc}_{4nq}(\eta, \hat{\eta}_q) \widehat{\Psi}(\hat{\eta}_q) \right)
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\widehat{W}(\eta_p) &= W^{bc}(\eta_p)/W^{bc}(\eta_1) \\
\widehat{\Psi}(\hat{\eta}_q) &= \Psi^{bc}(\hat{\eta}_q)/W^{bc}(\eta_1) \\
W_{\max} &= \max\left[\left|W_n^{bc}(\eta)\right|\right] \\
\Psi_{\max} &= \max\left[\left|\Psi_n^{bc}(\eta)\right|\right]
\end{aligned}$$

and Ω is replaced everywhere by Ω_n in all the functions comprising $C_{jnp}^{bc}(\eta, \eta_p)$. The expressions for $\widehat{W}(\eta_p)$ and $\widehat{\Psi}(\hat{\eta}_q)$ are obtained from (22) for $\Omega = \Omega_n$.

3. SPECIAL CASES

We now generate the special cases of D^{bc} and C^{bc} for four sets of boundary conditions by performing the following limiting operations:

Hinged–Hinged

$$\begin{aligned}
C_{jp}^{hh}(\eta, \eta_p) &= \lim_{\substack{c_1\to\infty \\ c_3\to\infty}} \frac{C_{jp}^{bc}(\eta, \eta_p, c_1, 0, c_3, 0)}{c_1 c_3} \qquad j = 1, 2, 3, 4 \\
D^{hh} &= \lim_{\substack{c_1\to\infty \\ c_3\to\infty}} \frac{D^{bc}(c_1, 0, c_3, 0)}{c_1 c_3}
\end{aligned} \tag{26}$$

Clamped–Clamped

$$\begin{aligned}
C_{jp}^{cc}(\eta, \eta_p) &= \lim_{\substack{c_1\to\infty \;\; c_2\to\infty \\ c_3\to\infty \;\; c_4\to\infty}} \frac{C_{jp}^{bc}(\eta, \eta_p, c_1, c_2, c_3, c_4)}{c_1 c_2 c_3 c_4} \qquad j = 1, 2, 3, 4 \\
D^{cc} &= \lim_{\substack{c_1\to\infty \;\; c_2\to\infty \\ c_3\to\infty \;\; c_4\to\infty}} \frac{D^{bc}(c_1, c_2, c_3, c_4)}{c_1 c_2 c_3 c_4}
\end{aligned} \tag{27}$$

Cantilever

$$\begin{aligned}
C_{jp}^{cf}(\eta, \eta_p) &= \lim_{\substack{c_1\to\infty \\ c_2\to\infty}} \frac{C_{jp}^{bc}(\eta, \eta_p, c_1, c_2, 0, 0)}{c_1 c_2} \qquad j = 1, 2, 3, 4 \\
D^{cf} &= \lim_{\substack{c_1\to\infty \\ c_2\to\infty}} \frac{D^{bc}(c_1, c_2, 0, 0)}{c_1 c_2}
\end{aligned} \tag{28}$$

Cantilever with Attachments

$$C_{jp}^{cc_3}(\eta,\eta_p) = \lim_{\substack{c_1\to\infty \\ c_2\to\infty}} \frac{C_{jp}^{bc}(\eta,\eta_p,c_1,c_2,c_3,0)}{c_1 c_2} \qquad j=1,2,3,4$$

$$D^{cc_3} = \lim_{\substack{c_1\to\infty \\ c_2\to\infty}} \frac{D^{bc}(c_1,c_2,c_3,0)}{c_1 c_2}. \tag{29}$$

The specific form of these functions created by the operations indicated in (26) to (29) will be given for only a few simple cases. In all cases, they will have been obtained by using the Symbolic toolbox in MATLAB to perform the indicated operations given above. For more details on this procedure see Magrab et al. (2005). The remaining cases will be given only in their high level algebraic form because, in most cases, their lower level forms are too lengthy.

(1) All in-span attachments removed

For this case, $A_p = B_q = 0$ and the natural frequency coefficients Ω_n are determined from (17); that is,

$$D^{bc} = 0. \tag{30}$$

The corresponding mode shapes are given by

$$W_n^{bc}(\eta) = \frac{1}{W_{\max}}\left(f_{1n}(\eta) + e_n f_{2n}(\eta)\right)$$

$$\Psi_n^{bc}(\eta) = \frac{1}{\Psi_{\max}}\left(g_{1n}(\eta) + e_n g_{2n}(\eta)\right) \tag{31}$$

where $f_{1n}, \ldots$ are given by (B1) and (B2) with Ω replaced by Ω_n and

$$e_n = -\frac{d_{11n}}{d_{12n}} = -\frac{d_{21n}}{d_{22n}}. \tag{32}$$

In general, $c_3 \to c_{3n}$ and $c_4 \to c_{4n}$. Referring to (17), we see that in equation (32) $d_{11n} = c_{3n} f_{1n}(1) - f'_{1n}(1) + g_{1n}(1)$, etc.

We now use (30) in (26)–(29) to obtain several special cases: hinged at both ends, clamped at both ends, clamped at one end and hinged at the other, and clamped at the left end and free at the right end (cantilever). For the latter case, we shall also examine the conditions for which there are attachments at the free end.

(a) Hinged at both ends

Upon using (26) and (30), we find that

$$D^{hh} = 0 \tag{33}$$

where

$$D^{hh} = \sinh\alpha\sin\beta. \tag{34}$$

The real roots of this equation are obtained when $\beta = n\pi$, $n = 1, 2, \ldots$. We then determine Ω_n from (11) to obtain

$$\Omega_n = \left[\frac{1}{2\gamma_{bs}R_o^4}\left\{D_n - \sqrt{D_n^2 - 4\gamma_{bs}R_o^4\left[(2\pi n)^4 + K_w\left(\gamma_{bs}R_o^2(n\pi)^2 + 1\right)\right]}\right\}\right]^{1/4} \tag{35}$$

where

$$D_n = 1 + (n\pi R_o)^2\left(1 + \gamma_{bs}\right) + \gamma_{bs}R_o^4K_w.$$

When the elastic foundation is not being considered, $K_w = 0$ in the above equations and the result, after notational differences are accounted for, agrees with (Huang, 1961).

(b) Clamped at both ends

Upon using (27) and (30), we find that

$$D^{cc} = 0 \tag{36}$$

where

$$\begin{aligned} D^{cc} &= 2 - 2\cosh\alpha\cos\beta + C_{cc}\sinh\alpha\sin\beta \\ C_{cc} &= \frac{1}{\alpha\beta b_o}\left[\left(\alpha^2+\beta^2\right)^2 + b_o\left(\alpha^2-\beta^2\right)\right] \end{aligned} \tag{37}$$

where

$$b_o = \frac{1}{\gamma_{bs}R_o^2}. \tag{38}$$

The algebraic form of C_{cc} differs from that given in (Huang, 1961), but when $K_w = 0$ numerically they are equal.

(c) Cantilever

Upon using (28) and (30), we find that

$$D^{cf} = 0 \tag{39}$$

where

$$
\begin{aligned}
D^{cf} &= 2 + C_{cf} \cosh\alpha \cos\beta + \left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right) \sinh\alpha \sin\beta \\
C_{cf} &= \frac{\left(\alpha^2 + \beta^2\right)^2}{b_o\left(\beta^2 - \alpha^2 - R_o^2\Omega^4\right)} - 2.
\end{aligned} \tag{40}
$$

The algebraic form of C_{cf} differs from that given in (Huang, 1961), but when $K_w = 0$ numerically they are equal.

(d) Cantilever with attachments at its free end

Upon using (29) and (30), we find that

$$
D^{cc_3} = 0 \tag{41}
$$

where

$$
\begin{aligned}
D^{cc_3} &= D^{cf} + c_3 \left\{C_{cf1} \cosh\alpha \sin\beta - C_{cf2} \sinh\alpha \cos\beta\right\} \\
C_{cf1} &= \frac{1}{\beta} \frac{\left(\beta^2 + \alpha^2\right)\left(\lambda^4 - \beta^2 - b_o\right)}{b_o\left(\beta^2 - \alpha^2 - R_o^2\Omega^4\right)} \\
C_{cf2} &= \frac{1}{\alpha} \frac{\left(\beta^2 + \alpha^2\right)\left(\lambda^4 + \alpha^2 - b_o\right)}{b_o\left(\beta^2 - \alpha^2 - R_o^2\Omega^4\right)}
\end{aligned} \tag{42}
$$

D^{cf} is given by (40) and, from (4),

$$
c_3 = \gamma_{bs} R_o^2 \left(-K_R + \Omega^4 M_R + \Omega^4 M_{BR,sdof} B_{BR}(\Omega)\right).
$$

We see that (41) is valid for any combination of the three attachments at $\eta = 1$: translational spring, mass (but not considering its rotational inertia), and single degree-of-freedom system. When there are no attachments at $\eta = 1$, then $c_3 = 0$ and (41) reduces to (39).

(e) Clamped at η = 0 and hinged at η = 1

To obtain the characteristic equation for these boundary conditions, we divide (41) by c_3 and take the limit as $c_3 \to \infty$ to arrive at

$$
D^{ch} = 0 \tag{43}
$$

where

$$
D^{ch} = C_{cf1} \cosh\alpha \sin\beta - C_{cf2} \sinh\alpha \cos\beta. \tag{44}
$$

(2) One in-span attachment

(i) Attachments with transverse motion only: $A_p \neq 0, B_q = 0$

For this case, (24) simplifies to

$$A_p C^{bc}_{1p}(\eta_p, \eta_p) - D^{bc} = 0 \quad p = 1, 2, 3. \tag{45}$$

From (2), we see that when $p = 1$ we have a mass attached at η_m and $A_1 = M\Omega^4$. When p = 2, we have a single degree-of-freedom system attached at η_{sdof} and $A_2 = M_{sdof}\Omega^4 A_{M_s}$. A special case of this is when $K_{sdof} \to \infty$, which results in $A_2 = M_{sdof}\Omega^4$; that is, this case is equivalent to the previous case where the mass is directly attached to the beam. When p = 3, we have a translational spring attached at η_k and $A_3 = -K_1$. When $K_1 \to \infty$, we have the case of a rigid in-span support for which the displacement is zero at η_k, which is equivalent to an intermediate hinged constraint. In this situation, (45) becomes

$$C^{bc}_{13}(\eta_3, \eta_3) = 0. \tag{46}$$

In (45), when $A_p = 0$ we obtain (30).

The natural frequency coefficients Ω_n are obtained from the solution to (45). The corresponding mode shapes are obtained from (19) as

$$\begin{aligned} W^{bc}_n(\eta) &= \frac{1}{W_{\max}} C^{bc}_{n1p}(\eta, \eta_p) \\ \Psi^{bc}_n(\eta) &= \frac{1}{\Psi_{\max}} C^{bc}_{n3p}(\eta, \eta_p) \end{aligned} \tag{47}$$

where $C^{bc}_{njp}(\eta, \eta_p)$, j = 1, 3, are given by (20) with Ω replaced everywhere by Ω_n in all the functions comprising $C^{bc}_{njp}(\eta, \eta_p)$ and W_{max} and Ψ_{max} are as defined in (25).

(ii) Attachments with rotational motion only: $A_p = 0, B_q \neq 0$

For this case, (24) simplifies to

$$B_q C^{bc}_{4q}(\hat{\eta}_q, \hat{\eta}_q) - D^{bc} = 0 \quad q = 1, 2 \tag{48}$$

where, from (2), when $q = 1$ we have a mass with rotational inertia J attached at η_m (but where the translational effects of its mass M are ignored) and $B_1 = J\Omega^4$. When $q = 2$, we have a torsion spring attached at η_t and $B_2 = -K_{t2}$. When $K_{t2} \to \infty$, we have the case of a rigid in-span support for which the rotation is zero at η_t; this is equivalent to an intermediate clamped-slider constraint. In this situation, (48) becomes

$$C^{bc}_{42}(\hat{\eta}_2, \hat{\eta}_2) = 0. \tag{49}$$

The natural frequency coefficients Ω_n are obtained from the solution to (48). The corresponding mode shapes are obtained from (25) as

$$\begin{aligned} W_n^{bc}(\eta) &= \frac{1}{W_{\max}} C_{n2q}^{bc}(\eta, \hat{\eta}_q) \\ \Psi_n^{bc}(\eta) &= \frac{1}{\Psi_{\max}} C_{n4q}^{bc}(\eta, \hat{\eta}_q) \end{aligned} \tag{50}$$

where $C_{njp}^{bc}(\eta, \hat{\eta}_p)$, j = 2, 4, is given by (20) with Ω replaced everywhere by Ω_n in all the functions comprising $C_{njp}^{bc}(\eta, \hat{\eta}_p)$.

(3) Two in-span attachments

(i) Attachments with transverse motion only: $A_p \neq 0, B_q = 0$

For this system, (24) becomes

$$\begin{aligned} &\left[A_p C_{1p}^{bc}(\eta_p, \eta_p) - D^{bc}\right]\left[A_s C_{1s}^{bc}(\eta_s, \eta_s) - D^{bc}\right] \\ -\;& A_s A_p C_{1p}^{bc}(\eta_s, \eta_p) C_{1s}^{bc}(\eta_p, \eta_s) = 0 \quad p, s = 1, 2, 3; \quad p \neq s. \end{aligned} \tag{51}$$

The quantities A_p and A_s can each represent a mass, translation spring, or a single degree-of-freedom system. It should be realized that in the notation used in (51), when A_p and A_s represent different types of attachments, the numerical values of η_p and η_s can be equal. For example, a translation spring and a mass can be attached to the beam at the same location. Conversely, two equal masses or two identical single degree-of-freedom systems, for example, can each be attached at a different location.

The natural frequency coefficients Ω_n are obtained from the solution to (51). The corresponding mode shapes are obtained from (25) as

$$\begin{aligned} W_n^{bc}(\eta) &= \frac{1}{W_{\max}}\left[A_{np} C_{n1p}^{bc}(\eta, \eta_p) - e_{1nsp} C_{n1s}^{bc}(\eta, \eta_s)\right] \\ \Psi_n^{bc}(\eta) &= \frac{1}{\Psi_{\max}}\left[A_{np} C_{n3p}^{bc}(\eta, \eta_p) - e_{1nsp} C_{n3s}^{bc}(\eta, \eta_s)\right] \end{aligned} \tag{52}$$

where

$$e_{1nsp} = \frac{A_{np} C_{n1p}^{bc}(\eta_p, \eta_p) - D_n^{bc}}{C_{n1s}^{bc}(\eta_p, \eta_s)} \tag{53}$$

and in the definitions of $C_{njp}^{bc}(\eta, \eta_p)$ and $C_{njs}^{bc}(\eta, \eta_s)$, j = 1, 3, D_n^{bc}, A_{np}, and A_{ns} Ω is replaced everywhere by Ω_n.

(ii) Attachments with rotational motion only: $A_p = 0, B_q \neq 0$

For this system, (24) becomes

$$\left[B_1 C_{41}^{bc}(\hat{\eta}_1, \hat{\eta}_1) - D^{bc}\right]\left[B_2 C_{42}^{bc}(\hat{\eta}_2, \hat{\eta}_2) - D^{bc}\right] - B_2 B_1 C_{41}^{bc}(\hat{\eta}_2, \hat{\eta}_1) C_{42}^{bc}(\hat{\eta}_1, \hat{\eta}_2) = 0. \tag{54}$$

This is the case when a mass with rotational inertia J attached at $\hat{\eta}_m$ (but where the translational effects of its mass M are ignored) and $B_1 = J\Omega^4$ and when a torsion spring is attached at $\hat{\eta}_t$ and for which $B_2 = -K_{t2}$.

The natural frequency coefficients Ω_n are obtained from the solution to (54). The corresponding mode shapes are obtained from (25) as

$$\begin{aligned} W_n^{bc}(\eta) &= \frac{1}{W_{\max}}\left[B_{n1}C_{n21}^{bc}(\eta,\hat{\eta}_1) - e_{2n21}C_{n22}^{bc}(\eta,\hat{\eta}_2)\right] \\ \Psi_n^{bc}(\eta) &= \frac{1}{\Psi_{\max}}\left[B_{n1}C_{n41}^{bc}(\eta,\hat{\eta}_1) - e_{2n21}C_{n42}^{bc}(\eta,\hat{\eta}_2)\right] \end{aligned} \tag{55}$$

where

$$e_{2n21} = \frac{B_{n1}C_{n41}^{bc}(\hat{\eta}_1,\hat{\eta}_1) - D_n^{bc}}{C_{n42}^{bc}(\hat{\eta}_1,\hat{\eta}_2)} \tag{56}$$

and in the definitions of $C_{njp}^{bc}(\eta,\hat{\eta}_p)$ and $C_{njs}^{bc}(\eta,\hat{\eta}_s)$, $j = 2, 4$, D_n^{bc}, B_{np}, and B_{ns} Ω is replaced everywhere by Ω_n.

(iii) Attachments with transverse motion and rotational motion: $A_p \neq 0, B_q \neq 0$

For this system, (24) becomes

$$\begin{aligned} &\left[A_pC_{1p}^{bc}(\eta_p,\eta_p) - D^{bc}\right]\left[B_qC_{4p}^{bc}(\hat{\eta}_q,\hat{\eta}_q) - D^{bc}\right] \\ &- \; B_qA_pC^{bc}(\hat{\eta}_q,\eta_p)C_{2p}^{bc}(\eta_p,\hat{\eta}_q) = 0 \quad p = 1,2,3 \quad q = 1,2. \end{aligned} \tag{57}$$

This case can be used to examine, for example, the effects of a mass and its rotational inertia attached at $\eta_p = \hat{\eta}_q$ and the case of a translation spring and a torsion spring attached at $\eta_p = \hat{\eta}_q$. Regarding this latter case, if we let the stiffness of the translation spring approach infinity; that is, $p = 3$, then (57) becomes

$$\begin{aligned} &B_q\left[C_{13}^{bc}(\eta_3,\eta_3)C_{43}^{bc}(\hat{\eta}_q,\hat{\eta}_q) - C_{33}^{bc}(\hat{\eta}_q,\eta_3)C_{23}^{bc}(\eta_3,\hat{\eta}_q)\right] \\ &- \; D^{bc}C_{13}^{bc}(\eta_3,\eta_3) = 0 \quad q = 1,2. \end{aligned} \tag{58}$$

One special case of this equation, a Euler beam hinged at both ends, has been investigated (Ginsberg and Pham, 1995). If, in addition, we let $B_q = 0$, then (58) reduces to (46).

The natural frequency coefficients Ω_n are obtained from the solution to (57). The corresponding mode shapes are obtained from (25) as

$$\begin{aligned} W_n^{bc}(\eta) &= \frac{1}{W_{\max}}\left[A_{np}C_{1np}^{bc}(\eta,\eta_p) - e_{3npq}C_{2nq}^{bc}(\eta,\hat{\eta}_q)\right] \\ \Psi_n^{bc}(\eta) &= \frac{1}{\Psi_{\max}}\left[A_{np}C_{3np}^{bc}(\eta,\eta_p) - e_{3npq}C_{4nq}^{bc}(\eta,\hat{\eta}_q)\right] \end{aligned} \tag{59}$$

where

$$e_{3npq} = \frac{A_{np} C^{bc}_{n1p}(\eta_p, \eta_p) - D^{bc}_n}{C^{bc}_{n2q}(\eta_p, \hat{\eta}_q)} \tag{60}$$

and in the definitions of $C^{bc}_{njp}(\eta, \eta_p), j = 1,3$ and $C^{bc}_{njq}(\eta, \hat{\eta}_q), j = 2, 4, D^{bc}_n, A_{np}$, and B_{nq} Ω is replaced everywhere by Ω_n.

4. NUMERICAL RESULTS

The numerical results were obtained in the following manner. For each C^{bc}_{jp} given by (20) and for D^{bc} given by (17) a symbolic object was created using the Symbolic toolbox in MATLAB. Then each of these five symbolic objects was evaluated for the four sets of limiting conditions given by (26)–(29). The resulting twenty symbolic expressions were then converted to MATLAB functions and used to obtain the natural frequency coefficients Ω_n and corresponding mode shapes for the following special cases of (24): equations (30), (45), (51), and (57).

The expressions developed for the Timoshenko beam also can be used directly to obtain numerically the natural frequency coefficients and mode shapes for the Euler (thin) beam. From the definition of R_o, we note that for a beam with a rectangular cross-section of height h that $R_o = 0.289(h/L)$ and for a beam of circular cross-section of diameter d that $R_o = 0.250(d/L)$. It has been found that numerically the Timoshenko beam results reduce to the Euler beam results very closely when $R_o < 0.01$. This corresponds roughly to $h/L < 0.04$ and $d/L < 0.04$. We shall use the value $R_o = 0.001$ to represent the Euler beam. This value provides agreement to within approximately 0.01% for the third natural frequency coefficients and better than that for the first and second frequency coefficients.

To place in context the stiffness ratios for the various springs K_α, we note that a value of 3 for a cantilever beam indicates that a spring attached at its free end has a stiffness equal to the stiffness of the beam itself. When the beam is hinged at both ends and the spring is attached at the midpoint, a value of K_α of 48 indicates that the spring stiffness and the beam stiffness are equal. Lastly, when the beam is clamped at both ends and the spring is attached at the midpoint, a value of K_α of 192 indicates that the spring stiffness and the beam stiffness are equal.

To show the wide range of results that are special cases of the solution obtained here, we have summarized numerous comparisons with results from the literature. The comparisons are given in Tables 3 to 5, where we have compared, when available, the first three natural frequency coefficients Ω_n/π. In Table 3, we have given the lowest three natural frequency coefficients for Euler beams with one in-span attachment and/or attachments at the boundary. Seventeen different combinations of attachment situations and boundary conditions are examined. In Table 4, we have presented the lowest three natural frequency coefficients for Timoshenko beams with one in-span attachment and/or attachments at the boundary. Lastly, in Table 5 we have given the natural frequency coefficients for Euler beams with two in-span attachments for four different scenarios. It is seen from the results in these tables that the agreement is excellent in all cases. We also see from the very wide range of beam systems cited from the literature the very wide range of applicability of the solution presented here.

Table 3. Natural frequency coefficients for an Euler beam with one in-span attachment and/or attachments at the boundary – comparisons with some results in the literature (R_o = 0.001).

Case	Boundary/In-span attachments*	Boundary conditions**	Ω_1/π	Ω_2/π	Ω_3/π	Source***
1	None	c-c	1.50556	2.49951	3.49941	Present work
			1.5056	2.4998	3.5000	(Balachandran and Magrab, 2004)
		c-f	0.59686	1.49413	2.50005	Present work
			0.5969	1.4942	2.5002	(Balachandran and Magrab, 2004)
		h-h	0.99999	1.99992	2.99973	Present work
			1.0000	2.0000	3.0000	(Balachandran and Magrab, 2004)
2	$M_R = 1.0$	c-f	0.39722	1.28312	2.27071	Present work
			0.3972	1.2832	2.2709	(Balachandran and Magrab, 2004)
3	$K_{BR} = 10$	c-f	0.83999	1.52586	2.5067	Present work
			0.8400	1.5259	2.5069	(Balachandran and Magrab, 2004)
4	$M_R = 1, K_{BR} = 1$	c-f	0.42680	1.28323	2.27072	Present work
			0.4281	–	–	(Gürgöze, 1996)
5	$M = 1$ at $\eta = 0.5$	c-c	1.09423	2.49951	3.11432	Present work
			1.0943	–	–	(Balachandran and Magrab, 2004)
	$M = 1$ at $\eta = 0.375$	c-f	0.57323	1.17931	2.26093	Present work
			0.5732	1.1793	2.2610	(Naguleswaran, 1999)
	$M = 1$ at $\eta = 0.2$	c-c	1.36294	2.03655	3.07464	Present work
			1.36302	–	–	(Turhan, 2000)
6	$K_w = 100$	c-c	1.5757	2.51579	3.50539	Present work
			1.5756	2.5159	3.5058	(De Rosa and Maurizi, 1998)
	$M_R = 1, K_w = 1$	c-f	0.40486	1.28422	2.27093	Present work
		(from graph $\rightarrow$)	0.398	–	–	(De Rosa and Maurizi, 1998)
7	$K_1 = 700$ at $\eta = 0.5$	h-h	1.88225	1.99992	3.13944	Present work
			1.8823	–	–	(Albarracín, et al., 2004)
	$K_1 = 400$ at $\eta = 0.5$	c-f	0.93025	1.93123	2.50018	Present work
			0.93025	–	–	(Balachandran and Magrab, 2004)
8	$M_{BR,sdof} = 1$,	c-f	0.37922	0.56604	0.86863	Present work
	$K_{BR,sdof} = 10$		0.37922	–	0.86866	(Gürgöze, 1996)
	$M_{sdof} = 0.2$,	c-f	0.52422	0.71288	1.4943	Present work
	$K_{sdof} = 3$ at $\eta = 0.75$		0.52422	0.71288	1.4943	(Gürgöze, 1998)
	$M_{sdof} = 10$,	c-f	0.40745	0.66776	1.54906	Present work
	$K_{sdof} = 50$ at $\eta = 0.35$		0.40746	–	–	(Balachandran and Magrab, 2004)
9	$M_{BR,sdof} = K_{BR,sdof} = 1$	c-f	0.29320	0.64702	1.49877	Present work
	$M_{sdof} = K_{sdof} = 1$ at $\eta = 0.5$		0.29320	0.64702	1.49881	(Gürgöze, 1998))

* The values of the attachments are all zero, except as indicated.

** c = clamped; h = hinged (simply supported, pinned); f = free.

*** More recent representative references have been selected; many of these references contain citations to earlier work.

Table 4. Natural frequency coefficients for Timoshenko beam with one in-span attachment and/or attachments at the boundary – comparisons with some results in the literature.

Case	Boundary/In-span attachments*	Boundary conditions§	Ω_1/π	Ω_2/π	Ω_3/π	Source
1	$M_{sdof}=1$, $K_{sdof}=10$	c-c	0.55611	1.40438	2.11933	Present work
	$\gamma_{bs}=3.12$, $\eta_{sdof}=0.5$,		0.55611	1.40438	2.11933	(Rossi et al., 1993)
	$R_o=0.05$					
2	$M_R=1$, $\gamma_{bs}=3.12$,	c-f	0.38706	1.08393	1.67285	Present work
	$R_o=0.1$		0.386	1.084	1.673	(Rossi et al., 1990)
3	$M_{sdof}=1$, $K_{sdof}=100$	h-h	0.73151	1.28611	1.90924	Present work
	$R_o=0.05$, $\gamma_{bs}=3.12$,		0.73174	1.28724	1.90968	(Posiadala, 1997)
	$\eta_{sdof}=2/3$					
4	$M_{sdof}=1.5$, $K_{sdof}=10^3$	c-f	0.54399	1.00245	1.91688	Present work
	$R_o=0.05$, $\gamma_{bs}=4$,	(from graph →)	0.528	0.984	1.875	(Kukla, 1997)
	$\eta_{sdof}=0.4$					

*The values of the attachments are all zero, except as indicated.

§ c = clamped; h = hinged (simply supported, pinned); f = free.

Table 5. Natural frequency coefficients for Euler beam with two in-span attachments – comparisons with some results in the literature (R_o = 0.001).

Case	Boundary/In-span attachments*	Boundary conditions§	Ω_1/π	Ω_2/π	Ω_3/π	Source
1	$K_1=10$, $\eta_k=0.7$	c-c	1.54057	2.50341	3.51867	Present work
	$K_{t2}=2$, $\eta_t=0.7$		1.54064	2.50365	3.51928	(Naguleswaran, 2003)
2	$M_{sdof}=1$, $K_{sdof}=100$,	c-c	0.89109	0.99661	1.67513	Present work
	$\eta_{sdof}=0.2$					
	$M_{sdof}=1$, $K_{sdof}=100$,	(from graph →)	0.89	–	–	(Jen and Magrab, 1993)
	$\eta_{sdof}=0.4$					
3	$K_1=1000$, $\eta_{sdof}=0.2$	c-c	2.0762	2.93966	3.63757	Present work
	$K_1=1000$, $\eta_{sdof}=0.4$	(from graph →)	2.07	–	–	(Jen and Magrab, 1993)
4	$M_R=0.2$, $M=0.2$, $\eta_m=1/3$	c-f	0.49784	1.23968	2.04200	Present work
	$M=0.2$, $\eta_m=2/3$		0.49784	1.23968	2.04204	(Farghaly, 1994)

* The values of the attachments are all zero, except as indicated.

§ c = clamped; h = hinged (simply supported, pinned); f = free.

It is difficult to compare the mode shapes obtained from this method with those obtained by other methods. However, for the few papers in which modes shapes are presented (Abramovich and Hamburger, 1991; Rossi et al., 1993; Lin and Chang, 2005) it appears that qualitatively the present method produces very similar shapes.

In Tables 6 and 7 new results for the Timoshenko beam are presented. In Table 6, we tabulate the natural frequency coefficients for a cantilever beam and in Table 7 we tabulate the natural frequency coefficients for a beam clamped at both ends. The purpose of Table 6 is to illustrate the effects of adding attachments to a cantilever beam. The purpose of Table 7 is to show the effects of including the rotational inertia of the attached mass and to show the effects of a torsion spring when added to a translational spring and attached at the same in-span location. In obtaining these results, we have used a value $\kappa = 5/6$ which, for a Poisson's ratio of 0.3, yields $\gamma_{bs} = 3.12$. For a discussion of the choice of κ see, for example, Stephen

Table 6. Natural frequency coefficients for a Timoshenko cantilever beam with various combinations of in-span attachments and attachments at its free end for R_o = 0.05 and γ_{bs} = 3.12. The numbers in parentheses are the values of $\hat{\omega}$.

	Attachments						Frequency coefficients		
Case	Boundary		In-span A_1 at $\eta = 0.6$		In-span A_2 at $\eta = 0.3$		Ω_1^2	Ω_2^2	Ω_3^2
1	K_R		K_1		K_1				
	0		0		0		3.43527	19.1036	46.6031
a	400		0		0		13.4635	35.842	60.473
							(0.9064)	(0.8315)	(0.7794)
b	400		400		0		28.5895	36.3495	64.5772
							(0.9054)	(0.8430)	(0.7953)
c	400		400		400		32.3409	44.4318	67.589
							(0.9382)	(0.8816)	(0.7989)
2		M_R		M		M			
a		1		0		0	1.53636	14.5623	39.7905
							(0.9865)	(0.8961)	(0.7819)
b		1		1		0	1.42614	8.70939	37.8307
							(0.9851)	(0.9072)	(0.7699)
c		1		1		1	1.41728	7.79575	19.8894
							(0.9847)	(0.8970)	(0.7637)
3	$K_{BR,sdof}$	$M_{BR,sdof}$	K_{sdof}	M_{sdof}	K_{sdof}	M_{sdof}			
a	400	1	0	0	0	0	1.53272	14.2288	20.0000*
							(0.9866)	(0.9029)	(1.0000)
b	400	1	400	1	0	0	1.42314	8.12202	20.0000*
							(0.98525)	(0.92088)	(1.0000)
c	400	1	400	1	400	1	1.41433	7.34691	14.4761
							(0.98478)	(0.90968)	(0.8882)

* Natural frequency coefficient of the single-degree-of-freedom system.

Table 7. Natural frequency coefficients for a Timoshenko beam clamped at both ends with various combinations of in-span attachments for R_o = 0.05 and γ_{bs} = 3.12. The numbers in parentheses are the values of $\hat{\omega}$.

	In-span Attachments		Frequency coefficients		
Case	A_1	B_1	Ω_1^2	Ω_2^2	Ω_3^2
	(both at $\eta = 0.6$)				
4	M	J			
	0	0	18.8371	44.33	75.0768
a	1	0.002	10.4875 (0.8466)	37.7116 (0.7552)	65.2659 (0.6991)
b	1	0.02	10.4173 (0.8535)	23.8024 (0.8570)	46.8822 (0.6942)
c	1	0.2	7.50674 (0.9168)	11.2221 (0.8070)	44.6753 (0.6879)
5	K_1	K_{t2}			
a	400	4	30.5189 (0.8624)	49.9753 (0.7413)	77.2151 (0.6203)
b	400	40	31.7013 (0.853)	52.6131 (0.7222)	80.6118 (0.5835)
c	400	400	32.1573 (0.8461)	53.8385 (0.7082)	82.6668 (0.5469)

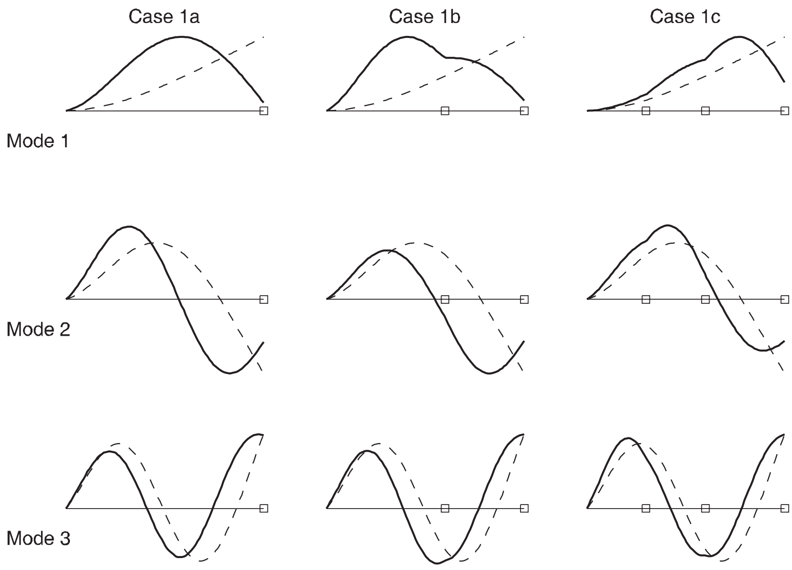

Figure 2. First three mode shapes of a Timoshenko cantilever beam with various spring attachments described by Cases 1 of Table 6. The curves with the dashed lines are the mode shapes for a Timoshenko beam without any attachments. The '□' indicates the locations of the springs.

(1997, 2002), Han et al. (1999), and Magrab (1979). It is noted that all values of Ω_n satisfy (12a). In the tables, we have also included the ratio $\hat{\omega} = \Omega_{nT}^2/\Omega_{nE}^2 = \omega_{nT}/\omega_{nE}$, where Ω_{nT}^2 and Ω_{nE}^2 are the square of the natural frequency coefficients of the Timoshenko and Euler beams, respectively, and ω_{nT} and ω_{nE} are the radian natural frequencies of the Timoshenko and Euler beams, respectively. Both of these natural frequency coefficients are obtained for the same set of boundary conditions and attachments.

The modes shapes corresponding to the values in the tables for the five cases are given in Figures 2 to 6. To convey the influence that these attachments have on each mode shape, we have included the corresponding mode shape for a cantilever beam without any attachments in Figures 2 to 4 and those for the beam clamped at both ends in Figures 5 and 6.

As expected, we see that the natural frequencies for the Timoshenko beam are always lower than those for the Euler beam. From the sampling of the cases presented in these tables, it is hard to draw too many conclusions, because the placement of the attachments and their magnitudes can greatly affect the results. However, for Cases 1, 2, and 3, we notice that the addition of one attachment to the free end of the beam dramatically changes the natural frequencies when compared to the case where there are no attachments. However, in general, the addition of one or two more attachments produces much smaller changes to the natural frequencies, and when the change is fairly substantial one can see from the mode shapes given in Figures 2 to 7 that the closeness of the locations of the attachments to the node points plays a significant role.

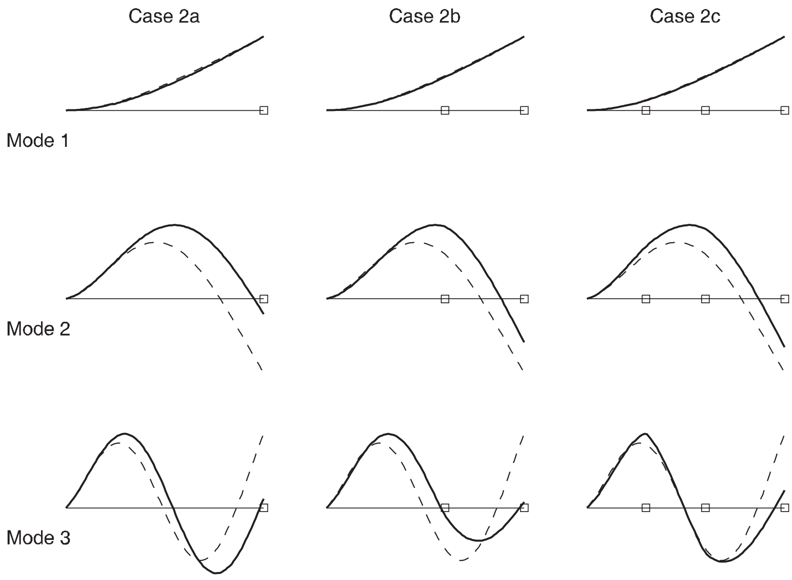

Figure 3. First three mode shapes of a Timoshenko cantilever beam with various mass attachments described by Cases 2 of Table 6. The curves with the dashed lines are the mode shapes for a Timoshenko beam without any attachments. The '□' indicates the locations of the masses.

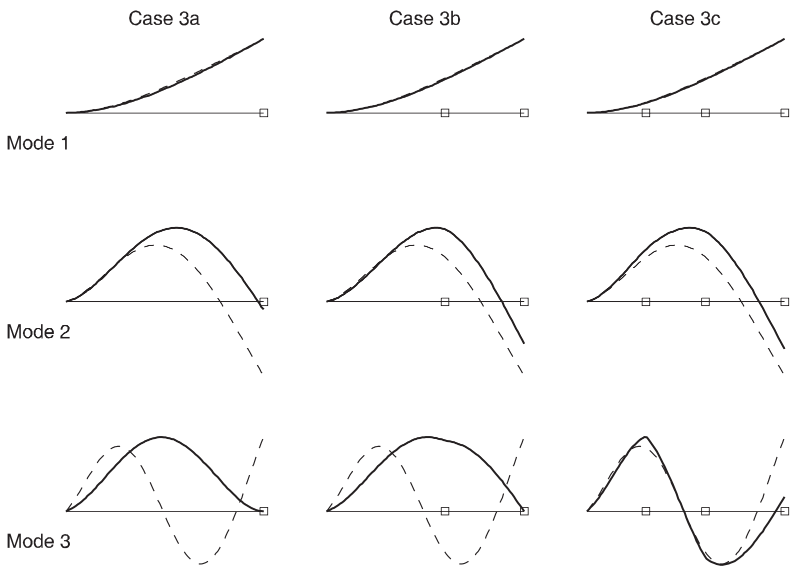

Figure 4. First three mode shapes of a Timoshenko cantilever beam with various single degree-of-freedom systems attached as described by Cases 3 of Table 6. The curves with the dashed lines are the mode shapes for a Timoshenko beam without any attachments. The '□' indicates the locations of the single degree-of-freedom systems.

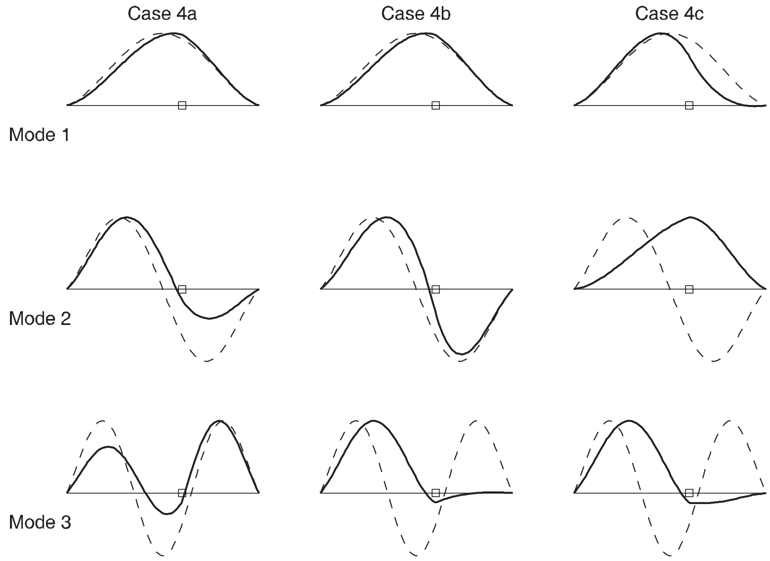

Figure 5. First three mode shapes of a Timoshenko beam clamped at both ends with various translation and rotational masses attached at the same point as described by Cases 4 of Table 7. The curves with the dashed lines are the mode shapes for a Timoshenko beam without any attachments. The '□' indicates the location of the mass.

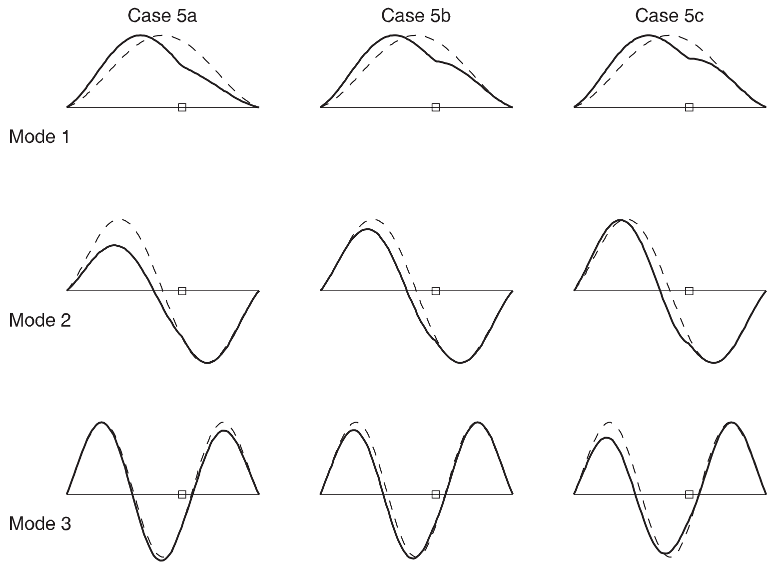

Figure 6. First three mode shapes of a Timoshenko beam clamped at both ends with various translation and rotation springs attached at the same point as described by Cases 5 of Table 7. The curves with the dashed lines are the mode shapes for a Timoshenko beam without any attachments. The '□' indicates the location of the translation and torsion springs.

5. SUMMARY

The Laplace transform method provided the means to obtain a solution to a vibrating system subject to a set of general boundary conditions and to four different types of in-span attachments applied simultaneously. The general results were then reduced in a straightforward manner to numerous special cases. This reduction was straightforward because the attachments on the boundaries of the beam could be treated independently of the number and type of in-span attachments. This uncoupling of the boundary conditions from the number and type of attachments is a direct consequence of the Laplace transform method, where one can ascribe a physical interpretation to the unknown constants appearing in the solution. This reduced the number of the unknown constants by a factor of two, thereby making it practical for one to obtain an analytical form of the solution that was amenable to computer-generated symbolic manipulation. The ability to perform this symbolic manipulation makes it possible to have the computer generate, if desired, the expressions for 93 special cases of the boundary conditions. This, combined with the 21 different combinations of in-span attachments allows one, in principle, to consider almost 2000 different scenarios.

When we compare the present solution method to the three major solution methods discussed in the Introduction, we find the following advantages of the Laplace transform method. The standard solution method can only obtain the solution for attachments at the end points and cannot obtain results for in-span attachments. The Laplace transform method removes this limitation. The beam partitioning method, for a given number of in-span attachments, generates a frequency determinant that can not easily handle the general boundary conditions and their special cases by using the straightforward limiting process discussed in this paper. However, the partitioning method has the advantage that within each partition, the constant cross-section of the beam can be different than the cross-section of its adjacent partition. When Green's functions are used, one must first independently determine them for each set of boundary conditions. When Lagrange multipliers are used, one must first obtain the natural frequencies and mode shapes for the system without the in-span attachments and then proceed to solve the system again with the attachments. In addition, convergence of the solution must be established. The Laplace transform method does not require one to obtain these boundary-condition-dependent intermediate solutions.

APPENDIX A. LAPLACE TRANSFORM PAIRS

The following transform pairs are obtained using partial fractions and a table of Laplace transform pairs (see, for example, Appendix A of Balachandran and Magrab, 2004).

$$L^{-1}\left(\frac{s^3}{\left(s^2-\alpha^2\right)\left(s^2+\beta^2\right)}\right) = Q(\eta) = \frac{1}{\alpha^2+\beta^2}\left[\alpha^2\cosh\alpha\eta+\beta^2\cos\beta\eta\right]$$

$$L^{-1}\left(\frac{s^2}{\left(s^2-\alpha^2\right)\left(s^2+\beta^2\right)}\right) = R(\eta) = \frac{1}{\alpha^2+\beta^2}\left[\alpha\sinh\alpha\eta+\beta\sin\beta\eta\right]$$

$$
\begin{aligned}
L^{-1}\left(\frac{s}{\left(s^2-\alpha^2\right)\left(s^2+\beta^2\right)}\right) &= S(\eta)=\frac{1}{\alpha^2+\beta^2}\left[\cosh\alpha\eta-\cos\beta\eta\right]\\
L^{-1}\left(\frac{1}{\left(s^2-\alpha^2\right)\left(s^2+\beta^2\right)}\right) &= T(\eta)=\frac{1}{\alpha^2+\beta^2}\left[\frac{1}{\alpha}\sinh\alpha\eta-\frac{1}{\beta}\sin\beta\eta\right] \qquad \text{(A1)}
\end{aligned}
$$

and

$$
\begin{aligned}
L^{-1}\left(\frac{s^3e^{-s\eta_j}}{\left(s^2-\alpha^2\right)\left(s^2+\beta^2\right)}\right) &= Q(\eta-\eta_j)u(\eta-\eta_j)\\
L^{-1}\left(\frac{s^2e^{-s\eta_j}}{\left(s^2-\alpha^2\right)\left(s^2+\beta^2\right)}\right) &= R(\eta-\eta_j)u(\eta-\eta_j)\\
L^{-1}\left(\frac{se^{-s\eta_j}}{\left(s^2-\alpha^2\right)\left(s^2+\beta^2\right)}\right) &= S(\eta-\eta_j)u(\eta-\eta_j)\\
L^{-1}\left(\frac{e^{-s\eta_j}}{\left(s^2-\alpha^2\right)\left(s^2+\beta^2\right)}\right) &= T(\eta-\eta_j)u(\eta-\eta_j) \qquad \text{(A2)}
\end{aligned}
$$

where $u(\eta)$ is the unit step function. It is seen that $R(0) = S(0) = T(0) = 0$ and $Q(0) = 1$.

The derivatives of $Q(\eta), \ldots, T(\eta)$ with respect to η are

$$
\begin{aligned}
Q'(\eta) &= V(\eta)=\frac{1}{\alpha^2+\beta^2}\left[\alpha^3\sinh\alpha\eta-\beta^3\sin\beta\eta\right]\\
R'(\eta) &= Q(\eta)\\
S'(\eta) &= R(\eta)\\
T'(\eta) &= S(\eta). \qquad \text{(A3)}
\end{aligned}
$$

It is seen that $Q'(0) = S'(0) = T'(0) = 0$ and $R'(0) = 1$.

APPENDIX B. DEFINITIONS OF f_j AND g_j AND THEIR DERIVATIVES

The quantities f_j are given by

$$
\begin{aligned}
f_1(\eta) &= Q(\eta)+c_1R(\eta)+R_o^2\Omega^4S(\eta)+c_1R_o^2b_\Omega T(\eta)\\
f_2(\eta) &= R(\eta)+c_2S(\eta)
\end{aligned}
$$

$$
\begin{aligned}
f_{3p}(\eta,\eta_p) &= R(\eta-\eta_p)u(\eta-\eta_p)+R_o^2 b_\Omega T(\eta-\eta_p)u(\eta-\eta_p) \quad p=1,2,3 \\
f_{4q}(\eta,\hat{\eta}_q) &= S(\eta-\hat{\eta}_q)u(\eta-\hat{\eta}_q) \quad q=1,2
\end{aligned} \tag{B1}
$$

and the quantities g_j are given by

$$
\begin{aligned}
g_1(\eta) &= -c_1 S(\eta)/\left(\gamma_{bs} R_o^2\right)+k_\Omega T(\eta) \\
g_2(\eta) &= Q(\eta)+c_2 R(\eta)+\gamma_{bs} R_o^2 k_\Omega S(\eta)+c_2\gamma_{bs} R_o^2 k_\Omega T(\eta) \\
g_{3p}(\eta,\eta_p) &= S(\eta-\eta_p)u(\eta-\eta_p) \quad p=1,2,3 \\
g_{4q}(\eta,\hat{\eta}_q) &= R(\eta-\hat{\eta}_q)u(\eta-\hat{\eta}_q)+\gamma_{bs} R_o^2 k_\Omega T(\eta-\hat{\eta}_q)u(\eta-\hat{\eta}_q) \quad q=1,2
\end{aligned} \tag{B2}
$$

where $Q(\eta), \ldots,$ are given by (A1) in Appendix A. It is noted that $f_{3p}(\eta_p,\eta_p) = g_{3p}(\eta_p,\eta_p) = f_{4p}(\hat{\eta}_q,\hat{\eta}_q) = g_{4p}(\hat{\eta}_q,\hat{\eta}_q) = 0$.

The derivatives of these quantities are, respectively,

$$
\begin{aligned}
f_1'(\eta) &= V(\eta)+c_1 Q(\eta)+R_o^2\Omega^4 R(\eta)+c_1 R_o^2 b_\Omega S(\eta) \\
f_2'(\eta) &= Q(\eta)+c_2 R(\eta) \\
f_{3p}'(\eta,\eta_p) &= Q(\eta-\eta_p)u(\eta-\eta_p)+R_o^2 b_\Omega S(\eta-\eta_p)u(\eta-\eta_p) \quad p=1,2,3 \\
f_{4q}'(\eta,\hat{\eta}_q) &= R(\eta-\hat{\eta}_q)u(\eta-\hat{\eta}_q) \quad q=1,2
\end{aligned} \tag{B3}
$$

and

$$
\begin{aligned}
g_1'(\eta) &= -c_1 R(\eta)/\left(\gamma_{bs} R_o^2\right)+k_\Omega S(\eta) \\
g_2'(\eta) &= V(\eta)+c_2 Q(\eta)+\gamma_{bs} R_o^2 k_\Omega R(\eta)+c_2\gamma_{bs} R_o^2 k_\Omega S(\eta) \\
g_{3p}'(\eta,\eta_p) &= R(\eta-\eta_p)u(\eta-\eta_p) \quad p=1,2,3 \\
g_{4q}'(\eta,\hat{\eta}_q) &= Q(\eta-\hat{\eta}_q)u(\eta-\hat{\eta}_q)+\gamma_{bs} R_o^2 k_\Omega S(\eta-\hat{\eta}_q)u(\eta-\hat{\eta}_q) \quad q=1,2
\end{aligned} \tag{B4}
$$

where we have used (A3). There should also be terms containing the derivatives of $u(\eta)$; however, since f'_j and g'_j will be evaluated only at $\eta = 1$ these terms will equal zero. Therefore, they have been omitted.

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