# Nonlinear solid mechanics in OpenFOAM

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# Nonlinear Solid Mechanics in OpenFOAM

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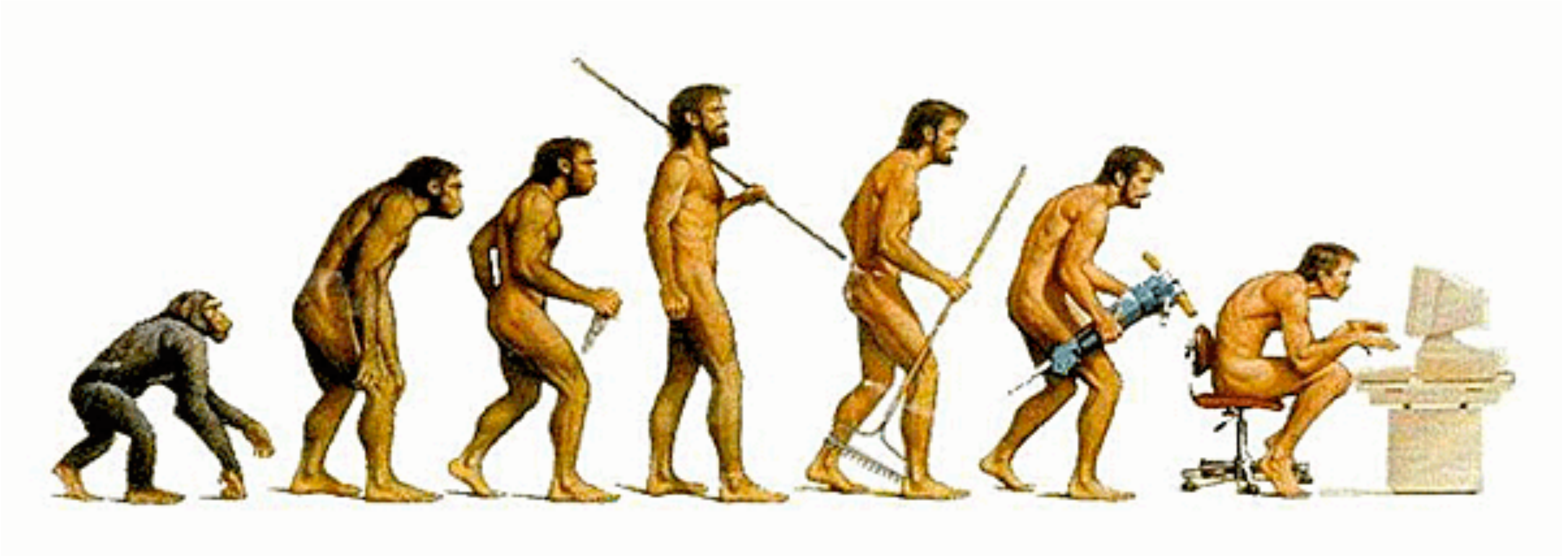


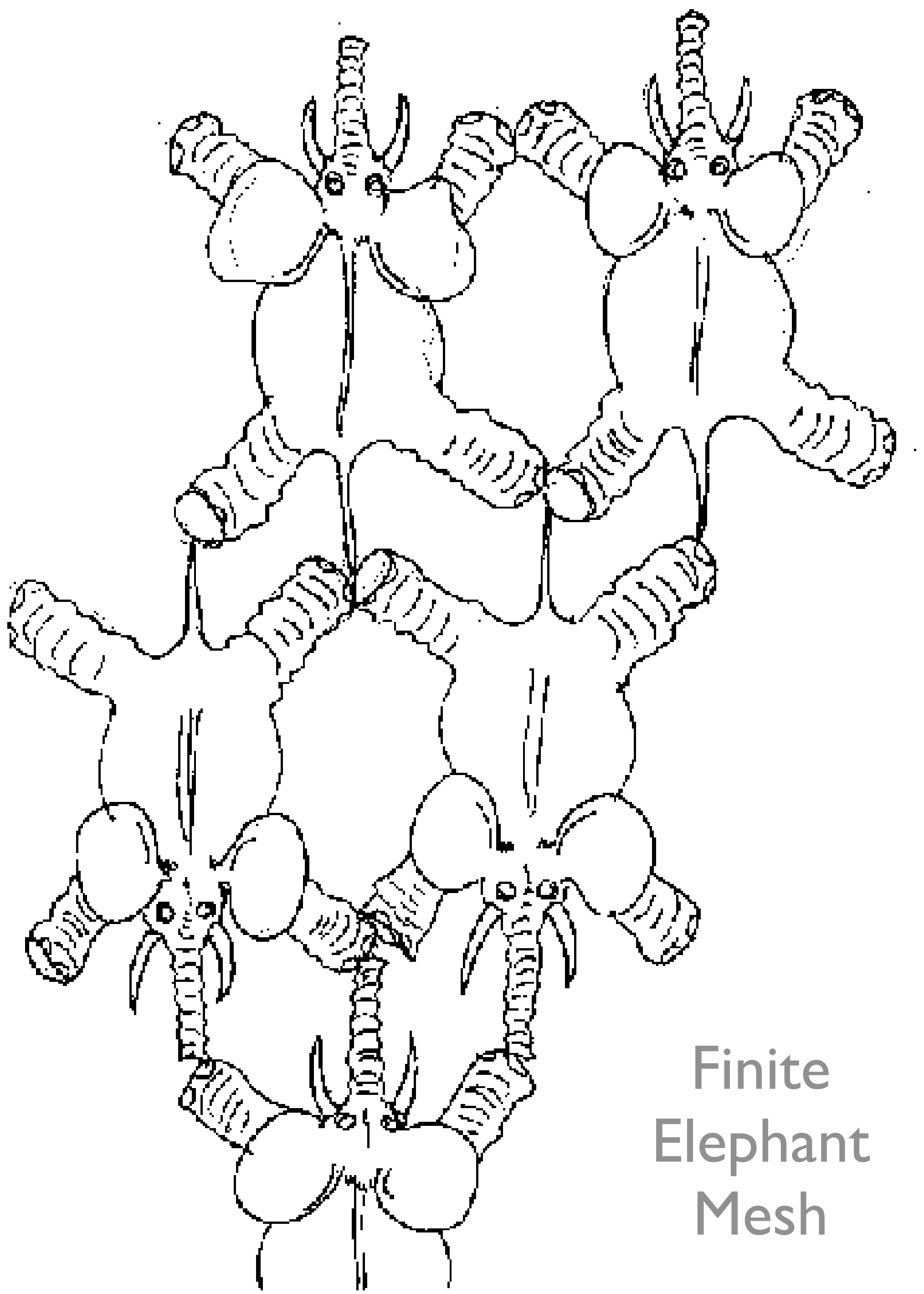
# Background & Motivation

# Background

Finite Element method is not the only way!

Finite Volume method can be applied to solids; it has slowly been gaining momentum since early 90's.

FV method developed for nonlinearity, so it should be suitable!



Finite
Elephant
Mesh

$$\oint_\Gamma \boldsymbol{n} \cdot \boldsymbol{D} \, d\Gamma - \int_\Omega Q \, d\Omega = 0$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \sigma$$

$$\rho C \frac{\partial T}{\partial t} = k \nabla^2 T$$

# Linear Finite Volume Solid Mechanics

# Governing Equations

Strong Form of Governing Equation **Momentum**

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}\Omega = \int_{\Omega}\nabla\cdot\boldsymbol{\sigma}\,\mathrm{d}\Omega + \int_{\Omega}\rho\boldsymbol{f}_b\,\mathrm{d}\Omega$$

Finite Volume Method directly uses strong form

Weak Form of Governing Equation **Momentum**

$$\frac{\partial}{\partial t}\int_\Omega \left(\rho\frac{\partial \boldsymbol{u}}{\partial t}\right)\cdot \boldsymbol{w}\;\mathrm{d}\Omega = \int_\Omega \boldsymbol{\sigma}\cdot\nabla\boldsymbol{w}\;\mathrm{d}\Omega + \int_\Omega \rho\boldsymbol{f}_b\cdot\boldsymbol{w}\;\mathrm{d}\Omega + \int_{\Gamma_t}\boldsymbol{t}\cdot\boldsymbol{w}\;\mathrm{d}\Gamma$$

Weak Form of Governing Equation **Momentum**

weighting function

$$\frac{\partial}{\partial t}\int_{\Omega}\left(\rho\frac{\partial \boldsymbol{u}}{\partial t}\right)\cdot \boldsymbol{w}\ \mathrm{d}\Omega = \int_{\Omega}\boldsymbol{\sigma}\cdot\nabla\boldsymbol{w}\ \mathrm{d}\Omega + \int_{\Omega}\rho\boldsymbol{f}_b\cdot\boldsymbol{w}\ \mathrm{d}\Omega + \int_{\Gamma_t}\boldsymbol{t}\cdot\boldsymbol{w}\ \mathrm{d}\Gamma$$

Finite Element Method uses weak form

# Development of the Mathematical Model

Isotropic Hooke's law assuming small strains

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\epsilon} + \lambda \mathrm{tr}(\boldsymbol{\epsilon})\mathrm{I} \qquad \boldsymbol{\epsilon} = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$$

$$\boldsymbol{\sigma} = \mu\nabla\boldsymbol{u} + \mu\nabla\boldsymbol{u}^T + \lambda \mathrm{tr}(\boldsymbol{u})\mathbf{I}$$

# Development of the Mathematical Model

**Insert constitutive relation into governing equation**

$$\frac{\partial}{\partial t}\int_\Omega \rho \frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}\Omega = \int_\Omega \nabla \cdot \underbrace{\left[\mu \nabla \boldsymbol{u} + \mu \nabla \boldsymbol{u}^T + \lambda \mathrm{tr}(\nabla \boldsymbol{u})\mathrm{I}\right]}_{\boldsymbol{\sigma}}\,\mathrm{d}\Omega + \int_\Omega \rho \boldsymbol{f}_b\,\mathrm{d}\Omega$$

$$\frac{\partial}{\partial t}\int_{\Omega}\rho\frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}\Omega=\int_{\Omega}\mu\nabla^{2}\boldsymbol{u}\,\mathrm{d}\Omega+\int_{\Omega}\nabla\cdot\left[\mu\nabla\boldsymbol{u}^{T}+\lambda\mathrm{tr}(\nabla\boldsymbol{u})\mathrm{I}\right]\,\mathrm{d}\Omega+\int_{\Omega}\rho\boldsymbol{f}_{b}\,\mathrm{d}\Omega$$

laplacian term *implicitly* approximated
using compact molecule

$$\frac{\partial}{\partial t}\int_\Omega \rho\frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}\Omega = \int_\Omega \mu\nabla^2\boldsymbol{u}\,\mathrm{d}\Omega + \int_\Omega \nabla\cdot\left[\mu\nabla\boldsymbol{u}^T + \lambda\mathrm{tr}(\nabla\boldsymbol{u})\mathrm{I}\right]\,\mathrm{d}\Omega + \int_\Omega \rho\boldsymbol{f}_b\,\mathrm{d}\Omega$$

laplacian term *implicitly* approximated
using compact molecule

$$\frac{\partial}{\partial t}\int_\Omega \rho \frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}\Omega = \boxed{\int_\Omega \mu \nabla^2 \boldsymbol{u}\;\mathrm{d}\Omega} + \int_\Omega \nabla \cdot \left[\mu \nabla \boldsymbol{u}^T + \lambda \mathrm{tr}(\nabla \boldsymbol{u})\mathrm{I}\right]\;\mathrm{d}\Omega + \int_\Omega \rho \boldsymbol{f}_b\,\mathrm{d}\Omega$$

$$\boxed{\int_\Omega (2\mu + \lambda)\nabla^2 \boldsymbol{u}\,\mathrm{d}\Omega - \int_\Omega (\mu + \lambda)\nabla^2 \boldsymbol{u}\,\mathrm{d}\Omega}$$

better convergence with
*over-relaxed* approach

# FEM & FVM Comparison **for solids**

## FVM

### Pros

attractively simple

strongly conservative

memory efficient

straight-forward nonlinearity

suitable for FSI/cracks

general polyhedral cells

### Cons

segregated solution not always efficient

higher order *elements* are more difficult

less development

# FEM

## Pros

efficient for linear problems

higher order elements

plethora of development

inclusion of discontinuities

## Cons

weakly conservative

memory hungry

different shape functions for every shape element

FSI

$$\oint_\Gamma \boldsymbol{n} \cdot \boldsymbol{D} \, d\Gamma - \int_\Omega Q \, d\Omega = 0$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \nabla \cdot \sigma$$

$$\rho C \frac{\partial T}{\partial t} = k \nabla^2 T$$

# Nonlinear Finite Volume Solid Mechanics

# Different Types of Nonlinearity

## Three Types

Geometric

Material

Loading

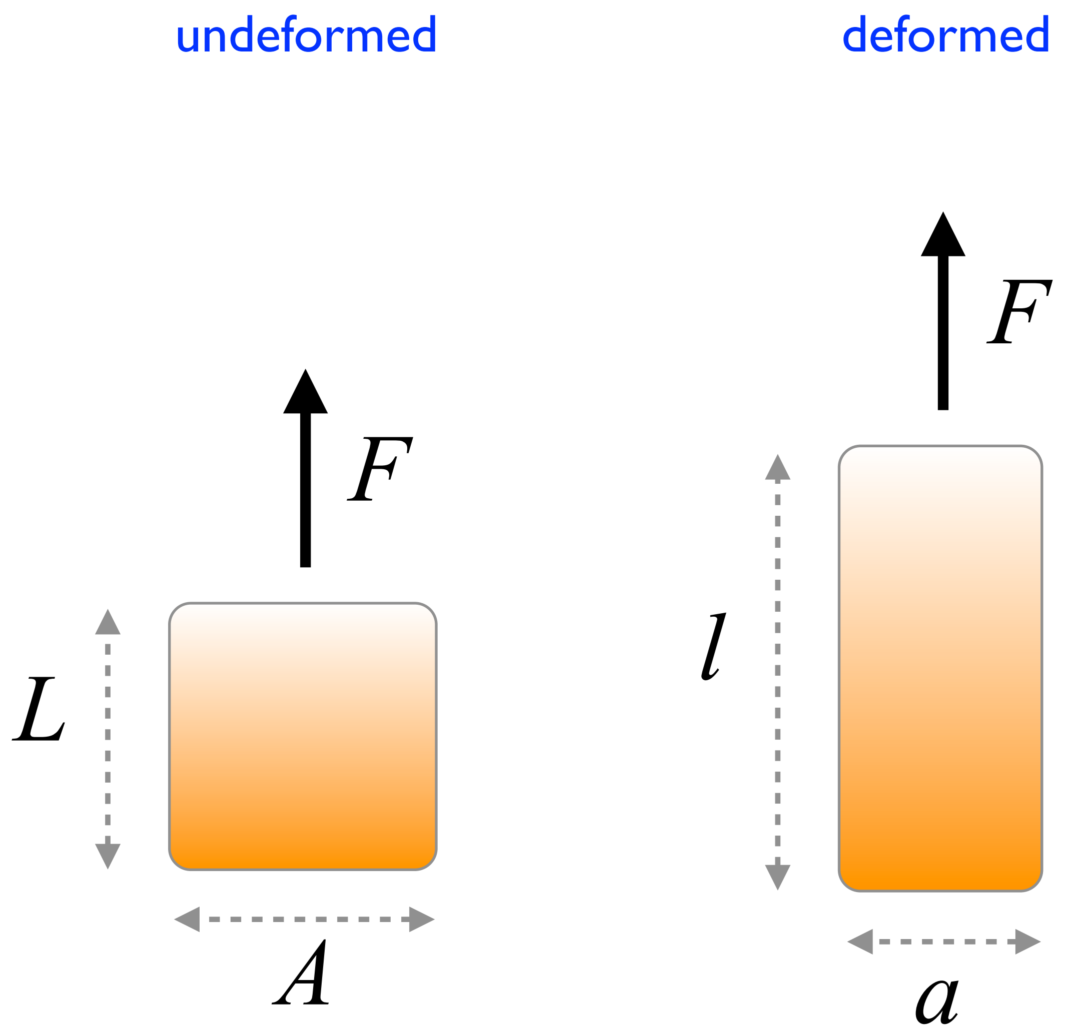
**System of Linear Equations**

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$$ linear

$$\boldsymbol{A}(x)\,\boldsymbol{x} = \boldsymbol{b}(x)$$ nonlinear

# Geometric

## large strains, changes in integration areas/volumes




engineering stress & strain

$$\boldsymbol{\sigma}_s = \frac{F}{A} \qquad \boldsymbol{\epsilon}_s = \frac{l - L}{L}$$

2nd Piola-Kirchhoff stress & Green strain

$$\boldsymbol{S} = \frac{F}{A}\frac{L}{l} \qquad \boldsymbol{\epsilon} = \frac{l^2 - L^2}{2L}$$

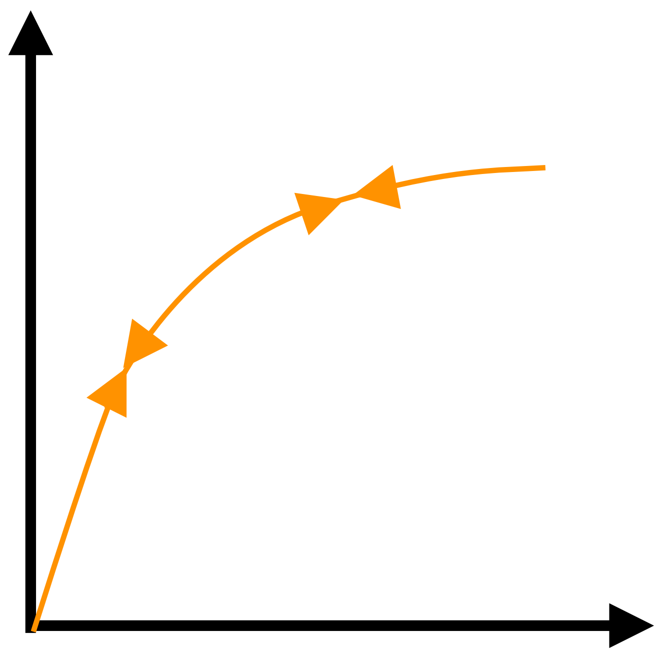
true stress & strain

$$\boldsymbol{\sigma} = \frac{F}{a} \qquad \boldsymbol{\epsilon} = \ln\frac{l}{L}$$

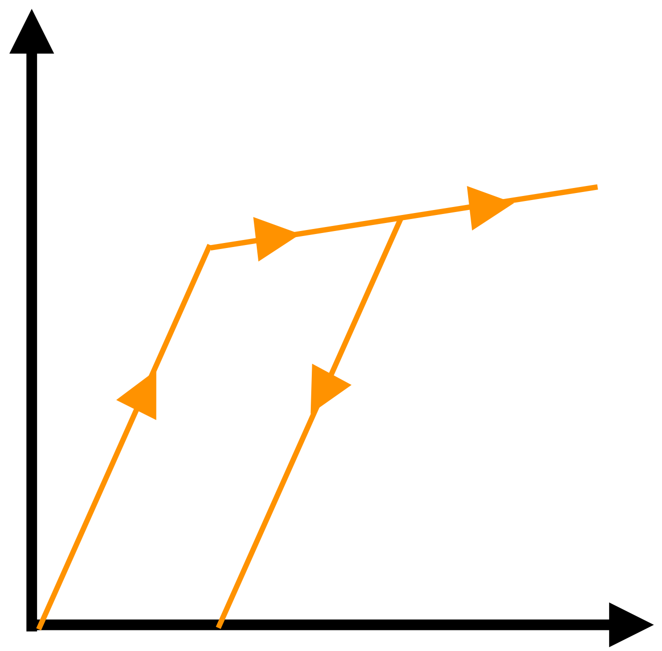
# Material

nonlinear stress strain curve e.g. plasticity

nonlinear elastic



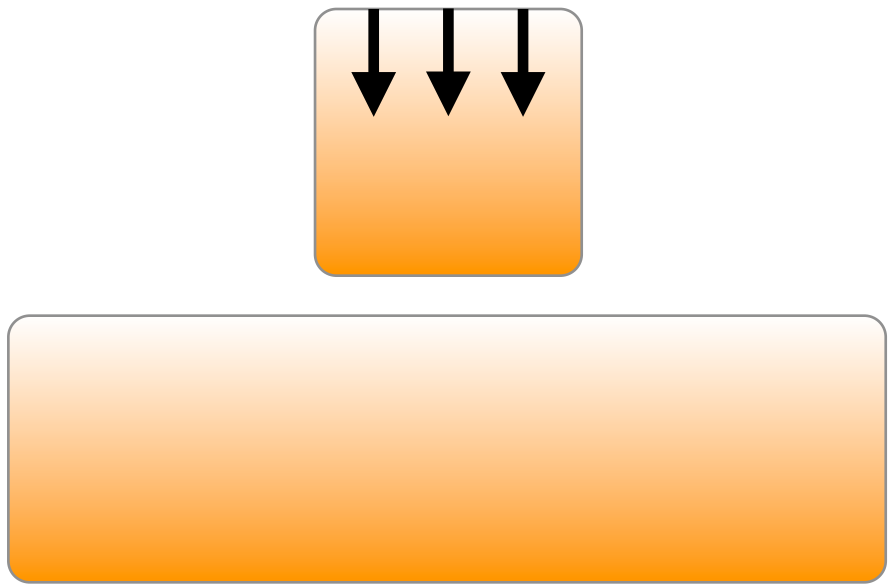
elastoplastic

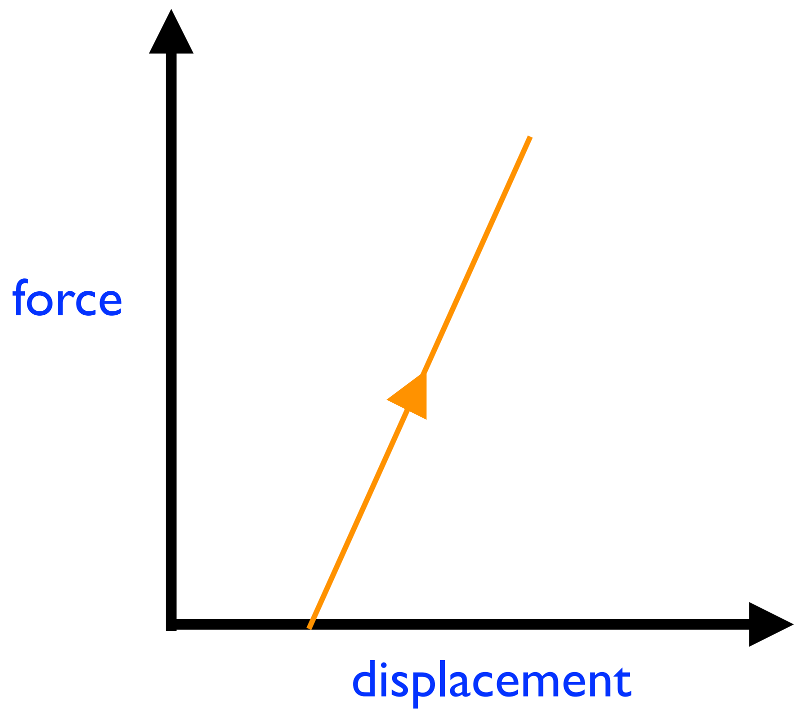


# Loading

## e.g. contact

displace top of block





# Solution Procedure

Update nonlinear terms

```
Segregated Solution
do
    solve (Aui = b) for i=1:3
    update A(ui) and b(ui)
while A(ui) && b(ui) not converged
```

No change to solution procedure

# Example: elasticPlasticSolidFoam

```
do
    solve
    (
        rho*fvm::ddt(DU)
        ==
        fvm::laplacian(2*mu + lambda, DU)
        + fvc::div
        (
            mu*gradDU.T() + lambda*tr(gradDU)*I
            - (mu + lambda)*gradDU
            - DEpsilonP
        )
    );

    update gradDU
    update DEpsilonP // plastic strain increment
while !converged
```

# Geometrically Nonlinear Mathematical Models

Two general methods to deal with geometric nonlinearity:

| | |
|---|---|
| Total Lagrangian | `elasticNonLinTLSolidFoam` |
| Updated Lagrangian | `elasticNonLinULSolidFoam` |

Choice between the two approaches typically depends on the constitutive model

# Total Lagrangian Mathematical Model

Linear momentum equation

neglecting inertia and body forces

$$\nabla \cdot \boldsymbol{\sigma} = 0$$

equivalent to

$$\oint_{\Gamma} \boldsymbol{n} \cdot \boldsymbol{\sigma} \, \mathrm{d}|\Gamma| = 0$$

$$\boldsymbol{\sigma} = \frac{1}{J}\boldsymbol{F} \cdot \boldsymbol{S} \cdot \boldsymbol{F}^T$$

Cauchy stress in terms of 2nd Piola Kirchhoff stress

$$\boldsymbol{\Gamma} = J\boldsymbol{F}^{-T} \cdot \boldsymbol{\Gamma}_o$$

Nanson's formula

Linear momentum equation may be written in terms of Cauchy stress or 2nd Piola-Kirchhoff stress

$$\oint_{\Gamma} \boldsymbol{n} \cdot \boldsymbol{\sigma} \, \mathrm{d}|\Gamma| = \oint_{\Gamma_o} \boldsymbol{n}_o \cdot \left( \boldsymbol{S} \cdot \boldsymbol{F}^T \right) \, \mathrm{d}|\Gamma_o| = 0$$

# Constitutive Law

St. Venant-Kirchhoff law *i.e.* Large strain/rotation form of Hooke's law

$$\boldsymbol{S} = 2\mu\boldsymbol{E} + \lambda\mathrm{tr}(\boldsymbol{E})\mathbf{I}$$

Green strain

$$\boldsymbol{E} = \frac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u} + \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^T\right)$$

# Total Lagrangian Mathematical Model

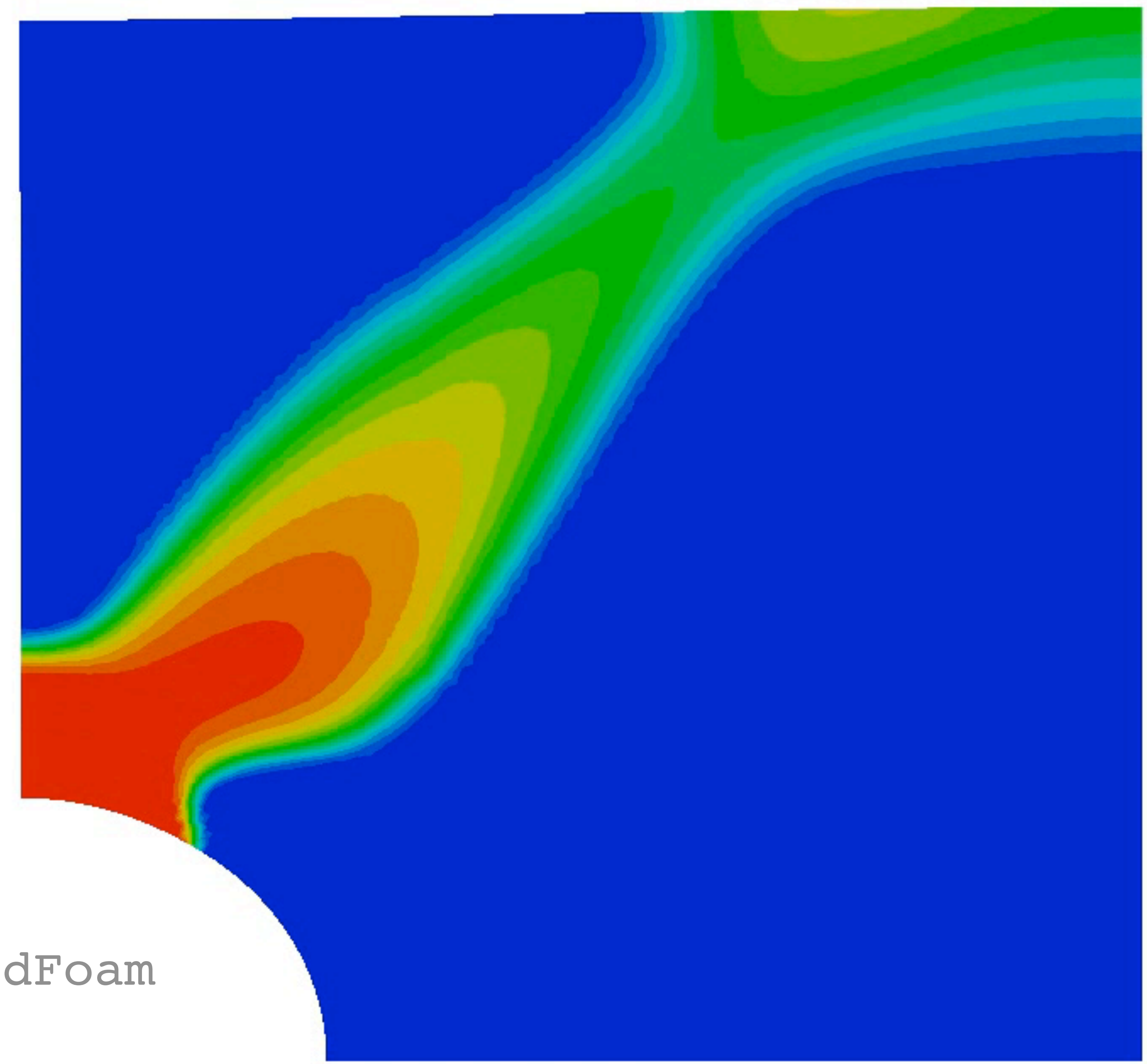
$$
\begin{aligned}
\rho\frac{\partial^2 \boldsymbol{u}}{\partial t} =& (2\mu+\lambda)\nabla^2\boldsymbol{u} \\
&+ \nabla\cdot\left(\mu\nabla\boldsymbol{u}^T + \lambda\mathrm{tr}(\nabla\boldsymbol{u})\mathbf{I} - (\mu+\lambda)\nabla\boldsymbol{u}\right) \\
&+ \nabla\cdot\left(\mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^T + \frac{1}{2}\lambda\mathrm{tr}(\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}^T)\mathbf{I}\right) \\
&+ \nabla\cdot\left(\boldsymbol{S}_{\mathrm{old}}\cdot\nabla\boldsymbol{u}\right)
\end{aligned}
$$

# elasticNonLinTLSolidFoam

```
fvVectorMatrix UEqn
(
    rho*fvm::d2dt2(U)
    ==
    fvm::laplacian(2*mu + lambda, U)
    + fvc::div
    (
        - (mu + lambda)*gradU
        + mu*gradU.T()
        + mu*(gradU & gradU.T())
        + lambda*tr(gradU)*I
        + 0.5*lambda*tr(gradU & gradU.T())*I
        + (sigma & gradU)
    )
);
```

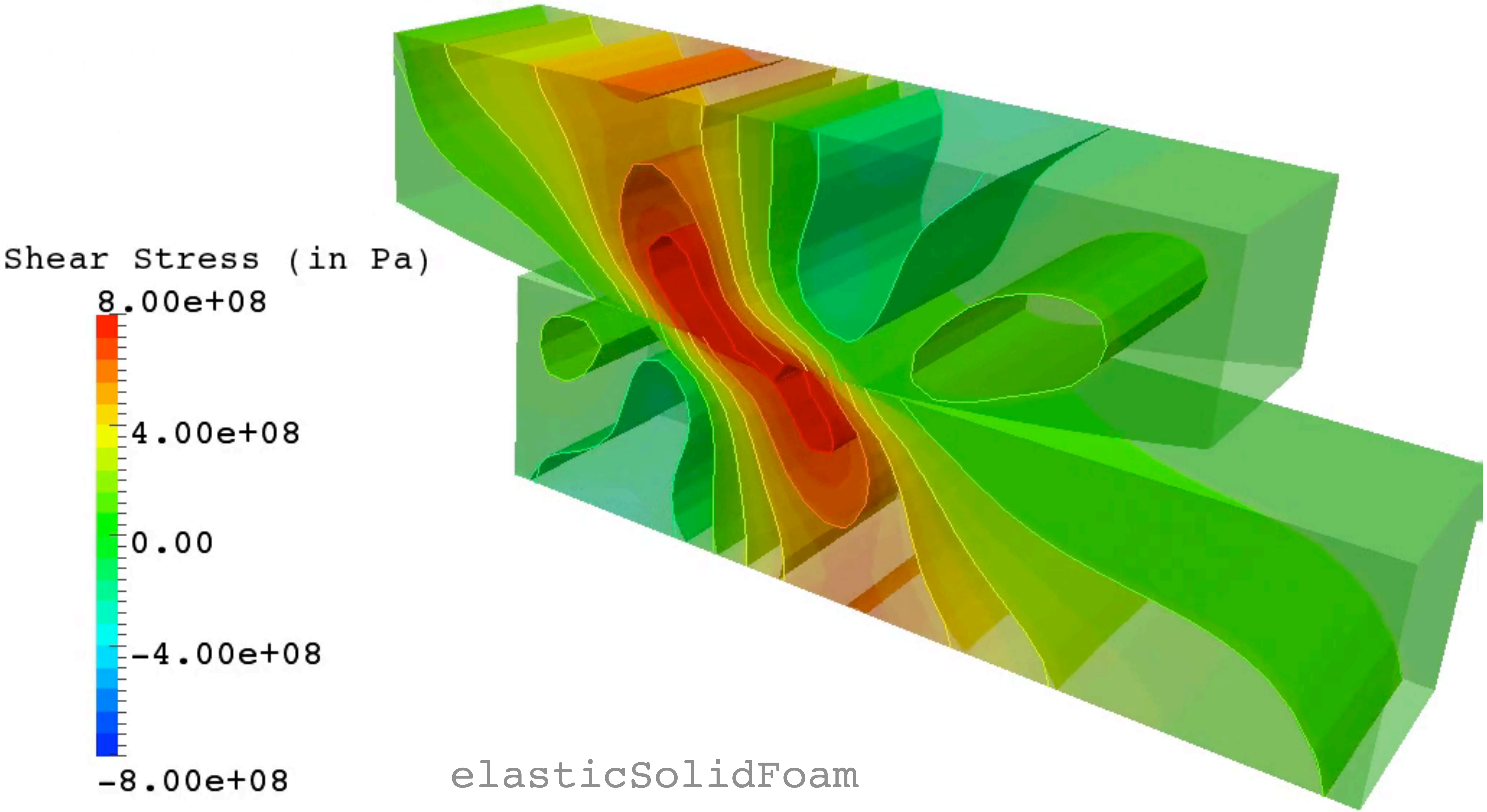
# Test Cases

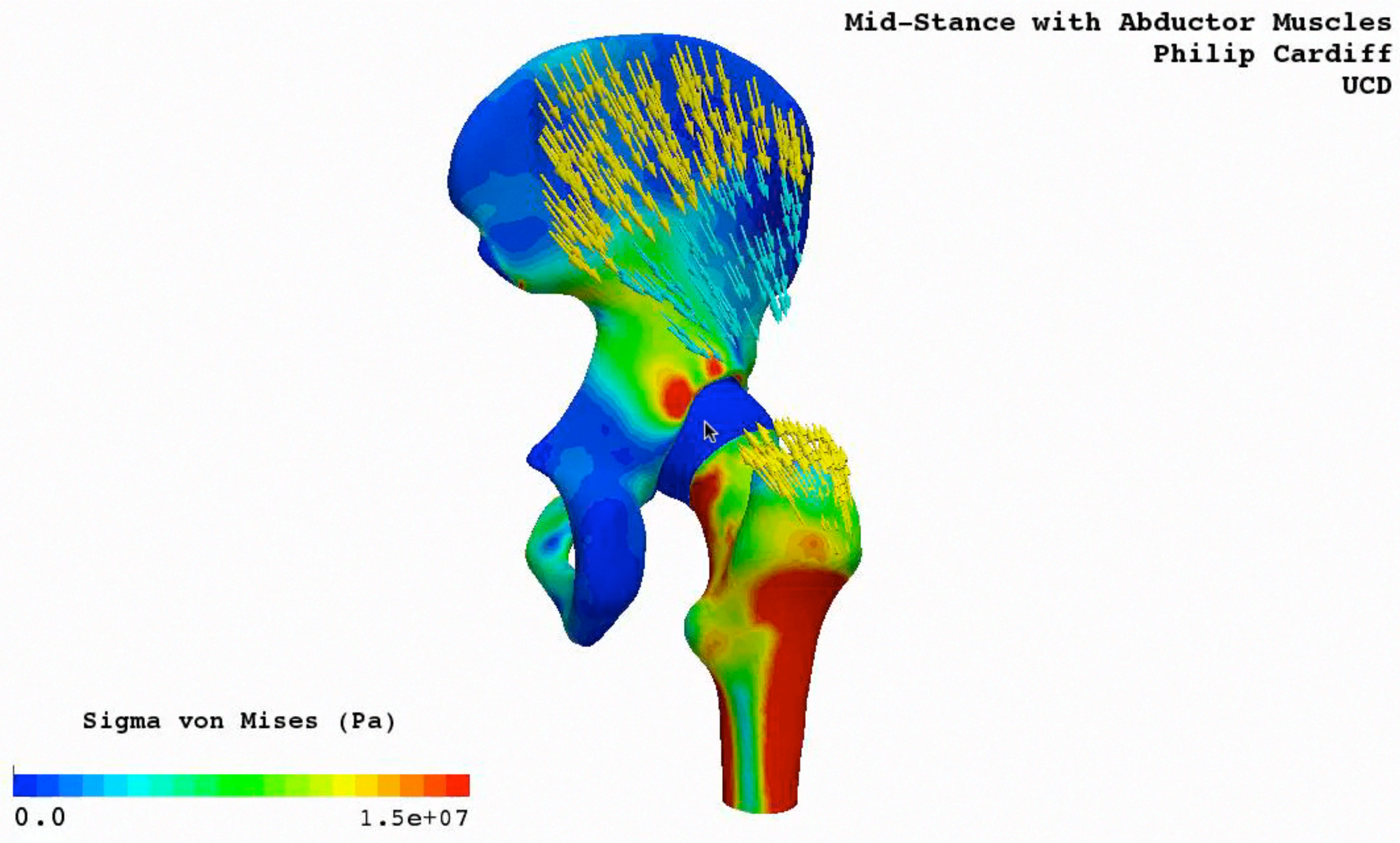
# Plasticity

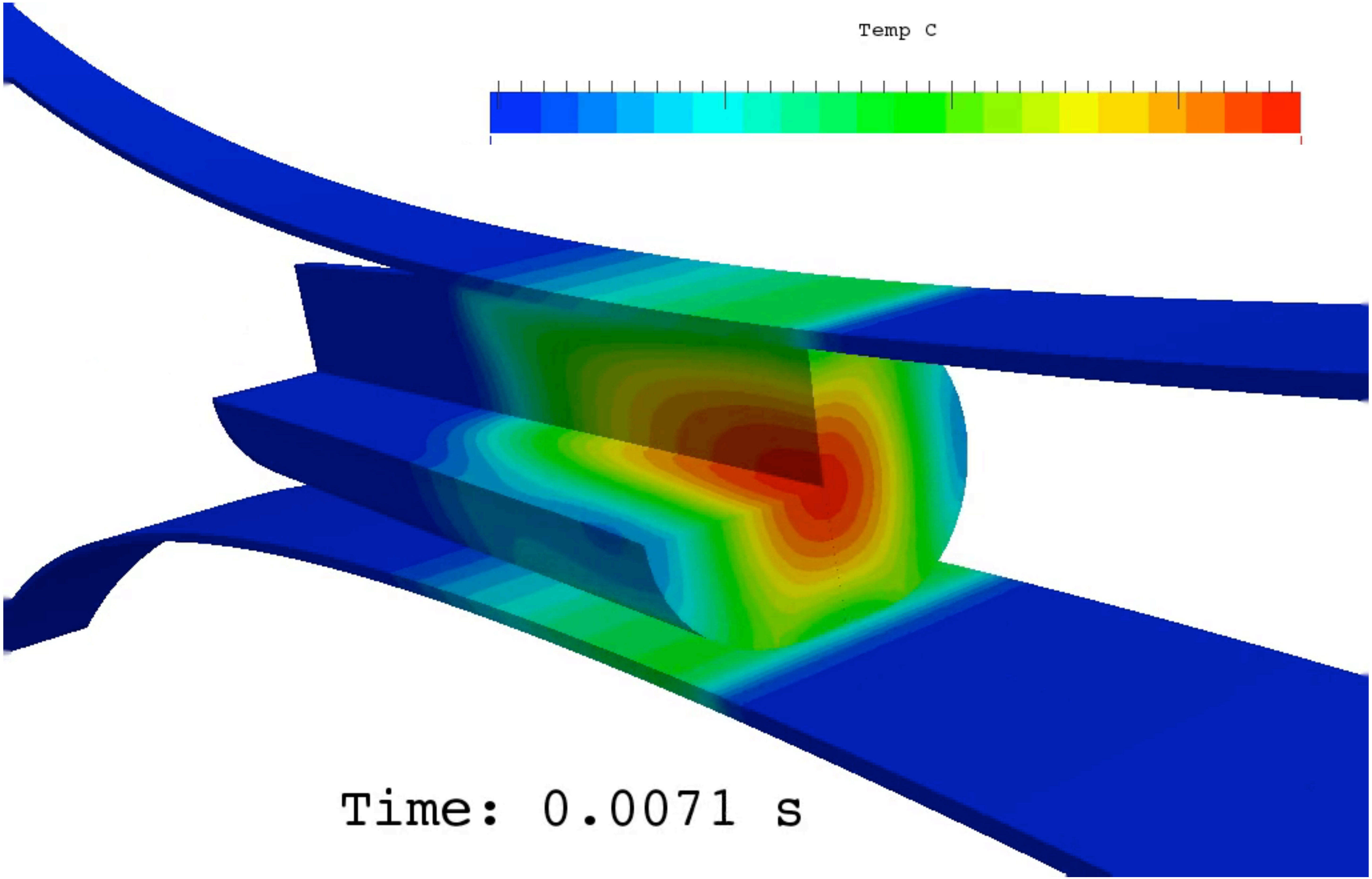


elasticPlasticSolidFoam

# Frictional Contact

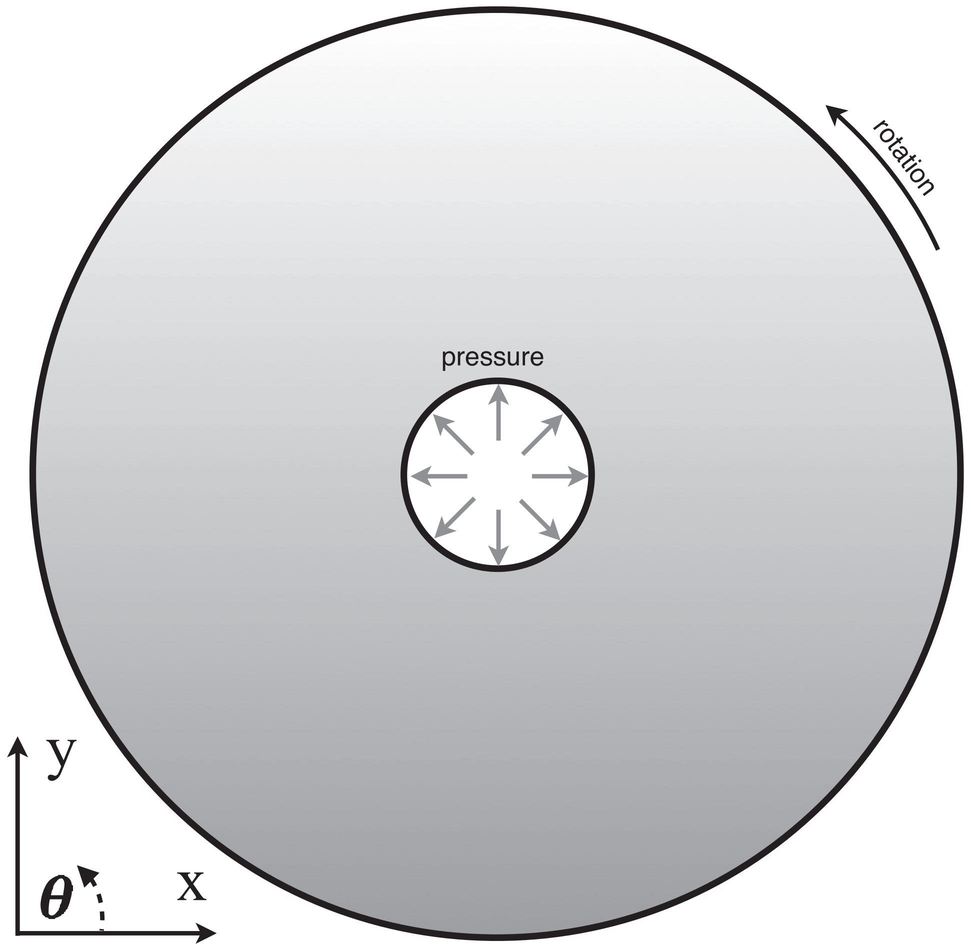


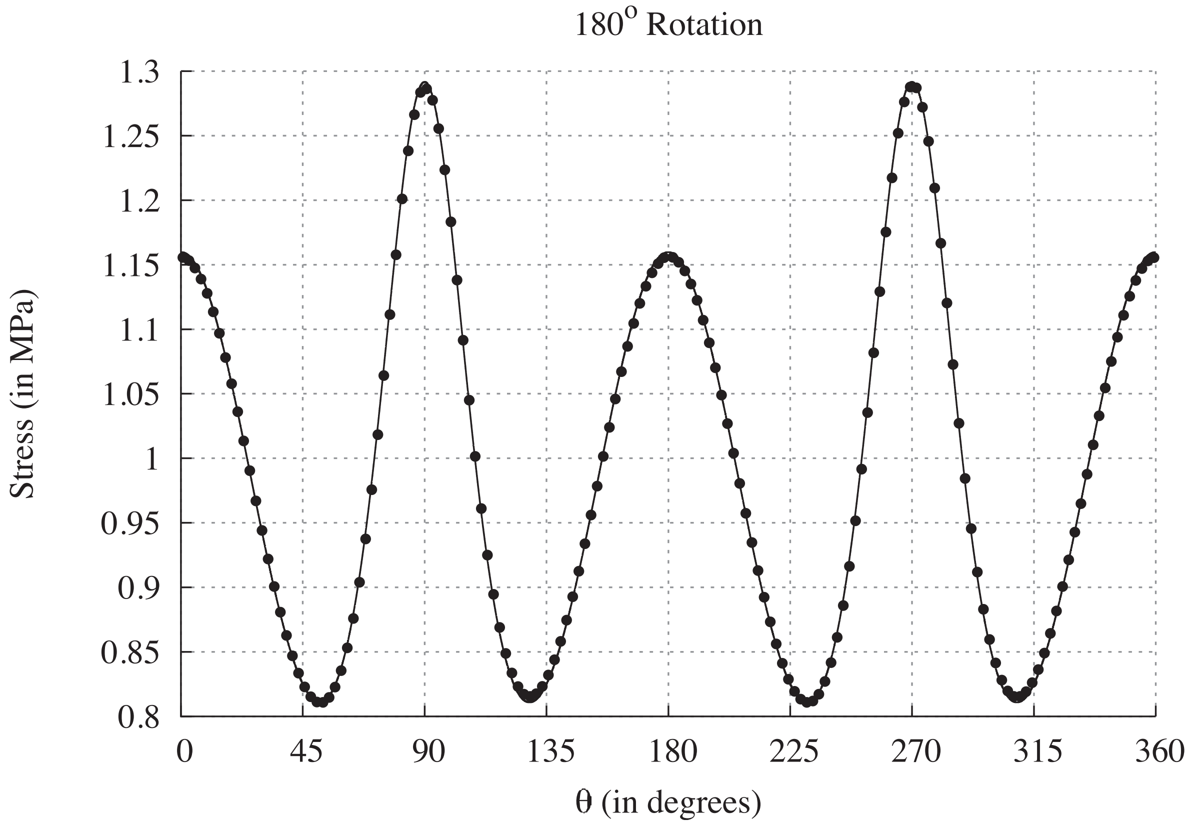

Mid-Stance with Abductor Muscles
Philip Cardiff
UCD
Sigma von Mises (Pa)
0.0
1.5e+07

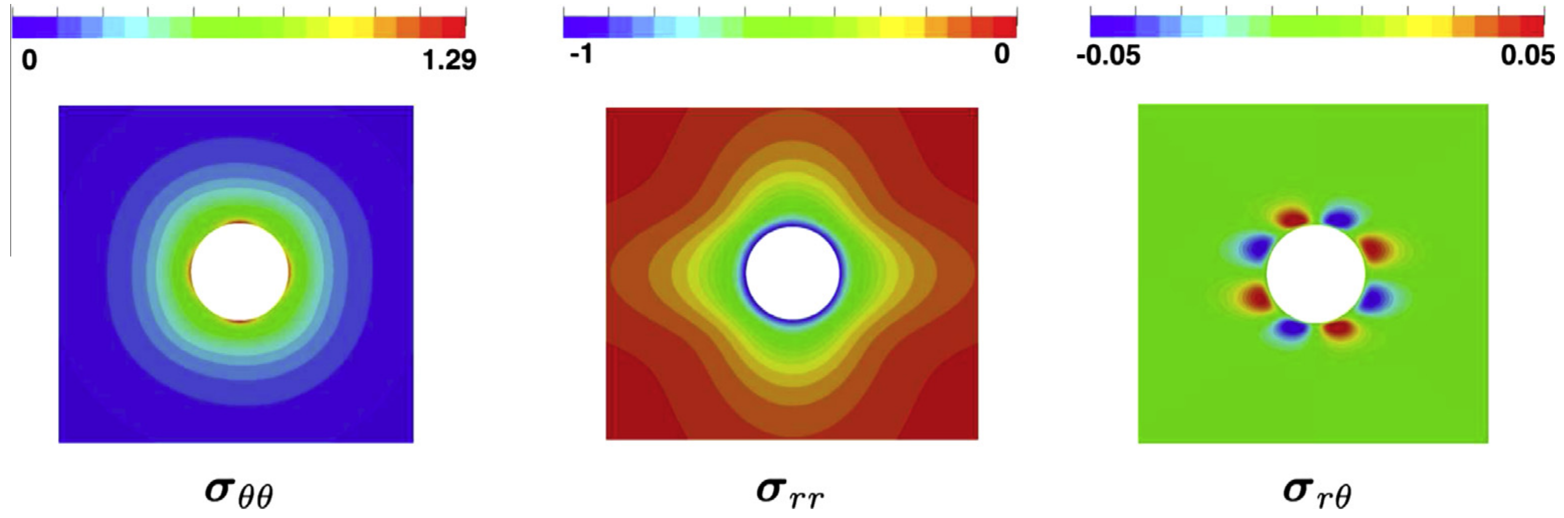

Temp C
Time: 0.0071 s

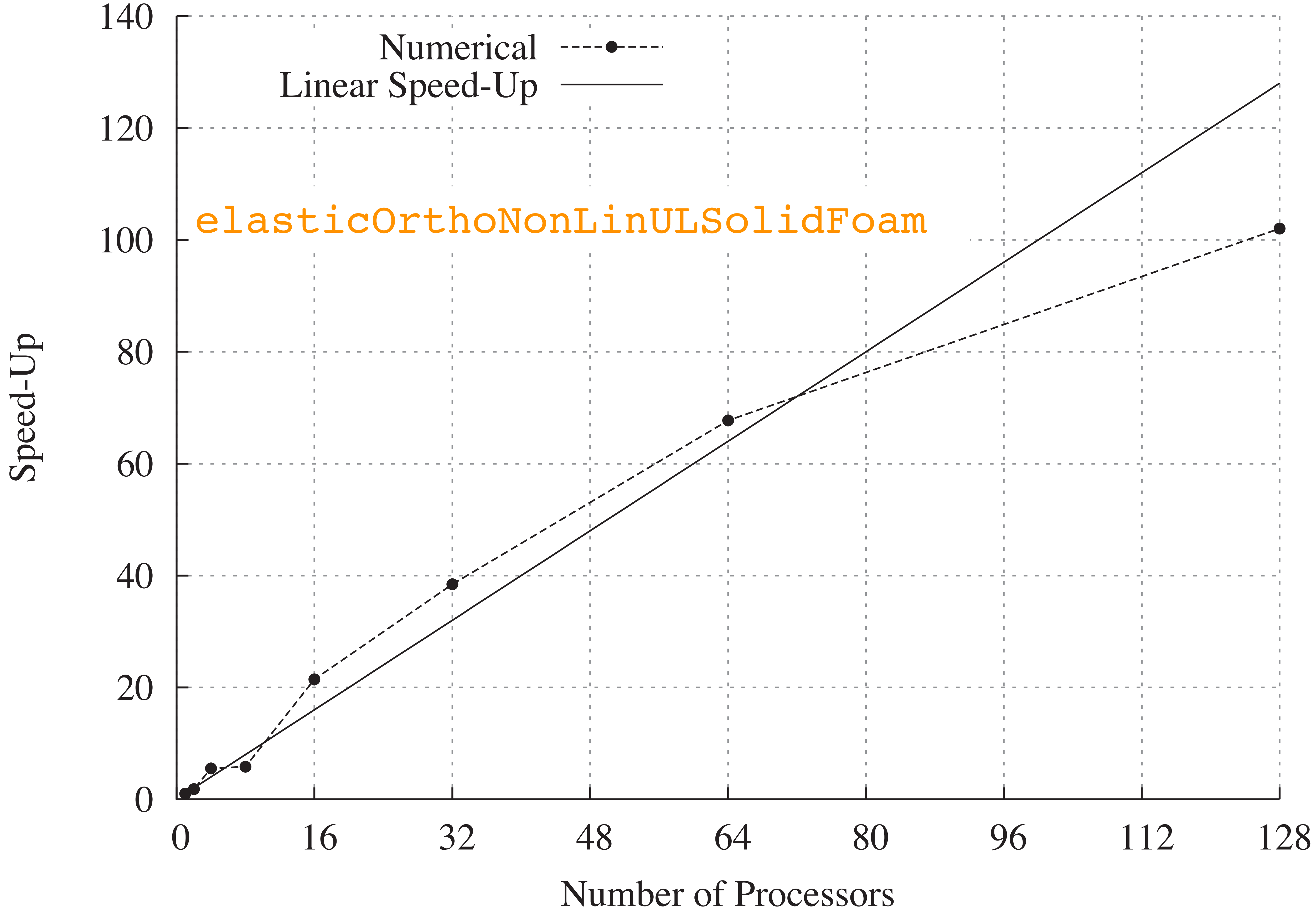
# Large Rotations & Orthotropic Elasticity

elasticOrthoNonLinULSolidFoam





elasticOrthoNonLinULSolidFoam
Numerical
Linear Speed-Up
Speed-Up
Number of Processors
0
20
40
60
80
100
120
140
0
16
32
48
64
80
96
112
128

# Summary & Conclusions

# Summary & Conclusions

OpenFOAM `solidMechanics` solvers capable of all forms of linearity with no change to underlying segregated solution procedure

Complex nonlinear problems treated in a implicit manner

Method is very memory efficient for large meshes, and shows good parallel speedup

## Future Steps

Consider the use of newly developed block-coupled solid solver for nonlinearity with Newton iterations, like in FE;

Clean integration of all methods into new FSI framework.

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