20 Stability

Objectives

The problem of "convergence" in a recurrent network was first raised in our discussion of the Hopfield network, in Chapter 3. It was noted there that the output of a recurrent network could converge to a stable point, oscillate, or perhaps even diverge. The "stability" of the steepest descent process and of the LMS algorithm were discussed in Chapter 9 and Chapter 10, respectively. The stability of Grossberg's continuous-time recurrent networks was discussed in Chapter 18.

In this chapter we will define stability more carefully. Our objective is to determine whether a particular set of nonlinear equations has points (or trajectories) to which its output might converge. To help us study this topic we will introduce Lyapunov's Stability Theorem and apply it to a simple, but instructive, problem. Then, we will present a generalization of the Lyapunov Theory: LaSalle's Invariance Theorem. This will set the stage for Chapter 21, where LaSalle's theorem is used to prove the stability of Hopfield networks.

Theory and Examples

Recurrent Networks

We first discussed recurrent neural networks, which have feedback connections from their outputs to their inputs, when we introduced the Hamming and Hopfield networks in Chapter 3. The Grossberg networks of Chapter 18 and Chapter 19 also contain recurrent connections. Recurrent networks are potentially more powerful than feedforward networks, since they are able to recognize and recall temporal, as well as spatial, patterns. However, the behavior of these recurrent networks is much more complex than that of feedforward networks.

For feedforward networks, the output is constant (for a fixed input) and is a function only of the network input. For recurrent networks, however, the output of the network is a function of time. For a given input and a given initial network output, the response of the network may converge to a stable output. However, it may also oscillate, explode to infinity, or follow a chaotic pattern. In the remainder of this chapter we want to investigate general nonlinear recurrent networks, in order to determine their long-term behavior.

We will consider recurrent networks that can be described by nonlinear differential equations of the form:

$$\frac{d}{dt}\mathbf{a}(t) = \mathbf{g}(\mathbf{a}(t), \mathbf{p}(t), \mathrm{t}) . \tag{20.1}$$

Here $\mathbf{p}(t)$ is the input to the network, and $\mathbf{a}(t)$ is the output of the network. (See Figure 20.1.)

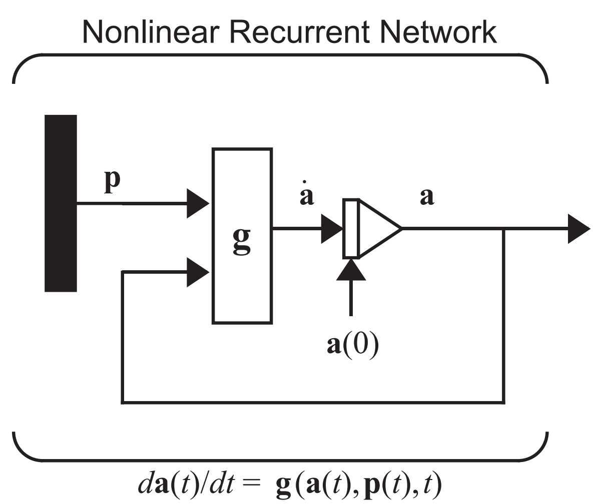
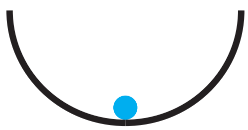


Figure 20.1 Nonlinear, Continuous-Time, Recurrent Network

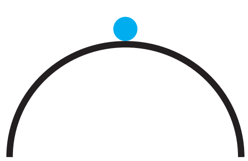
We want to know how these systems perform in the steady state. We will be most interested in those cases where the network converges to a constant output — a stable equilibrium point. A nonlinear system can have many stable points. For some neural networks these stable points represent stored prototype patterns. When possible, we would like to know where the stable points are, and which initial conditions $\mathbf{a}(0)$ converge to a given stable point (i.e., what is the basin of attraction for a given stable point?).

Stability Concepts



To begin our discussion, let's introduce some basic stability concepts with a simple, intuitive example. Consider the motion of a ball bearing, with dissipative friction, in a gravity field. In the adjacent figure, we have a ball bearing at the bottom of a trough (point $\mathbf{a}^*$). If we move the bearing to a different position, it will oscillate back and forth in the trough, but, because of friction, it will eventually settle back to the bottom of the trough. We will call this position an *asymptotically stable* point, which we will define more precisely in the next section.

Consider now the second figure in the left margin. Here we have a ball bearing positioned at the center of a flat surface. If we place the bearing in a different position, it will not move. The position at the center of the surface is not asymptotically stable, since the bearing does not move back to the center if it is moved away. However, it is stable in a certain sense, because at least the ball does not roll farther away from the center point. We call this kind of point *stable in the sense of Lyapunov*, which we will define in the next section.



Now consider the third figure in the left margin. The ball bearing is positioned at the top of a hill. This is an equilibrium point, since the ball will remain at the top of the hill, if we position it carefully. However, if the bearing is given the slightest disturbance, it will roll down the hill. This is an *unstable* equilibrium point.

In the next chapter we will try to design Hopfield neural networks, in which the stored prototype patterns will be asymptotically stable equilibrium points. We would also like the basins of attraction for these stable points to be as large as possible.

For example, consider Figure 20.2. We would like to design neural networks with large basins of attraction such as those of Case A. One can certainly imagine that if a ball that rolls with high friction is placed (with zero velocity) in any one of the basins of Case A, it will remain in that basin and will eventually find its way to the bottom (stable point). However, Case B is more complicated. If, for instance, one places a ball with friction at point P, it is not clear which stable point will eventually capture the ball. The ball may not come to rest at the stable point closest to P. It is also difficult to tell how large the basin of attraction is for a specific stable point.

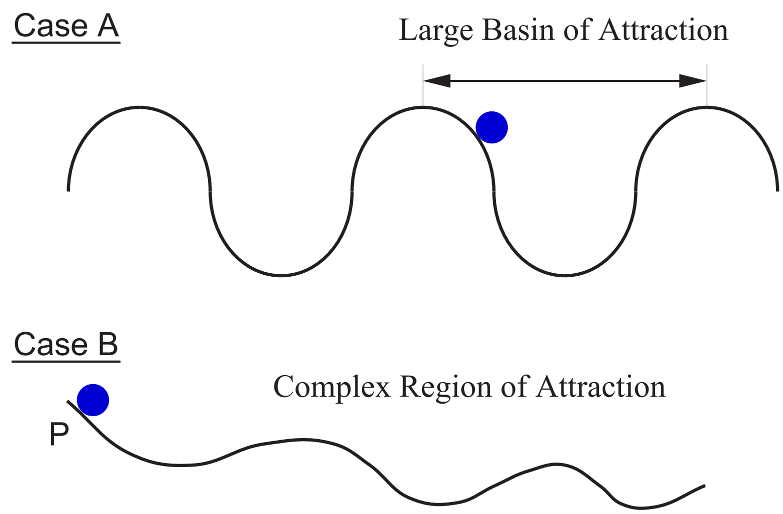


Figure 20.2 Basins of Attraction

Now that we have presented some intuitive notions of stability, we will pursue them with mathematical rigor in the remainder of this chapter.

Definitions

We will begin with specific mathematical definitions of the different types of stability discussed in the previous section. In these definitions we will be talking about the stability of an *equilibrium point*; a point $\mathbf{a}^*$ where the derivative in Eq. (20.1) is zero. For simplicity, we will talk specifically about the point $\mathbf{a}^* = \mathbf{0}$, which is referred to as the origin. This restriction does not affect the generality of our discussion.

Equilibrium Point

Stability

Definition 1: *Stability* (in the sense of Lyapunov)

The origin is a stable equilibrium point if for any given value $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that if $\|\mathbf{a}(0)\| < \delta$, then the resulting motion $\mathbf{a}(t)$ satisfies $\|\mathbf{a}(t)\| < \varepsilon$ for $t > 0$.

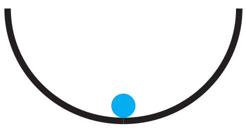
This definition says that the system output is not going to move too far away from a given stable point, so long as it is initially close to the stable point. Let's say that you want the system output to remain within a distance ε of the origin. If the origin is stable, then you can always find a distance δ (which may be a function of ε), such that if the system output is within δ of the origin at time $t = 0$, then it will always remain within ε of the origin. The position of the ball (with zero velocity) in the figure to the left is stable in the sense of Lyapunov, so long as the ball bearing has friction. If the ball bearing did not have friction, then any initial velocity would produce a trajectory $\mathbf{a}(t)$ in which the position would go to infinity. (The vector $\mathbf{a}(t)$ in this case would consist of the position and the velocity of the ball.)

Next, let's consider the stronger concept of asymptotic stability.

Asymptotic Stability

Definition 2: *Asymptotic Stability*

The origin is an asymptotically stable equilibrium point if there exists a number $\delta > 0$ such that whenever $\|\mathbf{a}(0)\| < \delta$ the resulting motion satisfies $\|\mathbf{a}(t)\| \to 0$ as $t \to \infty$.



This is a stronger definition of stability. It says that as long as the output of the system is initially within some distance δ of the stable point, the output will eventually converge to the stable point. The position of the ball (with zero velocity) in the diagram in the left margin is asymptotically stable, so long as the ball bearing has friction. If there is no friction, the position is only stable in the sense of Lyapunov.

We would like to build neural networks that have many specified asymptotically stable points, each of which represents a prototype pattern. This is the design objective we will use for building Hopfield networks in Chapter 21.

In addition to the stability definitions, there is another concept we will use in analyzing stability. It is the concept of a *definite* function. The next two definitions will clarify this concept.

Positive Definite

Definition 3: *Positive Definite*

A scalar function $V(\mathbf{a})$ is positive definite if $V(\mathbf{0}) = 0$ and $V(\mathbf{a}) > 0$ for $\mathbf{a} \neq \mathbf{0}$.

Positive Semidefinite

Definition 4: *Positive Semidefinite*

A scalar function $V(\mathbf{a})$ is positive semidefinite if $V(\mathbf{a}) \geq 0$ for all $\mathbf{a}$.

(These definitions can be modified appropriately to define the concepts *negative definite* and *negative semidefinite*.) Now that we have defined stability, let's consider a method for testing stability.

Lyapunov Stability Theorem

One of the most important approaches for investigating the stability of nonlinear systems is the theory introduced by Alexandr Mikhailovich Lyapunov, a Russian mathematician. Although his major work was first published in 1892, it received little attention outside Russia until much later. In this section we will discuss one of Lyapunov's most powerful techniques for stability analysis — the so-called *direct method*.

Consider the autonomous (unforced, no explicit time dependence) system:

$$\frac{d\mathbf{a}}{dt} = \mathbf{g}(\mathbf{a}) . \tag{20.2}$$

The Lyapunov stability theorem can then be stated as follows.

Theorem 1: *Lyapunov Stability Theorem*

If a positive definite function $V(\mathbf{a})$ can be found such that $dV(\mathbf{a})/dt$ is negative semidefinite, then the origin ($\mathbf{a} = \mathbf{0}$) is stable for the system of Eq. (20.2). If a positive definite function $V(\mathbf{a})$ can be found such that $dV(\mathbf{a})/dt$ is negative definite, then the origin ($\mathbf{a} = \mathbf{0}$) is asymptotically stable. In each case, V is called a Lyapunov function of the system.

You can think of $V(\mathbf{a})$ as a generalized energy function. The concept of the theorem is that if the energy of a system is continually decreasing ($dV(\mathbf{a})/dt$ negative definite), then it will eventually settle at some minimum energy state. Lyapunov's insight was to generalize the concept of energy, so that the theorem could be applied to systems where the energy is difficult to express or has no meaning.

We should note that the theorem only states that if a suitable Lyapunov function $V(\mathbf{a})$ can be found, the system is stable. It gives us no information about the stability of the system in those situations where we are unable to find such a function.

Pendulum Example

We can gain some insight into Lyapunov's stability theorem by applying it to a simple mechanical system. This system is very simple, and its operation is easy to visualize, and yet it illustrates important concepts that we will apply to neural network design in the next chapter. The example system we will use is the pendulum shown in Figure 20.3.

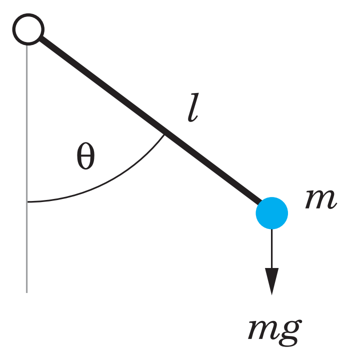


Figure 20.3 Pendulum

Using Newton's second law ($F = ma$), we can write the equation of operation of the pendulum as

$$ml\frac{d^2}{dt^2}(\theta) = -c\frac{d\theta}{dt} - mg\sin(\theta), \tag{20.3}$$

or

$$ml\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + mg\sin(\theta) = 0, \tag{20.4}$$

where θ is the angle of the pendulum, m is the mass of the pendulum, l is the length of the pendulum, c is the damping coefficient, and g is the gravitational constant.

The first term on the right side of Eq. (20.3) is the damping force, which is proportional to the velocity of the pendulum. It is this term that represents the energy dissipation in the system. The second term on the right side of Eq. (20.3) is the gravitational force, which is proportional to the sine of the angle of the pendulum. It is equal to zero when the pendulum is straight down and has its maximum value when the pendulum is horizontal.

If the damping coefficient is not zero, the pendulum will eventually come to rest hanging down in the vertical position. This solution might be viewed as $\theta = 0$, but more generally it is $\theta = 2\pi n$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. That is, given the appropriate initial conditions, the pendulum might simply settle to $\theta = 0$ or it might rotate once to give a solution of $\theta = 2\pi$, etc. There are many possible equilibrium solutions. (The positions $\theta = \pi n$, for odd values of n, are also equilibrium points, but they are not stable.)

To analyze the stability of this system, we will write the pendulum equation in state variable form, where it will appear as a pair of first-order differential equations. Let's choose the following state variables:

$$a_1 = \theta \text{ and } a_2 = \frac{d\theta}{dt}. \tag{20.5}$$

We can write equations for the pendulum in terms of these state variables as follows:

$$\frac{da_1}{dt} = a_2, \tag{20.6}$$

$$\frac{da_2}{dt} = -\frac{g}{l}\sin(a_1) - \frac{c}{ml}a_2. \tag{20.7}$$

Now we want to investigate the stability of the origin ($\mathbf{a} = \mathbf{0}$) for this pendulum system. (The origin corresponds to a pendulum angle of zero and a pendulum velocity of zero.) We first want to check that the origin is an equilibrium point. We do this by substituting $\mathbf{a} = \mathbf{0}$ into the state equations.

$$\frac{da_1}{dt} = a_2 = 0, \tag{20.8}$$

$$\frac{da_2}{dt} = -\frac{g}{l}\sin(a_1) - \frac{c}{ml}a_2 = -\frac{g}{l}\sin(0) - \frac{c}{ml}(0) = 0 \tag{20.9}$$

Since the derivatives are zero, the origin is an equilibrium point.

Next we need to find a Lyapunov function for the pendulum. For this example we will use the energy of the system as the Lyapunov function V. To obtain the total energy of the pendulum, we add the kinetic and potential energies.

$$V(\mathbf{a}) = \frac{1}{2}ml^2(a_2)^2 + mgl(1 - \cos(a_1)) \tag{20.10}$$

In order to test the stability of the system, we need to evaluate the derivative of V with respect to time.

$$\frac{d}{dt}V(\mathbf{a}) = [\nabla V(\mathbf{a})]^T \mathbf{g}(\mathbf{a}) = \frac{\partial V}{\partial a_1}\left(\frac{da_1}{dt}\right) + \frac{\partial V}{\partial a_2}\left(\frac{da_2}{dt}\right) \tag{20.11}$$

The partial derivatives of $V(\mathbf{a})$ can be obtained from Eq. (20.10), and the derivatives of the two state variables are given by Eq. (20.6) and Eq. (20.7). Thus we have

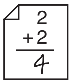
$$\frac{d}{dt}V(\mathbf{a}) = (mgl\sin(a_1))a_2 + (ml^2 a_2)\left(-\frac{g}{l}\sin(a_1) - \frac{c}{ml}a_2\right). \tag{20.12}$$

The $(mgl\sin(a_1))a_2$ terms cancel, which leaves only

$$\frac{d}{dt}V(\mathbf{a}) = -cl(a_2)^2 \le 0. \tag{20.13}$$

In order to prove that the origin ($\mathbf{a} = \mathbf{0}$) is asymptotically stable, we must show that this derivative is negative definite. The derivative is zero at the origin, but it also is zero for any value of a_1, as long as $a_2 = 0$. Thus, $dV(\mathbf{a})/\mathrm{dt}$ is negative semidefinite, rather than negative definite. From Lyapunov's theorem, then, we know that the origin is a stable point. However, we *cannot* say, from the theorem and this Lyapunov function, that the origin is asymptotically stable.

In this case we know that as long as the pendulum has friction, it will eventually settle in a vertical position, and, therefore, that the origin is asymptotically stable. However, Lyapunov's theorem, using our Lyapunov function, can only tell us that the origin is stable. To prove that the origin is asymptotically stable, we will need a refinement of Lyapunov's theorem, LaSalle's Invariance Theorem. We will discuss LaSalle's theorem in the next section.



First, let's investigate the pendulum further, by taking a specific numerical example. Let $g = 9.8, \quad m = 1, \quad l = 9.8, \quad c = 1.96$. Now we can rewrite the state equations for the pendulum as

$$\frac{da_1}{dt} = a_2 , \tag{20.14}$$

$$\frac{da_2}{dt} = -\sin(a_1) - 0.2a_2 . \tag{20.15}$$

Expressions for V and its derivative follow:

$$V = (9.8)^2\left[\frac{1}{2}(a_2)^2 + (1 - \cos(a_1))\right], \tag{20.16}$$

$$\frac{dV}{dt} = -(19.208)(a_2)^2 . \tag{20.17}$$

Note that dV/dt is zero for any value of a_1 as long as $a_2 = 0$.

Figure 20.4 displays the 3-D and contour plots of the energy surface, V, as the angle varies between -10 and +10 radians and the angular velocity varies between -2 and 2 radians per second. Note that in this range there are three possible minimum points of the energy surface, at 0 and $\pm 2\pi$.

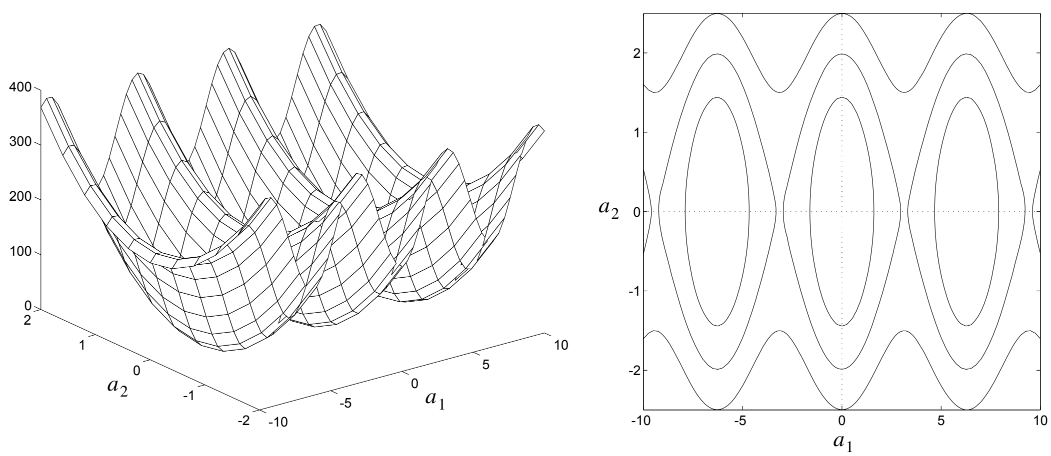


Figure 20.4 Pendulum Energy Surface

(We will find in Chapter 21 that the minimum points of the Lyapunov function can correspond to prototype patterns in an autoassociative neural network. The pendulum system, like recurrent neural networks, has many minimum points.)

Of course, the energy plots shown in Figure 20.4 do not tell us in what way, or by what route, the pendulum finds a particular energy minimum. To show this, we have plotted the energy contours, and one particular path for the pendulum, in Figure 20.5. The response trajectory, shown by the blue line, starts from an initial position, $a_1(0)$, of 1.3 radians (74°) and an initial velocity, $a_2(0)$, of 1.3 radians per second. The trajectory converges to the equilibrium point $\mathbf{a} = \mathbf{0}$.

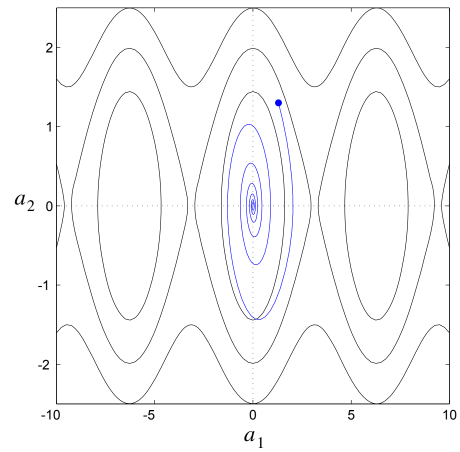


Figure 20.5 Pendulum Response on State Variable Plane

A time response plot of the two state variables is shown in Figure 20.6. Notice that, because the initial velocity is positive, the pendulum continues to move up initially. (Check to see if this agrees with Figure 20.5.) It reaches a maximum angle of about 2 radians before falling back down. The oscillations continue to decay as both state variables converge to zero.

In this case, both state variables converge to zero. However, this is not the only possible equilibrium point, as we will show later.

It is also interesting to plot the pendulum energy, V, as in Figure 20.7. Recall from Eq. (20.17) that the energy should never increase; this is consistent with Figure 20.7. Eq. (20.17) also predicts that the derivative of the energy curve should only be zero when the velocity, a_2, is zero. This is also verified if we compare Figure 20.7 with Figure 20.6. At those times where the a_2 graph crosses the zero axis, the slope of the energy curve is zero.

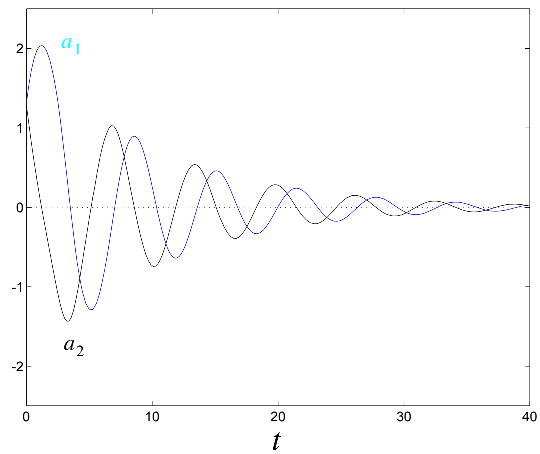


Figure 20.6 State Variables a_1 (blue) and a_2 vs. Time

Notice that, although there are points where the derivative of the energy curve is zero, the derivative does not remain zero until the energy is also zero. This observation will lead to LaSalle's Invariance Theorem, which we will discuss in the next section. The key idea of that theorem is to identify those points where the derivative of the Lyapunov function is zero, and then to determine if the system will be trapped at those points. (Those places where a trajectory can be trapped are called invariant sets.) If the only point that can trap the trajectory, and that has zero derivative, is the origin, then the origin is asymptotically stable.

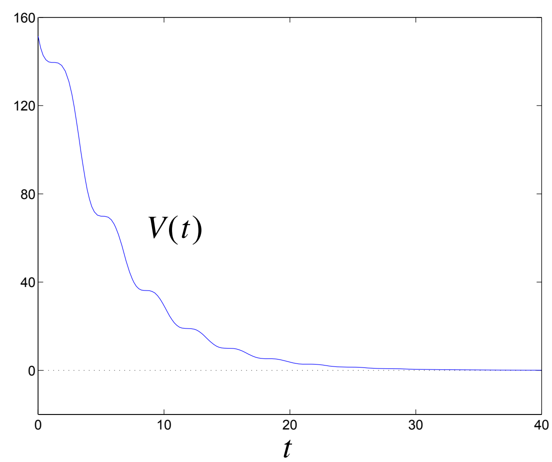


Figure 20.7 Pendulum Lyapunov Function (Energy) vs. Time

The particular pendulum behavior shown in the graphs in this section depends on the initial conditions of the two state variables. The choice of a different set of initial conditions may give results entirely different from those shown in these plots. We will expand on this in the next section.

To experiment with the pendulum, use the Neural Network Design Demonstration Dynamic System (`nnd17ds`).

LaSalle's Invariance Theorem

The pendulum example demonstrated a problem with Lyapunov's theorem. We found a Lyapunov function whose derivative was only negative semidefinite (not negative definite), and yet we know that the origin is asymptotically stable for the pendulum system. In this section we will introduce a theorem that clarifies this uncertainty in Lyapunov's theorem. It does so by defining those regions of the state space where the derivative of the Lyapunov function is zero, and then identifying those parts of that region that can trap the trajectory.

Before we discuss LaSalle's Invariance Theorem, we first need to introduce the following definitions.

Definitions

Lyapunov Function

Definition 5: *Lyapunov Function*

Let V be a continuously differentiable function from $\Re^n$ to $\Re$. If G is any subset of $\Re^n$, we say that V is a Lyapunov function on G for the system $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ if

$$\frac{dV(\mathbf{a})}{dt} = (\nabla V(\mathbf{a}))^T \mathbf{g}(\mathbf{a}) \tag{20.18}$$

does not change sign on G.

This is a generalization of our previous definition of the Lyapunov function, which we used in Theorem 1. Here we do not require that the function be positive definite. In fact, there is no direct requirement on the function itself (except that it be continuously differentiable). The only requirement is on the derivative of V. The derivative cannot change sign anywhere on the set G. Note that the derivative will not change sign if it is negative semidefinite or if it is positive semidefinite.

We should note here that we have not yet explained how to choose the set G. We will use the following definitions and theorems to help us select the best G for a given system.

Set Z

Definition 6: Set Z

$$Z = \{\mathbf{a}\colon dV(\mathbf{a})/dt = 0,\ \mathbf{a} \text{ in the closure of } G\}. \tag{20.19}$$

Here "the closure of G" includes the interior and the boundary of G. This is a key set. It contains all of those points where the derivative of the Lyapunov function is zero. Later we will want to determine where in this set the system trajectory can be trapped.

Invariant Set

Definition 7: *Invariant Set*

A set of points in $\Re^n$ is *invariant* with respect to $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ if every solution of $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ starting in that set remains in the set for all time.

If the system gets into an invariant set, then it can't get out.

Set L

Definition 8: *Set* L

L is defined as the largest invariant set in Z.

This set includes all possible points at which the solution might converge. The Lyapunov function does not change in L (because its derivative is zero), and the trajectory will be trapped in L (because it is an invariant set). Now, if this set has only one stable point, then that point is asymptotically stable. This is, in essence, what LaSalle's theorem will say.

Theorem

LaSalle's Invariance Theorem extends the Lyapunov Stability Theorem. We will use it to design Hopfield networks in the next chapter. The theorem proceeds as follows [Lasa67].

Theorem 2: *LaSalle's Invariance Theorem*

If V is a Lyapunov function on G for $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$, then each solution $\mathbf{a}(t)$ that remains in G for all $t > 0$ approaches $L^\circ = L \cup \{\infty\}$ as $t \to \infty$. (G is a basin of attraction for L, which has all of the stable points.) If all trajectories are bounded, then $\mathbf{a}(t) \to L$ as $t \to \infty$.

If a trajectory stays in G, then it will either converge to L, or it will go to infinity. If all trajectories are bounded, then all trajectories will converge to L.

There is a corollary to LaSalle's theorem that we will use extensively. It involves choosing the set G in a special way.

Corollary 1: *LaSalle's Corollary*

Let G be a component (one connected subset) of

$$\Omega_\eta = \{\mathbf{a}:\mathrm{V}(\mathbf{a}) < \eta\}\,. \tag{20.20}$$

Assume that G is bounded, $dV(\mathbf{a})/dt \le 0$ on the set G, and let the set $L^\circ = closure(L \cap G)$ be a subset of G. Then L° is an attractor, and G is in its region of attraction.

LaSalle's theorem, and its corollary, are very powerful. Not only can they tell us which points are stable (L°), but they can also provide us with a partial region of attraction (G). (Note that L° is constructed differently in the corollary than in the theorem.)

To clarify LaSalle's Invariance Theorem, let's return to the pendulum example we discussed earlier.

Example

Let's apply Corollary 1 to the pendulum example. The first step in using the corollary will be to choose the set Ω_η. This set will then be used to select the set G (a component of Ω_η).

For this example we will use the value $\eta = 100$, therefore Ω_{100} will be the set of points where the energy is less than or equal to 100.

$$\Omega_{100} = \{\mathbf{a}:V(\mathbf{a}) \le 100\} \tag{20.21}$$

This set is displayed in blue in Figure 20.8.

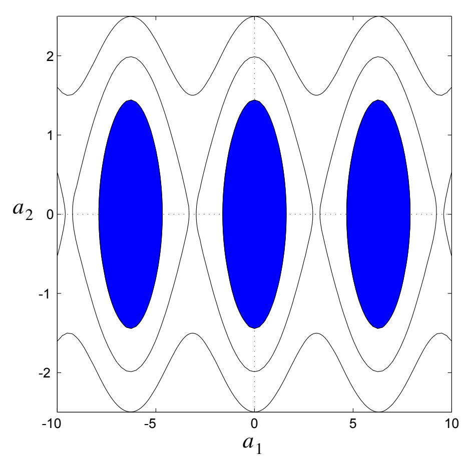


Figure 20.8 Illustration of the Set Ω_{100}

The next step in our analysis is to choose a component (connected subset) of Ω_{100} for the set G. Since we have been investigating the stability of the origin, let's choose the component of Ω_{100} that contains $\mathbf{a} = \mathbf{0}$. The resulting set is shown in Figure 20.9.

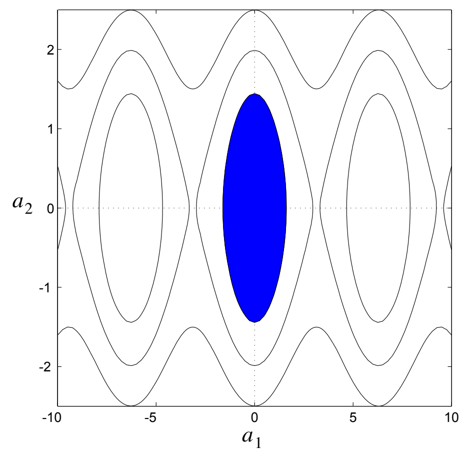


Figure 20.9 Illustration of the Set G

Now that we have chosen G, we need to check that the derivative of the Lyapunov function is less than or equal to zero on G. From Eq. (20.17) we know that $dV(\mathbf{a})/dt$ is negative semidefinite. Therefore it will certainly be less than or equal to zero on G.

We are now ready to determine the attractor set L°. We begin with the set L, which is the largest invariant set in Z.

$$Z = \{\mathbf{a}\colon dV(\mathbf{a})/dt = 0,\ \mathbf{a} \text{ in the closure of } G\}$$

$$= \{\mathbf{a}\colon a_2 = 0, \mathbf{a} \text{ in the closure of } G\}\ . \tag{20.22}$$

This can also be written as

$$Z = \{\mathbf{a}\colon a_2 = 0, -1.6 \le a_1 \le 1.6\}\ . \tag{20.23}$$

We know from Eq. (20.17) that the derivative of $V(\mathbf{a})$ is only zero when the velocity is zero, which corresponds to the a_1 axis. Therefore Z consists of the segment of the a_1 axis that falls within G. The set Z is displayed in Figure 20.10.

The set L is the largest invariant set in Z. To find L we need to answer the question: If we start the pendulum from an initial position between -1.6 and 1.6 radians, with zero initial velocity, will the velocity of the pendulum remain zero? Clearly the only such initial condition would be 0 radians (straight down). If we start the pendulum from any other position in Z, the pendulum will start to fall, so the velocity will not remain zero and the trajectory will move out of Z. Therefore, the set L consists only of the origin:

$$L = \{\mathbf{a}\colon \mathbf{a} = 0\}\ . \tag{20.24}$$

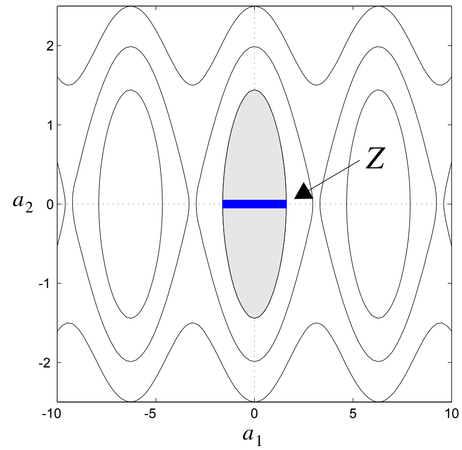


Figure 20.10 Illustration of the Set Z

The set L° is the closure of the intersection of L and G, which in this case is simply L:

$$L^\circ = closure(L \cap G) = L = \{\mathbf{a}: \mathbf{a} = 0\}. \tag{20.25}$$

Therefore, based on LaSalle's corollary, L° is an attractor (asymptotically stable point) and G is in its region of attraction. This means that any trajectory that starts in G will decay to the origin.

Now suppose that we had taken a bigger region for Ω_η, such as

$$\Omega_{300} = \{\mathbf{a}:(V(\mathbf{a}) < 300)\}. \tag{20.26}$$

This set is shown in gray in Figure 20.11.

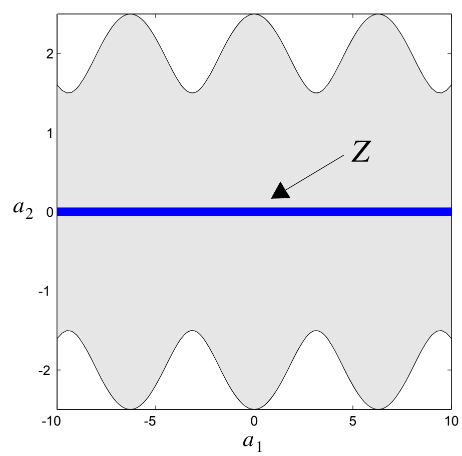


Figure 20.11 Illustration of $G = \Omega_{300}$ (Gray) and Z

We let $G = \Omega_{300}$, since Ω_{300} has only one component. The set Z is given by

$$Z = \{\mathbf{a}: a_2 = 0\}\,, \tag{20.27}$$

which is shown by the blue bar on the horizontal axis of Figure 20.11. Thus, it follows that

$$L^\circ = L = \{\mathbf{a}: a_1 = \pm n\pi,\ a_2 = 0\}\,. \tag{20.28}$$

This is because there are now several different positions within the set Z where we can place the pendulum, without causing the velocity to become nonzero. The pendulum can be pointing directly up or directly down. This corresponds to the positions $\pm n\pi$ for any integer n. If we place the pendulum in any of these positions, with zero velocity, then the pendulum will remain stationary. We can show this by setting the derivatives equal to zero in Eq. (20.14) and Eq. (20.15).

$$\frac{da_1}{dt} = a_2 = 0\,, \tag{20.29}$$

$$\left(\frac{da_2}{dt} = -\sin(a_1) - 0.2a_2 = -\sin(a_1) = 0\right) \Rightarrow (a_1 = \pm n\pi) \tag{20.30}$$

For this choice of $G = \Omega_{300}$ we can say very little about where the trajectory will converge. We tried to increase the size of our known region of attraction for the origin, but this G is a region of attraction for all of the equilibrium points. We made G too large. The set L° is illustrated by the blue dots in Figure 20.12.

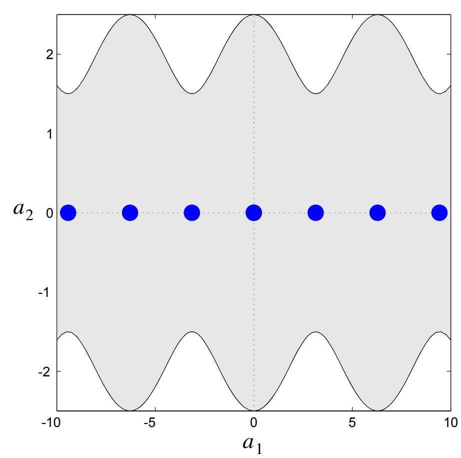


Figure 20.12 The Set L°

We cannot tell which of the equilibrium points (blue dots) will attract the trajectory. All we can say is that if we start somewhere in Ω_{300}, one of the equilibrium points will attract the system solution, but we cannot say for sure which one it will be. Consider, for instance, the trajectory shown in

Figure 20.13. This shows the pendulum response for an initial position of 2 radians and an initial velocity of 1.5 radians per second. This time the pendulum had enough velocity to go over the top, and it converged to the equilibrium point at 2π radians.

Now that we have discussed LaSalle's Invariance Theorem, you might want to experiment some more with the pendulum, in order to investigate the regions of attraction for the various stable points. To experiment with the pendulum, use the Neural Network Design Demonstration Dynamic System (`nnd17ds`).

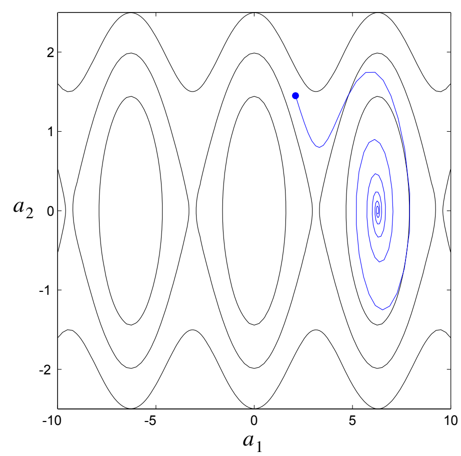


Figure 20.13 Pendulum Trajectory for Different Starting Conditions

Comments

The keys to LaSalle's theorem are the choices of the Lyapunov function V and the set G. We want G to be as large as possible, because that will indicate the region of attraction. However, we want to choose V so that the set Z, which will contain the attractor set, is as small as possible.

For instance, we could try $V = 0$. This is a Lyapunov function for the entire space $\Re^n$, since its derivative is zero (and therefore doesn't change sign) everywhere. However, it gives us no information since $Z = \Re^n$.

Notice that if V_1 and V_2 are both Lyapunov functions on G, and dV_1/dt and dV_2/dt have the same sign, then $V = V_1 + V_2$ is also a Lyapunov function, where $Z = Z_1 \cap Z_2$. If Z is smaller than both Z_1 and Z_2, then Z is a "better" Lyapunov function than either V_1 or V_2. V is always at least as good as either V_1 or V_2, since Z can never be larger than the smaller of Z_1 and Z_2. Therefore, if you have found two different Lyapunov functions and their derivatives have the same sign, then add them together and you may have a better function. The best Lyapunov function for a given system is the one that has the smallest attractor set and the largest region of attraction.

Summary of Results

Stability Concepts

Definitions

Definition 1: *Stability* (in the sense of Lyapunov)

The origin is a stable equilibrium point if for any given value $\varepsilon > 0$ there exists a number $\delta(\varepsilon) > 0$ such that if $\|\mathbf{a}(0)\| < \delta$, then the resulting motion $\mathbf{a}(t)$ satisfies $\|\mathbf{a}(t)\| < \varepsilon$ for $t > 0$.

Definition 2: *Asymptotic Stability*

The origin is an asymptotically stable equilibrium point if there exists a number $\delta > 0$ such that whenever $\|\mathbf{a}(0)\| < \delta$ the resulting motion satisfies $\|\mathbf{a}(0)\| \to 0$ as $t \to \infty$.

Definition 3: *Positive Definite*

A scalar function $V(\mathbf{a})$ is positive definite if $V(\mathbf{0}) = 0$ and $V(\mathbf{a}) > 0$ for $\mathbf{a} \neq \mathbf{0}$.

Definition 4: *Positive Semidefinite*

A scalar function $V(\mathbf{a})$ is positive semidefinite if $V(\mathbf{a}) \geq 0$ for all $\mathbf{a}$.

Lyapunov Stability Theorem

Consider the autonomous (unforced, no explicit time dependence) system

$$\frac{d\mathbf{a}}{dt} = \mathbf{g}(\mathbf{a}).$$

The Lyapunov stability theorem can then be stated as follows.

Theorem 1: *Lyapunov Stability Theorem*

If a positive definite function $V(\mathbf{a})$ can be found such that $dV(\mathbf{a})/dt$ is negative semidefinite, then the origin ($\mathbf{a} = \mathbf{0}$) is stable for this system. If a positive definite function $V(\mathbf{a})$ can be found such that $dV(\mathbf{a})/dt$ is negative definite, then the origin ($\mathbf{a} = \mathbf{0}$) is asymptotically stable. In each case, V is called a Lyapunov function of the system.

LaSalle's Invariance Theorem

Definitions

Definition 5: *Lyapunov Function*

Let V be a continuously differentiable function from $\Re^n$ to $\Re$. If G is any subset of $\Re^n$, we say that V is a Lyapunov function on G for the system $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ if

$$\frac{dV(\mathbf{a})}{dt} = (\nabla V(\mathbf{a}))^T \mathbf{g}(\mathbf{a})$$

does not change sign on G.

Definition 6: *Set* Z

$$Z = \{\mathbf{a}: dV(\mathbf{a})/dt = 0, \ \mathbf{a} \text{ in the closure of } G\} . \tag{20.31}$$

Definition 7: *Invariant Set*

A set of points G in $\Re^n$ is *invariant* with respect to $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ if every solution of $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$ starting in G remains in G for all time.

Definition 8: *Set* L

L is defined as the largest invariant set in Z.

Theorem

Theorem 2: *LaSalle's Invariance Theorem*

If V is a Lyapunov function on G for $d\mathbf{a}/dt = \mathbf{g}(\mathbf{a})$, then each solution $\mathbf{a}(t)$ that remains in G for all $t > 0$ approaches $L^\circ = L \cup \{\infty\}$ as $t \to \infty$. (G is a basin of attraction for L, which has all of the stable points.) If all trajectories are bounded, then $\mathbf{a}(t) \to L$ as $t \to \infty$.

Corollary 1: *LaSalle's Corollary*

Let G be a component (one connected subset) of

$$\Omega_\eta = \{\mathbf{a}: V(\mathbf{a}) < \eta\} . \tag{20.32}$$

Assume that G is bounded, $dV(\mathbf{a})/dt \le 0$ on the set G, and let the set $L^\circ = closure(L \cap G)$ be a subset of G. Then L° is an attractor, and G is in its region of attraction.

Solved Problems

P20.1 Test the stability of the origin for the following system.

$$da_1/dt = -a_1 + (a_2)^2$$

$$da_2/dt = -a_2(a_1 + 1)$$

The basic job here is to find a Lyapunov $V(\mathbf{a})$ that is positive definite and has a derivative that is negative semidefinite or, better yet, negative definite. (The latter is a stronger condition.)

Let us try $V(\mathbf{a}) = (a_1)^2 + (a_2)^2$. The derivative of $V(\mathbf{a})$ is

$$\frac{dV(\mathbf{a})}{dt} = (\nabla V)^T\left(\frac{d\mathbf{a}}{dt}\right) = \frac{\partial V}{\partial a_1}\left(\frac{da_1}{dt}\right) + \frac{\partial V}{\partial a_2}\left(\frac{da_2}{dt}\right),$$

or

$$\frac{dV(\mathbf{a})}{dt} = 2a_1(-a_1 + (a_2)^2) + 2a_2(-a_2(a_1 + 1)) = -2(a_1)^2 - 2(a_2)^2.$$

The derivative $dV(\mathbf{a})/dt$ is negative definite. Therefore, the origin is asymptotically stable.

P20.2 Test the stability of the origin for the following system.

$$da_1/dt = -(a_1)^5$$

$$da_2/dt = -5(a_2)^7$$

Let us try $V(\mathbf{a}) = (a_1)^2 + (a_2)^2$. Then we have

$$\frac{dV(\mathbf{a})}{dt} = 2a_1(-(a_1)^5) + 2a_2(-5(a_2)^7) = -2(a_1)^6 - 10(a_2)^8.$$

Here again, $dV(\mathbf{a})/dt$ is negative definite, and therefore the origin is asymptotically stable.

P20.3 Consider the mechanical system shown in Figure P20.1. This is a spring-mass-damper system, with a nonlinear spring. We will define $a_1 = x$ and $a_2 = dx/dt$. Then the equations of motion are

$$da_1/dt = a_2,$$

$$da_2/dt = -(a_1)^3 - a_2 \qquad \text{(nonlinear spring)}.$$

Consider the candidate Lyapunov function

$$V(\mathbf{a}) = \frac{1}{4}(a_1)^4 + \frac{1}{2}(a_2)^2.$$

Use the corollary to LaSalle's invariance theorem to provide as much information as possible about the equilibrium points and basins of attraction.

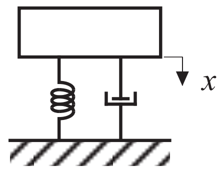


Figure P20.1 Mechanical System

First calculate the derivative of $V(\mathbf{a})$ as

$$\frac{dV(\mathbf{a})}{dt} = \frac{\partial V}{\partial a_1}\left(\frac{da_1}{dt}\right) + \frac{\partial V}{\partial a_2}\left(\frac{da_2}{dt}\right) = (a_1)^3 a_2 + a_2(-(a_1)^3 - a_2) = -(a_2)^2.$$

Thus, dV/dt does not change sign on $\Re^2$.

Now let us define

$$G = \Omega_\eta = \{\mathbf{a}: V(\mathbf{a}) \le \eta\}$$

and consider the case for $\eta = 1$. A contour plot of $V(\mathbf{a})$ is shown in Figure P20.2. The set Ω_1 is indicated in blue on the plot.

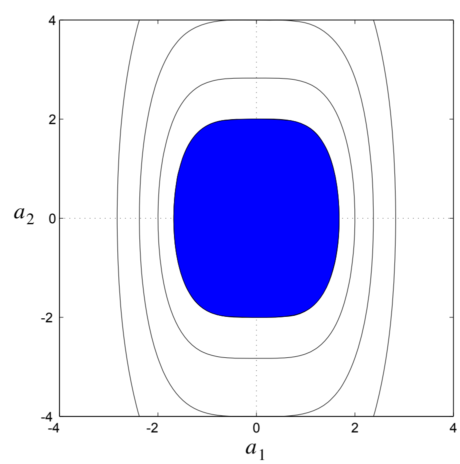


Figure P20.2 Contour Plot of $V(\mathbf{a})$ and Ω_1

Now we need to determine the set Z.

$$Z = \{\mathbf{a}: dV/dt = 0, \mathbf{a} \text{ in the closure of } G\} = \{\mathbf{a}: a_2 = 0, \mathbf{a} \text{ in the closure of } G\}$$

or

$$Z = \{\mathbf{a}: a_2 = 0, -\sqrt{2} \le a_1 \le \sqrt{2}\}$$

Next we find the set L. Since $\mathbf{a} = 0$ is the only invariant set,

$$L = \{\mathbf{a}: \quad a_1 = 0, \ a_2 = 0\} .$$

Therefore, the origin,

$$\mathbf{a} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

is an attractor and Ω_1 is in its region of attraction.

Further, we can increase η to show that the entire $\Re^2$ is the basin of attraction for the origin.

Figure P20.3 shows the response of the spring-mass-damper from an initial position of 2 and an initial velocity of 2. Note that the trajectory is parallel to the contour lines when the trajectory crosses the a_2 axis. This agrees with our earlier result, which showed that the derivative of the Lyapunov function was zero whenever $a_2 = 0$. Fortunately, the a_2 axis is not an invariant set (except for the origin); therefore the trajectory is only attracted to the origin.

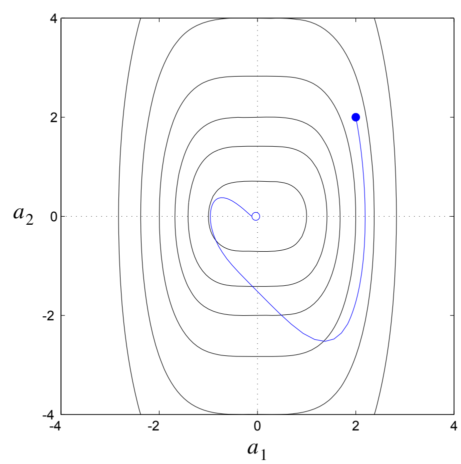


Figure P20.3 Spring-Mass-Damper Response

P20.4 Consider the following nonlinear system:

$$da_1/dt = a_1((a_1)^2 + (a_2)^2 - 4) - a_2$$

$$da_2/dt = a_1 + a_2((a_1)^2 + (a_2)^2 - 4) .$$

This system has two invariant sets, the origin

$$\{\mathbf{a}: \ \mathbf{a} = \mathbf{0} \ \} ,$$

and the circle

$$\{\mathbf{a}: (a_1)^2 + (a_2)^2 = 4 \ \} .$$

Assuming the candidate Lyapunov function

$$V(\mathbf{a}) = (a_1)^2 + (a_2)^2 ,$$

use LaSalle's Invariance Theorem to find out as much as you can about the region of attraction for the origin.

Our job, then, is to determine whether or not the given invariant sets represent a stable point or a stable trajectory. Let's first take a look at dV/dt. We recall that

$$\frac{dV(\mathbf{a})}{dt} = \frac{\partial V}{\partial a_1}\left(\frac{da_1}{dt}\right) + \frac{\partial V}{\partial a_2}\left(\frac{da_2}{dt}\right) ,$$

and substitute for the various terms to give

$$\frac{dV(\mathbf{a})}{dt} = 2a_1[a_1((a_1)^2 + (a_2)^2 - 4) - a_2] + 2a_2[a_1 + a_2((a_1)^2 + (a_2)^2 - 4)] .$$

This can be simplified to

$$\frac{dV(\mathbf{a})}{dt} = 2((a_1)^2 + (a_2)^2)((a_1)^2 + (a_2)^2 - 4) .$$

Thus, dV/dt is zero at $\mathbf{a} = \mathbf{0}$ and on the circle $(a_1)^2 + (a_2)^2 = 4$.

We now pick G, a region of attraction. Is there a change of sign of dV/dt over all $\Re^2$? Yes, there is. As we go from outside the circle of radius 2 to its interior, the sign of dV/dt changes from positive to negative. So dV/dt is negative semidefinite inside the circle $(a_1)^2 + (a_2)^2 = 4$. Let's pick a G inside this circle, so that the circle will not be included. The following set will do.

$$G = \Omega_1 = \{\mathbf{a}: V(\mathbf{a}) \le 1\}$$

Now we consider Ω_1. There are just two places that $dV/dt = 0$, and the only point inside Ω_1 is $\mathbf{a} = \mathbf{0}$. Therefore,

$$Z = \{\mathbf{a}: \ a_1 = 0, a_2 = 0\}$$

and

$$L^\circ = L = Z.$$

The origin is the attractor, and Ω_1 is in its region of attraction.We can use the same arguments to show that the region of attraction for the origin includes all points inside the circle $(a_1)^2 + (a_2)^2 = 4$.

Figure P20.4 displays two trajectories for this system, one that begins inside the circle $(a_1)^2 + (a_2)^2 = 4$, and one that begins outside the circle. Although the circle is an invariant set, it is not an attractor. The only attractor for this system is the origin.

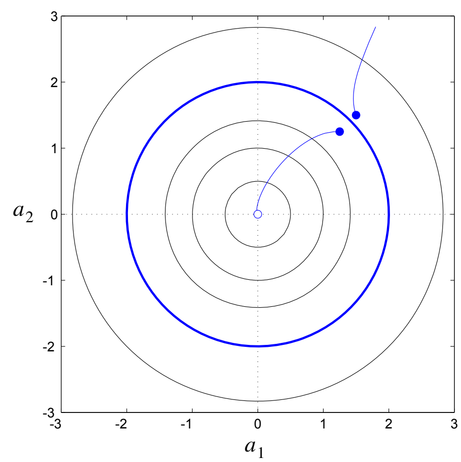


Figure P20.4 Sample Trajectories for Problem P20.4

P20.5 Consider the following nonlinear system.

$$da(t)/(dt) = -(a(t) - 1)(a(t) - 2)$$

i. Find any equilibrium points for this system.

ii. Use the following candidate Lyapunov function to obtain whatever information you can about the regions of attraction for the equilibrium points found in part (i). (Hint: Use the corollary to LaSalle's Invariance Theorem.)

$$V(\mathbf{a}) = (a-2)^2$$

i. To find the equilibrium points, we set $da(t)/dt = 0$.

$$0 = -(a-1)(a-2) \quad \Rightarrow \quad a = 1, a = 2 \quad \text{are equilibrium points}$$

ii. To use LaSalle's corollary, we need to find dV/dt.

$$\frac{dV}{dt} = \frac{\partial V}{\partial a}\left(\frac{da}{dt}\right) = 2(a-2)[-(a-1)(a-2)] = -2(a-1)(a-2)^2$$

Now we let

$$G = \Omega_\eta = \{a: \ V(a) < \eta\}.$$

For example, try $\eta = 0.5$. This gives

$$G = \Omega_{0.5} = \{a: (a-2)^2 < 0.5\}.$$

Note that a solution of $(a-2)^2 < 0.5$ yields

$$\pm(a-2) < \sqrt{0.5} \ \text{ or } \ 1.3 < a < 2.7.$$

Thus, dV/dt is negative definite on G.

Next we need to find the set Z, which contains those points within G where dV/dt is zero. There are two points where dV/dt is zero, $a = 1$ and $a = 2$. Only one of these falls within G. Therefore

$$Z = \{a: \ a = 2\}.$$

Now we need to find L, the largest invariant set in Z. There is only one point in Z, and it is an equilibrium point. Thus

$$L^\circ = L = Z.$$

This means that G is in the region of attraction for 2.

We can use the same arguments with values of η up to 1.0. So we can say that the region for attraction for $a = 2$ must include at least

$$\{a: \ 1 < a < 3\}.$$

What if we consider those regions where $\eta > 1$? Then Z includes both 1 and 2, and dV/dt will change sign on G. Therefore we cannot say anything about the region of attraction for $a = 1$, using this Lyapunov function and the corollary to LaSalle's Invariance Theorem.

Figure P20.5 displays some typical responses for this system. Here we can see that the equilibrium point $a = 1$ is actually unstable. Any initial condition above $a = 1$ converges to $a = 2$. Anything less than $a = 1$ goes to minus infinity.

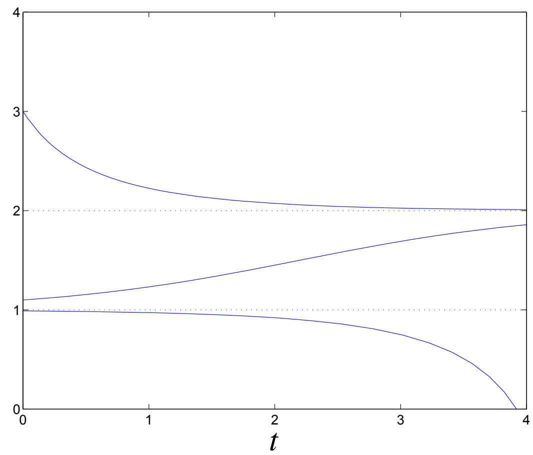


Figure P20.5 Stable and Unstable Responses for Problem P20.5

Epilogue

In this chapter we have presented the concept of stability, as applied to dynamic systems. For nonlinear dynamic systems, like recurrent neural networks, we do not talk about the stability of the system. Rather, we discuss the stability of certain system trajectories and, in particular, equilibrium points.

There were two main stability theorems discussed in this chapter. The first is the Lyapunov Stability Theorem, which introduces the concept of generalized energy — the Lyapunov function. The concept behind this theorem is that if a system's "energy" is always decreasing, then it will eventually stabilize at a point of minimum "energy."

The second theorem presented was LaSalle's Invariance Theorem, which is an enhancement of the Lyapunov Stability Theorem. There are two key improvements made by LaSalle. The first is a clarification of the cases in which the Lyapunov function does not decrease throughout the state space, but stays constant in some regions. LaSalle's theorem introduced the concept of an invariant set to identify those regions that can trap the system trajectory. The second improvement made by LaSalle's theorem is that, in addition to indicating the stability of equilibrium points, it also gave information about the regions of attraction of each stable point.

The ideas presented in this chapter are important tools for the analysis of recurrent neural networks, like the Grossberg networks of Chapters 18 and 19. (See [CoGr83] for an application of LaSalle's Invariance Theorem to recurrent neural networks.) In Chapter 21 we will use LaSalle's theorem to explain the operation of the Hopfield network.

Further Reading

[Brog91] W. L. Brogan, *Modern Control Theory,* 3rd Ed., Englewood Cliffs, NJ: Prentice-Hall, 1991.

This is a well-written book on the subject of linear systems. The first half of the book is devoted to linear algebra. It also has good sections on the solution of linear differential equations and the stability of linear and nonlinear systems. It has many worked problems.

[CoGr83] M. A. Cohen and S. Grossberg, "Absolute stability of global pattern formation and parallel memory storage by competitive neural networks," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 13, no. 5, pp. 815–826, 1983.

Cohen and Grossberg apply LaSalle's Invariance Theorem to the analysis of the stability of competitive neural networks. The network description is very general, and the authors show how their analysis can be applied to many different types of recurrent neural networks.

[Lasa67] J. P. LaSalle, "An invariance principle in the theory of stability," in *Differential Equations and Dynamic Systems*, J. K. Hale and J. P. LaSalle, eds., New York: Academic Press, pp. 277–286, 1967.

This article provides a unified presentation of Lyapunov's stability theory, including several extensions. It introduces LaSalle's Invariance Theorem and various corollaries.

[SlLi91] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*, Englewood Cliffs, NJ: Prentice-Hall, 1991.

This text is an introduction to nonlinear control systems. A significant portion of the book is devoted to the analysis of nonlinear dynamic systems. A number of stability theorems are presented and demonstrated.

Exercises

E20.1 Use Lyapunov's Stability Theorem to test the stability of the origin for the following systems.

i.

$$da_1/dt = -(a_1)^3 + a_2$$

$$da_2/dt = -a_1 - a_2$$

ii.

$$da_1/dt = -a_1 + (a_2)^2$$

$$da_2/dt = -a_2(a_1 + 1)$$

E20.2 Consider the following nonlinear system:

$$da_1/dt = a_2 - 2a_1((a_1)^2 + (a_2)^2),$$

$$da_2/dt = -a_1 - 2a_2((a_1)^2 + (a_2)^2).$$

i. Use Lyapunov's Stability Theorem and the candidate Lyapunov function shown below to investigate the stability of the origin.

$$V(\mathbf{a}) = \alpha(a_1)^2 + \beta(a_2)^2$$

```
» 2 + 2
ans =
     4
```

ii. Check your stability result from part (i) by writing a MATLAB M-file to simulate the response of this system for several different initial conditions. Use the `ode45` routine. Plot the responses.

E20.3 Consider the following nonlinear system:

$$da/dt = a(a + 1),$$

i. Find any equilibrium points.

ii. The following Lyapunov function is proposed. Show that this is a valid Lyapunov function for use in Lasalle's invariance theorem.

$$V(a) = -(2a^3 + 3a^2).$$

iii. Use the corollary to Lasalle's theorem and the proposed Lyapunov function to provide as much information as you can about the stable equilibrium points and their basins of attraction. Identify the sets Z, G and L. Use graphs wherever possible.

E20.4 Repeat E20.3 for the following systems and Lyapunov functions. (In some cases, it may be useful to sketch the Lyapunov functions.)

i. $da/dt = (a-2)(a+1)$, $V(a) = (a+1)^2$

ii. $da/dt = a(a+1)$, $V(a) = -(2a^3 + 3a^2)$

iii. $da/dt = a(a+2)$, $V(a) = -a^3/3 - a^2$

iv. $da/dt = -a(a-1)$, $V(a) = 2a^3 - 3a^2$

v. $da/dt = \cos(a)$, find a $V(a)$

vi. $da/dt = \sin(a)$, find a $V(a)$

E20.5 Consider the following nonlinear system:

$$da_1/dt = a_2,$$

$$da_2/dt = -a_2(1-a_2)^2 - a_1.$$

We want to use the corollary to Lasalle's invariance theorem to locate attractors and find out as much as we can about the basins of attraction, using the following Lyapunov function.

$$V(\mathbf{a}) = (a_1)^2 + (a_2)^2.$$

i. Find any equilibrium points.

ii. Find $dV(\mathbf{a})/dt$.

iii. Choose a set G.

iv. Find the corresponding set Z.

v. Find the set L.

vi. What have you learned about the attractors of this system and the basins of attraction? Can you learn more by modifying the set G? Explain.

```
» 2 + 2
ans =
     4
```

vii. Check your results by writing a MATLAB M-file to simulate the response of this system for several different initial conditions. Use the `ode45` routine. Plot the responses.

E20.6 Consider the following nonlinear system:

$$da_1/dt = a_2,$$

$$da_2/dt = -a_1 - (a_2)^3 .$$

i. Find any equilibrium points.

ii. Find as much information about the stability of the equilibrium points as possible, using the corollary to LaSalle's theorem and the candidate Lyapunov function

$$V(\mathbf{a}) = (a_1)^2 + (a_2)^2 .$$

iii. Check your results from parts (i) and (ii) by writing a MATLAB M-file to simulate the response of this system for several different initial conditions. Use the `ode45` routine. Plot the responses.

E20.7 Consider the following nonlinear system:

$$da/dt = (1 - a)(1 + a) = 1 - a^2 .$$

i. Find any equilibrium points.

ii. Find a suitable Lyapunov function. (Hint: Start with a form for dV/dt and work backward to find V.)

iii. Sketch the Lyapunov function.

iv. Use the corollary to LaSalle's theorem and the Lyapunov function of part (ii) to find as much information as possible about regions of attraction. Use graphs wherever possible.

(Hint: The graph shown in Figure E20.1 may be helpful.)

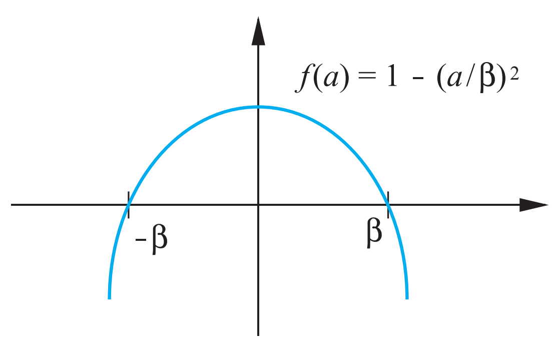
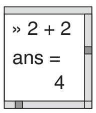


Figure E20.1 Helpful Function for Exercise E20.7

E20.8 Consider the following nonlinear system:

$$da_1/dt = a_2 - a_1((a_1)^4 + 2(a_2)^2 - 10),$$

$$da_2/dt = -(a_1)^3 - 3(a_2)^5((a_1)^4 + 2(a_2)^2 - 10).$$



i. Find any invariant sets. (You may want to simulate this system using MATLAB in order to help identify the invariant sets.)

ii. Using the candidate Lyapunov function shown below and the corollary to LaSalle's theorem, investigate the stability of the invariant sets you found in part (i).

$$V(\mathbf{a}) = ((a_1)^4 + 2(a_2)^2 - 10)^2$$

E20.9 Consider the following system:

$$\frac{d\mathbf{a}}{dt} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\mathbf{a} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

i. Find any equilibrium points.

ii. Find a Lyapunov function and identify attractors and basins of attraction. Use the corollary to Lasalle's theorem and carefully identify and graph the sets Ω_η, G, Z and L.

E20.10 For the nonlinear system

$$da_1/dt = a_2 + a_1(1 - (a_1)^2 - (a_2)^2),$$

$$da_2/dt = -a_1 + a_2(1 - (a_1)^2 - (a_2)^2),$$

we know that the following sets are invariant:

$$\{\mathbf{a} | \mathbf{a} = 0\},$$

$$\{\mathbf{a} | (a_1)^2 + (a_2)^2 = 1\}.$$

The following Lyapunov function is proposed:

$$V(\mathbf{a}) = ((a_1)^2 + (a_2)^2 - 1)^2.$$

Use the corollary to Lasalle's theorem to find out as much as possible about the basins of attraction for the two invariant sets given above, and graph the sets Ω_η, G, Z and L.

E20.11 Consider the system

$$da_1/dt = -\cos(a_2) - a_1 ,$$

$$da_2/dt = a_1 .$$

i. Find any equilibrium points.

ii. The following Lyapunov function is proposed. Show that this is a valid Lyapunov function for use in Lasalle's invariance theorem.

$$V(\mathbf{a}) = \frac{(a_1)^2}{2} + \sin(a_2)$$

iii. Use Lasalle's theorem to find out as much information as you can about the stable equilibrium points and their basins of attraction. (Make a rough sketch of the contour plot for $V(\mathbf{a})$ to assist you.)